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An exploration of weak Heyting algebras: Characterization and properties

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ABSTRACT

This paper explores weak Heyting algebras, an extension of complete Heyting algebras, focusing on characterizing this concept and identifying essential properties in terms of implication operators. The main emphasis is on unraveling the defining features and significance of the novel weak Heyting algebras. We further classify these structures within the context of a complete lattice and extend our findings to the Cartesian product. We facilitate comprehensive comparisons among these structures, by contributing to the broader understanding of weak Heyting algebras in mathematical research.

1. Introduction

Heyting algebras are mathematical structures with broad applications spanning computer science, logic, and topology. Originating as a generalization of Boolean algebras, they were introduced by A. Heyting in the 1930s ([26]). These algebras offer unique properties, making them particularly adept at modelling systems with partial or uncertain information. They have facilitated the development of robust tools for reasoning about computation and decision-making, as evidenced by notable works (e.g., [3,9,24,38]).

Defined as partially ordered sets with a binary operation representing implication, Heyting algebras satisfy a set of axioms that generalize those of Boolean algebras. For any elements x, y, and z in a Heyting algebra H, the following axioms hold:

1. Reflexivity: $x \leq x$.
2. Antisymmetry: If $x \leq y$ and $y \leq x$, then $x = y$.
3. Transitivity: If $x \leq y$ and $y \leq z$, then $x \leq z$.
4. There exists a bottom element 0 such that $0 \leq x$ for all $x \in H$.
5. There exists a top element 1 such that $x \leq 1$ for all $x \in H$.
6. $x \wedge y \leq z$ if and only if $x \leq y \to z$.

Heyting algebras boast a rich algebraic structure and share close connections with other mathematical structures, such as lattices. Their utility extends to formal systems for computation reasoning, including modal logic and type theory, as well as the study of topological spaces and their properties (e.g., [4,10,20,21,29,34]). Moreover, a Heyting algebra is a particular case of a Hilbert algebra. We strongly recommend the book [19] to deep in examples, properties and general information about Hilbert algebras.

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One of the most notable applications of Heyting algebras lies in the domain of constructive mathematics, which explores the foundational aspects of mathematics and the pivotal role of logic in mathematical reasoning. Heyting algebras furnish a natural framework for exploring constructive reasoning, which underscores the importance of constructive proofs and methods of proof. Additionally, they have been instrumental in the development of potent tools for analyzing decision-making and uncertainty, including fuzzy logic and rough sets (see, for instance, [1,25,28,32]).

A complete Heyting algebra is also useful in the study of Formal Concept Analysis ([40]) and uncertain information ([39]). In Formal Concept Analysis, there are two distinct ways to represent the information stored in a context: concepts and implications. When the latter is chosen, it becomes important to establish a sound and complete axiomatic system to work with them. To this end, the Simplification Logic was defined ([31]). Initially, the structure required to build such an axiomatic system was a Boolean algebra. Later, this system was extended to incorporate not only positive values but also multiple ones, creating a fuzzy framework. Implementing a fuzzy framework required a distributive lattice, and a complete Heyting algebra was the structure adopted for this purpose ([6]). However, when dealing with unknown information, non-distributive lattices may arise, making it impossible to define either a Boolean algebra or a complete Heyting algebra. It is worth noticing that there are other structures that allow to work with unknown information by using an intuitionistic logic, as l-Hemi-implicative semilattices ([16] or Hilbert algebras [14,15,22,23] that are useful when you are working with semilattice or even with partially ordered sets with a supremum [11]. However, in order to proof the soundness and completeness of the Simplification logic, it is necessary to consider algebras with some properties that those algebras do not fulfil. This problem highlights the importance of handling the weak Heyting algebra as opposed to other alternatives.

In addition to Hilbert algebras, other generalizations of Heyting algebras have been introduced recently from researchers. For example, in [13], the authors introduced the notion of a weak Heyting algebra in the sense of S. Celani and R. Jansana (originally called "weakly Heyting algebra"). In [37], a generalization introduced as "semi-Heyting algebras" were explored by H.P. Sankappanavar. However, both cases adhere to the requirement of a distributive lattice. Another generalization has been proposed for skew lattices ([2]).

In this paper, the algebraic structure well-known as weak Heyting algebras is explored. Firstly, we introduce fundamental concepts essential for navigating the discourse presented in this article. Through the exposition of preliminary findings and illustrative examples, we aim to furnish readers with a solid foundation in the subject matter. Also, we look for which properties are strictly required for the Simplification Logic. The central focus of our research is on characterizing weak Heyting algebras, a pivotal component of this manuscript. We embark on an exhaustive exploration of this algebraic structure, elucidating its defining properties. Section 4 is dedicated to classifying these structures within the context of a complete lattice. Leveraging the robust Theorem 17, we facilitate comprehensive comparisons among the weak Heyting algebras that can be defined on a fixed complete lattice. Section 5 extends our findings to the Cartesian product, while the final section encapsulates the concluding remarks of this article.

2. Preliminaries and initial results

Let us begin by revisiting fundamental concepts and preliminary results. Let us consider a partially ordered set $P = (L, \leq)$, that is, a set L and a partial order on L denoted by $\leq$. An example of partially ordered set is a lattice, which is defined as follows.

Definition 1 *([7])*. Let $(L, \leq)$ be a partially ordered set, we say that it is a lattice if each two elements $\{x, y\} \in L$ have a join, that is, a least upper bound, denoted by $a \vee b$ and a meet, that is, a greatest lower bound, denoted by $a \wedge b$.

Notice that a lattice $(L, \wedge, \vee)$ does not need to be bound. In our framework, we focus on complete lattices.

Definition 2 *([35])*. Let $(L, \wedge, \vee)$ be a lattice. We say that $(L, \wedge, \vee)$ is a complete lattice if for each $X \subseteq L$, there are $y, z \in L$ such that

$$y = \bigwedge_{x \in X} x \text{ and } z = \bigvee_{x \in X} x.$$

For a complete lattice, we use the notation $(L, \wedge, \vee, 0, 1)$, where

$$0 = \bigwedge_{x \in L} x \text{ and } 1 = \bigvee_{x \in L} x.$$

Therefore, a complete lattice always is bounded.

It is also important in this paper the notion of pseudocomplemented lattice [36].

Definition 3. A lattice $(L, \wedge, \vee, 0, 1)$ is pseudocomplemented if for all $x \in L$ in the lattice there exist $\bigvee\{z \in L \mid z \wedge x = 0\}$ and that element holds that $x \wedge \bigvee\{z \in L \mid z \wedge x = 0\} = 0$. $\bigvee\{z \in L \mid z \wedge x = 0\}$ is usually called the pseudocomplement of x.

In this paper, we focus on Heyting algebras, which are defined as follows.

Definition 4 *([35])*. A Heyting algebra is a bounded lattice $(L, \wedge, \vee, 0, 1)$ endowed by an operation $\to: L \times L \to L$ holding the following axioms:

[H1] $x \wedge (x \to y) = x \wedge y$ for all $x, y \in L$.
[H2] $x \wedge (y \to z) = x \wedge \big((x \wedge y) \to (x \wedge z)\big)$ for all $x, y \in L$.
[H3] $(x \wedge y) \to x = 1$ for all $x, y \in L$.

If $(L, \wedge, \vee, \to, 0, 1)$ is a complete lattice, then we say that the $(L, \wedge, \vee, \to, 0, 1)$ is a complete Heyting algebra. It is well-known that the above definition can be given in terms of an adjoint property.

Proposition 5 *([26]). Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice, $(L, \wedge, \vee, \to, 0, 1)$ is a complete Heyting algebra if and only if the implication operation $\to$ satisfies the following adjoint property for all $a, b, c \in L$:*

$$a \wedge b \le c \qquad \textit{if and only if} \qquad b \le a \to c. \tag{1}$$

Given a complete lattice, a necessary and sufficient condition to guarantee the existence of the implication operation $\to$ which provides a Heyting algebra is:

$$\bigvee \{x \in L \mid a \wedge x \le b\} \in \{x \in L \mid a \wedge x \le b\} \text{ for all } a, b \in L, \tag{2}$$

i.e., the supreme of the set is a maximum. Moreover, it is easy to check that Condition (1) is equivalent to:

$$a \to b = \max\{x \in L \mid a \wedge x \le b\} \text{ for all } a, b \in L. \tag{3}$$

Remark 6. As a consequence of Condition (3), each Heyting algebra is a distributive lattice. Thus, there is no lattice with a Heyting algebra structure that is not distributive.

In [17], the authors presented the following definition for weak Heyting algebras and proved that, considering the dual, is the necessary structure to prove the soundness and the completeness of the Simplification logic. Notice that "a weak Heyting algebra" in the sense of [17] and [33] is not the same of "a weak Heyting algebra" presented in [13] as we show in Example 18. Moreover, S. Celani and R. Jansana considered its definition only for a distributive framework. In this paper, we use weak Heyting algebra when we refer to "a weak Heyting algebra" in the sense of [17] and [33] and we will use weak Heyting algebra in the sense of S. Celani and R. Jansana when we want to refer to their definition.

Definition 7 *([33]).* An algebra $(L, \wedge, \vee, \to, 0, 1)$ is said to be a weak Heyting algebra if $(L, \wedge, \vee, 0, 1)$ is a complete lattice and the following axioms hold:

[wH1] $x \wedge y \neq 0$ implies $x \le y \to (x \wedge y)$ for all $x, y \in L$.
[wH2] $x \to y \ge y$ for all $x, y \in L$.
[wH3] $x \to y = 1$ if and only if $x \le y$ for all $x, y \in L$.
[wH4] $x \wedge (x \to y) = x \wedge y$ for all $x, y \in L$.

Indeed, the notion of a weak Heyting algebra introduced in [33] extends the notion of a complete Heyting algebra. For the sake of completeness, we provide a proof, which was not presented in the original article.

Proposition 8. *If $(L, \wedge, \vee, \to, 0, 1)$ is a complete Heyting algebra, then $(L, \wedge, \vee, \to, 0, 1)$ is a weak Heyting algebra.*

Proof. We have to prove that $(L, \wedge, \vee, \to, 0, 1)$ holds the properties [wH1], [wH2], [wH3] and [wH4].

1. Let us check [wH1]. Consider $x, y \in L$ satisfying $x \wedge y \neq 0$, by [H2],
$$x \wedge \big(y \to (x \wedge y)\big) = x \wedge \big((x \wedge y) \to (x \wedge x \wedge y)\big) = x \wedge \big((x \wedge y) \to (x \wedge y)\big).$$
By [H3], $(x \wedge y) \to (x \wedge y) = 1$, so we have that
$$x \wedge \big(y \to (x \wedge y)\big) = x \wedge 1 = x,$$
that is, $x \le y \to (x \wedge y)$.
2. In order to prove [wH2], assume $x, y \in L$. By [H2], we have that
$$y \wedge (x \to y) = y \wedge \big((x \wedge y) \to (x \wedge x)\big) = y \wedge \big((x \wedge y) \to x\big).$$
By [H3], $(x \wedge y) \to x = 1$. Therefore,
$$y \wedge (x \to y) = y \wedge 1 = y,$$

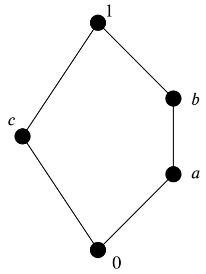

Fig. 1. The lattice N_5.

that is, $y \leq x \to y$.

3. If $x \leq y$, by [H3], $x \to y = 1$. Conversely, let us suppose that $x \to y = 1$. On the one hand,

$$x \wedge (x \to y) = x \wedge 1 = x.$$

On the other hand, by [H1], we have that

$$x \wedge (x \to y) = x \wedge y.$$

Therefore, $x = x \wedge y$ and consequently, $x \leq y$.

4. Finally, [wH4] is equal to [H1]. □

Consequently, each complete Heyting algebra is a weak Heyting algebra. In Examples 10 and 18, we illustrate weak Heyting algebras that do not meet the criteria of complete Heyting algebras, on both non-distributive and distributive lattices. In addition, in this paper, we proof that not all the weak Heyting algebra defined above are Hilbert algebras and not all the Hilbert algebras are weak Heyting algebra in the sense of [17] and [33].

A significative algebraic structure in the context of intuitionistic implications is an Hilbert algebra, which allows to define a residuum operation on the lattice without requiring a Heyting algebra, even if the lattice is not distributive (see [27]).

Definition 9 *([30]).* Let L be a set, $1 \in L$ his upper bound and $\to : L \to L$ an operation. We say that $(L, \to, 1)$ is an Hilbert algebra if it holds the following conditions:

[Hi1] $x \to (x \to y) = 1$ for all $x, y \in L$.
[Hi2] $(x \to (y \to z)) \to ((x \to y) \to (x \to z)) = 1$ for all $x, y, z \in L$.
[Hi3] $x \to y = 1$ and $y \to x = 1$ implies that $x = y$ for all $x, y \in L$.

Observe that a Hilbert algebra can be defined in all partially ordered set with an upper bound $(L, \leq, 1)$ by using the following operation $\to : L \to L$:

$$x \to y = \begin{cases} 1 & \text{if } x \leq y, \\ y & \text{otherwise.} \end{cases}$$

The operation $\to$ defined above is known as the implication given by the order [8,12]. Thus, there exists Hilbert algebras that are not Heyting algebras. However, each Heyting algebra is a Hilbert algebra [19].

Example 10. Let $(L, \wedge, \vee, 0, 1)$ be the Pentagon lattice shown on Fig. 1.

Consider the implication induced by order in N_5, that is, $\to : L \times L \to L$ that is defined by the table

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	c	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

It is an easy task to check that $(L, \wedge, \vee, \to, 0, 1)$ is a weak Heyting algebra and it is a Hilbert algebra but is not a complete Heyting algebra since the Pentagon lattice is not a distributive lattice and, as consequence, by Remark 6, it is not possible to define a complete Heyting algebra on L.

Fig. 2. On the left side, L. On the right side, L'.

3. Characterization of weak Heyting algebras in terms of a residuum operator

Given a complete lattice $(L, \wedge, \vee, 0, 1)$, the purpose of this section is to determine a necessary and sufficient condition on an implication operator $(\to)$ for constructing a weak Heyting algebra.

In order to carry out such a study, we are going to explore the sublattices. In particular, we are going to explore the filters generated by all the elements in the lattices (excluding the 0 which generates the own lattice).

Definition 11 *([7])*. Let $(L, \wedge, \vee)$ be a lattice we say that $F \subseteq L$ is a filter of L if $(F, \wedge, \vee)$ is a lattice and for any $a \in F$ and $b \in L$ we have that $a \vee b \in F$.

If $x \in L$, we denote the filter of L generated by the set $\{x\}$ by F_x. As we will prove, given a lattice L, a necessary and sufficient condition to define a weak Heyting algebra is that F_x has to be pseudocomplemented for all $x \in L$ with $x \neq 0$. To illustrate this, we present a lattice that possesses this property and another that does not.

Example 12. Let us consider the finite lattices L and L' in Fig. 2. First consider L. Observe that, given $x \neq 0 \in L$, if $x \in \{b, d, c, 1\}$ then F_x is a chain and it is a pseudocomplemented lattice. In the case of F_a it is also a pseudocomplemented lattice since the pseudocomplement of c is d, the pseudocomplement of d is c and the pseudocomplemented of a and 1 is a.

However, if we consider L' and take $y \in L'$ we have that F_y is not a pseudocomplemented lattice since if we take any point $x \in L'$ with $x \neq 1$ then x will have not pseudocomplement.

The following proposition justifies that, given a lattice L, a sufficient condition to be able to define a weak Heyting algebra is that F_x needs to be pseudocomplemented for all $x \neq 0 \in L$.

Proposition 13. *Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice assuming that F_x is pseudocomplemented for all $x \neq 0 \in L$. Let us consider the function $\to : L \times L \to L$ defined by:*

$$x \to y = \begin{cases} 1 & \text{if } x = y = 0, \\ 0 & \text{if } x \neq y = 0, \\ \bigvee_{z \in L} \{x \wedge z = x \wedge y\} & \text{if } y \neq 0 \text{ and } x \wedge y \in \{x, y\}, \\ y & \text{otherwise.} \end{cases}$$

Then, $(L, \wedge, \vee, \to, 0, 1)$ is a weak Heyting algebra.

Proof. We have to check that $(L, \wedge, \vee, \to, 0, 1)$ holds the properties [wH1], [wH2], [wH3] and [wH4].

1. In order to prove [wH1], assume $x, y \in L$ satisfying $x \wedge y \neq 0$. Notice that y and $(x \wedge y)$ are comparable elements, hence
$$y \to (x \wedge y) = \bigvee_{z \in L} \{y \wedge z = y \wedge (x \wedge y)\},$$
thus $x \leq y \to (x \wedge y)$.
2. Axiom [wH2] is straightforward from definition of $\to$.
3. Clearly, $0 \to 0 = 1$. If $x \leq y \neq 0$, then
$$x \to y = \bigvee_{z \in L} \{x \wedge z = x \wedge y\} = \bigvee_{z \in L} \{x \wedge z = x\} = 1.$$
4. Given $x, y \in L$, we must check that $x \wedge (x \to y) = x \wedge y$. If one of them is equal to 0, then the equality is trivial. Let us suppose that they are comparable elements, then

$$x \wedge (x \to y) = x \wedge (\bigvee_{z \in L} \{x \wedge z = x \wedge y\}).$$

If $0 \neq x < y$, the equality is straightforward. If $x \geq y \neq 0$, we have $x \wedge y = y$ and since F_y is pseudocomplemented and $\bigvee\{x \wedge z = y\}$ is the pseudocomplement of x in F_y,

$$x \wedge (\bigvee_{z \in L} \{x \wedge z = x \wedge y\}) = x \wedge y.$$

Otherwise, whenever they are not comparable elements, the equality is trivial.

Therefore, $(L, \wedge, \vee, \to, 0, 1)$ is a weak Heyting algebra. □

In order to prove that the condition given is also necessary, first the following lemma will help us to facilitate the understanding of the proof.

Lemma 14. *Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice and $y \neq 0 \in L$ such that F_y is not pseudocomplemented. In this case there exists a $x \in F_y$ such that*

$$x \wedge (\bigvee_{z \in L} \{x \wedge z = y\}) \neq y.$$

Proof. If F_y is not a pseudocomplemented lattice, then there is $x \in F_y$ satisfying one of the following two cases:

1) There does not exist $\bigvee\{z \in F_y \mid z \wedge x = y\}$.
2) $x \wedge \bigvee\{z \in F_y \mid z \wedge x = y\} \neq y$.

First, observe that option 1) is not possible since. Let $x \in F_y$ which means that $x \geq y$, as L is a complete lattice, $\bigvee\{z \in L \mid z \wedge x = y\}$ exist. In addition, if $z \wedge x = y$ and $x \geq y$, then $z \geq y$, as consequence, $z \in F_y$ thus, $\bigvee\{z \in F_y \mid z \wedge x = y\} = \bigvee\{z \in L \mid z \wedge x = y\}$ concluding that $\bigvee\{z \in F_y \mid z \wedge x = y\}$ exist for all $x \in F_y$. □

Proposition 13 establishes a sufficient condition on a complete lattice for the construction a weak Heyting algebra. As we prove in the following theorem, the condition is also necessary.

Theorem 15. *Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice. There exists a map $\to : L \times L \to L$ such that $(L, \wedge, \vee, \to, 0, 1)$ is a weak Heyting algebra if and only if F_y is pseudocomplemented for all $y \neq 0 \in L$.*

Proof. Let $(L, \wedge, \vee)$ be a lattice such that F_x is pseudocomplemented for all $x \neq 0 \in L$, considering the map $\to$ from Proposition 13, we have that $(L, \wedge, \vee, \to, 0, 1)$ is a weak Heyting algebra. Conversely, we assume that there exists a $y \neq 0$ in L such that F_y is not pseudocomplemented, as established Lemma 14, there is, at least, a $x \in F_y$ such that

$$x \wedge (\bigvee_{z \in L} \{x \wedge z = y\}) \neq y.$$

By reductio ad absurdum, let us suppose that there is a map $\to$ satisfying the properties [wH1], [wH2], [wH3] and [wH4]. Taking into account the axiom [wH4], we have that

$$x \wedge (x \to y) = x \wedge y = y,$$

hence $(x \to y) \in \{z \in L \mid x \wedge z = y\}$. Take $t \in \{z \in L \mid x \wedge z = y\}$, since $t \wedge x = y \neq 0$, using [wH1], we obtain that $t \leq x \to (t \wedge x)$, equivalently,

$$t \leq (x \to y).$$

This means that

$$\bigvee_{z \in L} \{x \wedge z = y\} = x \to y,$$

which contradicts that $x \wedge (\bigvee_{z \in L} \{x \wedge z = y\}) \neq y$. □

Remark 16. Observe that Theorem 15 also shows that there exists Hilbert algebras that are not weak Heyting algebra because given a complete lattice $(L, \wedge, \vee, 0, 1)$ such that F_y is not pseudocomplemented for a $y \neq 0 \in L$ then we can define a Hilbert algebra over L just using the implication given by the order. In addition, we can define Hilbert algebras over sets which are not complete lattices but this is not possible for weak Heyting algebras.

The following result provides a characterization of weak Heyting algebras in terms of its implication operator.

Theorem 17. *Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice and a map $\rightarrow: L \times L \longrightarrow L$. Then, $(L, \wedge, \vee, \rightarrow, 0, 1)$ is a weak Heyting algebra if and only if the function $\rightarrow$ holds the following assertions:*

a) $0 \rightarrow 0 = 1$.
b) *If $x \neq 0$, then $x \rightarrow 0 \in \{z \in L \mid x \wedge z = 0\}$.*
c) *If $y \neq 0$ and $x \wedge y \in \{x, y\}$, then $x \rightarrow y = \bigvee\{z \in L \mid x \wedge z = x \wedge y\}$.*
d) *If $x \wedge y \notin \{x, y\}$, then $x \rightarrow y \in \{z \in L \mid x \wedge z = x \wedge y$ and $y \leq z\}$.*

Proof. Let us suppose that $(L, \wedge, \vee, \rightarrow, 0, 1)$ is a weak Heyting algebra.

a) Straightforward from [wH3].
b) From [wH4] we have that

$$x \wedge (x \rightarrow 0) = x \wedge 0 = 0.$$

Therefore, if $x \neq 0$, necessarily $x \rightarrow 0 \in \{z \in L \mid x \wedge z = 0\}$.
c) If $y \neq 0$ and $x \wedge y \in \{x, y\}$ we distinguish two different cases.
Firstly, let us suppose $x \wedge y = x$. By [wH3] we have that

$$x \rightarrow y = 1 = \bigvee\{z \in L \mid x \wedge z = x \wedge y\}.$$

Secondly, let us suppose $x \wedge y = y$. By [wH4],

$$x \wedge (x \rightarrow y) = x \wedge y = y.$$

Hence, $x \rightarrow y \in \{z \in L \mid x \wedge z = x \wedge y\}$. Let us check that $x \rightarrow y$ is the maximum element considering an element $z \in L$ such that $x \wedge z = y$. Since $z \wedge x = y \neq 0$, using [wH1], we have that

$$z \leq x \rightarrow (z \wedge x) = x \rightarrow y,$$

that is, $x \rightarrow y = \bigvee\{z \in L \mid x \wedge z = x \wedge y\}$.
d) If $x \wedge y \notin \{x, y\}$, from [wH4] we have that $x \wedge (x \rightarrow y) = x \wedge y$, which implies that

$$x \rightarrow y \in \{z \in L \mid x \wedge z = x \wedge y\}.$$

Moreover, by [wH2], we know that $y \leq x \rightarrow y$.

Conversely, let us assume that $(L, \wedge, \vee, 0, 1)$ is a complete lattice and the map $\rightarrow: L \times L \rightarrow L$ fulfills *a)*, *b)*, *c)* and *d)*. Let us consider $x, y \in L$.

1. If $x \wedge y \neq 0$, then $x \neq 0$ and $y \neq 0$. We must check [wH1], i.e., that $x \leq y \rightarrow (x \wedge y)$. Notice that $0 \neq x \wedge y \leq y$, so we can use *c)* as follows

$$y \rightarrow (x \wedge y) = \bigvee\{z \in L \mid y \wedge z = y \wedge (x \wedge y)\}.$$

Therefore, for each $z \in L$ satisfying $y \wedge z = x \wedge y$, we have $z \leq y \rightarrow (x \wedge y)$. In particular, $z = x$.
2. Since $\rightarrow$ is completely defined considering the properties *a)*, *b)*, *c)* and *d)* and in each case $x \rightarrow y \geq y$ for all $x, y \in L$, we have that the axiom [wH2] holds.
3. Let us suppose $x \leq y$. If $y = 0$, using *a)*, we have that $x \rightarrow y = 1$. If $y \neq 0$, since $x \wedge y = x$, using *c)* we have that

$$x \rightarrow y = \bigvee\{z \in L \mid x \wedge z = x \wedge y = x\},$$

and trivially 1 is the maximum of the set.
Conversely, let us suppose $x \rightarrow y = 1$. If *a)* happens, clearly $x \leq y$. If $x \neq 0$, $x \rightarrow 0 \neq 1$ and consequently, case *b)* is impossible under our condition. If *c)* happens, then

$$1 = x \rightarrow y = \bigvee\{z \in L \mid x \wedge z = x \wedge y\},$$

hence $x \wedge 1 = x \wedge y$, equivalently, $x \leq y$. Finally, we check that case *d)* is impossible under our condition. By reductio ad absurdum, we assume that *d)* happens. Then,

$$1 = x \rightarrow y = \bigvee\{z \in L \mid x \wedge z = x \wedge y \text{ and } y \leq z\},$$

hence $x \wedge 1 = x \wedge y$, but $x \wedge y \notin \{x, y\}$ in case *d)*.

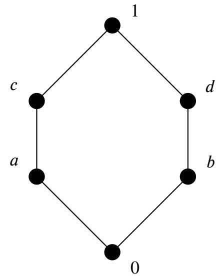

Fig. 3. Horizontal sum of two chains.

4. The proof of [wH4] is analogous to the given proof for [wH2]. □

This characterization can be employed to illustrate several examples of weak Heyting algebras. The following example highlights the distinctions among complete Heyting algebras ([35]), weak Heyting algebras ([33]), weak Heyting algebras in the sense of S. Celani and R. Jansana ([13]) and Hilbert algebras ([30]) on a distributive lattice. It is worth remembering that weak Heyting algebras in the sense of S. Celani and R. Jansana and complete Heyting algebras are defined exclusively on distributive lattices.

Example 18. Let $(L, \wedge, \vee, 0, 1)$ be the complete lattice in Fig. 3, which is a horizontal sum of two chains [5]. For our purposes, note that it is a distributive lattice. Let us consider the following four implication operators:

$\to_1$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	b	1	d	1
b	a	a	1	c	1	1
c	b	a	b	1	d	1
d	a	c	b	c	1	1
1	0	a	b	c	d	1

$\to_2$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	c	1	1
c	d	a	d	1	d	1
d	c	c	b	c	1	1
1	0	a	b	c	d	1

$\to_3$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	1	1	1	1	1	1
b	1	1	1	1	1	1
c	1	1	1	1	1	1
d	1	1	1	1	1	1
1	1	1	1	1	1	1

$\to_4$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	d	1	d	1
b	0	c	1	c	1	1
c	0	a	d	1	d	1
d	0	c	b	c	1	1
1	0	a	b	c	d	1

1. According to Theorem 17, we can conclude that $(L, \wedge, \vee, \to_1, 0, 1)$ is a weak Heyting algebra. However, it does not satisfy the properties of complete Heyting algebras, as evidenced by the equation:

$$b \to_1 0 = a \neq c = \max\{x \in L \mid b \wedge x \leq 0\}.$$

Furthermore, it does not meet the criteria for weak Heyting algebras in the sense defined by S. Celani and R. Jansana (see [13, Definition 3.1]) due to the failure of condition C_2:

$$(a \to_1 0) \wedge (c \to_1 0) = d \wedge b = b \neq c \to_1 0 = b.$$

Finally, it is not a Hilbert algebra as it does not fulfil the condition [Hi2]:

$$(c \to (d \to 0)) \to ((c \to d) \to (c \to 0)) = (c \to a) \to (d \to b) = a \to b = b \neq 1.$$

2. According to Equation (3), $(L, \wedge, \vee, \to_2, 0, 1)$ is the unique complete Heyting algebra that can be defined in L. Therefore, it is also a weak Heyting algebra, a weak Heyting algebra in the sense of [13, Definition 3.1] and Hilbert algebra.
3. Trivially, $(L, \wedge, \vee, \to_3, 0, 1)$ satisfies conditions C_1 to C_4 as defined in [13, Definition 3.1]. However, it does not qualify as a weak Heyting algebra, as indicated by Theorem 17 and the following fact:

$$b \to_3 0 = 1 \notin \{z \in L \mid b \wedge z = 0\}.$$

By Proposition 8, it is not a Heyting algebra. Finally, $(L, \wedge, \vee, \to_3, 0, 1)$ is not a Hilbert algebra as [Hi3] fails.
4. Notice that $(L, \wedge, \vee, \to_4, 0, 1)$ is not a Heyting algebra because Equation (3) fails:

$$b \to_4 0 = 0 \neq a = \max\{x \in L \mid b \wedge x \leq 0\}.$$

However, according to Theorem 17, it qualifies as a weak Heyting algebra, and the axioms C_1 to C_4 from [13, Definition 3.1] can be easily verified. Finally, observe that $(L, \wedge, \vee, \to_4, 0, 1)$ is not a Hilbert algebra as `[Hi2]` fails:

$$(c \to (d \to a)) \to ((c \to d) \to (c \to a)) =$$
$$(c \to c) \to (d \to a) = 1 \to c = c \neq 1.$$

As the above example shows, the term weak complete Heyting algebras defined in [33] are fully independent of the term weak Heyting algebra defined in [13]. The intersection of both structures is not empty and there are weak complete Heyting algebras that are not weak Heyting algebra in the sense of [13, Definition 3.1] and viceversa.

4. Properties of weak Heyting algebras

This section is dedicated to presenting properties of weak Heyting algebras and facilitating a comparison of these structures. One of these properties is given as a consequence of Theorem 17: an implication of a weak Heyting algebra is upper bounded by the implication of a complete Heyting algebra as the following proposition established.

Proposition 19. *Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice. If $(L, \wedge, \vee, \to_H, 0, 1)$ is a complete Heyting algebra and $(L, \wedge, \vee, \to_W, 0, 1)$ is a weak Heyting algebra, then*

$$x \to_W y \;\leq\; x \to_H y$$

for all $x, y \in L$.

Proof. Let us assume $x, y \in L$, taking into account Theorem 17, we are going to consider the following disjoint cases:

1. If $x \leq y$, clearly
$$x \to_H y = 1 = x \to_W y.$$
2. If $x \neq 0$ and $y = 0$, we have that
$$x \to_H 0 = \max\{z \in L \mid z \wedge x = 0\}$$
and
$$x \to_W 0 \in \{z \in L \mid x \wedge z = 0\}.$$
Therefore, $x \to_W 0 \leq x \to_H 0$.
3. If $0 \neq y < x$, we have that
$$x \to_H y = \max\{z \in L \mid z \wedge x \leq y\}$$
and
$$x \to_W y \in \{z \in L \mid x \wedge z = y\}.$$
Since $\{z \in L \mid x \wedge z = y\} \subseteq \{z \in L \mid z \wedge x \leq y\}$, we conclude that $x \to_W y \leq x \to_H y$.
4. Finally, if x and y are not comparable, we have that
$$x \to_H y = \max\{z \in L \mid z \wedge x \leq y\}$$
and
$$x \to_W y \in \{z \in L \mid x \wedge z = x \wedge y \text{ and } y \leq z\}.$$
Since $\{z \in L \mid x \wedge z = x \wedge y \text{ and } y \leq z\} \subseteq \{z \in L \mid z \wedge x \leq y\}$, we have that $x \to_W y \leq x \to_H y$. □

Conversely, we can build the infimum of the weak Heyting algebras.

Proposition 20. *Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice satisfying that F_y is pseudocomplemented for all $y \neq 0 \in L$. The following implication operator is the smallest one that generates a weak Heyting algebra:*

$$x \rightsquigarrow y = \begin{cases} 0 & \text{if } y = 0 \text{ and } x \neq 0, \\ y & \text{if they are not comparable,} \\ \bigvee\{z \in L \mid x \wedge z = x \wedge y\} & \text{otherwise.} \end{cases} \tag{4}$$

Proof. It is straightforward from Theorem 17. □

Consequently, given a complete Heyting algebra, we can provide a lower and upper bound of each implication operator of a weak Heyting algebra.

Corollary 21. *Let $(L, \wedge, \vee, \rightarrow_H, 0, 1)$ be a complete Heyting algebra and $(L, \wedge, \vee, \rightarrow_W, 0, 1)$ a weak Heyting algebra. For all $x, y \in L$, we have that*

$$x \rightsquigarrow y \leq x \rightarrow_W y \leq x \rightarrow_H y,$$

where $\rightsquigarrow$ is defined in the Equation (4).

In order to illustrate the maps $\rightsquigarrow$ and $\rightarrow_H$, we present the next example.

Example 22. Consider the finite lattice L presented in Fig. 2 (on the left side). We present the implication operations $\rightsquigarrow$ and $\rightarrow_H$.

$\rightsquigarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	1	1	1
b	0	a	1	c	1	1
c	0	d	b	1	d	1
d	0	c	b	c	1	1
1	0	a	b	c	d	1

$\rightarrow_H$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	b	1	b	1	1	1
b	c	c	1	c	1	1
c	b	d	b	1	d	1
d	0	c	b	c	1	1
1	0	a	b	c	d	1

Theorem 17 is illuminating. Fixing a complete lattice, thanks to this result, we are able to determine the number of weak Heyting algebras.

Proposition 23. *Let $(L, \wedge, \vee, 0, 1)$ be a complete lattice which F_y is pseudocomplemented for all $y \neq 0 \in L$. Let us consider the following values:*

1. *For each $x \in L$, the cardinal of the set $\{z \in L \mid x \wedge z = 0\}$ is denoted by $\mathcal{O}_x$.*
2. *For each $x, y \in L$, the cardinal of the set $\{z \in L \mid x \wedge z = x \wedge y$ and $y \leq z\}$ is denoted by $\mathcal{I}(x, y)$.*

Then, the number of maps $\rightarrow : L \times L \rightarrow L$ such that $(L, \wedge, \vee, \rightarrow, 0, 1)$ is a weak Heyting algebra is

$$\prod_{x \neq 0} \mathcal{O}_x \cdot \prod_{x \nsim y} \mathcal{I}(x, y),$$

where $x \nsim y$ denotes that x is not comparable to y.

Proof. According to Theorem 17, a map $\rightarrow$ satisfies:

a) $0 \rightarrow 0 = 1$.
b) If $x \neq 0$, then $x \rightarrow 0 \in \{z \in L \mid x \wedge z = 0\}$.
c) If $y \neq 0$ and $x \wedge y \in \{x, y\}$, then $x \rightarrow y = \bigvee \{z \in L \mid x \wedge z = x \wedge y\}$.
d) If $x \wedge y \notin \{x, y\}$, then $x \rightarrow y \in \{z \in L \mid x \wedge z = x \wedge y$ and $y \leq z\}$.

Therefore, the total number of implications defined on L is derived from all possible combinations. For each $x \neq 0$, Case b contributes $\mathcal{I}(x, y)$ implications. Case a and case c are deterministic. Consequently, the total number of weak Heyting algebras is obtained as their product:

$$\prod_{x \neq 0} \mathcal{O}_x \cdot \prod_{x \nsim y} \mathcal{I}(x, y). \quad \square$$

In the following example, the number of weak Heyting algebras is calculated for one of the complete lattices presented in Fig. 2.

Example 24. Let us consider the finite lattice L presented in Fig. 2 (on the left side). Firstly, we calculate the value of $\mathcal{O}_x$ assuming $x \neq 0$.

$$\begin{aligned}
\mathcal{O}_a &= |\{0,b\}| = 2,\\
\mathcal{O}_b &= |\{0,a,c\}| = 3,\\
\mathcal{O}_c &= |\{0,b\}| = 2,\\
\mathcal{O}_d &= |\{0\}| = 1,\\
\mathcal{O}_1 &= |\{0\}| = 1.
\end{aligned}$$

Now, we calculate the value of $\mathcal{I}(x,y)$ assuming $x \nsim y$.

$$\begin{aligned}
\mathcal{I}(a,b) &= |\{b\}| = 1,\\
\mathcal{I}(b,a) &= |\{a,c\}| = 2,\\
\mathcal{I}(b,c) &= |\{c\}| = 1,\\
\mathcal{I}(c,b) &= |\{b\}| = 1,\\
\mathcal{I}(c,d) &= |\{d\}| = 1,\\
\mathcal{I}(d,c) &= |\{c\}| = 1.
\end{aligned}$$

Therefore, the number of implications $\to$ which define a weak Heyting algebra $(L,\wedge,\vee,\to,0,1)$ is

$$\prod_{x\neq 0}\mathcal{O}_x \cdot \prod_{x\nsim y}\mathcal{I}(x,y) = 2\cdot 3\cdot 2\cdot 2 = 24.$$

5. Cartesian product of weak Heyting algebras

In this section, we study the Cartesian Product of weak Heyting algebras. First, given two complete lattices, $(L_1,\wedge_1,\vee_1,0_1,1_1)$ and $(L_2,\wedge_2,\vee_2,0_2,1_2)$ we denote by $L_1\times L_2$ to the Cartesian product of L_1 and L_2, that is,

$$L_1\times L_2 = \{(x,y) \mid x\in L_1, y\in L_2\}.$$

Notice that the orders in L_1 and L_2 define an order in $L_1\times L_2$. Thus, if $A=(x,y), B=(t,z)\in L_1\times L_2$, we say $A\sqsubseteq B$ if and only if $x\leq_1 t$ and $y\leq_2 z$. As a consequence, we can define pointwise $\sqcap$ and $\sqcup$ taking into account the binary operators $\wedge_1,\wedge_2$ and $\vee_1,\vee_2$, respectively. Moreover, $(L_1\times L_2,\sqcap,\sqcup,(0_1,0_2),(1_1,1_2))$ is a complete lattice (refer to [18]).

In the same way, given two weak Heyting algebras $(L_1,\wedge_1,\vee_1,\to_1,0_1,1_1)$ and $(L_2,\wedge_2,\vee_2,\to_2,0_2,1_2)$ we can define an implication on $(L_1\times L_2,\sqcap,\sqcup,(0_1,0_2),(1_1,1_2))$. Given $(x,y),(t,z)\in L_1\times L_2$, we consider the implication of $(x,y)\to(t,z)$ which is defined pointwise:

$$(x,y)\to(t,z) = (x\to_1 t, y\to_2 z).$$

In this section, we study the cases where $(L_1\times L_2,\sqcap,\sqcup,\to,(0_1,0_2),(1_1,1_2))$ is a weak Heyting algebra. In general, $(L_1\times L_2,\sqcap,\sqcup,\to,(0_1,0_2),(1_1,1_2))$ is not a weak Heyting algebra, as the following example shows.

Example 25. Let $(L,\wedge,\vee,\to,0,1)$ the weak Heyting algebra defined in Example 10. We define $(L\times L,\wedge,\vee,\to,(0,0),(1,1))$ such that $\wedge,\vee,\to$ are defined as pointwise extension of $\wedge,\vee,\to$. Please, notice that here we are choosing the same notation for the implication on L and the implication on $L\times L$. Let us see that $(L\times L,\wedge,\vee,\to,(0,0),(1,1))$ is not a weak Heyting algebra. Consider $(b,b)\in L\times L$ and $(0,b)\in L\times L$, trivially $(0,b)\sqcap(b,b)=(0,b)$ and, by Theorem 17, we have

$$(b,b)\to(0,b) = \bigvee\{z\in L\times L \mid (b,b)\wedge z = (0,b)\} = (c,1).$$

However,

$$(b,b)\to(0,b) = (b\to 0, b\to b) = (0,1).$$

Thus, it is not a weak Heyting algebra.

We have seen that the Cartesian product of weak Heyting algebras does not preserve the inner structure. The following result characterizes this fact.

Theorem 26. *Let $(L_1,\wedge_1,\vee_1,\to_1,0_1,1_1)$ and $(L_2,\wedge_2,\vee_2,\to_2,0_2,1_2)$ be two weak Heyting algebras. Then, $(L_1\times L_2,\sqcap,\sqcup,\to,(0_1,0_2),(1_1,1_2))$ is a weak Heyting algebra if and only if*

$$x\to_i 0_i = \max\{z\in L_i \mid z\wedge_i x = 0_i\} \text{ for all } x\in L \text{ satisfying } x\neq 0_i, \tag{5}$$

for each $i\in\{1,2\}$.

Proof. Without loss of generality, suppose that $\to_2$ does not hold Equation (5). We are going to prove that $(L_1\times L_2,\sqcap,\sqcup,\to,(0_1,0_2),(1_1,1_2))$ is not a weak Heyting algebra.

We consider $A=(y,0_2)$ and $B=(0_1,0_2)$, it is clear that $(0_1,0_2)\sqcap(y,0_2)=(0_1,0_2)$ and by Theorem 17, we have that

$$(0_1,y)\to(0_1,0_2) = \bigvee\{z\in L_1\times L_2 \mid (0_1,y)\wedge z = (0_1,0_2)\} = (1,\bigvee\{z\in L_2 \mid y\wedge z = 0_2\})$$

However, we have that

$$(0_1, y) \to (0_1, 0_2) = (0_1 \to_1 0_1, y \to_2 y),$$

which is a contradiction taking into account Theorem 17.

Conversely, let us suppose that $(L_i, \wedge_i, \vee_i, \to_i, 0_i, 1_i)$ is a weak Heyting algebras for each $i \in \{1,2\}$ holding Equation (5). We are going to prove that $(L_1 \times L_2, \sqcap, \sqcup, \to, (0_1, 0_2), (1_1, 1_2))$ is a weak Heyting algebra. In order to check it, we study the four assertions of Theorem 17.

a) Trivially, $(0_1, 0_2) \to (0_1, 0_2) = (0_1 \to_1 0_1, 0_2 \to_2 0_2) = (1_1, 1_2)$.

b) Assume $A = (x, y), B = (0_1, 0_2) \in L_1 \times L_2$. By Equation (5) and Theorem 17, we have that for all $x \in L_1$ and $y \in L_2$

$$(x, y) \to (0_1, 0_2) = (\bigvee\{z \in L_1 \mid x \wedge_1 z = 0_1\}, \bigvee\{t \in L_2 \mid y \wedge_2 t = 0_2\}).$$

Thus, $(x, y) \to (0_1, 0_2) \in \{z \in L_1 \times L_2 \mid (x, y) \wedge z = (0_1, 0_2)\}$.

c) Let us consider $A = (x, y), B = (z, t) \in L_1 \times L_2$ such that $B \neq (0_1, 0_2)$ and A is comparable with B. We distinguish two cases: if $z \neq 0_1$ and $t \neq 0_2$ or if $z = 0_1$ or $t = 0_2$. In the first case, by Theorem 17, we conclude that

$$(x, y) \to (z, t) = (\bigvee\{s \in L_1 \mid x \wedge_1 s = z\}, \bigvee\{r \in L_2 \mid y \wedge_2 r = t\}).$$

Equivalently,

$$(x, y) \to (z, t) = \bigvee\{r \in L_1 \times L_2 \mid r \sqcap (x, y) = (z, t)\}.$$

In the case that $z = 0_1$ or $t = 0_2$, we can use Equation (5) and Theorem 17 in order to conclude

$$(x, y) \to (z, t) = \bigvee\{r \in L_1 \times L_2 \mid r \sqcap (x, y) = (z, t)\}.$$

d) Let us consider $A = (x, y), B = (z, t) \in L_1 \times L_2$ such that A is not comparable with B. We distinguish three cases:

1) x is not comparable with z and y is not comparable with t.
Observe that $z \neq 0_1$ and $t \neq 0_2$ because 0_i is comparable with all the elements in the lattice L_i. In this case, we have that

$$(x, y) \to (z, t) = (x \to_1 z, y \to_2 t).$$

Using Theorem 17, we have that

$$x \to_1 z \in \{s \in L_1 \mid x \wedge_1 s = x \wedge_1 z \ \text{ and } \ z \leq_1 s\},$$

and

$$y \to_2 t \in \{r \in L_2 \mid y \wedge_2 r = y \wedge_2 t \ \text{ and } \ t \leq_2 r\}.$$

Thus, we can conclude that $(x, y) \to (z, t)$ belongs to the set

$$\{(s, r) \in L_1 \times L_2 \mid (x, y) \sqcap (s, r) = (x, y) \sqcap (z, t) \ \text{ and } \ (z, t) \sqsubseteq (s, r)\}.$$

2) x is comparable with z and y is not comparable with t or x not comparable with z and y comparable with t.
Without loss of generality, we will prove the case x comparable with z and y not comparable with t. We have that $(x, y) \to (z, t) = (x \to_1 z, y \to_2 t)$. We have that

$$y \to_2 t \in \{r \in L_2 \mid y \wedge_2 r = y \wedge_2 t \ \text{ and } \ t \leq_2 r\},$$

and for the value of $x \to_1 z$ we study two disjoint cases: If $z = 0_1$ by Equation (5),

$$x \to_1 z = \max\{s \in L_1 \mid s \wedge_1 x = 0_1\}.$$

If $z \neq 0_1$, using item *c*) of Theorem 17, we have the following equality:

$$x \to_1 z = \max\{s \in L_1 \mid s \wedge_1 x = x \wedge_1 z\}.$$

Thus, we can conclude that $(x, y) \to (z, t)$ belongs to the set

$$L_1 \times L_2 \mid (x, y) \sqcap (s, r) = (x, y) \sqcap (z, t) \ \text{ and } \ (z, t) \sqsubseteq (s, r)\}.$$

3) x comparable with z and y comparable with t. Under our premise, A is not comparable to B, therefore $x \neq z$ and $y \neq t$. Moreover, notice that just one of the following is possible:
i) $x >_1 z$ and $y <_2 t$.
ii) $x <_1 z$ and $y >_2 t$.

Since the proof for both of them is similar, let us assume $x <_1 z$ and $y >_2 t$. Firstly, observe that $z \neq 0_1$ because $x <_1 z$ and there is no element $x \in L$ with $x < 0_1$, thus, using Theorem 17, we have that

$$x \to_1 z = \max\{s \in L_1 \mid s \wedge_1 x = x \wedge_1 z\}.$$

If $t = 0_2$, using Equation (5), we have

$$y \to_2 t = \max\{r \in L_2 \mid r \wedge_2 y = 0_2\}.$$

If $t \neq 0_2$, using Theorem 17, we have

$$y \to_2 t = \max\{r \in L_2 \mid r \wedge_2 y = y \wedge_2 t\}.$$

Thus, we conclude that $(x, y) \to (z, t)$ belongs to the set

$$\{(s, r) \in L_1 \times L_2 \mid (x, y) \sqcap (s, r) = (x, y) \sqcap (z, t) \text{ and } (z, t) \sqsubseteq (s, r)\}. \quad \square$$

Trivially, the previous theorem can be extended to more components.

Corollary 27. *Let $\{(L_i, \wedge_i, \vee_i, \to_i, 0_i, 1_i)\}_{i=1}^n$ be a finite family of weak Heyting algebras. Then,*

$$\left(L_1 \times L_2 \times \dots L_n, \sqcap, \sqcup, \to, (0_1, \dots, 0_n), (1_1, \dots, 1_n)\right)$$

is a weak Heyting algebra if and only if

$$x \to_i 0_i = \max\{z \in L_i \mid z \wedge_i x = 0_i\} \text{ for all } x \in L \text{ satisfying } x \neq 0_i, \tag{6}$$

for each $i \in \{1, \dots, n\}$.

To end this section, we will provide an example of a weak Heyting algebra that satisfies Equation (5), and another one that does not.

Example 28. Let us consider the Pentagon N_5 presented in Fig. 1 and the following two implication operators on N_5:

$\to_1$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	c	1
b	0	a	1	c	1
c	0	a	b	1	1
1	0	a	b	c	1

$\to_2$	0	a	b	c	1
0	1	1	1	1	1
a	c	1	1	c	1
b	a	a	1	c	1
c	a	a	b	1	1
1	0	a	b	c	1

According to Theorem 17, $(L, \wedge, \vee, \to_i, 0, 1)$ qualifies as a weak Heyting algebra for both $i = 1$ and $i = 2$. In the case of $i = 1$, it does not satisfy Equation (5), whereas for $i = 2$, it does.

6. Conclusions

We have proof that, given a lattice, a condition to guarantee the existence of a weak Heyting algebra is given in terms of the filters generated by just one element of the lattice, in particular, the condition is that the filters need to be pseudocomplemented (Theorem 15) and obtained a characterization of them in terms of its implication (Theorem 17). We have provided a method to create this operator whenever all the filters generated by just one element of the lattice and Section 4 studies several properties of weak Heyting algebras. Moreover, we show some significant corollaries. For instance, fixed a complete lattice, we can determine the number of weak Heyting algebras (Proposition 23), and we can even determine an upper bound and lower bound of their implications (Corollary 21). The main result of the last section characterizes weak Heyting algebras in the Cartesian product (Theorem 26), which can be extended to more components (Corollary 27).

Due to the novel notion of a weak Heyting algebra, there are many open research lines. The authors suggest the following studies to delve deeper into the applicability of these algebraic structures:

1. It is well-known that Heyting algebras are fruitful in several fields. However, its existence depends on its domain. Weak Heyting algebras can be considered in wider domains, as non-distributive lattices. We hope that our characterization and results can be an useful tools to manage these new situations.
2. Theorem 26 can be useful to define a Simplification logic in a fuzzy framework by using intuitionistic intervals.
3. From a theoretical point of view, we have shown some differences among several extensions of a Heyting algebra (see Example 18). A more analytical study of their properties and how they are related is significantly interesting.

CRediT authorship contribition statement

Francisco Pérez-Gámez: Writing – review & editing, Writing – original draft, Validation, Resources, Methodology, Investigation, Conceptualization. **Carlos Bejines:** Writing – review & editing, Writing – original draft, Resources, Investigation, Conceptualization.

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Data availability

No data was used for the research described in the article.

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