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A Measure of Consistency for Fuzzy Logic Theories

Nicolás Madrid | Manuel Ojeda-Aciego*

[1]Depto. Matemática Aplicada, Universidad de Málaga, Málaga, Spain

Correspondence
*Manuel Ojeda-Aciego, ETSI Informática.
Blv. Louis Pasteur 35. 29071 Málaga, Spain.
Email: aciego@uma.es

Present Address
ETSI Informática. Blv. Louis Pasteur 35.
29071 Málaga, Spain

Abstract

Fuzzy logic has shown to be a suitable framework to handle contradictions in which, unsurprisingly, the notion of inconsistency can be defined in different ways. This paper starts with a short survey of different ways to define the notion of inconsistency in fuzzy logic systems. As a result, we provide a first notion of inconsistency by means of the absence of models. Subsequently, we define two measures of consistency that belong purely to the fuzzy paradigm; in the sense that both measures coincide with the crisp notion of consistency when the set of truth values is $\{0, 1\}$. Accordingly, we can state that the two provided measures of consistence are notions of consistence based on degrees, bringing back the spirit of fuzzy logic into the notion of consistency.

KEYWORDS:
Inconsistency; Fuzzy logic; Measure of consistency

1 | INTRODUCTION

Since its introduction, fuzzy sets and fuzzy logic have shown to be an interesting research topic. One can find lots of papers ranging from the development of algebraic theories of fuzzy structures or the underlying mathematics of fuzzy logic, to fuzzy modelling or automated control in terms of sets of fuzzy rules. Just to show a few examples, from the theoretical standpoint, in[1] a new method is presented for locating fuzzy optimal (nondominated) solutions of a fully fuzzy linear programming problem with inequality constraints and triangular fuzzy numbers; in[2] the notion of relational Galois connection is extended to be applied between transitive fuzzy directed graphs in a framework in which the components of the connection are crisp relations satisfying certain reasonable properties; in[3] the focus is on the study of systems of bipolar fuzzy relational equations, specifically, on the solvability and the algebraic structure of the set of solutions of these bipolar equations systems; in[4] the behavior of a construction method for an interval-valued fuzzy relation built from a fuzzy relation is studied. From the practical standpoint, in[5] one can find an effective method for obtaining a final conclusion from an inconsistent database of if-then rules; in[6] it is shown how a control application can leverage (even) from a set of inconsistent rules; and in[7] we can see a fuzzy logic-based mathematical model of a sequence of earthquakes using tools from fuzzy reasoning.

This work is concerned with the two latter applications, namely, the use of fuzzy reasoning and inconsistency. It is well-known that inconsistency collapses the system of classical logic because of the *ex falso quodlibet* principle (i.e. any statement can be proven from an inconsistent or contradictory set of premises). However, there are systems which are tolerant to inconsistency up to certain degree, and one of them is based on fuzzy logic. There are many inconsistency-tolerant approaches in fuzzy logic: just as an example, in[8] a number of expansions of the fuzzy logic MTL (monoidal t-norm logic) are introduced by means of primitive operators for consistency and inconsistency.

Although allowing inconsistency in fuzzy rule-bases is suitable for applications where refinement is unacceptable due to time constraints or impossible due to lack of additional data or knowledge[9], there are also works focused on repairing inconsistencies:

for instance[10] puts forward solutions to the problems of inconsistencies in fuzzy spatio-temporal XML documents, or[11] proposes a graphical method to visualize and repair inconsistencies for fuzzy reciprocal preference relations, or[12] proposes a method for repairing the inconsistency of a fuzzy preference relation and make it become one with weak transitivity.

Being the fuzzy realm a matter of degrees, a number of papers have focused on measuring the degree of inconsistency of a set of fuzzy rules, and a number of different inconsistency indices have been introduced. For instance[13] introduces the so-called knowledge-based consistency index for deriving priorities from fuzzy pairwise comparison matrices in multiple-criteria decision-making problems; other approaches introduce means for both measuring and repairing inconsistency, for example[14] presents a family of measures aimed at determining the amount of inconsistency in knowledge bases with graded truth and considers minimal adjustments in the truth-degrees of the propositions necessary to make the knowledge-base to be consistent within a given frame (in that case the Łukasiewicz semantics); last but not least[15] deals with the definition of measures of inconsistency in the residuated-logic-programming paradigm under the fuzzy answer set semantics and provides a soft mechanism to control the amount of information inferred, thus, controlling the inconsistencies by modifying slightly the truth values of some rules. The number of possible measures of inconsistency that can be found in the literature somehow suggests the existence of a problem with inconsistency in a fuzzy setting, namely, its definition: there is not a consensus on how to interpret inconsistency in a fuzzy system.

In this paper we briefly survey the main properties and equivalent characterisations of inconsistency in classical (crisp) logic and then, we focus on, under of point of view, the more natural way to define inconsistency in a logic theory, namely: the absence of models. This consideration as definition of inconsistency keeps some of the most important properties of inconsistency in the fuzzy paradigm, e.g., explosive reasoning. However, we also lose an important issue, we lose degrees; which is the soul of fuzzy logic. For such a reason, we propose a generalization of consistence by means of two measures of consistency. Specifically, we define two measures of consistency that belong purely to the fuzzy paradigm. In other words, both measures coincide with the crisp notion of consistency when the set of truth values is $\{0, 1\}$. Moreover, we provide a set of properties for both measures of consistency in order to motivate the use of them to represent the consistency of fuzzy logic theories.

The structure of the paper is the following: in Section 2 we provide the basic notions of fuzzy logic based on residuated lattices. Then, in Section 3 we present different possibilities to extend the notion of inconsistency in the fuzzy paradigm and motivate the absence of models as the strongest case of inconsistency. In Section 4 we define a measure of consistency that determines how much consistent is a formula with respect to a fuzzy logic theory. Subsequently, in Section 5, we define a measure of consistency for fuzzy logic theories. Finally, in Section 6, we resent some conclusions and future works.

2 | PRELIMINARIES

In this section we recall the basic notions of fuzzy logic. For the sake of the presentation, we restrict the description to the syntax and semantics of propositional logic and we avoid the description of different inference systems because it could bewilder the reader and lose the main point of the paper: the presentation of characteristic inconsistent measures in fuzzy logics.

A fuzzy logic system is usually founded on a residuated lattice, which is a tuple $(L, \leq, *, \rightarrow, 0, 1)$ such that:

- $(L, \leq)$ is a complete bounded lattice, with top and bottom element 1 and 0, respectively.
- $(L, *, 1)$ is a commutative monoid with unit element 1.
- $(*, \rightarrow)$ forms an adjoint pair, i.e.

$$z \leq (y \rightarrow x) \quad \text{if and only if} \quad y*z \leq x$$

Before presenting the syntax of propositional fuzzy logic, let us show some useful properties of residuated lattices.

Theorem 1 ([16]). Let $(L, \leq, *, \rightarrow, 0, 1)$ be a residuated lattice. Then for all $a, b, c \in L$:

i) $a \leq (b \rightarrow (a * b))$

ii) $a * (a \rightarrow b) \leq b$

iii) $a * (b \rightarrow c) \leq (a \rightarrow b) \rightarrow c$

iv) $a \rightarrow (b \rightarrow c) = (a * b) \rightarrow c = b \rightarrow (a \rightarrow c)$

v) $a * \left(\sup_{i\in\mathbb{I}} b_i\right) = \sup_{i\in\mathbb{I}} \left(a * b_i\right)$

vi) $a \to \left(\inf_{i\in\mathbb{I}} b_i\right) = \inf_{i\in\mathbb{I}} \left(a \to b_i\right)$

vii) $\wedge_{b\in L}((a \to b) \to b)) = a$

viii) $1 \to a = a$

ix) $a \to b = 1$ if and only if $a \leq b$.

The syntax of a fuzzy logic based on a residuated lattice $(L, \leq, *, \to, 0, 1)$ is given by the following inductive definition based on a set of propositional symbol Π:

- every propositional symbol in Π is a well-formed formula;
- every element of L is a well-formed formula;
- if ϕ and ψ are well-formed formula then:
  - $\varphi * \psi$ is a well-formed formula;
  - $\varphi \to \psi$ is a well-formed formula;
  - $\varphi \wedge \psi$ is a well-formed formula;
  - $\varphi \vee \psi$ is a well-formed formula;
  - $\neg\varphi$ is a well-formed formula.

The semantics is given as follows:

Definition 1. Let $(L, \leq, *, \to, 0, 1)$ be a residuated lattice. An *interpretation* is a mapping $I : \Pi \to L$. The domain of an interpretation I can be extended inductively to any well-formed formula as follows:

- $I(a) = a$ for all $a \in L$;
- $I(\varphi * \psi) = I(\varphi) * I(\psi)$;
- $I(\varphi \to \psi) = I(\varphi) \to I(\psi)$;
- $I(\varphi \wedge \psi) = \inf\{I(\varphi), I(\psi)\}$;
- $I(\varphi \vee \psi) = \sup\{I(\varphi), I(\psi)\}$;
- $I(\neg\varphi) = I(\varphi) \to 0$;

where φ and ψ are well-formed formulas.

The definition of theory, model of a theory and logical consequence in a theory are given as follows:

Definition 2. Let $(L, \leq, *, \to, 0, 1)$ be a residuated lattice.

- A *fuzzy logic theory* Γ is a set of well-formed formulas.
- A *model* of Γ is an interpretation M such that $M(\psi) = 1$ for all $\psi \in \Gamma$.
- We say that a formula ψ *is a consequence of* Γ (denoted by $\Gamma \models \psi$) if $M(\psi) = 1$ for all model M of Γ.

Fuzzy logic is a matter of degrees but by the definition above perhaps degrees do not look evident. By properties exposed in Theorem 1, we can represent the satisfiability of a formula ψ in at least a degree $a \in L$ by the new formula $\alpha \to \psi$. Note that an interpretation I is a model of $a \to \psi$ if and only if $M(p) \geq a$. Similarly, we can represent the satisfiability of a formula ψ in almost a degree $a \in L$ by the new formula $\psi \to a$. Hence, we can represent the satisfiability of a formula exactly in degree $a \in L$ by the theory $\Gamma = \{\psi \to a\ ,\ a \to \psi\}$.

The reader interested in a deeper knowledge of fuzzy logic is referred to[16,17]. In that references the reader can find out some complete and correct inference systems based on the Modus Ponens inference rule for the syntax and semantics described in this section. For that reason and for the sake of presentation, we use also the terminology *"ψ is inferred from Γ"* if $\Gamma \models \psi$.

In this paper, we restrict our attention to residuated lattices defined on the unit interval $[0,1]$. In the examples we will often use some of the main residuated lattices on $[0,1]$ given by the following operators:

- Gödel residuated lattice $([0,1], \leq, *_G, \rightarrow_G, 0, 1)$
$$x *_G y = \min\{x, y\} \qquad x \rightarrow_G y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise.} \end{cases}$$
- Product residuated lattice $([0,1], \leq, *_P, \rightarrow_P, 0, 1)$
$$x *_P y = x \cdot y \qquad x \rightarrow_P y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise.} \end{cases}$$
- Łukasiewicz residuated lattice $([0,1], \leq, *_Ł, \rightarrow_Ł, 0, 1)$
$$x *_Ł y = \max\{x + y - 1, 0\} \qquad x \rightarrow_Ł y = \min\{1 + y - x, 1\}$$

3 | INCONSISTENCY OF A FUZZY LOGIC THEORY IN THE STRONG SENSE

There are several ways to define an inconsistent logic theory in Boolean Logic, or *in the crisp case* as is usually stated in the fuzzy literature. For instance,[18] presents the following five possibilities;

- **explosive reasoning**: A theory is inconsistent if every formula can be inferred from it.
- **contradictory inference**: A theory is inconsistent if a contradictory set of formulas is inferred. The most usual example is the inference of two opposite literals p and $\neg p$.
- **the inference of a contradictory formula**: A theory is inconsistent if the formula $\bot$ is a consequence of it. We can change in this definition the formula $\bot$ by any contradictory formula i.e., a formula that is false for any interpretation; e.g. $p \wedge \neg p$.
- **trivial reasoning**: A theory is inconsistent if we can infer a formula such that is not a tautology and none of its propositional symbols appear in the theory.
- **lack of models**: A theory is inconsistent if it has no models.

All the definitions above are equivalent in the crisp case, but they are not so in other paradigms; for instance, in paraconsistent logics[19] it is possible to infer contradictory formulas without the burden of the explosive reasoning.

Although those definitions are not equivalent in general, there are clear relationships between them. For instance, if a logic theory (non necessarily crisp, but sound and complete) has no models, then it fulfils all the other definitions. This motivates the following notion of inconsistency in the strongest sense.

Definition 3. A logic theory Γ is said to be *inconsistent* if it has no models.

It is worth pointing out that the definition of inconsistency above is equivalent to the one related to *explosive reasoning* and *trivial reasoning*.

Theorem 2 ([16]). Let Γ be a fuzzy logic theory. Then the following statements are equivalent:

- Γ is inconsistent.
- $\Gamma \models \psi$ for all formula ψ.
- There exists a formula ψ whose propositional symbols do not appear in Γ such that $\Gamma \models \psi$ and $\emptyset \not\models \psi$.

An alternative definition of inconsistency could have been given based on contradictory formulas, but its fuzzy version is a little more controversial. The following definition is given in order to keep the equivalence of Definition 3 with the inference of a set of contradictory formulas.

Definition 4. Let $(L, \leq, *, \rightarrow)$ be a residuated lattice. We say that a set $\{\psi_i\}_{i\in\mathbb{N}}$ of formulas is *contradictory* if is inconsistent as a logic theory.

Firstly is worth noting that the previous definition depends on the residuated lattice considered.

Example 1. Let us consider the formulas $0.5 \rightarrow p$ and $0.5 \rightarrow \neg p$. If we analyze the theory $\Gamma = \{0.5 \rightarrow p\ ;\ 0.5 \rightarrow \neg p\}$ in the product logic then we have that Γ is inconsistent and therefore the formulas $0.5 \rightarrow p$ and $0.5 \rightarrow \neg p$ are contradictory. On the other hand, if we analyze Γ under the Łukasiewicz logic we have that the interpretation M given by $M(p) = 0.5$ is a model of Γ. As a result, $0.5 \rightarrow p$ and $0.5 \rightarrow \neg p$ are not contradictory in Łukasiewicz logic.

We say that a formula ψ is contradictory if the singleton $\{\psi\}$ is contradictory. As we have shown above, the contradictory formulas differs from the residuated lattice considered; however, there is a set of formulae that is contradictory with respect to every residuated lattice. Perhaps $\bot$ (i.e., falsehood) is the most representative among them. The following result shows the equivalence among Definition 3, the inference of a contradictory formula, and inference of a set of contradictory formulas.

Theorem 3. Let Γ be a fuzzy logic theory. Then the following statements are equivalent

a) Γ is inconsistent.

b) There exists a contradictory set of formulas $\{\psi_i\}_{i\in\mathbb{N}}$ such that $\Gamma \models \psi_i$ for all $i \in \mathbb{N}$.

c) There exists a contradictory formula ψ such that $\Gamma \models \psi$.

Proof. a) implies b) This is a direct consequence of Theorem 2.

b) implies c). Let us assume that there exists a contradictory set of formulas $\{\psi_i\}_{i\in\mathbb{N}}$ such that $\Gamma \models \psi_i$ for all $i \in \mathbb{N}$ and let us show that ψ defined by $\bigwedge_{i\in\mathbb{N}} \psi_i$ is a contradictory formula such that $\Gamma \models \psi$. Firstly, for all model M of Γ we have that $M(\psi_i) = 1$ for all $i \in \mathbb{N}$ which implies $M(\psi) = 1$; i.e., $\Gamma \models \psi$. Secondly, let I be an interpretation. Since $\{\psi_i\}_{i\in\mathbb{N}}$ is a contradictory set of formulas there exist $j \in \mathbb{N}$ such that $I(\psi_j) \neq 1$. As a result, there is not model of $\{\psi\}$ and therefore ψ is contradictory.

c) implies a). Let us assume by reduction ad absurdum that Γ has a model M. Then, from $\Gamma \models \psi$ we have that $M(\psi) = 1$, which implies that the set $\{\psi\}$ has a model (as a logic theory) and contradicts that ψ is contradictory. ☐

Corollary 1. Let Γ be a fuzzy logic theory. Then, $\Gamma \models \bot$ if and only if Γ is inconsistent.

4 | DEGREES OF CONSISTENCY FOR FORMULAS WITH RESPECT TO FUZZY LOGIC THEORIES

Roughly speaking, in the previous section we have shown that the notion of inconsistency, originally given for crisp logic theories, can be extended similarly for fuzzy logic theories; actually, Theorems 2 and 3 show that the behaviour of inconsistent fuzzy theories is similar to the behaviour of crisp logic theories under Definition 3 based on absence of models. However, such a procedure lacks of the underlying softness in fuzzy logic, namely, it is crisp notion and not a gradual one.

In the rest of the paper we aim at measuring "how far from being consistent an inconsistent theory is." In the context of crisp logic, some authors have followed such an idea by means of modifications of the set of formulas under consideration[18]. Note that in crisp logic theories the only reasonable way to measure how far is a logic theory to be consistent is by removing formulae (the more formulae you need to remove to make it consistent, the more inconsistency). However, the soul of fuzzy logic is quite different from that of crisp logic, and we can proceed also in a different way.

To better grasp the underlying idea in the measures of consistency presented hereafter, let us recall that in crisp logic, given a consistent logic theory Γ and a formula ψ such that $\{\psi\} \cup \Gamma$ is inconsistent, then $\Gamma \models \neg\psi$. The negation $\neg\psi$ is equivalent to the formula $\psi \rightarrow 0$, or in other words, $\Gamma \models \neg\psi$ is equivalent to ensure that the formula ψ has truth degree 0 for all the models of Γ. Whereas in crisp logic the only truth value different from 0 is 1, in fuzzy logic there are possibly infinitely many truth-values different from 0. As a consequence, we could consider different degrees of negation for ψ by means of formulas such as $\psi \rightarrow \alpha$. Moreover, note that $\Gamma \models \psi \rightarrow \alpha$ implies that ψ has at most truth degree α for all the models of Γ. Therefore, it makes sense to measure the inconsistency of ψ with respect to Γ by means of the minimum degree α such that $\Gamma \models \psi \rightarrow \alpha$. This way of measuring inconsistency is explained by the following idea: given a consistent fuzzy logic theory Γ and two inconsistent formulas ψ and φ with respect to Γ, we say that ψ is less inconsistent than φ if there are models of Γ that assign ψ truth-values close to 1 whereas all model of Γ assigns to φ truth-values close to 0.

In view of the comments in the previous paragraph, in this section we introduce the notion of α-feasible formula with respect to a theory. The value α is an element in the residuated lattice that determines an upper bound for the truth value of the formula. In that respect, a 0-feasible formula with respect to a logic theory Γ is a formula that have truth-degree 0 for all model of Γ whereas, a 1-feasible formula is evaluated to 1 by some model of Γ. In other words, the value α can be understood as a degree of compatibility. In the example below, we show the idea underlying the notion of α-feasibility with respect to empty theory.

Example 2. Let us consider the Łukasiewicz residuated lattice $([0,1], \leq, *_{Ł}, \rightarrow_{Ł}, 0, 1)$ and the following two formulas: $\bot$ and $p \wedge \neg p$. Both formulas are contradictory, since none of them has a model; i.e., for all interpretation I we have that $I(\bot) \neq 1$ and $I(p \wedge \neg p) \neq 1$. However, the inherent contradiction of the former formula can be considered stronger than in the latter, since for all interpretation I we have that $I(\bot) = 0$, which means it is in all cases completely false (0-feasible), whereas there are interpretations I such that $I(p \wedge \neg p) = 0.5$, which means that $p \wedge \neg p$ maybe be half true (at least 0.5-feasible).

The following definition introduces the notion of *α-feasible formula*, which will be used later to represent degrees of contradiction of formulas.

Definition 5. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and consider $\alpha \in L$. A formula ψ is said to be *α-feasible* w.r.t. Γ if $\Gamma \models \psi \rightarrow \alpha$. If $\Gamma = \emptyset$ we say that ψ is just *α-feasible*.

It is worth noting that a formula may be α-feasible for different values $\alpha \in L$; for instance all formula is 1-feasible. In order to better understand the degree of consistency provided below, it is important to note that the consistency is not given by only one α-feasibility, but by all the ones satisfied by a formula. The following results shows that if a formula ψ is α-feasible w.r.t Γ then, there is not model of Γ that assigns to ψ a truth-degree greater than α. In this way, the value α in a α-feasible formula ψ represents an upper bound of its "degree of satisfiability."

Proposition 1. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$. A formula ψ is *α-feasible* w.r.t Γ if and only if $M(\psi) \leq \alpha$ for all model M of Γ.

Proof. ψ is *α-feasible* w.r.t Γ if and only if $M(\psi \rightarrow \alpha) = 1$ for all model M of Γ, which is equivalent by Theorem 1 (item 9), to say that $M(\psi) \leq \alpha$ for all model M of Γ. $\square$

The following result shows that the set of contradictory formulas is precisely the complementary of the set of those formulas that are just 1-feasible formulas. Consequently, the set of consistent formulas coincides with the set of formulas that are only 1-feasible and moreover, for each $\alpha \neq 1$, the set formed by α-feasible formulas determines a class of contradictory formulas.

Corollary 2. A formula ψ is *α-feasible* with $\alpha \neq 1$ if and only if ψ is contradictory.

The notion of α-feasibility is related to the notion of inconsistency as follows.

Proposition 2. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and consider $\alpha \in L$. If ψ is *α-feasible* w.r.t Γ then, $\Gamma \cup \{\beta \rightarrow \psi\}$ is inconsistent for all $\beta > \alpha$.

Proof. Assume that ψ is *α-feasible* w.r.t Γ and $\beta > \alpha$. If Γ is inconsistent, the result is straightforward. Otherwise, let M be a model of Γ. Then, by Proposition 1 we have that $M(\psi) \leq \alpha$ and by hypothesis that $M(\psi) \leq \alpha < \beta$, which implies that $M(\beta \rightarrow \psi) \neq 1$. Therefore, $\Gamma \cup \{\beta \rightarrow \psi\}$ is inconsistent. $\square$

As a result, we can obtain the following proposition which shows a certain monotonicity with respect to the value α in the notion of α-feasibility.

Proposition 3. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$. If a formula ψ is *α-feasible* w.r.t Γ then, ψ is *β-feasible* w.r.t Γ for all $\beta \geq \alpha$.

Now, by using Propositions 2 and 3, we can consider the value α in the notion of α-feasibility as the opposite of a degree of contradiction and inconsistency. Actually, we can define the following measure of consistency.

Definition 6. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and let ψ be a well-formed formula. The *degree of consistency of ψ with respect to* Γ is defined as the value:

$$\mathrm{Mc}(\psi, \Gamma) = \min\{\alpha \mid \psi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\}$$

From the definition above, the measure Mc interprets the α-feasibility as *"the lower the value α may be, the less consistency between the formula and the logic theory"*. In this line, $\mathrm{Mc}(\psi, \Gamma) = 1$ means $\{\psi\} \cup \Gamma$ is consistent and $\mathrm{Mc}(\psi, \Gamma) < 1$ means inconsistency. Note also that a degree of inconsistency can be defined through a negation operator n and the value of $n(\mathrm{Mc}(\psi, \Gamma))$. However, we have decided to continue with the terminology and measure of consistency, despite the fact that literature focuses on the term inconsistency, in order to simplify the approach and make it as natural as possible (an avoiding the need to incorporate some negation operator. The first result aims at showing that such a measure is well defined, that is, that the minimum in the definition of Mc always exists.

Lemma 1. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and $\{\alpha_i\}_{i\in\mathbb{I}} \subseteq L$. If a formula ψ is α_i*-feasible* w.r.t Γ for all $i \in \mathbb{I}$, then ψ is $\left(\inf_{i\in\mathbb{I}} \alpha_i\right)$*-feasible* w.r.t Γ.

Proof. Let $\{\alpha_i\}_{i\in\mathbb{I}} \subseteq L$ and let ψ be an α_i*-feasible* formula w.r.t Γ for all $i \in \mathbb{I}$. That means that $\Gamma \models \psi \rightarrow \alpha_i$ for all $i \in \mathbb{I}$ or equivalently, that $M(\psi) \leq \alpha_i$ for all model M of Γ and all $i \in \mathbb{I}$. This implies that $M(\psi) \leq \inf_{i\in\mathbb{I}} \alpha_i$ for all model M of Γ, or equivalently that $\Gamma \models \psi \rightarrow \inf_{i\in\mathbb{I}} \alpha_i$; i.e., ψ is $\inf_{i\in\mathbb{I}} \alpha_i$*-feasible* w.r.t Γ. ☐

Corollary 3. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and let ψ be a well-formed formula, then $\min\{\alpha \mid \psi$ is α*-feasible* w.r.t $\Gamma\}$ exists.

From the definition of the measure Mc and Lemma 1 we have also the following straightforward result.

Corollary 4. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and let ψ be a well-formed formula, then ψ is $\mathrm{Mc}(\psi, \Gamma)$-feasible w.r.t. Γ.

As a consequence of the previous corollary and Proposition 1 we have the following interesting result, which provides a semantic meaning to the measure Mc.

Corollary 5. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and let ψ be a well-formed formula, then $M(\psi) \leq \mathrm{Mc}(\psi, \Gamma)$ for all model M of Γ.

The following theorem presents some interesting properties related to the measure Mc which support the use of the term "measure of consistency."

Theorem 4. Let Γ and Γ' be fuzzy logic theories defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and let ψ and ϕ be well-formed formulas. Then:

a) $\mathrm{Mc}(\psi, \Gamma) \geq \mathrm{Mc}(\psi, \Gamma \cup \Gamma')$; i.e., more formulas in the theory can only reduce the consistency of ψ.

b) $\mathrm{Mc}(\psi \wedge \varphi, \Gamma) \leq \inf\{\mathrm{Mc}(\psi, \Gamma), \mathrm{Mc}(\varphi, \Gamma)\}$;

c) If $\Gamma \models \varphi$, then $\mathrm{Mc}(\psi * \varphi, \Gamma) = \mathrm{Mc}(\psi, \Gamma)$;

d) If $\Gamma \models \varphi$, then $\mathrm{Mc}(\psi \wedge \varphi, \Gamma) = \mathrm{Mc}(\psi, \Gamma)$;

e) If $\Gamma \cup \{\psi\}$ is consistent then, $\mathrm{Mc}(\psi, \Gamma) = 1$;

f) If Γ is consistent and $\Gamma \models \psi$ then, $\mathrm{Mc}(\psi, \Gamma) = 1$;

g) If $\mathrm{Mc}(\psi, \Gamma) \neq 1$ then, $\Gamma \cup \{\psi\}$ is inconsistent;

h) If Γ is inconsistent then, $\mathrm{Mc}(\psi, \Gamma) = 0$ for all formula ψ;

i) $\mathrm{Mc}(\top, \Gamma) = 0$ if and only if Γ is consistent;

j) $\mathrm{Mc}(\bot, \Gamma) = 1$.

Proof. **a**) Let ψ be an α-feasible formula w.r.t. Γ. Then, for all model M of Γ we have that $M(\psi) \leq \alpha$. Since every model of $\Gamma \cup \Gamma'$ is by definition also a model of Γ, we have that ψ is an α-feasible formula w.r.t. $\Gamma \cup \Gamma'$ as well, i.e.,

$$\{\alpha \mid \psi \text{ is } \alpha\text{-}\textit{feasible} \text{ w.r.t } \Gamma\} \subseteq \{\alpha \mid \psi \text{ is } \alpha\text{-}\textit{feasible} \text{ w.r.t } \Gamma \cup \Gamma'\}.$$

As a result $\mathrm{Mc}(\psi, \Gamma) \geq \mathrm{Mc}(\psi, \Gamma \cup \Gamma')$.

b) Let $\alpha \in L$ such that ψ and φ are α-feasible. Then, by Proposition 1, for all model M of Γ we have that $M(\psi) \leq \alpha$ and $M(\varphi) \leq \alpha$. As a consequence, for all model M of Γ we have that $M(\psi \wedge \varphi) \leq \alpha$ as well. In other words, we have:

$$\{\alpha \mid \psi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\} \cap \{\alpha \mid \varphi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\} \subseteq \{\alpha \mid \psi \wedge \varphi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\}.$$

Therefore, by using that $\inf\{A \cap B\} = \inf\{\inf\{A\}, \inf\{B\}\}$, for all $A, B \subseteq L$, we have

$$\inf\Big\{\min\{\alpha \mid \psi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\}, \min\{\alpha \mid \varphi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\}\Big\} \geq \min\{\alpha \mid \psi \wedge \varphi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\}.$$

That is, the inequality $\mathrm{Mc}(\psi \wedge \varphi, \Gamma) \leq \inf\{\mathrm{Mc}(\psi, \Gamma), \mathrm{Mc}(\varphi, \Gamma)\}$.

c) Note firstly that given a model M of Γ we have

$$M(\psi * \varphi) = M(\psi) * M(\varphi) = M(\psi).$$

since $M(\varphi) = 1$ by hypothesis. Therefore, given $\alpha \in L$, we have that the inequality $M(\psi * \varphi) \leq \alpha$ is equivalent to $M(\psi) \leq \alpha$. As a result, by Proposition 1 we have that

$$\{\alpha \mid \psi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\} = \{\alpha \mid \psi * \varphi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\},$$

and then, by definition of Mc, we have that $\mathrm{Mc}(\psi * \varphi, \Gamma) = \mathrm{Mc}(\psi, \Gamma)$.

d) Identical to item **c**)

e) Let us assume that $\Gamma \cup \{\psi\}$ is consistent and let us show that ψ is α-feasible w.r.t. Γ if and only if $\alpha = 1$. If ψ is α-feasible w.r.t. Γ then, for all model M of Γ we have that $M(\psi) \leq \alpha$. Note that since $\Gamma \cup \{\psi\}$ is consistent, there exists a model M of Γ such that $M(\psi) = 1$. Therefore, $\alpha = 1$ necessarily. As a consequence

$$\{\alpha \mid \psi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\} = \{1\}$$

and $\mathrm{Mc}(\psi, \Gamma) = 1$.

f) It is a consequence of **c**) since in this case $\Gamma \cup \{\psi\}$ is consistent as well.

g) It is equivalent to **c**)

h) Assume that Γ is inconsistent and let us show that $\mathrm{Mc}(\psi, \Gamma) = 1$. Since Γ is inconsistent, from Theorem 2 we have that $\Gamma \models \varphi$ for all formula φ, specifically $\Gamma \models \psi \to 0$. As a result $0 \in \{\alpha \mid \psi \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\}$ and therefore $\mathrm{Mc}(\psi, \Gamma) = 0$.

i) One implication is a consequence of **c**) and the other is a consequence of **f**) by considering the case $\psi = \top$.

j) We only need to show that $0 \in \{\alpha \mid \bot \text{ is } \textit{α-feasible} \text{ w.r.t } \Gamma\}$, which is straightforward since $M(\bot) \leq 0$ holds for all model of Γ; i.e., $\Gamma \models \bot \to 0$ holds always (also in the case Γ is inconsistent). ☐

The following example shows that, as one could expect, the measure Mc depends on the underlying residuated lattice considered.

Example 3. Let us consider the theory $\Gamma = \{p \to q\ ,\ q \to 0.5\}$ defined on three different residuated lattices namely, Łukasiewicz, product and Gödel, and let us analyze the inconsistency of $0.7 \to p$ w.r.t. Γ in each residuated lattice. Firstly, let us note the following facts which will be used in the rest of the example.

1. In any of the three cases, the interpretation M_0 defined by $M_0(p) = 0$ and $M_0(q) = 0.5$ is a model of Γ, hence Γ is consistent.
2. The models of Γ have common features in the three residuated lattices. Specifically, every model M of Γ (independently of the underlying lattice considered) satisfies that $M(p) \leq M(q)$ and $M(q) \leq 0.5$; and as a result, $M(p) \leq 0.5$ as well.
3. Note also that $M_1(p) = M_1(q) = 0.5$ is a model of Γ.
4. In the three residuated lattices the formula $0.7 \to p$ is contradictory w.r.t. Γ; i.e., $\Gamma \cup \{0.7 \to p\}$ is inconsistent. This is because if M is a model of $\{p \to q\ ,\ q \to 0.5\ ,\ 0.7 \to p\}$, then necessarily M must satisfy the inequalities $M(p) \leq M(q)$, $M(q) \leq 0.5$ and $M(p) \geq 0.7$; which is impossible.

Gödel To measure the inconsistency of $0.7 \to p$ w.r.t. Γ in this logic we have to determine the minimum $\alpha \in [0,1]$ such that $0.7 \to p$ is α-feasible w.r.t. Γ; or in other words, we have to determine the set of those $\alpha \in [0,1)$ such that $M\big((0.7 \to p) \to \alpha\big) = 1$ for all model of Γ. Note that in Gödel logic we have:

$$M\big((0.7 \to p) \to \alpha\big) = 1 \iff M(0.7 \to p) \leq \alpha \iff M(p) \leq \alpha \text{ and } M(p) \leq 0.7 \iff M(p) \leq \alpha$$

where in the last equivalence we have used that for all model M we have that $M(p) \leq 0.5$. By the same inequality, we can assert that $0.7 \to p$ is α-feasible w.r.t. Γ for all $\alpha \geq 0.5$. Now, by the third remark given above, we know that the there exists a model such that $M(p) = 0.5$. As a result, we can conclude that if $\alpha < 0.5$, then $0.7 \to p$ is not α-feasible w.r.t. Γ since the inequality $M(p) \leq \alpha$ does not hold for the mentioned model. Therefore, the measure of consistency is given by the minimum of the interval $[0.5, 1]$, that is $\mathrm{Mc}(0.7 \to p, \Gamma) = 0.5$.

Product As in the previous case, we have to study firstly the equality $M\big((0.7 \to p) \to \alpha\big) = 1$ with $\alpha \in [0,1)$. In product logic, this is equivalent to

$$M\big((0.7 \to p) \to \alpha\big) = 1 \iff M(0.7 \to p) \leq \alpha \iff \frac{M(p)}{0.7} \leq \alpha \text{ and } M(p) \leq 0.7 \iff M(p) \leq \alpha \cdot 0.7$$

where in the last equivalence we have used that for all model M we have that $M(p) \leq 0.5$. By following a similar reasoning than for the Gödel logic, we can conclude that $0.7 \to p$ is α-feasible if and only if $\alpha \geq \frac{5}{7}$. As a result, under the product residuated lattice we have $\mathrm{Mc}(0.7 \to p, \Gamma) = \frac{5}{7}$.

Łukasiewicz In this case, given a model M of Γ in Łukasiewicz logic we have the equivalences:

$$M\big((0.7 \to p\big) \to \alpha) = 1 \iff M(0.7 \to p) \leq \alpha \iff \min\{0.3 + M(p), 1\} \leq \alpha$$

Since $M(p) \leq 0.5$ for all model of Γ and there is a model M of Γ such that $M(p) = 0.5$, the least value of α we can choose to guarantee $\Gamma \models (0.7 \to p) \to \alpha$ has to satisfy $0.3 + 0.5 = 0.8 \leq \alpha$; so in Łukasiewicz logic $\mathrm{Mc}(0.7 \to p, \Gamma) = 0.8$.

The following result shows that the consistency of $\Gamma \cup \{\psi\}$ is equivalent to $\mathrm{Mc}(\psi, \Gamma) = 1$ in many of the usual fuzzy frameworks, specifically, in finite and totally ordered lattices.

Proposition 4. Let $(L, \leq, *, \to)$ be a residuated lattice with L a finite and totally ordered lattice, let ψ be a well-formed formula and let Γ be a logic theory. Then $\mathrm{Mc}(\psi, \Gamma) = 1$ if and only if $\Gamma \cup \{\psi\}$ is consistent.

Proof. Note that one of the implications is a consequence of Theorem 4 item e).

For the other implication, let us assume that $\Gamma \cup \{\psi\}$ is inconsistent and we will show that $\mathrm{Mc}(\psi, \Gamma) \neq 1$ reasoning by cases:

- If Γ is inconsistent, then we have the result by Theorem 4 item h), since in this case $\mathrm{Mc}(\psi, \Gamma) = 0$.
- If Γ is consistent, let us consider the following value for α:

  $$\alpha = \sup\{M(\psi) \mid M \text{ is a model of } \Gamma\}.$$

  Note that the supremum above is defined on a non-empty subset of L because we are assuming that Γ is consistent. Now, since L is finite and totally ordered, we have that such a supremum is actually a maximum. Moreover, α is necessarily smaller than 1, since otherwise there would exist a model M of Γ such that $M(\psi) = 1$ contradicting the fact that $\Gamma \cup \{\psi\}$ is inconsistent. Finally, by the definition of α we have that $M(\psi) \leq \alpha$ for every model of M, which implies that $\Gamma \models \psi \to \alpha$; i.e., ψ is *α-feasible* w.r.t Γ and, therefore, $\mathrm{Mc}(\psi, \Gamma) \leq \alpha < 1$ as we wanted to prove.

□

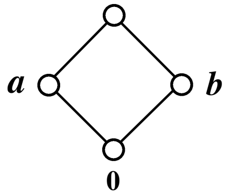
The following examples show that both conditions imposed on the residuated lattice above are necessary.

Example 4. Let us consider the diamond residuated lattice $(L, \wedge, *, \to)$ given by the Hasse diagram in the picture, together with the operators:

$$x * y = \inf\{x, y\} \qquad x \to y = \sup\{z \mid x * z \leq y\}$$

Let us consider also the theory Γ consisting in the following formula:

$$\big((p \wedge a) \to 0\big) \vee \big((p \wedge b) \to 0\big)$$



Note that Γ has three models namely, M_0, M_1 and M_2 given by $M_0(p) = 0$, $M_1(p) = a$ and $M_2(p) = b$, respectively. As a result, $\Gamma \cup \{1 \to p\}$ is inconsistent; despite this, let us show that $\mathrm{Mc}(1 \to p, \Gamma) = 1$.

For this, it is enough to show that $\Gamma \vDash (1 \to p) \to \alpha$ if and only if $\alpha = 1$. Firstly, for $\alpha = 1$ obviously $\Gamma \vDash (1 \to p) \to 1$. Secondly, let us show that for any $\alpha \neq 1$ there exists a model M of Γ such that $M((1 \to p) \to \alpha) \neq 1$.

Case $\alpha = a$: We have that $M_2((1 \to p) \to a) = (1 \to M_2(p)) \to a = (1 \to b) \to a = b \to a = 0$.

Case $\alpha = b$: We have that $M_1((1 \to p) \to b) = (1 \to M_1(p)) \to b = (1 \to a) \to b = a \to b = 0$.

Case $\alpha = 0$: We have that $M_1((1 \to p) \to 0) = (1 \to M_1(p)) \to b = (1 \to a) \to 0 = a \to 0 = b$.

Therefore, we can conclude that $\mathrm{Mc}(1 \to p, \Gamma) = 1$.

Example 5. Let us consider the Gödel residuated lattice $([0,1], \wedge, *_G, \to_G)$ and the logic theory Γ given by the following single formula

$$\bigvee_{n \in \mathbb{N}} \left(p \to \frac{n}{n+1}\right)$$

To begin with, let us show that any interpretation M such that $M(p) < 1$ is a model of Γ. It is easy to check that $M(p) = 1$ is not a model of Γ. Otherwise, assuming $M(p) < 1$, there exists $\bar{n} \in \mathbb{N}$ such that $M(p) \leq \frac{\bar{n}}{\bar{n}+1}$ and then,

$$M\left(p \to \frac{\bar{n}}{\bar{n}+1}\right) = 1 \qquad \Rightarrow \qquad M\left(\bigvee_{n \in \mathbb{N}} \left(p \to \frac{n}{n+1}\right)\right) = 1$$

Let us consider the formula $1 \to p$. Then, $\Gamma \cup \{1 \to p\}$ is obviously inconsistent, since any model M of $1 \to p$ should satisfy $M(p) = 1$. Let us show now that $\mathrm{Mc}(1 \to p, \Gamma) = 1$ which is equivalent to show that $\Gamma \vDash (1 \to p) \to \alpha$ if and only if $\alpha = 1$. Firstly, note that $\Gamma \vDash (1 \to p) \to 1$ holds straightforwardly. Secondly, assume $\alpha < 1$ and let us show that there is a model M of Γ such that $M\big((1 \to p) \to \alpha\big) < 1$. This is easy, because we can consider a model of Γ, say M, such that $M(p) > \alpha$. For such a model we have

$$M\big((1 \to p) \to \alpha\big) = \big(1 \to M(p)\big) \to \alpha = M(p) \to \alpha = \alpha < 1$$

Therefore we have $\mathrm{Mc}(1 \to p, \Gamma) = \min\{1\} = 1$.

5 | MEASURES OF CONSISTENCY FOR FUZZY LOGIC THEORIES.

In this section we follow a common procedure in the definition of measures of (in-)consistency in crisp logic: given a logic theory Γ, we consider subsets of consistent formulas contained in Γ. At this point, in crisp logic we can measure the inconsistency by considering the ratio or the absolute number of removed formulas. Interestingly enough, in a fuzzy environment, we can proceed differently: for instance, we can measure the consistency Mc of the removed formulas with respect to the remaining ones. In order to properly understand the rationale in the following definition, let us consider an arbitrary fuzzy logic theory of three formulas $\Gamma = \{\psi_1, \psi_2, \psi_3\}$. Assume that Γ is inconsistent and that ψ_1 is non-contradictory. The question is how consistent are formulas ψ_2 and ψ_3 in the logic theory $\Gamma^* = \{\psi_1\}$. One could think about measuring separately the consistence for both formulas and then aggregate them, but that is not possible because $\{\psi_1, \psi_2\}$ and $\{\psi_1, \psi_3\}$ could be consistent and, then, both measures would be 1. Therefore, the only reasonable option is to combine ψ_2 and ψ_3 into one formula. Note, that the consistency of $\{\psi_1, \psi_2, \psi_3\}$ is given by assuming on the one hand ψ_1 and, on the other hand, *both* ψ_2 *and* ψ_3 at the same time; the latter means that we are assuming $\psi_2 \wedge \psi_3$. Therefore, the consistency generated by ψ_2 and ψ_3 in the logic theory $\Gamma^* = \{\psi_1\}$ is $\mathrm{Mc}(\psi_2 \wedge \psi_3, \Gamma^*)$.

Definition 7. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \to)$, then we define the measure of consistency $\mathrm{Mc}^*(\Gamma)$ as:

$$\mathrm{Mc}^*(\Gamma) = \sup \left\{ \mathrm{Mc}\left(\bigwedge_{\psi_i \in \Gamma \setminus \Gamma^*} \psi_i \,, \Gamma^* \right) \mid \Gamma^* \subseteq \Gamma \text{ is consistent} \right\}.$$

At first sight, the reader may think that considering all the set of combinations of consistent sub-theories of a fuzzy logic theory Γ may be unpractical, however, the following result shows that only one consistent subtheory must be considered to compute the measure Mc^*, namely, the empty theory.

Theorem 5. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$, then:

$$\mathrm{Mc}^*(\Gamma) = \mathrm{Mc}\left(\bigwedge_{\psi_i \in \Gamma} \psi_i\ , \emptyset\right).$$

Proof. If Γ is consistent, then straightforwardly we have $\mathrm{Mc}^*(\Gamma) = 1 = \mathrm{Mc}\left(\bigwedge_{\psi_i \in \Gamma} \psi_i\ , \emptyset\right)$. Let us assume that Γ is inconsistent and let $\Gamma^* \subset \Gamma$ be a consistent fuzzy logic subtheory of Γ. Let us consider the three formulas

- $\psi = \bigwedge_{\psi_i \in \Gamma} \psi_i$
- $\varphi = \bigwedge_{\varphi \in \Gamma^*} \varphi$
- $\overline{\varphi} = \bigwedge_{\varphi \in \Gamma \setminus \Gamma^*} \varphi$

which, obviously, satisfy $\psi = \varphi \wedge \overline{\varphi}$ and $\Gamma^* \models \varphi$. Applying now Theorem 4 items a) and d) , we have

$$\mathrm{Mc}(\psi, \emptyset) = \mathrm{Mc}(\varphi \wedge \overline{\varphi}, \emptyset) \geq \mathrm{Mc}(\varphi \wedge \overline{\varphi}, \Gamma^*) = \mathrm{Mc}(\overline{\varphi}, \Gamma^*)$$

Now, let us note that both $\emptyset$ and Γ^* are consistent subtheories of Γ; as a result, $\mathrm{Mc}(\psi, \emptyset)$ and $\mathrm{Mc}(\overline{\varphi}, \Gamma^*)$ are elements of the set where the supremum is computed for the definition of Mc^* and that such inequality is true for all consistent subtheory $\Gamma^* \subseteq \Gamma$. Hence, we can conclude that such supremum is in fact a maximum reached in $\mathrm{Mc}(\psi, \emptyset)$; in other words, $\mathrm{Mc}^*(\Gamma) = \mathrm{Mc}(\psi, \emptyset)$. ☐

As a direct consequence of the previous theorem, we have the following corollary that shows that Mc^* satisfies those properties of a measure of consistency (i.e., the opposite properties of a measure of inconsistency).

Corollary 6. Let Γ and Γ' be fuzzy logic theories defined on a residuated lattice $(L, \leq, *, \rightarrow)$, then:

a) $\mathrm{Mc}^*(\Gamma) \geq \mathrm{Mc}^*(\Gamma \cup \Gamma')$;

b) If Γ is consistent then, $\mathrm{Mc}^*(\Gamma) = 1$;

c) If $\mathrm{Mc}^*(\Gamma) \neq 1$ then, Γ is inconsistent;

d) If L is finite and totally ordered, then $\mathrm{Mc}^*(\Gamma) = 1$ implies Γ is consistent.

Proof. Consequence of Theorem 4,Theorem 5 and Proposition 4. ☐

The measure of consistency Mc^* is related to the k-models which, in turn, are related to the so-called x-consistency[20] and α-cuts models[21]. The underlying idea in the k-models is to guarantee the satisfiability of formulas in at least truth-degree $k \in L$, and it is given in the following definition.

Definition 8. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$ and consider $k \in L$. We say that an interpretation M is a k-model of Γ if $M(\psi) \geq k$ for all $\psi \in \Gamma$.

The k-models were introduced in the context of Fuzzy Logic Programming aiming at providing "partial" models to a given inconsistent logic program (i.e., fuzzy logic theory). Later, it was proved that the existence of models is guaranteed by very general requirements in Fuzzy Logic Programming[22] and, then, k-models faded away. However, in the general context we are working on in this approach, the existence of models cannot be guaranteed easily and k-models may be valuable here. The following result relates the measure Mc^* with k-models.

Theorem 6. Let Γ be a fuzzy logic theory defined on a residuated lattice $(L, \leq, *, \rightarrow)$. If $\mathrm{Mc}^*(\Gamma) = \alpha$ then, there is not β-model of Γ with $\beta > \alpha$.

Proof. Let $\mathrm{Mc}^*(\Gamma) = \alpha$ and let $\beta > \alpha$. By Theorem 5, we have that $\mathrm{Mc}\left(\bigwedge_{\psi_i \in \Gamma} \psi_i\ , \emptyset\right) = \alpha$. Then, by applying Corollary 5 we have that for all interpretation $I\left(\bigwedge_{\psi_i \in \Gamma} \psi_i\right) \leq \alpha$, since any interpretation is a model of the empty logic theory and, hence, this prevents the possible existence of a β-model.

☐

In the following example we compute Mc^* for a certain fuzzy logic theory Γ.

Example 6. On the Gödel residuated lattice $([0,1], \leq, *_G, \rightarrow_G, 0, 1)$, let us consider the fuzzy logic theory formed by the following three formulas: $\psi_1 = p \rightarrow 0.4$, $\psi_2 = q \rightarrow p$ and $\psi_3 = 0.5 \rightarrow q$. To begin with, let us prove that the theory $\Gamma = \{\psi_1, \psi_2, \psi_3\}$ is inconsistent.

Assume that M is a model of Γ; since M satisfies ψ_1, then we have

$$M(\psi_1) = 1 \iff M(p \rightarrow 0.4) = 1 \iff M(p) \leq 0.4$$

since M also satisfies ψ_3 we have

$$M(\psi_3) = 1 \iff M(0.5 \rightarrow q) = 1 \iff M(q) \geq 0.5$$

as a result, we have $M(p) \leq 0.4 \leq 0.5 \leq M(q)$. This leads to the impossibility of M to be a model of ψ_2, since $M(\psi_2) = M(q \rightarrow p) = M(q) \rightarrow M(p) = M(p) \neq 1$; therefore Γ is inconsistent.

The previous chain of inequalities suggest that 0.4 could be the measure of consistence of Γ. In order to prove it we have to compute the minimum of the set $\{\alpha \mid \psi_1 \wedge \psi_2 \wedge \psi_3 \text{ is } \alpha\text{-feasible}\}$; that is, the minimum $\alpha \in [0,1]$ such that $\emptyset \models (\psi_1 \wedge \psi_2 \wedge \psi_3) \rightarrow \alpha$.

Let us consider $\alpha = 0.4$ and show that $\emptyset \models (\psi_1 \wedge \psi_2 \wedge \psi_3) \rightarrow 0.4$; i.e that $I((\psi_1 \wedge \psi_2 \wedge \psi_3) \rightarrow 0.4) = 1$ for all interpretation I. This amounts to prove that $I(\psi_1 \wedge \psi_2 \wedge \psi_3) \leq 0.4$ for all I, so let us consider an interpretation I and reason by cases:

- If $I(p) > 0.4$, then $I(\psi_1) = I(p \rightarrow 0.4) = 0.4$. That implies that $I(\psi_1 \wedge \psi_2 \wedge \psi_3) \leq 0.4$.
- If $I(p) \leq 0.4$ and $I(p) < I(q)$ then $I(\psi_2) = I(q \rightarrow p) = I(p) \leq 0.4$ and as a result $I(\psi_1 \wedge \psi_2 \wedge \psi_3) \leq 0.4$.
- If $I(p) \leq 0.4$ and $I(q) \leq I(p)$ then, necessarily $I(q) \leq 0.4$. Therefore, $I(\psi_3) = I(0.5 \rightarrow q) = I(q) \leq 0.4$, and then $I(\psi_1 \wedge \psi_2 \wedge \psi_3) \leq 0.4$.

as a result, we have that $\psi_1 \wedge \psi_2 \wedge \psi_3$ is 0.4-feasible.

Now, let us prove that if $\alpha < 0.4$, then $\emptyset \not\models (\psi_1 \wedge \psi_2 \wedge \psi_3) \rightarrow \alpha$; for this we have just to provide an interpretation such that $I((\psi_1 \wedge \psi_2 \wedge \psi_3) \rightarrow \alpha) \neq 1$. Let us consider the interpretation I given by $I(p) = I(q) = 0.5$. Then, we have:

- $I(\psi_1) = I(p) \rightarrow 0.4 = 0.5 \rightarrow 0.4 = 0.4$
- $I(\psi_2) = I(q) \rightarrow I(p) = 0.5 \rightarrow 0.5 = 1$
- $I(\psi_3) = 0.5 \rightarrow I(q) = 0.5 \rightarrow 0.5 = 1$

therefore $I(\psi_1 \wedge \psi_2 \wedge \psi_3) = 0.4 \not\leq \alpha$ and $I((\psi_1 \wedge \psi_2 \wedge \psi_3) \rightarrow \alpha) = \alpha \neq 1$.

As a conclusion, we can say that $\{\alpha \mid \psi_1 \wedge \psi_2 \wedge \psi_3 \text{ is } \alpha\text{-feasible}\} = [0.4, 1]$ and then $\mathrm{Mc}^*(\Gamma) = 0.4$.

6 | CONCLUSIONS AND FUTURE WORK

In this paper we have analyzed the notion of inconsistency in fuzzy logic theories. We have begun by presenting different and equivalent definitions of inconsistency in crisp logic which are not equivalent in other paradigms. At that point, we have presented the proposal of the notion of inconsistency in the strongest sense by means of the absence of models. We have shown that such a definition behaves in the fuzzy logic paradigm very similarly to the crisp paradigm. However, such a consideration lose the intrinsic gradualness of fuzzy logic since one theory is either consistent or inconsistent, nothing in the middle.

In order to provide a gradual notion of inconsistency in fuzzy logic theories, we have presented two different measures of consistency. The consideration of "consistency" instead of "inconsistency" lays on a technical issue: to provide a definition as simple as possible without artefacts as could be t-norms or negations operators non-associated with the underlying residuated lattice. The first measure of consistence measures how much compatible is a formula with respect to a given theory in the sense: the closer to 0, the more inconsistent; and the closer to 1, the more consistent. The second measure determine a degree of consistency of a logic theory by means of consistent subtheories. It is worth mentioning that those measures of consistency belong purely to the fuzzy paradigm. In other words, both definitions coincide with the standard notion of consistency when we restrict the set of truth values to $\{0, 1\}$; i.e., when we are in crisp logic, the measures of consistency defined in this paper have only two values: 0 (inconsistency) or 1 (consistency). Moreover, we have shown that both definitions satisfy convenient properties in order to be considered measures of consistency.

There are two main lines of future research. On the one hand it is convenient to keep digging up some measures of inconsistency in fuzzy paradigms. To have a notion of inconsistency based on degrees (as the ones proposed in this paper) may allow to incorporate a paraconsistent reasoning into inconsistent fuzzy logic theories without leaving the fuzzy paradigm aside. On the other hand, it is interesting to find out an application of the measures of consistence. For instance, we think they can be used to deal with contradictions in databases obtained from fails or system errors.

Author contributions

All authors contributed equally to the publication.

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Conflict of interest

The authors declare that there is no actual or potential conflict of interest in relation to this article.

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