On the added-mass force and moment and the vortex projection method. Application to thin airfoils

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Abstract

The force and moment on a solid body in non-uniform motion in an incompressible fluid are derived following the projection technique of Quartapelle and Napolitano (1983) and Chang (1992) in a compact vector form, identifying added-mass terms equivalent to those resulting from potential flow theory. The expressions are formally much simpler in a non-inertial frame fixed relative to the body, though general relations for an inertial frame are also given. Approximate analytical expressions are obtained for the force and moment on two-dimensional elliptic bodies at high Reynolds numbers, each one consisting of just two terms: an added-mass component and a vortex term (integration of a projection of Lamb's vector). The formulation is applied to a flat plate in steady motion and in non-uniform oscillatory motion, recovering classical results in both cases, but adding new physical insight about the origin of the different elements of the force and moment. It is shown that, in the non-steady motion, the added-mass components of the force and moment cancel out with part of their respective vortex terms, so that all the force and moment come from appropriate projections of Lamb's vector, including the supposedly added-mass force and moment. This result indicates that one has to be cautious about simplified aerodynamic models for flapping foils based on a simple separation of the added-mass and vortical components of the force and moment. The present results can be used to obtain analytically the force and moment on an ellipse with arbitrary motion through an incompressible fluid at high Reynolds numbers.

1 Introduction

Expressions for the aerodynamic force and moment on a solid body moving through a fluid in terms of the vortex flow structures are very useful both from a theoretical and from a practical or engineering point of view, and many works have been dedicated to obtain general, and simple, but theoretically sound, force and moment formulas based on vortex dynamics (for example, Burgers, 1920; Chang, 1992; Howe, 1995; Kambe, 1987; Lighthill, 1986; Noca *et al.*, 1997; Prandtl, 1918; Quartapelle & Napolitano, 1983; Wu, 1981, to cite only some relevant pionering works). The vortical approach is especially relevant in incompressible flows where, according to D'Alembert paradox, vorticity is, apart from unsteadiness, the only source of aerodynamic force.

Of special relevance is the well known vortical impulse theory, which was used, for example, by von Kármán & Sears (1938) to derive theoretically the lift and moment on a thin airfoil undergoing an arbitrary non-uniform motion in the inviscid and linear limit, and more recently to include the thrust force for the same limit in Fernandez-Feria (2016). The general impulse theory for the force and moment was derived by Wu (1981) for the arbitrary motion of any set of solid bodies in a viscous incompressible fluid, although previous contributions were made, in addition to von Kármán & Sears (1938), by Burgers (1920) and others (see, e.g., Biesheuvel & Hagmeijer, 2006; Wu *et al.*, 2018, for a review).

However, from a practical point of view, it is more useful the projection formulation due to Quartapelle & Napolitano (1983) and Chang (1992), which concentrates the effect of vorticity near the solid body, so that vortices far from the body have practically no effect on the force and moment. This is a clear advantage over the vortical impulse theory when computing the aerodynamic force and moment from vorticity distributions obtained either numerically or experimentally. For instance, in the case of a traveling airfoil one has to take into account the effect of the starting vortex far from the body when using the impulse theory. Or, in its defect, the vorticity flow associated to the wake exiting the control volume through a surface far from the body, which is difficult to obtain, particularly from experimental measurements, while this far vorticity has no effect on the computation of the forces and moment when using the projection theory (see, for example, Lee *et al.*, 2012; Martín-Alcántara & Fernandez-Feria, 2019; Martín-Alcántara *et al.*, 2015).

For high Reynolds numbers, in addition to neglect the surface viscous terms, one is tempted to simplify further the results from the projection method by considering the vorticity near the solid body as a bound vortex sheet surrounded by potential flow. This is what is done in the present paper for the particularly simple, but relevant, case of a two-dimensional incompressible flow. Before that, Chang's (1992) projection technique for the force is reformulated in a general compact vector form in §2, that it is also used to obtain the moment on the solid body straightforwardly, also in a vector compact form. The method separates neatly the added-mass contributions from the vortical ones coming from the projection of Lamb's vector. An alternative general separation, but using the impulse theory, was recently made by Limacher *et al.* (2018). In the present projection method, the added-mass terms are much simpler in a non-inertial reference frame fixed in relation to the solid body, though the expressions for an inertial reference frame are also given in Appendix A. Similar expressions were derived by Howe (1995) from a quite different approach for a rigid solid body in an inertial reference frame with velocity field vanishing at infinity, which was in turn generalized recently by Magnaudet (2011). However, Howe's approach made the assumption that vorticity is vanishingly small on the external surface far from the body enclosing the control volume. This greatly restricts its application for computing forces and moments from either numerical or experimental data in problems were vorticity is continuously shed by the moving body, since it would imply the use of unrealistic, exceedingly large control volumes. This problem is not present in the approach of Quartapelle & Napolitano (1983) and Chang (1992), where the integrals on the external surface decay very fast due to the selected projection functions, and no far-field flow assumptions have to be made on the external surface. The different force and moment elements coming from the projection approach of Quartapelle & Napolitano (1983) and Chang (1992) are shown here to be particularly simple in a non-inertial reference frame moving with the solid body.

The expressions for the force and moment on an elliptic body in a two-dimensional incompressible flow are obtained analytically in §3, with some mathematical details about the computations in elliptical coordinates summarized in Appendix B. Finally, the formulation is applied to a plate or thin airfoil in §4, both for steady and unsteady motions, recovering previously known formulas for the force and moment, but disclosing a rather unexpected result about the so-called added-mass contributions to the force and moment.

2 Summary of the projection formulas for the force and moment in an incompressible flow

Useful expressions for the force and moment on a solid body can be obtained by projecting the momentum equation with potential functions that vanish sufficiently fast far from the body and satisfy appropriate boundary conditions on the solid surface. The derivation summarized below is due to Quartapelle & Napolitano (1983) and Chang (1992), but now using compact vector notation.

Consider a rigid solid body of volume V_s, bounded by a surface S_s, moving through an incompressible fluid of density ρ and constant viscosity μ. The momentum equation can be written as

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\nabla}\left(p + \frac{1}{2}\rho v^2\right) + \rho \boldsymbol{\omega} \wedge \mathbf{v} + \mu \boldsymbol{\nabla} \wedge \boldsymbol{\omega} = \rho \mathbf{f}\,, \tag{1}$$

where $\mathbf{v}$ is the fluid velocity, $v^2 = \mathbf{v} \cdot \mathbf{v}$, p is the pressure, $\boldsymbol{\omega} = \boldsymbol{\nabla} \wedge \mathbf{v}$ is the vorticity field, and a body force per unit mass $\mathbf{f}$ has been added to account for a non-inertial reference frame (in an inertial frame, $\mathbf{f} = \mathbf{0}$, because the action of gravity is separated as a hydrostatic pressure that does not contribute to the aerodynamic force and moment, as usual). We use *vector* functions $\boldsymbol{\varphi}$ satisfying Laplace's equation and vanishing far from the origin of coordinates:

$$\nabla^2 \boldsymbol{\varphi} = \mathbf{0}\,, \quad \boldsymbol{\varphi} \to \mathbf{0} \quad \text{as} \quad |\mathbf{x}| \to \infty\,. \tag{2}$$

Taking into account that $\boldsymbol{\nabla} \cdot \mathbf{v} = 0$, that $(\boldsymbol{\nabla}\psi) \cdot \boldsymbol{\nabla}\boldsymbol{\varphi} = \boldsymbol{\nabla} \cdot (\psi \boldsymbol{\nabla}\boldsymbol{\varphi})$ and $(\boldsymbol{\nabla} \wedge \mathbf{A}) \cdot \boldsymbol{\nabla}\boldsymbol{\varphi} = \boldsymbol{\nabla} \cdot (\mathbf{A} \wedge \boldsymbol{\nabla}\boldsymbol{\varphi})$, for any scalar function ψ and vector function $\mathbf{A}$, multiplying (1) scalarly by $\boldsymbol{\nabla}\boldsymbol{\varphi}$ one gets

$$\boldsymbol{\nabla}\cdot\left[\rho\frac{\partial\mathbf{v}}{\partial t}\boldsymbol{\varphi}+\left(p+\frac{1}{2}\rho v^2\right)\boldsymbol{\nabla}\boldsymbol{\varphi}+\mu(\boldsymbol{\omega}\wedge\boldsymbol{\nabla})\boldsymbol{\varphi}\right]+\rho(\boldsymbol{\omega}\wedge\mathbf{v})\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}=\rho\mathbf{f}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}\,. \tag{3}$$

Thus, after integrating over a fluid control volume V_c bounded internally by the solid body surface S_s, and externally by a surface S_e sufficiently far away from the solid body (and from the origin of coordinates) for $\boldsymbol{\varphi}$ to vanish on S_e, one obtains

$$\int_{S_s} dS\left\{\mathbf{n}\cdot\left[\rho\frac{\partial\mathbf{v}}{\partial t}\boldsymbol{\varphi}+\left(p+\frac{1}{2}\rho v^2\right)\boldsymbol{\nabla}\boldsymbol{\varphi}\right]+\mu(\mathbf{n}\wedge\boldsymbol{\omega})\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}\right\}=\rho\int_{V_c} dV[\mathbf{f}-\boldsymbol{\omega}\wedge\mathbf{v}]\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}\,, \tag{4}$$

where $\mathbf{n}$ is the unit normal to the solid surface S_s pointing towards the solid. It is convenient to apply the divergence theorem to the body force term to write

$$\int_{S_s} dS\left\{\mathbf{n}\cdot\left[\rho\left(\frac{\partial\mathbf{v}}{\partial t}-\mathbf{f}\right)\boldsymbol{\varphi}+\left(p+\frac{1}{2}\rho v^2\right)\boldsymbol{\nabla}\boldsymbol{\varphi}\right]+\mu(\mathbf{n}\wedge\boldsymbol{\omega})\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}\right\}=-\rho\int_{V_c} dV[(\boldsymbol{\nabla}\cdot\mathbf{f})\boldsymbol{\varphi}+(\boldsymbol{\omega}\wedge\mathbf{v})\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}]\,. \tag{5}$$

Now, depending on the boundary condition for $\boldsymbol{\varphi}$ on the solid surface S_s, one can obtain from (4) an expression for the force or for the moment. Thus, if $\boldsymbol{\varphi}=\boldsymbol{\varphi}_v$ satisfies

$$\mathbf{n}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_v=\mathbf{n}\quad\text{on}\quad S_s\,,\quad\text{or}\quad\mathbf{n}_s\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_v=\mathbf{n}_s\quad\text{on}\quad S_s\,, \tag{6}$$

where $\mathbf{n}_s=-\mathbf{n}$ is the unit normal to the solid surface S_s pointing towards the fluid, since the force $\mathbf{F}$ that the incompressible fluid exerts on the closed solid body surface S_s can be written as

$$\mathbf{F}=-\int_{S_s} dS\,(\mathbf{n}_s p+\mu\mathbf{n}_s\wedge\boldsymbol{\omega})\;, \tag{7}$$

from (4) one obtains

$$\mathbf{F}=\mathbf{F}_a+\mathbf{F}_v+\mathbf{F}_\mu\,, \tag{8}$$

with

$$\mathbf{F}_a=\rho\int_{S_s} dS\left[\mathbf{n}_s\cdot\left(\frac{\partial\mathbf{v}}{\partial t}-\mathbf{f}\right)\boldsymbol{\varphi}_v+\mathbf{n}_s\frac{v^2}{2}\right]-\rho\int_{V_c} dV(\boldsymbol{\nabla}\cdot\mathbf{f})\boldsymbol{\varphi}_v\,, \tag{9}$$

$$\mathbf{F}_v=-\rho\int_{V_c} dV\,(\boldsymbol{\omega}\wedge\mathbf{v})\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_v\,, \tag{10}$$

$$\mathbf{F}_\mu=\mu\int_{S_s}(\mathbf{n}_s\wedge\boldsymbol{\omega})\cdot(\boldsymbol{\nabla}\boldsymbol{\varphi}_v-\mathsf{I})\,dS\,, \tag{11}$$

and where I is the unit tensor. All the contributions from $\mathbf{f}$ are included in $\mathbf{F}_a$, so that only Lamb's vector $\boldsymbol{\omega}\wedge\mathbf{v}$ contributes to the vortex force $\mathbf{F}_v$, as usual, but now conveniently concentrated near the solid body through the projection with the gradient of the auxiliary potential function. In a reference frame moving rigidly with the solid body, $\mathbf{v}=\mathbf{0}$ on the body surface S_s and the only contributions to $\mathbf{F}_a$ come from the inertial acceleration $\mathbf{f}$ (see below). Finally, the viscous term $\mathbf{F}_\mu$ is negligible for high Reynolds number flows (note that this term would be multiplied by the inverse of the Reynolds number, $Re^{-1}=\mu/(\rho VL)$, instead of μ, if the flow magnitudes are conveniently non-dimensionalized with a characteristic velocity V and a characteristic length L, while the non-dimensional forms of $\mathbf{F}_a$ and $\mathbf{F}_v$ would be (9)-(10) with $\rho=1.$).

Similarly, if $\boldsymbol{\varphi}=\boldsymbol{\varphi}_o$ satisfies

$$\mathbf{n}_s\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_o=(\mathbf{x}-\mathbf{x}_0)\wedge\mathbf{n}_s\quad\text{on}\quad S_s\,, \tag{12}$$

for a given point $\mathbf{x}_0$, since the moment $\mathbf{M}$ about the point $\mathbf{x}_0$ that the incompressible fluid exerts on the solid body surface S_s enclosing de volume V_s can be written as (see, e.g., Wu *et al.* (2006))

$$\mathbf{M}=-\int_{S_s}(\mathbf{x}-\mathbf{x}_0)\wedge\mathbf{n}_s p\,dS+\mu\int_{S_s}(\mathbf{x}-\mathbf{x}_0)\wedge(\boldsymbol{\omega}\wedge\mathbf{n}_s)\,dS-2\mu\int_{V_s}\boldsymbol{\omega}\,dV\,, \tag{13}$$

from (4) one obtains

$$\mathbf{M}=\mathbf{M}_a+\mathbf{M}_v+\mathbf{M}_\mu\,, \tag{14}$$

with

$$\mathbf{M}_a = \rho \int_{S_s} dS \left[\mathbf{n}_s \cdot \left(\frac{\partial \mathbf{v}}{\partial t} - \mathbf{f} \right) \boldsymbol{\varphi}_o + \frac{v^2}{2} (\mathbf{x} - \mathbf{x}_0) \wedge \mathbf{n}_s \right] - \rho \int_{V_c} dV (\boldsymbol{\nabla} \cdot \mathbf{f}) \boldsymbol{\varphi}_o \,, \tag{15}$$

$$\mathbf{M}_v = -\rho \int_{V_c} dV \, (\boldsymbol{\omega} \wedge \mathbf{v}) \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_o \,, \tag{16}$$

$$\mathbf{M}_\mu = \mu \left[\int_{S_s} (\mathbf{n}_s \wedge \boldsymbol{\omega}) \cdot \boldsymbol{\nabla} \boldsymbol{\varphi}_v dS - \int_{S_s} (\mathbf{x} - \mathbf{x}_0) \wedge (\mathbf{n}_s \wedge \boldsymbol{\omega}) dS - 2 \int_{V_s} \boldsymbol{\omega} dV \right] \,, \tag{17}$$

where now the viscous part $\mathbf{M}_\mu$ contains an additional term in relation to $\mathbf{F}_\mu$, which is written as an integral over de solid body volume V_s (it can also be written as $-2\mu \int_{S_s} dS \, \mathbf{n}_s \wedge \mathbf{v}$.)

These expressions are general for ρ and μ constants, with the only approximation that the surface S_e bounding the control volume V_c has to be sufficiently far away for $\boldsymbol{\varphi}_v$ and $\boldsymbol{\varphi}_o$ to vanish. As shown by Chang (1992), the auxiliary functions decay, in general, as $|\mathbf{x}|^{-2}$ for a three-dimensional flow, and as $|\mathbf{x}|^{-1}$ for a two-dimensional flow, so that their gradients decay as $|\mathbf{x}|^{-3}$ and $|\mathbf{x}|^{-2}$, respectively. In fact, as shown in Martín-Alcántara & Fernandez-Feria (2019), S_e need not to be too far away from S_s to obtain very accurate values for the force $\mathbf{F}$ on a flapping foil. Equations (8)-(11) for the force $\mathbf{F}$ coincide with the originally obtained by Chang (1992) when $\mathbf{f} = \mathbf{0}$, just written here in a vector compact form. Equations (14)-(17) for the moment, which are also written here in a vector compact form, is somewhat similar to that obtained by Howe (1995) (see below for a discussion).

2.1 Rigid solid body

For a rigid solid body with arbitrary translation, and rotating with angular velocity $\boldsymbol{\Omega}(t)$ about $\mathbf{x}_0$, we may use a non-inertial reference frame fixed relative to the moving body, so that $\mathbf{v} = \mathbf{0}$ on S_s and

$$\mathbf{f} = -\frac{d\mathbf{V}}{dt} - \frac{d\boldsymbol{\Omega}}{dt} \wedge \mathbf{x}' - \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{x}') - 2\boldsymbol{\Omega} \wedge \mathbf{v} \,, \qquad \boldsymbol{\nabla} \cdot \mathbf{f} = 2\boldsymbol{\Omega} \cdot \boldsymbol{\omega} + (k-1)\Omega^2 \,, \tag{18}$$

with

$$\mathbf{V} \equiv \frac{d\mathbf{x}_0}{dt} \,, \quad \mathbf{x}' \equiv \mathbf{x} - \mathbf{x}_0 \,, \tag{19}$$

and $k \equiv \boldsymbol{\nabla} \cdot \mathbf{x}$ is 3 for a three dimensional flow and 2 for a two dimensional flow. Thus, $\mathbf{F}_a$ and $\mathbf{M}_a$ can be written as

$$\mathbf{F}_a = -\mathsf{m}_v \cdot \frac{d\mathbf{V}}{dt} - \mathsf{m}_{vo} \cdot \frac{d\boldsymbol{\Omega}}{dt} - \rho \int_{S_s} \frac{|\boldsymbol{\Omega} \wedge \mathbf{x}'|^2}{2} \mathbf{n}_s dS - 2\rho\boldsymbol{\Omega} \cdot \int_{V_c} \boldsymbol{\omega} \boldsymbol{\varphi}_v dV \,, \tag{20}$$

$$\mathbf{M}_a = -\mathsf{m}_{vo} \cdot \frac{d\mathbf{V}}{dt} - \mathsf{m}_o \cdot \frac{d\boldsymbol{\Omega}}{dt} - \rho \int_{S_s} \frac{|\boldsymbol{\Omega} \wedge \mathbf{x}'|^2}{2} (\mathbf{x}' \wedge \mathbf{n}_s) dS - 2\rho\boldsymbol{\Omega} \cdot \int_{V_c} \boldsymbol{\omega} \boldsymbol{\varphi}_o dV \,, \tag{21}$$

where the following added-mass 3×3 symmetric tensors has been defined:

$$\mathsf{m}_v \equiv -\rho \int_{S_s} \boldsymbol{\varphi}_v \, \mathbf{n}_s dS = -\rho \int_{S_s} \boldsymbol{\varphi}_v \frac{\partial \boldsymbol{\varphi}_v}{\partial n_s} \, dS \,, \tag{22}$$

$$\mathsf{m}_{vo} \equiv -\rho \int_{S_s} \boldsymbol{\varphi}_v (\mathbf{x}' \wedge \mathbf{n}_s) dS = -\rho \int_{S_s} \boldsymbol{\varphi}_v \frac{\partial \boldsymbol{\varphi}_o}{\partial n_s} dS = -\rho \int_{S_s} \boldsymbol{\varphi}_o \frac{\partial \boldsymbol{\varphi}_v}{\partial n_s} dS \,, \tag{23}$$

$$\mathsf{m}_o \equiv -\rho \int_{S_s} (\mathbf{x}' \wedge \mathbf{n}_s) \boldsymbol{\varphi}_o \, dS = -\rho \int_{S_s} \frac{\partial \boldsymbol{\varphi}_o}{\partial n_s} \boldsymbol{\varphi}_o dS \,. \tag{24}$$

Actually,

$$\phi \equiv \boldsymbol{\varphi}_v \cdot \mathbf{V} + \boldsymbol{\varphi}_o \cdot \boldsymbol{\Omega} \tag{25}$$

is the velocity potential of the corresponding irrotational flow in an inertial frame, and the components of the vector $(\boldsymbol{\varphi}_v, \boldsymbol{\varphi}_o)^T$ are thus the six potentials ϕ_j defining the 6×6 added-mass symmetric tensor m for a potential and incompressible flow which, when expressed in body-fixed coordinates, depends only on the body geometry (for example, Newman (1977)). In this notation, for $i, j = 1, 2, 3$, $\mathsf{m}_{v_{ij}} = \mathsf{m}_{v_{ji}} = \mathsf{m}_{ij} = \mathsf{m}_{ji}$, $\mathsf{m}_{vo_{ij}} = \mathsf{m}_{vo_{ji}} = \mathsf{m}_{i,j+3} = \mathsf{m}_{j+3,i}$, and $\mathsf{m}_{o_{ij}} = \mathsf{m}_{o_{ji}} = \mathsf{m}_{i+3,j+3} = \mathsf{m}_{j+3,i+3}$.

The resulting expressions for $\mathbf{F}_a$ and $\mathbf{M}_a$ in an inertial reference frame are given in Appendix A.

Similar expressions to (8)-(11) for the force and (14)-(17) for the moment were derived by Howe (1995) (see also Howe, 2007) following a different approach, and for a rigid solid body in an inertial reference frame ($\mathbf{f} = \mathbf{0}$). In the derivation of the force, Howe used an additional (Kirchoff) vector potencial $\mathbf{X} \equiv \mathbf{x} - \boldsymbol{\varphi}_v$, which also satisfies Laplace's equation but with different boundary conditions. In fact, this function does not vanish at infinity and thus generates a different separation of the force elements. In addition, in Howe's derivation it is assumed that vorticity vanish on the external surface S_e, which in some problems implies an infinite volume V_c from which no wake vorticity generated at the body may exit through S_e. Just for comparison sake, Howe (1995) expresion of the force, in the present compact vector notation (and with $\mathbf{f} = \mathbf{0}$), can be written as

$$\mathbf{F} = \rho \frac{d}{dt} \int_{S_s} dS\, \mathbf{n}_s \cdot \mathbf{v} \boldsymbol{\varphi}_v + \rho \int_{S_s} dS \mathbf{n}_s \cdot \mathbf{v} \frac{D\mathbf{X}}{Dt} + + \rho \int_{V_c} dV\, (\boldsymbol{\omega} \wedge \mathbf{v}) \cdot \boldsymbol{\nabla} \mathbf{X} - \mu \int_{S_s} (\mathbf{n}_s \wedge \boldsymbol{\omega}) \cdot \boldsymbol{\nabla} \mathbf{X} dS\,. \quad (26)$$

The viscous term is exactly the same as $\mathbf{F}_\mu$ in (11), but the third term is not as 'compact' as the vortex force term $\mathbf{F}_v$ in (10), which is projected with $\boldsymbol{\nabla}\boldsymbol{\varphi}_v$ and decays as $|\mathbf{x}|^{-3}$ (or $|\mathbf{x}|^{-2}$ for a 2D flow), while now $\boldsymbol{\nabla}\mathbf{X}$ is unity at infinity. As a consequence of this substantial difference in the vortex force terms, together with the potential-flow assumption on the external surface S_e made in the derivation of (26), the two first terms in (26) are quite different from the term $\mathbf{F}_a$ in (9) (actually, from that given in Appendix A for a rigid body in a stationary frame).

Thus, the compactness of $\mathbf{F}_v$, which is Prandtl's vector-force conveniently concentred near the solid body by its projection through $\boldsymbol{\nabla}\boldsymbol{\varphi}_v$, is lost in Howe's formulation. This fact represents a strong limitation when using (26) to compute forces from vorticity data obtained either numerically or experimentally in problems were the vortical wake emanating from the body extends very far downstream (see, e.g., Martín-Alcántara & Fernandez-Feria, 2019). As in the force expressions resulting from the impulse theory (see, e.g., Wu *et al.*, 2018, for a review), (26) is more appropriate for computing forces in starting motions, or in any situation in which all vorticity remains relatively close to the solid body (Howe, 1995, 2007). The particular approximation described in the following sections would not be possible with Howe's approach. Instead, for a thin airfoil (Section 4), Howe's approach would be equivalent to that of von Kármán & Sears (1938) resulting from the impulse theory, which needs the vorticity distribution of the whole wake behind the non-stationary airfoil for computing the force and moment.

3 Two-dimensional inviscid flow around an ellipse

In a two dimensional flow the above expressions become greatly simplified. The vorticity has only one component, $\boldsymbol{\omega} = \omega \mathbf{e}_y$, say, for a motion in the $(x, z)-$plane with velocity field $\mathbf{v} = v_x \mathbf{e}_x + v_z \mathbf{e}_z$, and Lamb's vector is

$$\boldsymbol{\mathcal{L}} = \boldsymbol{\omega} \wedge \mathbf{v} = \omega (v_z \mathbf{e}_x - v_x \mathbf{e}_z)\,. \quad (27)$$

The body motion is given by $\mathbf{V} = V_x \mathbf{e}_x + V_z \mathbf{e}_z$ and $\boldsymbol{\Omega} = \Omega \mathbf{e}_y$, and the auxiliary potential functions are simplified to $\boldsymbol{\varphi}_v = \varphi_{v_x} \mathbf{e}_x + \varphi_{v_z} \mathbf{e}_z$ and $\boldsymbol{\varphi}_o = \varphi_{o_y} \mathbf{e}_y$.

We assume further that the solid body is an ellipse of semi-axes $c/2$ and $e/2$ along $\mathbf{e}_x$ and $\mathbf{e}_z$, respectively, in a reference frame fixed relative to the moving ellipse. Using elliptic coordinates (ξ, η),

$$x = \frac{c}{2} \frac{\cosh \xi}{\cosh \xi_0} \cos \eta\,, \quad z = \frac{e}{2} \frac{\sinh \xi}{\sinh \xi_0} \sin \eta\,, \quad \tanh \xi_0 = \frac{e}{c} \equiv \epsilon\,, \quad (28)$$

with $\xi = \xi_0$ the surface of the ellipse S_s for $0 \leq \eta \leq 2\pi$, these functions are, for a pivot point located at $\mathbf{x}_0 = a(c/2)\mathbf{e}_x + b(e/2)\mathbf{e}_z$ (see Appendix B):

$$\varphi_{v_x} = -\frac{e}{2} e^{\xi_0 - \xi} \cos \eta\,, \quad \varphi_{v_z} = -\frac{c}{2} e^{\xi_0 - \xi} \sin \eta\,, \quad (29)$$

$$\varphi_{o_y} = \frac{c^2}{16} (1 - \epsilon^2) e^{2(\xi_0 - \xi)} \sin(2\eta) + \frac{c^2}{4} e^{\xi_0 - \xi} (b \epsilon^2 \cos \eta - a \sin \eta)\,. \quad (30)$$

Incidentally, it is observed that these functions decay exponentially with the distance $\xi - \xi_0$ to the ellipse S_s. According to these functions, only the following added-mass coefficients do not vanish (see Appendix B):

$$m_{v_{xx}} = \pi\rho\frac{e^2}{4}\,, \quad m_{v_{zz}} = \pi\rho\frac{c^2}{4}\,, \tag{31}$$

$$m_{vo_{xy}} = m_{vo_{yx}} = -\pi\rho\frac{c^3}{8}b\epsilon^3\,, \quad m_{vo_{zy}} = m_{vo_{yz}} = \pi\rho\frac{c^3}{8}a\,, \tag{32}$$

$$m_{o_{yy}} = \pi\rho\frac{c^4}{16}\left[\frac{(1-\epsilon^2)^2}{8} + b^2\epsilon^4 + a^2\right]. \tag{33}$$

They include the case of a flat plate of chord length c by setting $e = 0$ ($\epsilon = 0$), and that of a circular cylinder of radius $R = e/2 = c/2$ ($\epsilon = 1$).

The integrals over the solid body surface S_s in (20)-(21) can be obtained by using elliptic coordinates, just as they are used to compute (31)-(33) in Appendix B. Thus, using the above-defined value of $\mathbf{x}_0$, one obtains

$$\mathbf{F}_a = -\pi\rho\frac{c^2}{4}\left[\epsilon^2\frac{dV_x}{dt}\mathbf{e}_x + \frac{dV_z}{dt}\mathbf{e}_z + \frac{c}{2}\frac{d\Omega}{dt}(-b\epsilon^3\mathbf{e}_x + a\mathbf{e}_z)\right] +$$

$$+\pi\rho\frac{c^3}{8}\Omega^2\epsilon(a\mathbf{e}_x + \epsilon b\mathbf{e}_z) - 2\rho\Omega\int_{V_c}\omega\left(\varphi_{v_x}\mathbf{e}_x + \varphi_{v_z}\mathbf{e}_z\right)dV\,, \tag{34}$$

$$M_a = \pi\rho\frac{c^3}{8}\left[b\epsilon^3\frac{dV_x}{dt} - a\frac{dV_z}{dt} - \frac{c}{2}\left(\frac{(1-\epsilon^2)^2}{8} + b^2\epsilon^4 + a^2\right)\frac{d\Omega}{dt}\right] - 2\rho\Omega\int_{V_c}\omega\varphi_{o_y}dV\,, \tag{35}$$

with $\mathbf{M}_a = M_a\mathbf{e}_y$.

On the other hand, using (27), the vortex force and moment (10) and (16) can be written as

$$\mathbf{F}_v = -\rho\int_{V_c}\boldsymbol{\mathcal{L}}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_v dV =$$

$$= -\rho\int_{V_c}\omega\left[\left(v_z\frac{\partial\varphi_{v_x}}{\partial x} - v_x\frac{\partial\varphi_{v_x}}{\partial z}\right)\mathbf{e}_x + \left(v_z\frac{\partial\varphi_{v_z}}{\partial x} - v_x\frac{\partial\varphi_{v_z}}{\partial z}\right)\mathbf{e}_z\right]dV\,, \tag{36}$$

$$\mathbf{M}_v = -\rho\int_{V_c}\boldsymbol{\mathcal{L}}\cdot\boldsymbol{\nabla}\boldsymbol{\varphi}_o\,dV = -\rho\mathbf{e}_y\int_{V_c}\omega\left(v_z\frac{\partial\varphi_{o_y}}{\partial x} - v_x\frac{\partial\varphi_{o_y}}{\partial z}\right)dV\,. \tag{37}$$

Finally, although the above expressions are valid for any value of the Reynolds number, we consider here the interesting case of high Reynolds number flows for which the viscous terms (11) and (17) can be neglected.

4 Application to a flat plate or thin airfoil at high Reynolds numbers

We consider the case of a very thin ellipse ($\epsilon \to 0$) or flat plate translating and rotating with a small angle of attack at any instant, and for high Reynolds numbers, so that there is no separation of the boundary layer except at the trailing edge. Thus, vorticity is concentrated in a thin region around the surface of the body S_s, which can be approximated by a bound vortex sheet of strength $\varpi_s(\mathbf{x},t)$, and in a thin wake starting from the trailing edge, which can be approximated by an almost planar free vortex sheet S_w with vortex strength $\varpi_e(\mathbf{x},t)$. On the other hand, since the projection functions $\boldsymbol{\varphi}_v$ and $\boldsymbol{\varphi}_o$ vanish very rapidly with the distance to S_s [see (29)-(30)], we can neglect the direct contribution from ϖ_e, except for that included in ϖ_s, to write (36)-(37) as

$$\mathbf{F}_v \simeq -\rho\int_{S_s}\varpi_s\left[\left(v_z\frac{\partial\varphi_{v_x}}{\partial x} - v_x\frac{\partial\varphi_{v_x}}{\partial z}\right)\mathbf{e}_x + \left(v_z\frac{\partial\varphi_{v_z}}{\partial x} - v_x\frac{\partial\varphi_{v_z}}{\partial z}\right)\mathbf{e}_z\right]dS\,, \tag{38}$$

$$M_v \simeq -\rho\int_{S_s}\varpi_s\left(v_z\frac{\partial\varphi_{o_y}}{\partial x} - v_x\frac{\partial\varphi_{o_y}}{\partial z}\right)dS\,, \tag{39}$$

with $\mathbf{M}_v = M_v\mathbf{e}_y$.

Similarly, in the limit $\epsilon \to 0$, the contributions (34) and (35) can be written as

$$\mathbf{F}_a \simeq -\pi\rho\frac{c^2}{4}\left(\frac{dV_z}{dt}+a\frac{c}{2}\frac{d\Omega}{dt}\right)\mathbf{e}_z - 2\rho\Omega\int_{S_s}\varpi_s\left(\varphi_{v_x}\mathbf{e}_x+\varphi_{v_z}\mathbf{e}_z\right)dS\,, \tag{40}$$

$$M_a \simeq -\pi\rho\frac{c^3}{8}\left[a\frac{dV_z}{dt}+\frac{c}{2}\left(\frac{1}{8}+a^2\right)\frac{d\Omega}{dt}\right] - 2\rho\Omega\int_{S_s}\varpi_s\varphi_{o_y}dS\,. \tag{41}$$

In a reference frame fixed relative to the solid body, the plate is located at $z=0$ between $x=-c/2$ and $x=c/2$, with $x=c/2$ the trailing edge. In this reference frame, $v_z=0$ on S_s, while v_x is the inviscid velocity on each side of the plate outside the bound vortex sheet. Using elliptic coordinates to integrate over S_s, one may write (38)-(41) as

$$\mathbf{F}_v \simeq \rho\int_0^{2\pi}\varpi_s\left(v_x\frac{\partial\varphi_{v_x}}{\partial z}\mathbf{e}_x+v_x\frac{\partial\varphi_{v_z}}{\partial z}\mathbf{e}_z\right)h_\eta d\eta\,, \tag{42}$$

$$M_v \simeq \rho\int_0^{2\pi}\varpi_s v_x\frac{\partial\varphi_{o_y}}{\partial z}h_\eta d\eta\,, \tag{43}$$

$$\mathbf{F}_a \simeq -\pi\rho\frac{c^2}{4}\left(\frac{dV_z}{dt}+a\frac{c}{2}\frac{d\Omega}{dt}\right)\mathbf{e}_z - 2\rho\Omega\int_0^{2\pi}\varpi_s\left(\varphi_{v_x}\mathbf{e}_x+\varphi_{v_z}\mathbf{e}_z\right)h_\eta d\eta\,, \tag{44}$$

$$M_a \simeq -\pi\rho\frac{c^3}{8}\left[a\frac{dV_z}{dt}+\frac{c}{2}\left(\frac{1}{8}+a^2\right)\frac{d\Omega}{dt}\right] - 2\rho\Omega\int_0^{2\pi}\varpi_s\varphi_{o_y}h_\eta d\eta\,, \tag{45}$$

where $h_\eta d\eta = (c/2)\sin\eta d\eta$ for $\xi=\xi_0$ and $\epsilon\to 0$. These integrals can also be computed using the Cartesian coordinate x by taking into account that

$$\int_{S_s} AdS = \int_0^{2\pi} Ah_\eta d\eta = \int_0^{\pi} A^+ h_\eta d\eta + \int_\pi^{2\pi} A^- h_\eta d\eta =$$

$$= -\int_{c/2}^{-c/2} A^+ dx + \int_{-c/2}^{c/2} A^- dx = \int_{-c/2}^{c/2}(A^- + A^+)dx\,, \tag{46}$$

for any quantity A, where the superscripts '$\pm$' refer to above and below the plate, respectively, and where use has been made of $x=(c/2)\cos\eta$ and $h_\eta d\eta = \mp dx$.

Near the plate ($\xi\to\xi_0$ or $|z|\to 0$) and for $\epsilon\to 0$, the functions (29)-(30) can be written as

$$\varphi_{v_x} \simeq -\frac{e}{2}(1+\xi_0-\xi)\cos\eta \simeq -\epsilon x\left(1-\frac{|z|}{\sqrt{(c/2)^2-x^2}}\right)\,, \tag{47}$$

$$\varphi_{v_z} \simeq -\frac{c}{2}(1+\xi_0-\xi)\sin\eta \simeq \mp\sqrt{(c/2)^2-x^2}+z\,, \tag{48}$$

$$\varphi_{o_y} \simeq \frac{c^2}{4}\sin\eta\left[\frac{\cos\eta}{2}-a+(\xi-\xi_0)(a-1)\right] \simeq \pm\frac{1}{2}\sqrt{(c/2)^2-x^2}(x-ac)+z\left(\frac{c}{2}a-x\right)\,. \tag{49}$$

To obtain their derivatives with respect to x and z at the plate $\xi=\xi_0$ it is convenient to use elliptic coordinates because some of them vanishes as the plate thickness ϵ tends to zero but with singularities at the leading- and trailing-edges, which may contribute to the above integrals. They are the following:

$$\frac{\partial\varphi_{v_x}}{\partial x} \simeq \frac{(\epsilon^2-\epsilon\tan^2\eta)}{\epsilon^2+\tan^2\eta}\,,\quad \frac{\partial\varphi_{v_x}}{\partial z} \simeq \frac{\epsilon\tan\eta}{\epsilon^2+\tan^2\eta}\,, \tag{50}$$

$$\frac{\partial\varphi_{v_z}}{\partial x} \simeq \frac{\tan\eta}{\epsilon^2+\tan^2\eta} \simeq \frac{1}{\tan\eta} = \frac{\pm x}{\sqrt{(c/2)^2-x^2}}\,,\quad \frac{\partial\varphi_{v_z}}{\partial z} \simeq \frac{-\epsilon+\tan^2\eta}{\epsilon^2+\tan^2\eta} \simeq 1\,, \tag{51}$$

$$\frac{\partial\varphi_{o_y}}{\partial x} \simeq \frac{c}{2}\frac{\frac{1}{2}-\cos^2\eta+a\cos\eta}{\sin\eta} \simeq \frac{\frac{c^2}{8}-x^2+\frac{c}{2}ax}{\pm\sqrt{(c/2)^2-x^2}}\,, \tag{52}$$

$$\frac{\partial \varphi_{o_y}}{\partial z} \simeq \frac{c}{2}\frac{\tan^2\eta(a-\cos\eta)}{\epsilon^2+\tan^2\eta} \simeq \frac{c}{2}(a-\cos\eta) \simeq \frac{c}{2}a - x\,. \quad (53)$$

Note that only in the cases in which the expressions do not vanish with ϵ have been written also in terms of the coordinate x.

The bound vortex sheet strength ϖ_s is related to $v_x^\pm$ through (e.g., Wu *et al.* (2015))

$$\varpi_s^\pm = \pm(v_x^\pm - \bar{v}_x)\,, \quad \bar{v}_x \equiv \frac{1}{2}(v_x^+ + v_x^-)\,, \quad \varpi_s^+ + \varpi_s^- = v_x^+ - v_x^-\,, \quad (54)$$

$$\varpi_s^+ = \varpi_s^- \equiv \varpi_s = \frac{1}{2}(v_x^+ - v_x^-)\,. \quad (55)$$

Thus, on using (46)-(53), equations (42)-(45) can be written as

$$F_{v_x} = \epsilon\rho\frac{c}{2}\int_0^{2\pi}\varpi_s(\eta)v_x(\eta)\frac{\tan\eta}{\epsilon^2+\tan^2\eta}\sin\eta d\eta\,, \quad F_{v_z} = \rho\int_{-c/2}^{c/2}\frac{1}{2}[(v_x^+)^2-(v_x^-)^2]dx\,, \quad (56)$$

$$F_{a_x} = \epsilon\rho\Omega\frac{c^2}{2}\int_0^{2\pi}\varpi_s(\eta)\cos\eta\sin\eta d\eta\,, \quad F_{a_z} = -\pi\rho\frac{c^2}{4}\left(\frac{dV_z}{dt} + a\frac{c}{2}\frac{d\Omega}{dt}\right)\,, \quad (57)$$

$$M_v = \rho\int_{-c/2}^{c/2}\frac{1}{2}[(v_x^+)^2-(v_x^-)^2]\left(\frac{c}{2}a - x\right)dx\,, \quad (58)$$

$$M_a = -\pi\rho\frac{c^3}{8}\left[a\frac{dV_z}{dt} + \frac{c}{2}\left(\frac{1}{8}+a^2\right)\frac{d\Omega}{dt}\right]\,. \quad (59)$$

It has been taken into account that, according to (46)-(49) and (55),

$$\int_0^{2\pi}\varpi_s\varphi_{v_z}h_\eta d\eta = \int_{c/2}^{c/2}\varpi_s(\varphi_{v_z}^+ + \varphi_{v_z}^-)dx = 0\,, \quad (60)$$

$$\int_0^{2\pi}\varpi_s\varphi_{o_y}h_\eta d\eta = \int_{c/2}^{c/2}\varpi_s(\varphi_{o_y}^+ + \varphi_{o_y}^-)dx = 0\,, \quad (61)$$

so that the terms proportional to Ω in F_{a_z} and M_a vanish for any vorticity didtribution ϖ_s.

The expressions for F_{v_x} and F_{a_x} are both multiplied by $\epsilon \to 0$, but they have not been neglected because ϖ_s and, consequently, v_x, are singular at the leading edge $\eta = \pi$ (also for this reason the integration has been maintained in the angular variable η). This singularity is of the form $\sin^{-1}\eta$, independently of the fact that the plate motion is steady or unsteady (e.g., von Kármán & Burgers (1935)); i.e.,

$$v_x^\pm \sim \frac{K(t)}{\sin\eta}\,, \quad \varpi_s \sim \frac{1}{2}\frac{K(t)}{\sin\eta} \quad \text{as } \eta \to \pi\,, \quad (62)$$

where $K(t)$ depends on the foil's motion. It is easy to check that the main contribution to the integrals comes from $|\eta - \pi| \sim \epsilon \ll 1$. Thus, writing $\eta = \pi + \epsilon\xi$, for $\xi = O(1)$ when $\epsilon \to 0$ one obtains

$$F_{v_x} \simeq -\rho\frac{c}{4}K^2\int_{-\infty}^{\infty}\frac{d\xi}{1+\xi^2} = -\pi\rho\frac{c}{4}K^2 \qquad \text{and} \quad F_{a_x} \to 0\,. \quad (63)$$

4.1 Stationary foil

It is instructive to check the above expressions by considering first a plate translating with constant velocity $\mathbf{V}$ and a small angle of attack α: $V_x = -V\cos\alpha$, $V_z = -V\sin\alpha$, and $\Omega = 0$, in the present reference frame attached to the plate. The potential velocity and vorticity strength on the plate are (e.g., Newman (1977)):

$$v_x^\pm = V\left(\cos\alpha + \sin\alpha\frac{1-\cos\eta}{\sin\eta}\right) = V\left(\cos\alpha \pm \sin\alpha\sqrt{\frac{c/2-x}{c/2+x}}\right)\,, \quad (64)$$

$$\varpi_s = \frac{1}{2}(v_x^+ - v_x^-) = \frac{1}{2}V\sin\alpha\frac{1-\cos\eta}{\sin\eta} = V\sin\alpha\sqrt{\frac{c/2-x}{c/2+x}}\,. \quad (65)$$

The added-mass contributions (57) and (59) obviously vanish, so that all the force and moment come from the vortex contributions (56) and (58). The force perpendicular to the plate is

$$F_z = F_{v_z} = 2\rho V^2 \cos\alpha \sin\alpha \int_{-c/2}^{c/2} \sqrt{\frac{c/2 - x}{c/2 + x}} dx = \pi c \rho V^2 \cos\alpha \sin\alpha = \rho V \Gamma \cos\alpha \,, \tag{66}$$

where $\Gamma \equiv \pi c V \sin\alpha$, which is the projection of the Kutta-Joukowski lift force perpendicular to $\mathbf{V}$ into the direction z normal to the plate.

According to (63), since $K = 2V \sin\alpha$ in this case, the component of the force parallel to the plate is

$$F_x = F_{v_x} = -\pi \rho c V^2 \sin^2\alpha = -\rho V \Gamma \sin\alpha \,, \tag{67}$$

which is the projection of the Kutta-Joukowski force into the direction x parallel to the plate, physically due to the pressure suction at the leading edge.

In relation to the moment,

$$M = M_v = 2\rho V^2 \cos\alpha \sin\alpha \int_{-c/2}^{c/2} \sqrt{\frac{c/2 - x}{c/2 + x}} \left(\frac{c}{2}a - x\right) dx =$$

$$= \pi \frac{c^2}{4}(1 + 2a)\rho V^2 \cos\alpha \sin\alpha = \frac{c(1 + 2a)}{4} F_z \,, \tag{68}$$

which is the well known result for a flat plate (e.g., Newman (1977); note that the moment $\mathbf{M} = M\mathbf{e}_y$ is positive here when clockwise).

4.2 Non-stationary foil. Oscillating plate

Before considering any particular motion of the thin foil or plate, one may obtain some general results for the non-stationary case.

Outside the boundary layers and vortex sheets, one may neglect viscous term in (1). Using $\mathbf{f}$ from (18) and

$$\mathbf{v} = \boldsymbol{\nabla}\phi - \mathbf{V} - \boldsymbol{\Omega} \wedge \mathbf{x}' \,, \tag{69}$$

where ϕ is the potential function (25) of the irrotational flow, Eq. (1) can be written as

$$\boldsymbol{\nabla}\left(\rho\frac{\partial\phi}{\partial t} + \rho\,\frac{v^2 - |\boldsymbol{\Omega} \wedge \mathbf{x}'|^2}{2} + p\right) = 0 \,. \tag{70}$$

Thus, outside viscous regions and vortex sheets, the quantity inside brackets does not depends on $\mathbf{x}$, which is a generalization of Bernoulli's equation for the present non-inertial reference frame rotating with the body. In particular, it can be applied at both sides of the plate to obtain the following relation for the inviscid flow magnitudes on each side:

$$\rho\frac{\partial\phi^+}{\partial t} + \rho\frac{(v_x^+)^2}{2} + p^+ = \rho\frac{\partial\phi^-}{\partial t} + \rho\frac{(v_x^-)^2}{2} + p^- \,, \tag{71}$$

where it has been taken into account that v_z and $|\boldsymbol{\Omega} \wedge \mathbf{x}'|^2$ are continuous across the plate located at $z = 0$.

Since, according to (25) and (47)-(49),

$$\rho\frac{\partial}{\partial t}(\phi^- - \phi^+) = \rho\frac{d\mathbf{V}}{dt} \cdot (\boldsymbol{\varphi}_v^- - \boldsymbol{\varphi}_v^+) + \rho\frac{d\boldsymbol{\Omega}}{dt} \cdot (\boldsymbol{\varphi}_o^- - \boldsymbol{\varphi}_o^+) =$$

$$= \rho\frac{dV_z}{dt} 2\sqrt{(c/2)^2 - x^2} - \rho\frac{d\Omega}{dt}\sqrt{(c/2)^2 - x^2}(x - ac) \,, \tag{72}$$

F_{v_z} and M_v in (56) and (58) can then be written, respectively, as

$$F_{v_z} = \rho\int_{-c/2}^{c/2} \frac{1}{2}[(v_x^+)^2 - (v_x^-)^2] dx = \int_{-c/2}^{c/2} (p^- - p^+) dx + \pi\rho\frac{c^2}{4}\left(\frac{dV_z}{dt} + a\frac{c}{2}\frac{d\Omega}{dt}\right) \,, \tag{73}$$

$$M_v = \rho \int_{-c/2}^{c/2} \frac{1}{2}[(v_x^+)^2 - (v_x^-)^2] \left(\frac{c}{2}a - x\right) dx =$$

$$= \int_{-c/2}^{c/2} (p^- - p^+) \left(\frac{c}{2}a - x\right) dx + \pi\rho \frac{c^3}{8} \left[a\frac{dV_z}{dt} + \frac{c}{2}\left(\frac{1}{8} + a^2\right)\frac{d\Omega}{dt}\right] . \quad (74)$$

This is a surprising result because it says that the supposedly added-mass contributions F_{a_z} and M_a in (57) and (59) cancel exactly with the second parts on the right hand sides of (73) and (74), so that all the lift and moment comes from the vortex contributions F_{v_z} and M_v, respectively. Note that these are given, by definition, by the first terms containing the pressure difference on the right hand sides of (73) and (74).

Substituting the vortex strength distribution $\varpi_s(x,t)$, first obtained by von Kármán & Sears (1938), one gets the well known expressions for the lift and moment of a non-stationary thin airfoil, which, for an oscillating foil were first obtained by Theodorsen (1935) computing the pressure difference $p^- - p^+$ directly from the unsteady potential function ϕ. The expressions for F_z and M contain a 'circulatory' part, associated to the well known Theodorsen function of the reduced frequency, and a 'non-circulatory' part sometimes dubbed added-mass contribution. But both parts comes from F_{v_z} and M_v once $[(v_x^+)^2 - (v_x^-)^2]/2$ is computed from the vortex strength distribution ϖ_s. In particular, each 'non-circulatory' part resulting from the vortex force and moment, F_{v_z} and M_v, contains an additional term in relation to (57) and (59). In the present notation they can be written, respectively, as

$$-\pi\rho\frac{c^2}{4}\left(\frac{dV_z}{dt} + a\frac{c}{2}\frac{d\Omega}{dt} - V_x\Omega\right) , \quad -\pi\rho\frac{c^3}{8}\left[a\frac{dV_z}{dt} + \frac{c}{2}\left(\frac{1}{8} + a^2\right)\frac{d\Omega}{dt} + \left(\frac{1}{2} - a\right)V_x\Omega\right] . \quad (75)$$

On the other hand, the horizontal or thrust force, which has only the vortex contribution (63), is just Garrick's (1936) propulsion force once $K(t)$ is extracted from the vortex strength distribution $\varpi_s(x,t)$ at the leading edge for this problem. To obtain a better approximation for the thrust force, one has to include the effect of the vorticity in the wake ϖ_e in the vicinity of the trailing edge.

5 Conclusion

The expression for the force obtained by Chang (1992), together with the equivalent for the moment, are summarized here in a compact vector form by projecting the momentum equation with appropriate vector potential functions. Some of their terms are related to the added-mass force and moment in a irrotational flow, which are written in the simplest form when using a reference frame fixed relative to the solid body.

For high Reynolds numbers, surface viscous terms can be neglected and the expressions for the force and moment contain just two terms, the added-mass terms and the vortex terms, the last ones consisting of integrals of Lamb's vector conveniently projected. In addition, one may approximate the vortex force and moment terms by computing Lamb's vector with just the vortex strength distribution of the bound vortex sheet surrounding the solid body, because the contributions from any other vorticity distribution detached from the solid body vanish quite rapidly when projected with the gradients of the auxiliary potential functions. This constitutes a great advantage in relation to other formulations, such as those based in the impulse theory, which need to consider the vorticity distribution even very far from the body to compute accurately the force and moment. The approximation is particularly simple for two-dimensional flows, which is developed here for a solid body with elliptic form, for which the auxiliary potential functions are obtained analytically. Once the bound vortex-sheet strength is computed for a given motion of the body by any standard method from the potential theory, the present approximation allows for the computation of the force and moment analytically. Moreover, the projection functions decay exponentially with the distance to the surface of the body, reinforcing the accurateness of the method in relation to Chang's (1992) original prediction for this decay as a negative power of the distance to the body.

The approximation is applied to a flat plate or thin airfoil (a very thin ellipse), recovering classical results for both a stationary motion of the plate and a plate undergoing a non-uniform motion, but adding new physical insight about the origin of the different force and moment terms. Thus, it is to be remarked as an interesting result that the present method shows that the so-called added-mass force and moment in a non-uniform motion of the foil come in fact from the vortex terms, i.e., from Lamb's vector on the body surface conveniently projected, and not from the supposedly added-mass terms resulting from the force and moment decomposition. Also, it is interesting to note that, although the analytical calculations for a thin airfoil are not simpler than, for instance, those from von Kármán & Sears (1938) approach, resulting from the impulse theory, conceptually it is simpler because only vorticity very close to the airfoil are relevant in the vortex force and moment, The present results can be used to obtain the force and moment on an ellipse with arbitrary motion through an incompressible fluid with just information of the vorticity distribution on its surface obtained either analytically, numerically or experimentally.

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A Added-mass terms for an inertial reference frame

In an inertial reference frame, $\mathbf{f} = \mathbf{0}$ and the fluid velocity on the solid body surface is

$$\mathbf{v} = \mathbf{V} + \boldsymbol{\Omega} \wedge \mathbf{x}' , \quad \mathbf{x} \in S_s , \tag{76}$$

where $\mathbf{V} = d\mathbf{x}_0/dt$. Since

$$\frac{D\mathbf{v}}{Dt} = \frac{d\mathbf{V}}{dt} + \frac{d\boldsymbol{\Omega}}{dt} \wedge \mathbf{x}' + \boldsymbol{\Omega} \wedge (\boldsymbol{\Omega} \wedge \mathbf{x}') \tag{77}$$

for a point moving with the solid body surface, on these points

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{D\mathbf{v}}{Dt} - \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = \frac{d\mathbf{V}}{dt} + \frac{d\boldsymbol{\Omega}}{dt} \wedge \mathbf{x}' + \mathbf{V} \wedge \boldsymbol{\Omega} . \tag{78}$$

Then, $\mathbf{F}_a$ and $\mathbf{M}_a$ can be written as

$$\mathbf{F}_a^i = -\mathsf{m}_v^i \cdot \frac{d\mathbf{V}}{dt} - \mathsf{m}_{vo}^i \cdot \frac{d\boldsymbol{\Omega}}{dt} - \mathsf{m}_V^i \cdot (\mathbf{V} \wedge \boldsymbol{\Omega}) + \rho \int_{S_s} \frac{|\boldsymbol{\Omega} \wedge (\mathbf{x} - \mathbf{x}_0)|^2}{2} \mathbf{n}_s dS , \tag{79}$$

$$\mathbf{M}_a^i = -\mathsf{m}_{vo}^i \cdot \frac{d\mathbf{V}}{dt} - \mathsf{m}_o^i \cdot \frac{d\boldsymbol{\Omega}}{dt} - \mathsf{m}_O^i \cdot (\mathbf{V} \wedge \boldsymbol{\Omega}) + \rho \int_{S_s} \frac{V^2 + |\boldsymbol{\Omega} \wedge (\mathbf{x} - \mathbf{x}_0)|^2}{2} (\mathbf{x}' \wedge \mathbf{n}_s) dS , \tag{80}$$

where the superscript i makes reference to the inertial frame and the new added-mass tensors are defined as

$$\mathsf{m}_V^i \equiv -\rho \int_{S_s} (\boldsymbol{\varphi}_v + \mathbf{x}') \, \mathbf{n}_s dS = \mathsf{m}_v^i - \rho \int_{S_s} \mathbf{x}' \mathbf{n}_s \, dS , \tag{81}$$

$$\mathsf{m}_O^i \equiv -\rho \int_{S_s} (\boldsymbol{\varphi}_v + \mathbf{x}') \, (\mathbf{x}' \wedge \mathbf{n}_s) dS = -\rho \int_{S_s} [\mathbf{n}_s \boldsymbol{\varphi}_o + \mathbf{x}'(\mathbf{x}' \wedge \mathbf{n}_s)] dS = \mathsf{m}_{vo}^i - \rho \int_{S_s} \mathbf{x}'(\mathbf{x}' \wedge \mathbf{n}_s) \, dS . \tag{82}$$

Note that the superscript i is also used in the different added-mass tensors because in the inertial frame $\boldsymbol{\varphi}_v$, $\boldsymbol{\varphi}_o$, $\mathbf{x}'$ and $\mathbf{n}_s$ depend on time as the solid rotates. The expressions (79)-(80), which are the more familiar expressions for the added-mass force and moment (Howe, 1995; Newman, 1977), are much more complex than (20)-(21) for the reference frame fixed to the body, though one has to project them into the inertial frame to calculate the motion of the body.

B Auxiliary potential functions for an ellipse and added-mass coefficients

To solve (2) for an elliptic body with semi-axes $c/2$ and $e/2$ along the x and z directions, respectively, we use the elliptic coordinates (28). The solution to Laplace's equation $\nabla^2 \phi = 0$ vanishing as $\xi \to \infty$ can be obtained by separation of variables (for example, Morse & Feshbach (1953)):

$$\phi(\xi, \eta) = \sum_{m=1}^{\infty} e^{-m\xi} [A_m \sin(m\eta) + B_m \cos(m\eta)] , \tag{83}$$

where the A_m and B_m are arbitrary constants.

According to (6), since $\mathbf{n}_s = \mathbf{e}_\xi$, the boundary conditions for the components φ_{v_x} and φ_{v_z} of $\boldsymbol{\varphi}_v$ can be written as

$$\left.\frac{\partial \varphi_{v_x}}{\partial \xi}\right|_{\xi=\xi_0} = \epsilon \frac{c}{2}\cos\eta\,, \quad \left.\frac{\partial \varphi_{v_z}}{\partial \xi}\right|_{\xi=\xi_0} = \frac{c}{2}\sin\eta\,. \tag{84}$$

Consequently, only the constant B_1 in (83) does not vanish for φ_{v_x}, while only A_1 is different from zero for φ_{v_z}, which can be written as

$$\varphi_{v_x} = -\frac{e}{2}e^{\xi_0-\xi}\cos\eta\,, \quad \varphi_{v_z} = -\frac{c}{2}e^{\xi_0-\xi}\sin\eta\,. \tag{85}$$

On the other hand, if we use $\mathbf{x}_0 = [a(c/2)\mathbf{e}_x + b(e/2)\mathbf{e}_z)$ for given values of a and b, since

$$[(\mathbf{x}-\mathbf{x}_0)\wedge \mathbf{n}_s]\cdot \mathbf{e}_y = [(\mathbf{x}-\mathbf{x}_0)\wedge \mathbf{e}_\xi]\cdot \mathbf{e}_y = (\mathbf{x}-\mathbf{x}_0)\cdot(\mathbf{e}_\xi \wedge \mathbf{e}_y) = (\mathbf{x}-\mathbf{x}_0)\cdot \mathbf{e}_\eta\,, \tag{86}$$

the boundary condition (12) for the component φ_{o_y} can be written as

$$\left.\frac{\partial \varphi_{o_y}}{\partial \xi}\right|_{\xi=\xi_0} = \frac{c^2}{4}\left[-\frac{1-\epsilon^2}{2}\sin 2\eta + a\sin\eta - b\epsilon^2\cos\eta\right]\,, \tag{87}$$

so that only the constants A_2, A_1 and B_1 in (83) do not vanish for φ_{o_y}, resulting

$$\varphi_{o_y} = \frac{c^2}{16}(1-\epsilon^2)e^{2(\xi_0-\xi)}\sin(2\eta) + \frac{c^2}{4}e^{\xi_0-\xi}(b\epsilon^2\cos\eta - a\sin\eta)\,. \tag{88}$$

According to (22)-(24), the non-vanishing added mass coefficients are the following:

$$m_{v_{xx}} = -\rho\int_0^{2\pi}\left(\varphi_{v_x}\frac{\partial\varphi_{v_x}}{\partial\xi}\right)_{\xi=\xi_0} d\eta = \pi\rho\frac{e^2}{4}\,, \quad m_{v_{zz}} = -\rho\int_0^{2\pi}\left(\varphi_{v_z}\frac{\partial\varphi_{v_z}}{\partial\xi}\right)_{\xi=\xi_0} d\eta = \pi\rho\frac{c^2}{4}\,, \tag{89}$$

$$m_{vo_{xy}} = -\rho\int_0^{2\pi}\left(\varphi_{o_y}\frac{\partial\varphi_{v_x}}{\partial\xi}\right)_{\xi=\xi_0} d\eta = -\pi\rho\frac{c^3}{8}b\epsilon^3\,, \quad m_{vo_{zy}} = -\rho\int_0^{2\pi}\left(\varphi_{o_y}\frac{\partial\varphi_{v_z}}{\partial\xi}\right)_{\xi=\xi_0} d\eta = \pi\rho\frac{c^3}{8}a\,, \tag{90}$$

$$m_{o_{yy}} = -\rho\int_0^{2\pi}\left(\varphi_{o_y}\frac{\partial\varphi_{o_y}}{\partial\xi}\right)_{\xi=\xi_0} d\eta = \pi\rho\frac{c^4}{16}\left[\frac{(1-\epsilon^2)^2}{8} + b^2\epsilon^4 + a^2\right]\,. \tag{91}$$

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