UNIVERSIDAD DE MÁLAGA

FACULTAD DE CIENCIAS

Programa de Doctorado en Matemáticas

PhD thesis (Tesis doctoral)

A_∞-persistence

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Málaga, 2015

Publicaciones y
Divulgación Científica

UNIVERSIDAD
DE MÁLAGA

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EDITA: Publicaciones y Divulgación Científica. Universidad de Málaga

Esta tesis ha sido parcialmente financiada por una beca para la Formación de Personal Investigador (FPI) del antiguo Ministerio de Ciencia e Innovación, el Proyecto de Investigación MTM2010-18089 del Ministerio de Economía y Competitividad, el Grupo de Investigación Consolidado FQM-213 de la Junta de Andalucía y la red ACAT de la European Science Foundation.

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Por la presente, como Director y Tutor de la tesis doctoral de D. Francisco Belchí Guillamón que lleva por título "*A_∞-persistence*",

AUTORIZO la presentación de la misma para que pueda ser defendida como Tesis Doctoral según la legislación vigente.

Asimismo,

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Francisco Belchí, Aniceto Murillo, "*A_∞-persistence*", Applicable Algebra in Engineering, Communication and Computing: Volume 26, Issue 1 (2015), 121-139. ISSN 0938-1279. doi: 10.1007/s00200-014-0241-4.

Hago constar por último que ningún contenido ni resultado de la anterior publicación ha sido utilizado en ninguna otra tesis ni trabajo de investigación.

Y para que así conste y surta los efectos oportunos, firmo el presente escrito en Málaga a 10 de Octubre de 2015

Aniceto Murillo Mas

En record de la Carme Guàrdia.

Sempre forta. Sempre positiva.

(In memory of Carme Guàrdia.

Always strong. Always positive.)

Agradecimientos (Acknowledgements)

Quiero dar las gracias a la gente que ha marcado más significativamente los años que he dedicado a esta tesis.

A mi familia, y en especial a mis padres, por haberme transmitido amor por encima de todas las cosas. A mi *familia de acogida* en Málaga: Lali, Carlos y Juan Carlos, por acogerme tan cálidamente (con y sin *terral*) durante los últimos tres años.

A Oihana, por lo personal y lo matemático. A nuestro Luis. A Marco, David y los demás compis del a veces superpoblado despacho. A PePe, por alegrarme el día tantas y tantas veces hablando de nuestras matemáticas. Ojalá podamos repetirlo (de manera *formal* o no) un número L_∞ de veces. A Diek; de mayor quiero tener lo que tú tienes con Espe (¡pero no con Espe!). A mis alfajores Nahuel y Nacho. Por muchos tumbos que dé, cuando estemos juntos siempre me sentiré en casa. A Mònica, Jorge y mi gente de Barcelona.

A mi director de tesis, Aniceto Murillo, porque sólo con él habría podido hacer una tesis y porque unas palabras suyas en persona siempre han bastado para motivarme, desbloquearme o darme un punto de vista nuevo. Gracias por tu paciencia y comprensión, y por haberme engañado para acabar cogiéndole el gustillo a la homotopía racional. ;-)

A todo el maravilloso grupo de personas en el departamento: Nieves (que confió en mí desde el principio, dándome la oportunidad de compartir docencia con ella), Viru (el gran sabio con un pasado oscuro), Luisa (siempre haciéndonos la vida más fácil) y todos los demás. Un gracias muy especial a Antonio Díaz, por seguir haciendo de mí un guerrero y por tantos comentarios útiles sobre mi tesis.

A los profesores de la Universitat de Barcelona que me formaron. En especial a Paco Guillén y a Carles Casacuberta. Si no me hubiera encontrado con ellos en el pasado, difícilmente habría acabado haciendo esta tesis.

I would like to express my sincere gratitude to Robert Ghrist and Graham Ellis for being so welcoming during my visit to their respective institutions. I would also like to thank Jim Stasheff for his suggestions on my first paper, Paweł Dłotko for mentioning Zigzag Persistence to me when I didn't know about it, Vladimir Dotsenko and Tornike Kadeishvili for pointing out to me what I was getting wrong about A_∞-structures, and Nick Scoville for his valuable feedback on this dissertation.

And of course, thanks to that spilt cup of milk.

Contents

Introduction **11**

1 Persistent Homology **23**

1.1 Filtrations 25

1.1.1 Simplicial complexes on a point cloud 26

1.1.2 Cubical complexes on a digital image 27

1.2 Persistent Homology 29

1.3 Zigzag persistence 37

1.3.1 Algorithmic approach 40

2 A_∞-structures **43**

2.1 Differential graded (co)algebras 44

2.1.1 Cup product and Alexander-Whitney diagonal 53

2.2 Basics of A_∞-(co)algebras 56

2.3 Algorithmic approach 62

2.4 The power of A_∞-structures 70

3 A_∞-persistence **73**

3.1 $\operatorname{Ker}\Delta_n$ & $\operatorname{Coker} m_n$ 74

3.2 Backing our stand 76

3.2.1 Lack of invariance 77

3.2.2 The invariant $\min\{n \mid \Delta_n \neq 0\}$ 78

3.2.3 Massey products 85

3.3 A first decomposition result 93

3.4 Coherent choices of A_∞-structure 97
3.5 Wake up every day (as long as you are alive) 99
3.6 A more general decomposition result 106
3.7 Algorithmic approach 112

4 Appendix: basics of rational homotopy theory 115

Resumen en Español 119

Introduction

This work is a mixture of two completely different worlds: *Persistent Homology* and A_∞-*(co)algebras* – the former, a newborn from the generation of Topological Data Analysis; the latter, a classical concept in Homological Perturbation Theory.

Persistent Homology (in the sense of [23,39,103]), a topological technique which has proved to be useful in areas such as digital imaging [89], sensor networks [33], molecular modelling [3], dynamical systems [38,88] and speech pattern analysis [15], to name a few, allows one to extract global structural information about data sets (specially high dimensional ones) which may contain noise.

At a closer look, with this theory we can approximate the Betti numbers of an unknown closed subset X of a metric space from a finite point sample of X [19, 28, 32, 82, 88]. In low dimensions, those numbers can sometimes be easy to guess (as in Figure 4.1), but a point cloud in a 100-dimensional

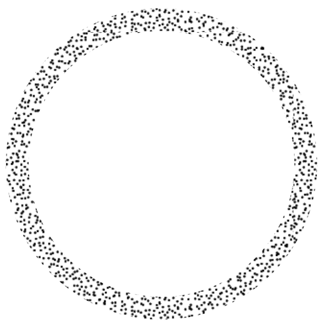


Figure 1: It is easy to see at first glance that this could be a point sample of an annulus.

space can be hard to visualize. Besides, in such a situation, we would need to consider not only number of components, tunnels and voids, but also say 99-dimensional features of the unknown underlying space X.

Why is this useful? Here is a toy example. Imagine we want to study some disease of which we do not know much and to do so we start by taking 100 measurements of a patient. These numbers we can see as coordinates, so we can visualize the data on this patient as a point in a Euclidean 100-dimensional space $\mathbb{E}^{100}$. We repeat the process with 70000 people and what we get is a bunch of 70000 points in $\mathbb{E}^{100}$, with each point representing a patient. Then, even if we do not ask any specific questions of the gathered data (either because we do not know enough of the disease to decide what to ask of because we do not want to restrict ourselves with answers conditioned by our previous knowledge), persistence will tell us about topological features of the space behind the point cloud and this can be translated into information about the disease.

Note that the more we know about this space underlying the point cloud, the more information we will get about the disease. This thesis has been carried out in the spirit of this consideration.

Specifically, since Persistent Homology, in its most basic form, can only contribute information at the level of Betti numbers, the challenge that started up this project was to enhance the power of persistence by finding tools that were able to distinguish, in several situations, between spaces with the same Betti numbers (and even isomorphic cohomology algebra) that are not homotopy equivalent; tools that could tell us, in some cases, about how the different components of a link are linked; and very importantly, tools that could admit a *persistent twist.*

The tools we propose to use are some structures with which the (co)homology of spaces can be endowed – namely, A_∞-algebras and A_∞-coalgebras [94].

We devote this thesis to back this stand and to develop the theoretical and algorithmic aspects of a persistence theory for A_∞-structures, or A_∞-*persistence,* for short.

We will explain the main constructions in our theory in terms of A_∞-

coalgebras on the homology of a space, but everything dualizes *mutatis mutandis* to A_∞-algebras on cohomology (see §3.1), yielding in particular a new approach to persistence of cup product [55].

For the rest of the introduction, let us work over a field of coefficients.

In general (see §2.2 for precise definition and details), an A_∞-coalgebra structure on a graded module C is a sequence of graded maps

$$\Delta_n : C \longrightarrow \underbrace{C \otimes \ldots \otimes C}_{n}, \qquad n \geq 1$$

of degree $|\Delta_n| = n - 2$, satisfying certain coassociativity relations. The "A" in A_∞-(co)algebra stands for *Associativity* and the "∞" indicates that for all $n \neq 1$ (without an upper bound on n), some (co)associativity axiom on $\Delta_1, \Delta_2, \ldots, \Delta_n$ holds up to the (co)chain homotopy Δ_{n+1}.

Given a topological space X, its homology $H_*(X)$ can always be endowed with several A_∞-coalgebra structures, all of them isomorphic, for which Δ_2 is the induced map in homology by any approximation to the diagonal and so that these A_∞-coalgebras satisfy a certain extra condition that makes them more topologically meaningful than a random A_∞-coalgebra on $H_*(X)$. We decided to call such A_∞-coalgebras *good* (see Def. 2.2.11).

We intend to study *good* A_∞-coalgebras in homology (and dually, *good* A_∞-algebras in cohomology) in a persistent way. To start motivating the choice of these structures, we prove:

Theorem 2.4.4. *Let X be a non-formal simply connected CW complex of finite type and let Y be the formal space associated to $H^*(X; \mathbb{Q})$. Then, X and Y have isomorphic rational homology groups and isomorphic rational cohomology algebras but their rational homologies do not admit isomorphic* good *A_∞-coalgebra structures.*

In Example 2.4.5 we show that there are spaces satisfying the hypotheses of this theorem.

This result is a first good sign, since we aim to build a theory of persistence that goes beyond Betti numbers.

Next step in this direction is to find some numbers, associated to topological spaces, that are susceptible of giving more information that Betti numbers and of admitting a persistent adaptation.

In §3.1 we see how the search for simplicity brings us to consider as candidates the following numbers. Given a *good* A_∞-coalgebra structure $\{\Delta_n\}_{n\geq 1}$ on the homology of X, we fix some $n \geq 1$ and some homology degree $p \geq 0$ and focus on the number

$$\dim \operatorname{Ker} \Delta_{n|H_p(X)}.$$

Similarly, given a *good* A_∞-algebra structure $\{m_n\}_{n\geq 1}$ on the cohomology of X, we fix some $n \geq 1$ and some cohomology degrees $p_1, \ldots, p_n \geq 0$ and focus on the number

$$\dim \operatorname{Coker} m_{n|H^{p_1}(X)\otimes \ldots \otimes H^{p_n}(X)},$$

where $\operatorname{Coker} f$ denotes the cokernel of the linear map $f\colon V \longrightarrow W$, *i.e.*, the quotient space $W/\operatorname{Im} f$.

It is worth noting that there are several *good* A_∞-coalgebra structures with which one can endow the homology of a space X. Furthermore, different choices of A_∞-coalgebra can yield different numbers $\dim \operatorname{Ker} \Delta_{n|H_p(X)}$, as shown in Example 3.2.1. The same thing happens with A_∞-algebras on $H^*(X)$ and the numbers $\dim \operatorname{Coker} m_{n|H^{p_1}(X)\otimes \ldots \otimes H^{p_n}(X)}$. To overcome this drawback, we prove various results showing that, in some cases, these numbers can still give information beyond Betti numbers and even beyond the cohomology algebra. Specifically, we first prove Theorem 3.2.2, a result that holds when working over a field of characteristic 0 that implies in particular that the numbers

$$k := \min\{n \mid \Delta_n \neq 0\}$$

and $\dim \operatorname{Ker} \Delta_{k|H_p(X)}$, for all $p \geq 0$, are invariants of the isomorphism class of minimal A_∞-coalgebras structures $\{\Delta_n\}_n$ on $H_*(X)$. An important consequence of this result is the following:

Corollary 3.2.3. *Let $\{\Delta_n\}_n$ and $\{\Delta'_n\}_n$ be two arbitrary* good *A_∞-coalgebra structures on the homology of a space X such that there exist some $m, m' \geq 1$*

for which $\Delta_m \neq 0$ *and* $\Delta'_{m'} \neq 0$*. Then, the numbers*

$$k := \min\{n \mid \Delta_n \neq 0\} \qquad \textit{and} \qquad \min\{n \mid \Delta'_n \neq 0\}$$

coincide,

$$(H_*(X), \{0, \ldots, 0, \Delta_k, 0, \ldots\}) \qquad \textit{and} \qquad (H_*(X), \{0, \ldots, 0, \Delta'_k, 0, \ldots\})$$

are isomorphic A_∞*-coalgebras and the numbers*

$$\dim Ker\, \Delta_{k|H_p(X)} \qquad \textit{and} \qquad \dim Ker\, \Delta'_{k|H_p(X)}$$

coincide too, for all $p \geq 0$*.*

In the same direction, we prove:

Proposition 3.2.12. *Let* L *denote the Borromean rings. Then, any* good A_∞*-algebra* $\{m_n\}_n$ *on* $H^*(\mathbb{S}^3 - L)$ *will satisfy*

$$m_{3|H^1 \otimes H^1 \otimes H^1} \not\equiv 0,$$

where H^1 *denotes* $H^1(\mathbb{S}^3 - L)$*.*

Note that we can obtain similar results for

$$m_{n|H^{p_1}(X) \otimes \ldots \otimes H^{p_n}(X)}$$

for arbitrary values of $n, p_1, \ldots, p_n$. In this sense we also prove Proposition 3.2.13. To state it, let $n = p + q + r$, $x = (x_1, \ldots, x_p), y = (y_1, \ldots, y_q)$, and $z = (z_1, \ldots, z_r)$. Consider three spaces S_1, S_2, S_3 in $\mathbb{R}^n$ (homeomorphic to three spheres) determined by the equations

$$\begin{aligned}
x &= 0, \qquad \|y\|^2 + \frac{\|z\|^2}{4} = 1 \qquad ((q + r - 1)\text{-dimensional sphere } S_1);\\
y &= 0, \qquad \|z\|^2 + \frac{\|x\|^2}{4} = 1 \qquad ((p + r - 1)\text{-dimensional sphere } S_2);\\
z &= 0, \qquad \|x\|^2 + \frac{\|y\|^2}{4} = 1 \qquad ((p + q - 1)\text{-dimensional sphere } S_3);
\end{aligned}$$

Notice that the case $p = q = r = 1$ yields Borromean rings.

In some sense more general than in the case of knots, these spaces S_i are not pairwise linked, which makes the obvious Massey product being defined, and moreover, denoting by K the disjoint union of S_1, S_2 and S_3, we prove the following:

Proposición 3.2.13. *With the notation above, any* good *A_∞-algebra $\{m_k\}_k$ on $H^*(\mathbb{S}^n - K)$ will have*

$$m_{3|H^p\otimes H^q\otimes H^r} \not\equiv 0,$$

where H^k denotes $H^k(\mathbb{S}^n - K)$.

All these results support our interest in studying the persistence of the mentioned numbers along a filtration, which we explain next in terms of A_∞-coalgebras and homology, although it also works for A_∞-algebras and cohomology.

Let

$$\mathcal{K}\colon \qquad K_0 \longrightarrow K_1 \longrightarrow \dots \longrightarrow K_N$$

be a sequence of topological spaces and continuous maps. In most applications, on works with *filtrations, i.e.,* $\mathcal{K}$ will consist of nested simplicial/cubical/CW subcomplexes and inclusions

$$K_0 \hookrightarrow K_1 \hookrightarrow \dots \hookrightarrow K_N.$$

For every $p \geq 0$ and $0 \leq i \leq j \leq N$, assume that the dimension of the p^{th} homology group of K_i is finite, $\dim H_p(K_i) < \infty$, and let

$$f_p^{i,j}\colon H_p(K_i) \longrightarrow H_p(K_j)$$

and

$$f^{i,j}\colon H_*(K_i) \longrightarrow H_*(K_j)$$

denote the maps induced in homology by the composition

$$K_i \longrightarrow K_{i+1} \longrightarrow \dots \longrightarrow K_j.$$

Classical persistence describes, for each $p \geq 0$, the evolution of the p^{th} Betti number $\beta_p(X) = \dim H_p(X)$ along $\mathcal{K}$ by studying the so called persistent groups

$$H_p^{i,j}(\mathcal{K}) := \operatorname{Im} f_p^{i,j}, \qquad 0 \leq i \leq j \leq N.$$

Specifically, it defines a multiset of intervals (*i.e.*, a set in which every element is an interval and has an assigned multiplicity), called the p^{th} barcode, that satisfies the following:

Theorem 1.2.4. [23, §3] **(Fundamental Theorem of Persistent Homology)** *For every $p \geq 0$, the p^{th} barcode of $\mathcal{K}$ is well-defined and, for all $0 \leq i \leq j \leq N$, the dimension of the persistent group $H_p^{i,j}(\mathcal{K})$ equals the number of intervals in the p^{th} barcode of $\mathcal{K}$ (counted with multiplicities) which contain the interval $[i, j]$.*

In particular, $\dim H_p(K_i)$ *equals the number of intervals in the p^{th} barcode of $\mathcal{K}$ (counted with multiplicities) which contain i.*

We give a new proof of this classical result in §1.2 (see the proofs of Lemma 1.2.7 and Theorem 1.2.4). This result guarantees easy computations (as discussed in §3.3) and allows a visual picture that captures the information of all the persistent groups at once. Moreover, results on stability [28] tell us that in a certain sense, the number of long intervals in the p^{th} barcode is robust with respect to noise and to small variations in the input and that this number encloses significant features of the original data.

Inspired by this, we do the following. Choose a *good* A_∞-coalgebra structure $\{\Delta_n^i\}_n$ on the homology of each term K_i, that we shall use along the rest of the introduction. For each $p \geq 0$ and for each $n \geq 1$, we want to describe the evolution of the number $\dim \operatorname{Ker} \Delta_{n|H_p(X)}$ along $\mathcal{K}$. For that purpose, we define what we call Δ_n*-persistent groups*

$$(\Delta_n)_p^{i,j}(\mathcal{K}) = \operatorname{Im} f_p^{i,j}|_{\cap_{k=i}^{j} \operatorname{Ker}\left(\Delta_n^k \circ f_p^{i,k}\right)}, \qquad 0 \leq i \leq j \leq N,$$

and a multiset of intervals, that we call the p^{th} Δ_n*-barcode*, that satisfy the following:

Corollary 3.6.5. *For every $p \geq 0$ and $n \geq 1$, the p^{th} Δ_n-barcode of $\mathcal{K}$ is well-defined and, for all $0 \leq i \leq j \leq N$, the dimension of the Δ_n-persistent group $(\Delta_n)_p^{i,j}(\mathcal{K})$ equals the number of intervals in the p^{th} Δ_n-barcode of $\mathcal{K}$ (counted with multiplicities) which contain the interval $[i,j]$.*

In particular, $\dim Ker\,\Delta^i_{n|H_p(K_i)}$ equals the number of intervals in the p^{th} Δ_n-barcode of $\mathcal{K}$ (counted with multiplicities) which contain i.

This is one of the most important results in the dissertation. Here is the path we followed to prove it.

First notice that if we choose a random basis of $\bigoplus_{i=0}^N H_p(K_i)$, computing the numbers $\dim H_p^{i,j}(\mathcal{K})$ by studying the evolution of the elements in that basis along $\mathcal{K}$ would be costly inasmuch as it would involve checking if the images by $f_p^{i,j}$ of linearly independent homology classes become linearly dependent. Theorem 1.2.4 can be restated in terms of the existence of a basis which allows a much simpler computation:

Theorem 1.2.8. [23, §3] *For every $p \geq 0$, there exists a basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$ such that, for all $0 \leq i \leq j \leq N$, the dimension of the persistent group $H_p^{i,j}(\mathcal{K})$ can be computed by counting the number of p-homology classes of the basis that do not vanish in the corresponding range:*

$$\dim H_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,j}\beta \neq 0\}.$$

In A_∞-persistence, choosing a random basis of $\bigoplus_{i=0}^N H_p(K_i)$ and computing the numbers $\dim(\Delta_n)_p^{i,j}(\mathcal{K})$ by studying the evolution of the elements in that basis along $\mathcal{K}$ would be even more costly, since it would involve checking further subtleties (see §3.3 for details).

In this direction, we start by proving:

Theorem 3.3.5. *Fix some integers $p \geq 0$ and $n \geq 1$. If $f_p^{i,i+1}(Ker\,\Delta_n^i) \subseteq Ker\,\Delta_n^{i+1}$ for all $0 \leq i < N$, then there exists a basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$ such that, for every $0 \leq i \leq j \leq N$, the dimension of the Δ_n-persistent group $(\Delta_n)_p^{i,j}(\mathcal{K})$ can be computed by counting the number of p-homology classes of the basis that do not Δ_n-fall asleep in the corresponding range:*

$$\dim(\Delta_n)_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \in Ker\,\Delta_n^k - \{0\}\, \textit{for all } k = i, \ldots, j\}.$$

Furthermore, this number equals

$$\#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,j}\beta \in Ker\,\Delta^j - \{0\}\}.$$

See Def. 3.3.1 for the meaning of Δ_n-falling asleep and §3.5 for the explanation of the chosen terminology.

It is worth pointing out that there are cases in which we can indeed apply Theorem 3.3.5:

Theorem 3.4.1. *With coefficients over the rationals $\mathbb{Q}$, let K_0 be a 1-connected CW complex and let N be an integer. If for all $0 < i \leq N$, K_i denotes a 1-connected space obtained attaching to K_{i-1} a cell such that*

$$\dim \bigoplus_{p \geq 0} H_p(K_i) = \dim \bigoplus_{p \geq 0} H_p(K_{i-1}) + 1$$

(and hence every attachment produces new homology), then there exists a good *A_∞-coalgebra $\left(\widetilde{H}_*(K_i), \{\Delta^i_n\}_n\right)$ for each $0 \leq i \leq N$ such that*

$$f^{i,i+1}\left(Ker\,\Delta^i_n\right) \subseteq Ker\,\Delta^{i+1}_n,$$

for every $n \geq 1$ and $0 \leq i < N$.

If we go further ahead and see what happens if we do not restrict to the assumptions in Theorem 3.3.5, we find a potential intermittent behaviour of the homology classes along $\mathcal{K}$ that we describe in §3.5. In essence, Theorem 3.5.1 shows how homology classes can, through the glasses of A_∞-persistence, appear and disappear several times along $\mathcal{K}$.

Finally, thanks to Zigzag Persistence [20], we demonstrate that we can still get a result in the style of Theorem 3.3.5 without the assumptions therein.

For every $0 \leq i \leq N$, fix an arbitrary *good* A_∞-coalgebra structure $\{\Delta^i_n\}_n$ on $H_*(K_i)$. Here is the statement.

Theorem 3.6.1. *For every $p \geq 0$ and $n \geq 1$, there exists a basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$ such that, for all $0 \leq i \leq j \leq N$, the dimension of the Δ_n-persistent group $(\Delta_n)^{i,j}_p(\mathcal{K})$ can be computed by counting the number of p-homology classes of the basis that do not Δ_n-fall asleep in the corresponding* *range:*

$$\dim(\Delta_n)_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \in Ker\,\Delta_n^k - \{0\}\, \textit{for all } k = i, \ldots, j\}.$$

Rearranging the information in the last result allows us to define the p^{th} Δ_n-barcode mentioned before (see §3.6 for details) and to conclude the appealing Corollary 3.6.5 stated above.

On the computational side, we do not implement any algorithms or conduct a performance analysis. Rather, on the one hand, we enhance an attractive algorithm by P. Real *et al.* [80] that enables us to compute *good* A_∞-coalgebra structures on the homology of CW complexes through the use of discrete vector fields (see Algorithm 2.3.5). On the other hand, we set up an abstract algorithm to compute the p^{th} Δ_n-barcodes of A_∞-persistence in the most general case (see Algorithm 3.7.1).

This thesis is organized as follows.

In Chapter 1 we recall the necessary concepts on Persistence and give a new proof for one of its most important theorems. First of all, we display in §1.1 how the initial sequences $\mathcal{K}$ are usually built in two very common applications – the study of point-cloud datasets and digital images. Secondly, in §1.2, we explain the basics of Persistent Homology and give an alternative proof of the Fundamental Theorem of Persistent Homology (see proofs of Lemma 1.2.7 and Theorem 1.2.4). Thirdly, we recall in §1.2 the Zigzag Decomposition heorem and algorithm we will need in Chapter 3.

Chapter 2 consist of the background on A_∞-(co)algebras plus some contributions of ours on how to compute them and on realizing how powerful they are. In §2.1 we set notation and recall the notion of a particular case of A_∞-(co)algebras that is crucial for our approach – *differential graded (co)algebras*. Next, we explore A_∞*-(co)algebras*: first, from the theoretical side in §2.2, as deformations of differential graded (co)algebras, and then computationally in §2.3, explaining our modification to an algorithm by P. Real *et al.* Finally, in §2.4 we show the power of A_∞-(co)algebras through some results, classical and new.

It is in Chapter 3 where we develop the core of our theory. In its introduction, we describe scenarios (that will be extended in §3.2.2 and §3.2.3) in which it is natural to consider the use of A_∞-persistence. We present in §3.1 the numbers (like $\dim \operatorname{Ker} \Delta^i_{n|H_p(X)}$) we want to study, given a *good* A_∞-coalgebra structure on the homology of a space X (or a *good* A_∞-algebra structure on its cohomology). If we take two different such structures on $H_*(X)$ (or on $H^*(X)$), their corresponding numbers can differ in general (§3.2.1), but we point out situations in which these numbers must necessarily be equal and situations in which they do not need to be equal but nevertheless provide information on the homotopy type of X (§3.2.2 and §3.2.3). These numbers will play the role that Betti numbers play in Persistent Homology in the sense that we will define a version of them for sequences of topological spaces and continuous maps (§3.3)

$$K_0 \longrightarrow K_1 \longrightarrow \ldots \longrightarrow K_N$$

and we will prove decomposition results that makes it easier to compute such "persistent numbers" and to visualize them all at once. Specifically, we first make some assumptions in order to prove one such result in §3.3 and present in §3.4 a construction that can be used under somewhat strong conditions to guarantee that we can apply such first decomposition result. If we get rid of all constraints, a curious intermittent behaviour (that justifies the *awake-asleep* terminology) can take place (§3.5) and we need a different approach to prove the most general decomposition theorem (§3.6). Finally, we make some remarks on how to compute A_∞-persistence in the most general case (§3.7).

In addition, as some of our results and examples lie heavily in classical facts from rational homotopy theory, we include for completeness an Appendix containing an overview of this theory and the results we use.

Chapter 1

Persistent Homology

Persistent Homology sprang independently from three different sources around the nineties. These three pillars consist of the *Size Theory* created by the group of M. Ferri and P. Frosini at the University of Bologna, the doctoral work of V. Robins at the University of Colorado Boulder, and the biogeometry project of H. Edelsbrunner at Duke University. These different sources led to two significantly different approaches to describe the *shape* of a space, so we should clarify which one this thesis follows.

On the one hand, Size Theory [8, 16, 32, 47] emphasizes that the meaning of the term "shape" depends on the context. If we want to study specific shape attributes of some objects, we must first think of continuous real-valued functions that take into account only the shape properties that are relevant to our problem, while disregarding the irrelevant ones. Then, we do the following with each of these functions: using a construction that involves 0-dimensional homology, the function induces a *signature* for each of the spaces to be studied – this is a simpler object associated to a space so that congruent shapes have the same signature. Some of these signatures are built by means of functions encoding distance from center of mass or from an axis, curvature or torsion, speed or greyscale colour tone, when applicable. Size theory has been used in this fashion to identify writers from their handwriting [44], letters from monograms [43] and also sign language gestures [53, 62, 100]; to classify leukocytes [45] and hepatic lesions [2]; to evaluate risk and intensity of cyclones [6] and more [9, 46, 76, 77].

On the other hand, the approach to Persistent Homology by V. Robins, H. Edelsbrunner et al. focuses only on shape properties that are preserved by homotopy equivalences. More specifically, they try to compute the homology groups (of all degrees, not only degree 0 as above) of spaces, usually unknown ones, described for instance by means of finite point samples. This approach has had notable applications in many areas such as digital imaging [89], sensor networks [33], molecular modelling [3], dynamical systems [38, 88] and speech pattern analysis [15].

In this dissertation we follow this second approach. We do not restrict to only (co)homology groups, but we study further structure on (co)homology as well. The idea would be to try to construct signatures for point-cloud datasets, induced by higher order operations in (co)homology.

We devote this chapter to present the necessary background on the theory of Persistence we use. Here are some classical references of the first steps in Persistent Homology [23, 39, 103], and some of the many surveys on the topic available in the literature [18, 37, 50, 102].

Consider an unknown closed subset X of a metric space and a known finite subset $\mathcal{P}$ of X. We would like to estimate topological properties of the space X out of the point sample $\mathcal{P}$. For instance, $\mathcal{P}$ may come from some statistical data and we may want to know about the space X modelling the data. Specifically, we are interested in the Betti numbers of the underlying space, $\{\beta_p(X)\}_{p\geq 0}$. The problem when trying to estimate them through a finite point sample is that these invariants are not at all robust to discontinuous changes in the input space. Persistent homology tackles this issue by constructing invariants which are robust in this sense but still have the kind of discriminatory power of Betti numbers.

Here we present an outline of this chapter as we roughly explain how Persistent Homology works.

Out of the point cloud $\mathcal{P}$ (or another input such as a digital image, for instance), one starts by building a sequence of nested simplicial subcomplexes

$$K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_N$$

so that several coarseness levels are considered at once. In Section 1.1 we describe how to get filtrations out of each of the two settings we are most interested in.

Then, one describes the evolution of the homology of the sequence through invariants of

$$H_*(K_0) \longrightarrow H_*(K_1) \longrightarrow \ldots \longrightarrow H_*(K_N)$$

called *barcodes* (Section 1.2). This statement is made precise in Theorem 1.2.4, the Fundamental Theorem of Persistent Homology. This result was originally proved by Carlsson and Zomorodian [23]. In Section 1.2, we will give our own proof of it, spot the main tool that lies behind each of the two approaches and mention some of the consequences of Stability, which give barcodes their importance.

In Section 1.3 we introduce zigzag persistence, a variant of persistent homology that we will use later on in Chapter 3.

Throughout this chapter, we will work over a field of coefficients $\Bbbk$. For instance, by $H_*(X)$ we will mean $H_*(X;\Bbbk)$, the homology of X with coefficients in $\Bbbk$.

1.1 Filtrations

The input of a persistent homology algorithm is usually a **filtration** [39, Section 3], [23, Section 4.2], *i.e.,* a sequence of nested simplicial/cubical/CW subcomplexes and inclusions

$$\mathcal{K}: \qquad K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_N.$$

Here we show classical ways of getting filtrations out of the two kinds of data we are most interested in. See for instance [37] to see how to get filtrations out of Morse or tame functions.

1.1.1 Simplicial complexes on a point cloud

Let X be a closed subset of a metric space M and fix a finite discrete set $\mathcal{P}$ of points in X, that we think of as a point-cloud sampling of X.

We are going to build a filtration of simplicial complexes out of $\mathcal{P}$ through which to study the topology of the space X. This is useful if we do not know X but only the point cloud $\mathcal{P}$.

Let $\varepsilon \geq 0$ be a parameter.

Definition 1.1.1. Given the point cloud $\mathcal{P}$ and a parameter $\varepsilon \geq 0$, the **Čech complex $\check{\mathcal{C}}_\varepsilon(\mathcal{P})$** is the abstract simplicial complex where the k-simplices correspond to collections of $k+1$ different points $x_0, \ldots, x_k$ in $\mathcal{P}$ such that the closed balls of radius $\varepsilon/2$ centered at these points have non-empty intersection,

$$\bigcap_{i=0}^{k} \bar{B}\left(x_i, \varepsilon/2\right) \neq \emptyset.$$

In Figure 1.1 we show an example of Čech complex.

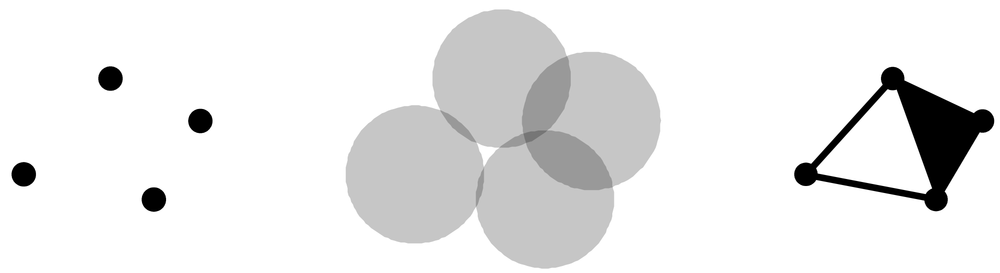


Figure 1.1: From left to right: a set $\mathcal{P}$ of four points in the real plane with the standard Euclidean distance, balls of radius ε around each of the four points, and the Čech complex $\check{\mathcal{C}}_{2\varepsilon}(\mathcal{P})$.

Theorem 1.1.2. [11, Corollary 3 in Section 9] **(Čech Theorem** or **Nerve Lemma)** $\check{\mathcal{C}}_\varepsilon(\mathcal{P})$ *has the homotopy type of the union of closed balls*

$$\bigcup_{x \in \mathcal{P}} \bar{B}\left(x, \varepsilon/2\right).$$

This result tells us that the Čech complex is an appropriate simplicial model for the topology of a point cloud, so we may wonder what the ideal parameter ε is (if it exists) in order to get the most accurate topological description of X out of $\check{\mathcal{C}}_\varepsilon(\mathcal{P})$. It turns out that such a parameter may be hard (or impossible!) to find, so one can consider instead several different values of ε and see how the topology evolves as the parameter ε grows. This is done by choosing positive numbers $\{\varepsilon_i\}_{i=0}^N$ such that $\varepsilon_i < \varepsilon_j$ whenever $i < j$ and the considering the sequence of inclusions of simplicial complexes

$$\mathcal{K}: \qquad \check{\mathcal{C}}_{\varepsilon_0}(\mathcal{P}) \hookrightarrow \check{\mathcal{C}}_{\varepsilon_1}(\mathcal{P}) \hookrightarrow \ldots \hookrightarrow \check{\mathcal{C}}_{\varepsilon_N}(\mathcal{P})$$

to which apply Persistent Homology.

In practise, one may use simplicial complexes other than Čech's and apply the same procedure. Here we mention just a couple of them. One reason for choosing alternative complexes is that, although the Čech complex is topologically faithful in the sense of Theorem 1.1.2, it is hard to compute and store. The **Vietoris-Rips complex** or simply **Rips complex** is a different kind of simplicial complex also used in this setting [17, 19, 22, 24, 33] because it is computationally much easier to handle, for it is determined by its 1-skeleton. It is less clear, though, which kind of topological information does the Rips complex encode from the point cloud. See [25] for more in this direction.

When, for instance, not just the Čech complex but even the Rips complex become too large, one may compute a simplicial complex on just a well chosen subsample of $\mathcal{P}$ – a so called *witness complex* [19].

1.1.2 Cubical complexes on a digital image

The most convenient models for digital images are given by cubical complexes [65], [31], [29], [30], which intuitively makes sense, since the information in a digital image is encoded in pixels or their higher dimensional analogues, which are cubes themselves. We use the approach of V. Robins et al. [89] for the following definitions.

A 3D greyscale digital image can be viewed as a function of the form

$$f\colon \mathcal{D} \longrightarrow \mathbb{R}$$

where

$$\mathcal{D} = \{(i,j,k) \in \mathbb{N}^3 \,|\, 1 \leq i \leq I, 1 \leq j \leq J, 1 \leq k \leq K\}.$$

A point $x \in \mathcal{D}$ is called a **voxel** (the analogue in 3D of a pixel). For instance, we can see f as a funciton

$$f\colon \mathcal{D} \longrightarrow [0,1] \subseteq \mathbb{R}$$

where the lower the value, the darker the voxel, so $f(x) = 0$ means that the voxel x is completely black and $f(x) = 1$ that it is completely white.

Notice that an RGB colour digital image would then consist of a function of the form

$$f\colon \mathcal{D} \longrightarrow \mathbb{R}^3,$$

where, for each $x \in \mathcal{D}$, $f(x) = (r(x), g(x), b(x))$ would consist of the red, green and blue components of the voxel x.

Let us stay with the greyscale case for simplicity. For any threshold $\varepsilon \geq 0$, we can associate to the image f the cubical complex $\mathcal{C}_\varepsilon$ where the n-cubes are determined as follows:

- 0-cubes (vertices): voxels $x \in \mathcal{D}$ whose greyscale value is less than or equal to the threshold, $f(x) \leq \varepsilon$.
- 1-cubes: unit edges determined by horizontally or vertically adjacent vertices.
- 2-cubes: unit squares determined by 2×2 voxel blocks whose 4 voxels are vertices.
- 3-cubes: unit cubes determined by $2 \times 2 \times 2$ voxel blocks whose 8 voxels are vertices.

This indeed determines a cubical complex and if we choose an increasing sequence of thresholds $\{\varepsilon_i\}_{0\leq i\leq N}$, we get a sequence of cubical subcomplexes

$$\mathcal{C}_{\varepsilon_0} \hookrightarrow \mathcal{C}_{\varepsilon_1} \hookrightarrow \ldots \hookrightarrow \mathcal{C}_{\varepsilon_N}.$$

1.2 Persistent Homology

Let

$$\mathcal{K}: \qquad K_0 \longrightarrow K_1 \longrightarrow \ldots \longrightarrow K_N$$

be a sequence of topological spaces and continuous maps. We are interested in finding out how homology evolves along this sequence.

As we have mentioned before, in most applications, $\mathcal{K}$ consists of nested simplicial/cubical/CW subcomplexes and inclusions

$$K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_N,$$

just as in §1.1. For every $p \geq 0$ and $0 \leq i \leq j \leq N$, assume that the dimension of the p^{th} homology group of K_i is finite, $\dim H_p(K_i) < \infty$, and let

$$f_p^{i,j} \colon H_p(K_i) \longrightarrow H_p(K_j)$$

and

$$f^{i,j} \colon H_*(K_i) \longrightarrow H_*(K_j)$$

denote the maps induced in homology by the composition

$$K_i \longrightarrow K_{i+1} \longrightarrow \ldots \longrightarrow K_j.$$

Definition 1.2.1. Let $0 \leq i \leq j \leq N$ and $p \geq 0$.

- A class $\alpha \in H_p(K_i)$ is **born** at K_i if

$$\alpha \neq 0 \qquad \text{and} \qquad \alpha \notin \operatorname{Im} f^{i-1,i}.$$

- A class $\alpha \in H_p(K_i)$ that is born at K_i is said to be **alive** at K_j (where $i \leq j$) if
$$f^{i,j}\alpha \notin \operatorname{Im} f^{i-1,j}. \tag{1.2.1}$$
Notice that (4.0.1) is equivalent to saying that, for all $i < k \leq j$,
$$f^{i,k}\alpha \notin \operatorname{Im} f^{i-1,k},$$
which in turn implies that $f^{i,k}\alpha \neq 0$ for all $i < k \leq j$.

- A class $\alpha \in H_p(K_i)$ that is born at K_i is said to **die** at K_j (where $i < j$) if it is alive at K_{j-1} but not at K_j, *i.e.*, if
$$f^{i,j-1}\alpha \notin \operatorname{Im} f^{i-1,j-1} \qquad \text{and} \qquad f^{i,j}\alpha \in \operatorname{Im} f^{i-1,j}.$$
This implies that either $f^{i,j}\alpha = 0$ or there exists some $\beta \neq 0 \in H_p(K_{i-1})$ such that
$$f^{i-1,j}\beta = f^{i,j}\alpha,$$
in which case we say that α and β have **merged,** that the class β is **older** (because it was born at K_k for some $k \leq i-1$, whereas α was born at K_i) and that β is the one among the two that represents the merged class (this is called the **Elder Rule**).

We will often say that a certain number n of classes are born at K_i actually meaning that n *linearly independent* classes are born at that stage.

Example 1.2.2. In this example we work on a point sample $\mathcal{P}$ of an annulus. In Figure 1.2 we show the topological space $\bigcup_{x\in\mathcal{P}} \bar{B}(x, \varepsilon/2)$ for some values of the parameter ϵ. In practice, we would study the Čech complex $\check{\mathcal{C}}_\varepsilon(\mathcal{P})$, which is a simplicial complex with the homotopy type of $\bigcup_{x\in\mathcal{P}} \bar{B}(x, \varepsilon/2)$ (see Theorem 1.1.2), so we will also use the notation $\check{\mathcal{C}}_\varepsilon(\mathcal{P})$ throughout the example. $\check{\mathcal{C}}_1(\mathcal{P})$ is homeomorphic to $\mathcal{P}$; it consists of a point for each point in $\mathcal{P}$ and it has no edges (see Figure 1.2a). When $\epsilon = 2$, several connected components merge in clusters and $\beta_1\left(\check{\mathcal{C}}_2(\mathcal{P})\right)$ is not zero anymore. Since it is hard to spot it in the pictures, we claim that $\beta_1\left(\check{\mathcal{C}}_2(\mathcal{P})\right) = 14$, and we have marked with arrows 14 non-trivial 1-cycles of $\bigcup_{x\in\mathcal{P}} \bar{B}\,(x, 2/2)$ whose homology classes are born at this step and generate $H_1\left(\bigcup_{x\in\mathcal{P}} \bar{B}\,(x, 2/2)\right)$ (see Figure 1.2b). For $\epsilon = 3$, all connected components have merged in a single one, and $H_1\left(\check{\mathcal{C}}_3\,(\mathcal{P})\right) \cong \Bbbk \oplus \Bbbk$, where one of these copies of the field of coefficients $\Bbbk$ is generated by the 1-cycle marked with an arrow and the other is generated by the big 1-cycle (see Figure 1.2c). Notice that the homology class of the small 1-cycle is born at this step, and not in the previous one $\varepsilon = 2$, because this class is not the image of a homology class of $\bigcup_{x\in\mathcal{P}} \bar{B}\,(x, 2/2)$, since all 1-dimensional homology classes of $\bigcup_{x\in\mathcal{P}} \bar{B}\,(x, 2/2)$ die entering $\bigcup_{x\in\mathcal{P}} \bar{B}\,(x, 3/2)$. For values of ε between 4 and 67, both inlcuded, the simplicial complex $\check{\mathcal{C}}_\epsilon\,(\mathcal{P})$ consists of one connected component and has $\beta_1\left(\check{\mathcal{C}}_\varepsilon\,(\mathcal{P})\right) = 1$, this Betti number being non-zero thanks to the central 1-cycle (see Figures 1.2d, 1.2e, 1.2f, 1.2g). For $\varepsilon \geq 68$, $\check{\mathcal{C}}_\epsilon\,(\mathcal{P})$ is contractible (see Figure 1.2h). In particular, there are both a 0-dimensional homology class and a 1-dimensional homology class that are alive for a wide range of values of ε.

Definition 1.2.3. For every $0 \leq i \leq j \leq N$ and $p \geq 0$ we define the $\boldsymbol{p}^{\mathbf{th}}$ **persistent homology group** of the sequence $\mathcal{K}$ between K_i and K_j as the vector space

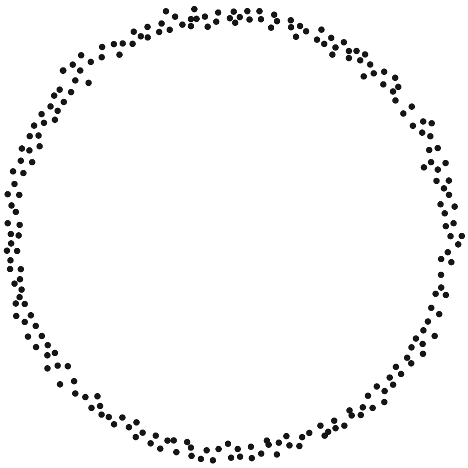
$$\boldsymbol{H_p^{i,j}(\mathcal{K})} := \operatorname{Im} f_p^{i,j}$$

and its corresponding **persistent Betti number** as

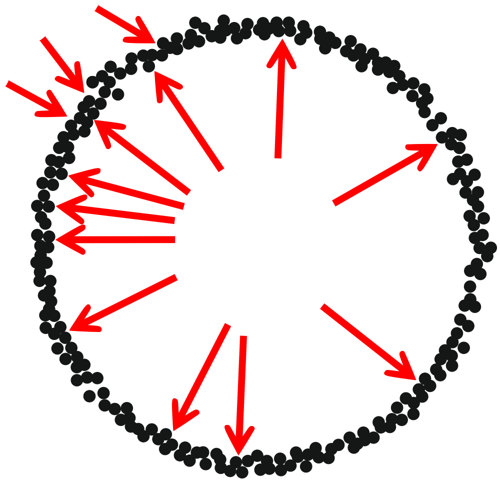
$$\boldsymbol{\beta_p^{i,j}(\mathcal{K})} := \dim H_p^{i,j}(\mathcal{K}).$$

We will usually omit the reference to the sequence $\mathcal{K}$ and simply write $\boldsymbol{H_p^{i,j}}$ and $\boldsymbol{\beta_p^{i,j}}$.

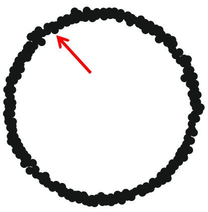
For each homology degree $p \geq 0$, we build a multiset of intervals to encode this information. Such a multiset is called the $\boldsymbol{p}^{\mathbf{th}}$ **barcode** and is defined as follows:



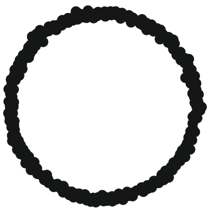
(a) $\epsilon = 1$



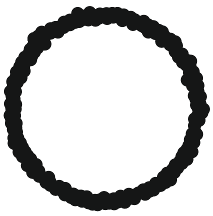
(b) $\epsilon = 2$



(c) $\epsilon = 3$

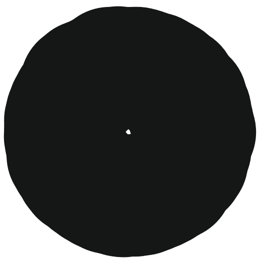


(d) $\epsilon = 4$

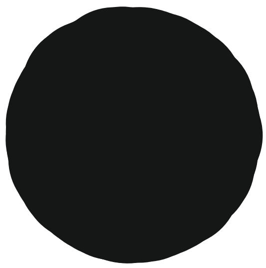


(e) $\epsilon = 5$

(f) $\epsilon = 44$



(g) $\epsilon = 65$



(h) $\epsilon = 68$

Figure 1.2: Given a point sample $\mathcal{P}$ of an annulus, we show some terms of a filtration consisting of topological spaces $\bigcup_{x\in\mathcal{P}} \bar{B}\,(x, \varepsilon/2)$ for a growing parameter ϵ. Figures 1.2a and 1.2b are scaled a little bigger to see them in more detail.

If at each stage we can attach an arbitrary finite number of cells, then two linearly independent classes can be born at the same time and merge some time later on, so we must have a way to choose which one *survives* and represents the merged class. To do so, it is enough to order the cells attached at each stage. In this general case, the p^{th} barcode consists of intervals of the form $[i, j)$ for $0 \leq i < j \leq N$, with **multiplicity**

$$\mu_p^{i,j} := \beta_p^{i,j-1} - \beta_p^{i-1,j-1} - \beta_p^{i,j} + \beta_p^{i-1,j}$$

and intervals of the form $[i, \infty)$ with multiplicity

$$\mu_p^{i,\infty} := \beta_p^{i,N} - \beta_p^{i-1,N}.$$

Theorem 1.2.4 proves that the construction of this multiset is well defined, *i.e.*, each $\mu_p^{i,j}$ and $\mu_p^{i,\infty}$ is greater or equal than zero.

In Figure 1.3 we show an example of a barcode.

Figure 1.3: Here we show a representation of the 1st barcode (the one associated to the evolution of the first Betti number, β_1) of the filtration in Example 1.2.2, where by each bar that starts at i and ends at j we mean an interval of the form $[i, j)$ in the barcode.

There is another graphical representation used to encode this information – that of *persistent diagrams* [39]. They give the exact same information as barcodes, so we will just speak in terms of the latter.

Next we proceed to give our own proof of the Fundamental Theorem of Persistent Homology, that we choose to restate as follows:

Theorem 1.2.4. [23, §3] **(Fundamental Theorem of Persistent Homology)** *For every $p \geq 0$, the p^{th} barcode of $\mathcal{K}$ is well defined and for all $0 \leq i \leq j \leq N$, the dimension of the persistent group $H_p^{i,j}(\mathcal{K})$ equals the number of intervals in the p^{th} barcode of $\mathcal{K}$ (counted with multiplicities) which contain the interval $[i, j]$.*

In particular, $\dim H_p(K_i)$ *equals the number of intervals in the p^{th} barcode of $\mathcal{K}$ (counted with multiplicities) which contain i.*

We use Figure 1.4 to illustrate Theorem 1.2.4.

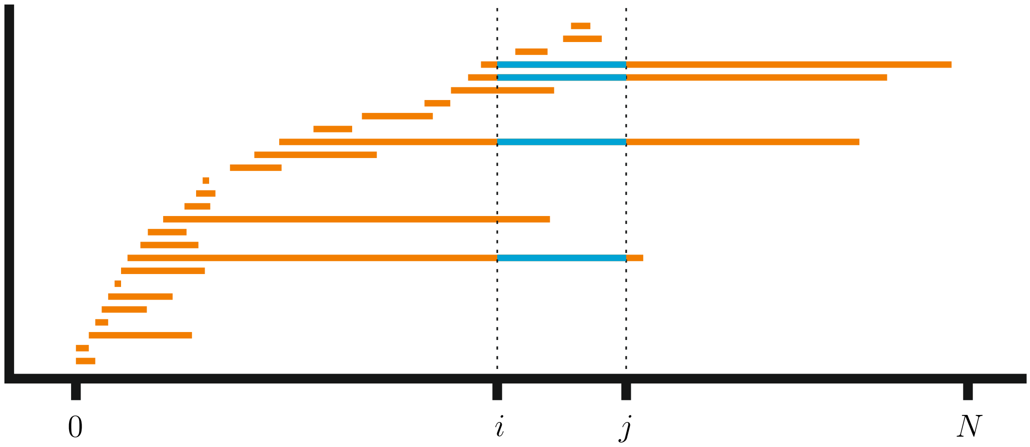


Figure 1.4: If this represents the p^{th} barcode of $\mathcal{K}$, Theorem 1.2.4 asserts that $\dim H_p^{i,j}(\mathcal{K}) = 4$, since there are four intervals containing $[i, j]$.

This result was first proved by Carlsson and Zomorodian in [23, §3] with a slightly different notation and vocabulary. They use the structure theorem for finitely generated modules over PIDs to show that this barcode construction is well-defined, and once this is done, it is straightforward to check that it implies the result. We will prove the same thing using Frobenius inequality on matrix ranks instead of the aforementioned structure theorem.

Proposition 1.2.5. ***(Frobenius inequality on matrix ranks)*** *Let A, B and C be matrices such that the products AB, ABC and BC are defined. Then*

$$\operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank}(B) + \operatorname{rank}(ABC).$$

First we prove a lemma that will have Theorem 1.2.4 as a corollary.

Definition 1.2.6. A **persistence module** is a sequence of finite dimensional vector spaces and linear maps between them of the form

$$V_0 \longrightarrow V_1 \longrightarrow \dots \longrightarrow V_N.$$

Lemma 1.2.7. *Let*

$$V_0 \xrightarrow{f^{0,1}} V_1 \xrightarrow{f^{1,2}} \dots \xrightarrow{f^{N-1,N}} V_N$$

be a persistence module and set

$$f^{i,j} := \begin{cases} f^{j-1,j} \circ \dots \circ f^{i,i+1}, & i+1 < j, \\ id_{V_i}, & i = j. \end{cases}$$

Then there always exists a multiset M of intervals of the form $[i,j)$ for $0 \leq i < j \leq N$, and of the form $[i,\infty)$ for $0 \leq i \leq N$ such that for all $0 \leq i \leq j \leq N$, the dimension $\dim \mathrm{Im} f^{i,j}$ can be computed as the sum of the multiplicity of each interval in M that contains $[i,j]$.

Proof. Let $\{d^{i,j}\}_{i \leq j}$ be a family of integers such that $d^{i,j} = 0$ for all i,j that do not satisfy $0 \leq i \leq j \leq N$. For all $0 \leq i \leq j \leq N$, define

$$N^{i,j} := d^{i,j} - d^{i-1,j} - d^{i,j+1} + d^{i-1,j+1}.$$

If $N^{i,j} \geq 0$ for every $0 \leq i \leq j \leq N$, then we can define a multiset M of intervals consisting of $N^{i,j-1}$ intervals of the form $[i,j)$, for all $0 \leq i < j \leq N$, and $N^{i,N}$ intervals of the form $[i,\infty)$.

Notice that for every $0 \leq i \leq j \leq N$, the sum of the multiplicity of each interval in M that contain $[i,j]$ equals $\sum_{k \leq i} \sum_{j \leq l} N^{k,l}$, which is precisely

$$\sum_{k \leq i} \sum_{j \leq l} \left(d^{k,l} - d^{k-1,l} - d^{k,l+1} + d^{k-1,l+1} \right) = \sum_{k \leq i} \left(d^{i,j} - d^{i-1,j} \right) = d^{i,j}.$$

This shows a constructive way of building, out of a family of integers $\{d^{i,j}\}_{i \leq j}$, a multiset M of intervals of the wanted form such that, for all $0 \leq i \leq j \leq N$,

$$\dim \mathrm{Im} f^{i,j} = \#\{\text{intervals in M that contain } [i,j]\},$$

as long as two conditions hold:

(a) $d^{i,j} = 0$ for all i, j that do not satisfy $0 \leq i \leq j \leq N$.

(b) $N^{i,j} = d^{i,j} - d^{i-1,j} - d^{i,j+1} + d^{i-1,j+1} \geq 0$, for all $0 \leq i \leq j \leq N$.

Finally, if we define

$$d^{i,j} := \begin{cases} \dim \operatorname{Im} f^{i,j}, & 0 \leq i \leq j \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

then condition (a) holds by construction and condition (b) reduces to Frobenius inequality on matrix ranks, finishing the proof. □

As a corollary, we prove Theorem 1.2.4.

Proof. Apply Lemma 1.2.7 to the p^{th} homology of the sequence $\mathcal{K}$,

$$H_p(K_0) \xrightarrow{f_p^{0,1}} H_p(K_1) \xrightarrow{f_p^{1,2}} \ldots \xrightarrow{f_p^{N-1,N}} H_p(K_N).$$

Since the multiset construction from the proof of Lemma 1.2.7 applied to this persistence module coincides with the p^{th} barcode defined above, we obtain the result. □

Notice that $H_p^{i,j}$ describes the p-dimensional homology classes that are born at or before K_i and die later than K_j, *i.e.*, the classes that are alive from K_i to K_j. If we take a random basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$, the persistent Betti numbers can then be therefore expressed as

$$\beta_p^{i,j} = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid \beta \text{ is alive at } K_k \text{ for } k = i, \ldots, j\},$$

but checking whether a class is alive at some point involves checking if it has merged with an older class, which is costly. A restatement of Theorem 1.2.4 guarantees there is always a good basis for which no checking mergings is needed:

Theorem 1.2.8. [23, §3] *For every $p \geq 0$, there exists a basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$ such that, for all $0 \leq i \leq j \leq N$, the dimension of the persistent group*

$H_p^{i,j}(\mathcal{K})$ can be computed by counting the number of p-homology classes of the basis that do not vanish in the corresponding range:

$$\dim H_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,j}\beta \neq 0\}.$$

Let X be a closed subset of a metric space and $\mathcal{P}$ a finite set of points in X. How do the barcodes of $\mathcal{P}$ change if we perturb it a little by either adding some noise points or by choosing a different point sample of X? Also, how do the barcodes of say a digital image of a lung change if we perturb it a little by either adding some noise points or by using a different method (maybe a different machine) to get another image of the same lung? Stability [28] rigurously proves that if these perturbations are small, then for every $p \geq 0$, the number of *long* intervals remain the same [28, Main Thm.]. Hence, these *invariant* quantities are the significant information in a barcode.

Furthermore, stability also guarantees that if $\mathcal{P}$ is a good enough point sample of X, we can compute the Betti numbers $\beta_p(X)$ out of the persistent homology of $\mathcal{P}$ [28, Homology Inference Thm.]. Chazal and Lieutier [26] independently obtained a similar result without the use of stability and [32, 82] proved more restrictive versions of this result.

Example 1.2.9. Consider the filtration in Example 1.2.2, created from a point sample $\mathcal{P}$ of an annulus. There are exactly one long interval in its 0^{th} barcode and one long interval in its 1^{st} barcode (see Figure 1.3), which agrees with the fact that the 0^{th} and 1^{st} Betti numbers of an annulus are $\beta_0 = \beta_1 = 1$.

1.3 Zigzag persistence

We devote Chapter 3 to describing our own new tool – persistence of A_∞-structures. A crucial tool to prove one of the main results in Chapter 3 (Theorem 3.6.1) is zigzag persistence, so we introduce this variant of persistence next.

In some situations, there is a persistence-flavoured kind of information we would like to obtain but we do not have a persistence module

$$U_1 \longrightarrow U_2 \longrightarrow \dots \longrightarrow U_r,$$

but rather just a sequence of the form

$$U_1 \longleftrightarrow U_2 \longleftrightarrow \dots \longleftrightarrow U_r,$$

in which each $\longleftrightarrow$ can be either a forward map $\longrightarrow$ or a backward map $\longleftarrow$ (see Section 3.6 for an example of this). Zigzag persistence, introduced by Carlsson and de Silva in [20], describes some barcodes one can still get out of a structure like this.

We now recall the basics of zigzag persistence [20]. A **zigzag module** $\mathbb{U}$ is a sequence of finite dimensional $\Bbbk$-vector spaces

$$U_1 \overset{p_1}{\longleftrightarrow} U_2 \overset{p_2}{\longleftrightarrow} \cdots \overset{p_{r-1}}{\longleftrightarrow} U_r$$

in which each p_i can be either a forward map $\xrightarrow{f_i}$ or a backward map $\xleftarrow{g_i}$. Hence, persistence modules can be seen as zigzag modules where all maps are oriented in the same direction. A **submodule** $\mathbb{X} \subset \mathbb{U}$ of $\mathbb{U}$ is a zigzag module

$$X_1 \longleftrightarrow X_2 \longleftrightarrow \dots \longleftrightarrow X_r$$

where each $X_i \subset U_i$ is a subspace, either

$$f_i(X_i) \subset X_{i+1} \qquad \text{or} \qquad g_i(X_{i+1}) \subset X_i,$$

and the maps are the restrictions of those of $\mathbb{U}$.

A submodule $\mathbb{X} \subset \mathbb{U}$ is a **direct summand**, or simply, a **summand** of $\mathbb{U}$, if there exists another submodule $\mathbb{Y} \subset \mathbb{U}$ such that $U_i = X_i \oplus Y_i$, for all i. We write $\mathbb{U} = \mathbb{X} \oplus \mathbb{Y}$. It is important to remark that submodules are not, in general, summands. For instance [20, Ex. 2.1], in the zigzag module

$$\Bbbk \xrightarrow{1} \Bbbk,$$

the map

$$0 \longrightarrow \Bbbk$$

constitutes a submodule but not a summand.

Given integers $1 \leq a \leq b \leq r$, an **interval of the form** $\mathbb{I}[\boldsymbol{a}, \boldsymbol{b}]$ denotes a zigzag module

$$I_1 \xleftrightarrow{p_1} I_2 \xleftrightarrow{p_2} \cdots \xleftrightarrow{p_{r-1}} I_r,$$

in which,

$$I_i = \begin{cases} \Bbbk, & a \leq i \leq b, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$p_i = \begin{cases} 1_{\Bbbk}, & a \leq i \leq b-1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the main result in zigzag persistence states:

Theorem 1.3.1. [48], [20, Theorem 2.5] **(Zigzag's Interval Decomposition Theorem)** *Every zigzag module* $\mathbb{U}$ *is isomorphic to a direct sum of intervals,*

$$\mathbb{U} \cong \oplus_j \mathbb{I}[a_j, b_j].$$

Gabriel's paper [48] (in German) is the original statement of the theorem with a different vocabulary, [34] contains a friendlier form, and in [20, §4.1] the reader can find an extended and constructive version of this statement which can be used to develop explicit algorithms for the computation of the involved intervals.

This result tells us we can encode all the information in a zigzag module

$$U_1 \longleftrightarrow U_2 \longleftrightarrow \ldots \longleftrightarrow U_r$$

by means of a multiset of integer intervals of the form

$$[a, b] \cap \{1, \ldots, r\},$$

$1 \leq a \leq b \leq r$, that we write as $[a, b]$ for simplicity. This multiset is called a **zigzag barcode.**

A notion of zigzag persistent diagram similar to that of the classical persistem diagrams can be constructed too, carrying the same information of the zigzag barcode.

Remark 1.3.2. Notice the difference in notation between ordinary barcodes and zigzag barcodes of a persistent module. An interval $[a, b+1)$ in the classical barcode is written as $[a, b]$ in the zigzag barcode, and one of the form $[a, \infty)$ in the classical becomes $[a, r]$.

On the one hand, the notation $[a, b)$ for classical barcodes is particularly natural if we think of the indexing parameter as continuous – the interval $[a, b)$ will correspond to a class that is alive at $b-1$ and not at b, but we do not know where it exactly died. On the other hand, the closed intervals are a good notation for zigzag barcodes because they allow us to keep symmetry between forward and backward maps.

Zigzag decompositions present an extra hazard non-existent in ordinary persistence. We will deal with that in Section 3.6.

1.3.1 Algorithmic approach

When explaining how to compute A_∞-persistence (§3.7), we will use an algorithm by Carlsson and de Silva [20] to compute the barcode decomposition of zigzag modules, so we include here such an algorithm.

Theorem 1.3.1 claimed that every zigzag module $\mathbb{U}$ is isomorphic to a direct sum of intervals

$$\mathbb{U} \cong \oplus_j \mathbb{I}[a_j, b_j].$$

We now rephrase Algorithm 4.4 in [20] to compute such a decomposition out of an arbitrary zigzag module.

The input is a zigzag module

$$\mathbb{V}\colon\ V_1 \overset{p_1}{\longleftrightarrow} V_2 \overset{p_2}{\longleftrightarrow} \cdots \overset{p_{n-1}}{\longleftrightarrow} V_n$$

in which each $V_k \overset{p_k}{\longleftrightarrow} V_{k+1}$ can be either a forward map $V_k \xrightarrow{f_k} V_{k+1}$ or a backward map $V_k \xleftarrow{g_k} V_{k+1}$. The output consists of two collections of numbers $\{b_i^k\}_{1\leq i\leq k\leq n}$ and $\{c_i^k\}_{1\leq i\leq k\leq n}$, whose meaning we will explain below.

Here is a way to write the algorithm.

Algorithm 1.3.3. [20, Algorithm 4.4]

$(R_0^1, R_1^1) := (0, V_1)$

$r_1^1 := \dim V_1$

$b_1^1 := 1$

`for` $k = 1$ `to` $n-1$ `do`

  `if` p_k `is a forward map` $V_k \xrightarrow{f_k} V_{k+1}$ `do`

    $\big(R_0^{k+1}, \dots, R_{k+1}^{k+1}\big) := \big(f_k(R_0^k), f_k(R_1^k), \dots, f_k(R_k^k), V_{k+1}\big)$

    $\big(b_1^{k+1}, \dots, b_{k+1}^{k+1}\big) := \big(b_1^k, \dots, b_k^k, k+1\big)$

  `else if` p_k `is a backward map` $V_k \xleftarrow{g_k} V_{k+1}$ `do`

    $\big(R_0^{k+1}, \dots, R_{k+1}^{k+1}\big) := \big(0, {g_k}^{-1}(R_0^k), \dots, {g_k}^{-1}(R_{k-1}^k), {g_k}^{-1}(R_k^k)\big)$

    $\big(b_1^{k+1}, \dots, b_{k+1}^{k+1}\big) := \big(k+1, b_1^k, \dots, b_k^k\big)$

  `end if`

  $(r_1, \dots, r_{k+1}) := \big(\dim R_1^{k+1}, \dots, \dim R_{k+1}^{k+1}\big)$

  $\big(r_1^{k+1}, \dots, r_{k+1}^{k+1}\big) := (r_1, r_2 - r_1, r_3 - r_2, \dots, r_{k+1} - r_k)$

  `for` $i = 1$ `to` k `do`

    `if` p_k `is a forward map` $V_k \xrightarrow{f_k} V_{k+1}$ `do`

      $c_i^k := r_i^k - r_i^{k+1}$

    `else if` p_k `is a backward map` $V_k \xleftarrow{g_k} V_{k+1}$ `do`

      $c_i^k := r_i^k - r_{i+1}^{k+1}$

    `end if`

  `end for`

$(c_1^n, \dots, c_n^n) := (r_1^n, \dots, r_n^n)$

After running the algorithm, we get two collections of numbers $\{b_i^k\}_{1 \le i \le k \le n}$ and $\{c_i^k\}_{1 \le i \le k \le n}$. The way to interpret them is the following [20, Theorem 4.1]. For all $1 \le i \le k \le n$, the interval $\mathbb{I}[b_i^k, k]$ appears with multiplicity c_i^k in the decomposition of $\mathbb{V}$, *i.e.*,

$$\mathbb{V} \cong \bigoplus_{1 \le i \le k \le n} c_i^k \, \mathbb{I}[b_i^k, k].$$

More on computations of zigzag decompositions can be found in [20, Sections 4 and 5] and in [21].

Chapter 2

A_∞-structures

In the sixties, J. D. Stasheff first introduced A_∞-algebras as a tool to study H-spaces [94]. In the seventies and eighties, A_∞-algebras were further developed with algebraic topology in mind [1, 10, 60, 72, 86, 91]. A decade later, the first connections between A_∞-algebras and fields like geometry and mathematical physics were established [49, 63, 73, 95]. Half a century after the invention of A_∞-(co)algebras, we resolve on applying them to Topological Data Analysis. Specifically, we have seen in Chapter 1 the kind of information Persistent Homology can provide us with, which is at the level of Betti numbers. A_∞-(co)algebras are structures that can give far more information than Betti numbers, and our goal is to add part of their discriminatory power to Persistence in Chapter 3. This is why we explore A_∞-structures in this chapter.

Here is an overview of what follows: to begin with, in Section 2.1 we recall the notion of a particular case of A_∞-(co)algebras that is crucial for our approach – *differential graded (co)algebras*. We also use this section to recall the standard DGA given by the cup product in singular cochains and its DGC dual in singular chains and to set plenty of necessary Homological Algebra notation. The reader familiar with these concepts can skip this section. Next, we explore *A_∞-(co)algebras*: first, from the theoretical side in Section 2.2, as deformations of differential graded (co)algebras, and then computationally in Section 2.3, explaining our modification to an algorithm by P. Real et al. Finally, in Section 2.4 we show the power of A_∞-(co)algebras through our results (Theorem 2.4.4 and Example 2.4.5) and a classical theorem by T. Kadeishvili.

Throughout this chapter and unless otherwise stated, we will work over $\boldsymbol{R}$, a commutative associative ring with unit, and we will generally not mention it nor will we include it in the notation. For instance, we will write *module* and *linear* instead of R-module and R-linear and we will denote by $C_*(X)$ the singular chain complex of X with coefficients in R. Also, we will consider 0 as a natural number, $0 \in \mathbb{N}$.

2.1 Differential graded (co)algebras

In this section we will recall concepts of Homological Algebra we will use in the rest of the chapter. In particular, we will present the ingredients needed to cook up the notions of *differential graded algebra* (DGA) and *differential graded coalgebra* (DGC).

Definition 2.1.1. A **graded module** is a direct sum of modules

$$M = \bigoplus_{p \in \mathbb{N}} M_p.$$

A graded module can also be defined over the indexing set $\mathbb{Z}$ instead of $\mathbb{N}$, but we will not need that.

For all $p \in \mathbb{N}$, elements in M_p will be called **homogeneous** elements of degree p, and the **degree** of a homogeneous element a will be denoted by $\boldsymbol{|a|}$.

Definition 2.1.2. Given graded modules M and N and an integer k, a **graded map of degree $\boldsymbol{k}$** from M to N

$$f\colon M \longrightarrow N$$

consists of a family of linear maps of the form

$$\{f_p\colon M_p \longrightarrow N_{p+k}\}_{p \in \mathbb{N}}.$$

The **degree** of a graded map f will be denoted by $\boldsymbol{|f|}$.

We will usually omit subscripts, simply writting

$$f\colon M_p \longrightarrow N_{p+k}$$

to denote f_p.

Composition of graded maps is defined degreewise.

A **morphism of graded modules** is a degree 0 graded map.

Definition 2.1.3. Given graded modules M and N, their **tensor product** $\boldsymbol{M \otimes N}$ is the graded module consisting of

$$(M \otimes N)_p = \bigoplus_{q+r=p} M_q \otimes N_r,$$

for all $p \in \mathbb{N}$.

For any family of objects for which the tensor product of a pair is defined (*e.g.,* graded module, dg module, graded map), we will define the tensor product of more than two of these objects inductively.

Notation 2.1.4. For any object M for which the tensor product is defined (*e.g.,* graded module, dg module, graded map), we will denote by $\boldsymbol{M^{\otimes n}}$ the n-fold tensor product

$$\underbrace{M \otimes \ldots \otimes M}_{n}$$

for all $n \geq 1$.

Definition 2.1.5. Let $f\colon M \longrightarrow N$ and $g\colon M' \longrightarrow N'$ be graded maps. Then we define a graded map of degree $|f| + |g|$,

$$\boldsymbol{f \otimes g}\colon M \otimes M' \longrightarrow N \otimes N',$$

by setting

$$(f \otimes g)(a \otimes b) := (-1)^{|g||a|} f(a) \otimes g(b), \tag{2.1.1}$$

for all $a \in M, b \in N$.

Notice that in (2.1.1) we have permuted the letters g and a and multiplied the sign by $(-1)^{|g||a|}$. Actually, every time we permute two objects (homogeneous elements or graded maps) of degrees p and q, we will multiply the sign by $(-1)^{pq}$. This is known as the **Koszul sign convention.**

Koszul's convention states at an algebraic level a rule we are somehow used to at a geometric level. Take for instance two vector spaces V and W of dimensions p, and q, respectively, and consider the orientation on V given by an ordered basis $(v_1, \ldots, v_p)$ of V and the orientation on W given by an ordered basis $(w_1, \ldots, w_q)$ of W. Then, the orientations of $V \times W$ and $W \times V$ are the same up to the sign of the permutation that sends $(v_1, \ldots, v_p, w_1, \ldots, w_q)$ to $(w_1, \ldots, w_q, v_1, \ldots, v_p)$, which is precisely $(-1)^{pq}$.

Notation 2.1.6. For any map f for which the tensor product is defined (*e.g.,* graded map, dg map), we will denote by $\boldsymbol{f^{\otimes n}}$ the n-fold tensor product

$$\underbrace{f \otimes \ldots \otimes f}_{n}$$

for all $n \geq 1$.

Definition 2.1.7. A **differential graded module** (or simply **dg module**) $\boldsymbol{(M, \partial)}$ consists of a graded module M and a differential ∂. A **differential**

$$\partial \colon M \longrightarrow M$$

is a graded map of degree -1 or $+1$ such that $\partial^2 = 0$. (M, ∂) is traditionally called a **chain complex** if $|\partial| = -1$ and a **cochain complex** if $|\partial| = +1$.

As usual, we use subscripts when dealing with chain complexes and superscripts when dealing with cochain complexes.

Definition 2.1.8. Given a chain complex

$$\ldots \xleftarrow{\partial_{p-1}} M_{p-1} \xleftarrow{\partial_p} M_p \xleftarrow{\partial_{p+1}} M_{p+1} \xleftarrow{\partial_{p+2}} \ldots,$$

for all $p \in \mathbb{N}$, the assumption $\partial_p \partial_{p+1} = 0$ makes the following inclusions hold

$$B_p \subset Z_p \subset M_p,$$

where Z_p and B_p are the modules

$$\boldsymbol{Z_p} := \operatorname{Ker} \partial_p \qquad \text{and} \qquad \boldsymbol{B_p} := \operatorname{Im} \partial_{p+1}.$$

Therefore we can define the quotient

$$\boldsymbol{H_p(M)} := \frac{Z_p}{B_p},$$

which is called the the $\boldsymbol{p}^{\text{th}}$ **homology module** of (M, ∂).

Similarly, if (M, ∂) is a cochain complex

$$\dots \xrightarrow{\partial^{p-2}} M^{p-1} \xrightarrow{\partial^{p-1}} M^p \xrightarrow{\partial^p} M^{p+1} \xrightarrow{\partial^{p+1}} \dots$$

define

$$\boldsymbol{Z^p} := \operatorname{Ker} \partial^p \qquad \text{and} \qquad \boldsymbol{B^p} := \operatorname{Im} \partial^{p-1}$$

and call the quotient

$$\boldsymbol{H^p(M)} := \frac{Z^p}{B^p}$$

the $\boldsymbol{p}^{\text{th}}$ **cohomology module** of (M, ∂).

Definition 2.1.9. Given dg modules $(M, \partial), (N, \partial')$ and an integer k, a **differential graded map (or simply dg map) of degree $\boldsymbol{k}$** from M to N

$$f\colon (M, \partial) \longrightarrow (N, \partial')$$

consists of a family of linear maps

$$\{f_p\colon M_p \longrightarrow N_{p+k}\}_{p \in \mathbb{N}}$$

such that the following diagram communtes up to the sign $(-1)^k$,

$$\begin{array}{ccccccccc}
\dots & \xleftarrow{\partial} & M_{p-1} & \xleftarrow{\partial} & M_p & \xleftarrow{\partial} & M_{p+1} & \xleftarrow{\partial} & \dots \\
 & & \downarrow{\scriptstyle f_{p-1}} & & \downarrow{\scriptstyle f_p} & & \downarrow{\scriptstyle f_{p+1}} & & \\
\dots & \xleftarrow{\partial'} & N_{p-1+k} & \xleftarrow{\partial'} & N_{p+k} & \xleftarrow{\partial'} & N_{p+1+k} & \xleftarrow{\partial'} & \dots
\end{array}$$

i.e.,

$$\partial' \circ f = (-1)^k f \circ \partial.$$

A **morphism of dg modules (or dg morphism)** is a degree 0 dg map.

Composition of dg maps is defined degreewise.

Every dg map $f\colon M \longrightarrow N$ induces a maps in (co)homology

$$H_p(M) \longrightarrow H_p(N)$$

or

$$H^p(M) \longrightarrow H^p(N)$$

(whichever appropriate), for all $p \in \mathbb{N}$.

Definition 2.1.10. Let

$$(M,d) \underset{g}{\overset{f}{\rightrightarrows}} (N,d)$$

be dg morphisms. A **dg homotopy** from f to g is a dg map

$$\phi\colon M \longrightarrow N$$

of degree $|\phi| = -|d|$ such that

$$f - g = d\phi + \phi d.$$

If there is a dg homotopy from f to g, we say that f and g are **dg homotopic** and write

$$f \simeq g,$$

or

$$f \underset{\phi}{\simeq} g$$

if we want to specify the dg homotopy ϕ.

A dg morphism

$$f\colon M \longrightarrow N$$

is called a **dg homotopy equivalence** if there is a dg morphism

$$g\colon N \longrightarrow M$$

such that

$$gf \simeq 1_M \qquad \text{and} \qquad fg \simeq 1_N.$$

In that case, we say the the dg modules M and N are **(dg homotopy) equivalent.**

Definition 2.1.11. Let (M, ∂) and (N, ∂') be dg modules. Their **tensor product $(M \otimes N, \delta)$** is the dg module consisting of

$$(M \otimes N)_p = \bigoplus_{q+r=p} M_q \otimes N_r$$

for all $p \in \mathbb{N}$, with differential

$$\delta\colon (M \otimes N)_p \longrightarrow (M \otimes N)_{p-1}$$

defined by

$$\delta(a \otimes b) = \partial a \otimes b + (-1)^q a \otimes \partial' b$$

on elementes $a \in M_q, b \in N_r$.

The **tensor product** of dg maps is defined as for graded maps.

Definition 2.1.12. A **graded algebra (A, μ)** is a graded module A equipped with a morphism of graded modules

$$\mu\colon A \otimes A \longrightarrow A$$

(called **multiplication** or **product**) that is associative, *i.e.*, such that the following diagram commutes

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes 1} & A \otimes A \\ {\scriptstyle 1 \otimes \mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array}$$

Notation 2.1.13. From now on, we will use ***ab*** to denote $\mu(a \otimes b)$ for any multiplication μ (associative or not). Thus, the morphism condition on the multiplication μ of a graded algebra can be expressed by saying that if $a \in A_p, b \in A_q$, then $ab \in A_{p+q}$.

Definition 2.1.14. Given graded algebras (A, μ) and (A', μ'), a **morphism of graded algebras** from A to A'

$$f\colon (A, \mu) \longrightarrow (A', \mu')$$

is a graded morphism that is multiplicative, *i.e.*, such that the following diagram commutes.

$$\begin{array}{ccc} A \otimes A & \xrightarrow{f \otimes f} & A' \otimes A' \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

Definition 2.1.15. A **differential graded algebra (DGA)** is a triple $\boldsymbol{(A, \partial, \mu)}$ in which

(i) (A, ∂) is a dg module,

(ii) (A, μ) is a graded algebra, and

(iii) $\mu\colon A \otimes A \longrightarrow A$ is a morphism of dg modules.

Remark 2.1.16. Notice that if (A, ∂, μ) is a DGA, then A_0 must be an algebra with associative multiplication

$$\mu\colon A_0 \otimes A_0 \longrightarrow A_0.$$

Definition 2.1.17. A **derivation of degree $\boldsymbol{k}$** on a graded algebra (A, μ) is a degree k graded map $d\colon A \longrightarrow A$ satisfying the Leibniz rule

$$d(ab) = d(a)b + (-1)^{k|a|}ad(b)$$

for all homogeneous $a, b \in A$.

By induction, this implies

$$d(a_1 \dots a_n) = \sum_{i=1}^{n} (-a)^{|a_1|+\dots+|a_{i-1}|} a_1 \dots d(a_i) \dots a_n$$

for all homogeneous $a_1, \dots, a_n \in A$.

Let (A, ∂) be a dg module and (A, μ) be a graded coalgebra. If we denote also by ∂ the differential on $A \otimes A$, then ∂ is a derivation on (A, μ) if and only if μ is a morphism of dg modules, which is equivalent to (A, ∂, μ) forming a DGA.

Definition 2.1.18. A **morphism of DGAs**

$$f\colon (A, \partial, \mu) \longrightarrow (A', \partial', \mu')$$

is a dg morphism

$$f\colon (A, \partial) \longrightarrow (A', \partial')$$

that is also a morphism of graded algebras

$$f\colon (A, \mu) \longrightarrow (A', \mu').$$

Definition 2.1.19. The cohomology DGA of a DGA (A, ∂, μ) with $|\partial| = 1$ is the DGA

$$(H^*(A), 0, \mu^*),$$

i.e., with zero differential and with the multiplication induced in cohomology.

Every morphism of DGAs induces a morphism of DGAs between the corresponding cohomology DGAs.

We next present dual definitions to some of the ones just listed.

Definition 2.1.20. A **graded coalgebra (C, Δ)** is a graded module C equipped with a morphism of graded modules

$$\Delta\colon C \longrightarrow C \otimes C$$

(called **comultiplication** or **coproduct**) that is coassociative, *i.e.*, such that the following diagram commutes

$$\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
{\scriptstyle\Delta}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta} \\
C \otimes C & \xrightarrow[\Delta\otimes 1]{} & C \otimes C \otimes C,
\end{array}$$

Definition 2.1.21. Given graded coalgebras (C, Δ) and (C', Δ'), a **morphism of graded coalgebras** from C to C'

$$f\colon (C, \Delta) \longrightarrow (C', \Delta')$$

is a graded morphism that is comultiplicative, *i.e.*, such that the following diagram commutes.

$$\begin{CD}
C @>{f}>> C' \\
@V{\Delta}VV @VV{\Delta'}V \\
C \otimes C @>>{f \otimes f}> C' \otimes C'
\end{CD}$$

Definition 2.1.22. A **differential graded coalgebra (DGC)** is a triple $\boldsymbol{(C, \partial, \Delta)}$ in which

(i) (C, ∂) is a dg module,

(ii) (C, Δ) is a graded coalgebra, and

(iii) $\Delta\colon C \longrightarrow C \otimes C$ is a morphism of dg modules.

Remark 2.1.23. Notice that if (C, ∂, Δ) is a DGC, then C_0 must be a coalgebra with coassociative comultiplication

$$\Delta\colon C_0 \longrightarrow C_0 \otimes C_0.$$

Definition 2.1.24. A **coderivation of degree $\boldsymbol{k}$** on a graded coalgebra (C, Δ) is a degree k graded map $d\colon C \longrightarrow C$ such that the following diagram commutes for all $p \in \mathbb{Z}$

$$\begin{CD}
C_p @>{\Delta}>> (C \otimes C)_p \\
@V{d}VV @VV{d \otimes 1 + 1 \otimes d}V \\
C_{p+k} @>>{\Delta}> (C' \otimes C')_{p+k}\,.
\end{CD}$$

Let (C, ∂) be a dg module and (C, Δ) be a graded coalgebra. If we denote also by ∂ the differential on $C \otimes C$, then ∂ is a coderivation on (C, Δ) if and only if Δ is a morphism of dg modules, which is equivalent to (C, ∂, Δ) forming a DGC.

Definition 2.1.25. A **morphism of DGCs**

$$f\colon (C,\partial,\Delta) \longrightarrow (C',\partial',\Delta')$$

is a dg morphism

$$f\colon (C,\partial) \longrightarrow (C',\partial')$$

that is also a morphism of graded coalgebras

$$f\colon (C,\Delta) \longrightarrow (C',\Delta').$$

2.1.1 Cup product and Alexander-Whitney diagonal

Let X be a topological space. Its singular chain complex $(C_*(X),\partial)$ has a standard DGA structure given by the cup product. Similarly, its singular cochain complex $(C^*(X),\delta)$ has a standard DGC structure given by the Alexander-Whitney diagonal as coproduct. This is no coincidence. In this section we recall the basic constructions that allow us to see some of the underlying connection between these operations. We will also point out that this has analogues in terms of simplicial, cellular and cubical complexes.

The main two ingredients in the construction of the two operations are the diagonal map

$$\begin{array}{rccc} D\colon & X & \longrightarrow & X\times X \\ & x & \longmapsto & (x,x) \end{array}$$

and the Alexander-Whitney map

$$AW\colon C_*(X\times Y) \longrightarrow C_*(X)\otimes C_*(Y),$$

that we will define next. The Alexander-Whitney diagonal is the composition

$$\Delta\colon C_*(X) \xrightarrow{D_\#} C_*(X\times X) \xrightarrow{AW} C_*(X)\otimes C_*(X)$$

and the cup product is the composition

$$\smile\colon C^*(X)\otimes C^*(X) \xrightarrow{\times} C^*(X\times X) \xrightarrow{D^\#} C^*(X),$$

where $\times$ denotes a cross product that can be defined through the map AW.

The Alexander-Whitney map AW is a natural chain homotopy equivalence of the form

$$AW\colon C_*(X\times Y)\longrightarrow C_*(X)\otimes C_*(Y)$$

for all spaces X, Y, whose chain homotopy inverse is the so called Eilenberg-Zilber map [40]. This map is unique up to homotopy equivalence.

In order to describe the map AW, let us set up some fairly common notation first. Let $p\in\mathbb{N}$ and let

$$e_i=(0,\dots,0,\underset{i}{1},0,\dots,0)\in\mathbb{R}^{p+1}$$

for every $0\leq i\leq p$. If we view the standard p-simplex as

$$\Delta^p=\left\{\sum_{i=0}^{p}t_ie_i\in\mathbb{R}^{p+1}\,|\,t_i\geq 0,\sum t_i=1\right\},$$

then we can define maps

$$\boldsymbol{f_k}\colon\Delta^k\longrightarrow\Delta^p\qquad\text{and}\qquad\boldsymbol{b_l}\colon\Delta^l\longrightarrow\Delta^p$$

on the vertices by

$$\begin{cases}f_k(e_i)=e_i, & \text{for all } 0\leq i\leq k\\ b_l(e_i)=e_{i+p-l}, & \text{for all } 0\leq i\leq l\end{cases}$$

and extend them by linearity. Thus the map f_k embeds Δ^k as the k-dimensional *front* face of Δ^p and b_l embeds Δ^l as the l-dimensinal *back* face of Δ^p.

Definition 2.1.26. Let X, Y be topological spaces. For any $p\in\mathbb{N}$, the **Alexander-Whitney map**

$$\boldsymbol{AW}\colon C_p(X\times Y)\longrightarrow\left(C_*(X)\otimes C_*(Y)\right)_p$$

is the linear map defined on p-simplices

$$\sigma\colon\Delta^p\longrightarrow X\times Y$$

with projections

$$\sigma_X\colon \Delta^p \xrightarrow{\pi_X\,\sigma} X \qquad \text{and} \qquad \sigma_Y\colon \Delta^p \xrightarrow{\pi_Y\,\sigma} Y$$

by

$$AW(\sigma) = \sum_{k+l=p} \sigma_X f_k \otimes \sigma_Y b_l.$$

Definition 2.1.27. The **Alexander-Whitney diagonal** is the composition of the Alexander-Whitney map and the map induced by the diagonal. With the notation above,

$$\boldsymbol{\Delta}\colon C_*(X) \xrightarrow{D_\#} C_*(X\times X) \xrightarrow{AW} C_*(X)\otimes C_*(X)$$

for every space X. Explicitely, for any $p\in\mathbb{N}$, and for any p-simplex $\sigma\colon \Delta^p \longrightarrow X$,

$$\Delta(\sigma) = \sum_{k+l=p} \sigma f_k \otimes \sigma b_l.$$

As we said before, we can also obtain the cup product out of the maps D and AW. Let us begin by defining a cross product

$$C^*(X)\otimes C^*(Y) \longrightarrow C^*(X\times Y)$$

for all spaces X and Y. Let $k,l\in\mathbb{N}$ and let $\alpha\in C^k(X), \beta\in C^l(Y)$. We define the **cross product** of α and β as the composition

$$\begin{array}{ccccc}
C_{k+l}(X\times Y) & & \overset{\boldsymbol{\alpha\times\beta}}{\dashrightarrow} & & R \\
{\scriptstyle AW}\downarrow & & & & \uparrow{\scriptstyle \cdot} \\
(C_*(X)\otimes C_*(Y))_{k+l} & \xrightarrow{\pi} & C_k(X)\otimes C_l(Y) & \xrightarrow{\alpha\otimes\beta} & R\otimes R,
\end{array}$$

where π is the projetion onto the indicated summand. Explicitely, for any $(k+l)$-simplex

$$\sigma\colon \Delta^{k+l} \longrightarrow X\times Y,$$

$$(\alpha\times\beta)(\sigma) = \alpha(\sigma_X f_k)\,\beta(\sigma_Y b_l),$$

and therefore,

$$D^\#(\alpha\times\beta)(\sigma) = \alpha(\sigma f_k)\,\beta(\sigma b_l) = (\alpha\smile\beta)(\sigma).$$

Hence, as mentioned at the beginning of this section, we can view the cup product as the composition

$$\smile: C^*(X) \otimes C^*(X) \xrightarrow{\times} C^*(X \times X) \xrightarrow{D^\#} C^*(X).$$

This section can be adapted to contexts other than singular. If we work with CW (resp., simplicial) complexes, then all can be done in a similar fashion by adding the use of cellular (resp., simplicial) approximations, and things keep being beautiful because no matter the approximations we choose, the result will be the same up to chain homotopy equivalence. The cubical case, interesting in some applications like digital imaging, is studied in [58] and [59].

2.2 Basics of A_∞-(co)algebras

In this section we recall the basic facts we shall use from A_∞-coalgebras and A_∞-algebras.

Definition 2.2.1. An A_∞-coalgebra $(C, \{\Delta_n\}_{n\geq 1})$ consists of a graded module C and a sequence of maps

$$\Delta_n \colon C \longrightarrow C^{\otimes n}$$

of degree $n-2$ such that, for all $n \geq 1$, the following so called *Stasheff identity* holds:

$$\mathbf{SI}(\boldsymbol{n})\colon \sum_{i=1}^{n}\sum_{j=0}^{n-i}(-1)^{i+j+ij}\left(1^{\otimes n-i-j} \otimes \Delta_i \otimes 1^j\right) \Delta_{n-i+1} = 0.$$

Notice that when these formulas are applied to elements, additional signs appear due to Koszul's convention (see *Definition 2.1.5*). Let us have a look at what they tell us for low values of n. SI(1) states that

$$\Delta_1\Delta_1 = 0,$$

meaning that we can see

$$\Delta_1 \colon C \longrightarrow C$$

as a differential on C, *i.e.,* (C, Δ_1) forms a chain complex. SI(2) says

$$\Delta_2\Delta_1 = (\Delta_1 \otimes 1 + 1 \otimes \Delta_1)\, \Delta_2,$$

so the coproduct

$$\Delta_2 \colon C \longrightarrow C \otimes C$$

is a chain map. Last, SI(3) translates into

$$(1 \otimes \Delta_2 - \Delta_2 \otimes 1)\, \Delta_2 = \Delta_3\Delta_1 + (\Delta_1 \otimes 1 \otimes 1 + 1 \otimes \Delta_1 \otimes 1 + 1 \otimes 1 \otimes \Delta_1)\, \Delta_3,$$

indicating that (C, Δ_1, Δ_2) is almost forced to form a DGC, but that the coassociativity of Δ_2, which is equivalent to the commutativity of the diagram

$$\begin{array}{ccc} C & \xrightarrow{\quad \Delta_2 \quad} & C \otimes C \\ {\scriptstyle \Delta_2}\downarrow & & \downarrow{\scriptstyle 1\otimes\Delta_2} \\ C \otimes C & \xrightarrow[\quad \Delta_2\otimes 1 \quad]{} & C \otimes C \otimes C, \end{array}$$

only needs to hold up to the chain homotopy

$$\Delta_3 \colon C \longrightarrow C \otimes C \otimes C.$$

Similarly to how Δ_2 is coassociative up to Δ_3, Δ_3 satisfies the pentagon identity up to Δ_4 and in general, Δ_n satisfies some axiom (that can be seen as a sort of coassociativity condition) up to Δ_{n+1}. The "A" in A_∞-(co)algebra stands for *Associativity* and the "∞" indicates that some (co)associativity axioms are relaxed up to higher homotopies without bound on the degree of the homotopies.

Example 2.2.2. Any DGC (C, ∂, Δ) can be seen as an A_∞-coalgebra

$$(C, \{\Delta_n\}_{n\geq 1}), \qquad \text{by setting} \qquad \Delta_n = \begin{cases} \partial, & n = 1 \\ \Delta, & n = 2 \\ 0, & n > 2. \end{cases}$$

Definition 2.2.3. An A_∞-coalgebra $(C, \{\Delta_n\}_{n\geq 1})$ is called **minimal** if

$$\Delta_1 = 0.$$

Remark 2.2.4. It is convenient to note that A_∞-coalgebra structures in the graded module C are in one-to-one correspondence with differentials in the complete tensor algebra $\widehat{T}(s^{-1}C) = \Pi_{n\geq 1} T^n(s^{-1}C)$ with $T^n(s^{-1}C) = (s^{-1}C)^{\otimes n}$. Here, $s^{-1}C$ denotes the desuspension of C, that is, $(s^{-1}C)_p = C_{p+1}$, the grading in $\widehat{T}(s^{-1}C)$ is given by

$$\left(\widehat{T}(s^{-1}C)\right)_p = \Pi_{n\geq 1}\left(T^n(s^{-1}C)\right)_p$$

and

$$\left(T^n(s^{-1}C)\right)_p = \bigoplus_{p_1+\ldots+p_n=p} \left(s^{-1}C\right)_{p_1} \otimes \ldots \otimes \left(s^{-1}C\right)_{p_n}$$

and the product giving the algebra structure is given by concatenation.

In fact, giving a differential $d\colon \widehat{T}(s^{-1}C) \longrightarrow \widehat{T}(s^{-1}C)$ in this algebra is equivalent to giving its restriction $d\colon s^{-1}C \longrightarrow \widehat{T}(s^{-1}C)$ and extending d as a derivation. Such $d\colon s^{-1}C \longrightarrow \widehat{T}(s^{-1}C)$ can be written as a sum $d = \sum_{n\geq 1} d_n$, with $d_n(s^{-1}C) \subset T^n(s^{-1}C)$, for $n \geq 1$. With this notation, the operators $\{\Delta_n\}_{n\geq 1}$ and $\{d_n\}_{n\geq 1}$ uniquely determine each other via

$$\begin{aligned} \Delta_n &= -s^{\otimes n} \circ d_n \circ s^{-1}\colon C \to C^{\otimes n}, \\ d_n &= -(-1)^{\frac{n(n-1)}{2}} (s^{-1})^{\otimes n} \circ \Delta_n \circ s\colon s^{-1}C \to T^n(s^{-1}C). \end{aligned} \tag{2.2.1}$$

Definition 2.2.5. Given an A_∞-coalgebra $(C, \{\Delta_n\}_n)$, the complete tensor algebra $\widehat{T}(s^{-1}C)$ with the differential given by (2.2.1) forms a DGA that is called the **cobar construction** of $(C, \{\Delta_n\}_n)$ and is usually denoted by ΩC.

Definition 2.2.6. A **morphism of A_∞-coalgebras**

$$f\colon (C, \{\Delta_n\}_{n\geq 1}) \to (C', \{\Delta'_n\}_{n\geq 1})$$

is a family of graded maps

$$f_{(k)}\colon C \longrightarrow C'^{\otimes k}, \qquad k \geq 1,$$

each of which is of degree $k-1$, such that for each $i \geq 1$, the following identity, that we will denote by $\mathbf{MI(}\boldsymbol{i}\mathbf{)}$, holds:

$$\sum_{\substack{p+q+k=i \\ q\geq 1, p,k\geq 0}} (1^{\otimes p} \otimes \Delta'_q \otimes 1^{\otimes k}) f_{(p+k+1)} = \sum_{\substack{k_1+\cdots+k_\ell=i \\ l, k_j\geq 1}} (f_{(k_1)} \otimes \cdots \otimes f_{(k_\ell)})\Delta_\ell.$$

This is equivalent to the existence of a morphism of DGAs, which we denote in the same way, between the corresponding cobar constructions

$$f\colon (\widehat{T}(s^{-1}C), d) \longrightarrow (\widehat{T}(s^{-1}C'), d').$$

Indeed, write $f_{|_{s^{-1}C}} = \sum_{k\geq 0} f_k$ with $f_k\colon s^{-1}C \to T^k(s^{-1}C')$ and consider the morphisms $f_{(k)}\colon C \to C'^{\otimes k}$ induced by f_k eliminating desuspensions. Then, it is straightforward to check that the equality $fd = d'f$ translates to the identities MI(i) above satisfied by the maps $\{f_{(k)}\}_{k\geq 0}$.

Definition 2.2.7. Let

$$f\colon C \longrightarrow C'$$

be a morphism of A_∞-coalgebras. Then we say that

- f is an **isomorphism** if $f_{(1)}$ is.
- f is a **quasi-isomorphism** if $f_{(1)}$ induces an isomorphism in homology.

Definition 2.2.8. A **transfer diagram** between chain (resp., cochain) complexes (A, ∂_A) and (M, ∂_M) is a diagram

$$\phi \circlearrowright (A, \partial_A) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (M, \partial_M)$$

in which π and ι are morphisms of dg modules and ϕ is a graded map of degree +1 (resp., -1) such that

- $\pi\iota = 1_M$,
- $\pi\phi = 0$,
- $\phi\iota = 0$,
- $\phi^2 = 0$ and
- $\iota\pi \underset{\phi}{\simeq} 1_A$.

The classical *Perturbation Lemma* [52, 56, 60], known nowadays as the *Homotopy Transfer Theorem* [64, 68], is a common tool to find A_∞-coalgebra structures as deformations of DGC's [57, 61]. For our purposes, the best adapted particular form of this theorem reads as follows:

Theorem 2.2.9. (Homotopy Transfer Theorem [64, 68] or Basic Perturbation Lemma [52, 56, 60]) *Let*

$$\phi \circlearrowright (C,\partial) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (C',\partial')$$

be a transfer diagram of chain complexes in which (C,∂,Δ) is a DGC. Then, there exist an explicit A_∞-coalgebra structure on C' and morphisms of A_∞-coalgebras,

$$C \underset{I}{\overset{P}{\rightleftarrows}} C' ,$$

such that $P_{(1)} = \pi$ and $I_{(1)} = \iota$.

We make explicit some formulas to compute the transferred A_∞-structure in Section 2.3.

Via this result, there is a standard and functorial way of endowing the simplicial / singular / cellular homology of a given complex with a structure of A_∞-coalgebra. For it, consider the DGC structure on the corresponding chains $C_*(X)$ given by an approximation to the diagonal (Section 2.1.1). Then, it is easy to find a transfer diagram of the form

$$\phi \circlearrowright (C_*(X),\partial) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (H_*(X),0).$$

Indeed, write $C_*(X) = A \oplus \partial A \oplus H$ with A such that $\partial\colon A \overset{\cong}{\to} \partial A$ and $H \cong H_*(X)$, choose ι to be the inclusion, and construct π and ϕ recursively so that they satisfy the properties in Definition 2.2.8. Finally, one just needs to apply Theorem 2.2.9.

We are interested in a special kind of A_∞-coalgebras we will call *good.* We choose this terminology because, as we will see in Sections 2.4 and 3.2, out of all the A_∞-coalgebra structures one could put on $H_*(X)$, the *good* ones can give particularly meaningful information about the homotopy type of X.

Remark 2.2.10. For instance, we just mentioned that Δ_3 measures the degree to which the coassociativity of Δ_2 can be relaxed in an A_∞-coalgebra $(C, \{\Delta_n\}_{n\geq 1})$. However, even if Δ_2 is strictly coassociative, Δ_3 may be non-trivial, and if the A_∞-coalgebra is *good*, this non-triviality can give crucial information in some cases (see §3.2).

Definition 2.2.11. We say that an A_∞-coalgebra $(H_*(X), \{\Delta_n\}_n)$ on the homology of a space X is a ***good* $\boldsymbol{A_\infty}$-coalgebra (induced by $\boldsymbol{X}$)** if it is minimal and quasi-isomorphic to the A_∞-coalgebra

$$(C_*(X), \{\partial, \Delta, 0, 0, \dots\}),$$

where $(C_*(X), \partial)$ denotes the corresponding chain complex and Δ denotes an approximation to the diagonal. We will drop the *induced by* X when no confusion is possible.

The stardard method just explained to build an A_∞-coalgebra structure on $H_*(X)$ shows that for every space X, there always exist *good* A_∞-coalgebras on $H_*(X)$ induced by X. Moreover, an immediate consequence from the last definition is that all *good* A_∞-coalgebras on $H_*(X)$ induced by X are isomorphic.

It is also straightforward to check that given a transfer diagram of the form

$$\phi \circlearrowright (C_*(X), \partial, \Delta) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (H_*(X), 0),$$

the transferred A_∞-coalgebra structure in $H_*(X)$ will be *good*.

The dual notion of an A_∞-coalgebra is that of an A_∞-algebra.

Definition 2.2.12. An A_∞-algebra $(A, \{m_n\}_{n\geq 1})$ consists of graded module A and a sequence of maps

$$m_n \colon A^{\otimes n} \longrightarrow A$$

of degree $2-n$ such that, for all $n \geq 1$, the following so called *Stasheff identity* hold:

$$\mathbf{SI}(\boldsymbol{n})\colon \sum_{\substack{n=r+s+t\\ s\geq 1\\ r,t\geq 0}} (-1)^{r+st} m_{r+1+t}\left(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}\right) = 0,$$

Everything said for A_∞-coalgebras dualizes to A_∞-algebras. In particular, in an A_∞-algebra $(A, \{m_n\}_{n\geq 1})$, m_3 measures the the associativity of m_2 but even if m_2 is strictly associative, m_3 may be non-trivial and give useful information (see Section 3.2). The Homotopy Transfer Theorem also works for DGAs and A_∞-algebras, and we define *good* A_∞-algebras analogously to Def. 2.2.11, using the cup product instead of approximations to the diagonal.

2.3 Algorithmic approach

There are several ways to obtain *good* A_∞-(co)algebras: by building a transfer diagram and using formulas [57, 61] from the Homotopy Transfer Theorem [64, 68] or the Basic Perturbation Lemma [52, 56, 60], by using constructions like the one in the original proof of Kadeishvili's Minimality Theorem [60, Theorem 1], [101] or like Merkulov's models built in [75, Section 3], by means of Quillen minimal models for 1-connected CW complexes (see the Appendix), etc.

Following the approach by P. Real *et al.* [79, 80], we will show a way to compute a *good* A_∞-coalgebra on the homology of a CW complex by building a discrete vector field that yields a transfer diagram. To do so, we have slightly modified the attractive Algorithm 1 in [80]. We chose to work with this algorithm because it is perfectly fitted for filtrations, so its adaptation to a persistent context is very natural (see §3.7). We show the algorithm (with the modifications incorporated) in Algorithm 2.3.5 and list all the changes made in Table 2.3.6.

In this section we will work with a finite CW complex (K, ∂) and, for simplicity, the ground ring will be the finite field of two elements $\mathbb{F}_2$, although what we will do can be generalized to other rings. Let $C_*(K)$ denote the cellular chain complex of K.

Definition 2.3.1. An **algebraic gradient vector field (algebraic gvf)** on $C_*(K)$ is a graded map of degree $+1$

$$\phi\colon C_*(K) \longrightarrow C_*(K)$$

such that $\phi^2 = 0$.

A **homology gvf** is an algebraic gvf which satisfies
$\phi\partial\phi = \phi$ and
$\partial\phi\partial = \partial$.

Example 2.3.2. Consider a filled square with the CW complex structure K givev by 0-cells $c_0, \ldots, c_3$, 1-cells $c_4, \ldots, c_7$ and 2-cell c_8 as in Figure 2.1.

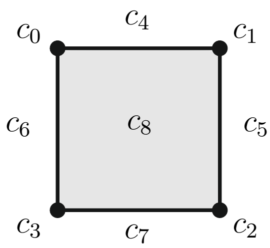


Figure 2.1: A CW complex desomposition K of a filled square.

The following assignments define a homology gvf ϕ on $C_*(K; \mathbb{F}_2)$:

$$\phi(c_0) = 0, \phi(c_1) = c_4, \phi(c_2) = c_4 + c_5, \phi(c_3) = c_6,$$

$$\phi(c_4) = \phi(c_5) = \phi(c_6) = 0, \phi(c_7) = c_8,$$

$$\phi(c_8) = 0.$$

Using Discrete Morse Theory pictorial language, a homology gvf can be expressed by means of arrows on the CW complex in the obvious manner. Figure 2.2 shows an example of this.

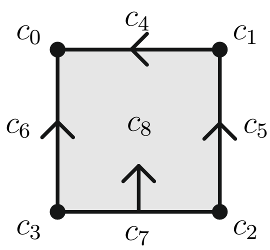


Figure 2.2: Arrows describing the homology gvf ϕ in Example 2.3.2.

The information that a homology gvf gives is equivalent to that of a transfer diagram:

Proposition 2.3.3. [80, Prop. 1] *A homology gvf*

$$\phi\colon C_*(K) \longrightarrow C_*(K)$$

yields a transfer diagram

$$\phi \circlearrowright (C_*(K), \partial) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (Im\,\pi, 0)$$

where

- $\pi := 1_{C_*(K)} - \partial\phi - \phi\partial\colon C_*(K) \longrightarrow Im\,\pi$, *and thus* $(Im\,\pi, 0)$ *becomes a chain subcomplex of* $C_*(K)$ *isomorphic to* $H_*(K)$, *and*
- $\iota\colon Im\,\pi \longrightarrow C_*(K)$ *is the inclusion.*

Conversely, given a transfer diagram of the form

$$\phi \circlearrowright (C_*(K), \partial) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (H_*(K), 0),$$

$\phi\colon C_*(K) \longrightarrow C_*(K)$ *is a homology gvf.*

We need a definition to understand the input of the algorithm.

Definition 2.3.4. Let (K, ∂) be a finite CW complex with $m+1$ cells. A **filter** is an ordered list

$$\mathcal{K}_m = \langle c_0, \ldots, c_m \rangle$$

of the cells of K so that, for all $0 \leq i \leq m$, the cells in the list

$$\mathcal{K}_i = \langle c_0, \ldots, c_i \rangle,$$

together with the boundary map ∂_i (the restriction of ∂ to those cells) form a subcomplex of K.

In Persistence, the filter usually matters a lot, but if we do not care about the filter, a simple way to get one out of an arbitrary finite CW complex is to first consider all its 0-cells in a certain order, then add all its 1-cells in a certain order and so on.

Given a finite finite CW complex (K, ∂) with $m+1$ cells, the input of Algorithm 2.3.5 is a filter

$$\mathcal{K}_m = \langle c_0, \ldots, c_m \rangle$$

and the boundary maps $\{\partial_i\}_{0 \leq i \leq m}$ as above.

And here is the algorithm, whose output will be a function ϕ_m defined on every element of the list

$$\mathcal{K}_m = \langle c_0, \ldots, c_m \rangle$$

so that its linear extension ϕ will be a homology gvf on $C_*(K)$:

ALGORITHM 2.3.5. Our adjustment of Algorithm 1 in [80].

$\mathcal{H}_0 := \{c_0\}$;

$\phi_0(c_0) := 0$;

`For` $i = 1$ `to` m`:`

  $\bar{c}_i := c_i + \phi_{i-1}\partial_i(c_i)$;

  $\mathcal{H}_i := \mathcal{H}_{i-1} \cup \{\bar{c}_i\}$;

  $\phi_i(c_i) := 0$;

  `If` $\partial_i(\bar{c}_i) = 0$`:`

    `For` $j = 0$ `to` $i - 1$`:`

      $\phi_i(c_j) := \phi_{i-1}(c_j)$;

    `End For;`

  `End If;`

  `If` $\partial_i(\bar{c}_i)$ `is a sum of the kind` $\sum_{j=1}^{r} u_j \neq 0$ `where each` u_j `is a different element of` $\mathcal{H}_{i-1}$`:`

    `Choose one of the summands` u_j `and let` k `be the subindex` j `of the chosen one;`

    $\tilde{\phi}(u_k) := \bar{c}_i$;

    $\tilde{\phi}(u) := 0$ `for every` $u \in \mathcal{H}_{i-1} - \{u_k\}$`;`

    `For` $j = 0$ `to` $i - 1$`:`

      $\phi_i(c_j) = (\phi_{i-1} + \tilde{\phi}(1_{\mathcal{K}_i} + \phi_{i-1}\partial_{i-1} + \partial_{i-1}\phi_{i-1}))(c_j)$;

    `End For;`

    $\mathcal{H}_i := \mathcal{H}_i - \{u_k, \bar{c}_i\}$`;`

  `End If;`

`End For;`

`END;`

Table 2.3.6. Here is a list with the changes made to Algorithm 1 in [80]. The lines refer to the lines in Algorithm 1 in [80], not to those in the algorithm above:

Line in Alg. 1 in [80]	**Change** `(Part I)`
1, 3 & 14	The computation of $\pi_i(c_j)$ has been removed, for all i and j. Reason: The value of $\pi_i(c_j)$ is not actually used in any step of the algorithm, and the final π obtained when the algorithm finishes is not the right one. Notice that one can compute it at the end, once one has the final ϕ, by setting $\pi := 1 - \partial\phi - \phi\partial$.
5 & 8	$(\partial_i + \partial_{i-1}\phi_{i-1}\partial_i)(c_i)$ has become $\partial_i(\bar{c}_i)$. Reason: Both things are the same, for ∂_i restricted to $\mathcal{K}_{i-1}$ equals ∂_{i-1}, and with this change we take advantage of $\bar{c}_i$ having already been computed.
9	$u_i \in \mathcal{H}_{i-1}$ has become $u_j \in \mathcal{H}_{i-1}$. Reason: just a typo.
9	$\sum_{j=1}^{r} \pi_{i-1}(e_{s_j})$ has been removed. Reason: The deleted part was just a reminder of the form that the elements in $\mathcal{H}_{i-1}$ have.
10	$\tilde{\phi}(u_k) := c_i$ has become $\tilde{\phi}(u_k) := \bar{c}_i$. Reason: If not, we do not obtain as OUTOUT a homology gvf, but simply an algebraic gvf. After this table, we show an example where the original algorithm would fail in providing a homology gvf.
13	$(\phi_{i-1} + \tilde{\phi}(1_{\mathcal{K}_i} + \phi_{i-1}\partial_{i-1} + \partial_{i-1}\phi_{i-1})(c_j)$ has become $(\phi_{i-1} + \tilde{\phi}(1_{\mathcal{K}_i} + \phi_{i-1}\partial_{i-1} + \partial_{i-1}\phi_{i-1}))(c_j)$. Reason: Bracket missing.

This table continues on the next page.

Line in Alg. 1 in [80]	**Change** `(Part II)`
15	The line has been placed inside (and at the end of) the last *If* clause. <u>Reason:</u> If not, whatever happened inside the two *If*, we would always do $\mathcal{H}_i := \mathcal{H}_i - \{u_k, \bar{c}_i\}$ after having set, in line 4, $\mathcal{H}_i := \mathcal{H}_{i-1} \cup \{\bar{c}_i\}$. This would mean the effect of a loop of the biggest *For* would always be equivalent to simply setting $\mathcal{H}_i := \mathcal{H}_{i-1} - \{u_k\}$, leaving the homology $\mathcal{H}_i$ with the only possibilities of remaining like $\mathcal{H}_{i-1}$ or decreasing its dimension in 1.

As the second comment on line 10 says, without the appropriate change, we would obtain algebraic gvf's, in general, but not homology gvf's. Here is an example of this. Consider the filtered simplicial complex (K, ∂) with filter

$$\mathcal{K}_4 = \langle [0], [1], [2], [1,2], [0,2] \rangle .$$

Using Algorithm 1 in [80], once we correct the typo in line 9 and the little mistake pointed out about line 15, if we do not correct what we remarked about line 10, then will not get a homology gvd, but simply one of the following algebraic gvf ϕ_4. Specifically,

- We will get $\mathcal{H}_4 = \{c_0\}$,
  $\phi_4(c_0) = 0$,
  $\phi_4(c_1) = c_4$,
  $\phi_4(c_2) = c_3 + c_4$,
  $\phi_4(c_3) = 0$,
  $\phi_4(c_4) = 0$,
  if we choose $u_k := c_1$,
- and we will get $\mathcal{H}_4 = \{c_1\}$,
  $\phi_4(c_0) = c_4$,
  $\phi_4(c_1) = 0$,

$\phi_4(c_2) = c_3,$
$\phi_4(c_3) = 0,$
$\phi_4(c_4) = 0,$
if we choose $u_k := c_0$.

These changes in the algorithm make us get as output a homology gvf on $C_*(K)$, which gives a transfer diagram

$$\phi \circlearrowright (C_*(K), \partial) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (\mathrm{Im}\,\pi, 0)$$

where $(\mathrm{Im}\,\pi, 0)$ is a dg module isomorphic to $(H_*(K), 0)$ (see Proposition 2.3.3), so we can use the following formulas, that can be deduced using the Basic Perturbation Lemma.

Proposition 2.3.7. *[52], [60], [57, Theorem 3.2]. Let (C, ∂_C, Δ) and (M, ∂_M) be a simply connected DGC and a dg module, respectively, and let*

$$\phi \circlearrowright (C, \partial_C) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (M, \partial_M)$$

be a transfer diagram between them. Then M is endowed with an A_∞-coalgebra structure given by the morphisms

$$\begin{aligned} \Delta_1 &= -\partial_M \\ \Delta_n &= (-1)^{[n/2]+n+1} \pi^{\otimes n} \Delta^{(n)} \phi^{[\otimes n-1]} \Delta^{(n-1)} \cdots \phi^{[\otimes 2]} \Delta^{(2)} \iota, \quad n \geq 2 \end{aligned}$$

where

$$\Delta^{(k)} = \sum_{i=0}^{k-2} (-1)^i 1^{\otimes i} \otimes \Delta \otimes 1^{\otimes k-i-2} : C^{\otimes k-1} \longrightarrow C^{\otimes k}$$

and

$$\phi^{[\otimes k]} := \sum_{j=0}^{k-1} 1^{\otimes j} \otimes \phi \otimes (\iota\pi)^{\otimes k-1-j} : C^{\otimes k} \longrightarrow C^{\otimes k},$$

for all $k \geq 2$.

Example 2.3.8. Consider the CW decomposition of the torus shown in Figure 2.3. Since the boundary of each cell is zero, the homology gvf that Algorithm 2.3.5 produces is $\phi = 0$. Applying the formulas in Proposition 2.3.7, we get a good A_∞-coalgebra $\{\Delta_n\}_n$ on the homology of the torus, where Δ_2 is the Alexander-Whitney diagonal and $\Delta_n = 0$ for all $n \neq 2$. In particular, this proves that the torus is a formal space.

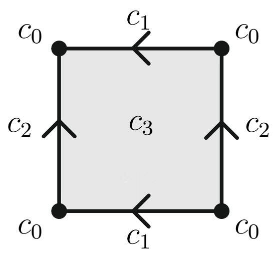


Figure 2.3: The simplest CW decomposition of the torus, consisting of one 0-cell, two 1-cells and one 2-cell. Here arrows do not represent a discrete vector field, but the identifications of the top and bottom faces of the square (both are the cell c_1) and of the left and right faces of the square (both are the cell c_2), respectively.

2.4 The power of A_∞-structures

In Chapter 3 we want to use A_∞-(co)algebras to improve the discriminatory power of persistence, that in its most basic form yields information at the level of Betti numbers. In this section we want to stress the power of A_∞-structures over that of Betti numbers and even over that of DGAs and DGCs.

We will start by proving Theorem 2.4.4, that along with Example 2.4.5, exhibits the existence of spaces with isomorphic homology groups and cohomology algebras but non-isomorphic *good* A_∞-coalgebras on their homologies.

To close this section, we name a classical argument (Theorem 2.4.6) that also makes our point in terms of A_∞-algebras, in relation with the cohomology ring of loop spaces $H^*(\Omega X)$.

Notice that in Section 3.2 we will give more results reinforcing the idea of the strength of A_∞-structures.

Let us recall a couple of definitions that will simplify the proof of Theorem 2.4.4. We refer the reader to the Appendix, which contains the notation and basic tools we will use from rational homotopy theory.

Definition 2.4.1. An A_∞-coalgebra $(C, \{\Delta_n\}_n)$ such that $\Delta_n = 0$ for all $n > 2$ is called **trivial.**

Definition 2.4.2. A minimal A_∞-coalgebra is called **degenerate** if it is isomorphic to a trivial one.

Definition 2.4.3. A space X is called **formal** if a (and therefore any) *good* A_∞-coalgebra on $H_*(X)$ induced by X is degenerate.

Here is the main result we proof in this section:

Theorem 2.4.4. *Let X be a non-formal simply connected CW complex of finite type and let Y be the formal space associated to $H^*(X;\mathbb{Q})$. Then, X and Y have isomorphic rational homology groups and isomorphic rational cohomology algebras but their rational homologies do not admit isomorphic* good *A_∞-coalgebra structures.*

Proof. Obviously, both spaces have isomorphic rational cohomology algebras, and in particular, isomorphic rational homology $H_*(X,\mathbb{Q}) \cong H_*(Y,\mathbb{Q})$ which we will denote by V.

Now, since Y is formal, every *good* A_∞-coalgebra on V induced by Y is degenerate, *i.e.,* isomorphic to a trivial one. On the other hand, since X is non-formal, no *good* A_∞-coalgebra on V induced by X can be isomorphic to a trivial one. Therefore, no *good* A_∞-coalgebra on $H_*(X,\mathbb{Q})$ (induced by X) can be isomorphic to a *good* A_∞-coalgebra on $H_*(Y,\mathbb{Q})$ (induced by Y). □

We now exhibit an example of spaces X and Y as in the theorem above:

Example 2.4.5. An explicit example given by the Theorem above is the following. Consider $X = (\mathbb{S}^3 \vee \mathbb{S}^3)^{(5)}$ to be the 5^{th} Postnikov stage of the wedge $\mathbb{S}^3 \vee \mathbb{S}^3$. Its rational cohomology can be easily computed via Sullivan approach to rational homotopy theory to yield,

$$H^*(X;\mathbb{Q}) \cong H^*(\Lambda(x_3, y_3, z_5), d),$$

where $\Lambda(x_3, y_3, z_5)$ denotes the free commutative (in the graded sense) algebra generated by two elements x_3, y_3 of degree 3 and one element z_5 of degree 5; the differential d is zero on x_3, y_3 and $dz_5 = x_3y_3$.

A straightforward computation shows that this cohomology algebra has a basis (as graded vector space)

$$\{1, \alpha_3, \beta_3, \gamma_8, \rho_8, \xi_{11}\}$$

in which subscripts denote degree, and the only non trivial products are

$$\alpha_3\rho_8 = -\beta_3\gamma_8 = -\xi_{11}.$$

A standard argument in rational homotopy theory shows that, if Y is the formal space associated to this cohomology algebra, the rational homotopy of Y is infinite dimensional contrary to the behaviour of X and thus, they do not have the same rational homotopy type. This shows that X is non-formal and the theorem above applies.

Last we mention a classical result due to T. Kadeishvili, this time in terms of A_∞-algebras. To show that any *good* A_∞-algebra on the cohomology of a space carries in general more information than simply the cohomology ring, we can notice that under mild conditions on X, any such *good* A_∞-algebra determines the cohomology of its loop space, $H^*(\Omega X)$, whereas the cohomology ring of X alone does not.

Theorem 2.4.6. [60, Proposition 2] *Let X be a simply connected space such that all its homology groups are free. Then, the homology of the bar construction (which is the dual of Definition 2.2.5) of any* good *A_∞-algebra on $H^*(X)$ is isomorphic to the cohomology of the loop space of X.*

Chapter 3

A_∞-persistence

What do we want to extend in the theory of persistence?

From an algebraic point of view, persistent homology allows us to study the evolution of homology along a filtration, at the level of Betti numbers. We want to enhance this tool in order to study that evolution at a deeper level, through the use of A_∞-structures (Theorems 3.3.5 and 3.6.1 and Corollary 3.6.5).

From a topological point of view, stability tells us that persistent homology can sometimes recover the Betti numbers of a space out of just a good finite point sample (see the last two paragraphs of §1.2). For instance, given an n-component link L in $\mathbb{S}^3$, one could expect persistent homology to detect the presence of n knots out of a good finite point sample of L or $\mathbb{S}^3 - L$, since

$$\beta_p(\mathbb{S}^3 - L) = \beta_p(L) = \begin{cases} n, & \text{if } p \in \{0, 1\} \\ 0, & \text{otherwise,} \end{cases}$$

but it would not detect (at least not in a non-*ad-hoc* way) any sort of information on how the n components are linked. Our wish is to use A_∞-structures to push persistence forward in situations like this, since they can sometimes give information in this direction (as explained in §3.2.3).

Similarly, good point samples of spaces that have different homotopy types but the same Betti numbers will be seen as equivalent through the lens of classical persistence. We regard A_∞-persistence as a potentially finer lens through which to look at such kinds of point clouds.

Notice that classical persistence is usually exposed in terms of homology but it can be applied to cohomology as well in the obvious manner. Analogously, we will explain our persistence of A_∞-coalgebras on homology but there is a completely dual theory of persistence of A_∞-algebras on cohomology.

Here is an outline of this chapter. Section 3.1 begins by explaining which numbers related to the higher (co)multiplications of *good* A_∞-(co)algebras we will focus on when preparing for persistence. Very importantly, such numbers are not in general invariants of the isomorphism class of A_∞-(co)algebras (Section 3.2.1), so we devote Sections 3.2.2 and 3.2.3 to display settings in which they do yield invariant information and hence it makes sense to compare them to try to distinguish spaces of different homotopy type. Section 3.3 gives the necessary definitions of our theory and proves one first decomposition result. Section 3.4 gives hypotheses under which to guarantee that we can apply such a decomposition. Section 3.5 explores the *intermittent* behaviour of homology classes with respect to the structural operations Δ_n or m_n along a filtration. When such behaviour takes place, we can no longer apply the first decomposition result. Section 3.6 shows that even in this situation we can still get a decomposition result. Section 3.7 closes the chapter with some notes on how to compute A_∞-persistence in the most general case.

Throughout this chapter we will work over a field of coefficients.

3.1 Ker Δ_n & Coker m_n

Similarly to how Persistent Homology studies the evolution along a filtration

$$K_0 \hookrightarrow K_1 \hookrightarrow \dots \hookrightarrow K_N$$

of non-zero classes in each $H_p(K_i)$, our approach consists of focusing on the evolution of non-zero classes in certain subspaces of each $H_p(K_i)$. In this section we explain which subspaces we consider.

In Section 3.2, we show different examples in which, given a space X, looking at an operation Δ_n from a *good* A_∞-coalgebra on $H_*(X)$ (or at an operation m_n from a *good* A_∞-algebra on $H^*(X)$) gives information about the homotopy type of X beyond Betti numbers and even beyond the standard cohomology algebra. Actually, in those examples, provided we know the dimensions of the domain and codomain of the linear map

$$\Delta_n \colon H_*(X) \longrightarrow H_*(X)^{\otimes n},$$

some of the extra information can be detected by computing the dimension of vector spaces like the ones below

$$\operatorname{Ker}\Delta_n, \operatorname{Im}\Delta_n, \operatorname{Coker}\Delta_n := \frac{H_*(X)^{\otimes n}}{\operatorname{Im}\Delta_n}, \operatorname{Coim}\Delta_n := \frac{H_*(X)}{\operatorname{Ker}\Delta_n}.$$

These numbers determine one another, so let us just choose one that seems simple to work with in our scenario. We were looking for a vector subspace of $H_*(X)$ to focus on, so let us discard the second and third of these vector spaces.

Now, let

$$K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_N$$

be a filtration of topological spaces, compute a *good* A_∞-coalgebra $\{\Delta_n^i\}_n$ on $H_*(K_i)$ for every $0 \leq i \leq N$, and fix some $n \geq 1$ and $p \geq 0$. If $[\alpha]$ is a non-zero class

$$0 \neq [\alpha] \in \operatorname{Coim}\Delta^i_{n|H_p(K_i)} = \frac{H_p(K_i)}{\operatorname{Ker}\Delta^i_{n|H_p(K_i)}},$$

then there are some non-zero a_k's such that

$$\Delta_n^i \alpha = a_1 \otimes \ldots \otimes a_n \neq 0.$$

Analogously, if $[f^{i,j}\alpha]$ is a non-zero class

$$0 \neq [f^{i,j}\alpha] \in \operatorname{Coim}\Delta^j_{n|H_p(K_j)},$$

then there are some non-zero b_k's such that

$$\Delta_n^j f^{i,j}\alpha = b_1 \otimes \ldots \otimes b_n \neq 0.$$

Observe that b_k might be different from $f^{i,j}a_k$ (see Theorem 3.5.1), since the equality

$$\left(f^{i,j}\right)^{\otimes n} \Delta_n^i = \Delta_n^j f^{i,j}$$

does not hold in general.

If we do not want to deal with how those a_k's and b_k's relate to one another, we can look at $\operatorname{Ker} \Delta_{n|H_p}$ instead of looking at $\operatorname{Coim} \Delta_{n|H_p}$, which is simpler in the sense that conditions like $0 \neq [\alpha] \in \operatorname{Ker} \Delta^i_{n|H_p(K_i)}$ and $0 \neq [f^{i,j}\alpha] \in \operatorname{Ker} \Delta^j_{n|H_p(K_j)}$ simply mean that $\Delta_n^i \alpha = 0$ and $\Delta_n^j f^{i,j}\alpha = 0$.

With this in mind, given a *good* A_∞-coalgebra structure $\{\Delta_n^i\}_n$ on each $H_*(K_i)$, the subspace of $H_p(K_i)$ we focus on is

$$\operatorname{Ker} \Delta^i_{n|H_p(K_i)}$$

(or minor modifications by restricting Δ_n^i to a smaller subspace). Similarly, given a *good* A_∞-algebra structure $\{m_n^i\}_n$ on each $H^*(K_i)$, the kind of subspaces of $H^p(K_i)$ we focus on is

$$\operatorname{Coker} m_{n|H^{p_1}(K_i)\otimes\ldots\otimes H^{p_n}(K_i)}$$

for $p_k's$ such that $\sum_{k=1}^n p_k = p + n - 2$.

3.2 Backing our stand

If we forget about the persistent approach for a moment, the idea we just discussed in Section 3.1 would consist of the following: given a space X and a *good* A_∞-algebra $(H^*(X), \{m_n\}_n)$ or a *good* A_∞-coalgebra $(H_*(X), \{\Delta_n\}_n)$, the point in computing numbers like

$$\dim \operatorname{Coker} m_{n|H^{p_1}(X)\otimes\ldots\otimes H^{p_n}(X)} \qquad \text{or} \qquad \dim \operatorname{Ker} \Delta_{n|H_p(X)}$$

(or minor modifications thereof) is to try to find extra information about X, beyond Betti numbers.

We now illustrate that despite these numbers not being invariants of the isomorphism type of minimal A_∞-(co)algebras in general if $n > 2$ (§3.2.1), there are settings in which some of these numbers are indeed invariants of that kind and provide information on the homotopy type of spaces beyond the standard cohomology algebra (§3.2.2 and §3.2.3).

3.2.1 Lack of invariance

The example below shows that degenerate A_∞-coalgebras do not need to be trivial and hence the homology of formal spaces can admit in general non-trivial *good* A_∞-coalgebra structures (see Def. 2.4.1, 2.4.2 and 2.4.3). In particular, numbers like $\dim \operatorname{Ker} \Delta_{n|H_p(X)}$ are not always necessarily the same in isomorphic minimal A_∞-coalgebras. We refer the reader again to the Appendix, which contains the notation and basic tools we will use from rational homotopy theory.

Example 3.2.1. Let $(\mathbb{L}(W), \partial)$ be the free DGL (subscripts will denote degree)

$$(\mathbb{L}(x_1, y_3, z_6), \partial)$$

where

$$\begin{aligned}\partial x_1 &= 0,\\ \partial y_3 &= [x_1, x_1],\\ \partial z_6 &= 2\left[[x_1, x_1], y_3\right].\end{aligned}$$

Consider the isomorphism of (non differential!) free Lie algebras appearing in [35, §2],

$$\psi\colon \mathbb{L}(x_1, y_3, u_6) \xrightarrow{\cong} \mathbb{L}(x_1, y_3, z_6)$$

which sends u_6 to $z_6 - [y_3, y_3]$ and any other generator to itself. Then, set in the free Lie algebra on the left the differential

$$\partial = \psi^{-1}\partial\psi$$

so that ψ becomes an isomorphism of DGL's. Set

$$(\mathbb{L}(V), \partial) = (\mathbb{L}(x_1, y_3, u_6), \partial).$$

A short computation shows that the only generator on this isomorphic DGL in which the differential has changed is:

$$\partial u_6 = 0.$$

Via Theorem 4.0.3, $(\mathbb{L}(V), \partial)$ is a Quillen minimal model of the space

$$X = (\mathbb{S}^2 \vee \mathbb{S}^7) \cup_g e^4$$

in which the (homotopy class of the) attaching map for the 4-cell is the Whitehead product

$$g = [id_{\mathbb{S}^2}, id_{\mathbb{S}^2}].$$

Since $(\mathbb{L}(V), \partial) \cong (\mathbb{L}(W), \partial)$ are isomorphic DGL's, $(\mathbb{L}(W), \partial)$ is also a Quillen minimal model of X, and in particular

$$V \cong W \cong s^{-1}\widetilde{H}_*(X; \mathbb{Q}).$$

A short computation shows that, in the universal enveloping algebra $(T(W), d) = U(\mathbb{L}(W), \partial)$ (see Appendix for its definition),

$$d_3 z_6 = 4x_1 \otimes x_1 \otimes y_3 + 4y_3 \otimes x_1 \otimes x_1.$$

In other words, in the A_∞-coalgebra structure on $H_*(X; \mathbb{Q})$ given by $(\mathbb{L}(W), \partial)$,

$$\Delta_3 \neq 0.$$

On the other hand, in the universal enveloping algebra $(T(V), d) = U(\mathbb{L}(V), \partial)$, $d_n = 0$ for all $n \neq 2$, which means that in the A_∞-coalgebra structure on $H_*(X; \mathbb{Q})$ given by $(\mathbb{L}(V), \partial)$,

$$\Delta_n = 0$$

for all $n \neq 2$; in particular, $\Delta_3 = 0$.

3.2.2 The invariant $\min\{n \mid \Delta_n \neq 0\}$

In this section we work over a field of characteristic 0 and prove that the numbers

$$k := \min\{n \mid \Delta_n \neq 0\}$$

and $\dim \operatorname{Ker} \Delta_k$ are invariants of the isomorphism class of minimal A_∞-coalgebras structures $\{\Delta_n\}_n$ on a graded module C (*e.g.*, $C = H_*(X)$). This might be a folklore result in rational homotopy theory but since we have not been able to find a proof of it, we provide one here.

Theorem 3.2.2. *Let C be a graded module and let $\{\Delta_n\}_n$ and $\{\Delta'_n\}_n$ be two isomorphic minimal A_∞-coalgebra structures on C such that there exist some $m, m' \geq 1$ for which $\Delta_m \neq 0$ and $\Delta'_{m'} \neq 0$. Then, the numbers*

$$k := \min\{n \mid \Delta_n \neq 0\} \qquad \textit{and} \qquad \min\{n \mid \Delta'_n \neq 0\}$$

coincide,

$$(C, \{0, \ldots, 0, \Delta_k, 0, \ldots\}) \qquad \textit{and} \qquad (C, \{0, \ldots, 0, \Delta'_k, 0, \ldots\})$$

are isomorphic A_∞-coalgebras and the numbers

$$\dim \mathit{Ker}\, \Delta_{k|C_p} \qquad \textit{and} \qquad \dim \mathit{Ker}\, \Delta'_{k|C_p}$$

coincide too, for all $p \geq 0$.

This turns out to be very useful for us because we are interested in computing *good* A_∞-coalgebra structures on $H_*(X)$, which are all minimal and therefore isomorphic in view of Theorem 2.2.9. Theorem 3.2.2 yields the following corollary:

Corollary 3.2.3. *Let $\{\Delta_n\}_n$ and $\{\Delta'_n\}_n$ be two arbitrary* good *A_∞-coalgebra structures on the homology of a space X such that there exist some $m, m' \geq 1$ for which $\Delta_m \neq 0$ and $\Delta'_{m'} \neq 0$. Then, the numbers*

$$k := \min\{n \mid \Delta_n \neq 0\} \qquad \textit{and} \qquad \min\{n \mid \Delta'_n \neq 0\}$$

coincide,

$$(H_*(X), \{0, \ldots, 0, \Delta_k, 0, \ldots\}) \qquad \textit{and} \qquad (H_*(X), \{0, \ldots, 0, \Delta'_k, 0, \ldots\})$$

are isomorphic A_∞-coalgebras and the numbers

$$\dim \mathit{Ker}\, \Delta_{k|H_p(X)} \qquad \textit{and} \qquad \dim \mathit{Ker}\, \Delta'_{k|H_p(X)}$$

coincide too, for all $p \geq 0$.

First we prove a preliminary result.

Lemma 3.2.4. *Let $(C, \{\Delta_n\}_n)$ be an A_∞-coalgebra such that $\Delta_n \neq 0$ for some $n \geq 1$. If we denote by k the lowest integer $n \geq 1$ such that $\Delta_n \neq 0$, then*

$$(C, \{0, \dots, 0, \Delta_k, 0, \dots\})$$

forms an A_∞-coalgebra too.

Proof. Let $(C, \{\Delta_n\}_n)$ and k be as in the statement and define

$$\Delta'_n := \begin{cases} 0, & n \neq k; \\ \Delta_k, & n = k. \end{cases}$$

Let us denote by

$$\mathrm{SI}(1), \mathrm{SI}(2), \dots, \mathrm{SI}(n), \dots$$

the Stasheff identities (Definition 2.2.1) on $\{\Delta_n\}_n$ and by

$$\mathrm{SI}'(1), \mathrm{SI}'(2), \dots, \mathrm{SI}'(n), \dots$$

the Stasheff identities on $\{\Delta'_n\}_n$.

$(C, \{\Delta'_n\}_n)$ forms an A_∞-coalgebra if and only if the identities $\mathrm{SI}'(n)$ hold for all $n \geq 1$. Notice that all of them except for $\mathrm{SI}'(2k-1)$ become the trivial identity $0 = 0$, and that $\mathrm{SI}'(2k-1)$ becomes

$$\sum_{j=0}^{k+1} (-1)^{k+j+kj} \left(1^{\otimes k-1-j} \otimes \Delta_k \otimes 1^j\right) \Delta_k = 0,$$

since this is the only $\mathrm{SI}'(n)$ in which Δ_k composes with Δ_k (tensored by identities), the only non-zero operation in $\{\Delta'_n\}$.

Let us now look at the identity $\mathrm{SI}(2k-1)$ on $\{\Delta_n\}_n$. If some Δ_n with $n > k$ appears in $\mathrm{SI}(2k-1)$, then Δ_n (or Δ_n tensored by identities) is pre or post composed with Δ_m for some $m < k$ (or with Δ_m tensored by identities). Since $\Delta_m = 0$, such a composition vanishes. Therefore, $\mathrm{SI}(2k-1)$ coincides with $\mathrm{SI}'(2k-1)$. Since we know that $(C, \{\Delta_n\}_n)$ forms an A_∞-coalgebra and hence that $\mathrm{SI}(2k-1)$ holds, this means that $\mathrm{SI}'(2k-1)$ holds too.

With this, we have shown that $(C, \{\Delta'_n\}_n)$ satisfies all Stasheff identities and thus forms an A_∞-algebra. □

We devote the rest of this section to proving Theorem 3.2.2, for which we will make use of the algebraic Milnor-Moore spectral sequence. The background on spectral sequences needed to follow the proof can be found in Sections 18 and 23(b) in [41] and Chapter 1 in [51].

Let $(C, \{\Delta_n\}_n)$ be a minimal A_∞-coalgebra and

$$\Omega C = (\widehat{T}(s^{-1}C), d)$$

its cobar construction (Def. 2.2.5).

Let us recall that we can write $d = \sum_{n\geq 1} d_n$, where each

$$d_n\colon \widehat{T}(s^{-1}C) \longrightarrow T^{\geq n}(s^{-1}C) = \Pi_{p\geq n} T^p(s^{-1}C)$$

satisfies

$$d_n\left(T^p(s^{-1}C)\right) \subseteq T^{p+n-1}(s^{-1}C)$$

for all $p \in \mathbb{N}$ and acts on $T^1(s^{-1}C) = s^{-1}C$ as the composition

$$d_n\colon s^{-1}C \xrightarrow{\ s\ } C \xrightarrow{-(-1)^{\frac{n(n-1)}{2}}\Delta_n} C^{\otimes n} \xrightarrow{(s^{-1})^{\otimes n}} T^n(s^{-1}C).$$

Let us denote by k the integer

$$k := \min\{n \,|\, \Delta_n \neq 0\},$$

which is obviously equal to

$$\min\{n \,|\, d_n \neq 0\}.$$

Since $(C, \{\Delta_n\}_n)$ is minimal, the number k must be at least 2.

Let us consider another minimal A_∞-coalgebra $(C, \{\Delta'_n\}_n)$ isomorphic to $(C, \{\Delta_n\}_n)$ and write $d' = \sum_{n\geq 1} d'_n$ as before. This means that if we denote by

$$\Omega' C = (\widehat{T}(s^{-1}C), d')$$

the cobar construction of $(C, \{\Delta'_n\}_n)$, then ΩC and $\Omega' C$ are isomorphic as DGA's.

We will prove that if we set

$$k' := \min\{n \,|\, \Delta'_n \neq 0\} \, (= \min\{n \,|\, d'_n \neq 0\}) \, ,$$

then $k = k'$ and the DGA's $(\widehat{T}(s^{-1}C), d_k)$ and $(\widehat{T}(s^{-1}C), d'_k)$ are isomorphic.

Assume $k \leq k'$ and let

$$\widehat{T}(s^{-1}C) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^p \supseteq F^{p+1} \supseteq \ldots$$

be the *word-length filtration* of ΩC, *i.e.*, for all $p \in \mathbb{N}$, F^p is the differential ideal given by

$$F^p = T^{\geq p}(s^{-1}C).$$

In the algebraic Milnor-Moore spectral sequence, E_0 is the graded algebra associated to this filtration, *i.e.*, for all $p \in \mathbb{N}$,

$$E_0^p = \frac{F^p}{F^{p+1}} \cong T^p(s^{-1}C)$$

and

$$E_0 = \bigoplus_{p \in \mathbb{N}} E_0^p \cong T(s^{-1}C).$$

The differential

$$\partial_0 = \bigoplus_{p \in \mathbb{N}} \partial_0^p \colon E_0 \longrightarrow E_0$$

is the map induced by d, thus for all $p \in \mathbb{N}$,

$$\partial_0^p \colon E_0^p \longrightarrow E_0^p$$

must be

$$d_1 \colon T^p(s^{-1}C) \longrightarrow T^p(s^{-1}C),$$

which is 0, for C is minimal.

Altogether,

$$(E_0, \partial_0) \cong (\widehat{T}(s^{-1}C), 0).$$

In the algebraic Milnor-Moore spectral sequence, for all $p \in \mathbb{N}$ we have

$$E_1^p \cong E_0^p \cong T^p(s^{-1}C),$$

and the map

$$\partial_1^p \colon E_1^p \longrightarrow E_1^{p+1}$$

induced by d is thus

$$d_2 \colon T^p(s^{-1}C) \longrightarrow T^{p+1}(s^{-1}C).$$

Hence,

$$(E_1, \partial_1) \cong (\widehat{T}(s^{-1}C), d_2).$$

Actually, we can easily verify that

$$(E_0, \partial_0) \cong \ldots \cong (E_{k-2}, \partial_{k-2}) \cong (\widehat{T}(s^{-1}C), 0)$$

and

$$(E_{k-1}, \partial_{k-1}) \cong (\widehat{T}(s^{-1}C), d_k).$$

The following result is a version of Theorem I in Section 1.3 in [51] in terms of DGA's.

Theorem 3.2.5. [51, Theorem I in Section 1.3] **(Comparison theorem)** *Let $\varphi \colon M \longrightarrow M'$ a filtered morphism of DGA's whose spectral sequences are convergent. Then, if there is some $i \in \mathbb{N}$ such that the induced map*

$$E_i(\varphi) \colon E_i(M) \longrightarrow E_i(M')$$

is an isomorphism of DGA's, then so are the maps

$$E_j(\varphi) \colon E_j(M) \longrightarrow E_j(M'),$$

for all $j \geq i$, and so is the map

$$H_*(\varphi) \colon H_*(M) \longrightarrow H_*(M').$$

Let

$$\varphi\colon \Omega C \longrightarrow \Omega C'$$

be an isomorphism of DGA's, that exists because $(C, \{\Delta_n\}_n)$ and $(C, \{\Delta'_n\}_n)$ are isomorphic A_∞-coalgebras. Since φ is filtered, it induces a morphism of spectral sequences

$$\{E_r(\varphi)\}_r$$

from the algebraic Milnor-Moore spectral sequence associated to ΩC to that associated to $\Omega' C$. Recall that the linear part of φ,

$$\varphi_1\colon s^{-1}C \longrightarrow s^{-1}C,$$

is defined by $\varphi(s^{-1}c) = \varphi_1(s^{-1}c) + \Phi$, where $\Phi \in T^{\geq 2}(s^{-1}C)$. Since φ is an isomorphism, φ_1 must also be an isomorphism. Now observe that $E_0(\varphi) = E_{k-2}(\varphi)$ is precisely the isomorphism

$$\widehat{T}(\varphi_1)\colon (\widehat{T}(s^{-1}C), 0) \longrightarrow (\widehat{T}(s^{-1}C), 0).$$

Applying the comparison criterion, we get that

$$E_{k-1}(\varphi)\colon \big(T(s^{-1}C), d_k\big) \longrightarrow \big(T(s^{-1}C), d'_k\big)$$

is an isomorphism of DGA's. In particular, d'_k cannot be zero. Hence $k = k'$ and the first claim in Theorem 3.2.2 holds.

For the second claim, that the A_∞-coalgebras (we know they are indeed A_∞-coalgebras thanks to Lemma 3.2.4)

$$(C, \{0, \ldots, 0, \Delta_k, 0, \ldots\}) \qquad \text{and} \qquad (C, \{0, \ldots, 0, \Delta'_k, 0, \ldots\})$$

are isomorphic, simply observe that the corresonding cobar constructions are $(\widehat{T}(s^{-1}C), d_k)$ and $(\widehat{T}(s^{-1}C), d'_k)$ which we have just seen are isomorphic DGA's via $E_{k-1}(\varphi)$.

Finally, this tells us there exists an isomorphism of A_∞-coalgebras

$$f\colon (C, \{0, \ldots, 0, \Delta_k, 0, \ldots\}) \longrightarrow (C, \{0, \ldots, 0, \Delta'_k, 0, \ldots\}).$$

The morphism identity MI(k) (see Def. 2.2.6) becomes

$$\Delta'_k f_{(1)} = f_{(1)}{}^{\otimes k} \Delta_k.$$

Since $f_{(1)}$ is an isomorphism, this implies that

$$\dim \operatorname{Ker} \Delta_{k|C_p} = \dim \operatorname{Ker} \Delta'_{k|C_p}$$

for all $p \geq 0$.

3.2.3 Massey products

In this section we will recall the notion of Massey products and give some reasons why we might want to have them in mind when computing Persistence of A_∞-structures the way we do it.

The triple and generalized Massey products were first introduced in [99] and [70], respectively, and a most clear general exposition of this topic can be found in [74].

Let (A, d, μ) be a DGA with $|d| = 1$. For our purposes, we may think of the usual singular cochain DGA of a space $(C^*(X), \delta, \smile)$. Let us consider its cohomology algebra $(H^*(A), \mu^*)$, use the notation ab to denote either $\mu(a \otimes b)$ or $\mu^*(a \otimes b)$, and let

$$\overline{a} := (-1)^{|a|+1} a$$

for any homogeneous $a \in A$.

The triple Massey product is only defined on classes $[u], [v], [w] \in H(A)$ such that

$$\begin{cases} [u][v] = 0 \\ [v][w] = 0. \end{cases}$$

Fixing such classes, if $[u] \in H^p(A), [v] \in H^q(A), [w] \in H^r(A)$, let us choose $u \in A^p, v \in A^q, w \in A^r$ representing $[u], [v], [w]$, respectively. Since $[u][v] = 0$, there exists some $s \in A^{p+q-1}$ such that $ds = \overline{u}v$. Analogously, since $[v][w] = 0$, there exists some $t \in A^{q+r-1}$ such that $dt = \overline{v}w$. We can form the element

$$\overline{s}w + \overline{u}t \in A^{p+q+r-1},$$

which satisfies

$$d(\overline{s}w + \overline{u}t) = 0,$$

so we can consider its cohomology class

$$[\overline{s}w + \overline{u}t] \in H^{p+q+r-1}(A).$$

Notice that, in order to create this cohomology class out of $[u], [v]$, and $[w]$, we have made 5 choices; namely, representants u, v, w and elements s, t. The **triple Massey product** of $[u], [v], [w]$, denoted by

$$\boldsymbol{\langle [u], [v], [w] \rangle},$$

is defined as the set of all possible classes of the form

$$[\overline{s}w + \overline{u}t] \in H^{p+q+r-1}(A),$$

where u, v, w, s, t vary over all possible choices as just explained.

Hence, $\langle [u], [v], [w] \rangle$ is a subset of $H^{p+q+r-1}(A)$. It is sometimes handy to see this as a single element in a group rather than as a subset. In this sense, it is easy to check that $\langle [u], [v], [w] \rangle$ is an element of the quotient group

$$\frac{H^{p+q+r-1}(A)}{[u]H^{q+r-1}(A) + H^{p+q-1}(A)[w]}. \tag{3.2.1}$$

An important idea is when we say that a Massey product vanishes (or equivalently, that it is trivial).

Definition 3.2.6. If the triple Massey product $\langle [u], [v], [w] \rangle$ is defined, we say that it is **trivial** (or **vanishing**) if the class it represents in the quotient (3.2.1) is the zero class, or equivalently, if $0 \in H^{p+q+r-1}(A)$ is an element of the set $\langle [u], [v], [w] \rangle$.

We say that the triple Massey product of (A, d, μ) is **trivial** (or **vanishing**) if whenever the triple Massey product of three classes is defined, it is trivial.

We have just shown how to define a 3-fold product set whenever certain 2-fold products vanish (namely, when some cohomology products are 0). Now we are going to show how to define a 4-fold product set whenever certain 3-fold products vanish (namely, when some triple Massey products contain the 0 class).

Let $[u], [v], [w], [x] \in H^*(A)$ be such that the triple Massey products

$$\langle [u], [v], [w] \rangle , \qquad \langle [v], [w], [x] \rangle$$

are defined and both contain 0. This means that there exist representatives u, v, w, x of $[u], [v], [w], [x]$, respectively, and homogeneous elements $t_0, t_1, t_2, Y_1, Y_2 \in A$, such that

$$dt_0 = \overline{u}v, \qquad dt_1 = \overline{v}w, \qquad dt_2 = \overline{w}x$$

and

$$dY_1 = \overline{t_0}w + \overline{u}t_1, \qquad dY_2 = \overline{t_1}x + \overline{v}t_2.$$

Then,

$$d\left(\overline{u}Y_2 + \overline{t_0}t_2 + \overline{Y_1}x\right) = 0$$

and therefore

$$[\overline{u}Y_2 + \overline{t_0}t_2 + \overline{Y_1}x] \in H^{|u|+|v|+|w|+|x|-2}(A).$$

We define the **4-fold Massey product** of the classes $[u], [v], [w], [x]$, denoted by $\langle [\boldsymbol{u}], [\boldsymbol{v}], [\boldsymbol{w}], [\boldsymbol{x}] \rangle$, as the set of all cohomology classes of the form

$$[\overline{u}Y_2 + \overline{t_0}t_2 + \overline{Y_1}x]$$

that can be formed by letting the representatives u, v, w, x and the homogeneous elements t_0, t_1, t_2, Y_1, Y_2 vary over all possible choices as just described.

More generaly, we can define an n-fold product whenever certain $(n-1)$-fold products vanish (in the sense, as before, that they contain the 0 class).

Definition 3.2.7. [67] Fix some $n \geq 3$ and let $[\alpha_i] \in H^{p_i}(A)$ for all $1 \leq i \leq n$. A **defining system** associated to $[\alpha_1], \ldots, [\alpha_n]$ is a set

$$\{a_{ij}\}_{1 \leq i \leq j \leq n, (i,j) \neq (1,n)} \subseteq A$$

satisfying, for all $1 \leq i \leq j \leq n$ with $(i, j) \neq (1, n)$, the following:

(i) $a_{ij} \in A^{p_i+p_{i+1}+\ldots+p_j+(i-j)}$,

(ii) a_{ii} is a representative of $[\alpha_i]$ in $H^{p_i}(A)$,

(iii) $d(a_{ij}) = \sum_{r=i}^{j-1} \overline{a_{ir}} a_{r+1,j}$.

To a defining system we associate the cocycle

$$\sum_{r=1}^{n-1} \overline{a_{1r}} a_{r+1,n} \in A^{p_1+\ldots+p_n+(2-n)}.$$

The n-**fold Massey product** of the classes $[\alpha_1], \ldots, [\alpha_n]$ is denoted by

$$\langle [\boldsymbol{\alpha_1}], \ldots, [\boldsymbol{\alpha_n}] \rangle$$

and defined as the set of cohomology classes of cocycles associated to all possible defining systems for $[\alpha_1], \ldots, [\alpha_n]$.

This indeed generalizes the definitions for $n = 3, 4$ given above. For $n = 3$, a choice of u, v, w, s, t amounts to a defining system where

$$\begin{pmatrix} a_{11} & a_{12} & \\ & a_{22} & a_{23} \\ & & a_{33} \end{pmatrix} = \begin{pmatrix} u & s & \\ & v & t \\ & & w \end{pmatrix}.$$

For $n = 4$, a choice of $u, v, w, x, t_0, t_1, t_2, Y_1, Y_2$ amounts to a defining system where

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \\ & a_{22} & a_{23} & a_{24} \\ & & a_{33} & a_{34} \\ & & & a_{44} \end{pmatrix} = \begin{pmatrix} u & t_0 & Y_1 & \\ & v & t_1 & Y_2 \\ & & w & t_2 \\ & & & x \end{pmatrix}.$$

We define triviality for an arbitrary n in a similar fashion as we did for the triple product.

Definition 3.2.8. If the n-fold Massey product $\langle [\alpha_1], \ldots, [\alpha_n] \rangle$ is defined, we say that it is **trivial** (or **vanishing**) if the set $\langle [\alpha_1], \ldots, [\alpha_n] \rangle$ contains the zero class.

We say that the n-fold Massey product of (A, d, μ) is **trivial** (or **vanishing**) if whenever the n-fold Massey product of n classes is defined, it is trivial.

For every topological space X, by its Massey products we mean the Massey products of the standard DGA $(C^*(X), \delta, \smile)$ of singular cochains.

The reason that makes triviality important is that it is a homotopy invariant in the following sense:

Proposition 3.2.9. *[66, Property 2.1] Given two homotopy equivalent spaces X and Y and any integer $n \geq 3$, the n-fold Massey product of X is trivial if and only if the n-fold Massey product of Y is trivial.*

Next we discuss some connections between A_∞-structures, Massey products and higher linking numbers. We will consider links in $\mathbb{S}^3$ (*i.e.*, collections of embeddings of a circle $\mathbb{S}^1$ into $\mathbb{S}^3$ which do not intersect) and embeddings of arbitrary spheres in a sphere of a higher dimension.

Let us start by recalling how cup product can detect linking number [85, §2.5], [71, §1], [90, §5.D]. We will use a version of Alexander duality:

Theorem 3.2.10. [4,5] **(Alexander Duality)** *Let X be a compact, locally contractible, non-empty, proper subspace of $\mathbb{S}^n$. Then, with coefficients in any abelian group, homology and cohomology relate through the following isomorphisms for all $p \in \mathbb{Z}$:*

$$\widetilde{H}^p\left(\mathbb{S}^n - X\right) \cong \widetilde{H}_{n-p-1}\left(X\right).$$

Hence, if we denote by K_1 and K_2 two disjoint knots in $\mathbb{S}^3$ and by X the disjoint union $K_1 \sqcup K_2$, we have

$$\widetilde{H}^p\left(\mathbb{S}^3 - X\right) \cong \widetilde{H}_{2-p}\left(X\right)$$

for all $p \in \mathbb{Z}$. Therefore, there are classes $\alpha_1, \alpha_2 \in H^1\left(\mathbb{S}^3 - X\right) \cong H_1\left(X\right) \cong \mathbb{Z} \oplus \mathbb{Z}$ that correspond to the fundamental classes $[K_1]$ and $[K_2]$, respectively. Their cup product

$$\alpha_1 \smile \alpha_2 \in H^2\left(\mathbb{S}^3 - X\right) \cong \widetilde{H}_0\left(X\right) \cong \mathbb{Z}$$

determines an integer which turns out to be the linking number of K_1 and K_2, up to a sign.

Moreover, Massey products can detect some higher order linking numbers. The connection between the $\bar{\mu}$-invariants of a link [78] and Massey products has been studied in [36, 42, 84, 93, 98], and more recent link invariants defined in [27, 83, 96] are closely related to Massey products too.

Let us take for instance the Borromean rings.

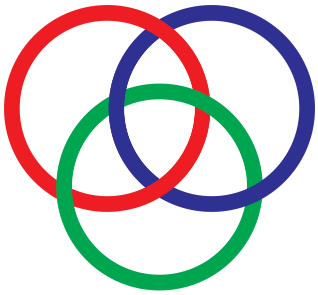


Figure 3.1: Borromean rings. In $\mathbb{S}^3$, every two of these three circles are unlinked yet it is impossible to pull the three of them apart.

Consider any two of the three circles and their corresponding fundamental classes. Since the two circles are unlinked, their linking number is 0 and the cup product of the Alexander duals of those fundamental classes vanish. Since this holds for any two of the three circles, the triple Massey product of the 3 generating 1-dimenisional cohomology classes in the complement is defined. Moreover, Massey proved that this triple product is non-trivial [71], giving a rigurous mathematical proof that, though pairwise unlinked, the 3 rings cannot be pulled apart.

One interesting relation between A_∞-structures and Massey products is that, given a topological space X, the higher A_∞-multiplications on any good A_∞-algebra on $H^*(X)$ compute Massey products up to a sign. More precisely, Palmieri et al. proved:

Theorem 3.2.11. *(Thm. 3.1 & Cor. A.5 in [69]) Let A be a DGA with differential of degree +1. For any* good *A_∞-algebra $\{m_n\}_n$ on its cohomology $H^*(A)$ and for any $n \geq 3$, if $\alpha_1, \ldots, \alpha_n \in H^*(A)$ are elements such that the*

Massey product $\langle\alpha_1,\ldots,\alpha_n\rangle$ is defined, then

$$(-1)^b m_n(\alpha_1\otimes\ldots\otimes\alpha_n)\in\langle\alpha_1,\ldots,\alpha_n\rangle\,,$$

where

$$b=1+|\alpha_{n-1}|+|\alpha_{n-3}|+|\alpha_{n-5}|+\ldots$$

Different choices inside the isomorphism class of *good* A_∞-algebras on $H^*(A)$ lead to higher multiplications m_n that may behave very differently, but this result gives us ways to take advantage of some particular choices of m_n as in our proposition below, that has a straighforward proof thanks to the fact that we are *standing upon the shoulders of giants.*

Proposition 3.2.12. *Let L denote the Borromean rings. Then, any* good *A_∞-algebra $\{m_n\}_n$ on $H^*(\mathbb{S}^3-L)$ will satisfy*

$$m_{3|H^1\otimes H^1\otimes H^1}\not\equiv 0,$$

where H^1 denotes $H^1(\mathbb{S}^3-L)$.

Proof. On the one hand, as we just said, Massey proved in [71] that the Alexander duals of the fundamental classes of the Borromean rings, which are some cohomology classes $\alpha,\beta,\gamma\in H^1(\mathbb{S}^3-L)$ that form a basis of $H^1(\mathbb{S}^3-L)$, satisfy

$$0\notin\langle\alpha,\beta,\gamma\rangle\,.$$

On the other hand, Theorem 3.2.11 tells us that

$$(-1)^b m_3(\alpha\otimes\beta\otimes\gamma)\in\langle\alpha,\beta,\gamma\rangle\,,$$

for a certain integer $b\in\mathbb{Z}$.

Both things together yield the conclusion

$$m_3(\alpha\otimes\beta\otimes\gamma)\neq 0.$$

□

This proposition gives an example of non-trivial operation m_3 on a strictly associative algebra $H^*(A)$ (supporting Remark 2.2.10 in terms of A_∞-algebras) and justifies a deep study of numbers like

$$\dim\left(\operatorname{Coker} m_{n|H^{p_1}(X)\otimes\ldots\otimes H^{p_n}(X)}\right)$$

for some $n, p_1, \ldots, p_n$, when working over a field, in spite of not being homotopy invariants.

Let us end this section by proving one last result, extending Proposition 3.2.12, and mentioning that other examples using

$$m_{n|H^{p_1}(X)\otimes\ldots\otimes H^{p_n}(X)}$$

for arbitrary values of $n, p_1, \ldots, p_n$ can be analogously obtained.

Let $n = p + q + r$, $x = (x_1, \ldots, x_p), y = (y_1, \ldots, y_q)$, and $z = (z_1, \ldots, z_r)$. Consider three spaces S_1, S_2, S_3 in $\mathbb{R}^n$ (homeomorphic to three spheres) determined by the equations

$$\begin{aligned}
&x = 0, \qquad \|y\|^2 + \frac{\|z\|^2}{4} = 1 \qquad ((q+r-1)\text{-dimensional sphere } S_1);\\
&y = 0, \qquad \|z\|^2 + \frac{\|x\|^2}{4} = 1 \qquad ((p+r-1)\text{-dimensional sphere } S_2);\\
&z = 0, \qquad \|x\|^2 + \frac{\|y\|^2}{4} = 1 \qquad ((p+q-1)\text{-dimensional sphere } S_3);
\end{aligned}$$

Notice that the case $p = q = r = 1$ yields Borromean rings.

Recall that for any space X homeomorphic to $\mathbb{S}^{n-1}$, $\mathbb{R}^n - X$ consists of two path components (Jordan-Brower separation theorem [12–14]), the *inside* and the *outside* of X. Note that any two of the spheres S_i are separated by a space homeomorphic to $\mathbb{S}^{n-1}$. For instance, S_1 lies outside the sphere

$$\frac{\|x\|^2}{3^2} + \frac{\|y\|^2}{(1/2)^2} + \frac{\|z\|^2}{(3/2)^2} = 1,$$

while S_3 lies inside it. Now we can use Alexander duality in the exact same way as before. Letting K be the disjoint union of S_1, S_2 and S_3, consider the cohomology classes

$$\alpha^p \in \widetilde{H}^p(\mathbb{S}^n - K) \cong \widetilde{H}_{q+r-1}(K),$$

$$\beta^q \in \widetilde{H}^q(\mathbb{S}^n - K) \cong \widetilde{H}_{p+r-1}(K),$$
$$\gamma^r \in \widetilde{H}^r(\mathbb{S}^n - K) \cong \widetilde{H}_{p+q-1}(K)$$

corresponding to the dual of the fundamental classes of S_1, S_2 and S_3. The fact that these spheres can be separated as explained makes the pairwise cup product of these cohomology classes vanish, so that their triple Massey product is defined. Furthermore, W. S. Massey also proved in [71, §4] that this Massey product is non-trivial. With the same proof of Proposition 3.2.12, *mutatis mutandis*, we can prove the following:

Proposition 3.2.13. *With the notation above, any* good A_∞*-algebra* $\{m_k\}_k$ *on* $H^*(\mathbb{S}^n - K)$ *will have*

$$m_{3|H^p \otimes H^q \otimes H^r} \not\equiv 0,$$

where H^k *denotes* $H^k(\mathbb{S}^n - K)$*.*

3.3 A first decomposition result

For the rest of the chapter, let

$$\mathcal{K}: \qquad K_0 \longrightarrow K_1 \longrightarrow \ldots \longrightarrow K_N$$

be a sequence of topological spaces and continuous maps such that $\dim H_p(K_i) < \infty$ for all $0 \leq i \leq N$ and for all $p \geq 0$. For $p \geq 0$ and $0 \leq i \leq j \leq N$, let

$$f_p^{i,j} \colon H_p(K_i) \longrightarrow H_p(K_j)$$

and

$$f^{i,j} \colon H_*(K_i) \longrightarrow H_*(K_j)$$

denote the maps induced in homology by the composition

$$K_i \hookrightarrow K_{i+1} \hookrightarrow \ldots \hookrightarrow K_j.$$

For the rest of the section, choose a *good* A_∞-coalgebra structure $\{\Delta_n^i\}_n$ on the homology of each term K_i $(0 \leq i \leq N)$.

Here is the counterpart in our theory of Definitions 1.2.1 and 1.2.3.

Definition 3.3.1. Let $0 \leq i \leq j \leq N$, $p \geq 0$ and $n \geq 1$.

- A class $\alpha \in H_p(K_i)$ **$\boldsymbol{\Delta}_n$-wakes up** at K_i if

$$\alpha \neq 0, \qquad \alpha \in \operatorname{Ker} \Delta_n^i \quad \text{ and } \quad \alpha \notin \operatorname{Im} f^{i-1,i}{}_{|\operatorname{Ker} \Delta_n^{i-1}}.$$

- A class $\alpha \in H_p(K_i)$ that Δ_n-wakes up at K_i is said to be **$\boldsymbol{\Delta_n}$-awake** at K_j if for all $i < k \leq j$,

$$f^{i,k}\alpha \in \operatorname{Ker} \Delta_n^k$$

and

$$f^{i,k}\alpha \notin \operatorname{Im} f^{i-1,k}{}_{|\bigcap_{l=i-1}^{k-1}\left(f^{i-1,l}\right)^{-1}\operatorname{Ker} \Delta_n^l}. \tag{3.3.1}$$

Notice that (4.0.2) implies that $f^{i,k}\alpha \neq 0$, for all $i < k \leq j$.

- A class $\alpha \in H_p(K_i)$ that Δ_n-wakes up at K_i is said to **$\boldsymbol{\Delta_n}$-fall asleep** at K_j if it is Δ_n-awake at K_i, K_{i+1}, ..., K_{j-1} but not at K_j, *i.e.,* if α is Δ_n-awake at K_i, K_{i+1}, ..., K_{j-1} and either

$$f^{i,j}\alpha \notin \operatorname{Ker} \Delta_n^j$$

or

$$f^{i,j}\alpha \in \operatorname{Im} f^{i-1,j}{}_{|\bigcap_{l=i-1}^{j-1}\left(f^{i-1,l}\right)^{-1}\operatorname{Ker} \Delta_n^l}. \tag{3.3.2}$$

Note that (4.0.3) includes the possibility $f^{i,j}\alpha = 0$.

Observe that if a class $\alpha \in H_p(K_i)$ is born at K_i and dies at K_l (see Def. 1.2.1), then if it is ever Δ_n-awake, then there exist j and k as in $i \leq j < k \leq l$ such that α Δ_n-wakes up at K_j and Δ_n-falls asleep at K_k. This starts to hint why we choose the *awake-asleep* terminology, and it will be completely apparent in Section 3.5.

Remark 3.3.2. Since every A_∞-coalgebra $(H_*(K_i), \{\Delta_n^i\}_n)$ in the definition above must be *good*, we have $\Delta_1^i = 0$ for all i, so the concepts of Δ_1-awakening and Δ_1-falling asleep coincide with those of birth and death in classical persistence.

Definition 3.3.3. For every $0 \leq i \leq j \leq N$, $p \geq 0$ and $n \geq 1$, we define the $\boldsymbol{p}^{\text{th}}$ $\boldsymbol{\Delta_n}$**-persistence group** between K_i and K_j as the vector space

$$\boldsymbol{(\Delta_n)_p^{i,j}(\mathcal{K})} = \operatorname{Im} f_p^{i,j}|_{\cap_{k=i}^{j}\operatorname{Ker}\left(\Delta_n^k \circ f_p^{i,k}\right)}.$$

We note that this definition includes that of classical persistence groups (see Def. 1.2.3):

Proposition 3.3.4. *$(\Delta_1)_p^{i,j}(\mathcal{K}) = H_p^{i,j}(\mathcal{K})$ for all $0 \leq i \leq j \leq N$ and $p \geq 0$.*

Proof. Since $\Delta_1^k = 0$ for each k, we have

$$(\Delta_1)_p^{i,j}(\mathcal{K}) = \operatorname{Im} f_p^{i,j} = H_p^{i,j}(\mathcal{K}).$$

□

It is obvious from the definition of the classical persistence groups that for any basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$,

$$\dim H_p^{i,j}(\mathcal{K}) = \#\left\{ \beta \in \mathcal{B} \cap H_p(K_i) \;\middle|\; \begin{array}{l} \text{there exist some } 0 \leq k \leq i \text{ and} \\ \alpha \in H_p(K_k) \text{ such that } \beta = f^{k,i}\alpha \\ \text{and } \alpha \text{ is alive at least from } K_i \text{ to } K_j \end{array} \right\}$$

for all $0 \leq i \leq j \leq N$ and $p \geq 0$.

Checking whether a class is alive is computationally costly because it involves checking whether it has merged with an older class. So the fundamental theorem of persistent homology, Theorem 1.2.8, tells us that there exists a basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$ that allows the much simpler computation

$$\dim H_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \neq 0 \text{ for all } k = i, \ldots, j\},$$

or even simpler,

$$\dim H_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,j}\beta \neq 0\},$$

for all $0 \leq i \leq j \leq N$ and $p \geq 0$.

On the other hand, concerning Δ_n-persistence (fixed $n \geq 1$), if we choose a basis of $\operatorname{Ker} \Delta^i_{n|H_p(K_i)}$ for each i and p, put them together to form a basis of $\bigoplus_i \operatorname{Ker} \Delta^i_{n|H_p(K_i)}$ and enlarge this one to get a basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$, then

$$\dim(\Delta_n)_p^{i,j}(\mathcal{K}) = \#\left\{ \beta \in \mathcal{B} \cap H_p(K_i) \;\middle|\; \begin{array}{l} \text{there exist some } 0 \le k \le i \text{ and} \\ \alpha \in H_p(K_k) \text{ such that } \beta = f^{k,i}\alpha \\ \text{and } \alpha \text{ is } \Delta_n\text{-awake at least from} \\ K_i \text{ to } K_j \end{array} \right\} \tag{3.3.3}$$

for all $0 \le i \le j \le N$ and $p \ge 0$.

Now, similarly to the classical case, checking if a class is Δ_n-awake is computationally costly. It is actually more costly than simply checking if it is alive, so a result that guarantees that we can simplify computations in the style of Theorem 1.2.8 is most needed. Indeed, we can prove the following.

Theorem 3.3.5. *Fix some integers $p \ge 0$ and $n \ge 1$. If $f_p^{i,i+1}(Ker\,\Delta_n^i) \subseteq Ker\,\Delta_n^{i+1}$ for all $0 \le i < N$, then there exists a basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$ such that*

$$\dim(\Delta_n)_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \in Ker\,\Delta_n^k - \{0\}\, \textit{for all } k = i, \dots, j\}$$

for every $0 \le i \le j \le N$. Furthermore, this number equals

$$\#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,j}\beta \in Ker\,\Delta^j - \{0\}\}.$$

Proof. Fix a *good* A_∞-coalgebra structure $\{\Delta_n^i\}_n$ on $H_*(K_i)$, for all $0 \le i \le N$ and fix two integers $n \ge 1$ and $p \ge 0$. Let us simply denote by Δ^i the restriction of the map Δ_n^i to $H_p(K_i)$, for all $0 \le i \le N$. If

$$f^{i,i+1}\left(\operatorname{Ker} \Delta^i\right) \subseteq \operatorname{Ker} \Delta^{i+1} \tag{3.3.4}$$

holds for all $0 \le i < N$, then the restriction of the maps in the persistence module

$$H_p(K_0) \xrightarrow{f_p^{0,1}} H_p(K_1) \xrightarrow{f_p^{1,2}} \dots \xrightarrow{f_p^{N-1,N}} H_p(K_N)$$

yields the persistence (sub)module

$$\operatorname{Ker}\Delta^0 \xrightarrow{f_p^{0,1}} \operatorname{Ker}\Delta^1 \xrightarrow{f_p^{1,2}} \dots \xrightarrow{f_p^{N-1,N}} \operatorname{Ker}\Delta^1. \tag{3.3.5}$$

Applying Lemma 1.2.7 to (3.3.5), we get that there exists a basis $\mathcal{B}'$ of $\bigoplus_i \operatorname{Ker}\Delta^i$ such that

$$\dim H_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B}' \cap H_p(K_i) \mid f^{i,j}\beta \neq 0\} \tag{3.3.6}$$

for all i, and if we enlarge $\mathcal{B}'$ to a basis $\mathcal{B}$ of $\bigoplus_i H_p(K_i)$, then the term on the right hand side in (3.3.6) coincides with

$$\#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,j}\beta \in \operatorname{Ker}\Delta^j - \{0\}\},$$

or equivalently (using (3.3.4)), with

$$\#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \in \operatorname{Ker}\Delta^k - \{0\} \text{ for all } k = i, \dots, j\}.$$

□

In §3.5 we show (along with the potentially intermittent behaviour of homology classes with respect to A_∞-persistence) that the assumption

$$f_p^{i,i+1}\left(\operatorname{Ker}\Delta_n^i\right) \subseteq \operatorname{Ker}\Delta_n^{i+1} \tag{3.3.7}$$

in Theorem 3.3.5 does not always hold, and we devote §3.6 to stating and proving a result establishing that there is always a barcode decomposition to describe the p-homology classes that are Δ_n-awake along the sequence $\mathcal{K}$, regardless of the assumption (3.3.7) holding or not.

3.4 Coherent choices of A_∞-structure

Under certain conditions, we can guarantee that the assumption

$$f^{i,i+1}\left(\operatorname{Ker}\Delta_n^i\right) \subseteq \operatorname{Ker}\Delta_n^{i+1}$$

of Theorem 3.3.5 will hold.

Theorem 3.4.1. *With coefficients over the rationals $\mathbb{Q}$, let K_0 be a 1-connected CW complex and let N be an integer. If for all $0 < i \leq N$, K_i denotes a 1-connected space obtained attaching to K_{i-1} a cell such that*

$$\dim \bigoplus_{p\geq 0} H_p(K_i) = \dim \bigoplus_{p\geq 0} H_p(K_{i-1}) + 1$$

(and hence every attachment produces new homology), then there exists a good *A_∞-coalgebra $\left(\widetilde{H}_*(K_i), \{\Delta_n^i\}_n\right)$ for each $0 \leq i \leq N$ such that*

$$f^{i,i+1}\left(Ker\,\Delta_n^i\right) \subseteq Ker\,\Delta_n^{i+1},$$

for every $n \geq 1$ and $0 \leq i < N$.

Proof. First of all, notice that the hypotheses turn the maps $f^{i,i+1}$ into inclusions.

Start the construction by choosing any Quillen minimal model $(\mathbb{L}(V), \partial)$ of K_0. Let $K_1 = K_0 \cup_f e^{m+1}$ be a 1-connected space obtained attaching an $m+1$-cell to K_0 via the map $f\colon \mathbb{S}^m \to K_0$. The condition

$$\dim \bigoplus_{p\geq 0} H_p(K_1) = \dim \bigoplus_{p\geq 0} H_p(K_0) + 1$$

guarantees that the class $[f] \in \pi_m(K_0) \otimes \mathbb{Q}$ is trivial, and hence so is the homology class in $H_{m-1}(\mathbb{L}(V), \partial)$ which is identified, via the isomorphism (ii) in the Appendix, with the homotopy class $[f]$. Therefore, since $0 \in \mathbb{L}(V)_{n-1}$ is a cycle representing the homology class $[0] \in H_{n-1}(\mathbb{L}(V), \partial)$, Theorem 4.0.3 tells us that the injection

$$(\mathbb{L}(V), \partial) \hookrightarrow (\mathbb{L}(V \oplus \mathbb{Q}a), \partial')$$

with

$$\begin{cases} \partial' v = \partial v, & \text{for all } v \in V \\ \partial' a = 0 \end{cases}$$

is a Quillen model of the inclusion

$$K_0 \hookrightarrow K_1.$$

Furthermore, since $\partial'_{1|V} = \partial_1 = 0$ and $\partial' a = 0$, we have that $\partial'_1 = 0$, so the model $(\mathbb{L}(V \oplus \mathbb{Q}a), \partial')$ is indeed minimal.

Applying this argument at each step, we get a Quillen minimal model

$$(\mathbb{L}(V_i), \partial^i)$$

for each K_i so that

$$\partial^{i+1} v = \partial^i v$$

for any $v \in V_i \subset V_{i+1}$ and for all $0 \leq i < N$. These Quillen minimal models then translate into *good* A_∞-coalgebras $\left(\widetilde{H}_*(K_i), \{\Delta^i_n\}_n\right)$ such that for all $n \geq 1$ and $0 \leq i < N$,

$$\Delta^{i+1}_n \alpha = \Delta^i_n \alpha$$

for any $\alpha \in \widetilde{H}_*(K_i) \subset \widetilde{H}_*(K_{i+1})$, and since the map $f^{i,i+1}$ is an inclusion, this means that the equality

$$\Delta^{i+1}_n f^{i,i+1} = \left(f^{i,i+1}\right)^{\otimes n} \Delta^i_n$$

holds and hence so does

$$f^{i,i+1}\left(\operatorname{Ker} \Delta^i_n\right) \subseteq \operatorname{Ker} \Delta^{i+1}_n.$$

□

3.5 Wake up every day (as long as you are alive)

If we are not in the restrictive hypotheses of Theorem 3.4.1, there might not exist A_∞-coalgebras on each $H_*(K_i)$ such that

$$f^{i,i+1}\left(\operatorname{Ker} \Delta^i_n\right) \subseteq \operatorname{Ker} \Delta^{i+1}_n.$$

Furthermore, even under the hypotheses of Theorem 3.4.1, if we do not follow the construction displayed in its proof, but allow instead more freedom in the choice of the different A_∞-structures, then they may fail to satisfy the assumptions

$$f^{i,i+1}\left(\operatorname{Ker}\Delta_n^i\right) \subseteq \operatorname{Ker}\Delta_n^{i+1}.$$

Theorem 3.5.1 proves this claim and exhibits an intermittent behaviour that can take place when we have a lot of freedom to choose the different A_∞-coalgebras. Namely, if $f^{i,i+1}(\operatorname{Ker}\Delta_n^i) \subseteq \operatorname{Ker}\Delta_n^{i+1}$ does not hold, then, by definition, a class that is Δ_n-awake at K_i can Δ_n-fall asleep at K_{i+1} without the need of dying at K_{i+1}. Moreover, Theorem 3.5.1 shows that classes can Δ_n-fall asleep and Δ_n-wake up several times along the filtration. This is, together with the fact that a class must be alive in order to be Δ_n-awake, the reason why we chose the *awake-asleep* terminology.

To prove Theorem 3.5.1, we will give an example of a class with such a behaviour using the rationals $\mathbb{Q}$ as field of coefficients. We refer the reader to the Appendix for the basics of rational homotopy theory.

Let $\mathcal{K}$ be the filtration of finite complexes

$$K_0 \overset{i_0}{\hookrightarrow} K_1 \overset{i_1}{\hookrightarrow} K_2 \overset{i_2}{\hookrightarrow} K_3$$

defined as follows:

- $K_0 = (\mathbb{S}^2_1 \vee \mathbb{S}^2_2 \vee \mathbb{S}^2_3 \vee \mathbb{S}^4) \cup_{g_1} e^6_1 \cup_{g_2} e^6_2$, in which $\mathbb{S}^2_1, \mathbb{S}^2_2, \mathbb{S}^2_3$ simply denote three different copies of $\mathbb{S}^2$ and the (homotopy classes of the) attaching maps for the 6-cells e^6_1, e^6_2 are the following Whitehead products:

$$g_1 = [id_{\mathbb{S}^4}, id_{\mathbb{S}^2_1}] + [id_{\mathbb{S}^2_1}, [id_{\mathbb{S}^2_1}, [id_{\mathbb{S}^2_1}, id_{\mathbb{S}^2_2}]]], \quad g_2 = [id_{\mathbb{S}^4}, id_{\mathbb{S}^2_2}].$$

- $K_1 = K_0 \cup_{g_3} e^4$, in which

$$g_3 = [id_{\mathbb{S}^2_1}, id_{\mathbb{S}^2_2}].$$

- $K_2 = K_1 \cup_{g_4} e^6$, in which

$$g_4 = [id_{\mathbb{S}^4}, id_{\mathbb{S}^2_1}] - [id_{\mathbb{S}^2_2}, [id_{\mathbb{S}^2_2}, [id_{\mathbb{S}^2_2}, id_{\mathbb{S}^2_3}]]].$$

- $K_3 = K_2 \cup_{g_5} e^4$, in which

$$g_5 = [id_{\mathbb{S}^2_2}, id_{\mathbb{S}^2_3}].$$

- The maps i_1, i_2, i_3 are the inclusions.

Then, we prove,

Theorem 3.5.1. *In*

$$H_*(K_0;\mathbb{Q}) \xrightarrow{f_0} H_*(K_1;\mathbb{Q}) \xrightarrow{f_1} H_*(K_2;\mathbb{Q}) \xrightarrow{f_2} H_*(K_3;\mathbb{Q}),$$

all the morphisms are injective and we can give a good *A_∞-coalgebra structure to each term $H_*(K_i;\mathbb{Q})$ so that the non zero homology class $\gamma_6 \in H_6(K_0)$, created by the cell e^6_1, is Δ_4-asleep at K_0, it Δ_4-wakes up at K_1, it is Δ_4-asleep again at K_2 and finally, it Δ_4-wakes up at K_3.*

Proof. Via Theorem 4.0.3 of the Appendix, the following is a Quillen minimal model of K_0 (from now on, subscripts will always denote degree):

$$(\mathbb{L}(V_0), \partial) = (\mathbb{L}(x_1, x_1', x_1'', y_3, z_5, z_5'), \partial)$$

where

$$\begin{aligned} \partial x_1 &= \partial x_1' = \partial x_1'' = \partial y_3 = 0, \\ \partial z_5 &= [y_3, x_1] + [x_1, [x_1, [x_1, x_1']]], \\ \partial z_5' &= [y_3, x_1']. \end{aligned}$$

Observe, either by direct computation or in light of the isomorphism (i) of the Appendix, that

$$\widetilde{H}_*(K_0;\mathbb{Q}) = \langle \alpha_2, \alpha_2', \alpha_2'', \beta_4, \gamma_6, \gamma_6' \rangle = sV_0.$$

Next, a short computation shows that, in the universal enveloping algebra $(T(V_0), d) = U(\mathbb{L}(V_0), \partial)$ (see Appendix for its definition),

$$d_4 z_5 = x_1 \otimes x_1 \otimes x_1' \otimes x_1 + x_1 \otimes x_1 \otimes x_1 \otimes x_1' - x_1' \otimes x_1 \otimes x_1 \otimes x_1 - x_1 \otimes x_1' \otimes x_1 \otimes x_1.$$

In other words, in the A_∞-coalgebra structure on $H_*(K_0;\mathbb{Q})$ given by $(\mathbb{L}(V_0),\partial)$,

$$\Delta_4\gamma_6 \neq 0,$$

and thus, γ_6 is Δ_4-asleep.

Next, again by Theorem 4.0.3, a Quillen model of the inclusion

$$i_0\colon K_0 \hookrightarrow K_1$$

is

$$(\mathbb{L}(x_1,x_1',x_1'',y_3,z_5,z_5'),\partial) \hookrightarrow (\mathbb{L}(x_1,x_1',x_1'',y_3,y_3',z_5,z_5'),\partial),$$

in which

$$\partial y_3' = [x_1,x_1'].$$

To describe in simpler terms the morphism induced in rational homology by the inclusion i_0, consider the isomorphism of (non differential!) free Lie algebras

$$\psi\colon \mathbb{L}(x_1,x_1',x_1'',y_3,y_3',u_5,z_5') \xrightarrow{\cong} \mathbb{L}(x_1,x_1',x_1'',y_3,y_3',z_5,z_5')$$

which sends u_5 to $z_5 - [x_1,[x_1,y_3']]$ and any other generator to itself. Then, set in the first free Lie algebra the differential

$$\partial = \psi^{-1}\partial\psi$$

so that ψ becomes an isomorphism of DGL's. A short computation shows that the only generator on this isomorphic DGL in which the differential has changed is:

$$\partial u_5 = [y_3,x_1].$$

To simplify the notation, set

$$(\mathbb{L}(V_1),\partial) = (\mathbb{L}(x_1,x_1',x_1'',y_3,y_3',u_5,z_5'),\partial),$$

and observe that the DGL morphism

$$(\mathbb{L}(V_0),\partial) \to (\mathbb{L}(V_1),\partial)$$

which sends z_5 to $u_5 + [x_1, [x_1, y_3']]$, and any other generator to itself, is again a Quillen minimal model of $i_0\colon K_0 \hookrightarrow K_1$. In particular, the map induced by this morphism on the suspension of the indecomposables $sV_0 \to sV_1$ is precisely the morphism $f_0 = H_*(i_0)$.

Thus, on the one hand,

$$f_0\colon \widetilde{H}_*(K_0;\mathbb{Q}) \longrightarrow \widetilde{H}_*(K_1;\mathbb{Q})$$

is naturally identified to the inclusion

$$sV_0 = \langle \alpha_2, \alpha_2', \alpha_2'', \beta_4, \gamma_6, \gamma_6' \rangle \hookrightarrow \langle \alpha_2, \alpha_2', \alpha_2'', \beta_4, \beta_4', \eta_6, \gamma_6' \rangle = sV_1,$$

for which $f_0(\gamma_6) = \eta_6$.

On the other hand, one sees that in the universal enveloping algebra $(T(V_1), d) = U(\mathbb{L}(V_1), \partial)$,

$$du_5 = d_2 u_5 = y_3 \otimes x_1 - x_1 \otimes y_3.$$

Equivalently, in the A_∞-coalgebra structure on $H_*(K_1;\mathbb{Q})$ given by $(\mathbb{L}(V_1), \partial)$,

$$\Delta_4 \eta_6 = \Delta_4 f_0(\gamma_6) = 0.$$

In other words, the class γ_6 Δ_4-wakes up at K_1.

In the next stage, Theorem 4.0.3 produces a Quillen model of the inclusion $i_1\colon K_1 \hookrightarrow K_2$,

$$(\mathbb{L}(x_1, x_1', x_1'', y_3, y_3', u_5, z_5'), \partial) \hookrightarrow (\mathbb{L}(x_1, x_1', x'', y_3, y_3', u_5, z_5', v_5), \partial),$$

in which

$$\partial v_5 = [y_3, x_1] - [x_1', [x_1', [x_1', x_1'']]].$$

Once again, to describe in simpler terms the morphism induced in rational homology by the inclusion i_1, consider the isomorphism of (non differential!) free Lie algebras

$$\mathbb{L}(x_1, x_1', x'', y_3, y_3', w_5, z_5', v_5) \xrightarrow{\cong} \mathbb{L}(x_1, x_1', x'', y_3, y_3', u_5, z_5', v_5)$$

which sends w_5 to $u_5 - v_5$ and any other generator to itself. Then endow the left hand side free Lie algebra with the differential so that the above becomes an isomorphism of DGL's. A short computation shows that the only generator on this isomorphic DGL in which the differential has changed is:

$$\partial w_5 = [x_1', [x_1', [x_1', x_1'']]].$$

Set

$$(\mathbb{L}(V_2), \partial) = (\mathbb{L}(x_1, x_1', x_1'', y_3, y_3', w_5, z_5', v_5), \partial),$$

and observe that the DGL morphism

$$(\mathbb{L}(V_1), \partial) \to (\mathbb{L}(V_2), \partial)$$

which sends u_5 to $w_5 + v_5$, and any other generator to itself, is a Quillen minimal model of $i_1 \colon K_1 \hookrightarrow K_2$. In particular the map induced by this morphism on the suspension of the indecomposables $sV_1 \to sV_2$ is precisely $f_1 = H_*(i_1)$.

Thus, on the one hand,

$$f_1 \colon \widetilde{H}_*(K_1; \mathbb{Q}) \longrightarrow \widetilde{H}_*(K_2; \mathbb{Q})$$

is naturally identified to the inclusion

$$sV_1 = \langle \alpha_2, \alpha_2', \alpha_2'', \beta_4, \beta_4', \eta_6, \gamma_6' \rangle \hookrightarrow \langle \alpha_2, \alpha_2', \alpha_2'', \beta_4, \beta_4', \rho_6, \gamma_6', \gamma_6'' \rangle = sV_2,$$

for which $f_1(\eta_6) = \rho_6$.

On the other hand, one sees that in the universal enveloping algebra $(T(V_2), d) = U(\mathbb{L}(V_2), \partial)$,

$$dw_5 = x_1' \otimes x_1' \otimes x_1'' \otimes x_1' + x_1' \otimes x_1' \otimes x_1' \otimes x_1'' - x_1'' \otimes x_1' \otimes x_1' \otimes x_1' - x_1' \otimes x_1'' \otimes x_1' \otimes x_1'.$$

Equivalently, in the A_∞-coalgebra structure on $H_*(K_2; \mathbb{Q})$ given by $(\mathbb{L}(V_2), \partial)$,

$$\Delta_4 \rho_6 = \Delta_4 f^{0,1}(\gamma_6) \neq 0.$$

In other words, the class γ_6 Δ_4-falls asleep at K_2.

Finally, we repeat this process once again, exhibiting via Theorem 4.0.3, a Quillen model of the inclusion $i_2\colon K_2 \hookrightarrow K_3$,

$$(\mathbb{L}(x_1, x_1', x'', y_3, y_3', w_5, z_5', v_5), \partial) \hookrightarrow (\mathbb{L}(x_1, x_1', x'', y_3, y_3', y_3'', w_5, z_5', v_5),$$

in which

$$\partial y_3'' = [x_1', x_1''].$$

As in previous steps, we get a DGL isomorphism

$$(\mathbb{L}(x_1, x_1', x'', y_3, y_3', y_3'', s_5, z_5', v_5), \partial) \xrightarrow{\cong} (\mathbb{L}(x_1, x_1', x'', y_3, y_3', y_3'', w_5, z_5', v_5), \partial),$$

which sends s_5 to $w_5 - [x_1', [x_1', y_3'']]$ and is such that $\partial s_5 = 0$. Set

$$(\mathbb{L}(V_3), \partial) = (\mathbb{L}(x_1, x_1', x_1'', y_3, y_3', y_3'', s_5, z_5', v_5), \partial),$$

and observe that the DGL morphism

$$(\mathbb{L}(V_2), \partial) \to (\mathbb{L}(V_3), \partial)$$

which sends w_5 to $s_5 + [x_1', [x_1', y_3'']]$, and any other generator to itself, is a Quillen minimal model of $i_2\colon K_2 \hookrightarrow K_3$. In particular the map induced by this morphism on the suspension of the indecomposables $sV_2 \to sV_3$ is precisely the morphism $f_2 = H_*(i_2)$.

Thus, on the one hand,

$$f_2\colon \widetilde{H}_*(K_2; \mathbb{Q}) \longrightarrow \widetilde{H}_*(K_3; \mathbb{Q})$$

is naturally identified to the inclusion

$$sV_2 = \langle \alpha_2, \alpha_2', \alpha_2'', \beta_4, \beta_4', \rho_6, \gamma_6', \gamma_6'' \rangle \hookrightarrow \langle \alpha_2, \alpha_2', \alpha_2'', \beta_4, \beta_4', \beta_4'', \varphi_6, \gamma_6', \gamma_6'' \rangle = sV_3,$$

for which $f_2(\rho_6) = \varphi_6$.

On the other hand, in the universal enveloping algebra $(T(V_3), d) = U(\mathbb{L}(V_3), \partial)$,

$$ds_5 = 0.$$

Equivalently, in the A_∞-coalgebra structure on $H_*(K_3; \mathbb{Q})$ given by $(\mathbb{L}(V_3), \partial)$,

$$\Delta_4 \varphi_6 = \Delta_4 f^{0,2}(\gamma_6) = 0,$$

and the class γ_6 Δ_4-wakes up again at K_3.

□

3.6 A more general decomposition result

Despite condition $f_p^{i,i+1}(\operatorname{Ker}\Delta_n^i) \subseteq \operatorname{Ker}\Delta_n^{i+1}$ not holding in general, we can still prove an extension of Theorem 3.3.5 to a more general case.

For every $0 \leq i \leq N$, fix an arbitrary *good* A_∞-coalgebra structure $\{\Delta_n^i\}_n$ on $H_*(K_i)$. Here is the statement.

Theorem 3.6.1. *For every $p \geq 0$ and $n \geq 1$, there exists a basis $\mathcal{B}$ of $\bigoplus_{i=0}^N H_p(K_i)$ such that, for all $0 \leq i \leq j \leq N$, the dimension of the Δ_n-persistent group $(\Delta_n)_p^{i,j}(\mathcal{K})$ can be computed by counting the number of p-homology classes of the basis that do not Δ_n-fall asleep in the corresponding range:*

$$\dim(\Delta_n)_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \in \mathit{Ker}\,\Delta_n^k - \{0\}\ \textit{for all}\ k = i, \dots, j\}.$$

Proof. To avoid extra notation, as n and p are fixed integers which do not depend on the methods of the proof, we will omit them and write Δ^i instead of the restriction of Δ_n^i to $H_p(K_i)$, $(\Delta)^{i,j}(\mathcal{K})$ instead of $(\Delta_n)_p^{i,j}(\mathcal{K})$, and $f^{i,j}$ instead of $f_p^{i,j}$.

Since $f^{i,i+1}(\operatorname{Ker}\Delta^i) \subseteq \operatorname{Ker}\Delta^{i+1}$ does not necessarily hold, we cannot consider the persistence module (3.3.5). Instead, let us consider the zigzag module $\mathbb{V}$:

$$\begin{array}{ccccccccccc}
V_0 & & & & V_1 & & & \cdots & & & V_N, \\
& \nwarrow^{\iota} & & \nearrow^{f^{0,1}} & & \nwarrow^{\iota} & & \nearrow^{f^{1,2}} \quad\quad \nwarrow^{\iota} & & \nearrow^{f^{N-1,N}} & \\
& & W_0 & & & & W_1 & \quad\quad\quad W_{N-1} & & &
\end{array}$$

in which ι denotes inclusion and

$$\begin{aligned}
V_i &= \operatorname{Ker}\Delta^i, \quad i = 0, \dots, N \\
W_i &= \operatorname{Ker}\Delta^i \cap \left(f^{i,i+1}\right)^{-1} \operatorname{Ker}\Delta^{i+1}, \quad i = 0, \dots, N-1.
\end{aligned}$$

First we must notice a subtlety. Let

$$\mathbb{U}\colon U_0 \longleftrightarrow U_1 \longleftrightarrow \dots \longleftrightarrow U_r$$

be a zigzag module and denote forward maps by $U_k \xrightarrow{f_k} U_{k+1}$ and backward map by $U_k \xleftarrow{g_k} U_{k+1}$. Theorem 1.3.1 tells us that $\mathbb{U}$ is isomorphic as zigzag module to a finite direct sum

$$\mathbb{U} \cong \bigoplus_m \mathbb{I}^m, \tag{3.6.1}$$

where each

$$\mathbb{I}^m \colon I_0^m \longleftrightarrow I_1^m \longleftrightarrow \dots \longleftrightarrow I_r^m$$

is of the form $\mathbb{I}^m = \mathbb{I}[a_m, b_m]$ for some $0 \leq a_m \leq b_m \leq r$, *i.e.,*

$$I_i^m = \begin{cases} \Bbbk, & a_m \leq i \leq b_m, \\ 0, & \text{otherwise}, \end{cases}$$

where the direction of the arrows coincides with that of $\mathbb{U}$, all maps

$$\Bbbk \longleftrightarrow \Bbbk$$

are identities and the rest are zero maps. Plus, the decomposition (3.6.1) is unique up to reordering.

Now, if all maps point forwards (*i.e.,* $\mathbb{U}$ is a persistence module), it is obvious that the existence of an element $x \in U_i - \{0\}$ such that

$$f_{j-1} \circ \dots \circ f_i(x) \neq 0$$

implies the existence of a zigzag module $\mathbb{I}[a, b]$ with $[i, j] \subseteq [a_m, b_m]$ in the decomposition (3.6.1).

However, as explained in [20], this intuition breaks down in the general case, for if $\mathbb{U}$ is an arbitrary zigzag module, then the existence of elements $x_k \in U_k - \{0\}$ for $i \leq k \leq j$ such that, for all $i \leq k < j$, either $x_{k+1} = f_k(x_k)$ or $x_k = g_k(x_{k+1})$, whichever is applicable, does not necessarily imply the existence of a zigzag module $\mathbb{I}[a, b]$ with $[i, j] \subseteq [a, b]$ in the decomposition (3.6.1).

Example 3.6.2. (Example 2.10 in [20]) Let

$$\mathbb{U}\colon \qquad U_0 \xleftarrow{g_0} U_1 \xrightarrow{f_1} U_2$$

be the zigzag module

$$\mathbb{U}\colon \qquad \Bbbk \xleftarrow{g_0} \Bbbk^2 \xrightarrow{f_1} \Bbbk$$

$$x \longleftarrow\!\shortmid (x,y) \longmapsto y.$$

If $\Bbbk \neq 0$, there must exist a non-zero element $x \in \Bbbk$, and this will satisfy

$$g_0(x,x) = x = f_1(x,x) \neq 0,$$

but $\mathbb{U}$ has no summand isomorphic to $\mathbb{I}[0,2]$ (although it has a submodule isomorphic to $\mathbb{I}[0,2]$), for the zigzag module $\mathbb{U}$ is isomorphic to the direct sum of the submodules

$$\Bbbk \longleftarrow \Bbbk \oplus 0 \longrightarrow 0$$

$$x \longleftarrow\!\shortmid (x,0) \longmapsto 0$$

and

$$0 \longleftarrow 0 \oplus \Bbbk \longrightarrow \Bbbk$$

$$0 \longleftarrow\!\shortmid (0,y) \longmapsto y$$

and therefore

$$\mathbb{U} \cong \mathbb{I}[0,1] \oplus \mathbb{I}[1,2].$$

Hence, in order to prove Theorem 3.3.5 it is key to show that if there are elements $x_k \in U_k - \{0\}$ for $i \leq k \leq j$ such that for all $i \leq k < j$, either $x_{k+1} = f_k(x_k)$ or $x_k = g_k(x_{k+1})$ (whichever is applicable), then there must be a zigzag module $\mathbb{I}[a,b]$ with $[i,j] \subseteq [a,b]$ in the decomposition (3.6.1) in the case of zigzag modules $\mathbb{U}$ that behave as our $\mathbb{V}$.

We begin by applying Zigzag's Interval Decomposition Theorem (Theorem 1.3.1) to obtain that $\mathbb{V}$ is isomorphic as zigzag module to a direct sum

$$\mathbb{I} = \bigoplus_{1 \leq m \leq M} \mathbb{I}^m,$$

for some $M \in \mathbb{N}$, where each $\mathbb{I}^m$ is an interval $\mathbb{I}[a_m, b_m]$ for some $0 \leq a_m \leq b_m \leq 2N-1$, of the form

$$\begin{array}{ccccccccccc} I_0^m & & & & I_1^m & & & & \cdots & & & & I_N^m. \\ & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & & \nwarrow & & \nearrow & \\ & & J_0^m & & & & J_1^m & & & & J_{N-1}^m & & \end{array}$$

Notice that, from the equality (3.3.3) and the isomorphism $\mathbb{V} \cong \mathbb{I}$, we can choose a basis of each $\operatorname{Ker} \Delta^i = V_i$, put them together to form a basis of $\bigoplus_i \operatorname{Ker} \Delta^i$ and enlarge this one to get a basis $\mathcal{B}$ of $\oplus_{i=0}^N H_p(K_i)$, so that

$$\#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \in \operatorname{Ker} \Delta_n^k - \{0\} \text{ for all } k = i, \ldots, j\} = N_{ij},$$

where N_{ij} is the number of intervals $\mathbb{I}^m$ of the form $\mathbb{I}[a, b]$ with $[2i, 2j] \subseteq [a, b]$. Hence, to prove Theorem 3.6.1 it suffices to show that

$$\dim(\Delta)^{i,j}(\mathcal{K}) = N_{ij}.$$

To this end, we introduce some notation. Choose an explicit zigzag isomorphism,

$$\Phi\colon \mathbb{V} \xrightarrow{\cong} \mathbb{I}$$

which consists of $\Bbbk$-linear isomorphisms,

$$\begin{aligned} \phi_k\colon V_k &\xrightarrow{\cong} \oplus_m I_k^m, \quad 0 \leq k \leq N, \\ \varphi_k\colon W_k &\xrightarrow{\cong} \oplus_m J_k^m, \quad 0 \leq k \leq N-1, \end{aligned}$$

making commutative all the corresponding squares:

$$\begin{array}{ccccc} V_k & \xleftarrow{\iota} & W_k & \xrightarrow{f^{k,k+1}} & V_{k+1} \\ \phi_k \downarrow \cong & & \cong \downarrow \varphi_k & & \phi_{k+1} \downarrow \cong \\ \oplus_m I_k^m & \longleftarrow & \oplus_m J_k^m & \longrightarrow & \oplus_m I_{k+1}^m. \end{array} \tag{3.6.2}$$

Denote by $\phi_k^{m'}$ the composition

$$\phi_k^{m'}\colon V_k \xrightarrow{\phi_k} \bigoplus_m \mathbb{I}_k^m \xrightarrow{\pi_{m'}} \mathbb{I}_k^{m'},$$

for all $0 \leq k \leq N$ and $1 \leq m' \leq M$, where $\pi_{m'}$ denotes the projection onto the m'-th summand. Define

$$\varphi_k^{m'} : W_k \longrightarrow J_k^{m'}$$

analogously for all $0 \leq k < N$.

We first show that $\dim(\Delta)^{i,j}(\mathcal{K}) \leq N_{ij}$. Let $0 \leq i \leq j \leq N$. A class $\alpha \in H_p(K_j)$ is in $(\Delta)^{i,j}(\mathcal{K}) = \operatorname{Im} f^{i,j}|_{\cap_{k=i}^{j}(f^{i,k})^{-1}\operatorname{Ker}\Delta^k}$ if and only if there exist elements

$$\alpha_\ell \in W_\ell \subset V_\ell, \quad \ell = i, \dots, j-1,$$

such that,

$$f^{\ell,\ell+1}(\alpha_\ell) = \alpha_{\ell+1}, \quad \ell = i, \dots, j-2, \quad \text{and} \quad f^{j-1,j}(\alpha_{j-1}) = \alpha.$$

Since ϕ_j is an isomorphism, if such $\alpha \in (\Delta)^{i,j}(\mathcal{K}) \subseteq V_j$ is non-zero, there must be some m such that $\phi_j^m(\alpha) \neq 0$, and therefore $I_j^m \cong \Bbbk$. As $\alpha = f^{j-1,j}(\alpha_{j-1})$, at the sight of diagram 3.6.2, $\varphi_{j-1}^m(\alpha_{j-1}) \neq 0$ and thus, $J_{j-1}^m \cong \Bbbk$ as well.

On the other hand, as $V_k \overset{\iota}{\hookleftarrow} W_k$ are inclusions and ϕ_k, φ_k are isomorphisms, also at the sight of 3.6.2 we have that $\phi_{j-1}^m(\alpha_{j-1}) \neq 0$, that is, $I_{j-1}^m \cong \Bbbk$.

Apply this process repeatedly to conclude that $\mathbb{I}^m$ is an interval of the form $\mathbb{I}[a,b]$ with $[2i,2j] \subseteq [a,b]$.

Remark 3.6.3. Moreover, this reasoning also shows that, in the decomposition $\mathbb{V} \cong \bigoplus_m \mathbb{I}^m$, for every m there are integers $0 \leq a \leq b \leq N$ such that $\mathbb{I}^m$ is an interval either of the form $\mathbb{I}[2a, 2b-1]$ or of the form $\mathbb{I}[2a, 2b]$.

Back with the proof, since

$$\phi_j : V_j \xrightarrow{\cong} \oplus_m I_j^m$$

is an isomorphism, we can chose a basis $\{\alpha_1, \dots, \alpha_l\}$ of $(\Delta)^{i,j}(\mathcal{K})$ (notice that $l = \dim(\Delta)^{i,j}(\mathcal{K}) \leq M$) such that

$$\phi_j(\alpha_k) = \phi_j^{m(k)}(\alpha_k) \in I_j^{m(k)}$$

for all $1 \leq k \leq l$ and so that the l numbers $m(k)$ are all different. Applying the process just described to each of the l linearly independent classes in $\{\alpha_1, \ldots, \alpha_l\}$ will generate l different intervals $\mathbb{I}^m$ of the form $\mathbb{I}[a, b]$ with $[2i, 2j] \subseteq [a, b]$. This shows that

$$\dim(\Delta)^{i,j}(\mathcal{K}) \leq N_{ij}.$$

A similar argument shows that $N_{ij} \leq \dim(\Delta)^{i,j}(\mathcal{K})$, and concludes the proof of Theorem 3.6.1. □

Definition 3.6.4. For every $p \geq 0$ and $n \geq 1$, we define the **p^{th} Δ_n-barcode of $\mathcal{K}$** as the multiset of intervals of the form $[a, b] \subseteq \mathbb{R}$, with $0 \leq a \leq b \leq N$, consisting of one $[a, b]$ for each interval of either the form $\mathbb{I}^m = \mathbb{I}[2a, 2b-1]$ or the form $\mathbb{I}^m = \mathbb{I}[2a, 2b]$ in the decomposition $\mathbb{V} \cong \bigoplus_m \mathbb{I}^m$ of the zigzag module $\mathbb{V}$ described in the proof.

With this definition, we can state a pleasant result in terms of similarity to the classical Theorem 1.2.4.

Corollary 3.6.5. *For every $p \geq 0$ and $n \geq 1$, the p^{th} Δ_n-barcode of $\mathcal{K}$ is well-defined and, for all $0 \leq i \leq j \leq N$, the dimension of the persistent group $(\Delta_n)_p^{i,j}(\mathcal{K})$ equals the number of intervals in the p^{th} Δ_n-barcode of $\mathcal{K}$ (counted with multiplicities) which contain the interval $[i, j]$.*

In particular, $\dim Ker \Delta^i_{n|H_p(K_i)}$ equals the number of intervals in the p^{th} Δ_n-barcode of $\mathcal{K}$ (counted with multiplicities) which contain i.

Proof. The Zigzag's Interval Decomposition Theorem (Theorem 1.3.1) says there is a the decomposition $\mathbb{V} \cong \bigoplus_m \mathbb{I}^m$ of the zigzag module $\mathbb{V}$ described in the proof of Theorem 3.6.1 and hence confirms that the p^{th} $\boldsymbol{\Delta_n}$-barcode of $\mathcal{K}$ is well-defined.

Remark 3.6.3 establishes that there is a bijection between the p^{th} Δ_n-barcode of $\mathcal{K}$ and the multiset of intervals $\mathbb{I}^m$ in the decomposition $\mathbb{V} \cong \bigoplus_m \mathbb{I}^m$.

Finally, the equality

$$\dim(\Delta)^{i,j}(\mathcal{K}) = N_{ij}$$

established in the proof of Theorem 3.6.1 corroborates the rest. □

3.7 Algorithmic approach

At this point, it should be obvious how to compute A_∞-persistence. Here is an abstract algorithm that takes as input integers $n \geq 1$, $p \geq 0$ and a filtration of CW complexes with finitely many cells

$$K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_N,$$

and produces the p^{th} Δ_n-barcode of the filtration given by some A_∞-coalgebra structures computed during the algorithm. This algorithm will serve as a skeleton for the more concrete algorithms developed later.

ALGORITHM 3.7.1.

1. `Compute a` ***`good`*** A_∞`-coalgebra structure` $\{\Delta_k^i\}_{k\geq 1}$ `on` $H_*(K_i)$`, for all` $0 \leq i \leq N$`; (`*`e.g.,`*` through Algorithm 2.3.5, when applicable, as explained below);`

2. `Using the integers` $n \geq 1$ `and` $p \geq 0$ `from the input, apply the construction in the proof of Theorem 3.6.1 to obtain a zigzag module` $\mathbb{V}$`;`

3. `Apply the zigzag decomposition algorithm (Algorithm 1.3.3) to` $\mathbb{V}$`;`

4. `Apply the construction in Definition 3.6.4;`

If we work over the field of two elements $\mathbb{F}_2$, we can use Algorithm 2.3.5 to carry out step number 1, which is computing a *good* A_∞-coalgebra $\{\Delta_k^i\}_k$ for each $H_*(K_i)$. Since Algorithm 2.3.5 is already designed to work with filtrations, we can apply it simply once and get all the homology gradient vector fields

$$\phi_i \colon C_*(K_i) \longrightarrow C_{*+1}(K_i)$$

we need to compute all those A_∞-structures.

Specifically, let $n \geq 1$ and $p \geq 0$ be integers and

$$K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_N$$

a filtration of CW complexes with finitely many cells. Order all cells in each K_i, denote the ordered cells of K_0 as

$$e_0^0, e_0^1, \ldots, e_0^{k_0}$$

and similarly, for each $0 < i \leq N$, say that K_i consists of the ordered cells in K_{i-1} plus the ordered cells

$$e_i^0, e_i^1, \ldots, e_i^{k_i}.$$

Then,

$$\left\langle e_0^0, e_0^1, \ldots, e_0^{k_0}, e_1^0, e_1^1, \ldots, e_1^{k_1}, \ldots, e_N^0, e_N^1, \ldots, e_N^{k_N} \right\rangle$$

forms a filter (Definition 2.3.4) and hence it can be used, along with the boundary maps induced by that of K_N, as input for Algorithm 2.3.5. Finally, for all $0 \leq i \leq N$, since

$$K_i = \left\langle e_0^0, e_0^1, \ldots, e_0^{k_0}, e_1^0, e_1^1, \ldots, e_1^{k_1}, \ldots, e_i^0, e_i^1, \ldots, e_i^{k_i} \right\rangle,$$

after running the step of Algorithm 2.3.5 in which the cell $e_i^{k_i}$ is added, we get a homology gvf on $C_*(K_i)$ and thus, using the formulae in Proposition 2.3.7, a *good* A_∞-coalgebra on $H_*(K_i)$.

Chapter 4

Appendix: basics of rational homotopy theory

In this appendix, for completeness, we recall the basic facts and classical results from rational homotopy theory used throughout the paper. For a compendium of this theory, we direct the reader to the standard reference [41].

A *differential graded Lie algebra* (DGL henceforth) is a differential graded vector space (L, ∂) in which:

- $L = \oplus_{p \in \mathbb{Z}} L_p$ in endowed with a linear operation, called Lie bracket,

  $$[\ ,\ ]\colon L_p \otimes L_q \longrightarrow L_{p+q}, \qquad p, q \in \mathbb{Z},$$

  satisfying *antisymmetry*,

  $$[x, y] = (-1)^{|x||y|+1}[y, x],$$

  and *Jacobi identity*,

  $$\big[x, [y, z]\big] = \big[[x, y], z\big] + (-1)^{|x||y|}\big[y, [x, z]\big],$$

  for any homogeneous elements $x, y, z \in L$. In other words, L is a graded Lie algebra.

- The differential ∂, of degree -1, satisfies the *Leibniz rule*,

$$\partial[x,y] = [\partial x, y] + (-1)^{|x|}[x, \partial y],$$

for any pair of homogeneous elements $x, y \in L$.

The tensor algebra $T(V) = \oplus_{n\geq 0} T^n(V)$ generated by the graded vector space V is endowed with a graded Lie algebra structure with the brackets given by commutators:

$$[a,b] = a \otimes b - (-1)^{|a||b|} b \otimes a$$

for any homogeneous elements $a, b \in T(V)$. Then, the *free Lie algebra* $\mathbb{L}(V)$ is the Lie subalgebra of $T(V)$ generated by V.

Observe that $\mathbb{L}(V)$ is filtered as follows,

$$\mathbb{L}(V) = \oplus_{n\geq 1} \mathbb{L}^n(V),$$

in which $\mathbb{L}^n(V)$ is the vector space spanned by Lie brackets of length n, that is $\mathbb{L}^n(V) = \mathbb{L}(V) \cap T^n(V)$. In particular, to set a differential in $\mathbb{L}(V)$, it is enough to define linear maps,

$$\partial_n \colon V \longrightarrow \mathbb{L}^n(V), \quad n \geq 1$$

in such a way that $\partial = \sum_{n\geq 1} \partial_n$ squares to zero and satisfies the Leibniz rule. Note that ∂_1, the so called *linear part of* ∂, is therefore a differential in V.

Remark 4.0.2. It is important to remark that, if $T(V)$ is endowed with a derivation d for which $(T(V), d)$ is a differential graded (non commutative) algebra, $\mathbb{L}(V)$ does not inherit a differential as d may not respect commutators. However, every time we have a DGL of the form $(\mathbb{L}(V), \partial)$, the differential ∂ can be extended as a derivation of graded algebras to $T(V)$ to make it a differential graded algebra. In other words, consider the functor

$$U \colon \mathbf{DGL} \longrightarrow \mathbf{DGA}$$

which associates to every DGL (L, ∂) its *universal enveloping algebra* UL which is the graded algebra

$$T(L)/\langle [x,y] - (x \otimes y - (-1)^{|a||b|} y \otimes x)\rangle, \quad x, y \in L,$$

with the differential induced by ∂. Then, whenever $(L, \partial) = (\mathbb{L}(V), \partial)$, one has,

$$U(\mathbb{L}(V), \partial) = (T(V), \partial).$$

In [87], D. Quillen associates to every 1-connected topological space X of the homotopy type of a CW complex, a particular differential graded Lie algebra $(\mathbb{L}(V), \partial)$ which is *reduced*, i.e., $V = \oplus_{p\geq 1} V_p$ for which:

(i) $H_*(V, \partial_1) \cong s^{-1}\widetilde{H}_*(X; \mathbb{Q}) = \widetilde{H}_{*-1}(X; \mathbb{Q})$.

(ii) $H_*(\mathbb{L}(V), \partial) \cong \pi_*(\Omega X) \otimes \mathbb{Q} \cong \pi_{*+1}(X) \otimes \mathbb{Q}$.

This is a *Quillen model of* X and it is called *minimal* whenever ∂ is decomposable, i.e., $\partial_1 = 0$. In this case, (i) becomes,

$$V \cong s^{-1}\widetilde{H}_*(X; \mathbb{Q}).$$

The Quillen minimal model is unique up to isomorphism. Moreover, its construction is functorial and it defines an equivalence between the homotopy category of 1-connected spaces of the homotopy type of rational CW complexes and that of reduced differential graded Lie algebras over $\mathbb{Q}$. Thus, two simply connected complexes have isomorphic Quillen minimal models if and only if they have the same rational homotopy type.

In particular, if $\varphi\colon (\mathbb{L}(U), \partial_U) \to (\mathbb{L}(V), \partial_V)$ is the Quillen minimal model of the map $f\colon X \to Y$, the morphism $H_*(\varphi)\colon H_*(\mathbb{L}(U), \partial_U) \to H_*(\mathbb{L}(V), \partial_V)$ is naturally identified to $\pi_*(\Omega f)\colon \pi_*(\Omega X) \to \pi_*(\Omega Y)$. In the same way, the morphism induced by φ at the "indecomposables" $\varphi_0\colon U \to V$ is naturally identified to $s^{-1}H_*(f)\colon s^{-1}\widetilde{H}_*(X; \mathbb{Q}) \to s^{-1}\widetilde{H}_*(Y; \mathbb{Q})$, the desuspension of the morphism induced by f in rational homology. To explicitly obtain φ_0, write, for each $u \in U$, $\varphi(u) = v + \Gamma$, with $v \in V$ and $\Gamma \in \mathbb{L}^{\geq 2}(V)$. Then $\varphi_0(u) = v$.

In the A_∞-language, Remarks 4.0.2 and 2.2.4 together with the isomorphism (i) above, assert that $\widetilde{H}_*(X;\mathbb{Q})$ can be functorially endowed with a structure of *good* A_∞-coalgebra induced by any Quillen minimal model $(\mathbb{L}(V),\partial)$ of X.

Given a 1-connected CW complex X, a (non necessarily minimal) Quillen model of X can be described in terms of a CW decomposition of X via the following,

Theorem 4.0.3. *[97, III.3.(6)] Let $Y = X \cup_f e^{n+1}$ be a 1-connected space obtained attaching an $(n{+}1)$-cell to X via the map $f\colon \mathbb{S}^n \to X$. Let $(\mathbb{L}(V),\partial)$ be a Quillen model of X and let $\Phi \in \mathbb{L}(V)_{n-1}$ be a cycle representing the homology class in $H_{n-1}(\mathbb{L}(V),\partial)$ which is identified, via the isomorphism (ii) above, with the homotopy class $[f] \in \pi_n(X)\otimes\mathbb{Q}$. Then, the injection*

$$(\mathbb{L}(V),\partial) \hookrightarrow (\mathbb{L}(V\oplus\mathbb{Q}a),\partial')$$

with

$$\begin{cases} \partial' v = \partial v, & \text{for all } v \in V \\ \partial' a = \Phi \end{cases}$$

is a Quillen model of the inclusion

$$X \hookrightarrow Y.$$

Finally, we recall that a simply connected complex is *formal* if its rational homotopy type depends only on its rational cohomology algebra. Given a commutative graded algebra H there is a formal simply connected complex X, unique up to rational homotopy, whose cohomology algebra $H^*(X;\mathbb{Q})$ is isomorphic to H.

Resumen en Español

Este trabajo es una mezcla entre dos mundos completamente distintos: Por un lado, la *Homología Persistente*, un recién nacido de la generación del Análisis Topológico de datos, y por el otro, las A_∞-*(co)álgebras*, una piezas clásicas en la Teoría de Perturbación Homológica.

La Homología Persistente (en el sentido de [23, 39, 103]), una técnica topológica que se ha aplicado con éxito en contextos tan dispares como imágenes digitales [89], redes de sensores [33], modelado molecular [3], sistemas dinámicos [38,88] y análisis del habla [15] entre muchos otros, permite extraer información global sobre la estructura de conjuntos de datos (especialmente complejos, que involucren muchas variables) que pueden contener pequeños errores (devido a la medición, por ejemplo).

Más de cerca, esta teoría nos permite aproximar los números de Betti de un subconjunto cerrado desconocido X de un espacio métrico a partir de una muestra finita de puntos de X [19,28,32,82,88]. En dimensiones bajas, estos números son a veces fáciles de calcular a simple vista (como en Figure 4.1), pero una nube de puntos en un espacio 100-dimensional puede resultar difícil de visualizar. Además, en tal situación, nos tendríamos que preocupar no sólo de el número de piezas (componentes conexas), túneles y cavidades del espacio X subyacente, sino también de rasgos 99-dimensionales, por ejemplo.

¿Por qué es esto útil? Veamos un ejemplo. Imaginemos que queremos estudiar una enfermedad de la cuál no sabemos mucho y para ello, empezamos por tomar 100 datos distintos de un paciente. Podemos ver estos números como coordenadas y visualizar los datos del paciente como un punto en un

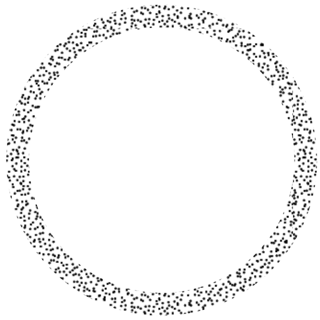


Figure 4.1: Es fácil ver a simple vista que ésta puede ser una muestra de puntos de una corona circular.

espacio euclídeo 100-dimensional $\mathbb{E}^{100}$. Repitiendo el proceso con 70000 personas obtenemos 70000 points in $\mathbb{E}^{100}$, donde cada punto representa a un paciente. Entonces, incluso si no le hacemos preguntas específicas al cúmulo de datos recopilados (bien porque no sepamos suficiente sober la enfermedad como para decidir qué preguntas hacer o bien porque no queramos que nuestro conocimiento previo condicione el tipo de información que podamos encontrar), la persistencia nos dará información topológica sobre el espacio que se esconde tras la nube de puntos y esto puede ser traducido luego en información sobre la enfermedad en cuestión.

Nótese que cuanto más sepamos sobre el espacio que se esconde tras la nube de puntos, más información conseguiremos acerca de tal enfermedad. El trabajo de la presente tesis se basa en esta observación.

Más concretamente, como la Homología Persistente, en su forma más básica, puede únicamente contribuir con información a nivel de los números de Betti, el desafío que puso en marcha este proyecto fue el de aumentar el poder de la persistencia encontrando herramientas que fueran capaces de distinguir, en diversas situaciones, entre espacios con los mismos números de Betti (e incluso con álgebras de cohomología isomorfas) que no fueran homotópicamente equivalentes; herramientas que puedieran darnos información, en algunos casos, sobre cómo están anudadas distintas componentes de un enlace; y lo más importante, herramientas que puedieran admitir un enfoque persistente.

Las herramientas que proponermos usar son unas estructuras con las que se puede dotar a la (co)homología de un espacio; a saber, A_∞-álgebras and A_∞-coálgebras [94].

Dedicamos esta tesis a respaldar este punto de vista y a desarrollar los aspectos teóricos y algorítmicos de una teoría de persistencia para A_∞-estructuras, o A_∞*-persistencia,* por simplicidad.

Explicaremos las principales construcciones de nuestra teoría en términos de A_∞-coálgebras en la homología de un espacio, pero todo dualiza *mutatis mutandis* para A_∞-álgebras en cohomología (véase §3.1), dando en particular un nuevo punto de vista al producto cup persistente [55].

Durante el resto de este resumen, trabajemos sobre un cuerpo de coeficientes.

En general, véase §2.2 para una definición precisa, una estructura de A_∞-coálgebra sobre un módulo graduado C es una familia de aplicaciones graduadas

$$\Delta_n : C \longrightarrow \underbrace{C \otimes \ldots \otimes C}_{n}, \qquad n \geq 1$$

de grado $|\Delta_n| = n - 2$, satisfaciendo ciertas relaciones de coasociatividad. La "A" en A_∞-(co)álgebra viene de la palabra *Asociatividad* y el "∞" indica que para toda $n \neq 1$ (sin restricciones en lo grande que sea n), unos axiomas de (co)asociatividad sobre $\Delta_1, \Delta_2, \ldots, \Delta_n$ se cumplen módulo la (co)cadena homotopía Δ_{n+1}.

Dado un espacio topológico X, su homología $H_*(X)$ siempre se puede dotar de varias estructuras de A_∞-coálgebra, todas ellas isomorfas, con Δ_2 siendo la aplicación inducida en homología por una aproximación a la diagonal y de modo que estas A_∞-coálgebras satisfacen una condición extra que las hace cargar con una información topológica mayor que una A_∞-coálgebra cualquiera en $H_*(X)$. Hemos dado en llamar a éstas, A_∞-coálgebras *buenas* (*good* A_∞-coalgebras).

Expliquemos lo que son estas estructuras con algo más de detalle:

Definición 2.2.1. Una A_∞-coálgebra $(C, \{\Delta_n\}_{n\geq 1})$ consiste en un módulo graduado C y una familia de aplicaciones graduadas

$$\Delta_n \colon C \longrightarrow C^{\otimes n}$$

de grado $n-2$ tales que, para todo $n \geq 1$, se tienen las siguentes identidades:

$$\mathbf{SI(}\boldsymbol{n}\mathbf{)}\colon \sum_{i=1}^{n}\sum_{j=0}^{n-i}(-1)^{i+j+ij}\left(1^{\otimes n-i-j}\otimes\Delta_i\otimes 1^j\right)\Delta_{n-i+1}=0.$$

Cuando estas fórmulas se aplican a elementos, hay que añadir los signos dados por el convenio de Koszul (véase *Def. 2.1.5*).

Ejemplo 2.2.2. Toda coálgebra graduada diferencial (DGC) (C,∂,Δ) puede verse como una A_∞-coálgebra

$$(C,\{\Delta_n\}_{n\geq 1}),\qquad \text{by setting}\qquad \Delta_n=\begin{cases}\partial, & n=1\\ \Delta, & n=2\\ 0, & n>2.\end{cases}$$

Definición 2.2.3. Una A_∞-coálgebra $(C,\{\Delta_n\}_{n\geq 1})$ se llama **minimal** si

$$\Delta_1=0.$$

Definición 2.2.6. Un **morfismo de A_∞-coálgebras**

$$f\colon (C,\{\Delta_n\}_{n\geq 1})\to(C',\{\Delta'_n\}_{n\geq 1})$$

es una familia de aplicaciones graduadas

$$f_{(k)}\colon C\longrightarrow C'^{\otimes k},\qquad k\geq 1,$$

cada una de grado $k-1$, de modo que para cada $i \geq 1$, se cumple la siguiente identidad:

$$\sum_{\substack{p+q+k=i\\ q\geq 1,p,k\geq 0}}(1^{\otimes p}\otimes\Delta'_q\otimes 1^{\otimes k})f_{(p+k+1)}=\sum_{\substack{k_1+\cdots+k_\ell=i\\ l,k_j\geq 1}}(f_{(k_1)}\otimes\cdots\otimes f_{(k_\ell)})\Delta_\ell.$$

Definición 2.2.7. Sea

$$f\colon C \longrightarrow C'$$

un morfismo de A_∞-coálgebras. Decimos entonces que

- f es un **isomorfismo** si lo es $f_{(1)}$.
- f is a **quasi-isomorfismo** si $f_{(1)}$ induce isomorfismo in homología.

Como hemos dicho, estamos interesados en un tipo especial de A_∞-coálgebra que llamaremos *buena.* Hemos escogido esta terminología porque, como vemos en las secciones 2.4 y 3.2, de entre todas las estructuras de A_∞-coálgebra que se pueden poner en $H_*(X)$, las *buenas* pueden dar infomcación particularmente significativa sobre el tipo de homotopía de X.

Observación 2.2.10. Por ejemplo, en toda A_∞-coálgebra $(C, \{\Delta_n\}_{n\geq 1})$, Δ_3 mide el grado hasta el cual se puede relajar la coasociatividad de Δ_2. Aún así, incluso si Δ_2 es estrictamente coasociativa, Δ_3 puede ser no trivial, y si el A_∞-coálgebra es *buena*, esta no trivialidad puede aportar jugosa información en algunos casos (véase §3.2).

Presentamos a continuación la definición de A_∞-coálgebra *buena.*

Definición 2.2.11. Decimos que una A_∞-coálgebra $(H_*(X), \{\Delta_n\}_n)$ en la homología de un espacio X es una **A_∞-coálgebra *buena* (inducida por X)** si es minimal y quasi-isomorfa al A_∞-coálgebra

$$(C_*(X), \{\partial, \Delta, 0, 0, \ldots\})\,,$$

donde $(C_*(X), \partial)$ denota el correspondiente complejo de cadenas y Δ denota una aproximación a la diagonal. Prescindiremos del *inducida por* X cuando no haya lugar a confusión.

Una consecuencia inmediata de la definiciónes que todas las A_∞-coálgebras *buenas* en $H_*(X)$ inducidas por X son isomorfas.

En §2.2 explicamos que siempre podemos conseguir una A_∞-coálgebra *buena* en la homología de un espacio. Una forma de ver cómo hacerlo es la siguiente.

Definición 2.2.8. Un **diagrama de transferencia** entre complejos de cadenas (resp., cocadenas) (A, ∂_A) y (M, ∂_M) es un diagrama

$$\phi \circlearrowright (A, \partial_A) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (M, \partial_M)$$

en el cual π y ι son morfismos de módulos graduados diferenciales y ϕ es una aplicación graduada de grado +1 (resp., -1) tal que

- $\pi\iota = 1_M$,
- $\pi\phi = 0$,
- $\phi\iota = 0$,
- $\phi^2 = 0$ y
- $\iota\pi \underset{\phi}{\simeq} 1_A$.

El clásico *Lema de Perturbación* [52, 56, 60], conocido hoy en día como el *Teorema de Transferencia Homotópica* [64, 68], es una herramienta muy usada para construir estructuras de A_∞-coálgebra como deformaciones de coálgebras graduadas diferenciales [57,61]. La versión que mejor se adapta a nuestros propósitos reza así:

Teorema 2.2.9. (Teorema de Transferencia Homotópica [64, 68] or Basic Perturbation Lemma [52, 56, 60]) *Sea*

$$\phi \circlearrowright (C, \partial) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (C', \partial')$$

un diagrama de transferencia entre complejos de cadenas en el que (C, ∂, Δ) forma una coálgebra graduada diferencial. Entonces, existe una estructura explícita de A_∞-coálgebra en C', así como morfismos de A_∞-coálgebras,

$$C \underset{I}{\overset{P}{\rightleftarrows}} C' ,$$

tales que $P_{(1)} = \pi$ y $I_{(1)} = \iota$.

En §2.3 damos fórmulas explícitas para calcular A_∞-estructuras transferidas.

Este resultado nos da una manera estándar y functorial de dotar a la homología simplicial / singular / celular de un cierto complejo, de una estructura de A_∞-coálgebra. Para ello, consideremos la coálgebra graduada diferencial en el correspondiente complejo de cadenas $C_*(X)$, dada por una aproximación de la diagonal (§2.1.1). Entonces, es fácil hayar un diagrama de transferencia de la forma

$$\phi \circlearrowright (C_*(X), \partial) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (H_*(X), 0).$$

En efecto, tomemos A y H tales que $C_*(X) = A \oplus \partial A \oplus H$, $\partial\colon A \overset{\cong}{\to} \partial A$ y $H \cong H_*(X)$. Pongamos como ι la inclusión to be the inclusion, y construyamos p y ϕ recursivamente para que satisfagan las propiedades que tocan.

Es sencillo comprobar también que dado un diagrama de transferencia de la forma

$$\phi \circlearrowright (C_*(X), \partial, \Delta) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (H_*(X), 0),$$

que ϕ sea 0 garantiza que la estructura de A_∞-coálgebra transferida en $H_*(X)$ sea *buena,* pero si $\phi = 0$, la estructura transferida en $H_*(X)$ no es necesariamente quasi-isomorfa a

$$(C_*(X), \{\partial, \Delta, 0, 0, \ldots\})\,.$$

En el último caso, el A_∞-coálgebra transferida forma de hecho una coálgebra graduada diferencial (DGC) – la DGC estándar en la homología de $(C_*(X), \partial, \Delta)$, que es el dual del álgebra de cohomología estándar.

Como hemos dicho, pretendemos estudiar las A_∞-coálgebras *buenas* en homología (y dualmente, las A_∞-álgebras *buenas* en cohomología) de una manera persistente. Para empezar a motivar esta elección de estructuras, probamos:

Teorema 2.4.4. *Sea X un CW complejo simplemente conexo, no formal, de tipo finito, y sea Y el espacio formal asociado a $H^*(X;\mathbb{Q})$. Entonces, X e*

Y tienen grupos de homología racional isomorfos y álgebras de cohomología racional isomorfos pero sus homologías racionales no admiten estructuras de A_∞-coálgebra buena *isomorfas.*

En el Ejemplo 2.4.5 mostramos espacios que satisfacen las hipótesis de este teorema.

Este resultado es una primera buena señal, puesto que nuestro objetivo es construir una teoría de persistencia que vaya más allá de los números de Betti. El siguiente paso es encontrar números, asociados a espacios topológicos, que sean susceptibles tanto de dar más información que los números de Betti como de admitir una adaptación persistente

En §3.1 vemos cómo la búsqueda de simplicidad nos lleva a considerar como candidatos los siguiente números. Dada una estructura de A_∞-coálgebra *buena* $\{\Delta_n\}_{n\geq 1}$ en la homología de X, fijamos un $n \geq 1$ y un grado de homología $p \geq 0$ y estudiamos el número

$$\dim \operatorname{Ker} \Delta_{n|H_p(X)}.$$

De manera similar, dada una estructura de A_∞-álgebra *buena* $\{m_n\}_{n\geq 1}$ en la cohomología de X, fijamos un $n \geq 1$ and varios grados de cohomología $p_1, \ldots, p_n \geq 0$ y estudiamos el número

$$\dim \operatorname{Coker} m_{n|H^{p_1}(X)\otimes\ldots\otimes H^{p_n}(X)},$$

donde $\operatorname{Coker} f$ denota el conúcleo de la aplicacion lineal $f\colon V \longrightarrow W$, *i.e.,* el espacio cociente $W/\operatorname{Im} f$.

Vale la pena hacer notar aquí que hay muchas estructuras de A_∞-coálgebra *buena* con las que se puede dotar a la homología de un espacio X. Es más, distintas elecciones de A_∞-coálgebra pueden dar distintos números $\dim \operatorname{Ker} \Delta_{n|H_p(X)}$:

Ejemplo 3.2.1. En este ejemplo damos dos estructuras de A_∞-coálgebra *buena* en $\widetilde{H}_*(X;\mathbb{Q})$, donde X es el espacio

$$X = (\mathbb{S}^2 \vee \mathbb{S}^7) \cup_g e^4$$

en el cual la (clase de homotopía de) la función de adjunción g es el producto de Whitehead

$$g = [id_{\mathbb{S}^2}, id_{\mathbb{S}^2}].$$

Estas estructuras son las inducidas, respectivamente, por dos modelos minimales de Quillen (véase el Apéndice para los detalles) de X: por un lado, el álgebra de Lie graduada diferencial (DGL)

$$(\mathbb{L}(W), \partial) = (\mathbb{L}(x_1, y_3, z_6), \partial),$$

donde los subíndices denotan el grado y la diferencial viene dada por

$$\begin{aligned} \partial x_1 &= 0, \\ \partial y_3 &= [x_1, x_1], \\ \partial z_6 &= 2\,[[x_1, x_1], y_3]\,. \end{aligned}$$

Por otro lado, el DGL

$$(\mathbb{L}(V), \partial) = (\mathbb{L}(x_1, y_3, u_6), \partial),$$

donde los subíndices denotan el grado y la diferencial viene dada por

$$\begin{aligned} \partial x_1 &= 0, \\ \partial y_3 &= [x_1, x_1], \\ \partial u_6 &= 0. \end{aligned}$$

En particular, la A_∞-coálgebra inducida por $(\mathbb{L}(W), \partial)$ tiene

$$\Delta_3 \neq 0,$$

mientras que la inducida por $(\mathbb{L}(V), \partial)$ tiene

$$\Delta_3 = 0.$$

De hecho, esta última tiene $\Delta_n = 0$ para todo $n > 2$, lo que muestra que el espacio X es formal sobre los racionales.

El mismo inconveniente ocurre con A_∞-álgebras en $H^*(X)$ y los números $\dim \operatorname{Coker} m_{n|H^{p_1}(X)\otimes\ldots\otimes H^{p_n}(X)}$. Para superar este obstáculo, probamos varios resultados que muestran escenarios en los que estos números pueden, a pesar de lo comentado, dar información más allá de los números de Betti e incluso más allá del álgebra de cohomología. Concretamente, primero demostramos el Teorema 3.2.2, un resultado que funciona cuando se trabaja sobre un cuerpo de característica 0, que implica en particular que los números

$$k := \min\{n \mid \Delta_n \neq 0\}$$

y $\dim \operatorname{Ker} \Delta_{k|H_p(X)}$, para todo $p \geq 0$, son invariantes de la clase de isomorfismo de estructuras de A_∞-coálgebra minimal $\{\Delta_n\}_n$ en $H_*(X)$. Una consecuencia importante de este resultado es el siguiente:

Corolario 3.2.3. *Sean $\{\Delta_n\}_n$ y $\{\Delta'_n\}_n$ dos estructuras cualesquiera de A_∞-coálgebra* buena *en la homología de un espacio X de modo que existan $m, m' \geq 1$ para los cuales $\Delta_m \neq 0$ y $\Delta'_{m'} \neq 0$. Entonces, los números*

$$k := \min\{n \mid \Delta_n \neq 0\} \qquad y \qquad \min\{n \mid \Delta'_n \neq 0\}$$

coinciden,

$$(H_*(X), \{0, \ldots, 0, \Delta_k, 0, \ldots\}) \qquad y \qquad (H_*(X), \{0, \ldots, 0, \Delta'_k, 0, \ldots\})$$

son A_∞-coálgebras isomorfas y los números

$$\dim Ker\, \Delta_{k|H_p(X)} \qquad y \qquad \dim Ker\, \Delta'_{k|H_p(X)}$$

también coinciden, para todo $p \geq 0$.

En la misma dirección, demostramos:

Proposición 3.2.12. *Denotemos por L los anillos de Borromeo. Entonces, cualquier A_∞-álgebra* buena *$\{m_n\}_n$ en $H^*(\mathbb{S}^3 - L)$ cumplirá*

$$m_{3|H^1\otimes H^1\otimes H^1} \not\equiv 0,$$

donde H^1 denota $H^1(\mathbb{S}^3 - L)$.

Obsérvese que podemos obtener resultados similares para

$$m_{n|H^{p_1}(X)\otimes\ldots\otimes H^{p_n}(X)}$$

con valores arbitrarios de $n, p_1, \ldots, p_n$. En esta dirección, probamos también Proposición 3.2.13. Para enunciar ésta, sean $n = p+q+r$, $x = (x_1, \ldots, x_p)$, $y = (y_1, \ldots, y_q)$, y $z = (z_1, \ldots, z_r)$ y consideremos tres espacios S_1, S_2, S_3 en $\mathbb{R}^n$ (homeomorfos a tres esferas) dados por las ecuaciones

$$\begin{aligned}
x &= 0, \qquad \|y\|^2 + \frac{\|z\|^2}{4} = 1 \qquad \text{(esfera } (q+r-1)\text{-dimensional } S_1\text{);}\\
y &= 0, \qquad \|z\|^2 + \frac{\|x\|^2}{4} = 1 \qquad \text{(esfera } (p+r-1)\text{-dimensional } S_2\text{);}\\
z &= 0, \qquad \|x\|^2 + \frac{\|y\|^2}{4} = 1 \qquad \text{(esfera } (p+q-1)\text{-dimensional } S_3\text{);}
\end{aligned}$$

Nótese que el caso $p = q = r = 1$ nos da unos anillos de Borromeo.

En un cierto sentido más general que para el caso de nudos, estos espacios S_i están no anudados dos a dos, lo que hace que el obvio producto de Massey esté definido, y es más, denotando por K la unión disjunta de S_1, S_2 y S_3, probamos lo siguiente:

Proposición 3.2.13. *Con esta notación, cualquier A_∞-álgebra* buena $\{m_k\}_k$ *en $H^*(\mathbb{S}^n - K)$ deberá cumplir*

$$m_{3|H^p\otimes H^q\otimes H^r} \not\equiv 0,$$

donde H^k denota $H^k(\mathbb{S}^n - K)$.

Todos estos resultados respaldan nuestro interés en el estudio de la persistencia de los mencionados números a lo largo de una filtración, que pasamos a explicar a continuación en términos de A_∞-coálgebras y homología, aunque funciona también para A_∞-álgebras and cohomología.

Sea

$$\mathcal{K}\colon \qquad K_0 \longrightarrow K_1 \longrightarrow \ldots \longrightarrow K_N$$

una sucesión de espacios topológicos y aplicaciones contínuas. A la hora de palicarlo en el mundo real, en la mayoría de los casos nos encontraremos con una filtración. Es decir, $\mathcal{K}$ consistirá en subcomplejos simpliciales/cúbicos/celulares anidados e inclusiones

$$K_0 \hookrightarrow K_1 \hookrightarrow \ldots \hookrightarrow K_N.$$

Para cada $p \geq 0$ y $0 \leq i \leq j \leq N$, supongamos que la dimensión del p-ésimo grupo de homología de K_i es finita, $\dim H_p(K_i) < \infty$, y denotemos por

$$f_p^{i,j} \colon H_p(K_i) \longrightarrow H_p(K_j)$$

y

$$f^{i,j} \colon H_*(K_i) \longrightarrow H_*(K_j)$$

las aplicaciones inducidas en homología por la composición.

$$K_i \longrightarrow K_{i+1} \longrightarrow \ldots \longrightarrow K_j.$$

La persistencia clásica describe, para cada $p \geq 0$, la evolución del p-ésimo número de Betti $\beta_p(X) = \dim H_p(X)$ a lo largo de $\mathcal{K}$ mediante el estudio de los llamados grupos persistentes

$$H_p^{i,j}(\mathcal{K}) := \operatorname{Im} f_p^{i,j}, \qquad 0 \leq i \leq j \leq N.$$

Más concretamente, define un multiconjunto (un conjunto donde cada elemento tiene asociada una multiplicidad) de intervalos, llamado el p-ésimo código de barras (barcode), que satisface lo siguiente:

Teorema 1.2.4. [23, §3] **(Teorema Fundamental de la Homología Persistente)** *Para cada $p \geq 0$, el p-ésimo código de barras de $\mathcal{K}$ está bien definido y, para todo $0 \leq i \leq j \leq N$, la dimensión del grupo persistente $H_p^{i,j}(\mathcal{K})$ es igual al número de intervalos en el p-ésimo código de barras de $\mathcal{K}$ (contados con multilicidad) que contienen al intervalo $[i, j]$.*

En particular, $\dim H_p(K_i)$ *es igual al número de intervalos en el p-ésimo código de barras de $\mathcal{K}$ (contados con multilicidad) que contienen al valor i.*

En §1.2, damos una nueva demostración de este resultado clásico. En particular, lo hacemos probando un Lema previo que tiene la siguiente forma.

Definición 1.2.6. Un **módulo de persistencia** es una sucesión finita de espacios vectoriales de dimensión finita y aplicaciones lineales entre ellos de la forma

$$V_0 \longrightarrow V_1 \longrightarrow \dots \longrightarrow V_N.$$

Lema 1.2.7. *Sea*

$$V_0 \xrightarrow{f^{0,1}} V_1 \xrightarrow{f^{1,2}} \dots \xrightarrow{f^{N-1,N}} V_N$$

un módulo de persistencia y definamos

$$f^{i,j} := \begin{cases} f^{j-1,j} \circ \dots \circ f^{i,i+1}, & i+1 < j, \\ id_{V_i}, & i = j. \end{cases}$$

Entonces, existe un multiconjunto M de intervalos de la forma $[i,j)$, para $0 \leq i < j \leq N$, y de la forma $[i,\infty)$, para $0 \leq i \leq N$ tal que, para cada $0 \leq i \leq j \leq N$, la dimensión $\dim \mathrm{Im} f^{i,j}$ se puede calcular como la suma de la multiplicidad de cada intervalo en M que contiene a $[i,j]$.

La primera prueba del Teorema 1.2.4, debida a Carlsson y Zomorodian [23, §3], utiliza el teorema de estructura de los módulos graduados finitamente generados sobre un dominio de ideales principales, mientras que nosotros usamos una desigualdad de Frobenius sobre el rango de productos de matrices (véase §Section:P.H.) para todos los detalles).

El Teorema 1.2.4 garantiza cierta facilidad en los cálculos (como explicamos en §3.3) y permite obtener una imagen visual que captura la información de todos los grupos persistentes a la vez. Es más, resultados de *estabilidad* [28] nos dicen que en un cierto sentido, el número de intervalos largos en el p-ésimo código de barras es robusto respecto a la introducción de ruido o de pequeñas variaciones en la nube de puntos que estemos estudiando (o más generalmente, en $\mathcal{K}$, provenga de donde provenga) y que este número atesora información sobre rasgos topológicos de nuestro objeto de estudio.

Inspirados por esto, procedemos como sigue. Elíjase una estructura de A_∞-coálgebra *buena* $\{\Delta^i_n\}_n$ en la homología de cada término K_i, que usaremos a lo largo del resto del resumen. Como para cada $p \geq 0$ y para cada

$n \geq 1$ queremos describir la evolución del número $\dim \operatorname{Ker} \Delta_{n|H_p(X)}$ a lo largo de $\mathcal{K}$, definimos lo que llamamos los *grupos* Δ_n*-persistentes* (Δ_n*-persistent groups*)

$$(\Delta_n)_p^{i,j}(\mathcal{K}) = \operatorname{Im} f_p^{i,j}|_{\cap_{k=i}^{j} \operatorname{Ker}\left(\Delta_n^k \circ f_p^{i,k}\right)}, \qquad 0 \leq i \leq j \leq N,$$

y un multiconjunto de intervalos, que llamamos el *p-ésimo* Δ_n*-código de barras* (p^{th} Δ_n*-barcode*), que cumplen lo siguiente:

Corolario 3.6.5. *Para cada* $p \geq 0$ *y* $n \geq 1$, *el p-ésimo* Δ_n*-código de barras de* $\mathcal{K}$ *está bien definido y, para todo* $0 \leq i \leq j \leq N$, *la dimensión del grupo* Δ_n*-persistente* $(\Delta_n)_p^{i,j}(\mathcal{K})$ *es igual al número de intervalos en el p-ésimo* Δ_n*-código de barras de* $\mathcal{K}$ *(contados con multiplicidades) que continenen al intervalo* $[i, j]$.

En particular, $\dim Ker\,\Delta^i_{n|H_p(K_i)}$ *es igual al número de intervalos en el el p-ésimo* Δ_n*-código de barras de* $\mathcal{K}$ *(contados con multiplicidades) que continenen al valor* i.

Este es uno de los resultados más importantes de esta tesis. A continuación mostramos el camino que seguimos para demostrarlo.

En primer lugar, hay que tener en cuenta que eligiendo una base arbitraria de $\bigoplus_{i=0}^{N} H_p(K_i)$, calcular los números $\dim H_p^{i,j}(\mathcal{K})$ estudiando la evolución a lo largo de $\mathcal{K}$ de los elementos en esa base sería muy costoso, puesto que incolucraría el comprobar si las imágenes por $f_p^{i,j}$ de clases de homología linealmente independientes son linealmente dependientes. El Teorema 1.2.4 se puede reescribir en términos de la existencia de una base que permita tal cálculo de una manera mucho más sencilla:

Teorema 1.2.8. [23, §3] *Para cada* $p \geq 0$, *existe una base* $\mathcal{B}$ *de* $\bigoplus_{i=0}^{N} H_p(K_i)$ *tal que, para todo* $0 \leq i \leq j \leq N$, *la dimensión del grupo persistente* $H_p^{i,j}(\mathcal{K})$ *se puede calcular contando el número de p-clases de homología de la base que no se anulan en el rango correspondiente:*

$$\dim H_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,j}\beta \neq 0\}.$$

En Homología Persistente, podemos describir la igualdad anterior en términos de clases que *sobreviven* a lo largo de un cierto rango, con la siguiente notación clásica:

Definición 1.2.1. Sean $0 \leq i \leq j \leq N$ y $p \geq 0$.

- Una clase $\alpha \in H_p(K_i)$ **nace** en K_i si

$$\alpha \neq 0 \qquad \text{y} \qquad \alpha \notin \operatorname{Im} f^{i-1,i}.$$

- Una clase $\alpha \in H_p(K_i)$ que nace en K_i se dice que está **viva** en K_j (donde $i \leq j$) si

$$f^{i,j}\alpha \notin \operatorname{Im} f^{i-1,j}. \tag{4.0.1}$$

  Observemos que (4.0.1) es equivalente a decir que, para toda $i < k \leq j$,

$$f^{i,k}\alpha \notin \operatorname{Im} f^{i-1,k},$$

  que a su vez implica que $f^{i,k}\alpha \neq 0$ para toda $i < k \leq j$.

- Una clase $\alpha \in H_p(K_i)$ que nace en K_i se dice que **muere** en K_j (donde $i < j$) si está viva en K_{j-1} pero no lo está en K_j, *i.e.*, si

$$f^{i,j-1}\alpha \notin \operatorname{Im} f^{i-1,j-1} \qquad \text{y} \qquad f^{i,j}\alpha \in \operatorname{Im} f^{i-1,j}.$$

  Esto implica que o bien $f^{i,j}\alpha = 0$ o bien existe algún $\beta \neq 0 \in H_p(K_{i-1})$ tal que

$$f^{i-1,j}\beta = f^{i,j}\alpha,$$

  en cuyo caso decimos que α y β se han fundido en la misma clase, que la clase β es **más antigua** (porque nació en K_k para alguna $k \leq i-1$, mientras que α nació en K_i) y que β es, de las dos, la clase que representa a la clase en la que se han fundido (esto se conoce como la regla del más antiguo, o **Elder Rule**).

De forma parecida, definimos los siguientes conceptos, que nos dotarán de un vocabulario adecuado para expresar ciertos comportamientos y resultados con comodidad y con propiedad:

Definición 3.3.1. Sean $0 \leq i \leq j \leq N$, $p \geq 0$ y $n \geq 1$.

- Una clase $\alpha \in H_p(K_i)$ **se $\mathbf{\Delta}_n$-despierta** en K_i si
$$\alpha \neq 0, \quad \alpha \in \operatorname{Ker} \Delta_n^i \quad \text{y} \quad \alpha \notin \operatorname{Im} f^{i-1,i}{}_{|\operatorname{Ker} \Delta_n^{i-1}}.$$
- Una clase $\alpha \in H_p(K_i)$ que se Δ_n-despierta en K_i decimos que **está $\mathbf{\Delta}_n$-despierta** en K_j si para toda $i < k \leq j$,
$$f^{i,k}\alpha \in \operatorname{Ker} \Delta_n^k$$
y
$$f^{i,k}\alpha \notin \operatorname{Im} f^{i-1,k}{}_{|\bigcap_{l=i-1}^{k-1}(f^{i-1,l})^{-1}\operatorname{Ker} \Delta_n^l}. \tag{4.0.2}$$
Notemos que (4.0.2) implica que $f^{i,k}\alpha \neq 0$, para toda $i < k \leq j$.
- Una clase $\alpha \in H_p(K_i)$ que se Δ_n-despierta en K_i decimos que **se $\mathbf{\Delta}_n$-duerme** en K_j si está Δ_n-despierto en K_i, K_{i+1}, …, K_{j-1} pero no lo está en K_j, *i.e.*, si α está Δ_n-despierta en K_i, K_{i+1}, …, K_{j-1} y o bien
$$f^{i,j}\alpha \notin \operatorname{Ker} \Delta_n^j,$$
o bien
$$f^{i,j}\alpha \in \operatorname{Im} f^{i-1,j}{}_{|\bigcap_{l=i-1}^{j-1}(f^{i-1,l})^{-1}\operatorname{Ker} \Delta_n^l}. \tag{4.0.3}$$
Observemos que (4.0.3) incluye la posibilidad $f^{i,j}\alpha = 0$.

Hacemos notar que si una clase $\alpha \in H_p(K_i)$ nace en K_i y muere en K_l, entonces, si está Δ_n-despierta en algún momento, deben existir j y k como en $i \leq j < k \leq l$ tales que α se Δ_n-despierta en K_j y se Δ_n-duerme en K_k. Con esto empezamos a vislumbrar el porqué de la elección de la terminología *dormido-despierto*, que se termina de aclarar en §3.5.

Además, por una de las propiedades de las A_∞-coálgebras *buenas*, tenemos lo siguiente:

Observación 3.3.2. Los conceptos de Δ_1-despertarse y Δ_1-dormirse coinciden con los de nacer y morir en persistencia clásica.

En A_∞-persistencia, elegir una base arbitraria de $\bigoplus_{i=0}^N H_p(K_i)$ y calcular los números $\dim(\Delta_n)_p^{i,j}(\mathcal{K})$ estudiando la evolución a lo largo de $\mathcal{K}$ de los elementos de esa base sería aún más costoso que en homología persistente, puesto que involucraría comprobaciones más sutiles (véase §3.3 para más detalles).

En esta dirección, empezamos por demostrar:

Teorema 3.3.5. *Fijemos enteros $p \geq 0$ y $n \geq 1$. Si $f_p^{i,i+1}(Ker\,\Delta_n^i) \subseteq Ker\,\Delta_n^{i+1}$ para todo $0 \leq i < N$, entonces existe una base $\mathcal{B}$ de $\bigoplus_{i=0}^N H_p(K_i)$ tal que, para cada $0 \leq i \leq j \leq N$, la dimensión del grupo Δ_n-persistente $(\Delta_n)_p^{i,j}(\mathcal{K})$ se puede calcular contando el número de p-clases de homología de la base se mantienen Δ_n-despiertas a lo largo del correspondiente rango:*

$$\dim(\Delta_n)_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \in Ker\,\Delta_n^k - \{0\} \text{ para toda } k = i, \ldots, j\}.$$

Más aún, este número es igual a

$$\#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,j}\beta \in Ker\,\Delta^j - \{0\}\}.$$

Vale la pena señalar que en efecto hay casos en los que se puede aplicar el Theorem 3.3.5:

Teorema 3.4.1. *Con coeficientes sobre los racionales $\mathbb{Q}$, sea K_0 un CW complejo simplemente conexo y sea $N \geq 0$ un entero. Si para todo $0 < i \leq N$, K_i denota un espacio simplemente conexo obtenido al adjuntarle a K_{i-1} una celda de modo que*

$$\dim \bigoplus_{p \geq 0} H_p(K_i) = \dim \bigoplus_{p \geq 0} H_p(K_{i-1}) + 1$$

(y por tanto cada adjunción produce nueva homología), entonces existe una A_∞-coálgebra buena $\left(\widetilde{H}_*(K_i), \{\Delta_n^i\}_n\right)$ *para todo $0 \leq i \leq N$, y se tiene*

$$f^{i,i+1}\left(Ker\,\Delta_n^i\right) \subseteq Ker\,\Delta_n^{i+1},$$

para cada $n \geq 1$ y $0 \leq i < N$.

Yendo más allá, si no nos restringimos a las hipótesis del Teorema 3.3.5, hallamos un potencial comportamiento de intermitencia en las clases de homología a lo largo de $\mathcal{K}$ que describimos en §3.5. Esencialmente, el Teorema 3.5.1 muestra que las clases de homología pueden Δ_n-dormirse y Δ_n-despertarse varias veces a lo largo de $\mathcal{K}$ (mientras estén vivas). Ésta es la otra razón por la que elegimos la terminología *dormido-despierto*.

Probamos este resultado construyendo un ejemplo en el que tal comportamiento tiene lugar usando los racionales $\mathbb{Q}$ como coeficientes. De nuevo, recordamos que en el Apéndice aparece todo los que necesitamos en lo referente a homotopía racional.

Para empezar, montamos una filtración $\mathcal{K}$ de CW complejos finitos

$$K_0 \overset{i_0}{\hookrightarrow} K_1 \overset{i_1}{\hookrightarrow} K_2 \overset{i_2}{\hookrightarrow} K_3$$

como sigue:

- $K_0 = (\mathbb{S}^2_1 \vee \mathbb{S}^2_2 \vee \mathbb{S}^2_3 \vee \mathbb{S}^4) \cup_{g_1} e^6_1 \cup_{g_2} e^6_2$, donde $\mathbb{S}^2_1, \mathbb{S}^2_2, \mathbb{S}^2_3$ denotan tres copiar distintas de $\mathbb{S}^2$ y las (clases de homotopía de las) funciones de adjunción de las 6-celdas e^6_1, e^6_2 son los siguientes productos de Whitehead:

$$g_1 = [id_{\mathbb{S}^4}, id_{\mathbb{S}^2_1}] + [id_{\mathbb{S}^2_1}, [id_{\mathbb{S}^2_1}, [id_{\mathbb{S}^2_1}, id_{\mathbb{S}^2_2}]]], \quad g_2 = [id_{\mathbb{S}^4}, id_{\mathbb{S}^2_2}].$$

- $K_1 = K_0 \cup_{g_3} e^4$, donde

$$g_3 = [id_{\mathbb{S}^2_1}, id_{\mathbb{S}^2_2}].$$

- $K_2 = K_1 \cup_{g_4} e^6$, donde

$$g_4 = [id_{\mathbb{S}^4}, id_{\mathbb{S}^2_1}] - [id_{\mathbb{S}^2_2}, [id_{\mathbb{S}^2_2}, [id_{\mathbb{S}^2_2}, id_{\mathbb{S}^2_3}]]].$$

- $K_3 = K_2 \cup_{g_5} e^4$, donde

$$g_5 = [id_{\mathbb{S}^2_2}, id_{\mathbb{S}^2_3}].$$

- Las aplicaciones i_1, i_2, i_3 son inclusiones.

Entonces, demostramos

Teorema 3.5.1. *En*

$$H_*(K_0;\mathbb{Q}) \xrightarrow{f_0} H_*(K_1;\mathbb{Q}) \xrightarrow{f_1} H_*(K_2;\mathbb{Q}) \xrightarrow{f_2} H_*(K_3;\mathbb{Q}),$$

todos los morfismos son inyectivos y podemos dotar a cada término $H_*(K_i;\mathbb{Q})$ *de una estructura de* A_∞*-coálgebra* buena *de modo que la clase no trivial* $\gamma_6 \in H_6(K_0)$*, creada por la celda* e_1^6*, está* Δ_4*-dormida en* K_0*, se* Δ_4*-despierta en* K_1*, se vuelve a* Δ_4*-dormir en* K_2 *y por último, se* Δ_4*-despierta en* K_3*.*

Finalmente, gracias a la herramienta de la Persistencia Zigzag [20], demostramos que podemos obtener un resultado del estilo del Teorema 3.3.5 con hipótesis mucho menos restrictivas.

Para cada $0 \leq i \leq N$, fijemos una estructura arbitraria de A_∞-coálgebra *buena* $\{\Delta_n^i\}_n$ en $H_*(K_i)$. El enunciado es como sigue:

Teorema 3.6.1. *Para cada* $p \geq 0$ *y* $n \geq 1$*, existe una base* $\mathcal{B}$ *de* $\bigoplus_{i=0}^N H_p(K_i)$ *tal que, para todo* $0 \leq i \leq j \leq N$*, la dimensión del grupo* Δ_n*-persistente* $(\Delta_n)_p^{i,j}(\mathcal{K})$ *se puede calcular contando el número de* p*-clases de homología de la base se mantienen* Δ_n*-despiertas a lo largo del correspondiente rango:*

$$\dim(\Delta_n)_p^{i,j}(\mathcal{K}) = \#\{\beta \in \mathcal{B} \cap H_p(K_i) \mid f^{i,k}\beta \in Ker\,\Delta_n^k - \{0\} \text{ para toda } k = i, \ldots, j\}.$$

Reordenando la información en el último resultado nos permite definir el p-ésimo Δ_n-código de barras mencionado con anterioridad (véase §3.6 para los detalles) y concluir con el atractivo resultado siguiente:

Corolario 3.6.5. *Para cada* $p \geq 0$ *y* $n \geq 1$*, el* p*-ésimo* Δ_n*-código de barras de* $\mathcal{K}$ *está bien definido y, para todo* $0 \leq i \leq j \leq N$*, la dimensión del grupo* Δ_n*-persistente* $(\Delta_n)_p^{i,j}(\mathcal{K})$ *es igual al número de intervalos en el* p*-ésimo* Δ_n*-código de barras de* $\mathcal{K}$ *(contados con multiplicidades) que continenen al intervalo* $[i,j]$*.*

En particular, $\dim Ker\,\Delta^i_{n|H_p(K_i)}$ *es igual al número de intervalos en el el* p*-ésimo* Δ_n*-código de barras de* $\mathcal{K}$ *(contados con multiplicidades) que continenen al valor* i*.*

Desde el punto de vista computacional, no implementamos algoritmos ni llevamos a cabo análisis de rendimiento. Lo que hacemos es, por un lado, realzar un atractivo algoritmo de P. Real *et al.* [80] que permite calcular estructuras de A_∞-coálgebra *buena* en la homología de CW complejos mediante el uso de campos vectoriales discretos (véase el Algoritmo 2.3.5). Por otro lado, mostramos el algoritmo básico para el cálculo de los Δ_n-códigos de barras de la A_∞-persistencia en el caso más general (Algoritmo 3.7.1).

Hemos organizado esta tesis como sigue:

En el Capítulo 1 recordamos los conceptos necesarios sobre Persistencia y damos una nueva demostración para uno de sus teoremas más importantes. En primer lugar, mostramos en §1.1 cómo se construyen las sucesiones $\mathcal{K}$ en dos de las aplicaciones más comunes: el estudio de datos en forma de nubes de puntos y el análisis de imágenes digitales. En segundo lugar, en §1.2, explicamos los principios básicos de la Homología Persistente y damos una prueba alternativa para el Teorema Fundamental de la Homología Persistente (véanse las demostraciones del Lema 1.2.7 y del Teorema 1.2.4). En tercer lugar, recordamos en §1.2 tanto el teorema como el algoritmo de Descomposición Zigzag que necesitaremos en el Capítulo 3.

El Capítulo 2 consiste en consiste en lo básico sobre A_∞-(co)álgebras más algunas contribuciones nuestras sobre cómo calcularlas y sobre cómo de poderosas son. En §2.1 establecemos notación y recordamos la noción de un caso particular de A_∞-(co)álgebra que es crucial para nuetro enfoque – *(co)álgebras graduadas diferenciales*. A continuación, exploramos las A_∞-*(co)álgebras*: primero desde un punto de vista teórico en §2.2, como deformaciones de (co)álgebras graduadas diferenciales, y luego desde un punto de vista computacional en §2.3, explicando nuestra modificación del algoritmo de P. Real *et al.* Finalmente, en §2.4 mostramos el poder de las A_∞-(co)álgebras a través de algunos resultados, clásicos y nuevos.

Es en el Capítuo 3 donde desarrollamos el grueso de nuestra teoría. En su introducción, describimos escenarios (que serán extendidos en §3.2.2 y §3.2.3) en los que resulta natural considerar el uso de la A_∞-persistencia. Presentamos en §3.1 los números (como $\dim \operatorname{Ker} \Delta^i_{n|H_p(X)}$) que queremos estudiar, dada una estructura de A_∞-coálgebra *buena* en la homologia de un espacio X (o una estructura de A_∞-álgebra *buena* en su cohomologia). Tomando dos tales estructuras diferentes en $H_*(X)$ (o en $H^*(X)$), sus correspondientes números pueden diferir en general (§3.2.1), pero señalamos situaciones en las que estos números deben necesariamente ser iguales y situaciones en las que aun no siendo necesariamente iguales, proporcionan información sobre el tipo de homotopía de X (§3.2.2 and §3.2.3). Estos números jugarán el papel que los números de Betti juegan en la Homología Persistente en el sentido de que definiremos una versión de ellos para sucesiones de espacios topológicos y aplicaciones contínuas (§3.3) $K_0 \longrightarrow K_1 \longrightarrow \ldots \longrightarrow K_N$ y demostraremos resultados de descomposición que harán más sencillo tanto el cálculo de tales "números persistentes" como la visualización de todos ellos a la vez. Concretamente, empezamos fijando ciertas hipótesis para probar un tal resultado en §3.3 y presentamos en §3.4 una construcción que bajo condiciones algo restrictivas garantiza que podamos aplicar este primer resultado de descomposición. Si relajamos mucho más las hipótesis, un curioso comportamiento intermitente (que justifica la terminología del estilo Δ_n-*despierto*) puede tener lugar (§3.5) y necesitamos un enfoque distinto para probar el teorema de descomposición más general (§3.6). Finalmente, discutimos brevemente cómo calcular A_∞-persistencia en el caso más general (§3.7).

Además, dado que varios de nuestros resultados y ejemplos se apoyan fuertemente en resultados clásicos de la Homotopía Racional, hemos incluido por completitud un "Apéndice" que contiene unas nociones básicas de esta teoría y los resultados que usamos.

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