THE EXCEPTIONAL LIE ALGEBRA G_2

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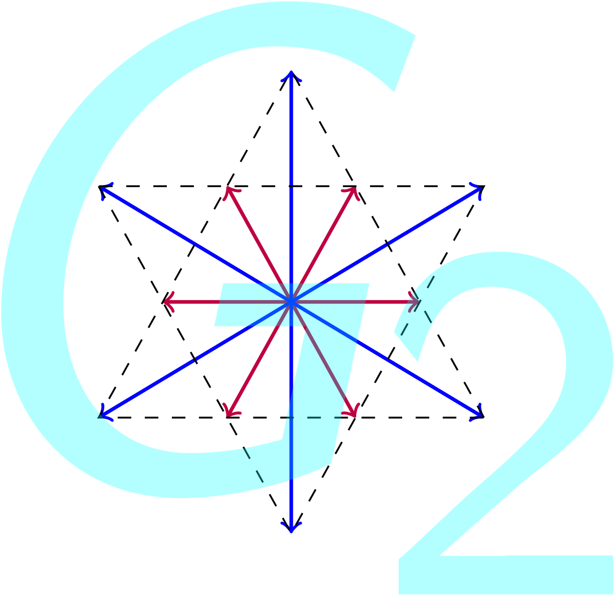
Non-associative Algebras and related topics


G2

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2. Quaternions, Octonions and cross products.
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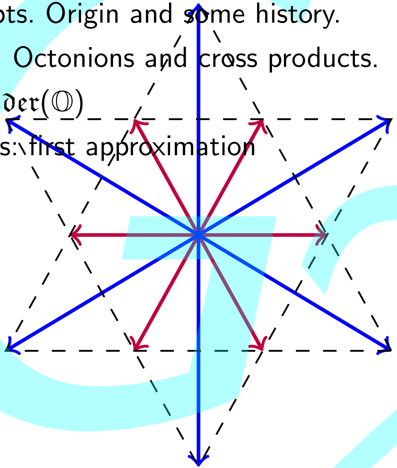


Lie algebras

Definition. A Lie algebra over a field $\mathbb{F}$ ($\mathbb{R}$ and $\mathbb{C}$ our setting) is an algebra $\mathfrak{g}$ (vector space over $\mathbb{F}$ and an $\mathbb{F}$-bilinear product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$) which satisfies

- ♡ Skewsymmetry: $[x, y] = -[y, x]$;
- ♡ Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Examples

- ⋆ $\mathfrak{g} = (\mathbb{F}^3, \times)$, $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$
- ⋆ $\mathrm{Mat}_n(\mathbb{F}) = \{A = (a_{ij}) : i, j \in \mathbb{F}\}$ is not Lie for $(A, B) \mapsto AB$
- ⋆ But for $(A, B) \mapsto [A, B] := AB - BA$ it is:
  $\mathfrak{gl}(n, \mathbb{F}) = (\mathrm{Mat}_n(\mathbb{F}), [\ ,\ ])$ general linear algebra
- ⋆ $\mathfrak{sl}(n, \mathbb{F}) = \{A \in \mathfrak{gl}(n, \mathbb{F}) : \mathrm{tr}(A) = 0\}$ special linear algebra
- ⋆ $\mathfrak{so}(n, \mathbb{F}) = \{A \in \mathfrak{gl}(n, \mathbb{F}) : A + A^T = 0\}$ orthogonal algebra

The founder of Lie group theory

Sophus Lie

Sophus Lie (1842-1899) discovered that

continuous transformation groups (now called, after him, *Lie groups*)
could be better understood by "linearizing" them,
and studying the corresponding generating vector fields
(called *infinitesimal generators* at that time).
The generators are subject to a linearized version of the group law,
now called the commutator bracket,
and have the structure of what is today called a Lie algebra.

The origins - I

Felix Klein

First inspiration: Klein (1849-1925)

After the appearance of the noneuclidean geometries, the following question was in the air:

what is really the geometry?

Erlangen programme: each geometry is the study of the invariants under the action of a group (*group of symmetries*):

Examples

- Euclidean Geometry $\rightarrow$ Rigid movements $\mathrm{O}(n) \ltimes \mathbb{R}^n$ (symm., rot., trans.)
- Projective Geometry $\rightarrow$ Projective group $\mathrm{PGL}(n+1) \cong \frac{\mathrm{GL}(n+1)}{\mathbb{R}^{\times}}$

The origins - II

Evariste Galois

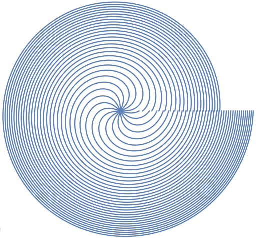
Algebraic inspiration: Galois (1811-1832)

Each algebraic equation is related to a group (Galois group, which permutes the roots) in such a way that

The equation can be solved by radicals when the group is solvable!

Can we relate to any differential equation a *differential Galois group* such that both solvabilities are equivalent?

Example of Lie's idea



$$\frac{dy}{dx} = \frac{y+x(x^2+y^2)}{x-y(x^2+y^2)} = \frac{g_2(x,y)}{g_1(x,y)}$$

$\rightsquigarrow x^2+y^2 = \frac{\frac{dy}{dx}-\frac{y}{x}}{1+\frac{dy}{dx}\frac{y}{x}} =$ tangent of angle at $P=(x,y)$ between the solution through P and the radius from $(0,0)$ to P $\rightsquigarrow$ invariant by rotations

Easy to check:

If $\alpha(s) = \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$ is a solution, and $\phi_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$ rotation of angle t $\Bigg\} \Rightarrow \phi_t \begin{pmatrix} x(s) \\ y(s) \end{pmatrix}$ is another solution!

$$\Rightarrow \phi\colon \mathbb{R} \to \mathrm{SO}(2),\ t \mapsto \phi_t, \text{ leaves the equation stable}$$

Take

$X_\phi(p) := \frac{\partial}{\partial t}\Big|_{t=0} \phi_t(p) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow X_\phi = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$

$Z = g_1\frac{\partial}{\partial x} + g_2\frac{\partial}{\partial y} \rightsquigarrow Z = (x-y(x^2+y^2))\frac{\partial}{\partial x} + (y+x(x^2+y^2))\frac{\partial}{\partial y}$ $\Bigg\} \Rightarrow [X_\phi, Z] = 0$

$\Rightarrow \quad (x^2+y^2)^{-1}$ integrating factor $\quad \Rightarrow \quad \frac{x^2+y^2}{2} + arctan\,\frac{x}{y} = C$ solutions

THE ORIGINS - III, Lie 1873, Analytic viewpoint

How can the knowledge of a stability group for a differential equation be utilized towards its integration?

$$\boxed{\frac{dy}{dx} = \frac{g_2(x,y)}{g_1(x,y)}} \quad \rightsquigarrow \quad g_2 dx - g_1 dy = 0$$

Def. A one-parameter subgroup $\phi\colon \mathbb{R} \to \mathit{Diff}(\mathbb{R}^2), 0 \mapsto \mathrm{id}, t \mapsto \phi_t$ (group hom.) leaves the equation stable if it permutes the solutions:

$$\alpha(s) = (x(s), y(s)) \text{ solution} \Rightarrow \phi_t(\alpha(s)) \text{ solution } \forall t$$

- Each one-parameter subgroup of diffeomorphisms of $\mathbb{R}^2$ has an induced vector field $X_\phi\colon \mathbb{R}^2 \to T\mathbb{R}^2$, $X_\phi(p) = \frac{\partial}{\partial t}\big|_{t=0} \phi_t(p)$

Lie Theorem:

$$\phi \text{ leaves the equation stable} \iff [X_\phi, Z] = \lambda Z \text{ for } Z = g_1 \frac{\partial}{\partial x} + g_2 \frac{\partial}{\partial y}, \quad \lambda \in \mathcal{C}^\infty(\mathbb{R}^2)$$

In this situation we can find an integrating factor (μ such that $\mu(g_2 dx - g_1 dy)$ exact):

If $X_\phi(p) = P\partial_x|_p + Q\partial_y|_p \Rightarrow \ \mu = (g_1 Q - g_2 P)^{-1}$

NOTION which has evolved to the notion of Lie group

Def: A group of continuous transformations in $\mathbb{R}^n$:

- The elements are mappings which transform a point $M = (x_1, \ldots, x_n)$ of a vector space into another point $M' = (x'_1, \ldots, x'_n)$ of the same space
- These transformations are (usually linear) invertible, including the identity, and the composition of two of them is another one
- The transformation depend upon r continuous parameters $x'_i = f_i(x_1, \ldots, x_n, t_1, \ldots, t_r)$

Above example SO(2): $\begin{cases} x'_1 = x_1 \cos t - x_2 \sin t = f_1(x_1, x_2, t) \\ x'_2 = x_1 \sin t + x_2 \cos t = f_2(x_1, x_2, t) \end{cases}$

Then $\left.\frac{d\phi_t}{dt}\right|_{t=0} = \left.\begin{pmatrix} -\sin t & -\cos t \\ -\cos t & \sin t \end{pmatrix}\right|_{t=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

is an infinitesimal generator (a basis of $\mathfrak{so}(2)$)

Complicating the example: SO(3)

$\mathrm{SO}(3) = \{A \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) : AA^t = I_3\}$, the special orthogonal group,

acts isometrically on $\mathbb{R}^3$ $\quad (\|v\| = \sqrt{v^t v} = \sqrt{v^t A^t A v} = \|Av\|)$

Most of its elements can be written as

$$\phi_{t_1,t_2,t_3} := R_{e_1,t_1} R_{e_2,t_2} R_{e_3,t_3}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t_1) & -\sin(t_1) \\ 0 & \sin(t_1) & \cos(t_1) \end{pmatrix} \begin{pmatrix} \cos(t_2) & 0 & -\sin(t_2) \\ 0 & 1 & 0 \\ \sin(t_2) & 0 & \cos(t_2) \end{pmatrix} \begin{pmatrix} \cos(t_3) & \sin(t_3) & 0 \\ -\sin(t_3) & \cos(t_3) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t_2)\cos(t_3) & \cos(t_2)\sin(t_3) & -\sin(t_2) \\ -\cos(t_3)\sin(t_1)\sin(t_2) - \cos(t_1)\sin(t_3) & \cos(t_1)\cos(t_3) - \sin(t_1)\sin(t_2)\sin(t_3) & -\cos(t_2)\sin(t_1) \\ \cos(t_1)\cos(t_3)\sin(t_2) - \sin(t_1)\sin(t_3) & \cos(t_3)\sin(t_1) + \cos(t_1)\sin(t_2)\sin(t_3) & \cos(t_1)\cos(t_2) \end{pmatrix}$$

(parametrization -not global- with 3 parameters: *Euler angles*)

The infinitesimal generators are $\frac{\partial \phi_{t_1,t_2,t_3}}{\partial t_i}\Big|_{t_1=t_2=t_3=0}$:

$$i = 1 : \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad i = 2 : \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad i = 3 : \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the generators of $\mathfrak{so}(3) = \{A \in \mathrm{Mat}_{3\times 3}(\mathbb{R}) : A + A^t = 0\}$

the algebra of skewsymmetric matrices which is the *tangent algebra* to $\mathrm{SO}(3)$

Lie algebras

These infinitesimal generators span a Lie algebra (the tangent algebra to the Lie group):

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- ♡ Skewsymmetry: $[x, y] = -[y, x]$;
- ♡ Jacobi identity: $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$.

Some natural questions

Q1 - What are all the possible Lie algebras?

Q2 - How can they appear as Lie algebras of vector fields?

TOWARDS A CLASSIFICATION of the algebraic structure

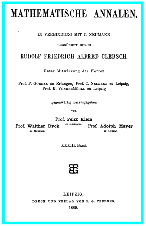
- Lie (1873-1876) did the classification for 2 complex variables
- Two families of complex simple Lie algebras were well-known at that time:

  - $\mathfrak{so}(n, \mathbb{C})$ = skew-symmetric complex matrices $\leftrightarrow$ orthogonal group
  - $\mathfrak{sl}(n, \mathbb{C})$ = trace-free matrices $\leftrightarrow$ groups preserving a volume form

  THE ONLY ONES?

  - $\mathfrak{sp}(n, \mathbb{C})$ $\leftrightarrow$ symplectic groups preserving a nondegenerate 2-form

- An astonishing discovery:



MATHEMATISCHE ANNALEN.

IN VERBINDUNG MIT C. NEUMANN
BEGRÜNDET DURCH
RUDOLF FRIEDRICH ALFRED CLEBSCH.

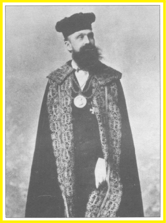
Unter Mitwirkung der Herren

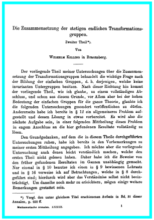
Prof. P. Gordan zu Erlangen, Prof. C. Neumann zu Leipzig,
Prof. K. VonderMühll zu Leipzig

gegenwärtig herausgegeben
von
Prof. Felix Klein
zu Göttingen.
Prof. Walther Dyck zu München. Prof. Adolph Mayer zu Leipzig.

XXXIII. Band.

LEIPZIG,
DRUCK UND VERLAG VON B. G. TEUBNER.
1889.





Die Zusammensetzung der stetigen endlichen Transformationsgruppen.

Zweiter Theil*).

Von

Wilhelm Killing in Braunsberg.

Der vorliegende Theil meiner Untersuchungen über die Zusammensetzung der Transformationsgruppen behandelt die wichtige Frage nach der Bildung der einfachen Gruppen, d. h. derjenigen, welche keine invarianten Untergruppen besitzen. Nach dieser Richtung hin kommt der vorliegende Theil, wie ich glaube, zu einem vollständigen Abschluss, und schon aus diesem Grunde, vor Allem aber bei der hohen Bedeutung der einfachen Gruppen für die ganze Theorie, glaubte ich die folgenden Untersuchungen gesondert veröffentlichen zu dürfen. Andererseits habe ich bereits in § 12 ein allgemeineres Problem aufgestellt und dessen Lösung in etwas vorbereitet. Es wird also die nächste Aufgabe sein, in einer folgenden Mittheilung dieses Problem in engem Anschluss an die hier gefundenen Resultate vollständig zu lösen.

Den Grundgedanken, auf dem die in diesem Theile durchgeführten Untersuchungen ruhen, habe ich bereits in den Vorbemerkungen zu meiner ersten Mittheilung angegeben. Ich möchte aber die vorliegende Untersuchung auch denen leicht verständlich machen, welche den ersten Theil nicht gelesen haben. Daher habe ich die Beweise von den früher gefundenen Resultaten im Ganzen unabhängig gemacht. Nur einmal in § 10 benutze ich einen in § 9 bewiesenen Lehrsatz und in § 16 verweise ich auf Betrachtungen, welche in § 8 durchgeführt sind; hierdurch wird aber das Verständniss selbst nicht beeinträchtigt. Um dasselbe noch mehr zu erleichtern, mögen einige weitere Bemerkungen gestattet sein.

*) Vergl. den unter gleichem Titel erschienenen Aufsatz in Bd. 31 dieser Annalen, p. 252 ff.

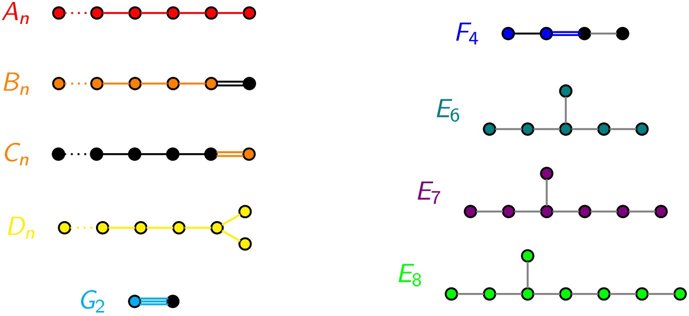
Mathematische Annalen. XXXIII. 1

KILLING 1887: Classification of f-d.simple Lie $\mathbb{C}$-alg.

Wilhelm Killing

"The greatest mathematical paper of all time"

- $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$
- A SURPRISE: of dimension 14, called $\mathfrak{g}_2$
- Four additional *exceptional* examples: $\mathfrak{f}_4$ (52), $\mathfrak{e}_6$ (78), $\mathfrak{e}_7$ (133), $\mathfrak{e}_8$ (248).



What is G2?

Killing found the structure constants of $\mathfrak{g}_2$, but this is not very useful!

	e_1	e_2	e_3	e_4	e_5	e_6	f_1	f_2	f_3	f_4	f_5	f_6	h_1	h_2
e_1	0	$2e_3$	e_4	0	0	$3e_2$	$-h_1$	$-f_6$	$-2f_2$	$-3f_3$	0	0	$-2e_1$	e_1
e_2	$-2e_3$	0	$-e_5$	0	0	0	e_6	$-h_2$	$2f_1$	0	$3f_3$	$-3e_1$	e_2	$-2e_2$
e_3	$-e_4$	e_5	0	0	0	0	$2e_2$	$-2e_1$	$-h_1-h_2$	$3f_1$	$-3f_2$	0	$-e_3$	$-e_3$
e_4	0	0	0	0	0	$-3e_5$	$3e_3$	0	$-3e_1$	$-6h_1-3h_2$	$3f_6$	0	$-3e_4$	0
e_5	0	0	0	0	0	0	0	$-3e_3$	$3e_2$	$-3e_6$	$-3h_1-6h_2$	$3e_4$	0	$-3e_5$
e_6	$-3e_2$	0	0	$3e_5$	0	0	0	$3f_1$	0	0	$-3f_4$	$3h_1-3h_2$	$3e_6$	$-3e_6$
f_1	h_1	$-e_6$	$-2e_2$	$-3e_3$	0	0	0	$2f_3$	f_4	0	0	$3f_2$	$2f_1$	$-f_1$
f_2	f_6	h_2	$2e_1$	0	$3e_3$	$-3f_1$	$-2f_3$	0	$-f_5$	0	0	0	$-f_2$	$2f_2$
f_3	$2f_2$	$-2f_1$	h_1+h_2	$3e_1$	$-3e_2$	0	$-f_4$	f_5	0	0	0	0	f_3	f_3
f_4	$3f_3$	0	$-3f_1$	$6h_1+3h_2$	$3e_6$	0	0	0	0	0	0	$-3f_5$	$3f_4$	0
f_5	0	$-3f_3$	$3f_2$	$-3f_6$	$3h_1+6h_2$	$3f_4$	0	0	0	0	0	0	0	$3f_5$
f_6	0	$3e_1$	0	0	$-3e_4$	$-3h_1+3h_2$	$-3f_2$	0	0	$3f_5$	0	0	$-3f_6$	$3f_6$
h_1	$2e_1$	$-e_2$	e_3	$3e_4$	0	$-3e_6$	$-2f_1$	f_2	$-f_3$	$-3f_4$	0	$3f_6$	0	0
h_2	$-e_1$	$2e_2$	e_3	0	$3e_5$	$3e_6$	f_1	$-2f_2$	$-f_3$	0	$-3f_5$	$-3f_6$	0	0

Even Killing was tired of proving the Jacobi identity for G_2!

What is G2?

First models as a local transformation group in $\mathbb{R}^n$ $\rightsquigarrow n = 5!$

Engel (1893) & Cartan (1893) G_2 is the stability group of the Pfaffian system:

$$\left.\begin{aligned} dx_3 + x_1 dx_2 - x_2 dx_1 &= 0 \\ dx_4 + x_3 dx_1 - x_1 dx_3 &= 0 \\ dx_5 + x_2 dx_3 - x_3 dx_2 &= 0 \end{aligned}\right\}$$

Cartan (1893) Symmetries of the solution space of $\left.\begin{aligned} f_{xx} &= \tfrac{4}{3}(f_{yy})^3 \\ f_{xy} &= (f_{yy})^2 \end{aligned}\right\}$

Elie Cartan

Friedrich Engel

The WHOLE 1893 Cartan's paper...

ANALYSE MATHÉMATIQUE. — *Sur la structure des groupes simples finis et continus.* Note de M. **Cartan**, présentée par M. Picard.

« Dans la résolution des équations algébriques, Galois détermine le nombre et la nature des équations auxiliaires au moyen de la structure

des groupes qui leur correspondent. D'une façon analogue, M. Lie a montré (*Ac. des Sc.*, Christiania, 1882) que, toutes les fois qu'un problème d'intégration dépend d'un *groupe fini et continu*, le nombre, le degré et la nature des équations auxiliaires sont déterminés par la structure de ce groupe. Les groupes de ces équations auxiliaires sont toujours, de même que chez Galois, des groupes *simples*.

» Les beaux travaux de MM. Picard et Vessiot ramènent la théorie des équations différentielles linéaires à l'étude de la structure d'un groupe.

» En conséquence, toutes les recherches sur la structure des groupes simples, continus ou discontinus, ont une extrême importance.

M. Lie a donné, entre autres (*Mat. Ann.*, B. XXV), une détermination complète de tous les groupes simples d'ordre r dont les plus grands sous-groupes sont d'ordre $r-1$, $r-2$ ou $r-3$. Il a de plus indiqué *quatre* grandes classes de groupes simples : le groupe projectif général à n variables, le groupe d'un complexe linéaire à $2n+1$ variables, et enfin le groupe projectif d'une surface du deuxième ordre à $2n$ et $2n+1$ variables.

» Dans une série de Notes parues dans les *Mathematische Annalen*, B. XXXI, XXXIII, XXXIV, XXXVI, M. Killing a publié des recherches étendues sur la structure des groupes. Il est arrivé, en particulier, au résultat extrêmement important que, à part les *quatre* grandes classes de groupes simples dont nous avons parlé plus haut, il n'y a que *cinq* groupes simples, qui ont respectivement 14, 52, 78, 133, 248 paramètres.

» Malheureusement, les considérations qui conduisent M. Killing à ces résultats manquent de rigueur. Il était, par suite, désirable de refaire ces recherches, d'indiquer ses théorèmes inexacts et de démontrer ses théorèmes justes. Je me permettrai d'indiquer rapidement les résultats auxquels je suis arrivé à cet égard.

» Dans la partie du Mémoire de M. Killing relative aux groupes simples se trouvent surtout deux lacunes importantes. En premier lieu, il ne considère que le cas où ce qu'il appelle l'*équation caractéristique* du groupe n'admet que des racines simples; il tente, il est vrai, de s'affranchir de cette restriction dans la troisième partie de son Mémoire, mais il s'appuie pour cela sur un théorème qu'il ne démontre que dans un cas particulier et qui, en général, est faux : à savoir que *si un groupe est son propre groupe dérivé, chaque transformation générale fait partie d'un sous-groupe formé d'autant de transformations échangeables entre elles, que l'équation caractéristique admet de racines identiquement nulles.* En second lieu, il ramène la détermination des groupes simples à la détermination de certains systèmes

de nombres entiers, mais il ne prouve pas du tout que toutes les racines de l'équation caractéristique ne dépendent que d'un seul de ces systèmes.

» Je suis parvenu à démontrer tous les résultats de M. Killing relatifs aux groupes simples. J'ai, de plus, déterminé complètement la structure des *cinq* groupes spéciaux cités plus haut (M. Killing en indique *deux* à 52 paramètres, mais ils sont identiques). J'ai trouvé, en particulier, pour le groupe à 14 paramètres, deux représentants dans un espace à cinq dimensions.

» Le premier est le plus grand groupe continu de transformations de contact de l'espace ordinaire, qui laisse invariant le système des deux équations aux dérivées partielles du deuxième ordre :

$$r = \tfrac{4}{3}t^3, \qquad s = t^2.$$

Ses fonctions caractéristiques sont

$$1,\quad x,\quad y,\quad p,\quad q,\quad z+xp,\quad 3z-yq,\quad 9zp-4q^3,\quad 3yp-4q^2,$$
$$y^2+4xq,\quad 3y^2p+12zq-8yq^2,\quad y^3-12xz+x^2p+xyq,$$
$$3yz-3xyp-y^2q+4xq^2,$$
$$36z^2-3(12xz+y^3)p-36yzq+12y^2q^2+16xq^3.$$

» Ce groupe laisse en même temps invariante une équation aux dérivées partielles du deuxième ordre, qui représente, dans l'espace (r, s, t), la développable dont le système défini plus haut représente l'arête de rebroussement.

» Le deuxième groupe est le plus grand groupe continu de l'espace à cinq dimensions qui laisse invariant le système des équations de Pfaff

$$dx_2 - x_4\,dx_1 = 0, \qquad dx_3 - x_2\,dx_1 = 0, \qquad dx_5 - x_4\,dx_2 = 0.$$

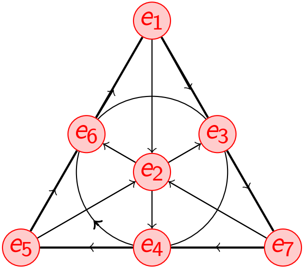
» Je demanderai à l'Académie la permission d'exposer, dans une prochaine Note, les résultats auxquels je suis arrivé, relativement à la structure des groupes en général. »

Hasta aqui fue la sesión 1, junto con algo de cuaternios.

Incluyo ahora un poquito de la sesión 3-4. El indice al final no se corresponde con el curso como fue

The supporting character, $\mathbb{O}$

The octonion $\mathbb{R}$-algebra is $\mathbb{O} = \sum_{i=0}^{7} \mathbb{R} e_i$, with $e_0 = 1_{\mathbb{O}}$ and



Fano plane

$$e_i e_{i+1} = e_{i+3} = -e_{i+1} e_i \pmod 7; \qquad e_i^2 = -1$$

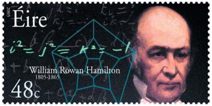
(each of the 7 lines behaves as $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ in $\mathbb{H}$)

A little bit of (well-known) history

Rotation of angle α in $\mathbb{R}^2 \leftrightarrow$ Multiplication by $e^{\mathbf{i}\alpha}$ in $\mathbb{C} = \mathbb{R} \oplus \mathbb{R}\mathbf{i}$.

$$\Rightarrow \mathrm{SO}(2) \cong \{z \in \mathbb{C} : |z| = 1\} = \mathbb{S}^1$$

Fascinated by the key role of $\mathbb{C}$ for describing rotations of the plane $\mathbb{R}^2$, W. Hamilton tried to find a multiplication in $\mathbb{R}^3$ for describing rotations of the space $\mathbb{R}^3$, no success!



Hamilton, 1843

Quaternion algebra:

$$\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$

Key properties of the Quaternion algebra

* Constructed by doubling $\mathbb{C}$: $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}\mathbf{j}$ with

$$(z_1 + z_2\mathbf{j})(w_1 + w_2\mathbf{j}) = (z_1 w_1 - \bar{w}_2 z_2) + (w_2 z_1 + z_2 \bar{w}_1)\mathbf{j}.$$

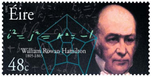
* Conjugation of $q = a_0 1 + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ is $\bar{q} = a_0 1 - a_1\mathbf{i} - a_2\mathbf{j} - a_3\mathbf{k}$.
* Trace is $q + \bar{q} = 2a_0 \in \mathbb{R}$ and norm is

$$n(q) = |q|^2 = q\bar{q} = a_0^2 + a_1^2 + a_2^2 + a_3^2,$$

  which satisfies $n(q_1 q_2) = n(q_1)n(q_2)$.
* So $q^{-1} = \frac{\bar{q}}{n(q)}$, it is a noncommutative division algebra!
* Relationship with cross-product: if $u, v \in \mathbb{H}_0 = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle \equiv \mathbb{R}^3$,

$$uv = -u \cdot v + u \times v$$

From Quaternions to Octonions (1843-1845)

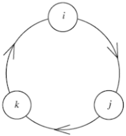


Hamilton, 1843

Quaternion algebra:

$$\mathbb{H} = \mathbb{R}1 \oplus \mathbb{R}\mathbf{i} \oplus \mathbb{R}\mathbf{j} \oplus \mathbb{R}\mathbf{k}$$

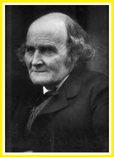
$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$



Letter from John Graves to Hamilton (1843):

There is still something in the system which gravels me. I have not yet any clear views as to the extent to which are at liberty to create imaginaries, and to endow them with supernatural properties. If with your alchemy you can make three pounds of gold, why should you stop there?

Graves, 1843



Cayley, 1845

Doubling again we get the octonions:

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell \quad \text{with}$$

$$(p_1 + p_2\ell)(q_1 + q_2\ell) = (p_1 q_1 - \bar{q}_2 p_2) + (q_2 p_1 + p_2 \bar{q}_1)\ell$$

(Cayley-Dickson doubling process)

Key properties of the Octonion algebra $\mathbb{O}$

- Constructed by doubling $\mathbb{H}$:

$$\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\} \leftrightarrow \{1, \mathbf{i}, \mathbf{j}, \ell, \mathbf{k}, \mathbf{j}\ell, -\mathbf{k}\ell, \mathbf{i}\ell\}$$

- It is noncommutative and NONASSOCIATIVE: $(\mathbf{ij})\ell = \mathbf{k}\ell \neq -\mathbf{k}\ell = \mathbf{i}(\mathbf{j}\ell)$
- But alternative: $(xx)y = x(xy)$...
- Conjugation of $x = a_0 1 + \sum_{i=1}^{7} a_i e_i$ is $\bar{x} := a_0 1 - \sum_{i=1}^{7} a_i e_i$.
- Trace is $t(x) := x + \bar{x} = 2a_0 \equiv 2\mathrm{Re}(x) \in \mathbb{R}$ and norm is

$$n(x) = \|x\|^2 = x\bar{x} = a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2,$$

  which satisfies $n(x_1 x_2) = n(x_1) n(x_2)$.
- It is a division algebra!: $x^{-1} = \frac{\bar{x}}{n(x)}$
- Relationship with cross-product in $\mathbb{R}^7$: if $u, v \in \mathbb{O}_0 \equiv \mathbb{R}^7 \equiv \mathrm{Im}(\mathbb{O})$,

$$uv = \underbrace{-u \cdot v}_{\in \mathbb{R}1} + \underbrace{u \times v}_{\in \mathbb{O}_0}$$

  which satisfies Lagrange identity: $(u \times v) \times v = (u \cdot v)v - (v \cdot v)u$.

Derivation algebras are Lie algebras

E. Cartan (1914):

$$\mathfrak{der}(\mathbb{O}) = \mathfrak{g}_2$$

$*$ If A associative algebra $\Rightarrow A^{-} := (A, [\ ,\ ])$ is a Lie algebra
$[a, b] := ab - ba$

Example. If V is a $\mathbb{F}$-vector space $\Rightarrow \mathrm{End}_{\mathbb{F}} V$ associative algebra

$\Rightarrow \mathfrak{gl}(V) := (\mathrm{End}_{\mathbb{F}} V)^{-}$ is a Lie algebra called *general linear algebra*

(the tangent Lie algebra to the general linear group $\mathrm{GL}(V)$)

$*$ If A alg. $\Rightarrow \mathfrak{der}(A) = \{d \colon A \to A : d(xy) = d(x)y + xd(y)\} \leq \mathfrak{gl}(A)$

Proof: $[d, d'] \in \mathfrak{der}(A)$:

$[d, d'](xy) = (dd' - d'd)(xy) = d(d'(x)y + xd'(y)) - d'(d(x)y + xd(y)) =$

$dd'(x)y + \cancel{d'(x)d(y)} + \cancel{d(x)d'(y)} + xdd'(y) - d'd(x)y - \cancel{d(x)d'(y)} - \cancel{d'(x)d(y)} - xd'd(y)$

$= [d, d'](x)y + x[d, d'](y)$

How to find concrete derivations

For any algebra A,

- $L_x : A \to A, y \mapsto xy$ left multiplication operator,
- $R_x : A \to A, y \mapsto yx$ right multiplication operator

If $d : A \mapsto A$,

$$d \in \mathfrak{der}(A) \iff [d, L_x] = L_{d(x)} \ \forall x \in A$$

Examples.

- If $L = A^-$ is Lie, $\mathrm{ad}(x) = L_x - R_x \in \mathfrak{der}(L)$,

  ($\mathrm{ad}(x)$ always a derivation on L Lie)

- If J is Jordan, $[L_x, L_y] \in \mathfrak{der}(J)$
- If A is alternative,

$$\boxed{D_{x,y} := [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \in \mathfrak{der}(A)}$$

$$D_{x,y}(z) = [[x, y], z] - 3(x, y, z)$$

$$\mathfrak{g}_2 \cong \mathfrak{der}(\mathbb{O}) = \{\textstyle\sum_i D_{x_i,y_i} : x_i, y_i \in \mathbb{O}\}$$

The Lie algebra $\mathfrak{der}(\mathbb{O})$ - I Remarkable elements.

A alg, $L_x\colon A \to A, y \mapsto xy$, $R_x\colon A \to A, y \mapsto yx$.

$d\colon A \mapsto A$ derivation $\Leftrightarrow [d, L_x] = L_{d(x)}\ \forall x \in A \Leftrightarrow [d, R_x] = R_{d(x)}\ \forall x \in A$

Example.

- If L is Lie, $R_x - L_x \in \mathfrak{der}(L)$,
- If J is Jordan, $[R_x, R_y] \in \mathfrak{der}(J)$

Lemma 1. A alternative $\Leftrightarrow$
$R_{xz} - R_z R_x = R_x R_z - R_{zx} = L_{zx} - L_z L_x = L_x L_z - L_{xz} = [R_z, L_x] = [L_z, R_x]$.
Moreover,

a) $R_x R_z + R_z R_x = R_{xz+zx}$

b) $[R_x, R_z] = -R_{[x,z]} - 2[L_x, R_z]$

c) $[L_x, L_z] = L_{[x,z]} - 2[L_x, R_z]$

d) $[R_y, [R_x, R_z]] = R_{[y,[x,z]]-2(x,y,z)}$

e) $D_{x,y} := [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \in \mathfrak{der}(A)$ and acts

$$D_{x,y}(z) = [[x, y], z] - 3(x, y, z)$$

THE LIE ALGEBRA $\mathfrak{der}(\mathbb{O})$ - II Basis and dimension.

- $\dim \mathfrak{g}_2 \leq \dim \mathfrak{gl}(\mathbb{O}) = 64$
- If $d \in \mathfrak{der}(\mathbb{O})$, then $d(1) = 0$, $d(\mathbb{O}_0) \subset \mathbb{O}_0$. $\dim \mathfrak{g}_2 \leq \dim \mathfrak{gl}(\mathbb{O}_0) = 49$
- If $d \in \mathfrak{der}(\mathbb{O})$, then $d|_{\mathbb{O}_0} \in \mathfrak{so}(\mathbb{O}_0, n)$. $\dim \mathfrak{g}_2 \leq \dim \mathfrak{so}(\mathbb{O}_0, n) = 21$
- $\mathrm{ad}(\mathbb{O}_0) \subset \mathfrak{so}(\mathbb{O}_0, n)$ and $\mathrm{ad}(\mathbb{O}_0) \cap \mathfrak{der}(\mathbb{O}) = 0$. $\dim \mathfrak{g}_2 \leq 21 - 7 = 14$
- There are 14 linearly independent elements in $\mathcal{D} = \{\sum_i D_{x_i, y_i} : x_i, y_i \in \mathbb{O}\} \leq \mathfrak{der}(\mathbb{O})$. $\dim \mathfrak{g}_2 = 14$ and $\mathcal{D} = \mathfrak{der}(\mathbb{O})$

Proof. ($D_{x,y} = -D_{y,x}$)

$$\left\{ \begin{array}{ccccccc} D_{i,j} & D_{i,\ell} & D_{i,k} & D_{i,i\ell} & D_{\ell,k} & D_{i,k\ell} & D_{j,k} \\ D_{\ell,k\ell} & D_{j,k\ell} & D_{\ell,j\ell} & D_{j,j\ell} & D_{i,j\ell} & D_{j,\ell} & D_{\ell,i\ell} \end{array} \right\}$$

Exercises

1. An algebra A is said G-graded if $A = \oplus_{g\in G} A_g$ vector subspaces s.t. $A_g A_h \subset A_{g+h}$.
   - In that case $E = \mathfrak{gl}(A)$ is G-graded, for $E_g = \{f \in E : f(A_h) \subset A_{g+h}\}$
   - And also $D = \mathfrak{der}(A)$ is G-graded, for $D_g = D \cap E_g$.
   - Check that $\mathbb{O}$ has a $\mathbb{Z}_2^3$-grading, determined by

$$\mathbb{O}_{(\bar{1},\bar{0},\bar{0})} = \mathbb{R}i, \quad \mathbb{O}_{(\bar{0},\bar{1},\bar{0})} = \mathbb{R}j, \quad \mathbb{O}_{(\bar{0},\bar{0},\bar{1})} = \mathbb{R}\ell.$$

   - Take the $\mathbb{Z}_2^3$-grading induced in $D = \mathfrak{der}(\mathbb{O})$, and check that, for $x \in (\mathbb{O}_0)_g, y \in (\mathbb{O}_0)_h$, then $D_{x,y} \in (\mathfrak{der}(\mathbb{O}))_{g+h}$
   - Check that the elements $D_{x,y}$ in the 7 columns in the previous slide belong to different homogeneous components.
   - So, the linear independence of the fourteen elements will be clear when checking that 2 elements in the same column are linearly independent. All the columns are undistinguishable. For instance, it is enough to check

$$D_{i,i\ell}(i) = 4i\ell, \quad D_{i,i\ell}(\ell) = 2j\ell, \quad D_{j,j\ell}(i) = 2i\ell, \quad D_{j,j\ell}(\ell) = 4j\ell$$

More exercises

2.

- If A is an alternative algebra, and $d \in \mathfrak{Der}(A)$, then $[d, D_{x,y}] = D_{d(x),y} + D_{x,d(y)}$.
- Hence the map $\mathbb{O}_0 \times \mathbb{O}_0 \to \mathfrak{Der}(\mathbb{O})$; $(x, y) \mapsto D_{x,y}$ is $\mathfrak{Der}(\mathbb{O})$-invariant. (For teachers, prove also that it is the only one up to isomorphism.)

3. If $(V, n, \times)$ is a cross product (field $\mathbb{F}$), prove that $A := \mathbb{F} \oplus V$,

- with product: 1 is the unit and $uv = -n(u, v)1 + u \times v$ if $u, v \in A$,
- and norm $\tilde{n}\colon A \to \mathbb{R}$: $\tilde{n}(1) = 1$, $\tilde{n}|_V = n$,

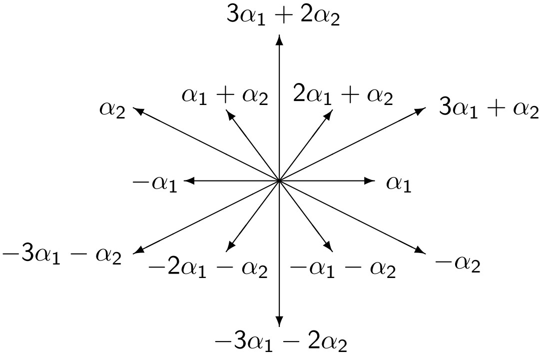
is a composition algebra (i.e. $\tilde{n}$ is multiplicative).

4. Prove that $\mathfrak{Der}(\mathbb{O}) \to \mathfrak{Der}(\mathbb{O}_0, \times)$, $d \mapsto d|_{\mathbb{O}_0}$, is a monomorphism of Lie algebras (in fact, it is an isomorphism).

5. Prove that
$\mathfrak{Der}(\mathbb{O}) \to \{f : \mathbb{O}_0 \to \mathbb{O}_0 \text{ lin} : n(f(u), v, w) + n(u, f(v), w) + n(u, v, f(w)) = 0\}$,
$d \mapsto d|_{\mathbb{O}_0}$, is a monomorphism of Lie algebras (in fact, it is an isomorphism).

From derivations to roots?

$\mathfrak{der}(\mathbb{O}^{\mathbb{C}}) = \mathfrak{g}_2^{\mathbb{C}}$ has to admit a root decomposition...



$\Phi_{G_2} = \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$

... but take care: $\mathfrak{der}(\mathbb{O})$ has not!

Its cousin: Split octonion algebra $\mathbb{O}_s$

Recall that we apply a doubling process $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}\ell$ with

$$(p_1 + p_2\ell)(q_1 + q_2\ell) = (p_1 q_1 - \bar{q}_2 p_2) + (q_2 p_1 + p_2 \bar{q}_1)\ell,$$

and $n(p_1 + p_2\ell) = n(p_1) + n(p_2)$ positive definite multiplicative norm.

And if we ask for $\ell^2 = 1$ instead of $\ell^2 = -1$?

Take $\mathbb{O}_s = \mathbb{H} \oplus \mathbb{H}\ell$ with

$$\boxed{(p_1 + p_2\ell)(q_1 + q_2\ell) = (p_1 q_1 + q_2 \bar{p}_2) + (q_2 p_1 + p_2 \bar{q}_1)\ell,}$$

Then the norm $\boxed{n(p_1 + p_2\ell) = n(p_1) - n(p_2)}$ is also multiplicative but no longer positive definite (with isotropic vectors = split).

Extended to $\mathbb{C}$-algebras, there is no difference between $\mathbb{O}$ and $\mathbb{O}_s$:

$$\mathbb{O}^{\mathbb{C}} \cong \mathbb{O}_s^{\mathbb{C}} \quad \text{(complex octonion algebra)}$$

basis of Split Octonions for roots

The basis of isotropic vectors

$$\mathcal{B}_{\text{good}} = \{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\} \text{ of } \mathbb{O}_s:$$

$$\begin{array}{lll} e_1 := \frac{1}{2}(1+\ell) & v_1 := \frac{1}{2}\mathbf{i}(1+\ell) & u_1 := \frac{1}{2}\mathbf{i}(1-\ell) \\ e_2 := \frac{1}{2}(1-\ell) & v_2 := \frac{1}{2}\mathbf{j}(1+\ell) & u_2 := \frac{1}{2}\mathbf{j}(1-\ell) \\ & v_3 := \frac{1}{2}\mathbf{k}(1+\ell) & u_3 := \frac{1}{2}\mathbf{k}(1-\ell) \end{array}$$

has all their products are 0 except for:

$$\begin{array}{llll} e_i^2 = e_i & e_1 u_i = u_i e_2 = u_i & u_i v_i = -e_1 & u_i u_j = \varepsilon_{ijk} v_k \\ & e_2 v_i = v_i e_1 = v_i & v_i u_i = -e_2 & v_i v_j = \varepsilon_{ijk} u_k \end{array}$$

and the matrix $[n]_{\mathcal{B}_{\text{good}}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 \\ 0 & 0 & I_3 & 0 \end{pmatrix}$

Zorn matrices

Zorn matrices are isomorphic to split octonions

$$\mathbb{O}_s \cong \left\{ \begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} : \alpha, \beta \in \mathbb{R}; u, v \in \mathbb{R}^3 \right\}$$

with product:

$$\begin{pmatrix} \alpha & u \\ v & \beta \end{pmatrix} \begin{pmatrix} \alpha' & u' \\ v' & \beta' \end{pmatrix} = \begin{pmatrix} \alpha\alpha' + u \cdot v' & \alpha u' + \beta' u - v \times v' \\ \alpha' v + \beta v' + u \times u' & \beta\beta' + v \cdot u' \end{pmatrix}$$

Proof:

$$e_1 \mapsto \begin{pmatrix} 1 & (0,0,0) \\ (0,0,0) & 0 \end{pmatrix} \quad e_2 \mapsto \begin{pmatrix} 0 & (0,0,0) \\ (0,0,0) & 1 \end{pmatrix}$$

$$v_1 \mapsto \begin{pmatrix} 0 & (0,0,0) \\ (1,0,0) & 0 \end{pmatrix} \quad u_1 \mapsto -\begin{pmatrix} 0 & (1,0,0) \\ (0,0,0) & 0 \end{pmatrix}$$

$$v_2 \mapsto \begin{pmatrix} 0 & (0,0,0) \\ (0,1,0) & 0 \end{pmatrix} \quad u_2 \mapsto -\begin{pmatrix} 0 & (0,1,0) \\ (0,0,0) & 0 \end{pmatrix}$$

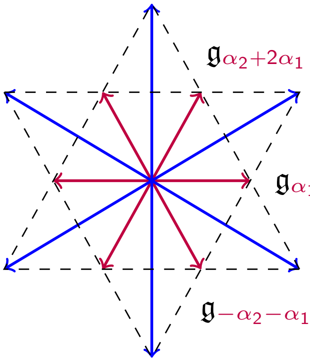
$$v_3 \mapsto \begin{pmatrix} 0 & (0,0,0) \\ (0,0,1) & 0 \end{pmatrix} \quad u_3 \mapsto -\begin{pmatrix} 0 & (0,0,1) \\ (0,0,0) & 0 \end{pmatrix}$$

Root decomposition of $\mathfrak{der}(\mathbb{O}_s) = \{\sum_i D_{x_i,y_i} : x_i, y_i \in \mathbb{O}_s\}$

$\mathfrak{h} = \langle D_{u_1,v_1}, D_{u_2,v_2}, D_{u_3,v_3}\rangle$ Cartan subalgebra and

the simultaneous diagonalization is:

$\mathfrak{g}_{2\alpha_2+3\alpha_1} = \langle D_{u_2,v_3}\rangle$



$\mathfrak{g}_{\alpha_2+2\alpha_1} = \langle D_{u_1,u_2}\rangle$

$\mathfrak{g}_{\alpha_2+3\alpha_1} = \langle D_{u_1,v_3}\rangle$

$\mathfrak{g}_{\alpha_1} = \langle D_{v_2,v_3}\rangle$

$\mathfrak{g}_{-\alpha_2} = \langle D_{u_1,v_2}\rangle$

$\mathfrak{g}_{-\alpha_2-\alpha_1} = \langle D_{u_1,u_3}\rangle$

$\mathfrak{g}_{-\alpha_1} = \langle D_{u_2,u_3}\rangle$ $\mathfrak{g}_{-\alpha_2-2\alpha_1} = \langle D_{v_1,v_2}\rangle$ $\mathfrak{g}_{\alpha_2+\alpha_1} = \langle D_{v_1,v_3}\rangle$

$\mathfrak{g}_{\alpha_2} = \langle D_{v_1,u_2}\rangle$ $\mathfrak{g}_{-\alpha_2-3\alpha_1} = \langle D_{v_1,u_3}\rangle$ $\mathfrak{g}_{-2\alpha_2-3\alpha_1} = \langle D_{v_2,u_3}\rangle$