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# On a control problem governed by a linear partial differential equation with a smooth functional

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## SUMMARY

In this paper, the optimal control problem for the Helmholtz equation with non-local boundary conditions is considered. The necessary and sufficient conditions of optimality in a maximum principle form have been obtained. We note that this problem is basically different from classical type problems because it is impossible to use the Green's formula and we cannot rewrite it in the variational form widely used in the literature. So it is impossible to use all the theory that has been developed for optimal control problems with classical boundary conditions.

KEY WORDS: Helmholtz equation, boundary value problems, optimality for control problems

## 1. Introduction

The first basic study of non-local boundary problem arises in connection with the mathematical modelling of plasma physics processes [1], where non-local problems were posed and analysed for a certain class of elliptic functions. In [2, 3] several results concerning the investigation of non-local boundary problems of the Bitsadze-Samarski type and its generalisations for some equations of mathematical physics, mainly elasticity and shell theory are presented.

Iterative methods were also suggested for solving such problems in the case of quite general elliptic equations.

The approximate solution of various optimisation problems takes an important place in the investigation of control processes with distributed parameters [4, 5, 6]. Part of these works are based on the use of nets. One of the actual problems of the nets method theory is to establish the coordinated estimations, where the order of the convergence is agreed with the smoothness of the differential problem solution. The investigation of the difference schemes

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*Correspondence to: [1] Department of Applied Physics. Polytechnic School. Campus El Ejido, s/n (29013). Malaga University. Spain. Fax/Phone +34952132013. fcaldeanuevactima.uma.es convergence for elliptic equations in the space of distribution solutions have been done in the works [7, 8, 9, 10].

The analysis of non-local boundary problems is motivated by theoretical interest as well as practical needs, and numerous interesting papers deals with these issues [11, 12, 13, 14, 15].

This paper scrutinises an unconstrained (i.e., without control constraints) optimal control problem governed by a two-dimensional linear PDEs with a smooth cost functional. The results of this paper may be of great interest for physical applications since control constraints are usually a reflection of physical limitations as well as to impose severe complications on the theoretical derivation of necessary and sufficient conditions.

As to analytical framework of this paper, it is assumed that each control function is a distribution defined on an open set $D$ and ranging in an open interval $V$ contained in $I\!R$. Hence the set of control function $V_{ad}$ is not complete, in any reasonable topology. This is not compatible with conventional existence theories of the optimal control. Other peculiarities of this paper is in the non-local boundary conditions for the PDEs involved, since by non-local boundary condition some general integral operator involved on the boundary is usually considered and the adjoint equation is an elliptic equation with a non-local transmission, this is to say, the solving of the adjoint system does not require the boundary condition, as well the non-convexity of “performance function” $F(x,y,u,v)$, which minimise a performance criterion assessed by an integral under certain conditions and is a given real function of the control variables and the corresponding solutions (state functions) to the PDE's governing the control problem. Therefore, these conditions make the topic slightly distinct to the existing literature on the control of elliptic PDEs.

## 2. Statement of the problem and main results

Let $\bar{D}$ be a rectangle, $\bar{D} = [0,l_1] \times [0,l_2]$, $\partial D$ the boundary of the rectangular domain, $\gamma = \{(l_1,y) : 0 \le y \le l_2\}$, $\gamma_0 = \{(\overset{\circ}{x},y) : 0 \le y \le l_2\}$, $\overset{\circ}{x}$ the fixed point of the interval $]0,l_1[$, $V_{ad}$ the set of control functions $V_{ad} = \{v : D \to V, v \in L_2(D)\}$, $V$ being an open interval contained in $I\!R$.

By $V_{ad} = L_2(\{D\},V)$ we denote the space of control functions $v : D \to V$ such that

$$\|v\|^2 = \iint_D |v(x,y)|^2 \, dx \, dy < \infty$$

We identify each function $v \in L_2(\{D\},V)$ with a distribution defined on $D$ and ranging in $V$.

Let us consider the non-local boundary problem for Helmholtz's equation for each fixed $v \in V_{ad}$ in the domain $D$ [1]:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(x,y)u = a(x,y)v + b(x,y), \qquad (x,y) \in D, \tag{1}$$

$$\begin{aligned} u(x,y) &= 0, && (x,y) \in \partial D \backslash \gamma, \\ u(l_1,y) &= \sigma u(\overset{\circ}{x},y), && 0 \le y \le l_2 \end{aligned} \tag{2}$$

where $a \in L_\infty(D)$, $b \in L_2(D)$, $0 \le q \in L_\infty(D)$, $0 < \sigma < 1$, $\sigma = \text{const}$.

As in [16], it can been shown that the solution of problem (1)-(2) exists, is unique and belongs to Sobolev's space $H^2(D)$ [17].

Let us consider the following functional:

$$I(v) = \iint_D F(x, y, u, v)\, dx\, dy \tag{3}$$

where $F : D \times I\!R^2 \to I\!R$, $u$ and $v$ being functions defined on $D$. We assume that function $F$ has continuous derivatives $F'_u$ and $F'_v$ for all $(x, y) \in D$ and is to be measurable on $D$ for all $(u, v) \in I\!R^2$.

Let us formulate the following control problem: finding the function $v_0 \in V_{ad}$, whose corresponding solution $u_0$ to the boundary value problem (1)-(2), together with $v_0$, results in minimal functional value (3).

Let us assume that a solution to the optimal control problem exists and investigate the necessary and sufficient conditions of optimality.

To obtain the necessary condition of optimality we follow the following scheme developed next.

Let us consider an arbitrary permissible control, $v_\epsilon \in V_{ad}$ and $u_\epsilon$ the corresponding solution to problem (1)-(2). Let us introduce the notation $\delta v = v_\epsilon - v_0$, $\delta u = u_\epsilon - u_0$; where $(u_0, v_0)$ is assumed to be an optimal pair. From here one obtains the following problem:

$$\begin{aligned} \frac{\partial^2 \delta u}{\partial x^2} + \frac{\partial^2 \delta u}{\partial y^2} - q(x,y)\delta u = a(x,y)\delta v, &\qquad (x,y) \in D \\ \delta u(x,y) = 0, &\qquad (x,y) \in \partial D \backslash \gamma, \\ \delta u(l_1, y) = \sigma \delta u(\overset{\circ}{x}, y), &\qquad 0 \le y \le l_2 \end{aligned} \tag{4}$$

Let function $\psi \neq 0$, $\psi \in H^2(D\backslash\gamma_0) \cap H^1(D)$ [17]. By multiplying equation (4) by $\psi$ and integrating over domain $D$, one gets:

$$\iint_D \psi(x,y) \left[ \frac{\partial^2 \delta u}{\partial x^2} + \frac{\partial^2 \delta u}{\partial y^2} - q(x,y)\delta u \right] dx\, dy = \iint_D a(x,y)\psi(x,y)\, \delta v\, dx\, dy \tag{5}$$

The increment of functional (3) with fixed $v_0$, $v_\epsilon$ is

$$\begin{aligned} \delta I = I(v_\epsilon) - I(v_0) = \iint_D [F(x,y,u_\epsilon,v_\epsilon) - F(x,y,u_0,v_0)]\, dx\, dy = \\ = \iint_D \left[ \frac{\partial F(\eta_1)}{\partial u} \delta u + \frac{\partial F(\eta_2)}{\partial v} \delta v \right] dx\, dy \end{aligned} \tag{6}$$

where $\eta_1 = (x, y, u_\epsilon + \theta_1 \delta u, v_\epsilon)$, $0 \le \theta_1 \le 1$, $\eta_2 = (x, y, u_0, v_\epsilon + \theta_2 \delta v)$, $0 \le \theta_2 \le 1$.

Taking into account (5) and (6), one obtains the following expression:

$$\begin{aligned} \delta I = \iint_D \psi(x,y) \left[ \frac{\partial^2 \delta u}{\partial x^2} + \frac{\partial^2 \delta u}{\partial y^2} - q(x,y)\delta u \right] dx\, dy - \\ - \iint_D a(x,y)\psi(x,y)\delta v\, dx\, dy + \iint_D \left[ \frac{\partial F}{\partial u}(\eta_1)\delta u + \frac{\partial F}{\partial v}(\eta_2)\delta v \right] dx\, dy \end{aligned} \tag{7}$$

To obtain the adjoint equation, let us make the following transformations:

$$\int_0^{l_1} \int_0^{l_2} \psi(x,y) \frac{\partial^2 \delta u}{\partial x^2}\, dx\, dy = \int_0^{l_2} \left( \int_0^{\overset{\circ}{x}^-} \psi(x,y) \frac{\partial^2 \delta u}{\partial x^2}\, dx + \int_{\overset{\circ}{x}^+}^{l_1} \psi(x,y) \frac{\partial^2 \delta u}{\partial x^2}\, dx \right) dy$$

Integrating partially the above equation one obtains:

$$\int_0^{l_2} \Big( \psi(l_1,y)\delta u_x(l_1,y) - \psi(0,y)\delta u_x(0,y) + (\psi(\overset{\circ}{x}^{-},y) - \psi(\overset{\circ}{x}^{+},y))\delta u_x(\overset{\circ}{x},y) +$$

$$+ (\psi_x(\overset{\circ}{x}^{+},y) - \psi_x(\overset{\circ}{x}^{-},y) - \sigma\psi_x(l_1,y))\delta u_x(\overset{\circ}{x},y) + \int_0^{l_1} \frac{\partial^2 \psi}{\partial x^2}\delta u(x,y)\,dx \Big)\,dy \qquad (8)$$

In a similar way one obtains

$$\int_0^{l_1}\int_0^{l_2} \psi(x,y)\frac{\partial^2 \delta u}{\partial y^2}\,dx\,dy =$$

$$= \int_0^{l_1}\left[\psi(x,l_2)\delta u_y(x,l_2) - \psi(x,0)\delta u_y(x,0) + \int_0^{l_2}\frac{\partial^2\psi}{\partial y^2}\delta u(x,y)\,dy\right]dx \qquad (9)$$

therefore, by grouping terms, the increment of the functional in (7) can be written in the following way:

$$\delta I = \iint_D \psi(x,y)\left[\frac{\partial^2 \delta u}{\partial x^2} + \frac{\partial^2 \delta u}{\partial y^2} - q(x,y)\delta u\right] -$$

$$- \iint_D a(x,y)\psi(x,y)\delta v\,dx\,dy + \iint_D \left[\frac{\partial F}{\partial u}(\eta_1)\delta u + \frac{\partial F}{\partial v}(\eta_2)\delta v\right]dx\,dy =$$

$$= \int_0^{l_2}\Big[\psi(l_1,y)\delta u_x(l_1,y) - \psi(0,y)\delta u_x(0,y) + \left(\psi(\overset{\circ}{x}^{-},y) - \psi(\overset{\circ}{x}^{+},y)\right)\,\delta u_x(\overset{\circ}{x},y) +$$

$$+ \left(\psi_x(\overset{\circ}{x}^{-},y) - \psi_x(\overset{\circ}{x}^{+},y) - \sigma\psi_x(l_1,y)\right)\,\delta u(\overset{\circ}{x},y) + \int_0^{l_1}\frac{\partial^2\psi}{\partial x^2}\delta u(x,y)\,dx\Big]\,dy +$$

$$+ \int_0^{l_1}\left[\psi(x,l_2)\delta u_y(x,l_2) - \psi(x,0)\delta u_y(x,0) + \int_0^{l_2}\frac{\partial^2\psi}{\partial y^2}\delta u(x,y)\,dy\right]dx +$$

$$+ \iint_D\left[\frac{\partial F}{\partial u}(\eta_1) - q(x,y)\psi\right]\delta u\,dx\,dy + \iint_D\left[\frac{\partial F}{\partial v}(\eta_2) - a(x,y)\psi(x,y)\right]\delta v\,dx\,dy \qquad (10)$$

As function $F$ is continuously differentiable with respect to $u, v$ then for $\delta u, \delta v \to 0$ one gets:

$$\left\|\frac{\partial F}{\partial u}(\eta_1) - \frac{\partial F}{\partial u}(x,y,u_0,v_0)\right\|_{L_2(D)} \to 0$$

$$\left\|\frac{\partial F}{\partial v}(\eta_2) - \frac{\partial F}{\partial v}(x,y,u_0,v_0)\right\|_{L_2(D)} \to 0$$

On the other hand, since $\psi \in H^2(D\backslash\gamma_0)\cap H^1(D)$, and taking into account the embedding Theorem [18] one has that $\psi(\overset{\circ}{x},y) \in C$. Therefore, proceeding from (10), one can conclude that, if $\psi$ is the solution to problem

$$\begin{aligned} \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - q(x,y)\psi &= -\frac{\partial F}{\partial u}(x,y,u_0,v_0), && (x,y)\in D\backslash\gamma_0, \\ \psi(x,y) &= 0 && (x,y)\in\partial D \\ \psi_x(\overset{\circ}{x}^{+},y) - \psi_x(\overset{\circ}{x}^{-},y) &= \sigma\psi_x(l_1,y), && 0\le y\le l_2 \end{aligned} \qquad (11)$$

then the increment of the functional $\delta I$ will take the form

$$\delta I = \iint_D \left[\frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi\right] \delta v \, dx \, dy \tag{12}$$

**Theorem 1.** *Let $\psi_0$ be a solution of the adjoint problem (11). Then the necessary condition for $(u_0, v_0)$ to be optimal is that the following relation is true almost everywhere*[†] *on $D$:*

$$\frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 = 0 \tag{13}$$

*Proof*

Let $(u_0, v_0)$ be the optimal pair. Let us show that condition (13) holds and then let us assume the contrary.

Let us suppose that $\frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 \neq 0$ on a set of positive Lebesgue's measure. Consequently,

$$\begin{aligned} 0 < \mu \left[\left\{(x, y) \in D | \frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 \neq 0\right\}\right] = \\ = \mu \left[\left\{(x, y) \in D \left| \frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 > 0 \right.\right\} \cup \right. \\ \left. \left\{(x, y) \in D \, | \frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 < 0\right\}\right] = \\ = \mu \left[\left\{(x, y) \in D \left| \frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 > 0 \right.\right\}\right] + \\ + \mu \left[\left\{(x, y) \in D \left| \frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 < 0 \right.\right\}\right] \end{aligned}$$

where $\mu$ is Lebesgue's measure on $D$.

Let us introduce the notation

$$D_+ = \left\{(x, y) \in D \left| \frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 > 0 \right.\right\}$$
$$D_- = \left\{(x, y) \in D \left| \frac{\partial F}{\partial v}(x, y, u_0, v_0) - a(x, y)\psi_0 < 0 \right.\right\}$$

According to our assumption: $\mu(D_+) + \mu(D_-) > 0$.

Let us consider two cases: $\mu(D_+) > 0$ and $\mu(D_-) > 0$.

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[†] We consider that two control functions $v$, $w$ are equal almost everywhere if

$$\{(x, y) \in D : v(x, y) \neq w(x, y)\}$$

is a null set.

According to Lebesgue's theory, if $v$ and $w$ are equal almost everywhere then integral of $|v - w|^2$ is zero and we must regard $v$ and $w$ as equal. Strictly speaking we should define the elements in $L^2(D)$ to be not functions but equivalent classes of function almost everywhere. Conventionally, however, one speaks as if the elements were functions, with equality interpreted as equality almost everywhere.

Let $\mu(D_+) > 0$ be. As $V$ is open, then there exists $k_0 > 0$ so that $v_\epsilon(x,y) = v_0(x,y) + k\xi_{D_+} \in V$ with $|k| \leq k_0$, where $\xi_{D_+}$ denotes the characteristic function of the set $D_+$.

Let us denote

$$T = \frac{\partial F}{\partial v}(x,y,u_0,v_0) - a(x,y)\psi_0$$

then there exists $\epsilon > 0$ so that $\iint_D T\,dx\,dy > \epsilon$.

Taking into account that $\delta v = k\xi_{D_+}$ and formula (12), one gets:

$$\iint_D \left[\frac{\partial F}{\partial v}(x,y,u_0,v_0) - a(x,y)\psi_0\right] \delta v\,dx\,dy = \iint_D Tk\xi_{D_+}\,dx\,dy = k\iint_{D_+} T\,dx\,dy \qquad (14)$$

Since $v_0$ is optimal, then for a sufficiently small increment $\delta v$

$$I(v_\epsilon) - I(v_0) = I(v_0 + \delta v) - I(v_0) \geq 0.$$

On the other hand, for $\delta v = k\xi_{D_+}$ and taking into account (14) we can choose $\delta > 0$ so that for $-\delta \leq k < 0$ we obtain

$$k\iint_{D_+} T\,dx\,dy < 0$$

that is for $-\delta \leq k < 0$ one obtains $I(v_0 + k\xi_{D_+}) - I(v_0) < 0$, this contradicts the optimality of $v_0$. Hence $\mu(D_+) = 0$.

$\mu(D_-) = 0$ can be proved in a similar way.

Thus, the necessary condition of optimality is proved. □

The upshot of optimal control problems is that under obtain conditions a problem of optimal control must satisfy the Maximum Principle of Pontrjagin [19].

Let us assume

1. $V$ is an open interval as it has previously been supposed.
2. $\frac{\partial F}{\partial v}(x,y,u_0(x,y),v_0(x,y)) > 0$, $\forall x,y \in D$.

Returning to the increment of the function $\delta I$ obtained in (10) and supposing that $\psi$ is the solution to the adjoin problem (11), then the increment of the function $\delta I$ can be written in the following way:

$$\delta I = \iint_D \left[\frac{\partial F}{\partial v}(\eta_2) - a(x,y)\psi(x,y)\right] \delta v\,dx\,dy \qquad 0 \leq \theta_2 \leq 1,$$
$$\eta_2 = (x,y,u_0(x,y),v_\epsilon + \theta_2\delta v) \qquad \delta v = v_\epsilon - v_0.$$

Taking into account the Mean Value Theorem, the above equation can be written as follows:

$$\delta I = \iint_D \left[F(x,y,u_0,v_\epsilon) - F(x,y,u_0,v_0) - a(x,y)\psi(x,y)\delta v\right]\,dx\,dy$$

Let us show that the two following relationships are equivalent:

1. $F(x,y,u_0,v_\epsilon) - a(x,y)\psi(x,y)v_\epsilon \geq F(x,y,u_0,v_0) - a(x,y)\psi(x,y)v_0$, $\forall v_\epsilon \in V_{ad}$
2. $\inf_{v_\epsilon \in V_{ad}} \left[F(x,y,u_0,v_\epsilon) - a(x,y)\psi(x,y)v_\epsilon\right] = F(x,y,u_0,v_0) - a(x,y)\psi(x,y)v_0$

*Proof*

**1⇒2** This implication stems from our hypothesis, that is to say, $V$ open and $\frac{\partial F}{\partial v}(x,y,u_0(x,y),v_0(x,y)) > 0$, which implies that function $F(x,y,u_0,v_0)$ is increasing on $V$.

**2⇒1** If 2 holds, then

$$F(x,y,u,v) - a(x,y)\psi(x,y)v \geq \inf_{v_\epsilon \in V_{ad}} [F(x,y,u_0,v_\epsilon) - a(x,y)\psi(x,y)v_\epsilon] =$$
$$= F(x,y,u_0,v_0) - a(x,y)\psi(x,y)v_0, \qquad \forall v \in V_{ad}$$

Therefore,

$$\delta I = I(v) - I(v_0) \geq 0, \qquad \forall v \in V_{ad}$$

i.e., $(u_0, v_0)$ is the optimal pair. □

The maximum principle is obtained, which can be formulated as the following theorem:

**Theorem 2 (Maximum Principle)** *Let the "cost functional" or "index of performance" $I$ be given by formula (3) and $\psi$ be solution to the adjoint problem (11), then for optimality of pair $(u_0, v_0)$ it is necessary and sufficient that the following relationship is true almost everywhere on $D$:*

$$\inf_{v \in V_{ad}} F(x,y,u_0,v) - a(x,y)\psi_0(x,y)v = F(x,y,u_0,v_0) - a(x,y)\psi(x,y)v_0$$

**Remark 1.** *For brevity's sake we have considered one fixed point of interval $(0, l_1)$.*

*In the case of several fixed points, this is to say, when considering the following non-local problem:*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - q(x,y)u = a(x,y)v + b(x,y), \qquad (x,y) \in D, \tag{15}$$

$$\begin{aligned} &u(x,y) = 0, && (x,y) \in \partial D \backslash \gamma, \\ &u(l_1,y) = \sum_{m=m_1}^{m_n} \sigma_m u(\overset{m}{x}, y), && 0 \leq y \leq l_2 \end{aligned} \tag{16}$$

*where $\overset{m}{x}$, $m = m_1, \ldots, m_n$, are fixed points of interval $(0, l_1)$.*

$$\sum \sigma_m < 1, \qquad \sigma_m = const > 0, \quad \forall m.$$

*The adjoint problem has the following form:*

$$\begin{aligned} &\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - q(x,y)\psi = -\frac{\partial F}{\partial u}(x,y,u_0,v_0), && \forall (x,y) \in D \backslash \gamma_0 \\ &\psi(x,y) = 0 && (x,y) \in \partial D \\ &\psi_x(\overset{m+}{x}, y) - \psi_x(\overset{m-}{x}, y) = \sigma_m \psi_x(l_1, y) && 0 \leq y \leq l_2, \quad m = m_1, \ldots, m_n. \end{aligned}$$

*The scheme developed in this paper together with all the considerations above mentioned allow us to obtain the same result for this problem.*

**Remark 2.** *Taking into account the Maximum Principle one has that for any neighbourhood sufficiently small of control $v_0$, that is to say, for any $v$ such that $\delta v = v - v_0 \to 0$, one can write the increment of the functional $\delta I$ in the following way:*

$$\delta I = \iint_D \left[ \frac{\partial F}{\partial v}(x,y,u_0,v_0) - a(x,y)\psi \right] \delta v \, dx \, dy \geq 0$$

*and since that* $V$ *is open, one has:*

$$\frac{\partial F}{\partial v}(x, y, u_0, v_0)a(x, y)\psi = 0.$$

*Again, we have obtained the necessary condition of optimality.*

## 3. Conclusion

The methodology used in this paper can be used for solving optimal control problems connected with elastic equilibrium of bodies under its pure shift (anti-plane strain) where mass force is taken as a control function, which minimises the potential energy of the body under consideration.

In the case of homogeneous bodies, this problem may be reduced to Poisson's equation with non-local boundary conditions.

## REFERENCES

1. Bitsadze A. V. and Samarskiĭ A. A. Some elementary generalizations of linear elliptic boundary value problems. *Soviet Math. Dokl.*, 1969; **10**:398–400.
2. Gordeziani.D. G. Accuracy of a variant in the theory of thin shells. *Sov. Phys., Dokl.*, 1974; **19** :385–386.
3. Gordeziani D. G. A class of nonlocal boundary value problems in elasticity theory and shell theory. In *Theory and numerical methods of calculating plates and shells, Vol. II (Russian) (Tbilisi)*; 106–127. Tbilis. Gos. Univ., Tbilisi, 1984.
4. Glowinski R., Lions J. L., and Trémolières R. Numerical analysis of variational inequalities, *Studies in Mathematics and its Applications*, 1981; **8**; North-Holland Publishing Co., Amsterdam.
5. Evseenko T. P. Approximate solution of optimal control problems by the difference method. In *Optimization of processes in systems with distributed parameters (Russian)*, 1976; **106**, pages 21–32, 106. Izdat. "Ilim", Frunze.
6. Lions J. L. *Optimal control of systems governed by partial differential equations.* Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York, 1971.
7. Lazarov R. D. and Makarov V. L. Convergence of the method of nets and the method of lines for multidimensional problems of mathematical physics in classes of generalized solutions. *Soviet Math. Dokl.*, 1981; **24** (1):69–73.
8. Lazarov R. D., Makarov V. L., and Samarskiĭ A. A.. Application of exact difference schemes for constructing and investigating difference schemes on generalized solutions. *Math. USSR-Sb.*, 1982; **45** (4):461–471.
9. Strikwerda J C. *Finite difference schemes and partial differential equations.* Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, second edition, 2004.
10. Samarskii A. A. *The theory of difference schemes*, volume 240 of *Monographs and Textbooks in Pure and Applied Mathematics.* Marcel Dekker Inc., New York, 2001.
11. Ashyralyev A. A note on the Bitsadze-Samarskii type nonlocal boundary value problem in a Banach space. *J. Math. Anal. Appl.*, 2008; **344** (1):557–573.
12. Il′in V. A. and Moiseev E. I. On the uniqueness of the solution of a mixed problem for the wave equation with nonlocal boundary conditions. *Differ. Equ.*, 2000; **36** (5):728–733.
13. Moiseev E. I. Solution of a nonlocal boundary value problem for the Poisson equation with mixed boundary conditions using the Green function. *Differ. Equ.*, 2005; **41** (10):1501–1504.
14. Paneyakh B. P. Some nonlocal boundary value problems for linear differential operators. *Mat. Zametki*, 1984; **35** (3):425–434.
15. Skubachevskiĭ A. L. Nonlocal boundary value problems for elliptic systems in angles. *J. Math. Sci.*, 2002; **114** (4):1561–1570.
16. Il′in V. A. and Moiseev E. I. A two-dimensional nonlocal boundary value problem for the Poisson operator in the differential and the difference interpretation. *Mat. Model.*, 1990; **2**(8):139–156.
17. Lions J. L. *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles.* Dunod, Paris, 1968.

18. Sobolev S. L. *Some applications of functional analysis in mathematical physics*, volume 90 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1991.
19. Boltjanskiĭ V. G., Gamkrelidze R. V., and Pontryagin L. S. The theory of optimal processes. I. The maximum principle. *Amer. Math. Soc. Transl. (2)*, 1961; **18**:341–382.