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A note on one-sided extrapolation of compactness and applications

by

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Abstract. In this paper one-sided counterparts of compactness extrapolation results of Hytönen and Lappas (2021) are provided. As a consequence, compactness results for one-sided singular integrals, commutators of one-sided fractional integrals and a certain class of L^r-Hörmander operators are given. The results for commutators of L^r-Hörmander operators seem new even in the classical setting.

1. Introduction and main results. We recall that an operator $T : X \to Y$ between Banach spaces is compact if $T(B_X)$ (where B_X is the closed unit ball in X) has compact closure in Y.

Within the framework of singular integrals and related operators, probably the first result on compact operators was obtained by Uchiyama [42], who showed that if T is a homogeneous singular integral with smooth kernel and $b \in \mathrm{CMO}$, that is, b is in the closure of $C_c^\infty(\mathbb{R}^n)$ in the BMO norm, then the commutator

$$[b,T] = b(x)Tf(x) - T(bf)(x)$$

is a compact operator on $L^p(\mathbb{R}^n)$ for every $1 < p < \infty$.

Since Uchiyama's paper, a number of authors have obtained results on compactness of several operators, such as Calderón–Zygmund operators [36, 44, 37, 38, 43] or commutators of CMO symbols with several operators [35, 14, 2, 3, 26, 45].

In the weighted setting, motivated by the study of the Beltrami equation, Clop and Cruz [10] provided a counterpart of Uchiyama's result. We recall that w is a *weight* if it is a positive and a.e. finite function, and that $w \in A_p$

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for $p>1$ if

$$\sup_Q\Big(\frac{1}{|Q|}\int_Q w\Big)\Big(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\Big)^{p-1}<\infty.$$

Clop and Cruz showed that if T is a Calderón–Zygmund operator and $b\in\mathrm{CMO}$ then for every $1<p<\infty$ and every $w\in A_p$ the operator$[b,T]$ is compact on $L^p(w)$.

In the last years, compactness in the weighted setting has been an active field of research; see for instance [41, 25, 18, 19, 17]. Even more recently, some results related to compactness of commutators in the two-weight setting have appeared [24, 32, 33].

Our aim in this paper is to provide compactness results in the one-sided weighted setting. As far as we know, just one paper has been devoted to this topic. García and Ortega [15] showed that if T is a one-sided Calderón–Zygmund operator with kernel supported in $(-\infty,0)$ and $b\in\mathrm{CMO}$, then $[b,T]$ is compact on $L^p(w)$ for $w\in A_p^+$ (see the definition of A_p^+ below).

Our main results concern extrapolation of compactness in the one-sided setting. Before presenting them, let us briefly discuss the state of the art in the classical setting. Very recently Hytönen and Lappas [20] settled an interesting result; we refer the reader also to [21] for a partial extension to the bilinear setting and [8] for a full extension to the multilinear setting and a number of applications. Hytönen and Lappas' result says that assuming that a linear operator T is bounded on $L^p(w_0)$ for some $p_0\in(\lambda,\infty)$ and every $w_0\in A_{p_0/\lambda}$ where $\lambda\geq 1$ (with operator norm dominated by an increasing function of $[w_0]_{A_{p_0/\lambda}}$), which is compact on $L^{p_1}(w_1)$ for some $p_1\in(\lambda,\infty)$ and just some $w_1\in A_{p_1/\lambda}$, it follows that T is compact on $L^p(w)$ for every $p\in(\lambda,\infty)$ and every $w\in A_{p/\lambda}$.

Our first main theorem is a one-sided version of that result. Recall that $w\in A_p^+$ for $1<p<\infty$ if

$$[w]_{A_p^+}=\sup_{a<b<c}\frac{1}{c-a}\int_a^b w\Big(\frac{1}{c-a}\int_b^c w^{-\frac{1}{p-1}}\Big)^{p-1}<\infty,$$

and analogously $w\in A_p^-$ if

$$[w]_{A_p^-}=\sup_{a<b<c}\frac{1}{c-a}\int_b^c w\Big(\frac{1}{c-a}\int_a^b w^{-\frac{1}{p-1}}\Big)^{p-1}<\infty.$$

THEOREM 1.1. *Let $\lambda\geq 1$ and $p_0,p_1\in(\lambda,\infty)$. Let T be a linear operator bounded on $L^{p_0}(w_0)$ for every $w_0\in A^+_{p_0/\lambda}$, with operator norm dominated by some increasing function of $[w_0]_{A^+_{p_0/\lambda}}$. Suppose in addition that T is compact on $L^{p_1}(w_1)$ for some $w_1\in A^-_{p_1/\lambda}$. Then T is compact on $L^p(w)$ for all $p\in(\lambda,\infty)$ and all $w\in A^+_{p/\lambda}$.*

In the one-sided setting, the natural hypothesis for w_1 is that $w_1 \in A^+_{p_1/\lambda}$. However, our approach leads to $w_1 \in A^-_{p_1/\lambda}$. In Remark 2.13 we comment on this. Finally, since $A_{p_1/\lambda} \subset A^-_{p_1/\lambda}$ we have the following corollary.

COROLLARY 1.2. *Let $\lambda \geq 1$ and $p_0, p_1 \in (\lambda, \infty)$. Let T be a linear operator bounded on $L^{p_0}(w_0)$ for every $w_0 \in A^+_{p_0/\lambda}$, with operator norm dominated by some increasing function of $[w_0]_{A^+_{p_0/\lambda}}$. Suppose in addition that T is compact on $L^{p_1}(w_1)$ for some $w_1 \in A_{p_1/\lambda}$. Then T is compact on $L^p(w)$ for all $p \in (\lambda, \infty)$ and all $w \in A^+_{p/\lambda}$.*

A particular case of interest that is quite useful in applications is when $w_1 \equiv 1$, which says that T is compact in the unweighted setting. We present that case in the next corollary.

COROLLARY 1.3. *Let $\lambda \geq 1$ and $p_0, p_1 \in (\lambda, \infty)$. Let T be a linear operator bounded on $L^{p_0}(w_0)$ for every $w_0 \in A^+_{p_0/\lambda}$, with operator norm dominated by some increasing function of $[w_0]_{A^+_{p_0/\lambda}}$. Suppose in addition that T is compact on L^{p_1}. Then T is compact on $L^p(w)$ for all $p \in (\lambda, \infty)$ and all $w \in A^+_{p/\lambda}$.*

Our next result is a one-sided counterpart of the corresponding off-diagonal result in [20]. Recall that given $1 < p \leq q < \infty$, $w \in A^+_{p,q}$ if

$$[w]_{A^+_{p,q}} = \sup_{a<b<c} \left(\frac{1}{c-a} \int_a^b w^q \right)^{1/q} \left(\frac{1}{c-a} \int_b^c w^{-p'} \right)^{1/p'} < \infty,$$

and analogously $w \in A^-_{p,q}$ if

$$[w]_{A^-_{p,q}} = \sup_{a<b<c} \left(\frac{1}{c-a} \int_b^c w^q \right)^{1/q} \left(\frac{1}{c-a} \int_a^b w^{-p'} \right)^{1/p'} < \infty.$$

THEOREM 1.4. *Let T be a linear operator bounded from $L^{p_0}(w_0^{p_0})$ to $L^{q_0}(w_0^{q_0})$ for some $1 < p_0 \leq q_0 < \infty$ and every $w_0 \in A^+_{p_0,q_0}$, with operator norm dominated by some increasing function of $[w_0]_{A^+_{p_0,q_0}}$. Assume additionally that T is compact from $L^{p_1}(w_1^{p_1})$ to $L^{q_1}(w_1^{q_1})$ for some $w_1 \in A^-_{p_1,q_1}$ where $1 < p_1 \leq q_1 < \infty$ with $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p_1} - \frac{1}{q_1}$. Then T is compact from $L^p(w^p)$ to $L^q(w^q)$ for every $1 < p \leq q < \infty$ with $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q}$ and all $w \in A^+_{p,q}$.*

As above, since $A_{p_1,q_1} \subset A^-_{p_1,q_1}$, we have the following corollary.

COROLLARY 1.5. *Let T be a linear operator bounded from $L^{p_0}(w_0^{p_0})$ to $L^{q_0}(w^{q_0})$ for some $1 < p_0 \leq q_0 < \infty$ and every $w_0 \in A^+_{p_0}$, with operator norm dominated by some increasing function of $[w_0]_{A^+_{p_0,q_0}}$. Assume additionally that T is compact from $L^{p_1}(w_1^{p_1})$ to $L^{q_1}(w_0^{q_1})$ for some $w_1 \in A_{p_1,q_1}$ where*

$1<p_1\le q_1<\infty$ with $\frac{1}{p_0}-\frac{1}{q_0}=\frac{1}{p_1}-\frac{1}{q_1}$. Then T is compact from $L^p(w^p)$ to $L^q(w^q)$ for every $1<p\le q<\infty$ with $\frac{1}{p_0}-\frac{1}{q_0}=\frac{1}{p}-\frac{1}{q}$ and all $w\in A^+_{p,q}$.

Since in most situations the most interesting choice for w_1 is $w_1\equiv 1$, we record that case in the following corollary.

Corollary 1.6. *Let T be a linear operator bounded from $L^{p_0}(w_0^{p_0})$ to $L^{q_0}(w^{q_0})$ for some $1<p_0\le q_0<\infty$ and every $w_0\in A^+_{p_0,q_0}$, with operator norm dominated by some increasing function of $[w_0]_{A^+_{p_0,q_0}}$. Assume additionally that T is compact from L^{p_1} to L^{q_1} where $1<p_1\le q_1<\infty$ with $\frac{1}{p_0}-\frac{1}{q_0}=\frac{1}{p_1}-\frac{1}{q_1}$. Then T is compact from $L^p(w^p)$ to $L^q(w^q)$ for every $1<p\le q<\infty$ with $\frac{1}{p_0}-\frac{1}{q_0}=\frac{1}{p}-\frac{1}{q}$ and all $w\in A^+_{p,q}$.*

The remainder of the paper is organized as follows. Section 2 is devoted to the proofs of the main results and in Section 3 we gather their applications, providing results on commutators and singular integrals. In particular, we give a result for certain L^r-Hörmander singular integrals that seems to be new even in the classical setting, and is related to [26].

Notation. We will write $a\lesssim b$ instead of $a\le Cb$ when $C>0$ is a numerical constant that does not involve the main parameters in the computation. Analogously, $a\simeq b$ will stand for $aC_1\le b\le C_2 a$ with constants $C_1,C_2>0$ independent of the main parameters involved in the definitions of a and b.

Given a ball B and $k>0$ we will denote by kB the ball concentric with B and of radius k times that of B.

2. Proofs of the main results.

In this section we prove Theorems 1.1 and 1.4. We follow the strategy of [20].

2.1. Preliminaries

2.1.1. *Extrapolation and interpolation results.* We begin by recalling two extrapolation results. The first of them can be obtained by minor modifications of [12, Theorem 3.25].

Theorem 2.1. *Let $\lambda\ge 1$ and $p_0\in(\lambda,\infty)$. Let T be a linear or sublinear operator bounded on $L^{p_0}(w_0)$ for every $w\in A^+_{p_0/\lambda}$, with operator norm bounded by some increasing function of $[w]_{A^+_{p_0/\lambda}}$. Then T is bounded on $L^p(w)$ for all $p\in(\lambda,\infty)$ and every $w\in A^+_{p/\lambda}$.*

The other extrapolation result we will rely upon is the following.

Theorem 2.2 ([31]). *Let T be a linear operator defined and bounded from $L^{p_0}(w_0^{p_0})$ to $L^{q_0}(w^{q_0})$ for some $1<p_0\le q_0<\infty$ and every $w_0\in A^+_{p_0,q_0}$, with operator norm bounded by some increasing function of $[w]_{A^+_{p_0,q_0}}$. Then T*

is bounded from $L^p(w^p)$ *to* $L^q(w^q)$ *for every* $1 < p \le q < \infty$ *such that* $\frac{1}{p_0} - \frac{1}{q_0} = \frac{1}{p} - \frac{1}{q}$, *and all* $w \in A^+_{p,q}$.

Another fundamental tool is the following abstract theorem due to Cwikel and Kalton [13]. Here we state the version contained in [20].

THEOREM 2.3. *Let* (X_0, X_1) *and* (Y_0, Y_1) *be Banach couples and let* T *be a linear operator such that* $T : X_0+X_1 \to Y_0+Y_1$ *and* $T : X_j \to Y_j$ *boundedly for* $j = 0, 1$. *Suppose moreover that* $T : X_1 \to Y_1$ *is compact. Let* $[\ ,\ ]_\theta$ *be the complex interpolation functor of Calderón. Then* $T : [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ *is also compact for* $\theta \in (0, 1)$ *under any of the following conditions:*

(1) X_1 *has the UMD* (*unconditional martingale differences*) *property;*
(2) X_1 *is reflexive, and* $X_1 = [X_0, E]_\alpha$ *for some Banach space* E *and* $\alpha \in (0, 1)$;
(3) $Y_1 = [Y_0, F]_\beta$ *for some Banach space* F *and* $\beta \in (0, 1)$;
(4) X_0 *and* X_1 *are both complexified Banach lattices of measurable functions on a common measure space.*

For the UMD property we direct the interested reader to [22, Chapter 4].

Since we are dealing with weighted L^p spaces, the following lemma, borrowed from [20, Lemma 3.5], will serve our purposes.

LEMMA 2.4. *If* $p_j \in [1, \infty)$ *and* w_j *are weights, then the spaces* $X_j = L^{p_j}(w_j)$ *satisfy condition* (4) *of Theorem* 2.3.

2.1.2. *Facts on one-sided* A_p *and* $A_{p,q}$ *weights.* Our first result is the following reverse Hölder inequality obtained in [11]. However, we shall use the version contained in [34]. Recall that $A^+_\infty = \bigcup_{p\ge1} A^+_p$ and $A^-_\infty = \bigcup_{p\ge1} A^-_p$.

LEMMA 2.5. *Assume that* $w \in A^+_\infty$. *Then there exists* $r = r_w > 1$ *such that*

$$\left(\frac{1}{b-a}\int_a^b w^r\right)^{1/r} \le 4\frac{1}{c-b}\int_b^c w$$

where $a < c$ *and* $b = \frac{c+a}{2}$. *Analogously, if* $w \in A^-_\infty$ *then there exists* $s = s_w > 1$ *such that*

$$\left(\frac{1}{c-b}\int_b^c w^s\right)^{1/s} \le 4\frac{1}{b-a}\int_a^b w.$$

The following result shows that it suffices to establish A_p-like conditions with a gap in order to show that the same condition holds without a gap.

LEMMA 2.6. *Let* $p, q > 1$ *and let* $t > 2$. *Assume that* v^q *and* u^{-p} *are nonnegative locally integrable functions. If for every interval* $I = (a, b)$ *we* *have*

$$\left(\frac{1}{l_I/t}\int_a^{a+l_I/t} v^q\right)^{1/q}\left(\frac{1}{l_I/t}\int_{b-l_I/t}^{b} u^{-p'}\right)^{1/p'} \leq K, \tag{2.1}$$

where $l_I = b-a$, *then*

$$\sup_{a<b<c}\left(\frac{v^q(a,b)}{c-a}\right)^{1/q}\left(\frac{u^{-p'}(b,c)}{c-a}\right)^{1/p'} \leq K.$$

Proof. It suffices to show that for every interval (a,c),

$$\left(\frac{v^q(a,b)}{c-a}\right)^{1/q}\left(\frac{u^{-p'}(b,c)}{c-a}\right)^{1/p'} \leq K.$$

Let $x\in(a,c)$. We define $x_0 = a$, $x_{k+1}-x_k = \frac{1}{t}(x-x_k)$. Then

$$(a,x) = \bigcup_{k=0}^{\infty}(x_k,x_{k+1}].$$

Now we fix k and let $y<x$ be such that $x-y = x_{k+1}-x_k$. Hence, by (2.1),

$$\begin{aligned} v^q(x_k,x_{k+1}) &\leq K^q(x_{k+1}-x_k)\left(\frac{x-y}{u^{-p'}(y,x)}\right)^{q/p'} \\ &\leq K^q(x_{k+1}-x_k)\big(M^-_{u^{-p'}}(u^{p'}\chi_{(a,x)})(x)\big)^{q/p'} \end{aligned}$$

where

$$M^-_\rho f(x) = \sup_{h>0}\frac{1}{\rho(x-h,x)}\int_{x-h}^{x} f(y)\rho(y)\,dy.$$

Summing over k gives

$$v^q(a,x) \leq K^q(x-a)\big(M^-_{u^{-p'}}(u^{p'}\chi_{(a,x)})(x)\big)^{q/p'}$$

and consequently

$$\frac{v^q(a,x)}{x-a} \leq K^q\big(M^-_{u^{-p'}}(u^{p'}\chi_{(a,x)})(x)\big)^{q/p'}.$$

Now let $b\in(a,c)$. For every $x\in(b,c)$ we have

$$\begin{aligned} \frac{v^q(a,b)}{c-a} \leq \frac{v^q(a,x)}{x-a} &\leq K^q\big(M^-_{u^{-p'}}(u^{p'}\chi_{(a,x)})(x)\big)^{q/p'} \\ &\leq K^q\big(M^-_{u^{-p'}}(u^{p'}\chi_{(a,c)})(x)\big)^{q/p'}. \end{aligned}$$

Since

$$(b,c) \subset \left\{x : M^-_{u^{-p'}}(u^{p'}\chi_{(a,c)})(x) \geq \left(\frac{v^q(a,b)}{(c-a)K^q}\right)^{p'/q}\right\},$$

by the weak type $(1,1)$ of $M^-_{u^{-p'}}$ with respect to the measure $u^{-p'}(x)dx$ (see [5]) we deduce that

$$u^{-p'}(b,c) \le u^{-p'}\left(\left\{x : M^-_{u^{-p'}}(u^{p'}\chi_{(a,c)})(x) \ge \left(\frac{v^q(a,b)}{(c-a)K^q}\right)^{p'/q}\right\}\right)$$
$$\le \left(\frac{K^q(c-a)}{v^q(a,b)}\right)^{p'/q}\int_{\mathbb{R}} u^{p'}\chi_{(a,c)}u^{-p'} = \left(\frac{K^q(c-a)}{v^q(a,b)}\right)^{p'/q}(c-a),$$

from which it readily follows that

$$\left(\frac{v^q(a,b)}{c-a}\right)^{1/q}\left(\frac{u^{-p'}(b,c)}{c-a}\right)^{1/p'} \le K. \blacksquare$$

PROPOSITION 2.7. *Let* $1 < p \le q < \infty$.

(1) $w \in A^+_{p,q}$, *iff* $w^q \in A^+_{1+q/p'}$ *iff* $w^{-p'} \in A^-_{1+p'/q}$.
(2) $w \in A^-_{p,q}$ *iff* $w^q \in A^-_{1+q/p'}$ *iff* $w^{-p'} \in A^+_{1+p'/q}$.

Proof. We provide a proof for $w \in A^+_{p,q}$; the other is analogous. To show the first equivalence, note that if $s = 1 + q/p'$ and $a < b < c$, then

$$\frac{1}{(c-a)^s}\int_a^b w^q\left(\int_b^c w^{q\frac{1}{1-s}}\right)^{s-1} = \frac{1}{c-a}\int_a^b w^q\left(\frac{1}{c-a}\int_b^c w^{q\frac{1}{1-(1+q/p')}}\right)^{1+\frac{q}{p'}-1}$$
$$=\frac{1}{c-a}\int_a^b w^q\left(\frac{1}{c-a}\int_b^c w^{-p'}\right)^{q/p'} = \left(\left(\frac{1}{c-a}\int_a^b w^q\right)^{1/q}\left(\frac{1}{c-a}\int_b^c w^{-p'}\right)^{1/p'}\right)^q.$$

Taking the supremum over $a < b < c$ yields

$$[w^q]_{A^+_{1+q/p'}} = [w]^q_{A^+_{p,q}}.$$

Analogously, let $r = 1 + p'/q$ and $a < b < c$. Then

$$\frac{1}{(c-a)^r}\int_b^c w^{-p'}\left(\int_a^b w^{-p'\frac{1}{1-r}}\right)^{r-1}$$
$$= \frac{1}{c-a}\int_b^c w^{-p'}\left(\frac{1}{c-a}\int_a^b w^{-p'\frac{1}{1-(1+p'/q)}}\right)^{1+p'/q-1} = \frac{1}{c-a}\int_b^c w^{-p'}\left(\frac{1}{c-a}\int_a^b w^q\right)^{p'/q}$$
$$= \left(\left(\frac{1}{c-a}\int_a^b w^q\right)^{1/q}\left(\frac{1}{c-a}\int_b^c w^{-p'}\right)^{1/p'}\right)^{p'}.$$

Again, taking the supremum over $a < b < c$ yields

$$[w^{-p'}]_{A^-_{1+p'/q}} = [w]^{p'}_{A^+_{p,q}}.$$

This ends the proof. $\blacksquare$

2.2. Proofs of the main theorems. We begin with the proof of Theorem 1.1, which relies upon the following lemma.

Lemma 2.8. *Let* $\lambda \in [1,\infty)$, *let* $q, q_1 \in (\lambda,\infty)$ *and* $v \in A^+_{q/\lambda}$, $v_1 \in A^-_{q_1/\lambda}$. *Then*

$$L^q(v) = [L^{q_0}(v_0), L^{q_1}(v_1)]_\gamma,$$

for some $q_0 \in (\lambda,\infty)$, $v_0 \in A^+_{q_0/\lambda}$, *and* $\gamma \in (0,1)$.

Assuming we have already established this lemma, we can prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 2.1, since T is a bounded operator on $L^{p_0}(w)$ for some $p_0 \in (\lambda,\infty)$ and every $w \in A^+_{p_0/\lambda}$, it is also bounded on $L^p(w)$ for all $p \in (\lambda,\infty)$ and all $w \in A^+_{p/\lambda}$. Additionally, we have assumed that T is a compact operator on $L^{p_1}(w_1)$ for some $p_1 \in (\lambda,\infty)$ and some $w_1 \in A^-_{p_1/\lambda}$. We need to show that T is compact on $L^p(w)$ for all $p \in (\lambda,\infty)$ and all $w \in A^+_{p/\lambda}$. Now, fix $p \in (\lambda,\infty)$ and $w \in A^+_{p/\lambda}$. By Lemma 2.8,

$$L^p(w) = [L^{p_0}(w_0), L^{p_1}(w_1)]_\theta$$

for some $p_0 \in (\lambda,\infty)$, some $w_0 \in A^+_{p_0/\lambda}$, some $\theta \in (0,1)$. Choosing $X_j = Y_j = L^{p_j}(w_j)$, we know that $T : X_0 + X_1 \to Y_0 + Y_1$, that $T : X_j \to Y_j$ is bounded, since T is bounded on all $L^q(w)$ with $q \in (\lambda,\infty)$ and $w \in A^+_{q/\lambda}$ as noted above, and $T : X_1 \to Y_1$ is compact by assumption. Lemma 2.4 ensures that condition (4) of Theorem 2.3 is also satisfied by the spaces $X_j = L^{p_j}(w_j)$. Hence, by Theorem 2.3, it follows that T is also compact on $[X_0, X_1]_\theta = [Y_0, Y_1]_\theta = L^p(w)$. ∎

For Theorem 1.4 the argument is analogous. We rely upon the following lemma.

Lemma 2.9. *Let* $1 < p_1 \le q_1 < \infty$, $1 < p \le q < \infty$, $w_1 \in A^-_{p_1,q_1}$, $w \in A^+_{p,q}$. *Then*

$$L^p(w^p) = [L^{p_0}(w_0), L^{p_1}(w_1)]_\gamma, \qquad L^q(w^q) = [L^{q_0}(w_0), L^{q_1}(w_1)]_\gamma,$$

for some $1 < p_0 \le q_0 < \infty$, $w_0 \in A^+_{p_0,q_0}$, *and* $\gamma \in (0,1)$.

Again, if we assume that we have already established this lemma, we can prove Theorem 1.4.

Proof of Theorem 1.4. By Theorem 2.2, if T is bounded from $L^{p_0}(w_0^{p_0})$ to $L^{q_0}(w^{q_0})$ for some $1 < p_0 \le q_0 < \infty$ and every $w_0 \in A^+_{p_0,q_0}$, then it is also bounded from $L^p(w^p)$ to $L^q(w^q)$ for every $1 < p \le q < \infty$ and all $w \in A^+_{p,q}$. We know additionally that T is compact from $L^{p_1}(w_1^{p_1})$ to $L^{q_1}(w_1^{q_1})$ and some $w_1 \in A^-_{p_1,q_1}$. We need to show that T is compact from $L^p(w^p)$ to $L^q(w^q)$ for all $1 < p \le q < \infty$ and every $w \in A^+_{p,q}$. Fix some $1 < p \le q < \infty$ and

$w \in A^+_{p,q}$. By Lemma 2.9,

$$L^p(w^p) = [L^{p_0}(w_0), L^{p_1}(w_1)]_\theta, \qquad L^q(w^q) = [L^{q_0}(w_0), L^{q_1}(w_1)]_\theta,$$

for some $1 < p_0 \le q_0 < \infty$, some $w_0 \in A^+_{p_0,q_0}$, and some $\theta \in (0,1)$. Choosing $X_j = L^{p_j}(w_j)$, $Y_j = L^{q_j}(w_j)$, we know that $T : X_0 + X_1 \to Y_0 + Y_1$, that $T : X_j \to Y_j$ is bounded, since T is bounded from $L^p(w^p)$ to $L^q(w^q)$ for every $1 < p \le q < \infty$ and every $w \in A^+_{p,q}$, and that $T : X_1 \to Y_1$ is compact by assumption. Lemma 2.4 ensures that condition (4) of Theorem 2.3 also holds for $X_j = L^{p_j}(w_j^{p_j})$. Hence, by Theorem 2.3, T is compact from $[X_0, X_1]_\theta = L^p(w^p)$ to $[Y_0, Y_1]_\theta = L^q(w^q)$ as claimed. ■

We devote the remainder of the section to proving Lemmas 2.8 and 2.9.

2.2.1. *Proofs of the key lemmas.* The first ingredient in the proofs of Lemmas 2.8 and 2.9 is the following theorem.

THEOREM 2.10. *If $q_0, q_1 \in [1,\infty)$ and w_0, w_1 are two weights, then for all $\theta \in (0,1)$ we have*

$$[L^{q_0}(w_0), L^{q_1}(w_1)]_\theta = L^q(w),$$

where

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \quad \text{and} \quad w^{1/q} = w_0^{(1-\theta)/q_0} w_1^{\theta/q_1}. \tag{2.2}$$

This result can be found in [4, Theorem 5.5.3], but it can be traced back to Stein and Weiss [40]. Having that result in mind, Lemmas 2.8 and 2.9 follow from Lemmas 2.11 and 2.12 respectively.

LEMMA 2.11. *Let $\lambda \in [1,\infty)$, $q_1, q \in (\lambda,\infty)$, $w_1 \in A^-_{q_1/\lambda}$, $w \in A^+_{q/\lambda}$. Then there exist $q_0 \in (\lambda,\infty)$, $w_0 \in A^+_{q_0/\lambda}$, and $\theta \in (0,1)$ such that (2.2) holds.*

LEMMA 2.12. *Let $1 < p_1 \le q_1 < \infty$, $1 < p \le q < \infty$, $w_1 \in A^-_{p_1,q_1}$, $w \in A^+_{p,q}$. Then there exist $1 < p_0 \le q_0 < \infty$, $w_0 \in A^+_{p_0,q_0}$, and $\theta \in (0,1)$ such that*

$$[L^{p_0}(w_0^{p_0}), L^{p_1}(w_1^{p_1})]_\theta = L^p(w^p), \qquad [L^{q_0}(w_0^{q_0}), L^{q_1}(w_1^{q_1})]_\theta = L^q(w^q),$$

where

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \qquad w = w_0^{1-\theta} w_1^{\theta}.$$

Before proving these lemmas we make a remark regarding the hypothesis in Lemma 2.11. A similar remark can be made for the $A^+_{p,q}$ case.

REMARK 2.13. Lemma 2.11 does not hold if we replace $w_1 \in A^-_{q_1/\lambda}$ by $w_1 \in A^+_{q_1/\lambda}$. Indeed, assume that $w_1(x) = e^{x^3} \in A^+_q$ and $w(x) = e^x \in A^+_q$. If the conclusion of the lemma held, then $w_0 = w_1^{-\frac{\theta}{q_1}\frac{q_0}{1-\theta}} w^{\frac{1}{q}\frac{q_0}{1-\theta}}$ for some

$\theta \in (0,1)$, that is,

$$w_0(x) = e^{-x^3 \frac{\theta}{q_1}\frac{q_0}{1-\theta}} e^{\frac{1}{q}\frac{q_0}{1-\theta}x} = e^{\frac{q_0}{1-\theta}\left(-\frac{\theta}{q_1}x^3 + \frac{1}{q}x\right)}$$

for some $\theta \in (0,1)$. Observe that for any $\theta \in (0,1)$,

$$\int_0^\infty w_0(x)\,dx < \infty. \tag{2.3}$$

However, such a property prevents w_0 from being an $A_{q_0}^+$ weight for any possible choice of q_0, since if $w_0 \in A_{q_0}^+$ then

$$\int_a^\infty w_0(x)\,dx = \infty$$

for every $a \in \mathbb{R}$. Indeed, if $w_0 \in A_{q_0}^+$ then

$$\int_0^x w_0\,dy \le c \int_x^{2x} w_0\,dy. \tag{2.4}$$

If additionally (2.3) holds, the right hand side of (2.4) is uniformly controlled by (2.3). Hence, by the dominated convergence theorem,

$$\int_0^\infty w_0(x)\,dx = \lim_{x\to\infty} \int_0^x w_0\,dy = 0,$$

which is a contradiction since w_0 is positive a.e.

The remainder of this subsection is devoted to proving Lemmas 2.11 and 2.12. We will use the following notation. Given any interval $I = (a,b)$, we denote

$$I_1^n = \left(a, a + \frac{l_I}{n}\right), \qquad I_2^n = \left(b - \frac{l_I}{n}, b\right).$$

2.2.2. *Proof of Lemma 2.11.* We adapt the argument in [20] to the one-sided setting. Given $\theta \in (0,1)$ we need to show that $w_0 \in A_{q_0}^+$, where

$$q_0(\theta) = \frac{1-\theta}{\frac{1}{q} - \frac{\theta}{q_1}}, \qquad w^{\frac{q_0(\theta)}{q(1-\theta)}} w_1^{-\frac{\theta q_0(\theta)}{q_1(1-\theta)}} = w_0(\theta) = w_0.$$

By the definition of q and q_1 we see that $q_0 \in (\lambda, \infty)$. Observe that for $p_i = \frac{q_i}{\lambda}$ and $p = \frac{q}{\lambda}$, the same relations hold:

$$p_0 = \frac{q_0}{\lambda} = \frac{1-\theta}{\frac{\lambda}{q} - \frac{\lambda\theta}{q_1}} = \frac{1-\theta}{\frac{1}{p} - \frac{\theta}{p_1}},$$

and hence it suffices to show that

$$[w_0]_{A_{p_0}^+} = \sup_{a<x_1<b} \left(\frac{1}{|(a,b)|}\int_a^{x_1} w_0\right)\left(\frac{1}{|(a,b)|}\int_{x_1}^b w_0^{-\frac{1}{p_0-1}}\right)^{p_0-1} < \infty.$$

The remainder of the argument reduces to showing the following claim:

CLAIM 2.14. *Given an interval I we have*

$$\left(\frac{1}{|I_1^4|}\int_{I_1^4} w_0\right)\left(\frac{1}{|I_2^4|}\int_{I_2^4} w_0^{-\frac{1}{p_0-1}}\right)^{p_0-1} \leq [w]_{A_p^+}^{\frac{p_1}{p_1-\theta p}}[w_1]_{A_{p_1}^-}^{\frac{\theta p}{p_1-\theta p}} < \infty$$

with $\theta \in (0,1)$ small enough and independent of I.

Lemma 2.11 readily follows by combining this claim with a direct application of Lemma 2.6. To establish Claim 2.14 it suffices to proceed as in [20, Lemma 4.1]. Since there are some differences in the way we apply the reverse Hölder inequality, we provide the full argument for the reader's convenience. For some $\varepsilon, \delta > 0$ to be chosen we find that, using the Hölder inequality with exponents $1+\varepsilon^{\pm 1}$ and $1+\delta^{\pm 1}$,

$$\begin{aligned}
&\left(\frac{1}{|I_1^4|}\int_{I_1^4} w_0\right)\left(\frac{1}{|I_2^4|}\int_{I_2^4} w_0^{-\frac{1}{p_0-1}}\right)^{p_0-1}\\
&= \left(\frac{1}{|I_1^4|}\int_{I_1^4} w^{\frac{p_0}{p(1-\theta)}} w_1^{-\frac{p_0\theta}{p_1(1-\theta)}}\right)\left(\frac{1}{|I_2^4|}\int_{I_2^4} w^{-\frac{p_0'}{p(1-\theta)}} w_1^{\frac{p_0'\theta}{p_1(1-\theta)}}\right)^{p_0-1}\\
&= \left(\frac{1}{|I_1^4|}\int_{I_1^4} w^{\frac{p_0}{p(1-\theta)}} (w_1^{-\frac{1}{p_1-1}})^{\frac{p_0\theta}{p_1'(1-\theta)}}\right)\left(\frac{1}{|I_2^4|}\int_{I_2^4} (w^{-\frac{1}{p-1}})^{\frac{p_0'}{p'(1-\theta)}} w_1^{\frac{p_0'\theta}{p_1(1-\theta)}}\right)^{p_0-1}\\
&\leq \left(\frac{1}{|I_1^4|}\int_{I_1^4} w^{\frac{p_0(1+\varepsilon)}{p(1-\theta)}}\right)^{\frac{1}{1+\varepsilon}}\left(\frac{1}{|I_1^4|}\int_{I_1^4} (w_1^{-\frac{1}{p_1-1}})^{\frac{p_0\theta(1+\varepsilon)}{p_1'\varepsilon(1-\theta)}}\right)^{\frac{\varepsilon}{1+\varepsilon}}\\
&\quad\times\left(\frac{1}{|I_2^4|}\int_{I_2^4} (w^{-\frac{1}{p-1}})^{\frac{p_0'(1+\delta)}{p'(1-\theta)}}\right)^{\frac{p_0-1}{1+\delta}}\left(\frac{1}{|I_2^4|}\int_{I_2^4} w_1^{\frac{p_0'\theta(1+\delta)}{p_1\delta(1-\theta)}}\right)^{\delta\frac{p_0-1}{1+\delta}}\\
&= \left(\frac{1}{|I_1^4|}\int_{I_1^4} w^{r(\theta)}\right)^{\frac{1}{1+\varepsilon}}\left(\frac{1}{|I_1^4|}\int_{I_1^4} (w_1^{-\frac{1}{p_1-1}})^{s(\theta)}\right)^{\frac{\varepsilon}{1+\varepsilon}}\\
&\quad\times\left(\frac{1}{|I_2^4|}\int_{I_2^4} (w^{-\frac{1}{p-1}})^{t(\theta)}\right)^{\frac{p_0-1}{1+\delta}}\left(\frac{1}{|I_2^4|}\int_{I_2^4} w_1^{u(\theta)}\right)^{\delta\frac{p_0-1}{1+\delta}} = (*).
\end{aligned}$$

where

$$r(\theta) = \frac{p_0(\theta)(1+\varepsilon)}{p(1-\theta)}, \qquad s(\theta) = \frac{p_0(\theta)\theta(1+\varepsilon)}{p_1'\varepsilon(1-\theta)},$$
$$t(\theta) = \frac{p_0'(\theta)(1+\delta)}{p'(1-\theta)}, \qquad u(\theta) = \frac{p_0'(\theta)\theta(1+\delta)}{p_1\delta(1-\theta)}.$$

Choosing $\varepsilon = \theta p/p_1'$ and $\delta = \theta p'/p_1$ we find that

$$r(\theta) = s(\theta) = \frac{p_0(\theta)(p_1' + \theta p')}{pp_1'(1-\theta)},$$
$$t(\theta) = u(\theta) = \frac{p_0'(\theta)(p_1 + \theta p')}{p'p_1(1-\theta)}.$$

By Lemma 2.5 there exists $\gamma_1 > 1$ such that

$$\left(\frac{1}{|I_1^4|}\int_{I_1^4} \rho^{\gamma_1}\right)^{1/\gamma_1} \leq 4\frac{1}{|I_1^2|}\int_{I_1^2} \rho,$$

where $\rho = w$ and $\rho = w_1^{-\frac{1}{p_1-1}}$ are A_∞^+ weights, and also there exists $\gamma_2 > 1$ such that

$$\left(\frac{1}{|I_2^4|}\int_{I_2^4} \rho^{\gamma_2}\right)^{1/\gamma_2} \leq 4\frac{1}{|I_2^2|}\int_{I_2^2} \rho,$$

where $\rho = w_1$ and $\rho = w^{-\frac{1}{p-1}}$ are A_∞^- weights. Note that $p_0(0) = p$ and $r(0) = t(0) = 1$. Hence, by continuity, we can find $\theta \in (0,1)$ small enough so that

$$1 < \max\{r(\theta), t(\theta)\} \leq \min\{\gamma_1, \gamma_2\}.$$

For that choice of θ, applying Lemma 2.5 we deduce that

$$(*) \lesssim \left(\frac{1}{|I_1^2|}\int_{I_1^2} w\right)^{\frac{p_0}{p(1-\theta)}} \left(\frac{1}{|I_1^2|}\int_{I_1^2} w_1^{-\frac{1}{p_1-1}}\right)^{\frac{\theta p_0}{p_1'(1-\theta)}}$$
$$\times \left(\frac{1}{|I_2^2|}\int_{I_2^2} w^{-\frac{1}{p-1}}\right)^{\frac{p_0'(p_0-1)}{p'(1-\theta)}} \left(\frac{1}{|I_2^2|}\int_{I_2^2} w_1\right)^{\frac{\theta p_0'(p_0-1)}{p_1(1-\theta)}} = (**)$$

and finally, rearranging terms,

$$(**) \lesssim \left(\left(\frac{1}{|I_1^2|}\int_{I_1^2} w\right)\left(\frac{1}{|I_2^2|}\int_{I_2^2} w^{-\frac{1}{p-1}}\right)^{p-1}\right)^{\frac{p_0}{p(1-\theta)}}$$
$$\times \left(\left(\frac{1}{|I_2^2|}\int_{I_2^2} w_1\right)\left(\frac{1}{|I_1^2|}\int_{I_1^2} w_1^{-\frac{1}{p_1-1}}\right)^{p_1-1}\right)^{\frac{\theta p_0}{p_1(1-\theta)}}$$
$$\leq [w]_{A_p^+}^{\frac{p_1}{p_1-\theta p}} [w_1]_{A_{p_1}^-}^{\frac{\theta p}{p_1-\theta p}},$$

as we wanted to show.

2.2.3. *Proof of Lemma 2.12.* By the choice of $\theta \in (0,1)$ we can determine p_0, q_0 and w_0 as follows:

$$p_0 = p_0(\theta) = \frac{1-\theta}{\frac{1}{p} - \frac{\theta}{p_1}},$$
$$q_0 = q_0(\theta) = \frac{1-\theta}{\frac{1}{q} - \frac{\theta}{q_1}},$$
$$w_0 = w_0(\theta) = w^{\frac{1}{1-\theta}} w_1^{-\frac{\theta}{1-\theta}},$$

and consequently it remains to show that one can choose $\theta \in (0,1)$ so that $1 < p_0 \le q_0 < \infty$ and $w_0 \in A^+_{p_0,q_0}$. As $1 < p_0(0) = p \le q = q_0(0) < \infty$, it is clear that choosing θ small enough the first condition holds. For the second, we follow the strategy used in the proof of Lemma 2.11, namely, given an interval $I = (a,b)$ we are going to show that

$$\left(\frac{1}{|I_1^4|}\int_{I_1^4} w_0^{q_0}\right)^{1/q_0}\left(\frac{1}{|I_2^4|}\int_{I_2^4} w_0^{-p_0'}\right)^{1/p_0'} \le [w]_{A^+_{p,q}}^{\frac{1}{1-\theta}}[w]_{A^-_{p_1,q_1}}^{\frac{\theta}{1-\theta}},$$

relying upon this estimate, a direct application of Lemma 2.6 yields the desired conclusion.

We use the Hölder inequality with exponents $1+\varepsilon^{\pm 1}$ and $1+\delta^{\pm 1}$ with $\delta, \varepsilon > 0$ to be chosen:

$$\begin{aligned}
(2.5)\qquad &\left(\frac{1}{|I_1^4|}\int_{I_1^4} w_0^{q_0}\right)^{1/q_0}\left(\frac{1}{|I_2^4|}\int w_0^{-p_0'}\right)^{1/p_0'}\\
&= \left(\frac{1}{|I_1^4|}\int_{I_1^4} w^{\frac{q_0}{1-\theta}} w_1^{-\frac{\theta q_0}{1-\theta}}\right)^{1/q_0}\left(\frac{1}{|I_2^4|}\int_{I_2^4} w^{-p_0'\frac{1}{1-\theta}} w_1^{p_0'\frac{\theta}{1-\theta}}\right)^{1/p_0'}\\
&\le \left(\frac{1}{|I_1^4|}\int_{I_1^4} w^{\frac{q_0(1+\varepsilon)}{1-\theta}}\right)^{\frac{1}{q_0(1+\varepsilon)}}\left(\frac{1}{|I_1^4|}\int_{I_1^4} w_1^{-\frac{\theta q_0}{1-\theta}\frac{1+\varepsilon}{\varepsilon}}\right)^{\frac{1}{q_0}\frac{\varepsilon}{1+\varepsilon}}\\
&\quad\times \left(\frac{1}{|I_2^4|}\int_{I_2^4} w^{-p_0'\frac{1}{1-\theta}(1+\delta)}\right)^{\frac{1}{p_0'(1+\delta)}}\left(\frac{1}{|I_2^4|}\int_{I_2^4} w_1^{p_0'\frac{\theta}{1-\theta}\frac{1+\delta}{\delta}}\right)^{\frac{1}{p_0'}\frac{\delta}{1+\delta}}\\
&= \left(\frac{1}{|I_1^4|}\int_{I_1^4} w^{qr(\theta)}\right)^{\frac{1}{q_0(1+\varepsilon)}}\left(\frac{1}{|I_1^4|}\int_{I_1^4} w_1^{-p_1's(\theta)}\right)^{\frac{1}{q_0}\frac{\varepsilon}{1+\varepsilon}}\\
&\quad\times \left(\frac{1}{|I_2^4|}\int_{I_2^4} w^{-p't(\theta)}\right)^{\frac{1}{p_0'(1+\delta)}}\left(\frac{1}{|I_2^4|}\int_{I_2^4} w_1^{q_1u(\theta)}\right)^{\frac{1}{p_0'}\frac{\delta}{1+\delta}},
\end{aligned}$$

where

$$\begin{aligned} r(\theta) &= \frac{q_0(\theta)(1+\varepsilon)}{q(1-\theta)}, \\ s(\theta) &= \frac{1}{p_1'} \frac{\theta q_0(\theta)}{1-\theta} \frac{1+\varepsilon}{\varepsilon}, \\ t(\theta) &= \frac{p_0'(\theta)}{p'} \frac{1}{1-\theta}(1+\delta), \\ u(\theta) &= \frac{p_0'(\theta)}{q_1} \frac{\theta}{1-\theta} \frac{1+\delta}{\delta}. \end{aligned}$$

Choosing $\varepsilon = q\theta/p_1'$ and $\delta = \theta p'/q_1$ we obtain

$$r(\theta) = s(\theta) = \frac{q_0(\theta)(p_1' + \theta q)}{q p_1'(1-\theta)}, \qquad t(\theta) = u(\theta) = \frac{p_0'(\theta)(q_1 + \theta p')}{q_1 p'(1-\theta)}.$$

Now by Proposition 2.7, we have $w^q \in A^+_{1+q/p'}$, $w^{-p'} \in A^-_{1+p'/q}$, $w_1^{-p_1'} \in A^+_{1+p_1'/q_1}$ and $w_1^{q_1} \in A^-_{1+q_1/p_1'}$, so that $w^q, w^{-p_1'} \in A^+_\infty$ and $w^{-p'}, w_1^{q_1} \in A^-_\infty$. Since for any $\eta > 0$, it is possible to choose θ close enough to 0 so that

$$1 < \max\{r(\theta), t(\theta)\} \leq 1 + \eta,$$

we can apply the reverse Hölder inequalities of Lemma 2.5 to each term on the right-hand side of (2.5) to obtain

$$\begin{aligned} &\left(\frac{1}{|I_1^4|}\int_{I_1^4} w^{qr(\theta)}\right)^{\frac{1}{q_0(1+\varepsilon)}} \left(\frac{1}{|I_1^4|}\int_{I_1^4} w_1^{-p_1's(\theta)}\right)^{\frac{1}{q_0}\frac{\varepsilon}{1+\varepsilon}} \\ &\qquad \times \left(\frac{1}{|I_2^4|}\int_{I_2^4} w^{-p't(\theta)}\right)^{\frac{1}{p_0'(1+\delta)}} \left(\frac{1}{|I_2^4|}\int_{I_2^4} w_1^{q_1u(\theta)}\right)^{\frac{1}{p_0'}\frac{\delta}{1+\delta}} \\ &\qquad \lesssim \left(\frac{1}{|I_1^2|}\int_{I_1^2} w^{q}\right)^{\frac{1}{q_0(1+\varepsilon)}r(\theta)} \left(\frac{1}{|I_1^2|}\int_{I_1^2} w_1^{-p_1'}\right)^{\frac{1}{q_0}\frac{\varepsilon}{1+\varepsilon}r(\theta)} \\ &\qquad \times \left(\frac{1}{|I_2^2|}\int_{I_2^2} w^{-p'}\right)^{\frac{1}{p_0'(1+\delta)}t(\theta)} \left(\frac{1}{|I_2^2|}\int_{I_2^2} w_1^{q_1}\right)^{\frac{1}{p_0'}\frac{\delta}{1+\delta}t(\theta)} \\ &\qquad = \left(\frac{1}{|I_1^2|}\int_{I_1^2} w^{q}\right)^{\frac{1}{q(1-\theta)}} \left(\frac{1}{|I_2^2|}\int_{I_2^2} w^{-p'}\right)^{\frac{1}{p'(1-\theta)}} \\ &\qquad \times \left(\frac{1}{|I_2^2|}\int_{I_2^2} w_1^{q_1}\right)^{\frac{1}{q_1}\frac{\theta}{1-\theta}} \left(\frac{1}{|I_1^2|}\int_{I_1^2} w_1^{-p_1'}\right)^{\frac{1}{p_1'}\frac{\theta}{1-\theta}} \leq [w]_{A^+_{p,q}}^{\frac{1}{1-\theta}} [w]_{A^-_{p_1,q_1}}^{\frac{\theta}{1-\theta}}, \end{aligned}$$

as we wanted to show.

3. Applications

3.1. Compactness of one-sided Calderón–Zygmund operators. Villarroya [43, Theorem 2.21] showed that a Calderón–Zygmund operator T extends to a compact operator on $L^p(\mathbb{R})$ for $1 < p < \infty$ if and only if T is associated with a compact Calderón–Zygmund kernel [43, Definition 2.3] and satisfies the weak compactness condition [43, Definition 2.12] (not to be confused with being weakly compact) and $T(1), T^*(1) \in \mathrm{CMO}(\mathbb{R})$.

This result combined with Corollary 1.3 leads to the following:

Corollary 3.1. *Let T be a Calderón–Zygmund operator associated to a compact Calderón–Zygmund kernel supported in*

$$\{(x, y) \in \mathbb{R}^2 : x < y\}$$

such that T satisfies the weak compactness condition and

$$T(1), T^*(1) \in \mathrm{CMO}(\mathbb{R}).$$

Then for every $1 < p < \infty$, T is compact on $L^p(w)$ with $w \in A_p^+$.

Proof. In [1, 9] it was shown that a Calderón–Zygmund operator with kernel supported on $\{(x, y) \in \mathbb{R}^2 : x < y\}$ is bounded on $L^p(w)$ for $w \in A_p^+$ and every $p \in (1, \infty)$. This fact combined with Villarroya's [43, Theorem 2.21], which ensures compactness in the unweighted setting, leads to the desired conclusion via Corollary 1.3. ∎

Very recently Mitkovski and Stockdale [36, Section 2] showed that a Calderón–Zygmund operator T extends to a compact operator on $L^2(\mathbb{R})$ if and only if T is weakly compact and $T(1), T^*(1) \in \mathrm{CMO}(\mathbb{R})$. Arguing as above we obtain the following corollary.

Corollary 3.2. *Let T be a Calderón–Zygmund operator associated to a compact Calderón–Zygmund kernel supported in $\{(x, y) \in \mathbb{R}^2 : x < y\}$ such that T is weakly compact and $T(1), T^*(1) \in \mathrm{CMO}(\mathbb{R})$. Then for every $1 < p < \infty$, T is compact on $L^p(w)$ with $w \in A_p^+$.*

3.2. Results on commutators

3.2.1. *Commutators of L^r-Hörmander operators.* In this section we will work in the n-dimensional case in the classical setting, since the results we present are unknown even in that setting.

We need the following definition, essentially borrowed from [7]. We refer the reader there for further related references.

Definition 3.3. We say that a locally integrable function

$$K : \{(x, y) \in \mathbb{R}^n : x \neq y\} \to \mathbb{R}$$

is an L^r-*Hörmander kernel* if there exists $\gamma > n/r'$ such that

$$\left(\int_{A_m(B)} |K(x,y)-K(x',y)|^r\,dy\right)^{1/r}+\left(\int_{A_m(B)} |K(y,x)-K(y,x')|^r\,dy\right)^{1/r} \leq C\frac{|x-x'|^{\gamma-n/r'}}{|B|^{\gamma/n}}2^{-m\gamma}$$

for every ball B and all $x, x' \in \frac{1}{2}B$ where $A_m(B) = 2^m B \setminus 2^{m-1}B$ with $m \geq 1$, and also

$$|K(x,y)| \leq C\frac{1}{|x-y|^n}.$$

We say that a linear operator T is an L^r-*Hörmander singular integral operator* if T is bounded on L^2, there exists an L^r-Hörmander kernel K such that for every $f \in L_c^\infty(\mathbb{R}^n)$,

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy \qquad \text{for } x \notin \operatorname{supp}(f),$$

and the limit $\lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x-y|<1} K(x,y)\,dy$ exists.

Note that under the conditions above (in fact, under weaker ones) Grafakos [16] showed that if T is an L^r-Hörmander operator with associated kernel K, then the maximal operator

$$T^*f(x) = \sup_{\varepsilon>0}\left|\int_{\varepsilon<|x-y|} K(x,y)f(y)\,dy\right|$$

is bounded on L^p. Consequently, if T is an L^r-Hörmander operator, then

$$Tf(x) = \lim_{\varepsilon\to 0^+}\int_{\varepsilon<|x-y|} K(x,y)f(y)\,dy$$

almost everywhere for $f \in L^p$.

Our compactness result seems to be new even in the classical setting:

Theorem 3.4. *Let $1 < r' < p < \infty$. Assume that K is an L^r-Hörmander kernel and T is the operator associated to K. If $b \in \mathrm{CMO}$, then for every $w \in A_{p/r'}$, $[b,T]$ is compact from $L^p(w)$ to $L^p(w)$. If $n = 1$ and additionally K is supported on $\{(x,y) \in \mathbb{R}^2 : x < y\}$, then for every $w \in A^+_{p/r'}$, $[b,T]$ is compact on $L^p(w)$.*

Theorem 3.4 follows from [20] in the standard setting, and from Theorem 1.3 in the one-sided setting, combined with the following result that we settle in this paper.

Theorem 3.5. *Let $1 < r' < p < \infty$. Let T be an L^r-Hörmander operator. If $b \in \mathrm{CMO}$, then $[b,T]$ is compact on L^p.*

We will prove Theorems 3.5 and 3.4 in the next subsection.

3.2.2. *Commutators of one-sided fractional integrals.* We recall that given $\alpha \in (0,1)$, the *one-sided fractional integral* I_α^+ is defined as

$$I_\alpha^+ f(x) = \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}}\, dy.$$

Our result deals with the compactness of the commutator of this operator.

THEOREM 3.6. *Let $1 < p < q < \infty$ and $\alpha \in (0,1)$ such that $\frac{1}{p} - \frac{1}{q} = \alpha$. If $b \in \mathrm{CMO}$ and $w \in A_p^+$ then $[b, I_\alpha^+]$ is compact from $L^p(w)$ to $L^q(w)$.*

The theorem above readily follows from Corollary 1.6 combined with the following consequence of [19, Theorem 1.5].

THEOREM 3.7. *Let $1 < p < q < \infty$ and $\alpha \in (0,1)$ be such that $\frac{1}{p} - \frac{1}{q} = \alpha$. If $b \in \mathrm{CMO}$ and $w \in A_p^+$ then $[b, I_\alpha^+]$ is compact from L^p to L^q.*

We give the proofs of these results in the next subsection.

3.3. Proofs of the results on commutators

3.3.1. *Proof of Theorem 3.5.* The following result will be useful.

LEMMA 3.8 ([46]). *Let $1 \le p < \infty$. A subset $H \subset L^p(\mathbb{R}^n)$ is precompact if and only if*

(1) *H is bounded in $L^p(\mathbb{R})$,*
(2) $\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h) - f(x)|^p\, dx = 0$ *uniformly in H,*
(3) $\lim_{M\to\infty} \int_{|x|>M} |f|^p = 0$ *uniformly in H,*

Before getting into the proof of Theorem 3.5 we need some preparations. Let $\varphi^\delta : \mathbb{R}^n \to [0,1]$ be a differentiable radial function such that

$$\begin{aligned} \operatorname{supp}(\varphi^\delta) &\subset \{|x| > \delta\}, \\ \varphi^\delta(x) &= 1, \qquad |x| \ge 2\delta, \\ \|\varphi^\delta\|_{L^\infty} &= 1, \\ \|\nabla \varphi^\delta\|_{L^\infty} &\simeq 1/\delta. \end{aligned}$$

Assume that K satisfies the L^r-Hörmander condition. For each $\delta \in (0,1)$ we set $K^\delta(x,y) = \varphi^\delta(x-y) K(x,y)$.

LEMMA 3.9. *Let $r > 1$ and $\delta \in (0,1)$. If K satisfies the L^r-Hörmander condition for some $\gamma > n/r'$, then*

$$|K^\delta(x,y)| \le C \frac{1}{|x-y|^n}, \qquad x \ne y,$$

and also

$$\Big(\int_{A_m(B)} |K^\delta(x,y) - K^\delta(x',y)|^r\,dy\Big)^{1/r} + \Big(\int_{A_m(B)} |K^\delta(y,x) - K^\delta(y,x')|^r\,dy\Big)^{1/r}$$
$$\le C\max\Big\{\frac{|x-x'|^{\gamma-n/r'}}{|B|^{\gamma/n}}2^{-m\gamma}, \frac{|x-x'|^{(1+n/r')-n/r'}}{|B|^{(1+n/r')/n}}2^{-(1+n/r')m}\Big\}$$

for every ball B and all $x, x' \in \frac{1}{2}B$, where $A_m(B) = 2^m B \setminus 2^{m-1}B$ if $m \ge 1$.

Proof. It is trivial that K^δ satisfies the size condition. For the smoothness condition it suffices to handle one of the two terms, since the condition is symmetric. We have

$$\Big(\int_{A_m(B)} |K^\delta(x,y) - K^\delta(x',y)|^r\,dy\Big)^{1/r} \le \sum_{i=1}^{3}\Big(\int_{C_j^m(B)} |K^\delta(x,y) - K^\delta(x',y)|^r\,dy\Big)^{1/r}$$

where

$$\begin{aligned} C_1^m &= A_m(B) \cap \{|x-y| > 2\delta\} \cap \{|x'-y| > 2\delta\},\\ C_2^m &= A_m(B) \cap \{|x-y| < 2\delta\} \cap \{|x'-y| > 2\delta\},\\ C_3^m &= A_m(B) \cap \{|x-y| > 2\delta\} \cap \{|x'-y| < 2\delta\}. \end{aligned}$$

We begin by observing that

$$\Big(\int_{C_1^m(B)} |K^\delta(x,y) - K^\delta(x',y)|^r\,dy\Big)^{1/r} \le \Big(\int_{A_m(B)} |K(x,y) - K(x',y)|^r\,dx\Big)^{1/r}$$

and hence the desired conclusion holds for this term. Since C_2^m and C_3^m are symmetric, it suffices to deal with C_2^m. Note that

$$\begin{aligned} &\Big(\int_{C_2^m} |K^\delta(x,y) - K^\delta(x',y)|^r\,dy\Big)^{1/r}\\ &\qquad = \Big(\int_{C_2^m} |K(x,y)\varphi^\delta(x-y) - K(x',y)\varphi^\delta(x'-y)|^r\,dy\Big)^{1/r}\\ &\qquad \le \Big(\int_{C_2^m} |K(x,y)\varphi^\delta(x-y) - K(x',y)\varphi^\delta(x-y)|^r\,dy\Big)^{1/r}\\ &\qquad\quad + \Big(\int_{C_2^m} |K(x',y)(\varphi^\delta(x'-y) - \varphi^\delta(x-y))|^r\,dy\Big)^{1/r}\\ &\qquad = I + II. \end{aligned}$$

For I, since $\|\varphi\|_{L^\infty} \le 1$,

$$I \le \Big(\int_{A_m(B)} |K(x,y) - K(x',y)|^r\,dy\Big)^{1/r},$$

and we are done by using the L^r-Hörmander condition for K.

For II,

$$II \leq \Big(|x-x'|^r \|\nabla\varphi^\delta\|_{L^\infty}^r \int_{C_2^m} |K(x',y)|^r\,dy\Big)^{1/r}$$
$$\leq C|x-x'|\bigg(\frac{1}{\delta^r}\int_{C_2^m} |K(x',y)|^r\,dy\bigg)^{1/r}$$
$$\leq C|x-x'|\bigg(\frac{1}{\delta^r}\int_{C_2^m} \frac{1}{|x'-y|^{nr}}\,dy\bigg)^{1/r}.$$

For $y \in C_2^m$,

$$\frac{1}{\delta} < \frac{2}{|x-y|}$$

and also for every $z \in \frac{1}{2}B$ and $y \in A_m(B)$, if we denote by c_B the center of the ball B, we have

$$\frac{1}{|z-y|} \leq \frac{2}{|c_B-y|}.$$

Hence, denoting by r_B the radius of B,

$$II \leq C|x-x'|\bigg(\int_{C_2^m} \frac{1}{|c_B-y|^{(n+1)r}}\,dy\bigg)^{1/r}$$
$$\leq C|x-x'|\bigg(\int_{A_m(B)} \frac{1}{|c_B-y|^{(n+1)r}}\,dy\bigg)^{1/r}$$
$$= C|x-x'|\bigg(\int_{2^{m-1}r_B}^{2^m r_B} \frac{1}{\rho^{(n+1)r}}\rho^{n-1}d\rho\bigg)^{1/r}$$
$$= C|x-x'|\bigg(\bigg[\frac{\rho^{n-(n+1)r}}{n-(n+1)r}\bigg]_{2^{m-1}r_B}^{2^m r_B}\bigg)^{1/r}$$
$$\leq C|x-x'|(2^m r_B)^{n/r-(n+1)} = C|x-x'|(2^m r_B)^{-n/r'-1}$$
$$= C|x-x'|{r_B}^{-n/r'}2^{-m}r_B^{-1}2^{-\frac{n}{r'}m} \simeq \frac{|x-x'|^{(1+n/r')-n/r'}}{|B|^{(1+n/r')/n}}2^{-(1+n/r')m}. \blacksquare$$

Armed with the lemma above we can present our proof of Theorem 3.5. We begin with some reductions. Note that since for every $p > r'$ we have

$$\|[b,T]f\|_{L^p} \leq C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}$$

(see for instance [27]), if $b \in \mathrm{CMO}$ then we can approximate b by functions $b_j \in C_c^\infty$ in the BMO norm and consequently

$$\|[b,T]f - [b_j,T]f\|_{L^p} = \|[b-b_j,T]f\|_{L^p} \lesssim \|b-b_j\|_{\mathrm{BMO}}\|f\|_{L^p}.$$

In particular, $[b_j, T] \to [b, T]$ in the L^p norm. Consequently, it suffices to prove compactness for commutators with smooth symbol. Now we shall consider T^δ, the operator associated to the truncation defined as above with $\delta \in (0,1)$.

LEMMA 3.10. *Let $b \in C_c^1(\mathbb{R}^n)$. There exists a constant $C > 0$ such that*

$$|[b,T]f(x)-[b,T^\delta]f(x)| \le C\delta\|\nabla b\|_{L^\infty} Mf(x) \quad \text{a.e. for every } \delta > 0. \tag{3.1}$$

Consequently, for every $p > 1$,

$$\lim_{\delta\to 0} \|[b,T]f(x) - [b,T^\delta]f(x)\|_{L^p} = 0.$$

Proof. Let $f \in L^p$. Let

$$[b,T_\varepsilon]f(x) = \int_{\varepsilon<|x-y|} (b(x)-b(y))K(x,y)f(y)\,dy.$$

First note that

$$[b,T]f(x) = \lim_{\varepsilon\to 0^+} [b,T_\varepsilon]f(x) \quad \text{a.e.} \tag{3.2}$$

since the corresponding result holds for T_ε and T. Let

$$T_\varepsilon^\delta f(x) = \int_{\varepsilon<|x-y|} K^\delta(x,y)f(y)\,dy.$$

If $\varepsilon > 2\delta$ then

$$|T_\varepsilon^\delta f(x) - T_\varepsilon f(x)| = 0.$$

If $0 < \varepsilon < 2\delta$ then

$$\begin{aligned} |T_\varepsilon^\delta f(x)-T_\varepsilon f(x)| &= \Bigg| \int_{\varepsilon<|x-y|<2\delta} K(x,y)f(y)\,dy - \int_{\delta<|x-y|<2\delta} K^\delta(x,y)f(y)\,dy \Bigg| \\ &\le |I(x)|+|II(x)|. \end{aligned}$$

For I we note that

$$\begin{aligned} |I(x)| &\le \Bigg| \int_{\varepsilon<|x-y|} K(x,y)f(y)\,dy \Bigg| + \Bigg| \int_{2\delta<|x-y|} K(x,y)f(y)\,dy \Bigg| \\ &\le 2\sup_{\rho>0} |T_\rho(f)(x)|. \end{aligned}$$

For II,

$$\begin{aligned} |II(x)| &\le \int_{\delta<|x-y|<2\delta} |K^\delta(x,y)|\,|f(y)|\,dy \le \int_{\delta<|x-y|<2\delta} \frac{|f(y)|}{|x-y|^n}\,dy \\ &\lesssim \frac{1}{|B(x,2\delta)|} \int_{|x-y|<2\delta} |f(y)|\,dy \le Mf(x). \end{aligned}$$

This yields

$$(T^\delta)^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon^\delta f(x)| \le \sup_{\varepsilon>0} |T_\varepsilon^\delta f(x) - T_\varepsilon f(x)| + \sup_{\varepsilon>0} |T_\varepsilon f(x)| \lesssim 2\sup_{\rho>0} |T_\rho f(x)| + Mf(x).$$

Then $(T^\delta)^*$ is bounded on L^p (actually, $\{(T^\delta)^*\}_{\delta>0}$ is uniformly bounded on L^p). Consequently, taking into account Lemma 3.9 and also the fact that for every x the limit

$$\lim_{\varepsilon\to 0^+} \int_{1>|x-y|>\varepsilon} K^\delta(x,y)\,dy$$

exists, we find that for every $f \in L^p$,

$$T^\delta f(x) = \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} K^\delta(x,y) f(y)\,dy \qquad \text{a.e.}$$

Therefore, if $f \in L^p$, then

$$[b, T^\delta] f(x) = \lim_{\varepsilon\to 0^+} \int_{|x-y|>\varepsilon} (b(x)-b(y)) K^\delta(x,y) f(y)\,dy \qquad \text{a.e.} \tag{3.3}$$

Now we are in a position to prove (3.1). For every $x \in \mathbb{R}^n$, if $\varepsilon < \delta$ then

$$\begin{aligned}|[b,T_\varepsilon]f(x) - [b,T_\varepsilon^\delta]f(x)| &= \Big| \int_{\varepsilon<|x-y|<2\delta} (b(x)-b(y))K(x,y)f(y)\,dy \\ &\quad - \int_{\delta<|x-y|<2\delta} (b(x)-b(y))K^\delta(x,y)f(y)\,dy \Big| \\ &\le |I_1(x)| + |I_2(x)|.\end{aligned}$$

By the smoothness of b and the size condition on K,

$$\begin{aligned}|I_1(x)| &\le \int_{\varepsilon<|x-y|<\delta} |b(x)-b(y)| \frac{1}{|x-y|^n} |f(y)|\,dy \\ &\le \|\nabla b\|_{L^\infty} \sum_{j=0}^\infty \int_{\frac{2\delta}{2^{j+1}}<|x-y|<\frac{2\delta}{2^j}} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \\ &\lesssim \|\nabla b\|_{L^\infty} \sum_{j=0}^\infty \frac{2\delta}{2^{j+1}} \frac{|B(0,1)|}{\left(\frac{2\delta}{2^{j+1}}\right)^n |B(0,1)|} \int_{2\delta/2^{j+1}<|x-y|<2\delta/2^j} |f(y)|\,dy \\ &\lesssim \|\nabla b\|_{L^\infty} \sum_{j=0}^\infty \frac{2\delta}{2^{j+1}} Mf(x) \lesssim \|\nabla b\|_{L^\infty} \delta Mf(x).\end{aligned}$$

For $I_2(x)$,

$$\begin{aligned}|I_2(x)| &\leq \int\limits_{\delta<|x-y|<2\delta} |b(x)-b(y)|\,|K^\delta(x,y)|\,|f(y)|\,dy \\ &\leq \|\nabla b\|_{L^\infty} \int\limits_{\delta<|x-y|<2\delta} \frac{|f(y)|}{|x-y|^{n-1}}\,dy \\ &\lesssim \|\nabla b\|_{L^\infty}\delta \frac{1}{|B(x,\delta)|}\int\limits_{|x-y|<\delta} |f(y)|\,dy \\ &\leq \|\nabla b\|_{L^\infty}\delta Mf(x).\end{aligned}$$

Then

$$|[b,T_\varepsilon]f(x)-[b,T_\varepsilon^\delta]f(x)| \lesssim \|\nabla b\|_{L^\infty}\delta Mf(x)$$

and taking into account (3.2) and (3.3), letting $\varepsilon \to 0$ we deduce that the desired inequality holds. The remaining assertion follows from the L^p boundedness of M by letting $\delta \to 0$. ∎

By the result above, to conclude the proof it suffices to show that if we fix $\delta > 0$ and $b \in C_c^1(\mathbb{R}^n)$, then $[b,T^\delta]$ is compact on L^p with $p > r'$. To do that, we are going to show that the conditions in Lemma 3.8 hold.

First, $H = \{[b,T^\delta]f : \|f\|_{L^p} \leq 1\}$ is bounded due to the boundedness of $[b,T^\delta]$ on L^p. Indeed, from the lemma above,

$$\begin{aligned}\|[b,T^\delta]f\|_{L^p} &\leq \|[b,T^\delta]f-[b,T]f\|_{L^p}+\|[b,T]f\|_{L^p} \\ &\leq c\|Mf\|_{L^p}+\|[b,T]f\|_{L^p} \lesssim \|f\|_{L^p}.\end{aligned}$$

We continue the proof by showing that

$$\lim_{h\to 0}\|[b,T^\delta]f(\cdot)-[b,T^\delta]f(\cdot+h)\|_{L^p} = 0 \tag{3.4}$$

uniformly in $f \in H$. We begin by writing

$$\begin{aligned}&[b,T^\delta]f(x)-[b,T^\delta]f(x+h) \\ &\quad= \int\limits_{\mathbb{R}^n}(b(x)-b(y))K^\delta(x,y)f(y)\,dy \\ &\qquad - \int\limits_{\mathbb{R}^n}(b(x+h)-b(y))K^\delta(x+h,y)f(y)\,dy \\ &\quad= (b(x)-b(x+h))\int\limits_{\mathbb{R}^n}K^\delta(x,y)f(y)\,dy \\ &\qquad + \int\limits_{\mathbb{R}^n}(b(x+h)-b(y))(K^\delta(x,y)-K^\delta(x+h,y))f(y)\,dy \\ &\quad= \int\limits_{\mathbb{R}^n}I_1(x,y,h)\,dy+\int\limits_{\mathbb{R}^n}I_2(x,y,h)\,dy.\end{aligned} \tag{3.5}$$

We treat each term separately. We begin with I_1,

$$\begin{aligned}\Big|\int I_1(x,y,h)\,dy\Big| &\le \|\nabla b\|_{L^\infty}|h|\,\Big|\int K^\delta(x,y)f(y)\,dy\Big| \\ &= \|\nabla b\|_{L^\infty}|h|\,\Big|\int_{|x-y|>\delta} K(x,y)f(y)\,dy\Big| \\ &\le \|\nabla b\|_{L^\infty}|h|T^*f(x),\end{aligned}$$

and hence, since as noted above, by [16, 39], T^*f is bounded on L^p, we see that if $\|f\|_{L^p}\le 1$ then

$$\begin{aligned}\Big\|\int_{\mathbb{R}^n} I_1(\cdot,y,h)\,dy\Big\|_{L^p} &\le \|\nabla b\|_{L^\infty}|h|\Big(\int_{\mathbb{R}^n}|T^*f(x)|^p\,dx\Big)^{1/p} \\ &\lesssim \|\nabla b\|_{L^\infty}|h|\,\|f\|_{L^p} \le \|\nabla b\|_{L^\infty}|h|\end{aligned}$$

and consequently

$$\Big\|\int_{\mathbb{R}^n} I_1(\cdot,y,h)\,dy\Big\|_{L^p}\to 0 \tag{3.6}$$

as $|h|\to 0$ uniformly on $\|f\|_{L^p}\le 1$. Now we deal with I_2. We split the integral of I_2 into four regions:

$$\begin{aligned}R_1 &= \{y\in\mathbb{R}^n : |x-y|>\delta,\ |x+h-y|>\delta\},\\ R_2 &= \{y\in\mathbb{R}^n : |x-y|>\delta,\ |x+h-y|<\delta\},\\ R_3 &= \{y\in\mathbb{R}^n : |x-y|<\delta,\ |x+h-y|>\delta\},\\ R_4 &= \{y\in\mathbb{R}^n : |x-y|<\delta,\ |x+h-y|<\delta\}.\end{aligned}$$

By the definition of K^δ, clearly $\int_{R_4} I_2(x,y,h)\,dy = 0$. Thus it suffices to study the integral over the remaining R_i. Let us begin with R_1. First note that if $|h|<\delta/2$, then $x, x+h\in\frac{1}{2}B(x,\delta)$. Consequently, taking into account Lemma 3.9,

$$\begin{aligned}&\Big|\int_{R_1} I_2(x,y,h)\,dy\Big| \\ &\le \int_{R_1}|(b(x+h)-b(y))(K^\delta(x,y)-K^\delta(x+h,y))f(y)|\,dy \\ &\le 2\|b\|_{L^\infty}\int_{|x-y|>\delta}|K^\delta(x,y)-K^\delta(x+h,y)|\,|f(y)|\,dy \\ &= 2\|b\|_{L^\infty}\sum_{m=1}^\infty\int_{2^mB(x,\delta)\setminus 2^{m-1}B(x,\delta)}|K^\delta(x,y)-K^\delta(x+h,y)|\,|f(y)|\,dy \\ &= 2\|b\|_{L^\infty}\sum_{m=1}^\infty|2^mB|\frac{1}{|2^mB|}\int_{2^mB(x,\delta)\setminus 2^{m-1}B(x,\delta)}|K^\delta(x,y)-K^\delta(x+h,y)|\,|f(y)|\,dy\end{aligned}$$

$$\begin{aligned}
&\leq 2\|b\|_{L^\infty} \sum_{m=1}^{\infty} \Bigg[|2^m B| \\
&\quad\times \bigg(\frac{1}{|2^m B|} \int\limits_{2^m B(x,\delta) \setminus 2^{m-1} B(x,\delta)} |K^\delta(x,y) - K^\delta(x+h,y)|^r \, dy \bigg)^{1/r} \\
&\quad\times \bigg(\frac{1}{|2^m B|} \int\limits_{2^m B(x,\delta)} |f(y)|^{r'} \, dy \bigg)^{1/r'} \Bigg] \\
&\leq 2\|b\|_{L^\infty} M_{r'} f(x) \sum_{m=1}^{\infty} |2^m B| \\
&\quad\times \bigg(\frac{1}{|2^m B|} \int\limits_{2^m B(x,\delta) \setminus 2^{m-1} B(x,\delta)} |K^\delta(x,y) - K^\delta(x+h,y)|^r \, dy \bigg)^{1/r} \\
&\leq 2c\|b\|_{L^\infty} M_{r'} f(x) \sum_{m=0}^{\infty} |2^m B|^{1/r'} \\
&\quad\times \max \left\{ \frac{|h|^{\gamma - n/r'}}{|B|^{\gamma/n}} 2^{-m\gamma}, \frac{|h|}{|B|^{(1+n/r')/n}} 2^{-(1+n/r')m} \right\} \\
&\leq 2c\|b\|_{L^\infty} M_{r'} f(x) \max\{|h|^{\gamma - n/r'}, |h|\} \\
&\quad\times \bigg(\sum_{m=0}^{\infty} \frac{|2^m B|^{1/r'}}{|B|^{\gamma/n}} 2^{-m\gamma} + \sum_{m=0}^{\infty} \frac{|2^m B|^{1/r'}}{|B|^{(1+n/r')/n}} 2^{-(1+n/r')m} \bigg) \\
&\lesssim \|b\|_{L^\infty} \max\{|h|^{\gamma - n/r'}, |h|\} \, M_{r'} f(x).
\end{aligned}$$

Consequently, if $\|f\|_{L^p} \leq 1$ then

$$\begin{aligned}
\bigg\| \int\limits_{R_1} I_2(\cdot, y, h) \, dy \bigg\|_{L^p} &\leq 2c_\delta \|b\|_{L^\infty} \max\{|h|^{\gamma - n/r'}, |h|\} \|M_{r'} f\|_{L^p} \\
&\lesssim \max\{|h|^{\gamma - n/r'}, |h|\}
\end{aligned}$$

and hence

$$\bigg\| \int\limits_{R_1} I_2(\cdot, y, h) \, dy \bigg\|_{L^p} \to 0 \tag{3.7}$$

as $|h| \to 0$ uniformly on $\|f\|_{L^p} \leq 1$.

The integrals over R_2 and R_3 are symmetric so we deal just with R_2. We may assume that $|h|$ is very small. Let $\rho > \delta + |h|$ be such that b vanishes outside $B_0 = B(0, \rho)$. Then $b(\cdot + h)$ has support in $2B_0$. Since $R_2 \subset B(x, |h| + \delta)$, for $|x| < 3\rho$ we have $R_2 \subset 4B_0$. Hence, using the size condition on K^δ,

$$\begin{aligned}
&\Big|\int_{R_2} I_2(x,y,h)\,dy\Big| \\
&\quad\le \Big|\int_{R_2} (b(x+h)-b(y))(K^\delta(x,y)-K^\delta(x+h,y))f(y)\,dy\Big| \\
&\quad= \Big|\int_{R_2} (b(x+h)-b(y))K(x,y)\varphi(x-y)f(y)\,dy\Big| \\
&\quad\le \int_{4B_0\cap R_2} \Big|(b(x+h)-b(y))K(x,y)\varphi(x-y)f(y)\Big|\,dy \\
&\quad\lesssim \|b\|_{L^\infty} \int_{R_2\cap 4B_0} \frac{|f(y)|}{|x-y|^n}\,dy \lesssim \frac{1}{\delta^n}\|b\|_{L^\infty}\frac{|4B_0|}{|4B_0|}\int_{R_2\cap 4B_0} |f(y)|\,dy \\
&\quad\le \frac{1}{\delta^n}\|b\|_{L^\infty}|4B_0|\bigg(\frac{1}{|4B_0|}\int_{4B_0}|f(y)|^{r'}\,dy\bigg)^{1/r'}\bigg(\frac{|R_2|}{|4B_0|}\bigg)^{1/r} \\
&\quad\lesssim \frac{\rho^{n/r'}}{\delta^n}\|b\|_{L^\infty}|R_2|^{1/r}M_{r'}f(x).
\end{aligned}$$

In the case $|x|\ge 3\rho$, since $|h|<\rho$ and $|x+h|>2\rho$, we have $b(x+h)=0$, and hence, arguing as above,

$$\begin{aligned}
\Big|\int_{R_2} I_2(x,y,h)\,dy\Big| &\le \int_{4B_0\cap R_2} |(b(x+h)-b(y))K(x,y)\varphi(x-y)f(y)|\,dy \\
&= \int_{4B_0\cap R_2} |b(y)K(x,y)\varphi(x-y)f(y)|\,dy \\
&\lesssim \|b\|_{L^\infty}\int_{R_2\cap 4B_0} \frac{|f(y)|}{|x-y|^n}\,dy \\
&\lesssim \frac{\rho^{n/r'}}{\delta^n}\|b\|_{L^\infty}|R_2|^{1/r}M_{r'}f(x)
\end{aligned}$$

Hence, gathering the estimates above, for $\|f\|_{L^p}\le 1$,

$$\begin{aligned}
\Big\|\int_{R_2} I_2(\cdot,y,h)\,dy\Big\|_{L^p} &\lesssim \frac{\rho^{n/r'}}{\delta^n}\|b\|_{L^\infty}\|M_{r'}f\|_{L^p}|R_2|^{1/r} \\
&\lesssim \frac{\rho^{n/r'}}{\delta^n}\|b\|_{L^\infty}|R_2|^{1/r}.
\end{aligned}$$

Since $|R_2|\to 0$ as $|h|\to 0$, we see that

$$\Big\|\int_{R_2} I_2(\cdot,y,h)\,dy\Big\|_{L^p}\to 0 \tag{3.8}$$

as $|h| \to 0$ uniformly for $\|f\|_{L^p} \le 1$. As we mentioned above, a similar argument yields

$$\Big\| \int_{R_3} I_2(\cdot, y, h)\, dy \Big\|_{L^p} \to 0 \tag{3.9}$$

as $|h| \to 0$ uniformly for $\|f\|_{L^p} \le 1$. Consequently, (3.4) follows by combining (3.5)–(3.9).

Finally, we study decay at infinity. Since b is compactly supported, there exists $R_0 > 0$ such that $\operatorname{supp} b \subset B(0, R_0)$. Let x be such that $|x| > R > 3R_0$. Then $x \notin \operatorname{supp} b$, so

$$\begin{aligned} |[b, T^\delta] f(x)| &= |b(x) T^\delta f(x) - T^\delta (fb)(x)| = |T^\delta (fb)(x)| \\ &= \Big| \int_{\operatorname{supp} b} b(y) K^\delta(x, y) f(y)\, dy \Big| \lesssim \|b\|_{L^\infty} \int_{\operatorname{supp} b} \frac{|f(y)|}{|x-y|^n}\, dy \\ &\lesssim \|b\|_{L^\infty} \frac{1}{|x|^n} \int_{B(0,R_0)} |f(y)|\, dy \le \|b\|_{L^\infty} \frac{1}{|x|^n} |B(0, R_0)|^{1/p'} \|f\|_{L^p}. \end{aligned}$$

Hence for $R > 3R_0$,

$$\begin{aligned} \|\chi_{\{|x|>R\}} [b, T^\delta] f\|_{L^p} &\lesssim \Big\| \frac{1}{|\cdot|^n} \chi_{\{|x|>R\}}(\cdot) \Big\|_{L^p} \|b\|_{L^\infty} |B(0, R_0)|^{1/p'} \|f\|_{L^p} \\ &\simeq \frac{1}{R^{n-n/p}} \|b\|_{L^\infty} |B(0, R_0)|^{1/p'} \|f\|_{L^p}, \end{aligned}$$

and letting $R \to \infty$ leads to the desired conclusion.

3.3.2. *Proof of Theorem 3.4.* Let T be an L^r-Hörmander operator. We begin by observing that the L^r-Hörmander condition presented here was borrowed from [7] and is implied by the one in [29, 30, 27, 23, 28]. In those papers it was shown that if $b \in \mathrm{BMO}$ then $[b, T]$ is bounded on $L^p(w)$ for every $w \in A_{p/r'}$ provided that $p > r'$ ($w \in A^+_{p/r'}$ in the one-sided setting). By inspection of the proofs it can be easily checked that the dependence of the boundedness constant on the $A_{p/r'}$ constant ($A^+_{p/r'}$ in the one-sided setting) is increasing. In particular, if we fix $p_0 > r'$ we find that $[b, T]$ is bounded on $L^{p_0}(w)$ for every $w \in A_{p_0/r'}$ ($w \in A^+_{p_0/r'}$ in the one-sided setting). Combining this fact, Theorem 3.5 and the main result of [20] or Corollary 1.3 in the one-sided setting yields the desired conclusion.

3.3.3. *Proof of Theorem 3.7.* It suffices to exploit the results in [19]. Let $\alpha \in (0, 1)$ and

$$T_{K_\alpha} f(x) = \int_{\mathbb{R}} K_\alpha(x, y) f(y)\, dy, \qquad x \notin \operatorname{supp} f,$$

with kernel K_α such that

$$|K_\alpha(x,y)| \le \frac{1}{|x-y|^{1-\alpha}} \tag{3.10}$$

and for $|x-y| > 2|x-x'|$,

$$|K_\alpha(x,y) - K_\alpha(x',y)| + |K_\alpha(y,x) - K_\alpha(y,x')| \le \rho\left(\frac{|x-x'|}{|x-y|}\right)\frac{1}{|x-y|^{1-\alpha}}, \tag{3.11}$$

where $\rho : [0,1] \to [0,\infty)$ is a modulus of continuity, that is, ρ is continuous, increasing, subadditive, $\rho(0) = 0$ and

$$\int_0^1 \rho(t)\,\frac{dt}{t} < \infty.$$

For that class of operators we have the following particular case of [19, Theorem 1.5].

THEOREM 3.11. *Let* $1 < p < q < \infty$*,* $0 \le \alpha < 1$*,* $\alpha = \frac{1}{p} - \frac{1}{q}$*. If* $b \in \mathrm{CMO}(\mathbb{R})$ *then* $[b, T_{K_\alpha}]$ *is compact from* L^p *to* L^q*.*

It is straightforward to check that Theorem 3.7 is a direct corollary of this result, by choosing

$$K_\alpha(x,y) = \frac{1}{(y-x)^{1-\alpha}}\chi_{\{z>0\}}(y-x).$$

The details are left to the reader.

3.3.4. *Proof of Theorem 3.6.* It was shown in [6] that if $1 < p < q < \infty$ with $\frac{1}{p} - \frac{1}{q} = \alpha$ then $[b, I_\alpha^+] : L^p(w^p) \to L^q(w^q)$ for every $w \in A_{p,q}^+$. By inspection of the proofs it can be easily checked that the dependence of the boundedness constant on the $A_{p,q}^+$ constant is increasing. In particular, if we fix $1 < p_0 < q_0 < \infty$ such that $\frac{1}{p_0} - \frac{1}{q_0} = \alpha$, we find that $[b, I_\alpha^+] : L^p(w^p) \to L^q(w^q)$ for every weight $w \in A_{p_0,q_0}^+$. Combining this fact with Theorem 3.7 and Corollary 1.6 yields the desired conclusion.

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