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A JORDAN CANONICAL FORM FOR NILPOTENT ELEMENTS IN AN ARBITRARY RING

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Abstract. In this paper we give an inductive new proof of the Jordan canonical form of a nilpotent element in an arbitrary ring. If $a \in R$ is a nilpotent element of index n with von Neumann regular a^{n-1}, we decompose $a = ea + (1-e)a$ with $ea \in eRe \cong \mathcal{M}_n(S)$ a Jordan block of size n over a corner S of R, and $(1-e)a$ nilpotent of index $< n$ for an idempotent e of R commuting with a. This result makes it possible to characterize prime rings of bounded index n with a nilpotent element $a \in R$ of index n and von Neumann regular a^{n-1} as a matrix ring over a unital domain.

Dedicated to the late Professor E. García Rus

Key words: von Neumann regular, nilpotent, Jordan canonical form.

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1. introduction

It is well-known that a square matrix over an algebraically closed field is similar to a matrix consisting of Jordan blocks. In [5] the authors study a similar result for nilpotent elements of the socle of a nondegenerate Jordan algebra, and in such Jordan context they proved that these elements admit, up to isomorphism, a Jordan canonical form. One of the main results of that work was the characterization of nilpotent indecomposable elements in the socle as nilpotent elements of index n and rank $n-1$.

Canonical forms of nilpotent elements in general rings, with the extra condition that all its powers von Neumann regular, have already been studied by several authors. In the book [14], O'Meara, Clark and Vinsonhaler remarked on the importance of the Weyr form, and characterized the possibility of obtaining a Weyr form of a nilpotent endomorphism t of any nonzero quasi-projective module P over an arbitrary ring R in terms of the von Neumann regularity of the powers of t in the endomorphism ring $\mathrm{End}_R(P)$, [14, Theorem 4.8.2]. In such situation, the module P decomposes as a direct sum of nonzero submodules, $P = A_1 \oplus A_2 \oplus \cdots \oplus A_r$, such that $t(A_1) = 0$ and $t(A_i) = A_{i-1}$ for $i = 2, \ldots, r$, see [7, Lemma 7.1] or [2, Lemma 3.5].

This Weyr decomposition of nilpotent endomorphisms was used by Beidar, O'Meara and Raphael to obtain the Jordan canonical form of a nilpotent element in a ring [2, Theorem 3.6 (iii)]. In their result they showed that if R is an algebra over a commutative ring Γ, and $a \in R$ is a nilpotent element of index r such that all the powers of a are von Neumann regular in R, then there exist integers

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$1 \leq n_1 < n_2 < \cdots < n_s = r$, an inner-inverse b of a, and ideals $I_1, I_2, \ldots, I_s$ of the ring Γ such that $\mathrm{Ann}_\Gamma(R) = \bigcap_j I_j$ and there is an algebra isomorphism

$$\Phi : \Gamma[a, b] \to \prod_{i=1}^{s} \mathcal{M}_{n_i}(\Gamma/I_i)$$

such that each canonical image of a in $\mathcal{M}_{n_i}(\Gamma/I_i)$ has the form of a Jordan matrix block. We believe that this very nice result fixes the skeleton of our Jordan canonical form.

Our arguments are ring-theoretic, rather than module-theoretic as in [2, Lemma 3.5], [2, Theorem 3.6], [14, Theorem 4.10.2] or [10], but they can easily be translated into the language of modules if the reader wishes, by using the fact that von Neumann regularity has a module-theoretic analog, see for example [16] or [6]. The first step in our induction process, which consists of splitting off a Jordan block of maximal size, only requires that $a^n = 0$ and a^{n-1} is von Neumann regular. In this case, we find an idempotent e in R that commutes with a and such that eRe is isomorphic to a ring of matrices of size $n \times n$ over a unital ring. The element ea under this isomorphism corresponds to a Jordan block of maximal size. If a itself coincides with ea, then a is called a block-element associated to the block-idempotent e.

We begin by finding an appropriate inner-inverse for a^{n-1}. The search of inner-inverses of regular elements satisfying families of additional monoid relations (related to powers of the original element) has been broadly dealt in the literature and is linked to inverse monoids, see for example [15] and [12]. We highlight the work of Nielsen and Stern [13, Theorem 4.8] (see also the related paper of Bergman [3]), where they show that if an element a and all its powers are von Neumann regular, an inner-inverse b can be produced such that a^k and b^k are inner-inverses of each other, for all k, and some other extra relations. For our purposes, the inner-inverse b needed for a^{n-1} must annihilate all the powers a^k for $k < n-1$ when multiplied on both sides, i.e., $ba^kb = 0$, together with a^{n-1} and b being inner-inverses of each other. We will call such an inner-inverse a Rus-inverse for a since a construction of such an element was done by the late Prof. E. García Rus in [5, Lemma 3.2] in the context of Jordan algebras of symmetric elements of an associative algebra with involution.

We show that every nilpotent element $a \in R$ for which all powers a^k are von Neumann regular can be decomposed (with respect to a family of orthogonal idempotents commuting with a) into a finite sum of nilpotent block-elements of decreasing indexes associated to those idempotents. These pieces are Jordan blocks and the sum of these Jordan blocks is what we call the Jordan canonical form of a nilpotent element.

As in the classical setting, if we turn to unital associative algebras A over an algebraically closed field, we can use the above decomposition to produce a Jordan canonical form of any algebraic element. If $m_a(X) = \prod_i (X - \lambda_i)^{n_i}$ is the minimal polynomial of an algebraic element $a \in A$ and all powers $(a - \lambda_i)^{k_i}$, $k_i \leq n_i$, are von Neumann regular, we build a family of orthogonal idempotents that decompose a into pieces that act as Jordan blocks.

We must remark that the block-idempotent associated to a nilpotent element a of index n with von Neumann regular a^{n-1} may not be unique. Nevertheless, we show that all of them coincide if they are central. In the particular case of rings with bounded index of nilpotency, we show that any nilpotent element a of maximal index n and regular a^{n-1} gives rise to a central — and therefore unique — block-idempotent. As an easy consequence of the Jordan canonical form, we recover and extend the descriptions in [7] of rings of bounded index n: for example, when R is prime and a is a nilpotent element of maximal index with von Neumann regular a^{n-1}, $R \cong \mathcal{M}_n(S)$ for a unital domain S, and when R is indecomposable and von Neumann regular, $R \cong \mathcal{M}_n(\Delta)$ for a division ring Δ.

2. Jordan Canonical Form

Throughout this paper we will deal with non-necessarily unital rings R. Expressions of type $(1-x)y$, for $x, y \in R$, will be allowed because they just mean $y - xy \in R$. R^1 will denote the Dorroh unitization of R, i.e., $R^1 = R + \mathbb{Z}1$.

2.1. An element $a \in R$ is said to be von Neumann regular if there exists $b \in R$ such that $aba = a$. If every element of R is von Neumann regular we say that R is a von Neumann regular ring. An element a in a unital ring R is called unit-regular if there exists an invertible $b \in R$ such that $aba = a$.

2.2. An element $a \in R$ is said to be nilpotent if there exists $n \in \mathbb{N}$ such that $a^n = 0$. The index of nilpotence of a nilpotent element a in a ring R is the least positive integer n such that $a^n = 0$. In particular, 0 is nilpotent of index 1. We say that a is nilpotent of maximal index if a is nilpotent of index n and there are no nilpotent elements in R of index m for $m > n$.

2.3. A ring R is said to be an abelian regular ring if R is von Neumann regular and every idempotent of R is contained in the center of R. It is well-known that R is abelian regular if and only if it is von Neumann regular and has no nonzero nilpotent elements [7, Theorem 3.2]. A non-necessarily unital ring R is called indecomposable if every central idempotent is either 0 or 1.

The following lemma is analogous to [5, Lemma 3.2], but we have slightly changed its proof to generalize the result to arbitrary rings with any torsion.

Lemma 2.4. *Let R be a ring and let $a \in R$ be a nilpotent element of index n such that a^{n-1} is von Neumann regular. Then there exists $b \in R$ such that $a^{n-1}ba^{n-1} = a^{n-1}$, $ba^{n-1}b = b$ and $ba^kb = 0$ for every $0 \le k \le n-2$. Such an element b will be called a Rus-inverse of a.*

Proof. Since a^{n-1} is von Neumann regular there exists $b \in R$ such that $a^{n-1}ba^{n-1} = a^{n-1}$. It is well-known that $b' = ba^{n-1}b \in R$, satisfies

$$a^{n-1}b'a^{n-1} = a^{n-1} \qquad b'a^{n-1}b' = b'.$$

We use a recursive argument: by decreasing induction on $s = n-2, \dots, 0$ suppose that there exists $b \in R$ such that for every $k = s+1, \dots, n-2$ we have that

$$a^{n-1}ba^{n-1} = a^{n-1}, \quad ba^{n-1}b = b \quad \text{and} \quad ba^kb = 0.$$

Define $d := 1 - a^{n-1-s}ba^s$. Then $c := db$ satisfies $a^k c = a^k b$, $k = s+1, \dots, n-1$, since $a^n = 0$, and

$$\begin{aligned} a^{n-1}ca^{n-1} &= a^{n-1}ba^{n-1} = a^{n-1}, \\ ca^{n-1}c &= ca^{n-1}b = dba^{n-1}b = db = c, \\ ca^k c &= ca^k b = dba^k b = 0, \text{ and} \\ ca^s c &= ca^s(1 - a^{n-1-s}ba^s)b = ca^s b - ca^s a^{n-1-s}ba^s b \\ &= dba^s b - dba^{n-1}ba^s b = 0. \end{aligned}$$

The Lemma now follows by recursion. □

2.5. The construction of elements associated to nilpotent elements a with regular a^{n-1} satisfying the extra properties of Lemma 2.4 was introduced in [5, Lemma 3.2] in the context of special Jordan algebras, i.e., Jordan algebras consisting on symmetric elements of associative algebras with involution. The main ideas of such lemma are due to the late Prof. E. García Rus, and that is the reason why we have decided to call them Rus-inverses.

Theorem 2.6. *Let R be a ring and $a \in R$ be a nonzero nilpotent element of index n such that a^{n-1} is von Neumann regular. Let b be a Rus-inverse of a. Then there exists n nonzero orthogonal idempotents $\{e_k\}_{k=1}^n$ in R (depending on b) such that if $e = \sum_{k=1}^n e_k$*

(a) *$ea = ae$,*
(b) *$a^{n-1}e = a^{n-1}$,*
(c) *$(1-e)a$ is nilpotent of index less than n,*
(d) *$eRe \cong \mathcal{M}_n(e_1Re_1)$, and if $\{e_{i,j}\}$ are the matrix units of the matrix ring eRe, $ea = eae = \sum_{k=1}^{n-1} e_{k+1,k}$.*
(e) *For every $s \in \{1, 2, \dots, n-1\}$, $(ea)^s$ is unit-regular in eRe: taking $d = \sum_{k=1}^{n-1} e_{k,k+1} + e_{n,1}$, then $(ea)^s d^s (ea)^s = (ea)^s$ and d^s is invertible in eRe.*

In the matrix representation of ea on item (d) we have that $e_{1,n} = eb = be$ is a Rus-inverse for ea with associated idempotent e.

We will say that ea is a block-element associated to the block-idempotent e (e may depend on the Rus-inverse b of a).

Proof. By Lemma 2.4 there exists a Rus-inverse $b \in R$ for a, i.e., $b \in R$ such that

$$a^{n-1}ba^{n-1} = a^{n-1}, \qquad ba^{n-1}b = b, \qquad ba^k b = 0, \ k \le n-2.$$

If we define $e_{i,j} := a^{i-1}ba^{n-j}$, for $i, j = 1, 2, \dots, n$, we have:

$$e_{u,v}e_{r,s} = a^{u-1}ba^{n-v}a^{r-1}ba^{n-s} = a^{u-1}ba^{n-v+r-1}ba^{n-s} = \delta_{v,r}a^{u-1}ba^{n-s} = \delta_{v,r}e_{u,s},$$

where $\delta_{v,r}$ denotes the Kronecker delta. Therefore $\{e_i := e_{i,i}\}_{i=1}^n$ is a set of orthogonal idempotents. Moreover, they are nonzero since $e_{n,i}e_ie_{i,1} = e_{n,1} = a^{n-1}ba^{n-1} = a^{n-1} \neq 0$.

(a) We have that

$$\begin{aligned} e_k a &= (a^{k-1}ba^{n-k})a = a(a^{k-2}ba^{n-k+1}) = ae_{k-1}, \quad \text{for} \quad k = 2, \dots n, \\ e_1 a &= 0 = ae_n. \end{aligned}$$

Therefore $e = \sum_{i=1}^n e_i$ is an idempotent of R that commutes with a:

$$ae = a\left(\sum_{i=1}^{n} e_i\right) = a\left(\sum_{i=1}^{n-1} e_i\right) = \left(\sum_{i=2}^{n} e_i\right) a = \left(\sum_{i=1}^{n} e_i\right) a = ea.$$

(b) $a^{n-1}e = a^{n-1}e_1 = a^{n-1}ba^{n-1} = a^{n-1}$.

(c) $((1-e)a)^{n-1} = (1-e)a^{n-1} = a^{n-1} - ea^{n-1} = 0$.

(d) The fact that eRe is isomorphic to a matrix ring follows directly from the fact that we have a full set of matrix units, as described in the first paragraph of the proof, see for example [11, (17.5) and (17.6)]. Moreover, $eae = (e_2 + \cdots + e_n)a(e_1 + \cdots + e_{n-1}) = \sum_{k=1}^{n-1} e_{k+1}ae_k = \sum_{k=1}^{n-1} e_{k+1,k}$.

(e) If $d := \sum_{k=1}^{n-1} e_{k,k+1} + e_{n,1}$ then $d^s = \sum_{k=1}^{n-s} e_{k,k+s} + \sum_{k=1}^{s} e_{n-s+k,k}$ is invertible in eRe for all $s \in \{1, \ldots, n-1\}$ and $(ea)^s d^s (ea)^s = (ea)^s$. □

Remark 2.7. In the particular case of a nilpotent von Neumann regular element a of index 2, a is a block-element associated to the idempotent $e = ab + ba$ for any Rus-inverse b of a. Notice that, in general, if a is nilpotent of index n, a^{n-1} is von Neumann regular and b is a Rus-inverse of a, the block-idempotent given in Theorem 2.6 is $e = (a+b)^n$.

Remark 2.8. Item (e) in Theorem 2.6 gives a representation of ea as a "Jordan block" (like the Jordan canonical form of a nilpotent element in a ring of endomorphisms of a finite dimensional vector space). Notice that ea and its powers are von Neumann regular (see item (f)), so the conditions of Theorem 2.6 are both necessary and sufficient. When a is a nilpotent block-element of index n, the von Neumann regularity of a^{n-1} implies the von Neumann regularity of every power of a.

The Jordan canonical form obtained in this theorem is not exactly the Jordan canonical form of a matrix ring over a field. In the classical setting, if $a \in \mathcal{M}_n(\mathbb{F})$ is a nilpotent matrix of index n, we get as many Jordan blocks of size n as the rank of the matrix a^{n-1}. In the Jordan decomposition given by Theorem 2.6(e), all Jordan blocks of the same size collapse into one.

Example 2.9. Let $\mathbb{F}$ be a field and let us consider the von Neumann regular ring $R = \mathcal{M}_6(\mathbb{F})$. The matrix

$$A := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{M}_6(\mathbb{F})$$

consists of two Jordan blocks of size 3×3 associated to the eigenvalue 0. If $\{e_{i,j}\}$ denote the matrix units in $\mathcal{M}_6(\mathbb{F})$, $A = e_{2,1} + e_{3,2} + e_{5,4} + e_{6,5}$. The element $A \in R$ is nilpotent of index 3. A Rus-inverse for A is $B = e_{1,3} + e_{4,6}$. If we denote by capital letters the elements $e_{i,j}$ defined in the proof of Theorem 2.6, we have

$$\begin{aligned} E_{1,1} &= BA^2 = e_{1,1} + e_{4,4}, \\ E_{2,2} &= ABA = e_{2,2} + e_{5,5}, \\ E_{3,3} &= A^2B = e_{3,3} + e_{6,6}, \text{ and} \\ E &= E_{1,1} + E_{2,2} + E_{3,3} = \mathrm{Id}, \end{aligned}$$

so $A = EA$ is a block-element associated to the identity matrix E in R. Notice that

$$\begin{aligned} &E_{2,1} = ABA^2 = e_{2,1} + e_{5,4}, \\ &E_{3,2} = A^2BA = e_{3,2} + e_{6,5}, \text{ hence} \\ &A = EA = E_{2,1} + E_{3,2}. \end{aligned}$$

Thus $R \cong \mathcal{M}_3(E_{1,1}RE_{1,1})$ with $E_{1,1}RE_{1,1} \cong \mathcal{M}_2(\mathbb{F})$ and

$$A = \begin{pmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \end{pmatrix} \quad \text{with} \quad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Corollary 2.10. *Let R be a ring and $a \in R$ be a nonzero nilpotent element of index n such that for every $s \in \mathbb{N}$, a^s is von Neumann regular. Then there exists a family $\{u_i\}_{i=1}^k$ of nonzero orthogonal idempotents that commute with a and such that $a = \sum_{i=1}^k u_i a$, and every $u_i a$ is a nilpotent block-element of index n_i with $n = n_1 > n_2 \cdots > n_k$ associated to the block-idempotent u_i.*

Proof. Let $a^{(1)} := a$. By Theorem 2.6 there exists a family of nonzero orthogonal idempotents $\{e_k^{(1)}\}$ such that if $u_1 := \sum e_k^{(1)}$, $u_1 a^{(1)} = u_1 a = a^{(1)} u_1$ is a nilpotent block-element of R of index $n_1 = n$, and $((1-u_1)a^{(1)})^{n-1} = 0$. Thus $a^{(2)} := (1-u_1)a^{(1)} = (1-u_1)a$ is nilpotent of index n_2 with $n > n_2$. Notice that $(a^{(2)})^{n_2-1}$ is von Neumann regular: since a^{n_2-1} is von Neumann regular there exists $c \in R$ with $a^{n_2-1}ca^{n_2-1} = a^{n_2-1}$ and if we multiply by $1-u_1$ on the both sides we obtain

$$(1-u_1)a^{n_2-1}ca^{n_2-1}(1-u_1) = (1-u_1)a^{n_2-1}(1-u_1),$$

so

$$\begin{aligned} &(a^{(2)})^{n_2-1}c(a^{(2)})^{n_2-1} = ((1-u_1)a)^{n_2-1}c((1-u_1)a)^{n_2-1} = \\ &= (1-u_1)a^{n_2-1}ca^{n_2-1}(1-u_1) = (1-u_1)a^{n_2-1}(1-u_1) = \\ &= ((1-u_1)a)^{n_2-1} = (a^{(2)})^{n_2-1}. \end{aligned}$$

We can apply Theorem 2.6 to the element $a^{(2)}$ and prove the corollary by a recursive argument. □

Now we follow the classical procedure to extend the former results to algebraic elements in a unital algebra A over a field $\mathbb{F}$.

Theorem 2.11. *Let A be a unital algebra over a field $\mathbb{F}$ and let a be an algebraic element in A such that its minimal polynomial $m_a(X)$ totally decomposes in $\mathbb{F}[X]$ as $\prod_{i=1}^k (X-\lambda_i)^{n_i}$ where $\lambda_i \neq \lambda_j$ if $i \neq j$. Suppose that $(a-\lambda_i)^{k_i}$ is von Neumann regular for every $k_i \leq n_i$, $i = 1, \ldots, k$. Then there exists a family of orthogonal idempotents $\{v_s\}_{s=1}^k$ and families of orthogonal idempotents $\{u_{s,i}\} \subseteq v_s R v_s$, all commuting with a, such that $a = \sum_{s,i} u_{s,i} a$, where each $u_{s,i}a - \lambda_s u_{s,i} \in u_{s,i} R u_{s,i}$ is a nilpotent block-element of index $n_{s,i} \leq n_s$ associated to the block-idempotent $u_{s,i}$,* *i.e.,* $u_{s,i}Ru_{s,i} \cong \mathcal{M}_{n_{s,i}}(S_{s,i})$ *for some unital algebra* $S_{s,i}$ *and* $u_{s,i}a$ *has the form*

$$\begin{pmatrix} \lambda_s & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_s & 0 & \cdots & & 0 \\ 0 & 1 & \lambda_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 1 & \lambda_s & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda_s \end{pmatrix}.$$

Proof. For every $s = 1, 2, \ldots, k$ let us define $q_s(X) := \prod_{i \neq s}(X - \lambda_i)^{n_i}$. Then $q_1(X), \ldots, q_k(X)$ are co-prime and we can find a Bezout identity for them: there exist polynomials $r_1(X), \ldots, r_k(X) \in \mathbb{F}[X]$ such that

$$r_1(X)q_1(X) + r_2(X)q_2(X) + \cdots + r_k(X)q_k(X) = 1.$$

If we define $v_s := r_s(a)q_s(a)$ we have that $\{v_s\}_{s=1}^k$ is a family of k-orthogonal idempotents that commute with a and such that $v_s(a - \lambda_s) \in v_sRv_s$ is a nilpotent element of R of index n_s. Now apply Corollary 2.10 to decompose each $v_s(a-\lambda_s) \in v_sRv_s$ into a sum of nilpotent block-elements (associated to a family of orthogonal idempotents $\{u_{s,i}\} \subseteq v_sRv_s$) of decreasing indexes $n_s = n_{s,1} > n_{s,2} >$ ... and then apply Theorem 2.6 to each element $u_{s,i}(a - \lambda_s)$ in $u_{s,i}R$: $u_{s,i}(a - \lambda_s)$ can be seen as a matrix of size $n_{s,i} \times n_{s,i}$ of the form

$$\begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \vdots & 1 & 0 \end{pmatrix},$$

so $u_{s,i}a$ can be seen as a matrix of size $n_{s,i} \times n_{s,i}$ of the form

$$\begin{pmatrix} \lambda_s & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_s & 0 & \cdots & & 0 \\ 0 & 1 & \lambda_s & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & 1 & \lambda_s & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda_s \end{pmatrix}.$$

□

Corollary 2.12. *Every algebraic element in an unital von Neumann regular algebra over an algebraically closed field admits a Jordan canonical form.*

Remark 2.13. In general, neither the Rus-inverse nor the associated idempotent in Theorem 2.6 are unique: If $R = \mathcal{M}_3(\mathbb{F})$ and $a = e_{1,2} \in R$, the element $b = e_{2,1} \in R$ is a Rus-inverse for a and a is a block-element with associated block-idempotent $e = e_{1,1} + e_{2,2}$; the element $b' = e_{2,1} + e_{3,1} \in R$ is another Rus-inverse for a and a is a block-element with associated block-idempotent $e' = e_{1,1} + e_{2,2} + e_{3,2}$.

Nevertheless, all block-idempotents agree when they are central. To prove that fact we need the following technical lemma.

The following result is (c)⇒(a) of [7, Theorem 7.2].

Lemma 2.14. *Let R be a non-necessarily unital ring and let $\{e_i\}_{i=1}^n$ be a set of nonzero orthogonal idempotents, where one of the e_i might be of the form $1-e \in R^1$ for an idempotent $e \in R$, and suppose that there exists $x_i \in R$, $i = 1, \ldots, n-1$, with $e_1 x_1 e_2 x_2 \cdots e_{n-1} x_{n-1} e_n \neq 0$. Then R contains a nilpotent element of index n.*

Proof. The element $a := \sum_{i=1}^{n-1} e_i x_i e_{i+1}$ is nilpotent of index n. □

2.15. We say that a nilpotent element $a \in R$ of index n with von Neumann regular a^{n-1} is block-maximal if one of its associated block-idempotents belongs to the center of R, i.e., if there exists a Rus-inverse of a such that the idempotent built in Theorem 2.6 is central.

Proposition 2.16. *All central block-idempotents associated to a block-maximal element coincide.*

Proof. Let a be a nilpotent element of R of index n such that a^{n-1} is von Neumann regular. Suppose that some block-idempotent e is in the center $Z(R)$ of R. By Theorem 2.6(e) $eRe \cong \mathcal{M}_n(S)$ for some unital ring S and $ea = \sum_{k=1}^{n-1} e_{k+1,k}$. Let us prove that the ideal $\langle a^{n-1} \rangle$ generated by a^{n-1} equals eR: since $a^{n-1} = ea^{n-1} = (ea)^{n-1} = e_{n,1}$, every $e_{i,j} = e_{i,n} e_{n,1} e_{1,j} \in \langle a^{n-1} \rangle$ and therefore $e \in \langle a^{n-1} \rangle$, hence $eR \subseteq \langle a^{n-1} \rangle$. On the other hand, $a^{n-1} = ea^{n-1} \in eR$, which implies, since eR is an ideal ($e \in Z(R)$), that $\langle a^{n-1} \rangle \subseteq eR$. Therefore, $\langle a^{n-1} \rangle$ is a unital ring and e is its unit element. Repeating the same argument with any other block-idempotent $e' \in Z(R)$ we also get that $e'R = \langle a^{n-1} \rangle$ and e' is the unit element of the unital ring $\langle a^{n-1} \rangle$, hence $e' = e$. □

Corollary 2.17. *Let R be a ring and let $a \in R$ be a nilpotent element of R of maximal index n such that a^{n-1} is von Neumann regular. Then a is block-maximal and all its associated block-idempotents coincide.*

Proof. By Theorem 2.6 there exists n orthogonal idempotents $\{e_i\}_{1=1}^n$ such that $e = \sum_{i=1}^n e_i$, $ea = ae \in eRe \cong \mathcal{M}_n(e_1 R e_1)$. Let us see that $eR(1-e) = 0 = (1-e)Re$. Suppose on the contrary that $eR(1-e) \neq 0$, so there exists $e_{k,k}$ and $x \in R$ with $e_{k,k} x (1-e) \neq 0$ and the chain

$$e_{1,1}\, e_{1,2}\, e_{2,2} \cdots e_{k-1,k-1}\, e_{k-1,k+1}\, e_{k+1,k+1} \cdots e_{n,n}\, e_{n,k} e_{k,k} x (1-e) \neq 0$$

implies by Lemma 2.14 that there exists a nilpotent element of index $n+1$, a contradiction. Similarly $(1-e)Re = 0$. Thus $e \in Z(R)$. We have shown that all block-idempotents associated to a are central, so all of them coincide by Proposition 2.16. □

Some authors have studied semiprime rings of bounded index of nilpotency via their complete right rings of quotients (see for example the works of Beidar and Mikhalev [4] and Hannah [8]). We also mention the result of Armendariz [1, Theorem 1], where he showed that a semiprime ring of maximal index n is a subdirect product of prime rings of indexes $\leq n$.

In the following result we describe the structure of rings with an element a of maximal index n and regular a^{n-1}. With some extra conditions on the regularity of the ring, we give more precise descriptions of coefficients of the involved rings of matrices. Some of the items of this theorem have already appeared in the literature (for example, (b) is [7, Lemma 7.17], and (e) is [7, Theorem 7.9] and is related to a former theorem of Kaplansky, see [9, Theorem 2.3]).

Theorem 2.18. *Let R be a ring and let $a \in R$ be a nilpotent element of R of maximal index n such that a^{n-1} is von Neumann regular. Then*

(a) *Any Rus-inverse of a gives rise to the same (central) idempotent e and $R = eRe \oplus (1-e)R(1-e)$. Moreover, $eRe \cong \mathcal{M}_n(S)$ where S is a ring without nonzero nilpotent elements.*
(b) *If R is von Neumann regular, $eRe \cong \mathcal{M}_n(S)$ where S is abelian regular.*
(c) *If R is indecomposable, $R \cong \mathcal{M}_n(S)$ where S is a unital ring without nilpotent elements.*
(d) *If R is prime, $R \cong \mathcal{M}_n(S)$ where S is a unital domain.*
(e) *If R is indecomposable and von Neumann regular, $R \cong \mathcal{M}_n(\Delta)$ where Δ is a division ring.*

In any case, as soon as R is indecomposable, $e = 1$ and a is a block-element.

Proof. By Theorem 2.6 there exist n orthogonal idempotents $\{e_i\}_{1=1}^n$ such that if $e = \sum_{i=1}^n e_i$, $ea = ae \in eRe \cong \mathcal{M}_n(e_1Re_1)$.

(a) By Corollary 2.17 there is only one (central) block-idempotent associated to a and $R = eRe \oplus (1-e)R(1-e)$. If there exists a nonzero nilpotent element $b \in e_1Re_1$ of index k, the element $be_{1,1} + \sum_{i=1}^{n-1} e_{i,i+1}$ is nilpotent of index $k+n-1$, a contradiction. Therefore $S = e_1Re_1$ is a ring without nonzero nilpotent elements.

(b) If R is von Neumann regular, $S = e_1Re_1$ is von Neumann regular without nilpotent elements, and therefore it is abelian regular.

(c) As soon as R is indecomposable, (a) implies that $e = 1$.

(d) If R is prime, then it is indecomposable and $e = 1$. Moreover, $S = e_1Re_1$ is also prime. A prime ring without nilpotent elements is a domain.

(e) Since R is indecomposable and von Neumann regular, $R \cong \mathcal{M}_n(S)$ where S is abelian regular. Let us see that S is a division ring: for any $0 \neq x \in S$ there exists $y \in S$ with $xyx = x$; then xy is a nonzero (central) idempotent of S, and therefore $\sum_{i=1}^n (xy)e_{ii} \in R$ is a nonzero central idempotent of R. Thus $xy = 1$, i.e., S a division ring. □

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