Manuscript

Aggregation of fuzzy subgroups

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Abstract

Given an aggregation function, and two fuzzy subgroups of a group, this study investigates whether the aggregation of them is also a fuzzy subgroup.

It is proven that if the cardinal of the group is a prime power, then it is always a fuzzy subgroup. For a general group, it is proven that the existence of a binary relation between the fuzzy subgroups determines if their aggregation is a fuzzy subgroup. Moreover, given two fuzzy subgroups, if the aggregation of them is a fuzzy subgroup for some strictly monotone aggregation function, then it is a fuzzy subgroup for any aggregation function. The present study also examines conditions on the aggregation function that characterise the case where the aggregation of two arbitrary fuzzy subgroups is also a fuzzy subgroup.

Keywords: Triangular norm, Aggregation function, Fuzzy subgroup, Lattice.

1. Introduction

The first definition of a fuzzy subgroup was given by A. Rosenfeld in [28], extending the subgroup concept of a group to a fuzzy environment. A fuzzy subgroup was defined as a fuzzy set μ of a group G satisfying $\mu(x) = \mu(x^{-1})$ and $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in G$. In 1981, P. Das studied fuzzy subgroups and characterized them using the level set notion (see [14]). The book [24] was devoted to the theory of fuzzy subgroups. In 1979, J. Anthony

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and H. Sherwood redefined fuzzy subgroups using an arbitrary t-norm instead of the minimum (T_M) as follows: A fuzzy subgroup μ of a group G with respect to a t-norm T (T-subgroup, for short) is a fuzzy set of G satisfying

$(G1)$ $\mu(e) = 1$, where e is the neutral element of G.

$(G2)$ $\mu(x) = \mu(x^{-1})$ for all $x \in G$.

$(G3)$ $\mu(xy) \geq T(\mu(x), \mu(y))$ for all $x, y \in G$.

Since then, many authors have studied the topic of fuzzy subgroups considering the latter definition (see, e.g. [4, 11, 15]).

Recently, research on fuzzy subgroups has followed different approaches. Some authors focus on the connection between T-subgroups and T-indistinguishability operators (see, e.g. [16, 18, 27]).

Consider a set S and its permutations group Σ_S, whose automorphisms group is denoted by $Aut(\Sigma_S)$. If E is a T-indistinguishability operator on S, F. Formato et al. proved that the fuzzy set

$$\mu_E(f) := \inf_{x \in S}\{E(x, f(x))\}, \text{ for all } f \in Aut(\Sigma_S),$$

is a T-subgroup on Σ_S. Conversely, given a T-subgroup μ on Σ_S, they proved that the binary relation

$$E_\mu(x, y) = \sup\{\mu(f) \mid f(x) = y\} \text{ for all } x, y \in S,$$

is a T-indistinguishability operator on S. For a set S representing a group that is invariant under translation, M. Demirci and J. Recasens proved that $E_{\mu_E} = E$ and $\mu_{E_\mu} = \mu$, by using the immersion of a group into its permutation group. Thus, results obtained under the framework of T-indistinguishabilities can be transferred to the framework of T-subgroups and vice versa.

Another important question about fuzzy subgroups is how to classify them. Given an equivalence relation between fuzzy subgroups of a group G, some properties of a fixed fuzzy subgroup can be extended to all the fuzzy subgroups in its equivalence class. An example is the sup property (see [8]). Several authors have studied equivalence relations on fuzzy subgroups according to different criteria (see, e.g. [19, 22, 26]). A natural equivalence relation on fuzzy subgroups is obtained by using the level sets: two fuzzy subgroups are equivalent if they have the same chain of levels. Other interesting equivalence relations on fuzzy subgroups are isomorphisms: two fuzzy subgroups μ and η are called isomorphic if there is an isomorphism f such that $f(\mu(x)) = \eta(x)$ for all $x \in G$. Other authors count the number of classes with respect to the equivalence relation they use (see, e.g. [5, 13, 30, 31]). The relationships among the most studied equivalence relations on fuzzy subgroups can be found in [9].

Another central research approach is the construction of new fuzzy subgroups from given ones (see [1]). For instance, if two T_M-subgroups have the same chain of level sets, their minimum is also a T_M-subgroup sharing the same chain of level sets (see [20]).

Throughout this study, we examine the aggregation of T-subgroups. The aggregation A of two fuzzy sets μ and η is defined pointwise by

$$A(\mu, \eta)(x) := A(\mu(x), \eta(x)).$$

If group G is isomorphic to either $(\mathbb{Z}_2, +)$ or $(\mathbb{Z}_3, +)$, the aggregation of two T-subgroups is a T-subgroup for any t-norm T. In the present study, we focus on the aggregation of T-subgroups of a group whose cardinal is greater than three.

This paper is organised as follows: Section 2 introduces the basic concepts. Section 3 is devoted to presenting the results. Our first significant result is that $A(\mu, \mu)$ is a T-subgroup for every aggregation function A and every T-subgroup μ if and only if T is equal to the minimum t-norm (see Theorem 3.7). Subsequently, the paper focuses on the aggregation of two T_M-subgroups (which are not necessarily equal). On this topic, we find conditions on the T_M-subgroups such that their aggregation through an aggregation function is also a T_M-subgroup (see Theorem 3.15).

Strictly monotone aggregations play an important role in the aggregation of two T_M-subgroups μ and η: if $A(\mu, \eta)$ is a T_M-subgroup for some strictly monotone aggregation function A, then $A(\mu, \eta)$ is a T_M-subgroup for any other aggregation function A (see Corollary 3.16). Moreover, for some families of T_M-subgroups, this result can be extended to aggregation functions that are strictly monotone on $(0,1] \times (0,1]$ and on $[0,1) \times [0,1)$ (see Corollary 3.17 and Corollary 3.19).

If the lattice of its subgroups is a chain, the aggregation of two T_M-subgroups is a T_M-subgroup (see Theorem 3.20). If the lattice of subgroups is not a chain, we find conditions on the aggregation function that characterise the case where the aggregation of two arbitrary T_M-subgroups is also a T_M-subgroup (see Theorem 3.21). The paper ends with some concluding remarks.

2. Preliminaries

We introduce some concepts that will be used throughout the manuscript.

Definition 2.1 ([21, 23])**.** *A triangular norm, t-norm for short, is a binary operation T on the unit interval $[0,1]$ such that for all $a, b, c \in [0,1]$ the following axioms are satisfied:*

$(T1)$ $\quad T(a,b) = T(b,a)$. *(Commutativity)*

$(T2)$ $\quad T(a, T(b,c)) = T(T(a,b),c)$. *(Associativity)*

$(T3)$ $\quad T(a,b) \leq T(a,c)$ *whenever* $b \leq c$. *(Monotonicity)*

$(T4)$ $\quad T(a,1) = a$. *(Boundary condition)*

Example 2.2. *The following binary operations are very well-known examples of t-norms:*

- $T_M(a,b) = \min(a,b)$ *for all* $a, b \in [0,1]$. *(minimum t-norm)*
- $T_P(a,b) = a \cdot b$ *for all* $a, b \in [0,1]$. *(product t-norm)*
- $T_L(a,b) = \max(a+b-1, 0)$ *for all* $a, b \in [0,1]$. *(Łukasiewicz t-norm)*
- $T_D(a,b) = \begin{cases} 0 & \text{if} \quad (a,b) \in [0,1)^2 \\ \min(a,b) & \text{otherwise.} \end{cases}$ *(drastic t-norm)*

Definition 2.3. *Let T_1 and T_2 be two t-norms. We say that T_1 is weaker than T_2 if $T_1(a,b) \leq T_2(a,b)$ for all $a,b \in [0,1]$. This is denoted as $T_1 \leq T_2$.*

Note that for any t-norm T, the following inequality holds: $T_D \leq T \leq T_M$.

Definition 2.4 ([3])**.** *Let G be a group and μ a fuzzy set of G. We say that μ is a fuzzy subgroup of G with respect to a t-norm T (T-subgroup for short) if it satisfies:*

($G1$) $\mu(e) = 1$, *where e is the neutral element of G.*

($G2$) $\mu(x) = \mu(x^{-1})$ *for all $x \in G$.*

($G3$) $\mu(xy) \geq T(\mu(x), \mu(y))$ *for all $x, y \in G$.*

Remark 2.5. *Let T_1 and T_2 be two t-norms satisfying $T_1 \leq T_2$. If μ is a T_2-subgroup of G, then μ is a T_1-subgroup of G. Consequently, if μ is a T_M-subgroup, then μ is a T-subgroup for every t-norm T.*

Definition 2.6. *Let G be a group and μ a fuzzy set of G. For each $t \in [0,1]$, the level set μ_t and the strict level set μ^t are defined as follows:*

$$\mu_t = \{x \in G \mid \mu(x) \geq t\} \qquad \mu^t = \{x \in G \mid \mu(x) > t\}$$

The strict level μ^0 is called the support of μ and is denoted by supp μ.

Level sets have been used extensively in the study of fuzzy subgroups (see for instance [17, 25]). P. Das characterised T_M-subgroups in terms of level sets in [14]. It is important to mention that P. Das did not consider the normalisation axiom ($G1$) in the definition of a T_M-subgroup. We rewrite his proposition taking this axiom into account.

Proposition 2.7. *Let G be a group and μ a fuzzy set of G. Then, μ is a T_M-subgroup of G if and only if all its non-empty level sets are subgroups of G and $\mu(e) = 1$.*

A consequence of Proposition 2.7 and Remark 2.5 is the following: If μ is a fuzzy set of a group G and its non-empty levels are subgroups of G, then μ is a T-subgroup of G with respect to any t-norm T.

Definition 2.8 ([12]). *An aggregation function is a binary operation A on the unit interval $[0,1]$ such that for all $r_1, r_2, s_1, s_2 \in [0,1]$ the following axioms are satisfied:*

(A1) $A(0,0) = 0$ *and* $A(1,1) = 1$. *(Boundary conditions)*

(A2) $A(r_1, s_1) \leq A(r_2, s_2)$ *whenever* $(r_1, s_1) \leq (r_2, s_2)$, *(Monotonicity)*
where $(r_1, s_1) \leq (r_2, s_2)$ *means* $r_1 \leq r_2$ *and* $s_1 \leq s_2$.

Definition 2.9 ([10]). *An aggregation function A is strictly monotone if*

$$A(r_1, s_1) < A(r_2, s_2)$$

for all $(r_1, s_1) \leq (r_2, s_2)$ *with* $(r_1, s_1) \neq (r_2, s_2)$.

Some well-known examples of aggregation functions are the arithmetic mean, the geometric mean, the harmonic mean, and the quadratic mean (see [10, 12]). Aggregation functions are classified into four broad classes: conjunctive, averaging, disjunctive, and mixed.

1. A conjunctive aggregation function A is an aggregation function such that $A(r,s) \leq \min\{r,s\}$ for all $r, s \in [0,1]$, for instance, t-norms.

2. An averaging aggregation function A is an aggregation function such that $\min\{r,s\} \leq A(r,s) \leq \max\{r,s\}$ for all $r, s \in [0,1]$, for instance, ordered weighted averaging operators (OWA operators).

3. A disjunctive aggregation function A is an aggregation function such that $\max\{r,s\} \leq A(r,s)$ for all $r, s \in [0,1]$, for instance, t-conorms.

4. If an aggregation function A does not belong to one of the classes above, it is called mixed. For instance: uninorms.

Averaging aggregation functions are called idempotents because they satisfy $A(r,r) = r$ for all $r \in [0,1]$. More information about aggregation functions can be found in [10, 12].

3. Aggregation of fuzzy subgroups

Given an aggregation function A and two fuzzy subsets μ and η of a group G, we can consider the fuzzy set $A(\mu, \eta)$ on G defined pointwise by

$$A(\mu, \eta)(x) := A(\mu(x), \eta(x)) \text{ for all } x \in G.$$

We will say that $A(\mu, \eta)$ is the aggregation of μ and η through A.

Lemma 3.1. *Let G be a group, T a t-norm, and A an aggregation function. If μ and η are T-subgroups, then $A(\mu, \eta)$ satisfies $(G1)$ and $(G2)$.*

Proof. $(G1)$ $A(\mu, \eta)(e) = A(\mu(e), \eta(e)) = A(1, 1) = 1$.

$(G2)$ $A(\mu, \eta)(x) = A(\mu(x), \eta(x)) = A(\mu(x^{-1}), \eta(x^{-1})) = A(\mu, \eta)(x^{-1})$.

□

However, given an arbitrary t-norm T, the aggregation of two T-subgroups μ and η is not a T-subgroup in general, as the following example shows:

Example 3.2. *Consider the group $G = (\mathbb{Z}_6, +)$ and the fuzzy sets μ and η defined in the table below. It is easy to verify that they are T_P-subgroups. Consider the aggregation function $A : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ defined by $A(x, y) = \max\{x + y - 1, 0\}$, that is, the Łukasiewicz t-norm.*

G	0	1	2	3	4	5
μ	1	0.5	0.7	0.6	0.7	0.5
η	1	0.6	0.8	0.7	0.8	0.6
$A(\mu, \eta)$	1	0.1	0.5	0.3	0.5	0.1

Since $A(\mu, \eta)(2 + 3) = 0.1$ and $T_P(A(\mu, \eta)(2), A(\mu, \eta)(3)) = T_P(0.5, 0.3) = 0.15$, we conclude that $A(\mu, \eta)$ is not a T_P-subgroup.

We first show the result of the aggregation of T-subgroups if group G has two or three elements.

Lemma 3.3. *Let G be a group with two or three elements, T a t-norm, and A an aggregation function. If μ and η are T-subgroups, then $A(\mu, \eta)$ is a T_D-subgroup.*

Proof. By hypothesis, $G = \{e, z, z^{-1}\}$, where $z = z^{-1}$ if G has two elements and $z \neq z^{-1}$ if G has three elements. We only need to prove axiom $(G3)$, that is,

$$A(\mu, \eta)(xy) \geq T_D(A(\mu, \eta)(x), A(\mu, \eta)(y)) \text{ for all } x, y \in G. \quad (1)$$

If $x = e$ or $y = e$, (1) holds.

Now, suppose that $x \neq e$ and $y \neq e$.

If $x \neq y$, then $x = y^{-1}$; hence,

$$A(\mu, \eta)(xy) = A(\mu, \eta)(e) = 1 \geq T_D(A(\mu, \eta)(x), A(\mu, \eta)(y)).$$

If $x = y$,

$$A(\mu, \eta)(xy) = A(\mu, \eta)(x^2) = A(\mu, \eta)(x^{-1}) = A(\mu, \eta)(x)$$

and by the monotonicity of T_D,

$$A(\mu, \eta)(x) \geq T_D(A(\mu, \eta)(x), A(\mu, \eta)(x)).$$

Hence, (1) holds and, therefore, $A(\mu, \eta)$ is a T_D-subgroup. □

Proposition 3.4. *Let G be a group with two or three elements, A an aggregation function, and T a t-norm. If μ, η are T-subgroups, then $A(\mu, \eta)$ is a T-subgroup.*

Proof. By Lemma 3.3, $A(\mu, \eta)$ is a T_D-subgroup. When group G has only two or three elements, the set of fuzzy subgroups with respect to a given t-norm is equal to the set of fuzzy subgroups with respect to any other t-norm (see [7]). Hence, $A(\mu, \eta)$ is a T-subgroup of G. □

The following two lemmas will be used in the proof of Theorem 3.7.

Lemma 3.5. *Let G be a group with at least four elements. If μ is a T_M-subgroup and A is an aggregation function, then $A(\mu, \mu)$ is a T_M-subgroup.*

Proof. We know that $A(\mu, \mu)$ satisfies $(G1)$ and $(G2)$. To show $(G3)$ holds, let $x, y \in G$ and suppose $\mu(x) \leq \mu(y)$. By the monotonicity of A, we have that $A(\mu, \mu)(x) \leq A(\mu, \mu)(y)$. Therefore,

$$A(\mu, \mu)(xy) = A(\mu(xy), \mu(xy)) \geq A(T_M(\mu(x), \mu(y)), T_M(\mu(x), \mu(y))) =$$

$$A(\mu(x), \mu(x)) = T_M(A(\mu(x), \mu(x)), A(\mu(y), \mu(y))) = T_M(A(\mu, \mu)(x), A(\mu, \mu)(y)).$$

□

Lemma 3.6. *Let T be a non-trivial t-norm, that is, $T \notin \{T_D, T_M\}$. If, for all $a, b \in (0, 1)$, $T(a, b) \in \{0, a \wedge b\}$, then there is $k \in (0, 1)$ satisfying $T(l, l) = l$ for all $l > k$ and $T(l, l) = 0$ for all $l < k$.*

Proof. Consider the following sets:

$$S_D = \{c \in (0, 1) \mid T(c, c) = 0\} \quad \text{and} \quad S_M = \{c \in (0, 1) \mid T(c, c) = c\}.$$

Let us check that they are non-empty sets. If $S_M = \emptyset$, then $T(c, c) = 0$ for all $c \in (0, 1)$. By monotonicity, $T(a, b) \leq T(a \vee b, a \vee b) = 0$ for all $a, b < 1$. This implies that $T = T_D$, which is a contradiction; hence, $S_M \neq \emptyset$. If $S_D = \emptyset$, then $T(c, c) = c$ for all $c \in (0, 1)$. By monotonicity, $a \wedge b = T(a \wedge b, a \wedge b) \leq T(a, b) \leq a \wedge b$ for all $a, b < 1$. This implies that $T = T_M$, which is a contradiction; hence, $S_D \neq \emptyset$.

Clearly, $S_D \cap S_M = \emptyset$ and $S_D \cup S_M = (0, 1)$. Moreover, by the monotonicity of T, the elements of S_D are smaller than those of S_M. Let $k = \sup S_D$; then, for all $a \in S_D$ and $b \in S_M$, we have that

$$a \leq k \leq b.$$

Therefore, for all $l < k$, $T(l, l) = 0$ and for all $l > k$, $T(l, l) = l$. □

The following theorem provides a characterisation for the aggregation of two T-subgroups μ and η whenever $\mu = \eta$.

Theorem 3.7. *Let G be a group with at least four elements and T a t-norm satisfying $T \neq T_D$. The following assertions are equivalent:*

1. $T = T_M$.

2. *For each T-subgroup μ and each aggregation function A, $A(\mu,\mu)$ is a T-subgroup.*

Proof. Suppose that $T = T_M$. Take an arbitrary T_M-subgroup μ and an aggregation function A. By Lemma 3.5, $A(\mu,\mu)$ is a T_M-subgroup.

Conversely, we will prove that if 1) is false, then 2) is also false. Let T be a t-norm satisfying $T \notin \{T_D, T_M\}$. We consider the following two cases for the t-norm:

$A)$ There are $a, b \in (0,1)$ satisfying $T(a,b) \notin \{0, a \wedge b\}$.

$B)$ For all $a, b \in (0,1)$, $T(a,b) \in \{0, a \wedge b\}$.

Case $A)$. Without loss of generality, suppose that $a \leq b$. Therefore, we have

$$0 < T(a,b) < a \leq b.$$

Consider the following function $A : [0,1]^2 \longrightarrow [0,1]$:

$$A(r,s) = \begin{cases} 0 & \text{if} \quad r \leq T(a,b) \text{ and } s \leq T(a,b) \\ 1 & \quad\text{otherwise.} \end{cases}$$

The boundary conditions and monotonicity of A are clear, so A is an aggregation function.

To define the T-subgroup μ, we consider two disjoint subcases for group G:

(i) There is $z \in G$ satisfying $o(z) \geq 4$.

(ii) For all $z \in G$, $o(z) \leq 3$.

We remember that $o(z)$ is the smallest natural n such that $z^n = e$.

For (i), we define a fuzzy set μ of G in the following way:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ a & \text{if} \quad x \in \{z, z^{-1}\} \\ T(a,a) & \quad \text{otherwise} \end{cases}$$

Note that, by the monotonicity of T, $T(a,a) \leq T(a,b)$, so $T(a,a) < a$. Since $o(z) \geq 4$, we have $z^2 \notin \{z, z^{-1}, e\}$ and $\mu(z^2) = T(a,a)$.

Let us prove that μ is a T-subgroup. It is clear that μ satisfies $(G1)$ and $(G2)$. To prove $(G3)$, let $x, y \in G$. We have three possible values for $\mu(xy)$. If $\mu(xy) = 1$, $(G3)$ clearly holds. If $\mu(xy) = a$, we have the following:

$$\mu(xy) = a \Rightarrow xy \in \{z, z^{-1}\} \Rightarrow x \neq e \text{ or } y \neq e \Rightarrow \mu(x) \neq 1 \text{ or } \mu(y) \neq 1.$$

By the definition of the fuzzy set μ and the monotonicity of T, we have that

$$T(\mu(x), \mu(y)) \leq a = \mu(xy).$$

If $\mu(xy) = T(a,a)$, since $\mu(x), \mu(y) \in \{1, a, T(a,a)\}$, we have $T(\mu(x), \mu(y)) \leq T(a,a) = \mu(xy)$, except whenever $\mu(x) = \mu(y) = 1$. However, if $\mu(x) = \mu(y) = 1$, then $x = y = e$, which implies $\mu(xy) = 1$. Hence, we conclude that μ is a T-subgroup.

Since

$$A(\mu, \mu)(z^2) = A(\mu(z^2), \mu(z^2)) = A(T(a,a), T(a,a)) = 0$$

and

$$T(A(\mu, \mu)(z), A(\mu, \mu)(z)) = T(A(a,a), A(a,a)) = T(1,1) = 1,$$

we conclude that $A(\mu, \mu)$ is not a T-subgroup.

For (ii), since $o(x) \leq 3$ for all $x \in G$, we have that the set $\{x, x^2, x^3\}$ is equal to the set $\{e, x, x^{-1}\}$. Observe that x could be identical to x^{-1} (in the case where $o(x) = 2$). Let $y \in G \setminus \{e\}$; since G has at least four elements, there exists $z \in G$ such that $z \notin \{y, y^2, y^3\} = \{e, y, y^{-1}\}$.

Two consequences, which will be used below, are:

- Since $y^{-2} \in \{y, y^2, y^3\}$, we have that $z \neq y^{-2}$.

- Since $o(z) \leq 3$, $z^2 \in \{e, z^{-1}\}$ and, $z^2 \neq y^{-1}$.

Define the fuzzy set μ of G as follows:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ a & \text{if} \quad x \in \{y, y^{-1}, z, z^{-1}\} \\ T(a,a) & \quad \text{otherwise} \end{cases}$$

Recall that $T(a,a) < a$.

Let us check that $\mu(yz) = T(a,a)$, or equivalently $yz \notin \{e, y, z, y^{-1}, z^{-1}\}$.

If $yz = e$, then $z = y^{-1}$.

If $yz = y$, then $z = e$.

If $yz = z$, then $y = e$.

If $yz = y^{-1}$, then $z = y^{-2}$.

Further, if $yz = z^{-1}$, then $z^2 = y^{-1}$. Therefore, each of the last five assumptions leads to a contradiction.

Proving that μ is a T-subgroup of G is similar to proving subcase (i).

Let us derive that $A(\mu,\mu)$ is not a T-subgroup.

Since

$$A(\mu,\mu)(yz) = A(\mu(yz), \mu(yz)) = A(T(a,a), T(a,a)) = 0$$

and

$$T(A(\mu,\mu)(y), A(\mu,\mu)(z)) = T(A(a,a), A(a,a)) = T(1,1) = 1,$$

we have that $A(\mu,\mu)(yz) < T(A(\mu,\mu)(y), A(\mu,\mu)(z))$. Hence, we conclude that $A(\mu,\mu)$ is not a T-subgroup.

Case B). Since for all $a, b \in (0,1)$, $T(a,b) \in \{0, a \wedge b\}$ by Lemma 3.6, there exists $k \in (0,1)$ satisfying $T(l,l) = l$ for all $l > k$ and $T(l,l) = 0$ for all $l < k$. Fix $y, z \in G \setminus \{e\}$ satisfying $y \neq z$ and $y \neq z^{-1}$. We define the fuzzy set μ of G as follows:

$$\mu(x) = \begin{cases} 1 & \text{if} \quad x = e \\ \frac{k}{2} & \text{if} \quad x \in \{y, y^{-1}\} \\ \frac{k}{3} & \text{if} \quad x \in \{z, z^{-1}\} \\ \frac{k}{4} & \text{if} \quad x \in \{yz, (yz)^{-1}\} \\ 0 & \quad\quad \text{otherwise} \end{cases}$$

It is clear that μ is well-defined and satisfies $(G1)$ and $(G2)$. To prove that $(G3)$ holds, let $u, v \in G$. If $u = e$ or $v = e$, it is clear that μ satisfies $(G3)$. Otherwise, by the selection of k,

$$T(\mu(u), \mu(v)) \leq T(\frac{k}{2}, \frac{k}{2}) = 0.$$

Then, $\mu(uv) \geq T(\mu(u), \mu(v)) = 0$. This implies that μ is a T-subgroup. Now, let us define a binary operation $A : [0,1]^2 \longrightarrow [0,1]$ as $A(r,s) = 1$ if $(r,s) \geq (\frac{k}{3}, \frac{k}{3})$ and $A(r,s) = 0$ otherwise. It is easy to check that A satisfies $(A1)$ and $(A2)$. Hence, A is an aggregation function. Moreover,

$$A(\mu,\mu)(yz) = A(\mu(yz), \mu(yz)) = A(\frac{k}{4}, \frac{k}{4}) = 0$$

and

$$T(A(\mu,\mu)(y), A(\mu,\mu)(z)) = T(A(\frac{k}{2}, \frac{k}{2}), A(\frac{k}{3}, \frac{k}{3})) = T(1,1) = 1.$$

Therefore, $A(\mu,\mu)(yz) = 0 < 1 = T(A(\mu,\mu)(y), A(\mu,\mu)(z))$ and $A(\mu,\mu)$ is not a T-subgroup. □

Remark 3.8. *As a consequence of Theorem 3.7, if $T \neq T_M$, there exists a T-subgroup μ and an aggregation function A such that $A(\mu,\mu)$ is not a T-subgroup.*

We showed in Example 3.2 that the aggregation of T_P-subgroups does not have to be a T_P-subgroup. We now provide an example showing that the aggregation of any two T_M-subgroups is not always a T_M-subgroup.

Example 3.9. *Consider the Klein group $G = (\mathbb{Z}_2 \times \mathbb{Z}_2, +)$ and the fuzzy sets μ and η defined in the table below:*

G	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
μ	1	0.5	0.7	0.5
η	1	0.6	0.6	0.8

It is clear that μ and η are T_M-subgroups. Take as aggregation function the Lukasiewicz t-norm T_L, defined as $T_L(x,y) = \max\{x+y-1, 0\}$.

Then,

G	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$T_L(\mu,\eta)$	1	0.1	0.3	0.3

Thus, we have that $T_L(\mu,\eta)$ is not a T_M-subgroup.

Some equivalence relations on T_M-subgroups ensure that the aggregation of two related T_M-subgroups is also a T_M-subgroup (see [8, 9]).

Definition 3.10 ([2]). *Let G be a group and μ, η two T_M-subgroups. We say that*

$$\mu \sim \eta \qquad \Longleftrightarrow \qquad \{\mu_t\}_{t \in \mathrm{Im}\ \mu} = \{\eta_s\}_{s \in \mathrm{Im}\ \eta},$$

that is, μ is equivalent to η if they have the same chain of level subgroups.

Moreover, A. Jain proved the following characterisation.

Proposition 3.11 ([20]). *Let G be a group and μ, η two T_M-subgroups. The following assertions are equivalent:*

1. $\mu \sim \eta$.
2. $\mu(x) < \mu(y)$ *if and only if* $\eta(x) < \eta(y)$ *for all* $x, y \in G$.

If two T_M-subgroups μ and η satisfy $\mu \sim \eta$, then their aggregation is always a T_M-subgroup (see [8]). No similar results exist for a t-norm $T \neq T_M$.

We now examine what conditions on T_M-subgroups ensure that their aggregation is also a T_M-subgroup for any aggregation function. We introduce the following concept:

Definition 3.12. *Let G be a group, and μ and η be two fuzzy sets of G. We denote by $\mathcal{L}(\mu, \eta)$ the following set:*

$$\mathcal{L}(\mu, \eta) := \{(x, y) \in G \times G \mid \mu(x) < \mu(y) \text{ and } \eta(x) > \eta(y)\}.$$

We can define a binary relation $\simeq$ on T_M-subgroups using this concept as follows:

$$\mu \simeq \eta \qquad \Longleftrightarrow \qquad \mathcal{L}(\mu, \eta) = \emptyset.$$

For each T_M-subgroup ν of a group G, we have

$$\mathcal{L}(\nu, \bar{k}) = \emptyset$$

where $\bar{k}$ is the T_M-subgroup defined as $\bar{k}(x) = 1$ if $x = e$ and $\bar{k}(x) = k$ if $x \neq e$, with $k \in [0, 1]$.

Note that if two T_M-subgroups μ and η have the same chain of level subgroups, that is, $\mu \sim \eta$ then $\mathcal{L}(\mu, \eta) = \emptyset$, but the binary relation $\simeq$ is not an equivalence relation, as the following example shows.

Example 3.13. *Take the two T_M-subgroups μ and η of Example 3.9. We have $\mu \simeq \bar{1}$ and $\bar{1} \simeq \eta$, but $\mu \not\simeq \eta$.*

Lemma 3.14. *Let G be a group, μ a T_M-subgroup of G, and $x, y \in G$. If $\mu(x) \neq \mu(y)$, then*

$$\mu(xy) = T_M(\mu(x), \mu(y)).$$

Proof. For a proof by reductio ad absurdum, suppose $\mu(xy) > T_M(\mu(x), \mu(y))$. Without loss of generality, let $\mu(x) < \mu(y)$. Then, $\mu(xy) > \mu(x)$ and since μ is a T_M-subgroup, we have

$$\mu(x) \geq T_M(\mu(xy), \mu(y^{-1})) = T_M(\mu(xy), \mu(y)).$$

However, $\mu(x) < \mu(y)$ and $\mu(x) < \mu(xy)$. □

Our next result characterises T_M-subgroups aggregated through every aggregation function.

Theorem 3.15. *Let G be a group and consider two T_M-subgroups μ and η of G. The following assertions are equivalent:*

1) $\mathcal{L}(\mu, \eta) = \emptyset$.

2) *For any aggregation function $A : [0,1]^2 \longrightarrow [0,1]$, we have that $A(\mu, \eta)$ is a T_M-subgroup.*

Proof. Suppose $\mathcal{L}(\mu, \eta) = \emptyset$ and take an arbitrary aggregation function A. Let us prove that $A(\mu, \eta)$ is a T_M-subgroup. By Lemma 3.1, $A(\mu, \eta)$ satisfies $(G1)$ and $(G2)$. To show that (G3) holds, let $x, y \in G$. We consider three cases:

1. $\mu(x) < \mu(y)$. By hypothesis, $\eta(x) \leq \eta(y)$. By monotonicity,

$$A(\mu(xy), \eta(xy)) \geq A(T_M(\mu(x), \mu(y)), T_M(\eta(x), \eta(y))) = A(\mu(x), \eta(x)).$$

Hence, $A(\mu, \eta)(xy) \geq A(\mu, \eta)(x) \geq T_M(A(\mu, \eta)(x), A(\mu, \eta)(y))$.

2. $\mu(x) > \mu(y)$. Therefore, since $(y,x) \notin \emptyset = \mathcal{L}(\mu,\eta)$, there must be $\eta(x) \geq \eta(y)$; otherwise $(y,x) \in \mathcal{L}(\mu,\eta)$ a contradiction. Then, by monotonicity,

$$A(\mu(xy),\eta(xy)) \geq A(T_M(\mu(x),\mu(y)),T_M(\eta(x),\eta(y))) = A(\mu(y),\eta(y)).$$

Hence, $A(\mu,\eta)(xy) \geq A(\mu,\eta)(y) \geq T_M(A(\mu,\eta)(x), A(\mu,\eta)(y))$.

3. $\mu(x) = \mu(y)$. We have $\eta(x) \leq \eta(y)$ or $\eta(x) \geq \eta(y)$. If $\eta(x) \leq \eta(y)$, by monotonicity,

$$A(\mu(xy),\eta(xy)) \geq A(T_M(\mu(x),\mu(y)),T_M(\eta(x),\eta(y))) = A(\mu(x),\eta(x))$$

Hence, $A(\mu,\eta)(xy) \geq A(\mu,\eta)(x) \geq T_M(A(\mu,\eta)(x), A(\mu,\eta)(y))$.
For $\eta(x) \geq \eta(y)$, the argument is similar.

Conversely, we will prove that if $\mathcal{L}(\mu,\eta) \neq \emptyset$, then there is an aggregation function A such that $A(\mu,\eta)$ is not a T_M-subgroup. Since $\mathcal{L}(\mu,\eta) \neq \emptyset$, there are $x, y \in G$ satisfying $\mu(x) < \mu(y)$ and $\eta(x) > \eta(y)$. Consider the aggregation function $A : [0,1]^2 \longrightarrow [0,1]$ defined as follows:

$$A(r,s) = \begin{cases} 1 & \text{if} \quad r \geq \mu(x) \text{ and } s \geq \eta(x) \\ 1 & \text{if} \quad r \geq \mu(y) \text{ and } s \geq \eta(y) \\ 0 & \qquad \text{otherwise.} \end{cases}$$

It is easy to check that A satisfies $(A1)$ and $(A2)$. Note that $A(\mu(x),\eta(x)) = 1 = A(\mu(y),\eta(y))$ and $A(\mu(x),\eta(y)) = 0$.

$$A(\mu,\eta)(xy) = A(\mu(xy),\eta(xy)) = A(\mu(x),\eta(y)) = 0$$

$$< 1 = \min(A(\mu,\eta)(x), A(\mu,\eta)(y)).$$

Then, by Lemma 3.14, we conclude that $(G3)$ does not hold. □

The following result describes the aggregation of two T_M-subgroups through strictly monotone aggregation functions.

Corollary 3.16. *Let G be a group, and μ and η be two T_M-subgroups of G. If A_{sm} is a strictly monotone aggregation function, then the following assertions are equivalent:*

1) $\mathcal{L}(\mu, \eta) = \emptyset$.

2) *$A_{sm}(\mu, \eta)$ is a T_M-subgroup.*

3) *$A(\mu, \eta)$ is a T_M-subgroup for every aggregation function A.*

Proof. In light of Theorem 3.15, we only need to prove 2) $\Rightarrow$ 1). Suppose that $A_{sm}(\mu, \eta)$ is a T_M-subgroup. For a proof by reductio ad absurdum, let there exist $x, y \in G$ such that $\mu(x) < \mu(y)$ and $\eta(y) < \eta(x)$. By Lemma 3.14, we have that $\mu(xy) = \mu(x) < \mu(y)$ and $\eta(xy) = \eta(y) < \eta(x)$. Let us take $k = \min\{A_{sm}(\mu, \eta)(x), A_{sm}(\mu, \eta)(y)\}$ and consider the level set $A(\mu, \eta)_k = \{z \in G \mid A_{sm}(\mu, \eta)(z) \geq k\}$. Clearly, $x, y \in A_{sm}(\mu, \eta)_k$. Since $A_{sm}(\mu, \eta)$ is a T_M-subgroup, $A_{sm}(\mu, \eta)_k$ is a subgroup of G, and then, $xy \in A_{sm}(\mu, \eta)_k$. However, since A_{sm} is strictly monotone, we have

$$\begin{aligned} A_{sm}(\mu, \eta)(xy) &= A_{sm}(\mu(xy), \eta(xy)) = A_{sm}(\mu(x), \eta(y)) \\ &< A_{sm}(\mu(y), \eta(y)) = A_{sm}(\mu, \eta)(y) \end{aligned}$$

and

$$\begin{aligned} A_{sm}(\mu, \eta)(xy) &= A_{sm}(\mu(xy), \eta(xy)) = A_{sm}(\mu(x), \eta(y)) \\ &< A_{sm}(\mu(x), \eta(x)) = A_{sm}(\mu, \eta)(x). \end{aligned}$$

This implies that $A_{sm}(\mu, \eta)(xy) < k$, which is a contradiction. We conclude that $\mathcal{L}(\mu, \eta) = \emptyset$. □

The OWAs, whose associated vector has no null components, are interesting examples of strictly monotone aggregation functions. The arithmetic mean is an example of an OWA (see [12]).

We now present a result that involves the aggregation of T_M-subgroups with aggregation functions that are strictly monotone on $(0, 1] \times (0, 1]$, for instance, strict t-norms (see [21]). The result provides a new characterisation of the aggregation of T_M-subgroups whose supports are the whole group.

Corollary 3.17. *Let G be a group and consider two T_M-subgroups μ and η of G such that supp μ = supp $\eta = G$. If A_0 is an aggregation function that is strictly monotone on $(0, 1] \times (0, 1]$, then the following assertions are equivalent:*

1) $\mathcal{L}(\mu, \eta) = \emptyset$.

2) *$A_0(\mu, \eta)$ is a T_M-subgroup.*

3) *$A(\mu, \eta)$ is a T_M-subgroup for all aggregation functions A.*

Proof. In light of Theorem 3.15, we only need to prove 2) $\Rightarrow$ 1). Suppose that $A_0(\mu, \eta)$ is a T_M-subgroup. For a proof by reductio ad absurdum, let there exist $x, y \in G$ such that $0 < \mu(x) < \mu(y)$ and $0 < \eta(y) < \eta(x)$. By Lemma 3.14, we have that $0 < \mu(xy) = \mu(x) < \mu(y)$ and $0 < \eta(xy) = \eta(y) < \eta(x)$. Let $k = \min\{A_0(\mu, \eta)(x), A_0(\mu, \eta)(y)\}$ and consider the level set $A_0(\mu, \eta)_k = \{z \in G \mid A_0(\mu, \eta)(z) \geq k\}$. Clearly, $x, y \in A_0(\mu, \eta)_k$. Since $A_0(\mu, \eta)$ is a T_M-subgroup by hypothesis, $A_0(\mu, \eta)_k$ is a subgroup of G, and then, $xy \in A_0(\mu, \eta)_k$. However, since A_0 is strictly monotone on $(0, 1] \times (0, 1]$ and $\mu(x) \neq 0 \neq \eta(y)$, we have

$$A_0(\mu, \eta)(xy) = A_0(\mu(xy), \eta(xy)) = A_0(\mu(x), \eta(y))$$

$$< A_0(\mu(y), \eta(y)) = A_0(\mu, \eta)(y)$$

and

$$A_0(\mu, \eta)(xy) = A_0(\mu(xy), \eta(xy)) = A_0(\mu(x), \eta(y))$$

$$< A_0(\mu(x), \eta(x)) = A_0(\mu, \eta)(x).$$

This implies that $A_0(\mu, \eta)(xy) < k$. Hence, $xy \notin A_0(\mu, \eta)_k$, which is a contradiction. We conclude that $\mathcal{L}(\mu, \eta) = \emptyset$. $\square$

The assumption about the support of the T_M-subgroups in the above Corollary is necessary, as the following example illustrates.

Example 3.18. *Let G be the Klein group, that is, $G = (\mathbb{Z}_2 \times \mathbb{Z}_2, +)$. Consider the fuzzy sets μ and η defined as follows:*

G	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
μ	1	0.7	0	0
η	1	0.5	0.4	0.4

By Proposition 2.7*,* μ *and* η *are* T_M*-subgroups. Let* A *be the geometric mean, that is,* $A(r,s) = \sqrt{r \cdot s}$*. Then, the values of* $A(\mu,\eta)$ *are given in the following table:*

G	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$
$A(\mu,\eta)$	1	$\sqrt{0.35}$	0	0

By Proposition 2.7*, we have that* $A(\mu,\eta)$ *is a* T_M*-subgroup. However, it is easy to check that* $\mathcal{L}(\mu,\eta) \neq \emptyset$*.*

There are many examples of aggregation functions that are strictly monotone on $[0,1) \times [0,1)$ but are not on $[0,1] \times [0,1]$, for instance, the dual t-conorms of the strict t-norms (see [21]). Another example is the probabilistic sum $\perp_{\text{sum}}$, defined as $\perp_{\text{sum}}(a,b) = a + b - a \cdot b$, which is the dual t-conorm of the product t-norm T_P. The following Corollary complements Corollary 3.17.

Corollary 3.19. *Let* G *be a group and consider two* T_M*-subgroups* μ *and* η *of* G *such that* $\mu_1 = \eta_1 = \{e\}$*. If* A_1 *is an aggregation function that is strictly monotone on* $[0,1) \times [0,1)$*, then the following assertions are equivalent:*

1) $\mathcal{L}(\mu,\eta) = \emptyset$.

2) $A_1(\mu,\eta)$ *is a* T_M*-subgroup.*

3) $A(\mu,\eta)$ *is a* T_M*-subgroup for all aggregation functions* A*.*

Proof. In light of Theorem 3.15, we only need to prove 2) $\Rightarrow$ 1). Suppose that $A_1(\mu,\eta)$ is a T_M-subgroup. For a proof by reductio ad absurdum, let there exist $x, y \in G$ such that $\mu(x) < \mu(y)$ and $\eta(y) < \eta(x)$. Note that $x \neq e \neq y$, and by hypothesis, $\mu(y) < 1$ and $\eta(x) < 1$. By Lemma 3.14, we have that $\mu(xy) = \mu(x) < \mu(y) < 1$ and $\eta(xy) = \eta(y) < \eta(x) < 1$. Let $k = \min\{A_1(\mu,\eta)(x), A_1(\mu,\eta)(y)\}$ and consider the level set $A(\mu,\eta)_k = \{z \in G \mid A_1(\mu,\eta)(z) \geq k\}$. Clearly, $x, y \in A_1(\mu,\eta)_k$. Since $A_1(\mu,\eta)$ is a T_M-subgroup, $A_1(\mu,\eta)_k$ is a subgroup of G, and then, $xy \in A_1(\mu,\eta)_k$. However, since A_1 is strictly monotone on $[0,1) \times [0,1)$ and $\mu(x) \neq 1 \neq \eta(y)$, we have

$$A_1(\mu,\eta)(xy) = A_1(\mu(xy),\eta(xy)) = A_1(\mu(x),\eta(y))$$

$$< A_1(\mu(y), \eta(y)) = A_1(\mu, \eta)(y)$$

and

$$A_1(\mu, \eta)(xy) = A_1(\mu(xy), \eta(xy)) = A_1(\mu(x), \eta(y))$$

$$< A_1(\mu(x), \eta(x)) = A_1(\mu, \eta)(x).$$

Therefore, $A_1(\mu, \eta)(xy) < k$. Hence, $xy \notin A_1(\mu, \eta)_k$, which is a contradiction. We conclude that $\mathcal{L}(\mu, \eta) = \emptyset$. □

Notice that the set of subgroups of a group endowed with the partial order given by the inclusion is a lattice. To find the aggregation functions that preserve the property of being a T_M-subgroup, we must consider the structure of this lattice. More precisely, given a group, we will distinguish between the following two cases:

- The lattice of subgroups is a chain.
- The lattice of subgroups is not a chain.

Let G be a finite group. If G is cyclic, then its lattice of subgroups is a chain if and only if its cardinal is a power of a prime number (see [6]). If G is not cyclic, by Sylow's theorems (see [29]), its lattice of subgroups is not a chain.

For a group G whose lattice of subgroups is a chain, the following result proves that every aggregation function A satisfies that $A(\mu, \eta)$ is a T_M-subgroup for all T_M-subgroups μ and η.

Theorem 3.20. *Let G be a group whose lattice of subgroups is a chain. If μ and η are T_M-subgroups, then $\mathcal{L}(\mu, \eta) = \emptyset$. Consequently, if A is an aggregation function, then $A(\mu, \eta)$ is a T_M-subgroup for all T_M-subgroups μ and η.*

Proof. Suppose that $\mathcal{L}(\mu, \eta) \neq \emptyset$. Then, there exist $x, y \in G$ such that $\mu(x) < \mu(y)$ and $\eta(x) > \eta(y)$. Consider the following subgroups of G:

$$H_1 := \eta_{\eta(x)} = \{z \in G \mid \eta(z) \geq \eta(x)\}$$

$$H_2 := \mu_{\mu(y)} = \{z \in G \mid \mu(z) \geq \mu(y)\}.$$

Since $x \in H_1$ and $x \notin H_2$, we have that $H_1 \not\subseteq H_2$ and since $y \in H_2$ and $y \notin H_1$, we have that $H_2 \not\subseteq H_1$. However, we have assumed that the lattice of subgroups of G is a chain. □

We now examine the case when the lattice of subgroups of G is not a chain.

Theorem 3.21. *Let G be a group and A an aggregation function. If the lattice of subgroups of G is not a chain, the following assertions are equivalent:*

1) $A(r,s) = T_M(A(1,s), A(r,1))$ *for all* $r,s \in [0,1]$.

2) $A(\mu,\eta)$ *is a* T_M*-subgroup for any* T_M*-subgroups* μ *and* η.

Proof. We begin by showing that condition 1) is equivalent to 1′):

1′) $A(r,s) = T_M(A(\alpha,s), A(r,\beta))$ for all $r,s \in [0,1]$ and for all $\alpha,\beta \in [0,1]$ such that $1 \geq \alpha \geq r$ and $1 \geq \beta \geq s$.

Clearly, 1′) implies 1). Conversely, by the monotonicity of A, given $r,s \in [0,1]$, for all $1 \geq \alpha \geq r$ and $1 \geq \beta \geq s$, we have that

$$A(r,s) \leq T_M(A(\alpha,s), A(r,\beta)) \leq T_M(A(1,s), A(r,1)).$$

By Condition 1), we have that

$$A(r,s) = T_M(A(\alpha,s), A(r,\beta)) = T_M(A(1,s), A(r,1)).$$

We now prove that 1′) if and only if 2). Suppose that A satisfies 1′) and take two arbitrary T_M-subgroups μ and η. By Lemma 3.1, $A(\mu,\eta)$ satisfies $(G1)$ and $(G2)$.

For $(G3)$, let $x,y \in G$ and assume, without loss of generality, that $\mu(x) \leq \mu(y)$.

If $\eta(x) \leq \eta(y)$, by the monotonicity $A(\mu,\eta)(x) \leq A(\mu,\eta)(y)$. Then,

$$A(\mu(xy), \eta(xy)) \geq A(T_M(\mu(x),\mu(y)), T_M(\eta(x),\eta(y))$$

$$= A(\mu(x),\eta(x)) = T_M(A(\mu,\eta)(x), A(\mu,\eta)(y)).$$

If $\eta(x) > \eta(y)$, since $\mu(x) \leq \mu(y)$, we have $\mu(x) = \mu(y)$ or $\mu(x) < \mu(y)$.

In the first case, $\mu(x) = \mu(y)$, and monotonicity ensures that $A(\mu, \eta)(y) \leq A(\mu, \eta)(x)$ holds. Then,

$$A(\mu(xy), \eta(xy)) \geq A(T_M(\mu(x), \mu(y)), T_M(\eta(x), \eta(y))$$

$$= A(\mu(y), \eta(y)) = T_M(A(\mu, \eta)(x), A(\mu, \eta)(y)).$$

In the second case, $\mu(x) < \mu(y)$, and by Lemma 3.14

$$A(\mu(xy), \eta(xy)) = A(\mu(x), \eta(y))$$

holds. By hypothesis $1')$,

$$A(\mu(x), \eta(y)) = T_M(A(\mu(y), \eta(y)), A(\mu(x), \eta(x))).$$

Hence, $A(\mu, \eta)(xy) = T_M(A(\mu, \eta)(x), A(\mu, \eta)(y))$ and $A(\mu, \eta)$ satisfies $(G3)$.

Conversely, we prove that 'NOT $1')$' implies 'NOT 2)'. 'NOT $1')$' means that there exist $r, s \in [0, 1]$ such that

$$A(r, s) \neq T_M(A(\alpha, s), A(r, \beta))$$

for some $1 \geq \alpha \geq r$ and $1 \ \geq \beta \geq s$. By the monotonicity of A,

$$A(r, s) < T_M(A(\alpha, s), A(r, \beta)).$$

Note that $\alpha \neq r$ and $\beta \neq s$. Since the lattice of subgroups of G is not a chain, we have that there exist $H_1, H_2 \leq G$ satisfying $H_1 \not\subset H_2$ and $H_2 \not\subset H_1$. Let us consider the following fuzzy sets:

$$\mu(z) = \begin{cases} 1 & \text{if} \quad z = e \\ \alpha & \text{if} \quad z \in H_1 \setminus \{e\} \\ r & \quad\ \text{otherwise} \end{cases} \qquad \eta(z) = \begin{cases} 1 & \text{if} \quad z = e \\ \beta & \text{if} \quad z \in H_2 \setminus \{e\} \\ s & \quad\ \text{otherwise} \end{cases}$$

Since the level sets of μ and η are subgroups of G, we have that μ and η are T_M-subgroups. Let $x \in H_2 \setminus H_1$ and $y \in H_1 \setminus H_2$. Then,

$$\mu(x) = r < \alpha = \mu(y) \qquad \text{and} \qquad \eta(x) = \beta > s = \eta(y).$$

Now, since $\mu(x) \neq \mu(y)$, by Lemma 3.14

$$A(\mu,\eta)(xy) = A(\mu(xy),\eta(xy)) = A(\mu(x),\eta(y)) = A(r,s)$$

and by hypothesis

$$A(r,s) < T_M(A(\alpha,s),A(r,\beta)) = T_M(A(\mu(y),\eta(y)),A(\mu(x),\eta(x)))$$

Hence, $A(\mu,\eta)(xy) < T_M(A(\mu,\eta)(x),A(\mu,\eta)(y))$.

Therefore, $A(\mu,\eta)$ is not a T_M-subgroup. □

An example of aggregation function satisfying Condition 1) of Theorem 3.21 is the following:

Example 3.22. *Consider a non-decreasing function* $f : [0,1] \longrightarrow [0,1]$ *such that* $f(0) = 0$ *and* $f(1) = 1$*. Then, the class of aggregation functions* $A_1, A_2 : [0,1]^2 \longrightarrow [0,1]$ *defined by*

$$A_1(r,s) = f(r) \qquad A_2(r,s) = f(s)$$

satisfies condition 1) *of Theorem 3.21. In particular, the projection of each component belongs to this class.*

Concluding Remarks

We study the aggregation of T-subgroups of a group G. As expected, the results depend on the aggregation function, on the T-subgroups, and on the structure of the group G.

We find a binary relation on the class of all T_M-subgroups that characterises the case where the aggregation of two related T_M-subgroups through an aggregation function is also a T_M-subgroup. Moreover, given two T_M-subgroups μ and η, if $A(\mu,\eta)$ is a T_M-subgroup for some strictly monotone aggregation function A, then the same is also true for every aggregation function.

If the lattice of subgroups of a group G is a chain, the aggregation of any two T_M-subgroups through any aggregation function is a T_M-subgroup. If the lattice is not a chain, the aggregation functions that preserve T_M-subgroups have been characterised.

Acknowledgments

The authors are grateful to the anonymous referees for their suggestions, which led to several improvements on the originally submitted version of this paper, including an important correction for Lemma 3.3.

The authors acknowledge the financial support of the Spanish Ministerio de Economía y Competitividad (Grant MTM 2016-79422-P and Grant PGC 2018-098623-B-I00).

The first author also thanks the support of the Asociación de Amigos of the University of Navarra.

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