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Permutation Representations and Automorphisms of Evolution Algebras

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Abstract. We prove that the natural permutation representation of highly transitive finite groups cannot be realized as the full automorphism group of an idempotent, finite-dimensional evolution algebra acting on the set of lines spanned by its natural elements. Specifically, for any sufficiently large integer n and $k \geq 4$, there does not exist an idempotent evolution algebra X of dimension n such that $\mathrm{Aut}(X)$ is isomorphic to a proper k-transitive subgroup of S_n. Nevertheless, we show that for any finite group G, any permutation representation $\xi\colon G \to S_n$, and any field $\Bbbk$, there exists an idempotent, finite-dimensional evolution $\Bbbk$-algebra X such that $\mathrm{Aut}(X) \cong G$, and the induced representation of $\mathrm{Aut}(X)$ on the natural idempotents of X is equivalent to ξ.

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1. Introduction

An n-dimensional evolution algebra X, defined over an arbitrary field $\Bbbk$, is a not necessarily associative and commutative algebra equipped with a natural basis $B = \{b_1, \ldots, b_n\}$ that satisfies the key property: $b_i b_j = 0$ whenever $i \neq j$. A notable feature of these algebras is that once the natural basis is fixed, the products b_i^2 for $i = 1, \ldots, n$ are encoded in a matrix $M_B(X)$, known as the structure matrix, that fully determines the structure of the evolution algebra.

First introduced by Tian [23,24], evolution algebras serve as a framework to model self-reproduction in non-Mendelian genetics, building upon the concept of genetic algebras originally defined by Etherington [20]. Since the publication of Tian's seminal works, they have attracted considerable attention, leading to significant progress in understanding their structure [3,5,7,10,12] and, in their classification for low-dimensional cases [4,6,8,9,18,22].

A central result in the study of automorphism groups of evolution algebras, with implications for their classification, is due to Elduque and Labra [19]. They showed that for an n-dimensional idempotent evolution algebra X

(i.e., $X^2 = X$), there exists an exact sequence:

$$1 \to D \to \mathrm{Aut}(X) \xrightarrow{\rho_X} S_n, \tag{1}$$

where $\mathrm{Aut}(X)$ denotes the automorphism group of the algebra, S_n is the symmetric group on n elements, and D is a finite abelian group related to unity roots in $\Bbbk$. Hence, a direct consequence of Eq. (1) is that for any finite-dimensional idempotent evolution $\Bbbk$-algebra X, $\mathrm{Aut}(X)$ is a finite group.

Conversely, it is known that any finite group is the automorphism group of an idempotent evolution algebra. More precisely, for any finite group G, and any field $\Bbbk$, there exists a finite-dimensional idempotent evolution $\Bbbk$-algebra X such that $\mathrm{Aut}(X) \cong G$ [14, Theorem 1.1]. This result was independently proved when $\mathrm{char}(\Bbbk) = 0$ [22, Theorem 3.1]. Moreover, when the units of $\Bbbk$ satisfy $|\Bbbk^*| \geq 2|G|$, then evolution algebra X can be chosen to be simple [16, Theorem A].

In the light of Eq. (1) and the preceding result [14, Theorem 1.1], it is natural to ask whether permutation representations of finite groups can be realized within this framework, so we formulate the following problem:

Question 1.1. *Let $\Bbbk$ be a field and let G be a finite group. For any permutation representation $\xi\colon G \to S_n$ with abelian kernel, does there exist a finite-dimensional idempotent evolution $\Bbbk$-algebra X such that $\mathrm{Aut}(X) \cong G$, and the induced permutation representation $\rho_X : \mathrm{Aut}(X) \to S_n$ in Eq. (1) is equivalent to ξ?*

Our first result shows that the answer to Question 1.1 may be negative. Specifically, a high transitivity of the subgroup $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ (see Definition 3.1) imposes restrictions on the behavior of ρ_X:

Theorem A. *Let $\Bbbk$ be a field and let X be an n-dimensional idempotent evolution $\Bbbk$-algebra.*

- (i) *For $n \geq 3$, if $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ is 2-transitive then ρ_X is faithful, that is $\mathrm{Aut}(X) \cong \rho_X\big(\mathrm{Aut}(X)\big)$.*
- (ii) *For $n \geq 5$, if $\rho\big(\mathrm{Aut}(X)\big) \leq S_n$ is 4-transitive then ρ_X is faithful and full, that is ρ_X induces an isomorphism $\mathrm{Aut}(X) \cong S_n$.*

In [22], it is claimed that n-dimensional idempotent evolution $\Bbbk$-algebras can be classified based on the isomorphism type of their automorphism group. Our result contributes to this classification problem by providing the following characterization, which complements [22, Theorem 3.2]:

Corollary B. *Let $\Bbbk$ be a field and let $G \leq S_n$ be a k-transitive group, $k \geq 4$. Then, G is the automorphism group of an n-dimensional idempotent evolution $\Bbbk$-algebra if and only if $G = S_n$.*

Permutation representations of the full automorphism group of a given finite-dimensional idempotent evolution $\Bbbk$-algebra X can be constructed in ways other than described in Eq. (1). Specifically, let $\widetilde{\mathcal{B}}_X$ be the set of all the idempotent natural elements of X (see Definition 2.7). Since $\mathrm{Aut}(X)$ acts naturally on $\widetilde{\mathcal{B}}_X$, it induces a permutation representation

$$\widetilde{\rho}_X : \mathrm{Aut}(X) \to S_m \tag{2}$$

where $m = |\widetilde{B}_X|$. Hence, similarly to Question 1.1, we raise the following problem:

Question 1.2. *Let $\Bbbk$ be a field and let G be a finite group. For any permutation representation $\xi\colon G \to S_m$, does there exist a finite-dimensional idempotent evolution $\Bbbk$-algebra X such that $\mathrm{Aut}(X) \cong G$, and the induced permutation representation $\widetilde{\rho}_X : Aut(X) \to S_m$ in Eq. (2) is equivalent to ξ?*

Unlike Question 1.1, we solve Question 1.2 affirmatively:

Theorem C. *Let $\Bbbk$ be a field and G be a finite group. For any permutation representation $\xi\colon G \to S_m$, there exist infinitely many non-isomorphic finite-dimensional idempotent evolution $\Bbbk$-algebras X such that $\mathrm{Aut}(X) \cong G$, and the induced permutation representation $\widetilde{\rho}_X$ in Eq. (2) is equivalent to ξ.*

Organization of this Paper

We start by giving an introduction to evolution algebras (Sect. 2) and multiply-transitive groups (Sect. 3). In Sect. 4 we prove Theorem A and Corollary B. The final section is dedicated to proving Theorem C, presented there as Theorem 5.2.

2. Basics of Evolution Algebras

In this section we review the necessary background on evolution algebras that is needed throughout this article. We refer to the monograph by Tian [23] for a thorough description on evolution algebras. However, most of our arguments are based on the results presented in [17, 19].

All the algebras considered in this work will be defined over a field $\Bbbk$ of arbitrary characteristic and have finite dimension. A $\Bbbk$-algebra is simply a vector space X endowed with a $\Bbbk$-bilinear map (multiplication) $X \times X \to X$ given by $(x, y) \mapsto xy$.

Definition 2.1. An n-dimensional evolution algebra is a $\Bbbk$-algebra X, provided with a basis $\mathcal{B} = \{b_1, \ldots, b_n\}$ satisfying $b_i b_j = 0$ for any $1 \le i \ne j \le n$. Such a basis is referred to as a natural basis.

Evolution algebras are non-associative and commutative [23, Corollary 1]. Although natural basis are not unique (see [17, Examples 2.4 and 2.5]), once a natural basis is chosen, the squares of its elements fully determine the structure of the algebra. This fact motivates the following definition:

Definition 2.2. Let X be an evolution $\Bbbk$-algebra with natural basis $\mathcal{B} = \{b_1, \ldots, b_n\}$. The structure matrix of X relative to $\mathcal{B}$ is defined as $M_{\mathcal{B}} := (\mu_{ij})$ where $b_i^2 = \sum_{j=1}^{n} \mu_{ij} b_j$, for $i = 1, \ldots, n$. The coefficients $\mu_{ij} \in \Bbbk$, $i, j = 1, \ldots, n$, are referred to as structure constants.

It is worth noting that some authors adopt a different notation for the structure constants, resulting in a structure matrix that is the transpose of ours. We will focus on a particularly special class of evolution algebras:

Definition 2.3. An evolution $\Bbbk$-algebra X is idempotent (or regular, or perfect) if $X^2 = X$.

A classical argument in linear algebra shows that an evolution algebra is idempotent if and only if its structure matrix, with respect to any natural basis, is invertible [17, Remark 4.2]. Idempotent evolution algebras are particularly convenient since they possess a unique natural basis, up to scaling and reordering of its elements:

Theorem 2.4 ([17, Theorem 4.4]). *Let X be an n-dimensional idempotent evolution $\Bbbk$-algebra and let $\mathcal{B} = \{b_1, \ldots, b_n\}$ and $\mathcal{B}' = \{b'_1, \ldots, b'_n\}$ be natural bases of X. Then, there exist nonzero scalars $\lambda_i \in \Bbbk$, and a permutation $\sigma \in S_n$, such that for $i = 1, \ldots, n$, $b'_i = \lambda_i b_{\sigma(i)}$.*

This leads to a characterization of the automorphisms of finite-dimensional idempotent evolution algebras:

Corollary 2.5. *Let X be an n-dimensional idempotent evolution $\Bbbk$-algebra, let $\mathcal{B} = \{b_1, \ldots, b_n\}$ be a natural basis of X and $\varphi \in \mathrm{Aut}(X)$. Then, there exist nonzero scalars $\lambda_i \in \Bbbk$ and a permutation $\sigma \in S_n$ such that $\varphi(b_i) = \lambda_i b_{\sigma(i)}$, $i = 1, \ldots, n$.*

Remark 2.6. If X is an idempotent evolution $\Bbbk$-algebra with natural basis $\mathcal{B} = \{b_1, \ldots, b_n\}$, Corollary 2.5 allows us to map each $\varphi \in \mathrm{Aut}(X)$ to a permutation $\rho_{(X,\mathcal{B})}(\varphi) := \sigma \in S_n$. This map is a group homomorphism and corresponds to the permutation representation of $\mathrm{Aut}(X)$ in Eq. (1). Notice that $\rho_{(X,\mathcal{B})}$ depends on the chosen natural basis $\mathcal{B}$, but if $\mathcal{B}'$ is a different basis, there exists a permutation $\tau \in S_n$ such that $\rho_{(X,\mathcal{B}')}(\varphi) = \tau^{-1} \cdot \rho_{(X,\mathcal{B})}(\varphi) \cdot \tau$. Therefore, the representations $\rho_{(X,\mathcal{B})}$ and $\rho_{(X,\mathcal{B}')}$ are equivalent (see [21, p. 35]) and we adopt the simplified notation ρ_X for the representation, ignoring the choice of basis.

We now proceed to define the permutation representation $\widetilde{\rho}_X$ in Eq. (2). We first define natural elements in an evolution algebra.

Definition 2.7 ([2, Definitions 2.2]). Let X be an evolution $\Bbbk$-algebra. A set $\mathcal{A} = \{x_1, \ldots, x_r\} \subset X$ is called an extending natural family if there exists a natural basis $\mathcal{B}$ of X such that $\mathcal{A} \subset \mathcal{B}$. We say that $x \in X$ is a natural element if $\{x\}$ is an extending natural family.

A very special example of an extending natural family is the set of all idempotent natural elements in an idempotent evolution algebra:

Proposition 2.8. *Let X be an n-dimensional idempotent evolution $\Bbbk$-algebra. The set of all idempotent natural elements in X, denoted $\widetilde{\mathcal{B}}_X$, forms an extending natural family. Specifically, $\widetilde{\mathcal{B}}_X$ is a finite set with size $m := |\widetilde{\mathcal{B}}_X| \leq n$.*

Proof. Let $\mathcal{B}' = \{b'_1, \ldots, b'_n\}$ be a natural basis of X with structure constants (μ'_{ij}). We focus on the elements of $\mathcal{B}'$ that satisfy ${b'_i}^2 = \mu'_{ii} b'_i$ with $\mu'_{ii} \neq 0$. After reordering, we can assume that the first m elements, $b'_1, \ldots, b'_m$, are the only ones in $\mathcal{B}'$ that meet this condition. If no such elements exist, we set $m = 0$. Let

$$b_i = \begin{cases} (\mu'_{ii})^{-1} b'_i & \text{for } i \leq m, \\ b'_i & \text{for } i > m. \end{cases}$$

This defines $\mathcal{B} = \{b_1, \dots, b_n\}$ as a natural basis of X with structure constants (μ_{ij}). For $i \leq m$,

$$b_i^2 = (\mu'_{ii})^{-2} {b'_i}^2 = (\mu'_{ii})^{-1} b'_i = b_i,$$

which shows that $\{b_1, \dots, b_m\} \subset \widetilde{\mathcal{B}}_X$. Additionally, by construction, we have that $\mu_{ii} = 1$, and $\mu_{ij} = 0$ for $i \neq j$ if and only if $i \leq m$.

We conclude by proving that $\widetilde{\mathcal{B}}_X \subset \{b_1, \dots, b_m\}$. Indeed, for any $x \in \widetilde{\mathcal{B}}_X$, since x is a natural element, there exists a natural basis $\mathcal{A}$ such that $x \in \mathcal{A} \subset X$. By Theorem 2.4, there is a nonzero scalar $\lambda_0 \in \Bbbk$ and an index $i_0 \in \{1, \dots, n\}$ such that $x = \lambda_0 b_{i_0}$. Since $x = x^2$, we have:

$$\lambda_0 b_{i_0} = (\lambda_0 b_{i_0})^2 = \lambda_0^2 \sum_{j=1}^{n} \mu_{i_0 j} b_j.$$

Comparing coefficients, we find that $\mu_{i_0 j} = 0$ for $j \neq i_0$ and $\mu_{i_0 i_0} \neq 0$. This implies that $i_0 \leq m$, and $\mu_{i_0 i_0} = 1$. Therefore, $\lambda_0 = 1$, and consequently, $x = b_{i_0} \in \{b_1, \dots, b_m\}$. □

We can now describe the permutation representation $\widetilde{\rho}_X$ from Eq. (2). Let X be an n-dimensional idempotent evolution $\Bbbk$-algebra, $\widetilde{\mathcal{B}}_X = \{b_1, \dots, b_m\}$ the set of all idempotent natural elements in X, and $\mathcal{B} = \{b_1, \dots, b_n\}$ the natural basis with $\widetilde{\mathcal{B}}_X \subset \mathcal{B}$, as described in the proof of Proposition 2.8.

We consider the permutation representation $\rho_X \colon \operatorname{Aut}(X) \to S_n$ constructed in Remark 2.6, which appears in Eq. (1). Given $\varphi \in \operatorname{Aut}(X)$, note that φ maps idempotent elements to idempotent elements. Thus, if $i \leq m$ (so that $b_i \in \widetilde{\mathcal{B}}_X$ and b_i is idempotent), we have $\varphi(b_i) = b_{\rho_X(\varphi)(i)}$, where $\rho_X(\varphi)(i) \leq m$. This defines a new permutation representation

$$\widetilde{\rho}_X \colon \operatorname{Aut}(X) \to S_m,$$

induced by the action of $\operatorname{Aut}(X)$ on $\widetilde{\mathcal{B}}_X$, which is given by

$$\widetilde{\rho}_X(\varphi)(i) = \rho_X(\varphi)(i) \quad \text{for } i = 1, \dots, m.$$

In other words, the permutation representation $\rho_X \colon \operatorname{Aut}(X) \to S_n$ can be decomposed as:

$$\begin{array}{ccc} \operatorname{Aut}(X) & \xrightarrow{\quad\rho_X\quad} & S_n \\ & \searrow^{\widetilde{\rho}_X \oplus \alpha} \qquad\qquad \nearrow & \\ & S_m \times \operatorname{Sym}(\{m+1, \dots, n\}) & \end{array}$$

Remark 2.9. As in the case of the representation ρ_X (see Remark 2.6), the definition of $\widetilde{\rho}_X$ is unique only up to equivalence. Indeed, while the set $\widetilde{\mathcal{B}}_X$ is unequivocally defined, and algebra morphisms necessarily chosen for the elements in $\widetilde{\mathcal{B}}_X$. Different orderings yield different representations, however these representations agree up to inner conjugation in S_m. In other words, any two representations arising from different orderings of $\widetilde{\mathcal{B}}_X$ are equivalent.

3. Multiply-Transitive Groups

We compile essential results on transitive finite groups that will be applied in the following sections. For a more in-depth treatment of this topic, the reader is encouraged to consult [11, Chap. 1] and [21, Chap. 7].

Definition 3.1. Let n and k be positive integers. A permutation group $P \leq S_n$ is k-transitive if for any two ordered sets of k distinct numbers $(x_1, \dots, x_k)$ and $(y_1, \dots, y_k)$, $1 \leq x_i, y_i \leq n$, there exists $\sigma \in P$ such that $\sigma(x_i) = y_i$ for $i = 1, \dots, k$.

An abstract finite group G is called k-transitive if there exists a permutation representation $\chi\colon G \to S_n$ such that the permutation group $\chi(G) \leq S_n$ is k-transitive.

Remark 3.2. Observe that every finite abstract group G is 1-transitive, as both its left and right regular representations define G as a 1-transitive permutation subgroup of $S_{|G|}$ [11, pp. 7–8]. Consequently, it is natural to focus on groups that are multiply transitive, that is, groups which are k-transitive for some $k \geq 2$. Moreover, note that if a finite abstract group G is k-transitive, then it is also l-transitive for every $l \leq k$.

From the Classification of Finite Simple Groups, the complete list of all possible multiply-transitive groups is known [11, Theorem 4.11]. Below, we recall the classification of 4-transitive groups:

Theorem 3.3. *All the k-transitive groups, $k \geq 4$ are:*

(i) *The symmetric group S_n is n-transitive.*
(ii) *The alternating group $A_n \leq S_n$ is $(n-2)$-transitive.*
(iii) *The Mathieu groups $M_n \leq S_n$, $n = 11, 12, 23, 24$, are 4-transitive for $n = 11, 23$ and 5-transitive for $n = 12, 24$.*

Lemma 3.4. *Let $P \leq S_n$ be a 4-transitive permutation group. Then, for any faithful permutation representation $\psi\colon P \to S_n$, the image $\psi(P) \leq S_n$ is also a 4-transitive permutation group.*

Proof. The result is straightforward for $P = A_n$ and $P = S_n$, as in these cases S_n contains only one conjugacy class of these subgroups. For the Mathieu groups, the claim follows from a case-by-case analysis of their possible permutation representations, as described in the *ATLAS of Finite Groups Representations* [1]. Specifically:

- For M_n with $n = 11, 23, 24$, there exists a unique copy of $M_n \leq S_n$ (up to inner conjugation within S_n).
- For M_{12}, there are two non-equivalent representations $M_{12} \leq S_{12}$, both of which are 5-transitive. $\square$

4. Proof of Theorem A

To ease the readability of this section, we have structured it as a series of lemmas, all of which share the assumptions stated in Theorem A. Throughout the discussion, we work over a ground field $\Bbbk$ and consider an idempotent n-dimensional evolution $\Bbbk$-algebra X with a fixed natural basis $\mathcal{B} = \{b_1, b_2, \dots, b_n\}$. The structure matrix associated to this basis is denoted by $M_{\mathcal{B}} = (\mu_{ij})_{i,j=1,\dots,n}$.

As stated in Corollary 2.5 and Remark 2.6, for any automorphism $\varphi \in \mathrm{Aut}(X)$, there exist nonzero scalars $\lambda_i \in \Bbbk$ for $i = 1, \dots, n$, and a permutation $\sigma \in S_n$ such that $\varphi(b_i) = \lambda_i b_{\sigma(i)}$ for all $i = 1, \dots, n$. Consequently, one can define the permutation representation $\rho_X : \mathrm{Aut}(X) \to S_n$ by setting $\sigma = \rho_X(\varphi)$.

Now, since $\varphi \in \mathrm{Aut}(X)$ is an algebra morphism, the equality $\varphi(b_i^2) = \varphi(b_i)^2$ holds for each $i = 1, \dots, n$. Expanding both sides, we obtain:

$$\varphi(b_i^2) = \varphi\left(\sum_j \mu_{ij} b_j\right) = \sum_j \mu_{ij}\lambda_j b_{\sigma(j)},$$

$$\varphi(b_i)^2 = (\lambda_i b_{\sigma(i)})^2 = \lambda_i^2 \sum_{\sigma(j)} \mu_{\sigma(i)\sigma(j)} b_{\sigma(j)}.$$

Comparing the coefficients of $b_{\sigma(j)}$, we deduce:

$$\lambda_j \mu_{ij} = \lambda_i^2 \mu_{\sigma(i)\sigma(j)}, \quad \text{for all } i, j = 1, \dots, n. \tag{3}$$

Conversely, given a permutation $\sigma \in S_n$ and coefficients $\lambda_i \neq 0$ for $i = 1, \dots, n$, one can construct a linear automorphism of X that maps each b_i to $\lambda_i b_{\sigma(i)}$. This linear map becomes an algebra automorphism if and only if Eq. (3) is satisfied for all $i, j = 1, \dots, n$.

Now, in view of Eq. (3), one infers that the way all possible $\sigma = \rho_X(\varphi) \in \rho_X\big(\mathrm{Aut}(X)\big)$ act on pairs (i, j) for $i, j = 1, \dots, n$, i.e. the 2-transitivity of the permutation group $\rho_X\big(\mathrm{Aut}(X)\big)$, should impose specific regularity conditions on the structure coefficients of X. This is formalized in the following result.

Lemma 4.1. *Let $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ be a 2-transitive group. Then, either $\mu_{ij} = 0$ for all $i \neq j$, or $\mu_{ij} \neq 0$ for all $i \neq j$. Similarly if $\rho_X\big(\mathrm{Aut}(X)\big)$ is transitive, then either $\mu_{ii} = 0$ for all $i \in \{1, \dots, n\}$, or $\mu_{ii} \neq 0$ for all $i \in \{1, \dots, n\}$.*

Proof. Let $i \neq j$ and $k \neq l$ be two arbitrary pairs of indices. By 2-transitivity, there exists $\varphi \in \mathrm{Aut}(X)$ such that $\sigma(i) = k$ and $\sigma(j) = l$, where $\sigma = \rho_X(\varphi)$. Using Eq. (3) for i, j, and noting that λ_i and λ_j are nonzero, it follows that $\mu_{ij} = 0$ if and only if $\mu_{kl} = 0$.

Similarly, if $\rho_X\big(\mathrm{Aut}(X)\big)$ is transitive, for any i and j, there exists $\phi \in \mathrm{Aut}(X)$ such that $\sigma(i) = j$ where $\sigma = \rho_X(\phi)$. Applying Eq. (3) for $i = j$, we deduce that $\mu_{ii} = 0$ if and only if $\mu_{jj} = 0$. □

We can prove that ρ_X is faithful if $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ is transitive, and if at least one diagonal structure coefficient, $\mu_{ii} = (M_{\mathcal{B}})_{ii}$, is nonzero for some $i \in \{1, \dots, n\}$.

Lemma 4.2. *Let $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ be a transitive group, and suppose that there exists $k \in \{1,\ldots,n\}$ such that $\mu_{kk} \neq 0$. Then, for any $\varphi \in \mathrm{Aut}(X)$, the nonzero scalars λ_i associated to φ by Corollary 2.5 are uniquely determined by $\sigma = \rho_X(\varphi)$. Namely,*

$$\lambda_i = \frac{\mu_{ii}}{\mu_{\sigma(i)\sigma(i)}},$$

for all $i \in \{1,\ldots,n\}$ and therefore ρ_X is faithful.

Proof. By Lemma 4.1, if $\mu_{kk} \neq 0$, for some $k \in \{1,\ldots,n\}$, then $\mu_{ii} \neq 0$ for all $i \in \{1,\ldots,n\}$. Then, by Eq. (3) with $i = j$ and $\sigma = \rho_X(\varphi) \in \rho_X\big(\mathrm{Aut}(X)\big)$, we obtain that $\lambda_i^2 = \dfrac{\mu_{ii}}{\mu_{\sigma(i)\sigma(i)}}\lambda_i$. Since $\lambda_i \neq 0$ we can simplify to $\lambda_i = \dfrac{\mu_{ii}}{\mu_{\sigma(i)\sigma(i)}}$. □

Moreover, if $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ is 2-transitive, with $n \geq 3$, we have the following:

Lemma 4.3. *Let $n \geq 3$, and let J denote the set of triples of pairwise distinct indices (i,j,k), where $i,j,k \in \{1,\ldots,n\}$. Suppose that $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ is a 2-transitive group and that $\mu_{ij} \neq 0$ for some $i \neq j$. Then, for any $\varphi \in \mathrm{Aut}(X)$, the nonzero scalars λ_i, associated to φ by Corollary 2.5, are uniquely determined by $\sigma = \rho_X(\varphi)$ and therefore ρ_X is faithful. More precisely, the function*

$$\begin{array}{rcl} J \times S_n & \xrightarrow{B} & \mathbb{k}^* \\ (i,j,k,\sigma) & \mapsto & \frac{\mu_{ik}}{\mu_{\sigma(i)\sigma(k)}}\frac{\mu_{\sigma(j)\sigma(k)}}{\mu_{jk}}\frac{\mu_{ji}}{\mu_{\sigma(j)\sigma(i)}} \end{array}$$

is well defined, and if $\varphi \in \mathrm{Aut}(X)$ with $\sigma = \rho_X(\varphi)$, then $\lambda_i = B(i,j,k,\sigma)$ for any $(i,j,k) \in J$.

Proof. First, notice that the map B is welldefined since, according to Lemma 4.1, $\mu_{ij} \neq 0$ for every $i \neq j$. Now, we can repeatedly apply Eq. (3), alternating the roles of i, j, and k (which are assumed to be distinct), and dividing by the corresponding structure constants, to obtain:

$$\lambda_k = \frac{\mu_{\sigma(j)\sigma(k)}}{\mu_{jk}}\lambda_j^2, \tag{4}$$

$$\lambda_j^2 = \frac{\mu_{ji}}{\mu_{\sigma(j)\sigma(i)}}\lambda_i, \tag{5}$$

$$\lambda_k = \frac{\mu_{\sigma(j)\sigma(k)}}{\mu_{jk}}\frac{\mu_{ji}}{\mu_{\sigma(j)\sigma(i)}}\lambda_i \text{ by (4) and (5)}, \tag{6}$$

$$\lambda_k = \frac{\mu_{\sigma(i)\sigma(k)}}{\mu_{ik}}\lambda_i^2, \tag{7}$$

$$\frac{\mu_{\sigma(i)\sigma(k)}}{\mu_{ik}}\lambda_i^2 = \frac{\mu_{\sigma(j)\sigma(k)}}{\mu_{jk}}\frac{\mu_{ji}}{\mu_{\sigma(j)\sigma(i)}}\lambda_i, \text{ by (6) and (7)}, \tag{8}$$

hence, since $\lambda_i \neq 0$, by (8), we finally obtain $\lambda_i = \dfrac{\mu_{ik}}{\mu_{\sigma(i)\sigma(k)}} \dfrac{\mu_{\sigma(j)\sigma(k)}}{\mu_{jk}} \dfrac{\mu_{ji}}{\mu_{\sigma(j)\sigma(i)}}$. □

Remark 4.4. The injectivity of ρ_X, as established in both Lemmas 4.2 and 4.3, can also be deduced using the arguments presented in [19]. Specifically, [19, Theorems 2.7 and 3.2] identify the kernel of ρ_X with the group of $(2^{\mathrm{b}(\Gamma(X,\mathcal{B}))}-1)$-roots of unity in $\Bbbk$, where $\Gamma(X,\mathcal{B})$ denotes the directed graph associated with X (see [7, Sect. 2.3]), and $\mathrm{b}(\Gamma)$ denotes the balance of the directed graph Γ (see [19, Sect. 2]). Now, if either $\mu_{ii} \neq 0$ for every $i \in \{1,\dots,n\}$ (which follows from the hypothesis of Lemma 4.2, using Lemma 4.1) or $n \geq 3$ and $\mu_{ij} \neq 0$ for every pairwise different $i,j \in \{1,\dots,n\}$ (which follows from the hypothesis of Lemma 4.3), then $\mathrm{b}(\Gamma(X,\mathcal{B})) = 1$ and the kernel of ρ_X is trivial.

Remark 4.5. Lemmas 4.2 and 4.3 not only establish the injectivity of ρ_X, but also provide explicit formulas for determining $\varphi \in \mathrm{Aut}(X)$ in terms of $\rho_X(\varphi)$ and the structure coefficients of X. This contrasts with the more indirect arguments presented in the previous remark. These formulas reveal that φ can be determined locally by the interaction of three elements from the natural bases and their images, rather than requiring a global analysis of the entire algebra structure. Such a local approach proves to be useful in the proof of Theorem A.

Now, we prove a slightly modified version of Theorem A(ii):

Theorem 4.6. *Let X be an idempotent n-dimensional evolution $\Bbbk$-algebra, and let the structure matrix $M_{\mathcal{B}}$ have a nonzero diagonal entry for some natural basis $\mathcal{B}$. If the group $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ is 2-transitive, then $\rho_X\big(\mathrm{Aut}(X)\big) = S_n$.*

Proof. To prove that $\rho_X\big(\mathrm{Aut}(X)\big) = S_n$, it suffices to show that for any $\tau \in S_n$, there exists an automorphism $\varphi \in \mathrm{Aut}(X)$ such that $\rho_X(\varphi) = \tau$. As we observed at the beginning of this section, this is equivalent to finding nonzero scalars λ_i^τ that satisfy Eq. (3) for each pair i,j.

By hypothesis $(M_{\mathcal{B}})_{ii} = \mu_{ii} \neq 0$, for some $i \in \{1,\dots,n\}$, if $\varphi \in \mathrm{Aut}(X)$ exists such that $\rho_X(\varphi) = \tau$, then applying Lemma 4.2, $\lambda_i^\tau = \frac{\mu_{ii}}{\mu_{\tau(i)\tau(i)}} \neq 0$ for all $i \in \{1,\dots,n\}$. It then remains to verify that these nonzero scalars λ_i^τ satisfy Eq. (3) for each pair of distinct indices i,j.

Let $i,j \in \{1,\dots,n\}$ be distinct indices. Since $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ is 2-transitive, there exists $\phi \in \mathrm{Aut}(X)$ such that $\rho_X(\phi) = \sigma \in S_n$, with $\sigma(i) = \tau(i)$ and $\sigma(j) = \tau(j)$. Now, let $\phi \in \mathrm{Aut}(X)$ be such that $\rho_X(\phi) = \sigma$ with coefficients λ_k^σ, $k \in \{1,\dots,n\}$. By Lemma 4.2, we have that $\lambda_i^\tau = \lambda_i^\sigma$ and $\lambda_j^\tau = \lambda_j^\sigma$. Therefore, we obtain the following equalities:

$$\lambda_j^\tau \mu_{ij} = \lambda_j^\sigma \mu_{ij} = (\lambda_i^\sigma)^2 \mu_{\sigma(i)\sigma(j)} = (\lambda_i^\tau)^2 \mu_{\tau(i)\tau(j)},$$

where the second equality follows from the fact that $\phi \in \mathrm{Aut}(X)$. Thus, the nonzero scalars λ_i^τ satisfy Eq. (3), and they define an automorphism $\varphi \in \mathrm{Aut}(X)$ such that $\tau = \rho_X(\varphi) \in \rho_X\big(\mathrm{Aut}(X)\big)$. □

We now have all the necessary ingredients to prove our main theorem:

Proof of Theorem A. Observe that Theorem A(i) is essentially a corollary of Lemmas 4.2 and 4.3. We now turn our attention to proving Theorem A(ii).

Recall from Lemma 4.1 that either $\mu_{ij} = 0$ for all $i \neq j$, or $\mu_{ij} \neq 0$ for all $i \neq j$. Similarly, either $\mu_{ii} = 0$ for all $i \in \{1, \ldots, n\}$, or $\mu_{ii} \neq 0$ for all i. If $\mu_{ii} \neq 0$ for every i, then Theorem A(ii) follows directly from Theorem 4.6. Thus, we can assume that $\mu_{ij} \neq 0$ for $i \neq j$. Additionally, since S_n has no proper 4-transitive subgroups when $n < 5$, we further assume that $n \geq 5$.

We now proceed with an argument similar to that in the proof of Theorem 4.6: given $\tau \in S_n$, we construct $\varphi \in \mathrm{Aut}(X)$ such that $\rho_X(\varphi) = \tau$, by selecting nonzero scalars λ_i^τ that satisfy Eq. (3), and defining $\varphi(b_i) = \lambda_i^\tau b_{\tau(i)}$.

By Lemma 4.3, for φ to be an automorphism, the scalars must be of the form $\lambda_i^\tau = B(i,j,k,\tau) \neq 0$ for some $i \neq j \neq k \neq i$. Moreover, they must be independent of the specific choices of j and k: we select $j' \notin \{i,j,k\}$ and show that $B(i,j,k,\tau) = B(i,j',k,\tau)$. Indeed, since $\rho_X\big(\mathrm{Aut}(X)\big) \leq S_n$ is 4-transitive, there exists $\phi \in \mathrm{Aut}(X)$ such that $\sigma = \rho_X(\phi)$ agrees with τ on the set $\{i,j,j',k\}$. Therefore, we have the following equalities:

$$
\begin{aligned}
B(i,j,k,\tau) &= B(i,j,k,\sigma), \quad \text{because } \tau(i,j,k) = \sigma(i,j,k),\\
B(i,j,k,\sigma) &= B(i,j',k,\sigma), \quad \text{because} \sigma \text{is induced by an automorphism of} X,\\
B(i,j',k,\sigma) &= B(i,j',k,\tau), \quad \text{because } \tau(i,j',k) = \sigma(i,j',k).
\end{aligned}
$$

By combining all three equations, we obtain that $B(i,j,k,\tau) = B(i,j',k,\tau)$. The argument showing that the definition of the scalars λ_i^τ is independent of k follows similarly.

To complete the proof, it only remains to check Eq. (3) for the scalars λ_i^τ to obtain that $\varphi \in \mathrm{Aut}(X)$. Given $i,j \in \{1,\ldots,n\}$ (which may be equal), we select two additional auxiliary indices $k \neq l \in \{1,\ldots,n\}$, distinct from i and j. We then proceed as in the previous paragraph: we choose $\phi \in \mathrm{Aut}(X)$ such that $\sigma = \rho_X(\phi)$ agrees with τ when restricted to the set $\{i,j,k,l\}$. Let λ_i^σ denote the nonzero scalars associated to ϕ by Corollary 2.5, then we have:

$$
\begin{aligned}
\lambda_j^\tau \mu_{ij} &= B(j,k,l,\tau)\mu_{ij} = B(j,k,l,\sigma)\mu_{ij}\\
&= \lambda_j^\sigma \mu_{ij}, \text{ because } \tau(j,k,l) = \sigma(j,k,l),\\
\lambda_j^\sigma \mu_{ij} &= (\lambda_i^\sigma)^2 \mu_{\sigma(i)\sigma(j)}, \text{ because } \sigma = \rho_X(\phi),\\
(\lambda_i^\sigma)^2 \mu_{\sigma(i)\sigma(j)} &= B(i,k,l,\sigma)^2 \mu_{\sigma(i)\sigma(j)}\\
&= B(i,k,l,\tau)^2 \mu_{\tau(i)\tau(j)}\\
&= (\lambda_i^\tau)^2 \mu_{\tau(i)\tau(j)}, \text{ because } \tau(j,k,l) = \sigma(j,k,l).
\end{aligned}
$$

By combining these equations, we obtain

$$
\lambda_j^\tau \mu_{ij} = (\lambda_i^\tau)^2 \mu_{\tau(i)\tau(j)},
$$

which is precisely Eq. (3) for i, j, and τ. Repeating this argument for all pairs i, j, we conclude that $\varphi \in \mathrm{Aut}(X)$, thus completing the proof. □

We finish this section proving Corollary B:

Proof of Corollary B. Let $G \leq S_n$ be a k-transitive group with $k \geq 4$, and assume $\mathrm{Aut}(X) \cong G$. Recall that the kernel of the permutation representation $\mathrm{Aut}(X) \xrightarrow{\rho_X} S_n$ is a normal abelian subgroup [19, Theorem 3.2]. However, by the classification in Theorem 3.3, G has no proper abelian normal subgroups. Hence, ρ_X must be injective. Therefore, $\rho_X(\mathrm{Aut}(X)) \leq S_n$ and by Lemma 3.4, it follows that $\rho_X(\mathrm{Aut}(X)) \leq S_n$ is k-transitive. Applying Theorem A, we conclude that $\mathrm{Aut}(X) \cong S_n$. □

Remark 4.7. Corollary B shows that there is no n-dimensional idempotent evolution $\Bbbk$-algebra X, with $\Bbbk$ any field, such that $\mathrm{Aut}(X) \cong A_n$ for $n \geq 6$ or $\mathrm{Aut}(X) \cong M_n$ for $n = 11, 12, 23, 14$. In particular, the standard permutation representation $A_n \hookrightarrow S_n$ for $n \geq 6$ and $M_n \hookrightarrow S_n$ for $n = 11, 12, 23, 14$ cannot be realized as ρ_X for any X over any ground field $\Bbbk$. This illustrates that the answer to Question 1.1 is negative for some representations.

5. Actions on Idempotents

We have observed (see Remark 4.7) that certain permutation representations cannot be realized as ρ_X for any idempotent evolution algebra X, indicating that the answer to Question 1.1 is negative in some cases. However, in this section, we show that the scenario changes completely when considering the permutation representation $\widetilde{\rho}_X$. Specifically, we establish a positive answer to Question 1.2 by proving Theorem C.

To that purpose, we build upon the ideas in [14] and construct idempotent evolution algebras from finite simple graphs. We begin by recalling [13, Theorem 2.1], a refinement of [15, Theorem 3.1], which shows that permutation representations can be realized in the context of graphs:

Theorem 5.1 [13, Theorem 2.1]. *Let G be a finite group, V be a finite non-empty set and $\xi : G \to \mathrm{Sym}(V)$ a permutation representation of G on V. Then, there are infinitely many non-isomorphic (simple, undirected) finite connected graphs $\mathcal{G}$ such that:*

(i) *$V \subset V(\mathcal{G})$ and V is $\mathrm{Aut}(\mathcal{G})$-invariant.*
(ii) *Every vertex in $\mathcal{G}$ has degree at least 2.*
(iii) *$\mathrm{Aut}(\mathcal{G}) \cong G$;*
(iv) *the restriction $G \cong \mathrm{Aut}(\mathcal{G}) \to \mathrm{Sym}(V)$ is precisely ξ.*

To clarify how we use Theorem 5.1, we restate Theorem C as follows:

Theorem 5.2. *Let $\Bbbk$ be a field, G a finite group, V a finite non-empty set, and $\xi : G \to \mathrm{Sym}(V)$ a permutation representation of G on V. There exists an idempotent evolution $\Bbbk$-algebra X with a natural basis $\mathcal{B}$ such that:*

(1) *The elements of $\widetilde{B}_X$, the set of natural idempotent elements of X, are indexed by elements of V, hence we may identify $\widetilde{B}_X \cong V$. Recall that $\widetilde{B}_X$ is $\mathrm{Aut}(X)$-invariant.*
1. $\mathrm{Aut}(X) \cong G$.
(2) *The permutation representation $\widetilde{\rho}_X : G \cong \mathrm{Aut}(X) \to \mathrm{Sym}(\widetilde{B}_X) \cong \mathrm{Sym}(V)$ is precisely ξ.*

Proof. We apply Theorem 5.1 to first obtain a graph $\mathcal{G}$ from which we can explicitly derive an evolution algebra with the desired properties, following a procedure similar to the one outlined in [14].

Let $\mathcal{G}$ be the given graph, with vertices labeled $V(\mathcal{G}) = \{v_1, \ldots, v_n\}$ in such a way that $V = \{v_1, \ldots, v_r\} \subset V(\mathcal{G})$. We now define the algebra X by giving a natural basis $\mathcal{B} = \{b_{v_i} : v_i \in V(\mathcal{G})\} \cup \{b_{e_j} : e_j \in E(\mathcal{G})\}$ and the squares of its elements:

- $b_{v_i}^2 = b_{v_i}$, for $v_i \in V$;
- $b_{v_i}^2 = b_{v_i} + \sum_{w \in V} b_w$, for $v_i \in V(\mathcal{G}) \setminus V$;
- $b_{e_i}^2 = b_{e_i} + b_{v_a} + b_{v_b}$, for $e_i \in E(\mathcal{G})$ and $e_i = \{v_a, v_b\}$.

With this definition, we identify a vertex $v \in$ with b_v, so that $V \equiv \widetilde{\mathcal{B}}_X = b \in \mathcal{B} : b^2 = b$ as shown in (1). Next we analyze the group $\mathrm{Aut}(X)$ to check that (2) and (3) are also satisfied. First, note that by ordering $\mathcal{B}$ as defined earlier, the coefficient matrix $M_{\mathcal{B}}$ becomes upper-triangular with ones along the diagonal. This ensures that $M_{\mathcal{B}}$ is invertible, implying that X is an idempotent evolution algebra. Then, by Corollary 2.5, we know that any automorphism of X permutes the elements of $\mathcal{B}$ up to multiplication by nonzero scalars.

Since the number of nonzero coefficients, in each row of the coefficients matrix, is invariant under automorphisms, in the case where $r \neq 2$, we deduce that for any $\varphi \in \mathrm{Aut}(X)$, the following holds: for any $v \in V$, $\varphi(b_v) \in \Bbbk^* b_w$ for some $w \in V$; for any $u \in V(\mathcal{G}) \backslash V$, $\varphi(b_u) \in \Bbbk^* b_w$ for some $w \in V(\mathcal{G}) \backslash V$; and for any $e \in E(\mathcal{G})$, $\varphi(b_e) \in \Bbbk^* b_{e'}$ for some $e' \in E(\mathcal{G})$.

In the case where $r = 2$, this argument fails, as it could be that for some $v \in V(\mathcal{G}) \backslash V$, $\varphi(b_v) \in \Bbbk^* b_e$ with $e = \{u, w\}$. However, we can rule this out as follows. Since each vertex of $\mathcal{G}$ has degree at least 2, there exists an edge $f = \{v, v'\}$. Then, we have

$$\varphi(b_f)^2 = \varphi(b_f^2) = \varphi(b_f + b_v + b_{v'}) = \varphi(b_f) + \varphi(b_v) + \varphi(b_{v'}) = \varphi(b_f) + \lambda b_e + \varphi(b_{v'})$$

for some $\lambda \neq 0$. By Corollary 2.5, $\varphi(b_f)$ is a scalar multiple of a basis element. According to the structure constants, the square of this basis element can only have a nonzero b_e component if $\varphi(b_f) = \mu b_e$ for some $\mu \neq 0$. This is a contradiction, as it would imply that φ is not injective. Similarly, $\varphi(b_{v'})$ cannot contribute to the b_e component. Therefore, even in the case $r = 2$, the same conclusion holds as in the previous paragraph.

We deduce that there exists a permutation $\sigma \in \mathrm{Sym}(V(\mathcal{G}))$ that preserves V such that $\varphi(b_v) \in \Bbbk^* b_{\sigma(v)}$ for vertices, and a permutation of the edges that we also denote σ such that $\varphi(b_e) \in \Bbbk^* b_{\sigma(e)}$ for edges. For this automorphism φ, we obtain scalars $\lambda_v, \lambda_e \in \Bbbk^*$ for each $v \in V(\mathcal{G})$ and $e \in E(\mathcal{G})$, such that $\varphi(b_v) = \lambda_v b_{\sigma(v)}$ and $\varphi(b_e) = \lambda_e b_{\sigma(e)}$.

Now, for $v \in V$, we have:

$$\lambda_v b_{\sigma(v)} = \varphi(b_v) = \varphi(b_v^2) = \varphi(b_v)^2 = \lambda_v^2 b_{\sigma(v)}^2 = \lambda_v^2 b_{\sigma(v)}.$$

Thus, $\lambda_v = 1$.

Next, for $v \in V(\mathcal{G}) \setminus V$, we consider:

$$\varphi(b_v^2) = \varphi(b_v + \sum_{w \in V} b_w) = \lambda_v b_{\sigma(v)} + \sum_{w \in V} \lambda_w b_{\sigma(w)} = \lambda_v b_{\sigma(v)} + \sum_{w \in V} b_w,$$

and

$$\varphi(b_v)^2 = (\lambda_v b_v)^2 = \lambda_v^2 \left(b_{\sigma(v)} + \sum_{w \in V} b_w \right).$$

Thus, $\lambda_v = 1$ for all $v \in V(\mathcal{G})$.

Now, let $e \in E(\mathcal{G})$, with $e = \{v, w\}$ and $\sigma(e) = \{v', w'\}$. We have

$$\varphi(b_e^2) = \varphi(b_e + b_v + b_w) = \lambda_e b_{\sigma(e)} + \lambda_v b_{\sigma(v)} + \lambda_w b_{\sigma(w)},$$

and

$$\varphi(b_e)^2 = (\lambda_e b_e)^2 = \lambda_e^2 (b_{\sigma(e)} + b_{v'} + b_{w'}).$$

Therefore, $\lambda_e = 1$ and $\{v', w'\} = \{\sigma(v), \sigma(w)\}$. This shows that the permutation σ on the vertices is a graph isomorphism, while the permutation on the edges is the induced one.

With this, we have proved that an automorphism of X induces an automorphism of $\mathcal{G}$.

The converse is also true: consider an automorphism of $\mathcal{G}$, which can be described by a permutation σ of the vertices, inducing a permutation of the edges (denoted by the same name) such that $\sigma(\{v, w\}) = \{\sigma(v), \sigma(w)\}$. We define an automorphism $\varphi \in \mathrm{Aut}(X)$ on the natural basis by setting $\varphi(b_v) = b_{\sigma(v)}$ for $v \in V(\mathcal{G})$ and $\varphi(b_e) = b_{\sigma(e)}$ for $e \in E(\mathcal{G})$. To verify that this is an automorphism, we check that it preserves the squares of the basis elements. Indeed, for $v \in V$, we have that $\sigma(v) \in V$, so $\varphi(b_v^2) = \varphi(b_v) = b_{\sigma(v)} = (b_{\sigma(v)})^2$. Now, for $v \in V(\mathcal{G}) \setminus V$, we have that $\varphi(b_v^2) = \varphi(b_v + \sum_{w \in V} b_w) = b_{\sigma(v)} + \sum_{w \in V} b_{\sigma(w)} = b_{\sigma(v)} + \sum_{w \in V} b_w = (b_{\sigma(v)})^2$ using the fact that σ permutes V. Finally, for $e = \{u, v\} \in E(\mathcal{G})$, we have that $\varphi(b_e^2) = \varphi(b_e + b_u + b_v) = b_{\sigma(e)} + b_{\sigma(u)} + b_{\sigma(v)} = (b_{\sigma(e)})^2$, since σ is a graph automorphism. Thus, $\varphi \in \mathrm{Aut}(X)$.

The maps we have constructed between $\mathrm{Aut}(X)$ and $\mathrm{Aut}(\mathcal{G})$ are clearly mutual inverses and respect composition. Therefore, we conclude that $\mathrm{Aut}(X) \cong \mathrm{Aut}(\mathcal{G})$ as groups. Moreover, the restriction of the action to V commutes with this isomorphism, thus completing the proof of the theorem. □

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Declarations

Conflict of interest The authors declare no conflict of interest.

References

[1] Abbott, R., Bray, J., Linton, S., Nickerson, S., Norton, S., Parker, R., Suleiman, I., Tripp, J., Walsh, P., Wilson, R.: Atlas of Finite Group Representations—Version 3 (2024). https://brauer.maths.qmul.ac.uk/Atlas/v3/

[2] Boudi, N., Cabrera Casado, Y., Siles Molina, M.: Natural families in evolution algebras. Publ. Mat. Barc. **66**(1), 159–181 (2022)

[3] Cabrera Casado, Y., Cardoso Gonçalves, M.I., Gonçalves, D., Martín Barquero, D., Martín González, C.: Chains in evolution algebras. Linear Algebra Appl. **622**, 104–149 (2021)

[4] Cabrera Casado, Y., Martín Barquero, D., Martín González, C.: Two-dimensional perfect evolution algebras over domains. J. Algebr. Comb. **58**(2), 569–587 (2023)

[5] Cabrera Casado, Y., Martín Barquero, D., Martín González, C., Tocino, A.: Tensor product of evolution algebras. Mediterr. J. Math. **20**(1), 31 (2023)

[6] Cabrera Casado, Y., Martín Barquero, D., Martín González, C., Tocino, A.: On simple evolution algebras of dimension two and three. Constructing simple and semisimple evolution algebras. Linear Multilinear Algebra (2024). https://doi.org/10.48550/arXiv.2206.13912

[7] Cabrera Casado, Y., Siles Molina, M., Velasco, M.V.: Evolution algebras of arbitrary dimension and their decompositions. Linear Algebra Appl. **495**, 122–162 (2016)

[8] Cabrera Casado, Y., Siles Molina, M., Velasco, M.V.: Classification of three-dimensional evolution algebras. Linear Algebra Appl. **524**, 68–108 (2017)

[9] Calderón Martín, A.J., Fernández Ouaridi, A., Kaygorodov, I.: Non-degenerate evolution algebras. J. Pure Appl. Algebra **228**(6), 107594 (2024)

[10] Camacho, L.M., Gómez, J.R., Omirov, B.A., Turdibaev, R.M.: Some properties of evolution algebras. Bull. Korean Math. Soc. **5**, 1481–1494 (2013)

[11] Cameron, P.J.: Permutation Groups. Lond. Math. Soc. Stud. Texts, vol. 45. Cambridge University Press, Cambridge (1999)

[12] Casas, J.M., Ladra, M., Omirov, B.A., Rozikov, U.A.: On nilpotent index and dibaricity of evolution algebras. Linear Algebra Appl. **43**(9), 90–105 (2013)

[13] Costoya, C., Gomes, R., Viruel, A.: Realization of permutation modules via Alexandroff spaces. Result Math. **79**(4), 14 (2024)

[14] Costoya, C., Ligouras, P., Tocino, A., Viruel, A.: Regular evolution algebras are universally finite. Proc. Am. Math. Soc. **150**(3), 919–925 (2022)

[15] Costoya, C., Méndez, D., Viruel, A.: Representability of permutation representations on coalgebras and the isomorphism problem. Mediterr. J. Math. **17**, 157 (2020)

[16] Costoya, C., Muñoz, V., Tocino, A., Viruel, A.: Automorphism groups of Cayley evolution algebras. Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM **117**(2), 11 (2023)

[17] Elduque, A., Labra, A.: Evolution algebras and graphs. J. Algebra Appl. **14**(7), 1550103 (2015)

[18] Elduque, A., Labra, A.: On nilpotent evolution algebras. Linear Algebra Appl. **505**, 11–31 (2016)

[19] Elduque, A., Labra, A.: Evolution algebras, automorphisms, and graphs. Linear Multilinear Algebra **69**(1), 1–12 (2019)

[20] Etherington, I.M.H.: Genetic algebras. Proc. R. Soc. Edinb. **59**, 242–258 (1939)

[21] Robinson, D.J.S.: A Course in the Theory of Groups. Grad. Texts Math., vol. 80, 2nd edn. Springer-Verlag, New York, NY (1995)

[22] Sriwongsa, S., Zou, Y.M.: On automorphism groups of idempotent evolution algebras. Linear Algebra Appl. **641**, 143–155 (2022)

[23] Tian, J.P.: Evolution Algebras and Their Applications. Lecture Notes in Mathematics, vol. 1921. Springer, Berlin (2008)

[24] Tian, J.P., Vojtěchovský, P.: Mathematical concepts of evolution algebras in non-Mendelian genetics. Quasigr. Relat. Syst. **14**(1), 111–122 (2006)

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