

Classifying adjoint pairs and adjoint triples in an Atanassov L -fuzzy framework

Ondrej Krídlo Manuel Ojeda-Aciego

Abstract—We study and classify a family of adjoint pairs and adjoint triples for Atanassov L -fuzzy framework based on a complete residuated lattice satisfying the double negation law.

Index Terms—Adjoint pair; adjoint triple; Girard monoid; Atanassov L -fuzzy sets

I. INTRODUCTION

Residuated structures are fundamental for the algebraic treatment of some non-classical logics, such as multiple-valued logics or fuzzy logics. The key notion in residuation is that of *adjoint pair*, i.e. a pair of operations (\otimes, \rightarrow) which satisfy the so-called adjoint condition

$$a \otimes b \leq c \iff a \leq b \rightarrow c.$$

If the operation \otimes is not commutative, then the previous equivalence depends on which argument of $a \otimes b$ is displaced and, in general, two sided residuated implications enter the scene. Now, the general form of the adjoint condition is

$$a \otimes b \leq c \iff a \leq b \rightarrow_1 c \iff b \leq a \rightarrow_2 c$$

and, we would be talking about an *adjoint triple* $(\otimes, \rightarrow_1, \rightarrow_2)$.

The simultaneous use of several adjoint pairs led to the multi-adjoint framework, which has found applications in several areas, such as generalized logic programming [1] or Formal Concept Analysis [2]. Similarly, in contexts in which commutativity does not necessarily hold, the adjoint triples have taken the leading role, for instance, being the main subject of study [3] or as a tool for providing alternative constructions of concept lattices [4], [5].

The other important notion here is that of Atanassov fuzzy set. This type of fuzzy sets were introduced in [6] by considering, for all elements x a membership degree $\mu(x)$ together with a non-membership degree $\nu(x)$ such that $\mu(x) + \nu(x) \leq 1$, somehow allowing an *indetermination degree* about x in the case of strict inequality. It is worth mentioning that if the indetermination degree equals zero then the theory collapses to that of standard fuzzy sets.

The construction on the unit interval was later generalized by using a complete lattice instead of the unit interval as underlying set of truth-values [7], [8]. According to the number of papers in the recent years, this research area is very active, and a lot of researchers are still studying and applying this kind of sets [9], [10], [11].

O. Krídlo is with the Institute of Informatics, University of Pavol Jozef Šafárik, Košice, Slovakia (email: ondrej.kridlo@upjs.sk), and M. Ojeda-Aciego is with the Dept. Matemática Aplicada, Univ. de Málaga, Biv. Louis Pasteur 35, 29071 Málaga, Spain. (email: aciego@uma.es).

Previous work on defining adjoint pairs or adjoint triples in an Atanassov L -fuzzy framework has been published in [12], [13], [4] and, in this paper, we aim at a more thorough treatment by systematically considering, firstly, a family of reasonable definitions and, then, studying their feasibility. Our required underlying framework is that of a residuated lattice satisfying the double negation law (such a structure has been already called a *Girard monoid* [14]); this is not a very strict requirement since it holds, for instance, for the whole class of MV-algebras.

The structure of the paper is the following: in Section II we give the preliminary definitions related to Atanassov L -fuzzy sets, residuated lattices, double negation law and related results; then in Section III, our framework for building adjoint pairs and adjoint triples is introduced, and provide the initial set of admissible conjunctions to work with in the rest of the paper; Sections IV and V focus on the construction of the adjoint pairs (resp. triples) and the proofs of non-existence; finally, in Section VI we draw some conclusions and provide prospects for future work.

II. PRELIMINARY DEFINITIONS

As stated above, we will be primary dealing with truth-values not necessarily belonging to the unit interval, but to a complete residuated lattice (see [15], [16], [17] for details).

Definition 1. An algebra $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow \rangle$ is said to be a complete residuated lattice if

- 1) $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice where 0 and 1 are the bottom and top elements respectively.
- 2) $\langle L, \otimes, 1 \rangle$ is a commutative monoid.
- 3) $\langle \otimes, \rightarrow \rangle$ is an adjoint pair, i.e.

$$a \otimes b \leq c \quad \text{if and only if} \quad a \leq b \rightarrow c$$

for all $a, b, c \in L$, where \leq is the ordering generated by \wedge and \vee .

Let us recall now the notion of L -fuzzy set defined on a complete lattice, as introduced in [18].

Definition 2. Given a complete lattice L together with an involutive order reversing operation $N: L \rightarrow L$, and a universe set E . An Atanassov L -fuzzy set (ILF set) A in E is defined as an object having the form:

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in E \}$$

where the functions $\mu_A: E \rightarrow L$ and $\nu_A: E \rightarrow L$ define the degree of membership and the degree of non-membership,

respectively, to A of the elements $x \in E$, and for every $x \in E$, the following inequality holds:

$$\mu_A(x) \leq N(\nu_A(x)). \quad (1)$$

When the previous inequality is strict, there is a certain indetermination degree on the knowledge about x .

Note that, when the underlying lattice is residuated, we already have a negation operator defined by $\neg x = x \rightarrow 0$. As a result, we can define the ILF-lattice associated with a given residuated lattice \mathcal{L} as follows:

Definition 3. Given a complete residuated lattice $\mathcal{L} = \langle L, \wedge, \vee, 0, 1, \otimes, \rightarrow \rangle$, we can consider the lattice of Atanassov L -truth-values

$$\mathcal{L}_{ALV} = \langle \{ \langle k_1, k_2 \rangle \in L \times L \mid k_1 \leq \neg k_2 \}, \leq \rangle$$

where the ordering on \mathcal{L}_{ALV} is defined by $\langle k_1, k_2 \rangle \leq \langle m_1, m_2 \rangle$ if and only if $k_1 \leq m_1$ and $k_2 \geq m_2$.

Note that this construction of \mathcal{L}_{ALV} is used in [19], considering \mathcal{L} as the underlying set of truth-values instead of the unit interval. We will often denote any element of \mathcal{L}_{ALV} as $\bar{a} = \langle a_1, a_2 \rangle$. The following result is straightforward, and proves that \mathcal{L}_{ALV} is, indeed, a complete lattice.

Note that the lattice of Atanassov L -truth-values is a complete sublattice of $L \times L^d$. So if we would like to make a residuated lattice out of $L \times L^d$, the first two operations together with the top and bottom elements are already defined, the aim here is to define conjunction and implication in a natural way (or three corresponding operations in a non-commutative case). We will be dealing with whether the new defined "operations" are closed within \mathcal{L}_{ALV} .

Lemma 1 ([19]). $\langle \mathcal{L}_{ALV}, \leq \rangle$ forms a complete lattice in which the meet and join are defined by

$$\bigwedge_{i \in I} \bar{a}_i = \left\langle \bigwedge_{i \in I} a_{i1}; \bigvee_{i \in I} a_{i2} \right\rangle \quad \bigvee_{i \in I} \bar{a}_i = \left\langle \bigvee_{i \in I} a_{i1}; \bigwedge_{i \in I} a_{i2} \right\rangle$$

Our working framework needs that the underlying residuated lattice satisfy one extra condition.

Definition 4 ([14]). A Girard monoid is a residuated lattice \mathcal{L} satisfying the law of double negation, namely, the equality $x = (x \rightarrow 0) \rightarrow 0$ holds for all $x \in L$.

This notion represents one of the several variants in which one can find residuated lattices, it was used by Girard in his development programme for linear logics, and a study of its structure in the particular case of the unit interval can be found in [20]. Other well-known enriched versions of residuated lattices include, for instance, Heyting algebras (satisfying $x \otimes y = x \wedge y$), BL-algebras (satisfying divisibility, i.e. $x \wedge y = x \otimes (x \rightarrow y)$), and the prelinearity, i.e. $(x \rightarrow y) \vee (y \rightarrow x) = 1$), or MV-algebras (BL-algebra satisfying the law of double negation).

The definition of the conjunctive in \mathcal{L}_{ALV} (to be introduced in the next section) will make use of the following operator:

Definition 5. The operator $\oplus: L \times L \rightarrow L$ is defined by

$$a \oplus b = \neg a \rightarrow b = (a \rightarrow 0) \rightarrow b.$$

Under the double negation law, it is not difficult to check that \oplus is both commutative and associative. Furthermore, the De Morgan laws between \otimes and \oplus , and also between \vee and \wedge , and contraposition law also hold:

Lemma 2 ([21]). These equalities hold in a Girard monoid

- $\neg(a \otimes b) = \neg a \oplus \neg b$
- $\neg(a \oplus b) = \neg a \otimes \neg b$
- $\neg(a \vee b) = \neg a \wedge \neg b$
- $\neg(a \wedge b) = \neg a \vee \neg b$
- $a \rightarrow b = \neg b \rightarrow \neg a$
- $\neg(a \rightarrow b) = a \otimes \neg b$

Hereafter, we will assume that \mathcal{L} is a Girard monoid.

III. A FIRST POSET OF POSSIBLE CONJUNCTIONS

The analysis of the reasonable conjunctions in the lattice of Atanassov's truth-values \mathcal{L}_{ALV} leads to two parameters, each one governing the way in which the membership and the non-membership functions are built. As a result, those conjunctions will be denoted by \boxtimes_{ij} in which the indexes i and j describe, respectively, the construction of the membership and non-membership function.

Taking into account the particularities of the truth-values in \mathcal{L}_{ALV} , we consider the following five possibilities to define the first component (membership function) for the conjunction for all $\langle a_1, a_2 \rangle, \langle b_1, b_2 \rangle \in \mathcal{L}_{ALV}$:

- i=1 $a_1 \otimes b_1$, i.e. conjunction of both membership values.
- i=2 $a_1 \otimes \neg b_2$, i.e. conjunction of first membership value and the negation of second non-membership value.
- i=3 $\neg a_2 \otimes b_1$, i.e. conjunction of the negation of first non-membership value and second membership value.
- i=4 $(a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1)$, join of two previous approaches.
- i=5 $\neg a_2 \otimes \neg b_2$, conjunction of the negation of both non-membership values.

By duality, there are also five possibilities for the second component:

- j=1 $a_2 \oplus b_2$
- j=2 $(a_2 \oplus \neg b_1) \wedge (\neg a_1 \oplus b_2)$
- j=3 $\neg a_1 \oplus b_2$
- j=4 $a_2 \oplus \neg b_1$
- j=5 $\neg a_1 \oplus \neg b_1$

For instance, \boxtimes_{24} refers to the operator defined as

$$\bar{a} \boxtimes_{24} \bar{b} = \langle a_1 \otimes \neg b_2, a_2 \oplus \neg b_1 \rangle$$

All the previous possibilities for the membership, resp. non-membership, functions can be represented in a Hasse diagram corresponding to the inherited ordering (just take into account the monotonicity of \otimes and \oplus and Atanassov's inequation $a_2 \leq \neg a_1$ for all $\langle a_1, a_2 \rangle \in \mathcal{L}_{ALV}$). The resulting lattices for the membership, resp. non-membership, functions are shown in Figure 1; note that the ordered set of second components is dually isomorphic to that of the first components.

Finally, since Atanassov's inequation (1) should be preserved, not every possible combination of both components is admissible. The only combinations which satisfy Atanassov's inequation and, hence, those we will consider in the rest of the paper are the following (note that the subscripts refer to the numbering of vertices in Figure 1):

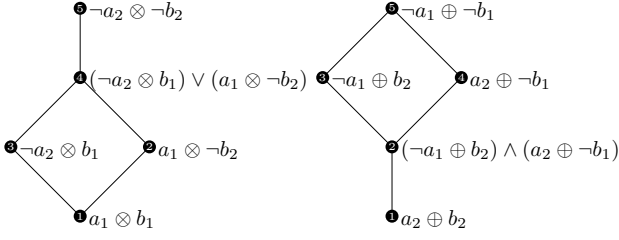


Fig. 1. Lattices of possible membership (left) and non-membership (right) functions for Atanassov conjunction $\bar{a} \boxtimes \bar{b}$.

- \boxtimes_{1j} for all $j \in \{1, 2, 3, 4, 5\}$
 - $j = 5$ $\neg(a_1 \otimes b_1) = \neg a_1 \oplus \neg b_1$
 - $j = 4$ $\neg(a_1 \otimes b_1) = \neg a_1 \oplus \neg b_1 \geq a_2 \oplus \neg b_1$
 - $j = 3$ $\neg(a_1 \otimes b_1) = \neg a_1 \oplus \neg b_1 \geq \neg a_1 \oplus b_2$
 - $j = 2$ $\neg(a_1 \otimes b_1) = \neg a_1 \oplus \neg b_1 \geq (a_2 \oplus \neg b_1) \wedge (\neg a_1 \oplus b_2)$
 - $j = 1$ $\neg(a_1 \otimes b_1) = \neg a_1 \oplus \neg b_1 \geq a_2 \oplus b_2$
- \boxtimes_{2j} for $j \in \{1, 2, 3\}$
 - $j = 3$ $\neg(a_1 \otimes \neg b_2) = \neg a_1 \oplus b_2$
 - $j = 2$ $\neg(a_1 \otimes \neg b_2) = \neg a_1 \oplus b_2 \geq (a_2 \oplus \neg b_1) \wedge (\neg a_1 \oplus b_2)$
 - $j = 1$ $\neg(a_1 \otimes \neg b_2) = \neg a_1 \oplus b_2 \geq a_2 \oplus b_2$
- Counterexample for \boxtimes_{2j} where $j \in \{4, 5\}$
 $\langle 1, 0 \rangle \boxtimes_{2j} \langle 0, 0 \rangle = \langle 1, 1 \rangle$ and $\langle 1, 1 \rangle$ is not in \mathcal{L}_{ALV} .
- \boxtimes_{3j} for $j \in \{1, 2, 4\}$
 - $j = 4$ $\neg(\neg a_2 \otimes b_1) = a_2 \oplus \neg b_1$
 - $j = 2$ $\neg(\neg a_2 \otimes b_1) = a_2 \oplus \neg b_1 \geq (a_2 \oplus \neg b_1) \wedge (\neg a_1 \oplus b_2)$
 - $j = 1$ $\neg(\neg a_2 \otimes b_1) = a_2 \oplus \neg b_1 \geq a_2 \oplus b_2$
- Counterexample for \boxtimes_{3j} where $j \in \{3, 5\}$
 $\langle 0, 0 \rangle \boxtimes_{3j} \langle 1, 0 \rangle = \langle 1, 1 \rangle$ and $\langle 1, 1 \rangle$ is not in \mathcal{L}_{ALV} .
- \boxtimes_{4j} for $j \in \{1, 2\}$
 - $j = 2$ $\neg((a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1)) = (\neg a_1 \oplus b_2) \wedge (a_2 \oplus \neg b_1)$
 - $j = 1$ $\neg((a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1)) = (\neg a_1 \oplus b_2) \wedge (a_2 \oplus \neg b_1) \geq (a_2 \oplus b_2)$
- Counterexample for \boxtimes_{4j} where $j \in \{3, 5\}$
 $\langle 0, 0 \rangle \boxtimes_{4j} \langle 1, 0 \rangle = \langle 1, 1 \rangle$ and $\langle 1, 1 \rangle$ is not in \mathcal{L}_{ALV} .
- Counterexample for \boxtimes_{4j} where $j \in \{4, 5\}$
 $\langle 1, 0 \rangle \boxtimes_{4j} \langle 0, 0 \rangle = \langle 1, 1 \rangle$ and $\langle 1, 1 \rangle$ is not in \mathcal{L}_{ALV} .
- \boxtimes_{51}
 - $\neg(\neg a_2 \otimes \neg b_2) = a_2 \oplus b_2$
- Counterexample for \boxtimes_{5j} where $j \in \{2, 3, 4, 5\}$
 $\langle 0, 0 \rangle \boxtimes_{5j} \langle 0, 0 \rangle = \langle 1, 1 \rangle$ and $\langle 1, 1 \rangle$ is not in \mathcal{L}_{ALV} .

All possible pairs of first and second components of admissible (hence, Atanassov L -fuzzy) conjunctions form a lattice, which is shown in Figure 2. Recall that the ordering is that $\boxtimes_{ij} \leq \boxtimes_{kl}$ if and only if for any two $\bar{a}, \bar{b} \in \mathcal{L}_{ALV}$ holds $(\bar{a} \boxtimes_{ij} \bar{b})_1 \leq (\bar{a} \boxtimes_{kl} \bar{b})_1$ and $(\bar{a} \boxtimes_{ij} \bar{b})_2 \geq (\bar{a} \boxtimes_{kl} \bar{b})_2$.

Note that all the Atanassov conjunctions in the central axis of the lattice are commutative (hence, can generate adjoint pairs) whereas those in the left and right parts are non-commutative (so, they can generate adjoint triples). As the conjunctions in the left and in the right part are somehow symmetric, for instance $\bar{a} \boxtimes_{13} \bar{b} = \bar{b} \boxtimes_{14} \bar{a}$, their behaviour is essentially identical.

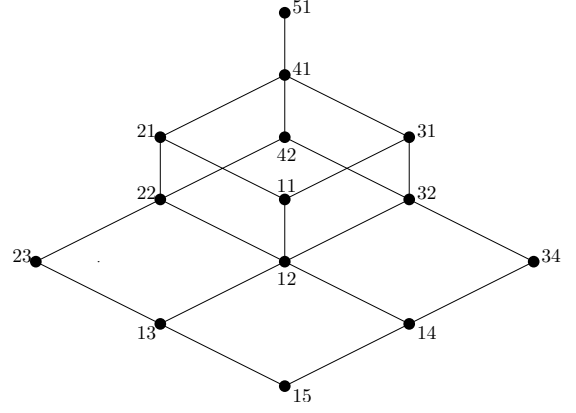


Fig. 2. Lattice of admissible conjunctions.

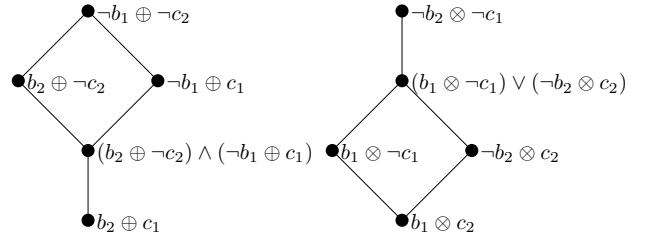


Fig. 3. Lattices of possible membership (left) and non-membership (right) functions for Atanassov implication $\bar{b} \rightrightarrows \bar{c}$.

Remark 1. The construction of the Atanassov L -fuzzy implications associated with each conjunction is straightforward for the first component, since it has to be the (standard) residuated implication; however, there is certain leeway with respect to the non-membership function. For convenience, these possibilities are depicted in Figure 3. Note that, once again, not every combination is admissible, the restrictions imposed by the Atanassov's inequality in this case are exactly the same as those for the conjunction; this is so because of the double negation law and the fact that the possibilities for membership and non-membership functions are interchanged.

Last but not least, let us highlight that, not every admissible conjunction can be part of an adjoint pair or triple, this will be proved in Propositions 6 and 11 in subsequent sections.

IV. ADJOINT PAIRS

As stated above, the set $\{\boxtimes_{51}, \boxtimes_{41}, \boxtimes_{42}, \boxtimes_{11}, \boxtimes_{12}, \boxtimes_{15}\}$ corresponds to commutative conjunctions. In this section we will introduce the corresponding implications which create adjoint pairs. Initial results were obtained in [12], [13], [4].

Proposition 1. $\langle \boxtimes_{51}, \rightrightarrows_{51} \rangle$ forms an adjoint pair on \mathcal{L}_{ALV} where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

$$\bar{a} \boxtimes_{51} \bar{b} = \langle \neg a_2 \otimes \neg b_2; a_2 \oplus b_2 \rangle \quad \bar{b} \rightrightarrows_{51} \bar{c} = \langle b_2 \oplus c_1; \neg b_2 \otimes \neg c_1 \rangle$$

Proof. It is obvious that the output values of \rightrightarrows_{51} are in \mathcal{L}_{ALV} . Given elements $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$

- Assume $\bar{a} \boxtimes_{51} \bar{b} \leq \bar{c}$. This means $\neg a_2 \otimes \neg b_2 \leq c_1$ and $c_2 \leq a_2 \oplus b_2$.

We have that $\neg a_2 \otimes \neg b_2 \leq c_1 \iff \neg a_2 \leq \neg b_2 \rightarrow c_1 = b_2 \oplus c_1$ and, in addition, $a_1 \leq \neg a_2$. Hence, we obtain $a_1 \leq b_2 \oplus c_1$.

Moreover, from $\neg a_2 \leq b_2 \oplus c_1$ above we obtain $a_2 \geq \neg(b_2 \oplus c_1) = \neg b_2 \otimes \neg c_1$.

The two inequalities imply $\bar{a} \leq \bar{b} \rightrightarrows_{51} \bar{c}$.

- Conversely, assuming $\bar{a} \leq \bar{b} \rightrightarrows_{51} \bar{c}$ we have, in particular, $\neg b_2 \otimes \neg c_1 \leq a_2$, which is equivalent to $\neg c_1 \leq \neg b_2 \rightarrow a_2 = a_2 \oplus b_2$. From this inequality we obtain two consequences:

- (i) Applying negation, we have $c_1 \geq \neg a_2 \otimes \neg b_2$.
- (ii) Recalling $c_2 \leq \neg c_1$, we obtain $c_2 \leq a_2 \oplus b_2$.

Therefore, we have proved $\bar{a} \boxtimes_{51} \bar{b} \leq \bar{c}$. \square

Proposition 2. $\langle \boxtimes_{41}, \rightrightarrows_{41} \rangle$ is an adjoint pair on \mathcal{L}_{ALV} where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined as follows:

- $\bar{a} \boxtimes_{41} \bar{b} = \langle (a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1); a_2 \oplus b_2 \rangle$
- $\bar{b} \rightrightarrows_{41} \bar{c} = \langle b_2 \oplus c_1; (b_1 \otimes \neg c_1) \vee (\neg b_2 \otimes c_2) \rangle$

Proof. It is not difficult to check that the output values of \rightrightarrows_{41} are in \mathcal{L}_{ALV} .

Given elements $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$

- Assume $\bar{a} \boxtimes_{41} \bar{b} \leq \bar{c}$. This means $(a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1) \leq c_1$ and $c_2 \leq a_2 \oplus b_2 = \neg b_2 \rightarrow a_2$.

From $(a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1) \leq c_1$, using the first disjunct we obtain $a_1 \leq \neg b_2 \rightarrow c_1 = b_2 \oplus c_1$.

From second disjunct we obtain $\neg a_2 \leq b_1 \rightarrow c_1$, equivalently, $b_1 \otimes \neg c_1 \leq a_2$. Now, using $c_2 \leq \neg b_2 \rightarrow a_2$ we get $\neg b_2 \otimes c_2 \leq a_2$. Hence, we have proved $(b_1 \otimes \neg c_1) \vee (\neg b_2 \otimes c_2) \leq a_2$.

The two paragraphs above imply that $\bar{a} \leq \bar{b} \rightrightarrows_{41} \bar{c}$.

- Conversely, assuming $\bar{a} \leq \bar{b} \rightrightarrows_{41} \bar{c}$ we have, in particular, $a_1 \leq b_2 \oplus c_1 = \neg b_2 \rightarrow c_1$, which is equivalent to $a_1 \otimes \neg b_2 \leq c_1$.

Our current assumption also leads to $(b_1 \otimes \neg c_1) \vee (\neg b_2 \otimes c_2) \leq a_2$. From this inequality we obtain two consequences:

- (i) Using the first disjunct, we have $b_1 \otimes \neg c_1 \leq a_2$, which is equivalent to $\neg a_2 \otimes b_1 \leq c_1$.
- (ii) From $\neg b_2 \otimes c_2 \leq a_2$, we obtain $c_2 \leq a_2 \oplus b_2$.

Therefore, we have proved $\bar{a} \boxtimes_{41} \bar{b} \leq \bar{c}$. \square

Proposition 3. $\langle \boxtimes_{42}, \rightrightarrows_{42} \rangle$ is an adjoint pair on \mathcal{L}_{ALV} where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

- $\bar{a} \boxtimes_{42} \bar{b} = \langle (a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1); (a_2 \oplus \neg b_1) \wedge (\neg a_1 \oplus b_2) \rangle$
- $\bar{b} \rightrightarrows_{42} \bar{c} = \langle b_2 \oplus c_1; b_1 \otimes \neg c_1 \rangle$

Proof. It is not difficult to check that the output values of \rightrightarrows_{42} are in \mathcal{L}_{ALV} .

Given elements $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$

- Assume $\bar{a} \boxtimes_{42} \bar{b} \leq \bar{c}$. In particular, this means $(a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1) \leq c_1$.

Using the first disjunct we obtain $a_1 \leq \neg b_2 \rightarrow c_1 = b_2 \oplus c_1$. From second disjunct we obtain $\neg a_2 \leq b_1 \rightarrow c_1$, equivalently, $b_1 \otimes \neg c_1 \leq a_2$.

The paragraph above implies that $\bar{a} \leq \bar{b} \rightrightarrows_{42} \bar{c}$.

- Conversely, assume $\bar{a} \leq \bar{b} \rightrightarrows_{42} \bar{c}$, then:

On the one hand, we have $a_1 \leq b_2 \oplus c_1 = \neg b_2 \rightarrow c_1$, which is equivalent to $a_1 \otimes \neg b_2 \leq c_1$. Moreover, applying negation to the previous inequality, and using $c_2 \leq \neg c_1$, we have $c_2 \leq \neg a_1 \oplus b_2$.

On the other hand, we also have $b_1 \otimes \neg c_1 \leq a_2$, which is equivalent to $b_1 \otimes \neg a_2 \leq c_1$. Moreover, by using $c_1 \leq \neg c_2$ and applying negation, we obtain $c_2 \leq \neg b_1 \oplus a_2$.

Summarizing, by using the results in the two previous paragraphs together with the properties of meet and join, have proved both $(a_1 \otimes \neg b_2) \vee (\neg a_2 \otimes b_1) \leq c_1$ and $c_2 \leq (a_2 \oplus \neg b_1) \wedge (\neg a_1 \oplus b_2)$, that is, $\bar{a} \boxtimes_{42} \bar{b} \leq \bar{c}$. \square

Proposition 4 ([12]). $\langle \boxtimes_{11}, \rightrightarrows_{11} \rangle$ is an adjoint pair on \mathcal{L}_{ALV} where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

- $\bar{a} \boxtimes_{11} \bar{b} = \langle a_1 \otimes b_1; a_2 \oplus b_2 \rangle$
- $\bar{b} \rightrightarrows_{11} \bar{c} = \langle (b_1 \rightarrow c_1) \wedge (\neg b_2 \rightarrow \neg c_2); \neg b_2 \otimes c_2 \rangle$

Proposition 5. $\langle \boxtimes_{12}, \rightrightarrows_{12} \rangle$ is an adjoint pair on \mathcal{L}_{ALV} where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

- $\bar{a} \boxtimes_{12} \bar{b} = \langle a_1 \otimes b_1; (a_2 \oplus \neg b_1) \wedge (\neg a_1 \oplus b_2) \rangle$
- $\bar{b} \rightrightarrows_{12} \bar{c} = \langle (\neg b_1 \oplus c_1) \wedge (b_2 \oplus \neg c_2); b_1 \otimes c_2 \rangle$

Proof. It is not difficult to check that the output values of \rightrightarrows_{12} are in \mathcal{L}_{ALV} .

Consider arbitrary elements $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, and assume $\bar{a} \boxtimes_{12} \bar{b} \leq \bar{c}$; this means $a_1 \otimes b_1 \leq c_1$ and $c_2 \leq (\neg a_1 \oplus b_2) \wedge (a_2 \oplus \neg b_1)$.

- $a_1 \otimes b_1 \leq c_1$ is equivalent to $a_1 \leq b_1 \rightarrow c_1 = \neg b_1 \oplus c_1$.
- $c_2 \leq \neg a_1 \oplus b_2$ is equivalent to $a_1 \leq c_2 \rightarrow b_2 = \neg b_2 \rightarrow \neg c_2 = b_2 \oplus \neg c_2$.
- $c_2 \leq a_2 \oplus \neg b_1 = b_1 \rightarrow a_2$ is equivalent to $b_1 \otimes c_2 \leq a_2$.

We have proved that $\bar{a} \leq \bar{b} \rightrightarrows_{12} \bar{c}$.

For the converse we have just to read the previous equivalences from right to left. \square

It is remarkable that not all the Atanassov L -fuzzy conjunctions are compatible with the construction of residuated implication in this extended framework. The last result in this section is a proof of non-existence of adjoint pair in the Atanassov L -fuzzy case for any compatible conjunction.

Proposition 6. The conjunction $\bar{a} \boxtimes_{15} \bar{b} = \langle a_1 \otimes b_1; \neg a_1 \oplus \neg b_1 \rangle$ does not have a residuated implication in Atanassov's sense.

Proof. According to the intended behaviour of the Atanassov L -fuzzy residuated implication, its first component should be the corresponding residuated implication at the underlying residuated lattice level and, furthermore, its second component should be compatible with Atanassov's inequality, i.e. there are just two possibilities which are compatible

$$\bar{b} \rightrightarrows \bar{c} = \langle \neg b_1 \oplus c_1, b_1 \otimes \neg c_1 \rangle$$

$$\bar{b} \rightrightarrows \bar{c} = \langle \neg b_1 \oplus c_1, b_1 \otimes c_2 \rangle$$

and we will show that none of them is, in fact, the residuated implication in the Atanassov L -fuzzy sense by showing an example in which the adjoint property fails.

On the one hand, we have

$$\langle 0, 0 \rangle \boxtimes_{15} \langle 1, 0 \rangle = \langle 0 \otimes 1, \neg 0 \oplus \neg 1 \rangle = \langle 0, 1 \rangle.$$

whereas, on the other hand, for any of the two possible implications above we have that

$$\langle 1, 0 \rangle \rightrightarrows \langle 0, 1 \rangle = \langle 0, 1 \rangle$$

and, as a result $\langle 0, 0 \rangle \boxtimes_{15} \langle 1, 0 \rangle \leq \langle 0, 1 \rangle$ is not equivalent to $\langle 0, 0 \rangle \leq \langle 1, 0 \rangle \rightrightarrows \langle 0, 1 \rangle$ since $\langle 0, 0 \rangle > \langle 0, 1 \rangle$ in \mathcal{L}_{ALV} . \square

V. ADJOINT TRIPLES

In the following we will try to find adjoint triples for non-commutative conjunctions defined on \mathcal{L}_{ALV} . Recall the adjoint triple condition for $(\otimes, \rightarrow_1, \rightarrow_2)$

$$x \leq y \rightarrow_1 z \iff x \otimes y \leq z \iff y \leq x \rightarrow_2 z$$

Proposition 7. $\langle \boxtimes_{21}, \rightrightarrows_{211}, \rightrightarrows_{212} \rangle$ is an adjoint triple where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

- $\bar{a} \boxtimes_{21} \bar{b} = \langle a_1 \otimes \neg b_2; a_2 \oplus b_2 \rangle$
- $\bar{b} \rightrightarrows_{211} \bar{c} = \langle b_2 \oplus c_1; \neg b_2 \otimes c_2 \rangle$
- $\bar{a} \rightrightarrows_{212} \bar{c} = \langle (\neg a_1 \oplus c_1) \wedge (a_2 \oplus \neg c_2); (\neg a_2 \otimes c_2) \vee (a_1 \otimes \neg c_1) \rangle$

Proof. It is not difficult to check that both implications satisfy Atanassov's inequality.

Given elements $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$

$$\bar{a} \boxtimes_{21} \bar{b} \leq \bar{c} \iff \bar{a} \leq \bar{b} \rightrightarrows_{211} \bar{c}$$

By definition, $\bar{a} \boxtimes_{21} \bar{b} \leq \bar{c}$ means $a_1 \otimes \neg b_2 \leq c_1$ and $c_2 \leq a_2 \oplus b_2$

- (i) $a_1 \otimes \neg b_2 \leq c_1$ is equivalent to $a_1 \leq \neg b_2 \rightarrow c_1$
- (ii) $c_2 \leq a_2 \oplus b_2$ is equivalent to $\neg b_2 \otimes c_2 \leq a_2$,

and the two inequalities above say that $\bar{a} \leq \bar{b} \rightrightarrows_{211} \bar{c}$.

- $\bar{a} \boxtimes_{21} \bar{b} \leq \bar{c} \implies \bar{b} \leq \bar{a} \rightrightarrows_{212} \bar{c}$
 - (i) From $a_1 \otimes \neg b_2 \leq c_1$ we have $\neg b_2 \leq a_1 \rightarrow c_1$, which is equivalent to $a_1 \otimes \neg c_1 \leq b_2$, and from $b_1 \leq \neg b_2$, we have $b_1 \leq a_1 \rightarrow c_1$.
From $c_2 \leq a_2 \oplus b_2$ we obtain $\neg a_2 \otimes \neg b_2 \leq \neg c_2$ and $\neg b_2 \leq \neg a_2 \rightarrow \neg c_2$, from $b_1 \leq \neg b_2$, we have $b_1 \leq \neg a_2 \rightarrow \neg c_2$.
Hence, we have $b_1 \leq (a_1 \rightarrow c_1) \wedge (\neg a_2 \rightarrow \neg c_2)$.
 - (ii) From $a_1 \otimes \neg b_2 \leq c_1$ we have $\neg c_1 \leq \neg a_1 \oplus b_2 = a_1 \rightarrow b_2$, which is equivalent to $a_1 \otimes \neg c_1 \leq b_2$.
From $c_2 \leq a_2 \oplus b_2 = \neg a_2 \rightarrow b_2$ we obtain $c_2 \otimes \neg a_2 \leq b_2$.
Hence, we have $(a_1 \otimes \neg c_1) \vee (\neg a_2 \otimes c_2) \leq b_2$.

From the previous inequalities we have $\bar{b} \leq \bar{a} \rightrightarrows_{212} \bar{c}$.

- $\bar{a} \boxtimes_{21} \bar{b} \leq \bar{c} \iff \bar{b} \leq \bar{a} \rightrightarrows_{212} \bar{c}$
By $\bar{b} \leq \bar{a} \rightrightarrows_{212} \bar{c}$ we have $(\neg a_2 \otimes c_2) \vee (a_1 \otimes \neg c_1) \leq b_2$:
 - (i) From $\neg a_2 \otimes c_2 \leq b_2$ we obtain $c_2 \leq \neg a_2 \rightarrow b_2 = a_2 \oplus b_2$.
 - (ii) From $a_1 \otimes \neg c_1 \leq b_2$ we obtain $a_1 \otimes \neg b_2 \leq c_1$.
From the previous inequalities we obtain $\bar{a} \boxtimes_{21} \bar{b} \leq \bar{c}$. \square

Proposition 8. $\langle \boxtimes_{31}, \rightrightarrows_{311}, \rightrightarrows_{312} \rangle$ is an adjoint triple where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

- $\bar{a} \boxtimes_{31} \bar{b} = \langle \neg a_2 \otimes b_1; a_2 \oplus b_2 \rangle$
- $\bar{b} \rightrightarrows_{311} \bar{c} = \langle (\neg b_1 \oplus c_1) \wedge (b_2 \oplus \neg c_2); (\neg b_2 \otimes c_2) \vee (b_1 \otimes \neg c_1) \rangle$

$$\bullet \bar{a} \rightrightarrows_{312} \bar{c} = \langle a_2 \oplus c_1; \neg a_2 \otimes c_2 \rangle$$

Proof. Similar to previous one. \square

Proposition 9 ([13]). $\langle \boxtimes_{22}, \rightrightarrows_{221}, \rightrightarrows_{222} \rangle$ is an adjoint triple where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

- $\bar{a} \boxtimes_{22} \bar{b} = \langle a_1 \otimes \neg b_2; (\neg a_1 \oplus b_2) \wedge (a_2 \oplus \neg b_1) \rangle$
- $\bar{b} \rightrightarrows_{221} \bar{c} = \langle b_2 \oplus c_1; b_1 \otimes c_2 \rangle$
- $\bar{a} \rightrightarrows_{222} \bar{c} = \langle (\neg a_1 \oplus c_1) \wedge (a_2 \oplus \neg c_2); a_1 \otimes \neg c_1 \rangle$

Proposition 10. $\langle \boxtimes_{32}, \rightrightarrows_{321}, \rightrightarrows_{322} \rangle$ is an adjoint triple where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

- $\bar{a} \boxtimes_{32} \bar{b} = \langle \neg a_2 \otimes b_1; (\neg a_1 \oplus b_2) \wedge (a_2 \oplus \neg b_1) \rangle$
- $\bar{b} \rightrightarrows_{321} \bar{c} = \langle (\neg b_1 \oplus c_1) \wedge (b_2 \oplus \neg c_2); b_1 \otimes \neg c_1 \rangle$
- $\bar{a} \rightrightarrows_{322} \bar{c} = \langle a_2 \oplus c_1; a_1 \otimes c_2 \rangle$

Proof. Similar to the previous one. \square

Proposition 11. Conjunctions $\boxtimes_{13}, \boxtimes_{23}, \boxtimes_{14}, \boxtimes_{34}$ given by

- $\bar{a} \boxtimes_{13} \bar{b} = \langle a_1 \otimes b_1; \neg a_1 \oplus b_2 \rangle$
- $\bar{a} \boxtimes_{23} \bar{b} = \langle a_1 \otimes \neg b_2; \neg a_1 \oplus b_2 \rangle$
- $\bar{a} \boxtimes_{14} \bar{b} = \langle a_1 \otimes b_1; a_2 \oplus \neg b_1 \rangle$
- $\bar{a} \boxtimes_{34} \bar{b} = \langle \neg a_2 \otimes b_1; a_2 \oplus \neg b_1 \rangle$

cannot be part of an adjoint triple in the Atanassov's sense.

Proof. The idea is the same as that of Proposition 6, namely, we will show that none of the admissible possibilities is, in fact, the residuated implication in the Atanassov L -fuzzy sense by showing an example in which the adjoint property fails.

For cases $\boxtimes_{13}, \boxtimes_{23}$

$$\langle 0, 0 \rangle \boxtimes_{13} \langle 1, 0 \rangle = \langle 0, 0 \rangle \boxtimes_{23} \langle 1, 0 \rangle = \langle 0, 1 \rangle.$$

Now, any possible implication $\bar{b} \rightrightarrows \bar{c}$ is smaller than $\langle \neg b_1 \oplus \neg c_2, b_1 \otimes c_2 \rangle$ (see Remark 1), hence we have that

$$\langle 1, 0 \rangle \rightrightarrows \langle 0, 1 \rangle \leq \langle \neg 1 \oplus \neg 1, 1 \otimes 1 \rangle = \langle 0, 1 \rangle.$$

as a result $\langle 0, 0 \rangle \boxtimes_x \langle 1, 0 \rangle \leq \langle 0, 1 \rangle$ (for $x \in \{13, 23\}$) is not equivalent to $\langle 0, 0 \rangle \leq \langle 1, 0 \rangle \rightrightarrows \langle 0, 1 \rangle$ since $\langle 0, 0 \rangle > \langle 0, 1 \rangle$ in \mathcal{L}_{ALV} .

For cases \boxtimes_{14} and \boxtimes_{34} the construction of the counterexample can be obtained similarly. \square

Despite of the non-existence result above, it is indeed possible to define kind of a *lateral* residuated implication for all those non-commutative conjunctions. This feature is useful in order to build isotone Galois connections as done in [13], hence providing concept-forming operators for suitable generalizations of formal concept analysis [22], [23], [24].

Lemma 3. The pairs $\langle \boxtimes_{13}, \rightrightarrows_{13} \rangle$ and $\langle \boxtimes_{23}, \rightrightarrows_{23} \rangle$ satisfy the adjoint property

$$\bar{a} \boxtimes \bar{b} \leq \bar{c} \iff \bar{b} \leq \bar{a} \rightrightarrows \bar{c}$$

where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

$$\bar{a} \boxtimes_{13} \bar{b} = \langle a_1 \otimes b_1; \neg a_1 \oplus b_2 \rangle \quad \bar{a} \boxtimes_{23} \bar{b} = \langle a_1 \otimes \neg b_2; \neg a_1 \oplus b_2 \rangle$$

$$\bar{a} \rightrightarrows_{13} \bar{c} = \langle \neg a_1 \oplus c_1; a_1 \otimes c_2 \rangle \quad \bar{b} \rightrightarrows_{23} \bar{c} = \langle \neg a_1 \oplus c_1; a_1 \otimes \neg c_1 \rangle$$

Lemma 4. The pairs $\langle \boxtimes_{14}, \rightrightarrows_{14} \rangle$ and $\langle \boxtimes_{34}, \rightrightarrows_{34} \rangle$ satisfy the adjoint property

$$\bar{a} \boxtimes \bar{b} \leq \bar{c} \iff \bar{a} \leq \bar{b} \rightrightarrows \bar{c}$$

where, for any $\bar{a}, \bar{b}, \bar{c} \in \mathcal{L}_{ALV}$, the operations are defined by

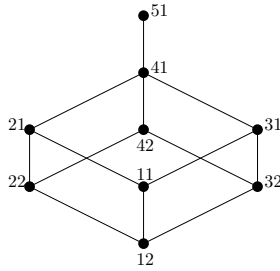


Fig. 4. Lattice of conjunctions with Atanassov L -fuzzy adjoint pair or triple.

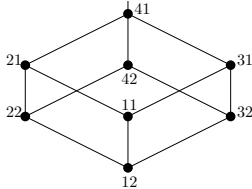


Fig. 5. Conjunctions with strict Atanassov's inequality

$$\begin{aligned} \bar{a} \boxtimes_{14} \bar{b} &= \langle a_1 \otimes b_1; a_2 \oplus \neg b_1 \rangle & \bar{a} \boxtimes_{34} \bar{b} &= \langle \neg a_2 \otimes b_1; a_2 \oplus \neg b_1 \rangle \\ \bar{b} \Rightarrow_{14} \bar{c} &= \langle \neg b_1 \oplus c_1; b_1 \otimes c_2 \rangle & \bar{b} \Rightarrow_{34} \bar{c} &= \langle b_1 \rightarrow c_1; b_1 \otimes \neg c_1 \rangle \end{aligned}$$

Figure 4 shows the sublattice of admissible conjunctions which actually do generate adjoint pairs or triples.

VI. CONCLUSIONS AND FUTURE WORK

A study of adjoint pairs and adjoint triples under the Atanassov L -fuzzy standpoint have been presented. Interestingly, the requirement to work on the lattice \mathcal{L}_{ALV} causes that some standard conjunctors for which an adjoint pair is known do not behave adequately when analysed in the new extended framework (Proposition 6). It is also remarkable that some proofs did not require the use of Atanassov's inequality. We would like to further study these situations in order to better understand the reasons of this behaviour. From a theoretical standpoint we will also consider another possibilities to generate new adjoint pairs and triples in this Atanassov L -fuzzy framework.

Analysing the conjunctions in Figure 4 we get that both components of the adjoint pair $(\boxtimes_{51}, \Rightarrow_{51})$ have zero indetermination degree, i.e. Atanassov's inequality is, in fact, an equality. This fact suggests that the top element in Figure 4 is not properly Atanassov L -fuzzy and, hence, the lattice would have the shape of a cube (see Figure 5). Moreover, should we wish to avoid using any operator with zero indetermination degree (so that we have a proper extension to Atanassov's framework), then three more conjunctions would be filtered out, leading to the diamond-shaped lattice of Figure 6. As future work, we will further analyse specific properties of the corresponding adjoint pairs/triples, and analyse their usefulness for the multi-adjoint framework [25], [26], [27].

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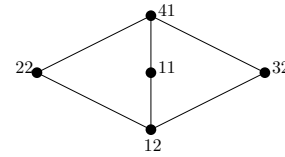


Fig. 6. After filtering out any operator with zero indetermination degree.

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