LOG-GAUSSIAN COX PROCESSES IN INFINITE-DIMENSIONAL SPACES

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ABSTRACT. This paper introduces new results on doubly stochastic Poisson processes, with log-Gaussian Hilbert-valued random intensity (LGHRI), defined from the Ornstein-Uhlenbeck process (O-U process) in Hilbert spaces. Sufficient conditions are derived for the existence of a counting measure on ℓ^2, for this type of doubly stochastic Poisson processes. Functional parameter estimation and prediction is achieved from the discrete-time approximation of the Hilbert-valued O-U process by an autoregressive Hilbertian process of order one (ARH(1) process). The results derived are applied to functional prediction of spatiotemporal log-Gaussian Cox processes, and an application to functional disease mapping is developed. The numerical results given, from the conditional simulation study undertaken, are compared to those ones obtained, when the random intensity is assumed to be a spatiotemporal long-range dependence (LRD) log-Gaussian process (see [19]).

Keywords: ARH(1) process; Hilbert-valued O-U process; Infinite dimension; Parameter estimation and prediction; Spatiotemporal log-Gaussian Cox process.

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1. Introduction

The Doubly Stochastic Poisson Processes (DSPPs) were introduced by Cox [9], considering a positive random variable in the definition of the intensity instead of a deterministic function. This generalization allows the exogenous process to the model to influence transitions in the point process. The standard construction, and main probabilistic properties of DSPPs were introduced in [24]. Alternatively, in the books [8] and [13], martingale theory is applied for the description of DSPPs. The drawback of the mentioned approaches is the reduced number of applications they open, since they focus on general properties of DSPP, without exploring the functional form of the intensity of the process. The analytical expressions of the probability density functions for different types of DSPPs are obtained, when the functional form of the random intensity is explicitly defined (see [14]). In this context, restrictions and limitations, arising from the non-negativity condition of the intensity, were overcome in [3], and [26], suggesting a lognormal model for the intensity process. The shot-noise type process, guaranteeing the non-negativity of intensity, is proposed in [15]. On the other hand, in [38] is assumed that the intensity is governed by a one-dimensional Feller process, and the functional form of the probability density function for the corresponding DSPP is also derived. Feller processes were introduced and studied in the financial literature, after the pioneer work [10]. One of the properties of Feller process is that it lives in $\mathbb{R}_+$. Hence, the non-negativity condition is fulfilled.

In the present paper, we adopt the log-Gaussian intensity model approach to define Cox processes in real separable Hilbert spaces, under certain spectral diagonalization conditions. Particularly, ℓ^2 space is considered, given the well-known isometry with real separable Hilbert spaces. We assume that the logarithmic transform of the intensity is controlled by a Hilbert-valued diffusion process, the Hilbert-valued O-U process (see, for example, [18] and [22], where Feller processes in one or d dimensions are incorporated as intensity models). In that sense, the family of processes introduced here extends, in certain aspects, the models studied in [3], [26] and [38]. The above described literature has emerged, motivated, in part, by the interest of point processes in finance, specially of DSPPs, inducing a considerably progress at the end of the 1990s, with the development of models of managing and pricing the credit risk. Furthermore, the theory of point processes have applications in several additional areas (e.g., biostatistics and reliability theory). Just to mention an example, we refer to the reader to [23], where point process theory is applied to study the size of tumors in rats over a period of time; to [29], where the description of occurrences of credit events is achieved from point processes; to [12], where testing and validation of software is performed in reliability theory.

In [34], log-Gaussian Cox processes, with intensity being constant within square quadrants, satisfying a conditional autoregression are introduced (see also [5]). Gibbs sampling can be applied to explore these discretized models. However, such models do not converge to anything reasonable as the sides of the quadrants tend to zero (see [34]). In [31] this difficulty is overcome. Several properties of this class of processes are analyzed, e.g., the complete characterization of their distribution by the intensity, and the pair correlation function. Higher-order moment properties can also be expressed in terms of both functions. In addition, predictability properties are enhanced by the fact that the distribution of a log-Gaussian Cox process, restricted to a bounded subset, is known. They can be observed within a bounded window in absence of edge effects. Log-Gaussian Cox processes then offer, in particular, a suitable framework for modeling and estimation in disease mapping.

This paper introduces log-Gaussian Cox processes in a real separable Hilbert space H, deriving sufficient conditions for the proper definition of a counting measure on the space ℓ^2 isometric to H. The main ingredient required is the diagonal spectral representation, in terms of a common system of eigenvectors, of the operators involved in the definition of the auto-covariance operator of a Hilbert-valued O-U process. The discrete-time approximation of the H-valued O-U process, in terms of an ARH(1) process, is also obtained from projection into such an eigenvector system. The componentwise estimation of the autocorrelation operator, and of the auto-covariance operator of the innovation process of an ARH(1) process then leads to the corresponding parameter estimation of the approximated O-U process. Plug-in prediction results then follow from the integral form of the H-valued O-U process, and from the corresponding formulation of the ℓ^2-valued predictor for the H-valued log-Gaussian Cox process. An application to disease mapping is showed. Specifically, a conditional simulation study is undertaken from breast, prostate, and brain cancer data in the provinces of Spain. A comparative study with the LRD log-Gaussian intensity case is also developed, applying the wavelet-based parameter estimation methodology presented in [19], for spatiotemporal zero-mean LRD Gaussian processes.

The outline of the paper is as follows. Preliminary elements are introduced in Section 2. In Section 3, numerical projection methods are applied, under suitable conditions, for a diagonal spectral representation of the integral equation satisfied by O-U process in Hilbert spaces. Parameter estimation and prediction of Hilbert-valued O-U process, from its discrete-time approximation in terms of an ARH(1) process, is achieved in Section 4. Definition and prediction of log-Gaussian Cox process in real separable Hilbert spaces is provided in Section 5. As an application, estimation and prediction results for spatiotemporal log-Gaussian Cox process are obtained in Section 6. A conditional simulation study is undertaken in Section 7, to illustrate the results derived for short-dependence log-Gaussian intensities in comparison with the LRD log-Gaussian case, in the context of functional disease mapping. Final comments are given in Section 8.

In the remaining sections, we will consider that all the stochastic processes introduced below are defined on the basic probability space $(\Omega, \mathcal{A}, P)$.

2. Preliminaries

The preliminary definitions and results required for the subsequent development of this paper are now provided.

2.1. **Homogeneous Poisson process in ℓ^2 space.** The Poisson process in ℓ^2 space, and its classical parameter estimation and prediction is now introduced (see [7]).

Let $\{N_{t,j},\ t \in \mathbb{R}_+,\ j \geq 1\}$ be a sequence of independent homogeneous Poisson processes with respective intensities $\lambda_j > 0$, $j \geq 1$, such that $\sum_j \lambda_j < \infty$. Since

$$E\left(\sum_j N_{t,j}^2\right) = \sum_j E\left(N_{t,j}^2\right) = \sum_j \left[\lambda_j t + (\lambda_j t)^2\right] < \infty,$$

it follows that $\sum_j N_{t,j}^2 < \infty$ almost surely. Then, $M_t = \{N_{t,j},\ j \geq 0\}$ defines a random variable with values in ℓ^2, and satisfying $E\|M_t\|_{\ell^2}^2 < \infty$. Thus, $\{M_t,\ t \geq 0\}$ is a continuous time ℓ^2- valued process.

Classical prediction. Assume that M_T is observed and we want to predict M_{T+h} $(h > 0)$. For this purpose, we first define a MLE for $\{\lambda_j,\ j \geq 1\}$. Let $\mathcal{N} \subset \ell^2$ be the family of sequences $\{x_j,\ j \geq 1\} = (x_j)$ such that x_j is an integer for each j, and $x_j = 0$, for sufficiently large j. We may write

$$\mathcal{N} = \bigcup_k \mathcal{N}_k,$$

where $\mathcal{N}_k = \{(x_j) :\ (x_1, \ldots x_k) \in \mathbb{N}^k;\ x_j = 0,\ j > k\}$. Clearly, $\mathcal{N}_k$ is countable for every k. It then follows that $\mathcal{N}$ is countable since it is the countable union of countable sets. Thus, one may define the counting measure μ on $\mathcal{N}$, and extend it by setting $\mu(l^2 - \mathcal{N}) = 0$. The obtained measure is then σ- finite.

Now M_t is considered as $\mathcal{N}$-valued random variable. Actually, since $N_{t,j}^2$ is an integer, $\sum_j N_{t,j}^2 < \infty$ (a.s.) implies that $N_{t,j} = 0$, a.s. for j large enough. More precisely: There exists Ω_0 such that $P(\Omega_0) = 1$, and for all $\omega \in \Omega_0$, there exists a $j_0(\omega, \lambda, T)$ such that $N_{T,j}(\omega) = 0$, for $j > j_0$. Therefore, one may define the likelihood of M_T with respect to μ by setting

$$L(M_T(\omega), \lambda) = \prod_{j=1}^{j_0(\omega,\lambda,T)} \exp\left(-\lambda_j T\right) \frac{(\lambda_j T)^{N_{T,j}(\omega)}}{N_{T,j}(\omega)!}.$$

Hence, the MLE of (λ) is given by:

$$(\widehat{\lambda})_T = \left(\frac{N_{T,j}}{T},\ j \geq 1\right) = (\widehat{\lambda}_{T,j},\ j \geq 1). \tag{1}$$

Clearly, $\sum_{j=1}^{\infty} \widehat{\lambda}_{T,j} < \infty$ (a.s) and $(\widehat{\lambda})_T$ is unbiased ((λ) being considered as a parameter with values in ℓ^2). It follows that

$$f(M_T) = \frac{T+h}{T} M_T \tag{2}$$

is an unbiased efficient predictor of $E_\lambda\left(M_{T+h} | M_T\right) = h(\lambda) + M_T$, $(\lambda) \in \ell^2$ (see [7]).

2.2. **Strongly cylindrical Wiener process induced by a Hilbert-valued Wiener process.** We restrict here out attention to the real separable Hilbert space case. Let H and H^* respectively be a real separable Hilbert space and its dual. Denote by $\mathcal{B}(H)$ the Borel σ-algebra in H. For $f_1^*, \dots, f_n^* \in \Gamma \subseteq H^*$, and $B \in \mathcal{B}(\mathbb{R}^n)$, we define a cylindrical set or cylinder with respect (H, Γ) as follows:

$$\mathcal{Z}(f_1^*, \dots, f_n^*, B) := \{g \in H : \ (\langle g, f_1^* \rangle, \dots, \langle g, f_n^* \rangle) \in B\}.$$

The set of all cylindrical sets is denoted by $Z(H, \Gamma)$, which turns out to be an algebra. The generated σ-algebra is denoted by $C(H, \Gamma)$, and it is called cylindrical σ-algebra with respect to (H, Γ). If $\Gamma = H^*$, we write $C(H) := C(H, \Gamma)$. Since H is separable, then, both the Borel $\mathcal{B}(H)$ and the cylindrical σ-algebra $C(H)$ coincide.

A function $\mu : C(H) \to [0, \infty]$ is called cylindrical measure on $C(H)$, if for each finite subset $\Gamma \subseteq H^*$, the restriction of μ on the σ–algebra $C(H, \Gamma)$ is a measure. A cylindrical measure is called finite if $\mu(H) < \infty$. For every function $\varphi : H \to \mathbb{R}$, being measurable with respect to $C(H, \Gamma)$, for a finite subset $\Gamma \subseteq H^*$, the integral $\int f(u)\mu(du)$ is well defined as a real-valued Lebesgue integral if it exists.

Given a probability space $(\Omega, \mathcal{A}, P)$ with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$, a cylindrical random variable on H is defined as a linear map $X : \ H^* \to \mathcal{L}^0(\Omega)$, with $\mathcal{L}^0(\Omega)$ denoting the linear space of all real valued random variables, which are defined from $X(f)(\mu)$, the image of the cylindrical measure μ on H, under the map $X(f)$, for each $f \in H$.

A cylindrical process X in H is a family $\{X_t = X(t), \ t > 0\}$ of cylindrical random variables in H. Note that the characteristic function $\varphi_X : H^* \to \mathbb{C}$, with $\varphi_X(f^*) = E[\exp(iXf^*)]$, $f^* \in H^*$, of a cylindrical random variable is positive-definite, and continuous on finite subspaces. Then, there exists a cylindrical measure μ_X with the same characteristic function. We call μ_X the cylindrical distribution of X. Reciprocally, for every cylindrical measure μ on $C(H)$, there exists a probability space $(\Omega, \mathcal{A}, P)$, and a cylindrical random variable $X : \ H^* \to \mathcal{L}^0(\Omega)$, such that $\mu = \mu_X$ is the cylindrical distribution of X. A strongly Gaussian cylindrical measure μ, has characteristic function φ given by

$$\varphi(f^*) = \exp\left(-\frac{1}{2}\langle Qf^*, f^* \rangle)\right), \quad \forall f^* \in H^*,$$

for certain symmetric positive operator $Q : \ H^* \to H$. A weakly cylindrical Wiener process $\{W(t) : \ t > 0\}$ is strongly cylindrical if the cylindrical distribution $\mu_{W(1)}$ of $W(1)$ is strongly Gaussian (see, for example, Definition 6.7 in [35]).

An adapted H-valued stochastic process $\{W(t) : \ t > 0\}$ is called a Wiener process if
(a) $W(0) = 0$ P-a.s.;
(b) W has independent, stationary increments;
(c) there exists Q, a symmetric positive operator such that $Q : \ H^* \longrightarrow H$, and $W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q)$, for all $0 \leq s \leq t$. A strongly cylindrical Wiener process $\{W(t) : \ t > 0\}$, with covariance operator $Q = i_Q i_Q^*$, is induced by an H-valued Wiener process if and only if i_Q is Hilbert-Schmidt, where i_Q denotes the continuous inclusion of the range of Q, with the norm $\|f\|_Q = Q(f)(f) = \langle Q(f^*), f \rangle_H$, $f \in H$, into H (see Theorem 8.1 in [35], for a wider characterization of this class of strongly cylindrical Wiener processes).

2.3. **The Ornstein-Uhlenbeck process in Hilbert spaces.** Let H be a real separable Hilbert space. Consider the Ornstein-Uhlenbeck process (O-U process), defined by

$$X(t, f) = \exp(tA)(f) + \int_0^t \exp((t-s)A)dW(s), \quad t \geq 0, \quad f \in H, \tag{3}$$

where W is a strongly cylindrical Wiener process in H, induced by an H-valued Wiener process (see Theorems 8.1 in [35]), with covariance operator $Q : H^* \longrightarrow H$, such that, for $0 \leq s \leq t$,

$$W(t) - W(s) \sim \mathcal{N}(0, (t-s)Q).$$

The following assumptions are considered (see [11]; [22]):

(i) The operator A is the infinitesimal generator of a strongly continuous semigroup $\exp(tA)$, $t \geq 0$, of linear operators in H satisfying $\|\exp(tA)\| \leq M\exp(-\omega t)$, $t \geq 0$, for some M and $\omega > 0$.

(ii) Q is a continuous, linear self-adjoint, nonnegative operator in H.

(iii) Considering, for $f \in H$,

$$Q_t(f) = \int_0^t \exp(sA)Q\exp(sA^*)(f)ds \tag{4}$$

we have $\sup_{t>0} \mathrm{Tr}Q_t < \infty$, where Tr denotes the trace.

The stochastic integral (3) takes its values in H, and is understood in the Itô sense. The H-valued random variable $X(t, f)$ has Gaussian distribution with mean $\exp(tA)f$, and covariance operator Q_t for each $t > 0$. The associated Gaussian measure is denoted as $\mu_{\exp(tA)f,Q_t}$. The Ornstein-Uhlenbeck transition semigroup P_t is then given by

$$(P_t\phi)(f) = \int_H \phi(y)\mu_{\exp(tA)f,Q_t}(dy), \quad t > 0, \quad f \in H,$$

for every bounded measurable real-valued function ϕ on H. The operator $Q_\infty(f) = \int_0^\infty \exp(tA)Q\exp(tA^*)(f)dt$, $f \in H$, is in the trace class under Assumption (i), and the Gaussian measure in H, μ_{0,Q_∞}, is the unique invariant measure for P_t. The explicit form, and the space where the probability density, the Radon-Nikodym derivative of μ_{0,Q_∞}, lies are obtained, under certain conditions, in [22].

2.4. **ARH(1) processes.** In Section 4.1, we consider the discrete-time approximation of the above-introduced O-U process in Hilbert spaces, in terms of an ARH(1) process. The main definitions and elements on ARH(1) processes are now given (see [6]).

Definition 1. *Let H be a real separable Hilbert space. A sequence $Y = \{Y_n,\ n \in \mathbb{Z}\}$ of H-valued random variables on a basic probability space $(\Omega, \mathcal{A}, P)$ is called an autoregressive Hilbertian process of order one, associated with $(\eta, \varepsilon, \rho)$, if it is stationary and satisfies*

$$X_n = Y_n - \eta = \rho(Y_{n-1} - \eta) + \varepsilon_n = \rho(X_{n-1}) + \varepsilon_n, \quad n \in \mathbb{Z}, \tag{5}$$

where the expectation $\eta = E[Y_n] \in H$ is constant, $\varepsilon = (\varepsilon_n,\ n \in \mathbb{Z})$ is a Hilbert-valued white noise in the strong sense (i.e., a zero-mean stationary sequence of independent $H-$valued random variables with $E\|\varepsilon_n\|_H^2 = \sigma^2 < \infty$, for every $n \in \mathbb{Z}$), and $\rho \in \mathcal{L}(H)$, with $\mathcal{L}(H)$ being the space of linear bounded operators on H. For each $n \in \mathbb{Z}$, ε_n and X_{n-1} are assumed to be uncorrelated.

If there exists a positive $j_0 \geq 1$ such that $\|\rho^{j_0}\|_{\mathcal{L}(H)} < 1$, then, the ARH(1) process $X = \{X_n,\ n \in \mathbb{Z}\}$ in (5) is standard, and there exists a unique stationary solution to equation (5) (see, for example, [6], Chapter 3).

The auto-covariance and cross-covariance operators are given by

$$\begin{aligned} C_X &= E[X_n \otimes X_n] = E[X_0 \otimes X_0], \quad n \in \mathbb{Z}, \\ D_X &= E[X_n \otimes X_{n+1}] = E[X_0 \otimes X_1], \quad n \in \mathbb{Z}, \end{aligned} \tag{6}$$

where, for $f, g \in H$,

$$f \otimes g(h) = f \langle g, h \rangle_H, \quad \forall h \in H,$$

defines a Hilbert-Schmidt operator on H. Operator C_X is assumed to be in the trace class. In particular, $E\|X_n\|_H^2 < \infty$, for all $n \in \mathbb{Z}$.

3. Projection of Hilbert-valued O-U process

Let $X = \{X_t,\ t \geq 0\}$ be a Hilbert-valued process satisfying equation (3), and the conditions (i)–(iii) formulated in Section 2.3. Here, we also assume the following additional condition:

Assumption A1 Operators A, in equation (3), and Q, the covariance operator characterizing the distribution of the strongly cylindrical Wiener W, in such an equation, are positive operators, admitting a diagonal spectral decomposition, with respect to a common orthonormal basis of eigenvectors $\{\phi_j\}_{j\geq 1}$ in H. That is,

$$A = \sum_{j=1}^{\infty} \lambda_j(A)\phi_j \otimes \phi_j \tag{7}$$

$$Q = \sum_{j=1}^{\infty} \lambda_j(Q)\phi_j \otimes \phi_j. \tag{8}$$

Remark 1. ***Assumption A1*** *holds, in particular, when operators A and Q are related by a continuous function (see* [16]*, pp. 119-140). Unconditional basis, like wavelet bases, also allow the diagonal spectral series representation, for example, of the distributional kernels of Calderón-Zygmund operators (see, for example,* [25]*, pp. 664–665).*

In particular,

$$\exp(tA)(f) = \sum_{j=1}^{\infty} \exp(-\lambda_j(A)t) \langle f, \phi_j \rangle_H \phi_j, \quad \forall f \in H. \tag{9}$$

From Theorems 7.1 and 8.1 in [35], under **Assumption A1**, from equations (7)–(9), keeping in mind that

$$\{\psi_j = \lambda_j^{1/2}(Q)\phi_j\}_{j\geq 1} \tag{10}$$

defines an orthonormal basis of the reproducing kernel Hilbert space of Q, W admits the following series expansion:

$$W(t)(g^*) = \sum_{j=1}^{\infty} \langle i_Q(\psi_j), g \rangle_H B_j(t), \quad \forall g^* \in U^* = [\overline{Q^{1/2}(H)}^{\|\cdot\|_Q}]^*, \tag{11}$$

where U^* denotes the dual space of $U = \overline{Q^{1/2}(H)}^{\|\cdot\|_Q}$, the completion of $Q^{1/2}(H)$, with respect to the norm $\|\cdot\|_Q$, induced by Q, i.e., $\|f\|_Q = Q(f)(f) = \langle Q(f^*), f \rangle_H$, for all $f \in H$, with, as before, $f^* \in H^*$ denoting the dual element of f, given by Riesz Representation Theorem. Here, $\{B_j(t),\ t \geq 0\}_{j\geq 1}$ are independent standard real-valued Wiener processes, and, as before, i_Q is the continuous inclusion mapping of the range of Q into H.

Theorem 1. *Under the conditions (i)–(iii) assumed in Section 2.3, and* ***Assumption A1****, for every $j \geq 1$, $\{Y_p(t) = \langle X(t, \phi_j), \phi_p \rangle_H,\ t \in \mathbb{R}_+\}_{p\geq 1}$ defines a sequence of independent real-valued O-U processes, with respective parameters $\{\lambda_p(A)\}_{p\geq 1}$ and $\{\lambda_p^{1/2}(Q)\}_{p\geq 1}$, such that $Y_p(0) \equiv 1$, for each $p \geq 1$.*

Proof. Under **Assumption A1**, from equations (9) and (11), projection of equation (3) into the orthonormal eigenvector system $\{\phi_j\}_{j\geq 1}$ leads to the following identities: For each $t > 0$, and $p \geq 1$, and for any $j \geq 1$,

$$\begin{aligned}
\langle X(t,\phi_j),\phi_p\rangle_H &= \langle \exp(tA)(\phi_j),\phi_p\rangle_H + \left\langle \int_0^t \exp((t-s)A)dW(s),\phi_p \right\rangle_H \\
&= \sum_{k=1}^{\infty} \exp(-\lambda_k(A)t)\,\langle \phi_k,\phi_j\rangle_H\,\langle \phi_k,\phi_p\rangle_H \\
&+ \sum_{k=1}^{\infty} \langle \phi_k,\phi_p\rangle_H \sum_{l=1}^{\infty} \int_0^t \exp(-\lambda_k(A)(t-s))\,\langle \phi_k, i_Q(\psi_l)\rangle_H\, B_l(ds) \\
&= \exp\left(-\lambda_p(A)t\right) + \lambda_p^{1/2}(Q)\int_0^t \exp(-\lambda_p(A)(t-s))B_p(ds),
\end{aligned} \tag{12}$$

where the integral is interpreted as a one-parameter Wiener-Itô stochastic integral with respect to the Wiener measure $B_p(ds)$, for each $p \geq 1$. The last identity of equation (12) means that, for each $p \geq 1$, and for every $j \geq 1$, $\{Y_p(t) = \langle X(t,\phi_j),\phi_p\rangle_H\,,\ t \in \mathbb{R}_+\}$ is a real-valued O-U process satisfying the Langevin equation:

$$dY_p(t) = -\lambda_p(A)Y_p(t)dt + \lambda_p^{1/2}(Q)B_p(dt). \tag{13}$$

That is, for each $p \geq 1$, $\{Y_p(t),\ t \in \mathbb{R}_+\}$ is an O-U process starting at $t = 0$, with initial condition $Y_p(0) \equiv 1$, and parameters $\lambda_p(A)$ and $\lambda_p^{1/2}(Q)$. Indeed, $\{Y_p\}_{p\geq 1}$ is a sequence of independent real-valued O-U processes.

4. Parameter estimation and prediction of Hilbert-valued O-U process

In this section, we consider the discrete-time ARH(1) approximation of H-valued O-U process introduced in Section 2.3, for its componentwise parameter estimation, and plug-in prediction.

4.1. **Discrete-time ARH(1) approximation of the Hilbert-valued O-U.**

In the discrete-time approximation of equation (3), several difficulties related to the definition of a suitable concept of derivative (e.g., total derivative like Fréchet derivative, directional derivative like Gâteaux derivative, H-derivative, etc.) can be found. This is the reason why we do not adopt the approach related to considering, in equation (5), the operator $\gamma = I - \rho$, and the innovation process ε_n defined as $\varepsilon_n = \sigma(W(n) - W(n-1))$, for all $n \in \mathbb{N}_*$, with, as before, W denoting a strongly cylindrical and H-valued Wiener process, leading to

$$X_{t+1} - X_t = -\gamma X_t + \sigma(W_{t+1} - W_t), \quad t = 1,2,\dots \tag{14}$$

However, under **Assumption A1**, Theorem 1 allows us to obtain a discrete-time projection approximation, in terms of a sequence of independent AR(1) processes, defined from equation (13). Specifically, for each $t = 1,2,\dots$, and $p \geq 1$,

$$Y_p(t+1) - Y_p(t) = -\lambda_p(A)Y_p(t) + \lambda_p^{1/2}(Q)(B_p(t+1) - B_p(t)), \tag{15}$$

where, for each $\tau > 0$, we can now consider

$$Y_{p,\tau}(n+1) = \gamma_\tau Y_{p,\tau}(n) + \sqrt{\tau}\varepsilon_n, \tag{16}$$

with $\gamma_\tau = 1 - \lambda_p(A)\tau$. For each $t > 0$, and $p \geq 1$, we then have

$$\begin{aligned} Y_{p,\tau}([t/\tau]) &= \gamma_\tau^{[t/\tau]} Y_p(0) + \lambda_p^{1/2}(Q)\sqrt{\tau}\gamma^{[t/\tau]-1}\int_0^{[t/\tau]} \gamma^{-[s]} dB_p(s) \\ &\sim \exp(-\lambda_p(A)t) + \lambda_p^{1/2}(Q)\exp(-\lambda_p(A)t)\int_0^t \exp(\lambda_p(A)s)dB_p(s), \end{aligned} \tag{17}$$

where, in the last step, the scaling property of Brownian motion is used $\sqrt{\tau}B_p(s) \sim B_p(\tau s)$. Note that, $\gamma_\tau \sim (1-\lambda_p(A))^\tau = \gamma^\tau$, as $\tau \to 0$. This argument can be stated in a rigorous way to arrive to the weak-convergence of the process $Y_{p,\tau}([t/\tau])$ to the continuous real-valued O-U process, when $\tau \to 0$ (see [21] and [32]) for more details). The conclusion is that if the interval between successive observations is taken to going to zero, and the coefficients are adjusted properly, then, each element (15)–(16) of the obtained sequence of AR(1)-processes has the continuous real-valued Ornstein-Uhlenbeck process (see (13) and (17)) as its scaling limit.

On the other hand, under the following assumption, the AR(1) process sequence introduced in (16) also defines an ARH(1) process. This is the reason why we refer to the ARH(1) discrete-time approximation of H-valued O-U process in this section.

Assumption A2. The autocorrelation operator ρ, in equation (5), admits the following diagonal spectral representation, in the weak sense: For all $f \in H$,

$$\rho(f) = \sum_{k=1}^{\infty} \lambda_k(\rho) \langle \phi_k, f\rangle_H \phi_k, \tag{18}$$

where the series (18) is finite for every $f \in H$, and, as before, $\{\phi_j\}_{j\geq 1}$ denotes the system of eigenvectors of the auto-covariance operator C_X in (6).

From equations (15)–(17), **Assumption A1** implies **Assumption A2**, when the ARH(1) process considered is constructed as the discrete-time approximation of an H-valued O-U process, i.e., when both processes are characterized by a sequence of independent AR(1) processes as given in equations (15)–(16), and equation (19) below in the following result.

Lemma 1. *Under **Assumption A2**, projection of the ARH(1) equation (5) into $\{\phi_j\}_{j\geq 1}$ leads to the following expression: For each $p \geq 1$,*

$$\begin{aligned} X_p(n) &= \langle X_n, \phi_p\rangle_H = \lambda_p(\rho)\langle X_{n-1}, \phi_p\rangle_H + \langle \varepsilon_n, \phi_p\rangle_H \\ &= \lambda_p(\rho) X_p(n-1) + \varepsilon_p(n), \quad n \in \mathbb{Z}, \end{aligned} \tag{19}$$

where $\{X_p(n),\ n \in \mathbb{Z}\}$ is an AR(1) process.

Proof. The proof follows straightforward from the weak-sense diagonal spectral representation of ρ in equation (18), and from the series expansion

$$C_X(h) = \sum_{k=1}^{\infty} \lambda_k(C_X)\langle \phi_k, h\rangle_H \phi_k, \quad \forall h \in H,$$

which holds from the trace property of C_X.

4.2. **Parameter estimation and prediction of H-valued O-U.** The aim of this section is the formulation of respective componentwise parameter estimators of operators A and Q, characterizing the distribution of H-valued O-U process in equation (3), from the moment-based parameter estimation of the AR(1) process sequence introduced in (16), under **Assumptions A1**. Note that, under this assumption, the eigenvalues of operators A and Q, in (7)–(8), with respect to $\{\phi_j\}_{j\geq 1}$, characterize the probability distribution of $\{Y_p = \langle X(t,\phi.), \phi_p\rangle_H\}_{p\geq 1}$(see Theorem 1). Furthermore, from Lemma 1, under **Assumption A2**, the AR(1) process sequence (16) also defines an ARH(1) process as given in equation (19).

Under the assumption that the eigenvectors of C_X, the auto-covariance operator of ARH(1) process in equation (5), are known, the following componentwise estimators for ρ and $R_\varepsilon = E[\varepsilon_1 \otimes \varepsilon_1] = E[\varepsilon_n \otimes \varepsilon_n]$, for every $n \in \mathbb{N}_*$, based on a sample of size T, are considered (see, for example, [2] and [37]):

$$\begin{aligned}
\widehat{\rho}_{k_T} &= \sum_{k=1}^{k_T} \widehat{\lambda}_{T,k}(\rho)\phi_k \otimes \phi_k, \quad \widehat{\lambda}_{T,k}(\rho) = \frac{\frac{1}{T-1}\sum_{i=0}^{T-2} X_k(i)X_k(i+1)}{\frac{1}{T}\sum_{i=0}^{T-1} X_k^2(i)}, \\
&\qquad k = 1,\dots k_T, \\
\widehat{R}_\varepsilon &= \sum_{k=1}^{k_T} \widehat{\lambda}_{T,k}(R_\varepsilon)\phi_k \otimes \phi_k, \quad \widehat{\lambda}_{T,k}(R_\varepsilon) = \frac{1}{T}\left(\sum_{i=0}^{T-1} X_k^2(i)\right)\left(1 - [\widehat{\lambda}_{T,k}(\rho)]^2\right), \\
&\qquad k = 1,\dots k_T,
\end{aligned} \tag{20}$$

where $X_k(i)$ satisfies equation (19), for $i = 0,\dots,T-1$, and for each $k = 1,\dots k_T$. The truncation order k_T should satisfy suitable conditions in relation to the relative magnitude of $\lambda_{k_T}(C_X)$ and T in order to ensure consistency, since $k_T \to \infty$, when $T \to \infty$ (see, for example [2] and [6]).

Finally, under **Assumptions A1**–**A2**, from equations (15) and (19), operators A and Q, in equations (7) and (8), characterizing the distribution of H-valued O-U process in (3), can be approximated as follows:

$$\widehat{A}_T = \sum_{k=1}^{k_T}(1 - \widehat{\lambda}_{T,k}(\rho))\phi_k \otimes \phi_k \tag{21}$$

$$\widehat{Q}_T = \sum_{k=1}^{k_T} \widehat{\lambda}_{T,k}(R_\varepsilon)\phi_k \otimes \phi_k, \tag{22}$$

where, in this case, $\{\phi_j\}_{j\geq 1}$ denotes the common system of eigenvectors of A and Q in equations (7) and (8), and equations (18) and (19), considering $\rho = I - A$. Note that, under **Assumption A1**, $\{\phi_j\}_{j\geq 1}$ also coincides with the known eigenvector system involved in the series representation (11) of the H-valued Wiener process W.

From equations (3), (21) and (22), the following plug-in predictor of H-valued O-U process can be considered:

$$\widehat{X}_T(t,f) = \exp(t\widehat{A}_T)(f) + \int_0^t \exp((t-s)\widehat{A}_T)d\widehat{W}_T(s), \quad t \geq 0, \quad f \in H, \tag{23}$$

where $\widehat{W}_T(t) - \widehat{W}_T(s) \sim \mathcal{N}(\mathbf{0}, (t-s)\widehat{Q}_T)$, i.e., $\widehat{W}_T(t) - \widehat{W}_T(s)$ has Gaussian probability distribution $\mu_{\mathbf{0},(t-s)\widehat{Q}_T}$ in H, with zero mean and covariance operator $(t-s)\widehat{Q}_T$, with $\widehat{Q}_T$ given in (22).

5. Log-Gaussian Cox process in infinite dimensional spaces

This section provides the main elements and theoretical results, involved in the introduction, parameter estimation and prediction of Cox processes with log-Gaussian intensity, defined from H-valued O-U process.

First, sufficient conditions are derived for the definition of $\{\boldsymbol{\mathcal{C}}(t),\ t \in \mathbb{R}_+\} = \{\mathcal{C}_p(t),\ p \geq 1,\ t \in \mathbb{R}_+\}$ as an ℓ^2-valued doubly stochastic Poisson process, with ℓ^2-valued random intensity $\{\lambda_t = \exp(Y_p(t)),\ p \geq 1,\ t \in \mathbb{R}_+\}$, where Y_p satisfies equations (12) and (13), for each $p \geq 1$. To compute the variance of $\Lambda_p(t) = \int_0^t \exp(Y_p(s))ds$, in Proposition 1 below, the system of Hermite polynomials is considered.

It is well-known that Hermite polynomials form a complete orthogonal system of the Hilbert space $L_2(\mathbb{R}, \varphi(u)du)$, the space of square integrable functions with respect to the standard normal density φ. They are defined as follows:

$$H_k(u) = (-1)^k e^{\frac{u^2}{2}} \frac{d^k}{du^k} e^{-\frac{u^2}{2}}, \quad k = 0, 1, \ldots.$$

In particular, for a zero-mean Gaussian process Y, for $k \geq 1$,

$$\mathrm{E}\ H_k(Y(\mathbf{x})) = 0, \quad \mathrm{E}\ \left(H_k(Y(\mathbf{x}))\ H_m(Y(\mathbf{y}))\right) = \delta_{m,k}\ m!\ \left(\mathrm{E}[Y(\mathbf{x})Y(\mathbf{y})]\right)^m \tag{24}$$

(see, for example, [33]).

Proposition 1. *Assume that*

$$\lambda_p(Q) < 2\lambda_p(A), \quad p \geq 1. \tag{25}$$

Then, for each $p \geq 1$, considering the conditional covariance of Y_p, and assuming that t is sufficiently large, to ensure that the difference with the unconditional covariance of the real-valued O-U process Y_p is negligible, the following identity then holds:

$$\begin{aligned} &Var\left(\int_0^t \exp(Y_p(s))\, ds\right) \\ &= \sum_{k=1}^{\infty} \frac{C_k^2}{k!} \int_0^t \int_0^t \left[\frac{\lambda_p(Q)}{2\lambda_p(A)} \exp(-\lambda_p(A)|u-v|)\right]^k dudv < \infty, \end{aligned} \tag{26}$$

where $C_k = \int_{\mathbb{R}} \exp(u) H_k(u)\varphi(u)du$, $k \geq 1$, and $G(u) = \exp(u) = \sum_{k=1}^{\infty} \frac{C_k}{k!} H_k(u)$, for all $u \in \mathbb{R}$, in $L^2(\mathbb{R}, \varphi(u)du)$.

Proof. Equation (26) follows straightforward from (24), and the definition of Y_p in equation (13), for each $p \geq 1$. In addition, $G(u) = \exp(u) \in L^2(\mathbb{R}, \varphi(u)du)$ implies

$$\|G\|^2_{L^2(\mathbb{R},\varphi(u)du)} = \sum_{k=1}^{\infty} \left[\frac{C_k}{k!}\right]^2 < \infty. \tag{27}$$

From Cauchy–Schwarz inequality, equation (27), and condition (25), leading to $\left[\frac{\lambda_p(Q)}{2\lambda_p(A)}\right]^2 < 1$, we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{C_k^2}{k!}\int_0^t\int_0^t\left[\frac{\lambda_p(Q)}{2\lambda_p(A)}\exp\left(-\lambda_p(A)|u-v|\right)\right]^k dudv\\
&\leq \sqrt{\sum_{k=1}^{\infty}\left[\frac{C_k^2}{k!}\right]^2}\sqrt{\sum_{k=1}^{\infty}\left[\int_0^t\int_0^t\left[\frac{\lambda_p(Q)}{2\lambda_p(A)}\exp\left(-\lambda_p(A)|u-v|\right)\right]^k dudv\right]^2}\\
&\leq \sqrt{\sum_{k=1}^{\infty}\left[\frac{C_k^2}{[k!]^2}\right]^2}\sqrt{\sum_{k=1}^{\infty}\left[\int_0^t\int_0^t\left[\frac{\lambda_p(Q)}{2\lambda_p(A)}\exp\left(-\lambda_p(A)|u-v|\right)\right]^k dudv\right]^2}\\
&\leq \widetilde{K}\sqrt{\sum_{k=1}^{\infty}\left[\frac{C_k}{k!}\right]^2}\,t^2\sqrt{\sum_{k=1}^{\infty}\left[\frac{\lambda_p(Q)}{2\lambda_p(A)}\right]^{2k}} = \sqrt{\sum_{k=1}^{\infty}\left[\frac{C_k}{k!}\right]^2}\frac{t^2}{\sqrt{1-\left[\frac{\lambda_p(Q)}{2\lambda_p(A)}\right]^2}} < \infty,
\end{aligned} \tag{28}$$

where $\widetilde{K}$ is a positive constant, keeping in mind that (27) implies the sequence $\left\{\left[\frac{C_k}{k!}\right]^2\right\}_{k\geq 1}$ is convergent to zero, and, in particular, for k sufficiently large, $\left[\frac{C_k}{k!}\right]^4 < \left[\frac{C_k}{k!}\right]^2 < 1$.

The following condition is now considered.

Assumptions A3 The following asymptotic holds:

$$\frac{2\lambda_p(A)}{\sqrt{[2\lambda_p(A)]^2-[\lambda_p(Q)]^2}} = \mathcal{O}\left(p^{-\alpha}\right), \quad p\to\infty, \quad \alpha > 1.$$

Proposition 2. *Under the conditions (i)–(iii) formulated in Section 2.3, considering also* ***Assumptions A1, A3****, and (25) to hold, we have, for every* $t\in\mathbb{R}_+$,

$$\sum_{p=1}^{\infty}\mathcal{C}_p^2(t) < \infty, \quad a.s.$$

Proof. From equation (28), under condition (25), and **Assumptions A1** and **A3**,

$$\begin{aligned}
E\left[\sum_{p=1}^{\infty}\mathcal{C}_p^2(t)\right] &= \sum_{p=1}^{\infty}E\left[\mathcal{C}_p^2(t)\right]\\
&\leq \widetilde{K}\sum_{p=1}^{\infty}\sqrt{\sum_{k=1}^{\infty}\left[\frac{C_k}{k!}\right]^2}\frac{t^2}{\sqrt{1-\left[\frac{\lambda_p(Q)}{2\lambda_p(A)}\right]^2}}\\
&= \widetilde{K}t^2\sqrt{\sum_{k=1}^{\infty}\left[\frac{C_k}{k!}\right]^2}\sum_{p=1}^{\infty}\left[1-\left[\frac{\lambda_p(Q)}{2\lambda_p(A)}\right]^2\right]^{-1/2}\\
&= \widetilde{K}t^2\sqrt{\sum_{k=1}^{\infty}\left[\frac{C_k}{k!}\right]^2}\sum_{p=1}^{\infty}\frac{2\lambda_p(A)}{\sqrt{[2\lambda_p(A)]^2-[\lambda_p(Q)]^2}} < \infty.
\end{aligned} \tag{29}$$

From Proposition 2, $\boldsymbol{\mathcal{C}} = \{\mathcal{C}_p(t),\ p\geq 1,\ t\in\mathbb{R}^+\}$ is a ℓ^2-valued doubly stochastic Poisson process. With a similar reasoning to the case of ℓ^2-valued homogeneous Poisson process in Section 2.1, we arrive to the existence of a σ-finite counting measure $\mu_{\boldsymbol{\mathcal{C}}}$ associated with such a process, with respect to which, in particular, the likelihood of $\{\mathcal{C}_p(T),\ p \geq 1\}$ can be defined for every $T > 0$. Furthermore, for each $p \geq 1$, and for any $0 < t_1 < t_2$, the following identity holds, given the values $Y_p(\omega, s)$, $\omega \in \Omega$, and $t_1 \leq s \leq t_2$, with, as before, Y_p satisfying (13) for each $p \geq 1$,

$$\begin{aligned}&\Pr\left(\mathcal{C}_p(t_2) - \mathcal{C}_p(t_1) = k | Y_p(\omega, s);\ t_1 \leq s \leq t_2\right)\\ &= \frac{\exp\left(-\int_{t_1}^{t_2} \exp(Y_p(\omega, s)ds\right)\left[\int_{t_1}^{t_2} \exp(Y_p(\omega, s))ds\right]^k}{k!}.\end{aligned}$$

Hence, assuming that conditionally to $\{\mathbf{Y}(t),\ t \in \mathbb{R}_+\} = \{Y_p(t),\ p \geq 1,\ t \in \mathbb{R}_+\}$, $\{\boldsymbol{\mathcal{C}}(t),\ t \in \mathbb{R}_+\} = \{\mathcal{C}_p(t),\ p \geq 1,\ t \in \mathbb{R}_+\}$ defines a sequence of independent non-homogeneous Poisson processes, we have, for $\mathbf{k} = (k_1, \dots, k_n, \dots) \in \mathbb{N}^\infty$

$$\begin{aligned}&\Pr\left(\boldsymbol{\mathcal{C}}(t_1) - \boldsymbol{\mathcal{C}}(t_2) = \mathbf{k} | \mathbf{y}_{t_1,t_2}\right)\\ &= \Pr\left(\boldsymbol{\mathcal{C}}(t_1) - \boldsymbol{\mathcal{C}}(t_2) = \mathbf{k} | \mathbf{Y}(\omega, s) = \mathbf{y}(s);\ t_1 \leq s \leq t_2\right)\\ &= \prod_{p=1}^{p_0(\omega, \mathbf{y}_{t_1,t_2}, t_1, t_2)} \frac{\exp\left(-\int_{t_1}^{t_2} \exp(Y_p(\omega, s)ds\right)\left[\int_{t_1}^{t_2} \exp(Y_p(\omega, s))ds\right]^{k_p}}{(k_p)!}.\end{aligned} \tag{30}$$

Moreover, given the observations $\mathbf{y}_T = \{Y_p(\omega, t) = y_p(t),\ p \geq 1,\ 0 \leq t \leq T\}$, for certain $\omega \in \Omega$, with, as before, $Y_p(t) = \langle X(t, \phi.), \phi_p \rangle_H$, satisfying equation (13), for each $p \geq 1$, and any $j \geq 1$, we can compute, from (23) (see Sections 4.1 and 4.2), the plug-in predictor

$$\begin{aligned}\widehat{Y}_p(T+h) &= \left\langle \widehat{X}_T(T+h, \phi_j), \phi_p \right\rangle_H\\ &= \left\langle \exp((T+h)\widehat{A}_T)(\phi_j) + \int_0^{T+h} \exp((t-s)\widehat{A}_T) d\widehat{W}_T(s), \phi_p \right\rangle_H,\end{aligned} \tag{31}$$

for $h \in \mathbb{R}_+$. Thus, for any $h > 0$, the following ℓ^2-valued predictor of log-Gaussian Cox process $\boldsymbol{\mathcal{C}}$ is formulated:

$$E_{\widehat{\mathbf{Y}}_T(\omega)}\left(\boldsymbol{\mathcal{C}}(T+h) | \boldsymbol{\mathcal{C}}(\omega, T)\right) = \prod_{p=1}^{p_0(\omega, \widehat{\mathbf{Y}}_T(\omega), T, T+h)} \int_T^{T+h} \exp\left(\widehat{Y}_p(\omega, s)\right) ds + \boldsymbol{\mathcal{C}}(\omega, T),$$

where $\widehat{\mathbf{Y}}_T(\omega) = \{\widehat{Y}_p(\omega, s),\ p \geq 1,\ T \leq s \leq T+h\}$, with $\widehat{Y}_p(\cdot)$ being defined in (31), for every $p \geq 1$, from the observations

$$\begin{aligned}\{\mathbf{Y}(\omega, s),\ 0 \leq s \leq T\} &= \{Y_p(\omega, s),\ p \geq 1,\ 0 \leq s \leq T\}\\ &= \{\langle X(\omega, s, \phi.), \phi_p \rangle_H,\ p \geq 1,\ 0 \leq s \leq T\}\\ &= \{y_p(s),\ p \geq 1,\ 0 \leq s \leq T\}.\end{aligned}$$

6. Spatiotemporal log-Gaussian Cox process in functional disease mapping

The results given in Section 5 are applied here to the particular case where $H = L^2(D)$, with $D \subseteq \mathbb{R}^2$. Indeed, we are motivated by the problem of functional estimation and prediction of disease maps. In the next section, in the conditional simulation study undertaken, we will focus on estimation and prediction of breast, prostate, and brain cancer risk maps in Spain.

6.1. **Observation of the functional rate of disease incidences.** Let $\{\mathcal{C}_t,\ t \in \mathbb{R}_+\}$ be a log-Gaussian Cox process with $L^2(D)$-valued random intensity

$$\{\lambda_t = E_t \exp(\beta_t + X_t),\ t \in \mathbb{R}_+\}, \tag{32}$$

with X_t being $L^2(D)$-valued O-U process, satisfying the conditions assumed in the previous section, and, with, for each $t \in \mathbb{R}_+$, E_t being a $L^2(D)$-valued deterministic function, providing the map of expected cases of the disease in the analyzed sub-regions of D, in that time t. Hence, E_t has functional values being positive on D, and β_t is a $L^2(D)$-valued fixed effect parameter, for each $t \in \mathbb{R}_+$ (see [36]). In particular, for each $t \in \mathbb{R}_+$, and for every $\mathbf{x}^i = (x_1^i, x_2^i)$, representing the spatial location of the subregion i analyzed in D,

$$\ln(\lambda_t(\mathbf{x}^i)) \quad = \quad \ln(E_t(\mathbf{x}^i)) + \beta_t(\mathbf{x}^i) + X_t(\mathbf{x}^i), \quad i = 1, \dots, N. \tag{33}$$

Our $L^2(D)$-valued observation model is obtained from equation (33) by adding an observation noise, as follows: For $i = 1, \dots, N$,

$$Z_t(\mathbf{x}^i) = \ln(\lambda_t(\mathbf{x}^i)) + \epsilon_t(\mathbf{x}^i) = \ln(E_t(\mathbf{x}^i)) + \beta_t(\mathbf{x}^i) + X_t(\mathbf{x}^i) + \epsilon_t(\mathbf{x}^i),$$

where, for each $t \in \mathbb{R}_+$, $Z_t, \lambda_t, E_t, \epsilon_t, \beta_t, X_t \in L^2(D)$. In particular, $\{\epsilon_t,\ t \in \mathbb{R}_+\}$ is a zero-mean $L^2(D)$-valued Gaussian process with independent components, and diagonal auto-covariance operator.

For any $h > 0$, the functional estimate of a disease risk map at time $T + h$, based on functional intensity observations up to time T, is constructed from

$$\frac{\mathbf{\Phi}(\boldsymbol{\mathcal{C}}\widehat{(T+h)})(\mathbf{x}^i)}{E_{T+h}(\mathbf{x}^i)}, \quad i = 1, \dots, N,$$

where $\mathbf{\Phi}$ denotes the inverse of the projection operator into the known eigenvector system $\{\phi_j\}_{j \geq 1}$ of Q and A, and

$$\boldsymbol{\mathcal{C}}\widehat{(T+h)} \quad = \quad \prod_{p=1}^{p_0(\omega, \widehat{\mathbf{Y}}_T(\omega), T, T+h)} \int_T^{T+h} \widehat{\lambda}_p(\omega, s) ds + \boldsymbol{\mathcal{C}}(\omega, T), \tag{34}$$

with, for every $p \geq 1$,

$$\{\widehat{\lambda}_p(\omega, s) = \langle E_s, \phi_p(\cdot) \rangle_H \exp\left(\left\langle \widehat{\beta_s(\omega, \cdot)}, \phi_p(\cdot) \right\rangle_H + \left\langle \widehat{X}_T(\omega, s, \phi_\cdot), \phi_p \right\rangle_H \right),$$

for $\omega \in \Omega$ denoting, as before, the sample point associated with the observed realization of the functional random sample. Here for each $s \in (T, T + h]$, $\widehat{\beta_s}$ is a nonparametric functional estimate of β_s (see equation (37) below), and $\widehat{X}_T$ is defined as in equation (23), based on functional observations up to time T.

7. Conditional simulation study

Three functional samples are generated, under **Assumptions A1, A2** and **A3**, conditionally to the available records on expected cases of breast, prostate, and brain cancer diseases in the provinces of Spain. Moreover, from (32)–(33), our observation model is defined from the following equation, derived by considering the Taylor expansion of ln of function $\frac{\mathcal{C}_t(\mathbf{x}^i)}{E_t(\mathbf{x}^i)}$ at the observed value $\lambda_t(\mathbf{x}^i)$, for each $t = 1, \dots, T$, and for any

$i = 1, \dots, N,$

$$\begin{aligned}
\widetilde{Z}_t(\mathbf{x}^i) &= \ln\left(\frac{\mathcal{C}_t(\mathbf{x}^i)}{E_t(\mathbf{x}^i)}\right) = \ln(\mathcal{C}_t(\mathbf{x}^i)) - \ln\left(E_t(\mathbf{x}^i)\right) \\
&= \ln(\lambda_t(\mathbf{x}^i)) + \frac{\mathcal{C}_t(\mathbf{x}^i) - \lambda_t(\mathbf{x}^i)}{\lambda_t(\mathbf{x}^i)} - \ln\left(E_t(\mathbf{x}^i)\right) \\
&= \ln\left(E_t(\mathbf{x}^i)\right) + \beta_t(\mathbf{x}^i) + X_t(\mathbf{x}^i) - \ln\left(E_t(\mathbf{x}^i)\right) + \frac{\mathcal{C}_t(\mathbf{x}^i) - \lambda_t(\mathbf{x}^i)}{\lambda_t(\mathbf{x}^i)} \\
&= \beta_t(\mathbf{x}^i) + X_t(\mathbf{x}^i) + \frac{\mathcal{C}_t(\mathbf{x}^i) - \lambda_t(\mathbf{x}^i)}{\lambda_t(\mathbf{x}^i)} = \beta_t(\mathbf{x}^i) + X_t(\mathbf{x}^i) + \xi_t(\mathbf{x}^i),
\end{aligned} \tag{35}$$

where, from Gaussian approximation of the Poisson distribution, for each $t = 1, \dots, T$, and for any $i = 1, \dots, N$,

$$\xi_t(\mathbf{x}^i) = \left\{\frac{\mathcal{C}_t(\mathbf{x}^i) - \lambda_t(\mathbf{x}^i)}{\lambda_t(\mathbf{x}^i)},\ i = 1, \dots, N,\ t = 1, \dots, T\right\} \tag{36}$$

are approximately distributed as independent normal random variables with respective variances

$$\left\{\frac{1}{\lambda_t(\mathbf{x}^i)},\ i = 1, \dots, N,\ t = 1, \dots, T\right\}.$$

As before, the ARH(1) approach is adopted to obtain the functional parameter estimation of the $L^2(D)$-valued O-U process X_t, with D being the Iberian peninsula (see Section 4, and in particular, Sections 4.1 and 4.2). In the last identity in (35), for each $t \in \mathbb{R}_+$, $L^2(D)$-valued parameter β_t is estimated, in a nonparametric framework, from the following equation:

$$\widehat{\beta}_t = \mathbf{\Phi}(\mathbf{\Phi}^*(\widehat{Q}_t + \widehat{C}_{\xi_t})^{-1}\mathbf{\Phi})^{-1}\mathbf{\Phi}^*(\widehat{Q}_t + \widehat{C}_{\xi})^{-1}\widetilde{Z}_t, \tag{37}$$

where, as before, $\widetilde{Z}_t$ is defined from (35), $\mathbf{\Phi}^*$ denotes the projection operator into the known eigenvector system $\{\phi_j\}_{j\geq 1}$ of Q and A, which also coincides with the common eigenvector system of the autocorrelation operator ρ, and the autocovariance operator C_X of the ARH(1) process, providing the discrete-time approximation of the H-valued O-U process. Here, $\widehat{Q}_t$ denotes the componentwise estimator of the autocovariance operator Q_t, in equation (4), of X_t, computed from the componentwise estimators (21) and (22) of operators A and Q, respectively. Finally, $\widehat{C}_{\xi_t}$ denotes the empirical finite-dimensional approximation of the auto-covariance operator C_{ξ_t} of the $L^2(D)$-valued process ξ_t, introduced in equation (36), based on the observed values $\left\{\frac{1}{\lambda_t(\mathbf{x}^i)},\ i = 1, \dots, N,\ t = 1, \dots, T\right\}$. Note that, in practice, a truncation order is considered in the implementation of equation (37), which should be smaller than the functional sample size T (see, for example, [6]).

The Kalman filtering is applied to obtain the prediction of the functional values of the ARH(1) process at times $t = T + h$, $h > 0$, approximating the corresponding H-values of O-U process. Specifically, for a given truncation order $M < T$, considering the diagonal approximation (19) of the ARH(1) state equation, the following finite-dimensional model is formulated: For $n = 1, \dots, T$,

$$\mathbf{X}_{M\times 1}(n) = \mathbf{\Lambda}_{M\times M}(\rho)\mathbf{X}_{M\times 1}(n-1) + \boldsymbol{\varepsilon}_{M\times 1}(n), \tag{38}$$

where, for each $t = 1, \dots, T$, $\mathbf{X}_{M\times 1}(t)$ is a $M \times 1$ vector with entries $X_p(t) = \langle X_t, \phi_p \rangle_H$, $p = 1, \dots, M$, $\mathbf{\Lambda}_{M\times M}(\rho)$ is a $M \times M$ diagonal matrix with entries $\lambda_p(\rho)$, $p = 1, \dots, M$, and

$\boldsymbol{\varepsilon}_{M\times 1}(t)$ is a $M\times 1$ vector with entries $\varepsilon_p(t) = \langle \varepsilon_t, \phi_p\rangle_H$, $p = 1,\dots,M$. Kalman filtering equations are then implemented, from (35) and (38), as follows:

$$\widehat{\mathbf{X}}(t \mid t) = \widehat{\mathbf{X}}(t \mid t-1) + \mathbf{K}_t \left(Z_t - \mathbf{\Phi}_M \widehat{\mathbf{X}}(t \mid t-1)\right)$$

where $\widehat{\mathbf{X}}(t \mid t) = E(\mathbf{X}(t) \mid Z_t, ..., Z_1)$ is the (a-posteriori) updated projection estimate of the random coefficients $X_p(t)$, $p = 1,\dots,M$, at time t, and $\widehat{\mathbf{X}}(t \mid t-1) = E(\mathbf{X}(t) \mid Z_{t-1}, ..., Z_1)$ is the corresponding (a-priori) projection estimate, previously computed, for time t. Here, Z_t is obtained from the observed values (35) by subtracting the nonparametric estimate (37) at that time t. Operator $\mathbf{K}_t$ denotes the *operator of earnings*, reflecting the smoothing performed on functional data, given by

$$\mathbf{K}_t = \mathbf{P}_{t|t-1}\mathbf{\Phi}^*_M \left(\mathbf{C}_{\xi_t} + \mathbf{\Phi}_M \mathbf{P}_{t|t-1}\mathbf{\Phi}^*_M\right)^{-1}$$

where $\mathbf{\Phi}^*_M$ and $\mathbf{C}_{\xi_t}$ respectively denote the projection operator into $\phi_1,\dots,\phi_M$, and the auto-covariance operator of ξ_t, the observation noise. Furthermore,

$$\mathbf{P}_{t|t-1} = \mathrm{Cov}(\mathbf{X}(t) \mid Z_{t-1}, ..., Z_1) = \mathbf{\Lambda}_{M\times M}(\rho)\mathbf{P}_{t-1|t-1}\mathbf{\Lambda}_{M\times M}(\rho) + \mathbf{R}_{\varepsilon_t}$$

with $\mathbf{R}_{\varepsilon_t}$ being the covariance matrix of the random vector $\boldsymbol{\varepsilon}_{M\times 1}(t)$, with components $\varepsilon_p(t) = \langle \varepsilon_t, \phi_p\rangle_H$, $p = 1,\dots,M$, and

$$\mathbf{P}_{t|t} = E\left((\mathbf{X}(t) - \widehat{\mathbf{X}}(t \mid t))(\mathbf{X}(t) - \widehat{\mathbf{X}}(t \mid t))^*\right) = \mathbf{P}_{t|t-1} - \mathbf{K}_t\mathbf{\Phi}_M\mathbf{P}_{t|t-1}.$$

Thus, the finite-dimensional approximation of the Kalman filter predictor, for each time t of interest, is given by

$$\widehat{X}_t = \mathbf{\Phi}_M\widehat{\mathbf{X}}(t \mid t-1) = \mathbf{\Phi}_M\mathbf{\Lambda}(\rho)\widehat{\mathbf{X}}(t-1 \mid t-1). \quad (39)$$

The initial values considered are

$$\begin{aligned} \widehat{\mathbf{X}}(0 \mid 0) &= \mathbf{0} \\ P_{0|0} &= E\left[\mathbf{X}_{M\times 1}(0)[\mathbf{X}_{M\times 1}(0)]^T\right], \end{aligned}$$

with $\mathbf{X}_{M\times 1}(0) = (X_1(0),\dots,X_M(0))^T = (\langle X_0, \phi_1\rangle_H, \dots, \langle X_0, \phi_M\rangle_H)^T$.

In practice, $\mathbf{\Lambda}_{M\times M}(\rho)$ is estimated by the diagonal matrix with entries $\widehat{\lambda}_{T,k}(\rho)$, $k = 1,\dots,M$, defined in (20), and $M\times M$ matrix $\mathbf{R}_{\varepsilon_t}$ is approximated by a diagonal matrix with entries $\widehat{\lambda}_{T,k}(R_\varepsilon)$, $k = 1,\dots,M$, also defined in (20).

Functional prediction of risk maps, at each time t of interest, is then obtained from

$$\widehat{r}_t = \exp(\widehat{\beta}_t + \widehat{X}_t), \quad (40)$$

with $\widehat{\beta}_t$ in (37), and $\widehat{X}_t$ computed from the above Kalman filtering equations, in terms of the projected residuals, associated with the estimation of $L^2(D)$-valued fixed effect parameter β_t, for each time t of interest. The spatial averaged empirical mean-square errors, for each one of the years studied, computed from 100 realizations of the Kalman predictor (39), and the fixed effect nonparametric estimator (37), are displayed in Table 1, for the three types of cancer data generated, breast, prostate, and brain cancer data. It can be observed that Prostate and Breast cancer data are better generated and estimated from the log-Gaussian ARH(1) intensity approach.

7.1. **LRD spatiotemporal intensity.** In this section, we consider the case where the random intensity (32), $\{\lambda_t,\ t\in\mathbb{R}_+\}$, of the log-Gaussian Cox process studied is defined from a LRD spatiotemporal Gaussian process, satisfying the following integral equation, in the mean-square sense:

$$X_t(\mathbf{x}^i) \underset{\text{m.s.}}{=} \int_{\mathbb{R}_+}\int_D K(t,s,\mathbf{x}^i,\mathbf{y},\boldsymbol{\theta})V(s,\mathbf{y})ds d\mathbf{y}, \quad (41)$$

for $i = 1,\dots,N$, and $t\in\mathbb{R}_+$. Here,

Year	Prostate Cancer	Breast Cancer	Brain Cancer
1	4.587e-002	4.197e-002	9.337e-001
2	4.150e-002	3.516e-002	6.712e-001
3	3.410e-002	3.001e-002	3.264e-001
4	3.586e-002	3.089e-002	4.651e-001
5	3.227e-002	2.688e-002	3.823e-001
6	3.216e-002	3.058e-002	1.766e-001
7	3.391e-002	2.797e-002	4.192e-001
8	2.891e-002	3.304e-002	3.921e-001
9	3.128e-002	2.886e-002	3.182e-001
10	3.096e-002	3.127e-002	1.744e-001
11	3.070e-002	3.137e-002	2.977e-001
12	2.989e-002	3.248e-002	2.250e-001
13	3.052e-002	3.263e-002	6.456e-001
14	2.809e-002	3.462e-002	2.350e-001
15	2.866e-002	3.489e-002	1.209e+000
16	2.814e-002	3.419e-002	2.742e-001
17	2.627e-002	3.397e-002	3.685e-001
18	2.858e-002	3.629e-002	1.244e-001
19	3.101e-002	3.637e-002	2.864e-001
20	3.556e-002	3.677e-002	1.857e-001
21	3.528e-002	3.497e-002	6.199e-001
22	3.715e-002	3.167e-002	3.545e-001
23	3.337e-002	3.344e-002	2.227e-001
24	3.432e-002	2.605e-002	1.223e+000
25	3.405e-002	3.184e-002	1.396e-001
26	3.171e-002	2.634e-002	
27	3.165e-002	3.126e-002	
28	3.156e-002	2.438e-002	
29	2.998e-002	2.771e-002	
30	2.892e-002	2.809e-002	
31	2.593e-002	2.387e-002	
32	2.671e-002		
33	2.459e-002		
34	2.206e-002		

TABLE 1. Spatial averaged Empirical Mean Square Errors (EMSEs) by year, obtained from 100 generations of the functional sample of size T of the ARH(1) intensity process, for the three cancer-type data generated.

$$K(t,s,\mathbf{x}^i,\mathbf{y},\boldsymbol{\theta}) = |t-s|^{-1+\nu}|x_1^i - y_1|^{-1+\varpi_1}|x_2^i - y_2|^{-1+\varpi_2},$$

with $\boldsymbol{\theta} = (\nu,\varpi_1,\varpi_2) \in (0,1)^3$ (see [19]). Process V is regular zero-mean Gaussian process such that the integral (41) is finite in the mean-square sense. Data are then generated, considering the observation model (35), but replacing X_t by process given in (41). $L^2(D)$-valued parameter β_t is estimated from equation (37). The least-squared predictor is then computed, from the associated residuals as follows:

$$\widehat{X}_t = R_{X_t(\widetilde{Z}_t-\widehat{\beta}_t)}R^{-1}_{(\widetilde{Z}_t-\widehat{\beta}_t)(\widetilde{Z}_t-\widehat{\beta}_t)}(\widetilde{Z}_t-\widehat{\beta}_t), \quad \forall t \in \mathbb{R}_+, \tag{42}$$

where

$$R_{X_t(\widetilde{Z}_t-\widehat{\beta}_t)} = E[X_t \otimes \widetilde{Z}_t - \widehat{\beta}_t]$$

$$R^{-1}_{(\widetilde{Z}_t-\widehat{\beta}_t)(\widetilde{Z}_t-\widehat{\beta}_t)} = \left[E[(\widetilde{Z}_t - \widehat{\beta}_t) \otimes (\widetilde{Z}_t - \widehat{\beta}_t)]\right]^{-1}, \quad (43)$$

with $\widetilde{Z}_t$ being defined as in equation (35), and with X_t being given in equation (41). Here, as usual, $\otimes$ denotes the tensorial product of functions in $L^2(D)$. Parameter $\boldsymbol{\theta} = (\nu, \varpi_1, \varpi_2)$, involved in the definition of the kernels characterizing the integral operators $R_{X_t(\widetilde{Z}_t-\widehat{\beta}_t)}$ and $R^{-1}_{(\widetilde{Z}_t-\widehat{\beta}_t)(\widetilde{Z}_t-\widehat{\beta}_t)}$, has been estimated from the original breast, prostate, and brain cancer data records in the provinces of Spain, by applying the wavelet-based methodology proposed in [19] (see also [20]). The presence of strong correlations in time is more evident than in space for the three types of cancer data analyzed (see Table 2).

	$\hat{\nu}$	$\hat{\beta}_1$	$\hat{\beta}_2$
Prostate Cancer	0.6997	0.4103	0.5483
Breast Cancer	0.6667	0.4441	0.4356
Brain Cancer	0.6883	0.4555	0.55435

TABLE 2. LRD parameter estimates.

The spatial averaged empirical mean-square errors, for each one of the years studied, computed from 100 realizations of the least-squared predictor (42), and the fixed effect nonparametric estimator (37), are displayed in Table 3, for the three types of cancer data conditionally generated, breast, prostate, and brain cancer data. It can be observed that Brain cancer data are better generated and estimated from the LRD log-Gaussian intensity approach.

7.1.1. *Mortality log-risk maps.* An over smoothing from the LRD log-Gaussian intensity approach is observed, while a weaker smoothing is obtained from the application of the log-Gaussian ARH(1)-based estimation, allowing a larger degree of heterogeneity in the spatial patterns. Consequently, it is expected, in the generations and estimations performed, a larger number of false negatives, when the LRD log-Gaussian intensity approach is implemented. While a larger number of false positives are expected, from the implementation of the ARH(1) discrete-time approximation of the log-Gaussian H-valued-O-U-based random intensity approach (see Figures 1 and 2).

Year	Prostate Cancer	Breast Cancer	Brain Cancer
1	2.715e-001	4.238e+000	2.539e-001
2	2.492e+001	5.445e-002	7.854e-002
3	2.742e+000	2.530e-002	6.705e-002
4	6.400e+000	6.835e-002	6.445e-002
5	8.250e-001	2.399e-001	1.089e-001
6	4.614e-002	1.549e+001	9.699e-002
7	4.589e-001	3.204e-002	2.633e-001
8	5.280e-001	7.784e-002	9.773e-002
9	4.348e-001	5.195e-002	7.650e-002
10	5.723e-002	3.386e+000	6.673e-002
11	1.919e-002	1.168e-001	7.677e-002
12	1.427e+000	2.654e-002	8.259e-002
13	1.610e-002	2.723e-002	7.083e-002
14	3.094e+000	4.607e-002	5.450e-002
15	1.495e-002	2.165e+000	8.851e-002
16	3.012e-001	6.044e-002	4.365e-002
17	4.050e-002	4.107e-002	5.732e-002
18	1.409e-002	1.848e-002	5.499e-002
19	2.185e-001	1.178e+000	3.053e-002
20	2.088e-002	2.170e-001	3.951e-002
21	1.027e+000	4.749e-002	2.638e-002
22	1.574e-002	4.337e-002	5.691e-002
23	9.894e-002	6.459e-002	2.416e-002
24	1.254e-002	1.791e+000	4.403e-002
25	1.391e-001	3.138e-002	2.506e-002
26	2.559e-001	3.077e-002	
27	1.532e-002	2.021e-002	
28	4.820e-002	2.019e-002	
29	2.951e-002	2.623e-001	
30	1.298e-001	2.447e-002	
31	3.505e-002	1.050e-002	
32	3.021e-002		
33	7.394e-002		
34	5.161e-002		

Table 3. Spatial averaged Empirical Mean Square Errors (EMSEs) by year, obtained from 100 generations of the functional sample of size T of the LRD random intensity model, for the three cancer-type data generated.

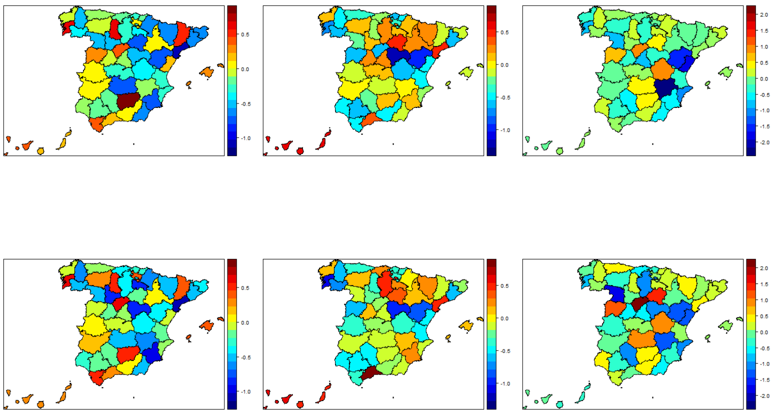

FIGURE 1. *Log-Gaussian O-U-based framework.* O-U-based generated Prostate (year 1975), Breast (year 1975) and Brain (year 1986) cancer mortality log-risk maps in Spain (from top-left to top-right), and their respective Kalman filtering estimations (from bottom-left to bottom-right).

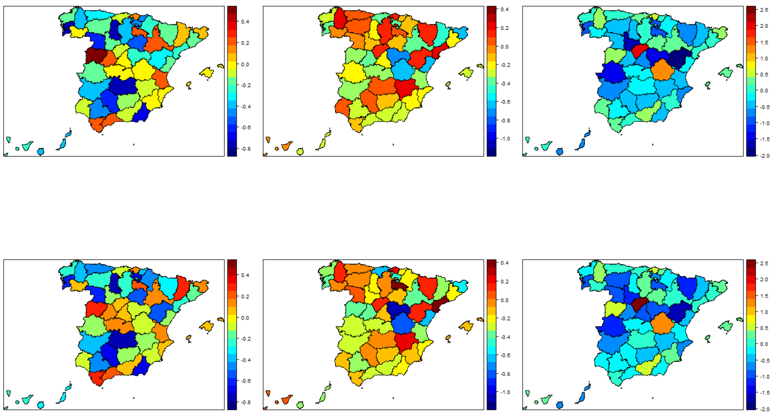

FIGURE 2. *LRD Log-Gaussian framework.* LRD-based generated Prostate (year 1975), Breast (year 1975) and Brain (year 1986) cancer mortality log-risk maps in Spain (from top-left to top-right), and their respective LRD-least-squared-based estimations (from bottom-left to bottom-right).

8. Final Comments

This paper presents new results, in the context of doubly stochastic Poisson processes in infinite-dimensional spaces, to cover some limitations arising in the parameter estimation and prediction of log-Gaussian Cox processes in space and time (see, for example, [17]). Note that, **Assumption A1**, which plays a key role in the approach adopted in this paper, in the derivation of the main results (i.e., Theorem 1, and Propositions 1 and 2), is satisfied, for a wide class of operators, as reflected in Remark 1. To illustrate the application of the theoretical results derived, in this paper, to the context of spatiotemporal log-Gaussian Cox processes, a conditional simulation study is undertaken, where log-Gaussian ARH(1) and LRD random intensity frameworks are compared, in the analysis of spatiotemporal prostate, breast and brain cancer data from the provinces of Spain.

The presented approach can also be extended to the framework of fractional Poisson processes (see, for example, [4], [27] and [30]). Indeed, the infinite-dimensional framework can be considered for reformulation of recent results in this topic, like the ones presented in [1] and [28], and their extension to the doubly stochastic fractional Poisson process context.

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References

[1] G. Aletti, N.N. Leonenko and E. Marzbach *Fractional Poisson fields and martingales.* http://arxiv.org/pdf/1601.08136.pdf (2016).

[2] J. Álvarez-Liébana, D. Bosq and M. D. Ruiz-Medina, *Asymptotic properties of a componentwise ARH(1) plug-in predictor*, Journal of Multivariate Analysis (to appear).

[3] S. Basu and A. Dassios, *A Cox process with log-normal intensity*, Insurance: Mathematics and Economics **31** (2002), 297-302.

[4] L. Beghin and E. Orsingher, Fractional Poisson processes and related planar random motions, *Electron. J. Probab.* **14** (2009), 1790–1827.

[5] J. E. Besag, *Spatial interaction and the statistical analysis of lattice systems*, Journal of the Royal Statistical Society: Series B **36** (1974), 192–225.

[6] D. Bosq, *Linear Processes in Function Spaces*, Springer-Verlag, New York, 2000.

[7] D. Bosq and M. D. Ruiz-Medina, *Bayesian estimation in a high dimensional parameter framework*, Electronic Journal of Statistics **8** (2014), 1604-1640.

[8] P. Brémaud, *Point Processes and Queues: Martingale Dynamics*, SpringerVerlag, New York, 1972.

[9] D. R. Cox, *Some statistical methods connected with series of events*, Journal of Royal Statistical Society B **17** (1955), 129-164.

[10] J. Cox, J. Ingersoll and S. Ross, *A theory of the term structure of interest rates*, Econometrica **53** (1985), 385-408.

[11] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions, Encyclopedia of Mathematics and its Applications, Vol. 44*, Cambridge University Press, Cambridge, 1992.

[12] S. Dalal and A. McIntosh, *When to stop testing for large software systems with changing code*, IEEE Transactions on Software Engineering **20** (1994), 318-323.
[13] D. J. Daley and D. Vere-Jones, *An Introduction to Theory of Point Processes*, Springer-Verlag, New York, 1988.
[14] A. D. G. Dario and A. Simonis, *Properties of doubly stochastic Poisson process with affine intensity.* arXiv:1109.2884v2 (2011).
[15] A. Dassios and J. Jang, *The distribution of the interval between events of a Cox process with shot noise intensity*, Journal of Applied Mathematics and Stochastic Analysis (2008), 1–14. (ID 367170, doi:10.1155/2008/367170).
[16] R. Dautray and J. -L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology Volume 3: Spectral Theory and Applications*, Springer, New York, 1990.
[17] P.J. Diggle, P. Moraga, B. Rowlingson and B.M. Taylor, *Spatial and spatio-temporal log-Gaussian Cox processes: Extending the Geostatistical paradigm*, Statistica Science **28** (2013), 542–563.
[18] D. Duffie and R. Kan, *A yield-factor model of interest rates*, Mathematical Finance **6** (1996), 379-406.
[19] M. P. Frías and M. D. Ruiz-Medina, *Computing functional estimators of spatiotemporal long-range dependence parameters in the spectral-wavelet domain*, Journal of Statistical Planning and Inference **141** (2011), 2417-2427.
[20] M. P. Frías, M. D. Ruiz-Medina and V. V. Anh, *Wavelet-based estimation of anisotropic spatiotemporal long-range dependence*, Stochastic Analysis and Applications **31** (2013), 359-380.
[21] J. P. Fouque, G. Papanicolaou and K. R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, 2000.
[22] M. Fuhrman, *A note on the nonsymmetric Ornstein-Uhlenbeck process in Hilbert spaces*, Applied Mathematics Letters **8** (1995), 19-22.
[23] M. Gail, T. Santner and C. Brown, *An analysis of comparative carcinogenesis experiments based on multiple times to tumor*, Biometrics **36** (1980), 255-266.
[24] J. Grandell, *Doubly Stochastic Process*, Springer-Verlan, New York, 1976.
[25] C. Heil and F. Walnut, *Fundamental Papers in Wavelet Theory*, Princeton University Press, Oxford, 2006.
[26] Y. V. Kozachenko and O. O. Pogorilyak, *A method of modelling log Gaussian Cox process*, Theory of Probability and Mathematical Statistics **77** (2008), 91-105.
[27] N. Leonenko and E. Merzbach, *Fractional Poisson fields*, Methodology and Computing in Applied Probability **17** (2015), 155-168.
[28] N. Leonenko, E. Scalas and M. Trinh, *The fractional non-homogeneous Poisson process.* http://arxiv.org/abs/1601.03965 (2016).
[29] D. Lando, *On Cox processes and credit risky securities*, Review of Derivatives Research **2** (1998), 99-120.
[30] M. M. Meerschaert, E. Nane and P. Vellaisamy, *The fractional Poisson process and the inverse stable subordinator*, Electron. J. Probab. **16** (2011), 1600–1620.
[31] J. Moller, A. R. Syversveen and R. Waagepetersen, *Log Gaussian Cox processes*, *Scandinavian Journal of Statistics* **25** (1998), 451-482.
[32] D. B. Nelson, *ARCH models as diffusion approximations*, Journal of Econometrics **45** (1990), 7-38.
[33] G. Peccati and M. S. Taqqu, *Wiener Chaos: Moments, Cumulants and Diagrams*, Springer, New York, 2011.
[34] S. L. Rathbun and N. Cressie, *A space-time survival point process for a longleaf pine forest in Southern Georgia*, Journal of the American Statistical Association **89** (1994), 1164-1174.
[35] M. Riedle, *Cylindrical Wiener processes*, Research Report No. 7 Probability and Statistics Group School of Mathematics, The University of Manchester, Manchester, 2008.
[36] M. D. Ruiz-Medina, *Functional analysis of variance for Hilbert-valued multivariate fixed effect models*, Statistics **50** (2016), 689-715.
[37] M. D. Ruiz-Medina, E. Romano and R. Fernández-Pascual, *Plug-in interval prediction for a special class of standard ARH(1) processes*, Journal of Multivariate Analysis **146** (2016), 138-150.
[38] G. Wei, P. Clifford and J. Feng, *Population death sequences and Cox processes driven by interacting Feller diffusions*, Journal of Physics A: Mathematical and General **35** (2002), 9-31.

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