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ORIGINAL PAPER

Operators of Hilbert and Cesàro type acting on Dirichlet spaces

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Abstract

If $\alpha > -1$ the space of Dirichlet type $\mathcal{D}_\alpha^2$ consists of those functions f which are analytic in the unit disc $\mathbb{D}$ such that f' belongs to the weighted Bergman space A_α^2. The space D_0^2 is the classical Dirichlet space $\mathcal{D}$. If g is an analytic function in $\mathbb{D}$, we study the generalized Hilbert operator $\mathcal{H}_g$ defined by

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz)\,dt$$

acting on the spaces $\mathcal{D}_\alpha^2$ ($0 \le \alpha \le 1$). We obtain a characterization of those g for which $\mathcal{H}_g$ is bounded, compact, or Hilbert-Schmidt on the Dirichlet space $\mathcal{D}$. In addition to this, we use our results concerning the operators $\mathcal{H}_g$ to study certain Cesàro-type operators $\mathcal{C}_{(\eta)}$ acting on the spaces $\mathcal{D}_\alpha^2$ ($0 \le \alpha \le 1$). We give also a characterization of the positive finite Borel measures μ in $[0, 1)$ for which a certain Cesàro type operator $\mathcal{C}_\mu$ associated to μ is bounded on the Bergman space A_α^1 ($\alpha > -1$). This is an extension of the previously known results for the spaces A_α^p with $p > 1$ and $\alpha > -1$.

Keywords Dirichlet spaces · Hilbert-type operators · Cesàro-type operators

Mathematics Subject Classification 47B91 · 30H20 · 30H25

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1 Introduction

The Hilbert matrix

$$\mathcal{H} = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & . & . \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & . & . \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & . & . \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix},$$

induces the *Hilbert operator*, denoted also by $\mathcal{H}$ on certain spaces of complex sequences by matrix multiplication:

$$\mathcal{H}\left(\{a_n\}_{n=0}^{\infty}\right) = \left\{\sum_{k=0}^{\infty} \frac{a_k}{n+k+1}\right\}_{n=0}^{\infty}.$$

This operator is well defined in ℓ^1 but it is not bounded from ℓ^1 into itself. Hilbert's inequality

$$\left(\sum_{n=0}^{\infty}\left|\sum_{k=0}^{\infty}\frac{b_k}{n+k+1}\right|^p\right)^{1/p} \le \frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{k=0}^{\infty}|b_k|^p\right)^{1/p}, \quad 1 < p < \infty,$$

(see Chapter IX of [23]) shows that $\mathcal{H}$ is bounded from ℓ^p into itself for $1 < p < \infty$.

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$ and let $\mathrm{Hol}(\mathbb{D})$ be the space of all analytic functions in $\mathbb{D}$.

If $0 < r < 1$ and $f \in \mathrm{Hol}(\mathbb{D})$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{it})|^p\,dt\right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r, f) = \sup_{|z|=r}|f(z)|.$$

If $0 < p \le \infty$, the Hardy space H^p consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty$.

If $0 < p < \infty$ and $\alpha > -1$, the weighted Bergman space A^p_α consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1)\int_{\mathbb{D}}|f(z)|^p(1-|z|^2)^\alpha\,dA(z)\right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p. Here, $dA(z) = \frac{1}{\pi}dx\,dy$ denotes the normalized Lebesgue area measure in $\mathbb{D}$. We refer to [12] for the theory of Hardy spaces, and to [13], [24] and [43] for Bergman spaces.

If $0 < p < \infty$ and $\alpha > -1$ the space of Dirichlet type $\mathcal{D}^p_\alpha$ consists of all primitives of functions in A^p_α. Hence, if f is analytic in $\mathbb{D}$, then $f \in \mathcal{D}^p_\alpha$ if and only if

$$\|f\|^p_{\mathcal{D}^p_\alpha} \stackrel{\text{def}}{=} |f(0)|^p + \|f'\|^p_{A^p_\alpha} < \infty.$$

The space $\mathcal{D}^2_0$ is the classical Dirichlet space $\mathcal{D}$ and we shall write $\|f\|_{\mathcal{D}}$ for $\|f\|_{\mathcal{D}^2_0}$. Hence, if $f\in\mathcal{D}$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ ($z\in\mathbb{D}$), then

$$\|f\|^2_{\mathcal{D}} = |f(0)|^2 + \int_{\mathbb{D}}|f'(z)|^2\,dA(z) = |a_0|^2 + \sum_{n=1}^{\infty}n|a_n|^2.$$

Let us recall also that $D^2_1 = H^2$. We mention [2] and [14] as excellent references for the Dirichlet space and some other related spaces.

The Hilbert operator can be also viewed as an operator acting on spaces of analytic function in $\mathbb{D}$, by its action on the sequences of Taylor coefficients: if $f(z)=\sum_{k=0}^{\infty}a_kz^k\in \mathrm{Hol}(\mathbb{D})$ we define

$$\mathcal{H}(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a_k}{n+k+1}\right)z^n, \tag{1}$$

whenever the right hand side makes sense and defines and analytic function in $\mathbb{D}$.

Hardy's inequality [12, page 48] guarantees that the transformed power series in (1) converges on $\mathbb{D}$ and defines there an analytic function $\mathcal{H}(f)(z)$ whenever $f\in H^1$. In other words, $\mathcal{H}(f)$ is a well defined analytic function for every $f\in H^1$. The Hilbert operator $\mathcal{H}$ is bounded from H^p to H^p, whenever $1<p<\infty$ but $\mathcal{H}$ is not bounded on H^1 [9, Theorem 1.1]. In [11] the norm of $\mathcal{H}$ acting on Hardy spaces was computed. Concerning the Bergman spaces A^p, the operator $\mathcal{H}: A^p\to A^p$ is bounded if and only if $2<p<\infty$, [10]. But $\mathcal{H}$ is not even defined in A^2, for it was shown in [11] that there exist functions $f\in A^2$ such that the series defining $\mathcal{H}(f)(0)$ is divergent. Jevtić and Karapretović [25] found the pairs (p,α) for which $\mathcal{H}$ is bounded on the spaces A^p_α and $\mathcal{D}^p_\alpha$. Lanucha, Nowak, and Pavlović [28] considered the question of finding subspaces of H^1 and of A^2 which are mapped by $\mathcal{H}$ into H^1 and A^2, respectively, and also the action of the operator $\mathcal{H}$ on the Bloch space and on Besov spaces.

The Fejér–Riesz inequality [12, page 46] shows that if $f\in H^1$ then

$$\int_0^1|f(t)|\,dt < \infty. \tag{2}$$

This yields that if $f\in H^1$, $\mathcal{H}(f)$ can be written also in the form,

$$\mathcal{H}(f)(z)=\sum_{n=0}^{\infty}\left(\int_0^1 t^nf(t)\,dt\right)z^n = \int_0^1 f(t)\frac{1}{1-tz}\,dt,\quad z\in\mathbb{D}, \tag{3}$$

or, equivalently,

$$\mathcal{H}(f)(z)=\int_0^1 f(t)g'(tz)\,d\zeta\ \ (z\in\mathbb{D})\quad \text{with } g(z)=\log\frac{1}{1-z}. \tag{4}$$

Actually, if $f\in\mathrm{Hol}(\mathbb{D})$ satisfies (2), then $\mathcal{H}(f)$ is well defined and (3) holds.

In collaboration with J. Á. Peláez and A. Siskakis, the authors of this paper considered in [19], for any given $g\in\mathrm{Hol}(\mathbb{D})$, the generalized Hilbert operator $\mathcal{H}_g$ defined by

$$\mathcal{H}_g(f)(z) = \int_0^1 f(t)g'(tz)\,dt. \tag{5}$$

If $f\in\mathrm{Hol}(\mathbb{D})$ satisfies (2) then the integral in (5) converges absolutely and, hence, for any $g\in\mathrm{Hol}(\mathbb{D})$, $\mathcal{H}_g(f)$ is a well defined analytic function in $\mathbb{D}$. In particular, this is true for any $f\in H^1$.

As noted above, $\mathcal{H} = \mathcal{H}_g$ with $g(z) = \log \frac{1}{1-z}$.

Let us remark that if $f \in \mathrm{Hol}(\mathbb{D})$ satisfies (2), then $\mathcal{H}_g(f)$ has a representation in terms of the Taylor coefficients similar to (1):
If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ $(z \in \mathbb{D})$, then

$$\begin{aligned}\mathcal{H}_g(f)(z) &= \sum_{k=0}^{\infty} \left((k+1) b_{k+1} \int_0^1 t^k f(t)\, dt \right) z^k \\ &= \sum_{k=0}^{\infty} \left((k+1) b_{k+1} \sum_{n=0}^{\infty} \frac{a_n}{n+k+1} \right) z^k.\end{aligned}$$

The paper [19] was devoted to obtain characterizations of those functions $g \in \mathrm{Hol}(\mathbb{D})$ for which $\mathcal{H}_g$ is bounded on the Hardy spaces H^p, the Bergman spaces A^p_α and on the spaces of Dirichlet type $\mathcal{D}^p_\alpha$ $(1 < p < \infty, \alpha > -1)$. Mean Lipschitz spaces played a basic role in these characterizations.

For $1 \le p \le \infty$ and $0 < \alpha \le 1$, we let Λ^p_α be the space of those $f \in \mathrm{Hol}(\mathbb{D})$ having a non-tangential limit at almost every point of $\partial\mathbb{D}$ and so that $\omega_p(\cdot, f)$, the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f, satisfies $\omega_p(\delta, f) = \mathrm{O}(\delta^\alpha)$, as $\delta \to 0$. When $p = \infty$ we write Λ_α instead of Λ^∞_α. This is the usual Lipschitz space of order α. More precisely, a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to Λ_α if and only if it has a continuous extension to the closed unit disc and its boundary values satisfy a Lipschitz condition of order α. Classical results of Hardy and Littlewood (see [6] and [12, Chapter 5]) show that whenever $0 < \alpha \le 1$ and $1 \le p \le \infty$ we have that $\Lambda^p_\alpha \subset H^p$ and that

$$\Lambda^p_\alpha = \left\{ f \text{ analytic in } \mathbb{D} : M_p(r, f') = \mathrm{O}\left((1-r)^{\alpha - 1}\right), \quad \text{as } r \to 1 \right\}. \tag{6}$$

The corresponding "little oh" spaces are denoted by λ^p_α.

If $1 < p < \infty$ and $(1/p) < \alpha \le 1$, then $\Lambda^p_\alpha \subset \Lambda_{\alpha - (1/p)}$ and, hence, each function in Λ^p_α has a continuous extension to the closed unit disc [6, p. 88]. This is not true for the space $\Lambda^p_{1/p}$. This follows easily noticing that the function $f(z) = \log(1-z)$ is an unbounded function which belongs to $\Lambda^p_{1/p}$ for all $p \in (1, \infty)$. Bourdon, Shapiro, and Sledd [6] proved that $\Lambda^p_{1/p} \subset BMOA$ for all $p \in (1, \infty)$ (we refer to [20] for the theory of $BMOA$-functions).

The main results in [19] are summarized in the following theorem.

Theorem A *Let g be an analytic function in $\mathbb{D}$.*

(i) *If $1 < p \le 2$, then $\mathcal{H}_g$ is bounded from H^p into H^p if and only if $g \in \Lambda^p_{1/p}$.*
(ii) *If $2 < p < \infty$ and $\mathcal{H}_g$ is bounded from H^p into H^p, then $g \in \Lambda^p_{1/p}$.*
(iii) *If $2 < p < \infty$ and $g \in \Lambda^q_{1/q}$ for some $q \in (1, p)$, then $\mathcal{H}_g$ is bounded from H^p into H^p.*
(iv) *If $1 < p < \infty$ and $-1 < \alpha < p - 2$, then $\mathcal{H}_g$ is bounded from A^p_α into A^p_α if and only if $g \in \Lambda^p_{1/p}$.*
(v) *If $1 < p < \infty$ and $p - 2 < \alpha \le p - 1$, then $\mathcal{H}_g$ is bounded from $\mathcal{D}^p_\alpha$ into $\mathcal{D}^p_\alpha$ if and only if $g \in \Lambda^p_{1/p}$.*

As noted in [19], the condition $-1 < \alpha < p-2$ in (iv) is not a real restriction. It is needed to insure that any function $f \in A^p_\alpha$ satisfies (2), which is necessary for the operator $\mathcal{H}_g$ being well defined on A^p_α for every $g \in \mathrm{Hol}(\mathbb{D})$. Let us remark that, in particular, the Bergman space $A^2 = A^2_0$ is not among those included in (iv). In fact, the function f given by

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{\log(n+1)}, \quad z \in \mathbb{D},$$

belongs to A^2, and we easily see that if $n \in \mathbb{N}$ and g is the monomial $g(z) = z^n$ $(z \in \mathbb{D})$, then $\mathcal{H}_g(f)$ is not well defined.

The Dirichlet space $\mathcal{D} = \mathcal{D}_0^2$ is not among those appearing in (v). However, $\mathcal{D} \subset H^2 \subset H^1$ and, hence, any $f \in \mathcal{D}$ satisfies (2). Consequently, the operator $\mathcal{H}_g$ is well defined in $\mathcal{D}$ for any $g \in \mathrm{Hol}(\mathbb{D})$.

Our main objective in Sect. 2 will be characterizing those $g \in \mathrm{Hol}(\mathbb{D})$ for which $\mathcal{H}_g$ is bounded or compact from $\mathcal{D}$ into itself. Section 3 will be mainly devoted to study certain generalized Cesàro operators acting on the Dirichlet spaces $\mathcal{D}_\alpha^2$ $(0 \le \alpha \le 1)$.

In Sect. 4 we do not deal with Dirichlet spaces. Instead, it will be devoted to characterize those finite positive Borel measures μ concentrated on the radius $[0, 1)$ for which a Cesàro type operator $\mathcal{C}_\mu$ associated to μ is bounded on the spaces A_α^1, extending results of [15] where the corresponding question for the spaces A_α^p, with $1 < p < \infty$ and $\alpha > -1$, was solved.

Throughout the paper we shall be using the convention that $C = C(p, \alpha, q, \beta, \dots)$ will denote a positive constant which depends only upon the displayed parameters $p, \alpha, q, \beta \dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Furthermore, for two real-valued functions K_1, K_2 we write $K_1 \lesssim K_2$, or $K_1 \gtrsim K_2$, if there exists a positive constant C independent of the arguments such that $K_1 \le C K_2$, respectively $K_1 \ge C K_2$. If we have $K_1 \lesssim K_2$ and $K_1 \gtrsim K_2$ simultaneously, then we say that K_1 and K_2 are equivalent and we write $K_1 \asymp K_2$.

2 The generalized Hilbert operators $\mathcal{H}_g$ acting on the Dirichlet space

Taking $p = 2$ in part (v) of Theorem A, we see that:

$$\begin{aligned} &\text{If } 0 < \alpha \le 1 \text{ and } g \in \mathrm{Hol}(\mathbb{D}), \text{ then,} \\ &\mathcal{H}_g \text{ is bounded from } \mathcal{D}_\alpha^2 \text{ into itself if and only if } g \in \Lambda_{1/2}^2. \end{aligned} \tag{7}$$

It is natural to ask whether or not this remains true for $\alpha = 0$. The answer is negative. We obtain next the substitute of (7) for the Dirichlet space $\mathcal{D} = \mathcal{D}_0^2$.

Theorem 1 *Let g be an analytic function in $\mathbb{D}$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$ $(z \in \mathbb{D})$. Then the following two conditions are equivalent:*

(i) *The operator $\mathcal{H}_g$ is bounded from $\mathcal{D}$ into itself.*
(ii) $\sum_{n=N}^{\infty} (n+1)|b_{n+1}|^2 = \mathrm{O}\left(\frac{1}{\log N}\right)$.

Proof Let us start proving that (i) implies (ii). So, suppose that $\mathcal{H}_g$ is bounded from $\mathcal{D}$ into itself. For $0 < a < 1$ set

$$f_a(z) = \left(\log \frac{1}{1-a}\right)^{-1/2} \log \frac{1}{1-az} = \left(\log \frac{1}{1-a}\right)^{-1/2} \sum_{k=1}^{\infty} \frac{a^k}{k} z^k, \quad z \in \mathbb{D}.$$

Clearly $f_a \in \mathcal{D}$, for all $a \in (0, 1)$, and

$$\|f_a\|_{\mathcal{D}}^2 \asymp 1. \tag{8}$$

We have

$$\mathcal{H}_g(f_a)(z) = \left(\log\frac{1}{1-a}\right)^{-1/2}\sum_{k=0}^{\infty}\left((k+1)b_{k+1}\sum_{n=1}^{\infty}\frac{a^n}{n(n+k+1)}\right)z^k, \quad z\in\mathbb{D},$$

and, hence,

$$\|\mathcal{H}_g(f_a)\|_{\mathcal{D}}^2 \gtrsim \left(\log\frac{1}{1-a}\right)^{-1}\sum_{k=0}^{\infty}(k+1)^3|b_{k+1}|^2\left(\sum_{n=1}^{\infty}\frac{a^n}{n(n+k+1)}\right)^2. \tag{9}$$

For $N\in\mathbb{N}$, $N\ge 2$, set $a_N = 1-\frac{1}{N}$. Using (9), and the facts that $a_N^n \gtrsim 1$, for $1\le n\le N$, and $\frac{k+1}{N+k+1}\gtrsim\frac{1}{2}$, for $k\ge N$, we obtain

$$\begin{aligned}\|\mathcal{H}_g(f_{a_N})\|_{\mathcal{D}}^2 &\gtrsim \frac{1}{\log N}\sum_{k=N}^{\infty}(k+1)|b_{k+1}|^2(k+1)^2\left(\sum_{n=1}^{N}\frac{a_N^n}{n(n+k+1)}\right)^2\\ &\gtrsim \frac{1}{\log N}\sum_{k=N}^{\infty}(k+1)|b_{k+1}|^2(k+1)^2\left(\sum_{n=1}^{N}\frac{1}{n(n+k+1)}\right)^2\\ &\gtrsim \frac{1}{\log N}\sum_{k=N}^{\infty}(k+1)|b_{k+1}|^2\frac{(k+1)^2}{(N+k+1)^2}\left(\sum_{n=1}^{N}\frac{1}{n}\right)^2\\ &\gtrsim (\log N)\sum_{k=N}^{\infty}(k+1)|b_{k+1}|^2.\end{aligned}$$

So

$$\sum_{k=N}^{\infty}(k+1)|b_{k+1}|^2 \lesssim \frac{1}{\log N}\|\mathcal{H}_g(f_{a_N})\|_{\mathcal{D}}^2. \tag{10}$$

This, (i), and (8) give (ii). □

Let us turn to prove the other implication. So, suppose (ii). Of course, this implies that $g\in\mathcal{D}$. Take $f\in\mathcal{D}$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ ($z\in\mathbb{D}$). We have

$$\begin{aligned}\|\mathcal{H}_g(f)\|_{\mathcal{D}}^2 &\le \sum_{k=0}^{\infty}(k+1)^3|b_{k+1}|^2\left(\sum_{n=0}^{\infty}\frac{|a_n|}{k+n+1}\right)^2\\ &\lesssim |a_0|^2\left(\sum_{k=0}^{\infty}(k+1)|b_{k+1}|^2\right) + \sum_{k=0}^{\infty}(k+1)^3|b_{k+1}|^2\left(\sum_{n=1}^{\infty}\frac{|a_n|}{k+n+1}\right)^2\\ &\lesssim \|f\|_{\mathcal{D}}^2\|g\|_{\mathcal{D}}^2 + I + II \end{aligned}\tag{11}$$

where

$$I = \sum_{k=0}^{\infty}(k+1)^3|b_{k+1}|^2\left(\sum_{n=1}^{k}\frac{|a_n|}{k+n+1}\right)^2,$$

$$II = \sum_{k=0}^{\infty}(k+1)^3|b_{k+1}|^2\left(\sum_{n=k+1}^{\infty}\frac{|a_n|}{k+n+1}\right)^2.$$

Let us estimate I. Using (ii) and the fact that $\max(j,m) \asymp j+m+1$ ($j,m \in \mathbb{N}$), we obtain

$$\begin{aligned} I &= \sum_{k=0}^{\infty}(k+1)|b_{k+1}|^2(k+1)^2\left(\sum_{n=1}^{k}\frac{|a_n|}{k+n+1}\right)^2 \\ &\le \sum_{k=0}^{\infty}(k+1)|b_{k+1}|^2\left(\sum_{n=1}^{k}|a_n|\right)^2 \\ &= \sum_{k=0}^{\infty}(k+1)|b_{k+1}|^2\left(\sum_{j=1}^{k}\sum_{m=1}^{k}|a_j||a_m|\right) \\ &= \sum_{j,m=1}^{\infty}|a_j||a_m|\sum_{k=\max(j,m)}^{\infty}(k+1)|b_{k+1}|^2 \\ &\lesssim \sum_{j,m=1}^{\infty}\frac{|a_j||a_m|}{\log(j+m+1)}. \end{aligned} \tag{12}$$

Let us recall Schur test [36] (see also [8, p. 9] and [2, Appendix B]).

Proposition B (Schur test) *Let* $\left(\alpha_{j,m}\right)_{j,m=0}^{\infty}$ *be an infinite matrix of complex numbers and suppose that there exist* $c_1, c_2 \ge 0$ *and a sequence of positive numbers* $\{p_m\}_{m=0}^{\infty}$ *such that*

$$\sum_{j=0}^{\infty}\alpha_{j,m}p_j \le c_1 p_m, \quad \text{for all } m,$$

$$\sum_{m=0}^{\infty}\alpha_{j,m}p_m \le c_2 p_j, \quad \text{for all } j,$$

then

$$\left|\sum_{j,m=0}^{\infty}\alpha_{j,m}z_j w_m\right|^2 \le c_1c_2\left(\sum_{j=0}^{\infty}|z_j|^2\right)\left(\sum_{j=0}^{\infty}|w_j|^2\right)$$

for all pair of sequences $\{z_j\}_{j=0}^{\infty}$, $\{w_j\}_{j=0}^{\infty} \in \ell^2$.

Let us go back to (12). Argue as in [37, pp. 813–814]. Set

$$\alpha_{j,m} = [(j+1)(m+1)]^{-1/2}[\log(j+m+1)]^{-1}, \quad p_j = [(j+1)\log(j+1)]^{-1/2},$$

$$z_j = w_j = j^{1/2}|a_j|.$$

Using Lemma 2 of [37] and Schur test, we obtain

$$\sum_{j,m=0}^{\infty}\frac{|a_j||a_m|}{\log(j+m+1)} \lesssim \sum_{j=0}^{\infty}j|a_j|^2 \le \|f\|_{\mathcal{D}}^2.$$

Hence

$$I \lesssim \|f\|_{\mathcal{D}}^2. \tag{13}$$

The following elementary lemma will be used to estimate II and also later on to obtain some other results.

Lemma 1 *For $a > 2$, we have*

$$\int_1^a \frac{1}{x(x+a)^2}\,dx \asymp \left(\frac{\log a}{a^2}\right) \quad \textit{and} \quad \int_a^\infty \frac{dx}{x(x+a)^2}\,dx \asymp \left(\frac{1}{a^2}\right).$$

So, let us estimate II. Using the Cauchy–Schwarz inequality, we see that

$$\begin{aligned} II &= \sum_{k=0}^\infty (k+1)^3 |b_{k+1}|^2 \left(\sum_{n=k+1}^\infty \frac{\sqrt{n}|a_n|}{\sqrt{n}(k+n+1)}\right)^2 \\ &\le \|f\|_{\mathcal{D}}^2 \sum_{k=0}^\infty (k+1)^3 |b_{k+1}|^2 \left(\sum_{n=k+1}^\infty \frac{1}{n(k+n+1)^2}\right). \end{aligned}$$

Since $\sum_{n=k+1}^\infty \frac{1}{n(k+n+1)^2} \asymp \int_{k+1}^\infty \frac{dx}{x(x+k+1)^2}$, Lemma 1 implies that

$$II \lesssim \|f\|_{\mathcal{D}}^2 \sum_{k=0}^\infty (k+1)^3 |b_{k+1}|^2 \frac{1}{(k+1)^2} \le \|f\|_{\mathcal{D}}^2 \|g\|_{\mathcal{D}}^2.$$

This, (11), and (13) give that $\|\mathcal{H}_g(f)\|_{\mathcal{D}}^2 \lesssim \|f\|_{\mathcal{D}}^2$. Hence, (i) is proved.

Remark 1 Clearly, the functions g which satisfy condition (ii) belong to the Dirichlet space $\mathcal{D}$. It is natural to ask whether or not they must be bounded. The answer is negative. Indeed, let

$$g(z) = \sum_{n=1}^\infty \frac{z^n}{n\log(n+1)}, \quad z \in \mathbb{D}.$$

Then g satisfies (ii) and $g \notin H^\infty$.

Remark 2 Condition (ii) can be expressed in terms of the speed at which g is approached by its sequence of partial sums $\{S_N(g)\}_{N=1}^\infty$ in the Dirichlet norm. Indeed, if

$$S_N(g)(z) = \sum_{k=0}^N b_k z^k, \quad z \in \mathbb{D}, \quad N = 0, 1, 2, \ldots,$$

then (ii) is equivalent to saying that

$$\|g - S_N(g)\|_{\mathcal{D}}^2 = \mathrm{O}\left(\frac{1}{\log(N+1)}\right).$$

Next we obtain the substitute of Theorem 1 regarding compactness.

Theorem 2 *Let g be an analytic function in $\mathbb{D}$, $g(z) = \sum_{n=0}^\infty b_n z^n$ ($z \in \mathbb{D}$). Then the following two conditions are equivalent:*

(i) *The operator $\mathcal{H}_g$ is compact from $\mathcal{D}$ into itself.*
(ii) $\sum_{n=N}^\infty (n+1)|b_{n+1}|^2 = \mathrm{o}\left(\frac{1}{\log N}\right)$.

Proof Suppose (i). For $N \in \mathbb{N}$, $N \geq 2$, let a_N be defined as in the proof of Theorem 1. Then $\|f_{a_N}\|_{\mathcal{D}} \asymp 1$ and $\{f_{a_N}\} \to 0$, as $N \to \infty$, uniformly in compact subsets of $\mathbb{D}$. Using Lemma 3.7 of [41], we deduce that

$$\|\mathcal{H}_g(f_{a_N})\|_{\mathcal{D}}^2 \to 0, \quad \text{as } N \to \infty.$$

This and (10) give (ii).

Let us prove the other implication. So, suppose that (ii) holds. Set

$$A_N = \sup_{p\geq N} \left(\log(p+1) \sum_{k=p}^{\infty} (k+1)|b_{k+1}|^2 \right), \quad N \geq 1. \tag{14}$$

We have that $\{A_N\} \to 0$, as $N \to \infty$. Using Theorem 1, we see that $\mathcal{H}_g$ is bounded from $\mathcal{D}$ into itself.

For $N \in \mathbb{N}$ let $\mathcal{H}_{g,N}$ be defined as follows:

If $f \in \mathcal{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, we set

$$\mathcal{H}_{g,N}(f)(z) = \sum_{k=0}^{N} \left((k+1) b_{k+1} \sum_{n=0}^{\infty} \frac{a_n}{n+k+1} \right) z^k.$$

Clearly, the operators $\mathcal{H}_{g,N}$ are finite rank operators from $\mathcal{D}$ into itself. Then the compactness of $\mathcal{H}_g$ will follow from the fact that

$$\mathcal{H}_{g,N} \to \mathcal{H}_g, \quad \text{in the operator norm.} \tag{15}$$

So, take $f \in \mathcal{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. We have

$$\|\mathcal{H}_{g,N}(f) - \mathcal{H}_g(f)\|_{\mathcal{D}}^2 \leq \sum_{k=N}^{\infty} (k+1)^3 |b_{k+1}|^2 \left(\sum_{n=0}^{\infty} \frac{|a_n|}{n+k+1} \right)^2 = I + II \tag{16}$$

where,

$$I = \sum_{k=N}^{\infty} (k+1)^3 |b_{k+1}|^2 \left(\sum_{n=0}^{k} \frac{|a_n|}{n+k+1} \right)^2.$$

$$II = \sum_{k=N}^{\infty} (k+1)^3 |b_{k+1}|^2 \left(\sum_{n=k+1}^{\infty} \frac{|a_n|}{n+k+1} \right)^2.$$

Now,

$$\begin{aligned} I &\leq \sum_{k=N}^{\infty} (k+1)|b_{k+1}|^2 \left(\sum_{n=0}^{k} |a_n| \right)^2 \\ &\leq \sum_{k=N}^{\infty} (k+1)|b_{k+1}|^2 \left(\sum_{n=0}^{N} |a_n| \right)^2 + \sum_{k=N}^{\infty} (k+1)|b_{k+1}|^2 \left(\sum_{n=N}^{k} |a_n| \right)^2. \end{aligned} \tag{17}$$

We have

$$\begin{aligned}\left(\sum_{n=0}^{N}|a_n|\right)^2 &\lesssim |a_0|^2 + \left(\sum_{n=1}^{N}\frac{\sqrt{n}\,|a_n|}{\sqrt{n}}\right)^2\\ &\lesssim |a_0|^2 + \left(\sum_{n=1}^{N} n|a_n|^2\right)\left(\sum_{n=1}^{N}\frac{1}{n}\right)\\ &\lesssim \|f\|_{\mathcal{D}}^2[\log(N+1)],\end{aligned}$$

and then

$$\sum_{k=N}^{\infty}(k+1)|b_{k+1}|^2\left(\sum_{n=0}^{N}|a_n|\right)^2 \lesssim A_N\|f\|_{\mathcal{D}}^2. \tag{18}$$

Arguing as in the proof of Theorem 1, we obtain

$$\sum_{k=N}^{\infty}(k+1)|b_{k+1}|^2\left(\sum_{n=0}^{N}|a_n|\right)^2 \lesssim \sum_{j,m=N}^{\infty}|a_j||a_m| \sum_{k=\max(j,m)}^{\infty}(k+1)|b_{k+1}|^2.$$

Using the facts that the sequence $\{A_N\}$ is decreasing and that $\max(j,m) \asymp j+m$, and using Schur test as above, we deduce that

$$\begin{aligned}\sum_{k=N}^{\infty}(k+1)|b_{k+1}|^2\left(\sum_{n=N}^{k}|a_n|\right)^2 &\lesssim A_N \sum_{j,m=N}^{\infty}\frac{|a_j||a_m|}{\log(j+m+1)}\\ &\le A_N \sum_{j,m=N}^{\infty}\frac{|a_{j+N}||a_{m+N}|}{\log(j+m+1)}\\ &\le A_N \sum_{j=0}^{\infty} j|a_{j+N}|^2\\ &\le A_N\|f\|_{\mathcal{D}}^2.\end{aligned}$$

This, (17), and (18) give that

$$I \lesssim A_N\|f\|_{\mathcal{D}}^2. \tag{19}$$

Let us turn our attention to II. Using the Cauchy–Schwarz inequality and Lemma 1, we see that

$$\sum_{n=k+1}^{\infty}\frac{|a_n|}{n+k+1} = \sum_{n=k+1}^{\infty}\frac{\sqrt{n}|a_n|}{\sqrt{n}(n+k+1)} \le \|f\|_{\mathcal{D}}\left(\sum_{n=k+1}^{\infty}\frac{1}{n(n+k+1)^2}\right)^{1/2} \lesssim \frac{\|f\|_{\mathcal{D}}}{k+1},$$

and then it follows that $II \lesssim A_N\|f\|_{\mathcal{D}}^2$. This and (19) give that

$$\|\mathcal{H}_{g,N}(f) - \mathcal{H}_g(f)\|_{\mathcal{D}}^2 \lesssim A_N\|f\|_{\mathcal{D}}^2$$

and (15) follows. □

Let us recall that if H is a Hilbert space, a linear operator $T : H \to H$ is said to be a Hilbert-Schmidt operator if $\sum_{j\in I}\|T(e_j)\|^2 < \infty$ for some (equivalently, for all) orthonormal basis

$\{e_j\}_{j\in I}$ of H. The spaces $\mathcal{D}^2_\alpha$ $(\alpha > 1)$ are Hilbert spaces. In particular, the Dirichlet space $\mathcal{D}$ is a Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle f, g\rangle = f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA(z), \quad f, g \in \mathcal{D}.$$

Theorem 6 of [19] shows that, for $0 < \alpha \le 1$, $\mathcal{H}_g$ is Hilbert-Schmidt on $\mathcal{D}^2_\alpha$ if and only if $g \in \mathcal{D}$. Next we give a characterization of those g for which the operator $\mathcal{H}_g$ is a Hilbert-Schmidt operator on $\mathcal{D}$.

Theorem 3 *Let g be an analytic function in $\mathbb{D}$, $g(z) = \sum_{n=0}^\infty b_n z^n$ $(z \in \mathbb{D})$. Then the following conditions are equivalent:*

(i) *The operator $\mathcal{H}_g$ is a Hilbert-Schmidt operator in $\mathcal{D}$.*
(ii) $\sum_{n=0}^\infty n|b_{n+1}|^2 \log(n+1) < \infty$.
(iii) $\int_{\mathbb{D}} |g'(z)|^2 \log \frac{1}{1-|z|}\, dA(z) < \infty$.

Proof Set $e_0(z) = 1$ $(z \in \mathbb{D})$ and

$$e_j(z) = \frac{z^j}{\sqrt{j}}, \quad z \in \mathbb{D}, \quad j = 1, 2, \dots.$$

Then $\{e_0, e_1, e_2, \dots\}$ is an orthonormal basis of $\mathcal{D}$.

We have

$$\mathcal{H}_g(e_0)(z) = \sum_{k=0}^\infty (k+1) b_{k+1} \frac{z^k}{k+1}, \quad z \in \mathbb{D}.$$

Hence,

$$\|\mathcal{H}_g(e_0)\|^2_{\mathcal{D}} \le \sum_{k=0}^\infty k|b_{k+1}|^2. \tag{20}$$

For every $j \ge 1$, we have

$$\mathcal{H}_g(e_j)(z) = \sum_{k=0}^\infty \frac{(k+1)b_{k+1}}{\sqrt{j}\,(k+j+1)} z^k$$

and, hence,

$$\|\mathcal{H}_g(e_j)\|^2_{\mathcal{D}} \asymp \frac{1}{j}\sum_{k=0}^\infty \frac{(k+1)^3|b_{k+1}|^2}{(k+j+1)^2}.$$

Then, using Lemma 1, we obtain

$$\begin{aligned}
\sum_{j=1}^\infty \|\mathcal{H}_g(e_j)\|^2_{\mathcal{D}} &\asymp \sum_{j=1}^\infty \frac{1}{j}\sum_{k=0}^\infty \frac{(k+1)^3|b_{k+1}|^2}{(k+j+1)^2} \\
&\asymp \sum_{k=0}^\infty (k+1)^3|b_{k+1}|^2 \sum_{j=1}^\infty \frac{1}{(j+1)(k+j+1)^2} \\
&\asymp \sum_{k=0}^\infty (k+1)|b_{k+1}|^2 \log(k+1).
\end{aligned}$$

Using this and (20), the equivalence (i) $\Leftrightarrow$ (ii) follows.

Now, since $\int_0^1 r^n \log \frac{1}{1-r}\, dr \asymp \frac{\log n}{n}$, we have

$$\int_{\mathbb{D}} |g'(z)|^2 \log \frac{1}{1-|z|}\, dA(z) = 2\int_0^1 \sum_{n=1}^{\infty} n^2 |b_n|^2 r^{2n-1} \log \frac{1}{1-r}\, dr \asymp \sum_{n=1}^{\infty} n|b_n|^2 \log n.$$

This shows that (ii) and (iii) are equivalent. □

3 Operators of Cesàro type acting on Dirichlet spaces

The Cesàro operator $\mathcal{C}$ is defined over the space $\mathcal{S}$ of all complex sequences as follows: If $(a) = \{a_k\}_{k=0}^{\infty} \in \mathcal{S}$ then

$$\mathcal{C}\,((a)) = \left\{ \frac{1}{n+1} \sum_{k=0}^{n} a_k \right\}_{n=0}^{\infty}.$$

Classical results of Hardy [21] and Landau [27] (see also [23, Theorem 326, p. 239]) insure that $\mathcal{C}$ is bounded from ℓ^p to ℓ^p for $1 < p \le \infty$.

Just as it happens with the Hilbert operator, identifying any given function $f \in \mathrm{Hol}(\mathbb{D})$ with the sequence $\{a_k\}_{k=0}^{\infty}$ of its Taylor coefficients, the Cesàro operator $\mathcal{C}$ becomes a linear operator from $\mathrm{Hol}(\mathbb{D})$ into itself as follows:

If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ $(z \in \mathbb{D})$, then

$$\mathcal{C}(f)(z) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

The Cesàro operator is bounded on H^p for $0 < p < \infty$. For $1 < p < \infty$, this follows from a result of Hardy on Fourier series [22] together with the M. Riesz's theorem on the conjugate function [12, Theorem 4.1]. Siskakis [38] used semigroups of composition operators to give an alternative proof of this result and to extend it to $p = 1$. Siskakis [39] gave also direct proof of the boundedness on H^1. Miao [31] dealt with the case $0 < p < 1$. Stempak [40] gave a proof valid for $0 < p \le 2$ and Andersen [1] provided another proof valid for all $p < \infty$. More results on the Cesàro operator acting on Hardy spaces, as well as on some other related spaces, have been obtained by Blasco in [5].

Next we shall consider certain generalized Cesàro operators. Let us notice that

$$\frac{1}{n+1} = \int_0^1 t^n dt, \quad n = 0, 1, 2, \ldots.$$

If μ is a complex Borel measure on $\mathbb{D}$, and $n \ge 0$, we let μ_n denote the moment of order n of μ, that is,

$$\mu_n = \int_{\mathbb{D}} w^n \, d\mu(w), \quad n = 0, 1, 2, \ldots,$$

and we define the operator $\mathcal{C}_\mu : \mathrm{Hol}(\mathbb{D}) \to \mathrm{Hol}(\mathbb{D})$ as follows: If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$, we set

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

Since the sequence of moments $\{\mu_n\}$ is bounded, $\mathcal{C}_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ for every $f \in \mathrm{Hol}(\mathbb{D})$.

Notice that $\mathcal{C} = \mathcal{C}_\mu$ when μ is the Lebesgue measure concentrated on the radius $[0, 1)$. Thus the operators $\mathcal{C}_\mu$ are a natural generalization of the Cesàro operator $\mathcal{C}$.

The operators $\mathcal{C}_\mu$ with μ being a finite positive Borel measure concentrated in the radius $[0, 1)$ have been studied in [15], [17], and [4]. In particular, the work [15] includes a complete characterization of the finite positive Borel measures μ concentrated in $[0, 1)$ for which $\mathcal{C}_\mu$ is bounded on any of the spaces H^p $(1 \le p \le \infty)$, A^p_α $(1 < p < \infty, \alpha > -1)$, $BMOA$, or the Bloch space $\mathcal{B}$. Necessary and sufficient conditions for the boundedness of $\mathcal{C}_\mu$ in the Besov spaces $B^p = \mathcal{D}^p_{p-2}$ are presented in [17]. Blasco works in [5] with the operators $\mathcal{C}_\mu$ with μ being a complex Borel measure on $[0, 1)$.

The complex Borel measures μ on $\mathbb{D}$ for which $\mathcal{C}_\mu$ is bounded on H^2 or on the Bergman space A^2_α are characterized in [16], namely, such measures are those for which $F_\mu \in \Lambda^2_{1/2}$, where F_μ is the function defined by $F_\mu(z) = \sum_{n=0}^\infty \mu_n z^n$ $(z \in \mathbb{D})$. The corresponding result for the Dirichlet spaces $\mathcal{D}^2_\alpha$ was left open in [16].

A further generalization of the Cesàro operator is the following one.

If $(\eta) = \{\eta_n\}_{n=0}^\infty$ is a sequence of complex numbers, let $\mathcal{C}_{(\eta)} = \mathcal{C}_{\{\eta_n\}}$ be the matrix

$$\mathcal{C}_{\{\eta_n\}} = \begin{pmatrix} \eta_0 & 0 & 0 & 0 & . & . \\ \eta_1 & \eta_1 & 0 & 0 & . & . \\ \eta_2 & \eta_2 & \eta_2 & 0 & . & . \\ \eta_3 & \eta_3 & \eta_3 & \eta_3 & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \end{pmatrix}.$$

When $\{\eta_n\} = \{\frac{1}{n+1}\}$, $\mathcal{C}_{\{\eta_n\}}$ is the Cesàro matrix. Let us remark that this generalized Cesàro matrices are also called Rhaly matrices [29], [33] because Rhaly considered particular cases of them (see [34, 35]).

The matrix $\mathcal{C}_{(\eta)}$ induces an operator, denoted also by $\mathcal{C}_{(\eta)}$ or $\mathcal{C}_{\{\eta_n\}}$, on the space $\mathcal{S}$ of complex sequences by matrix multiplication: If $(a) = \{a_n\}_{n=0}^\infty$ is a sequence of complex numbers then

$$\mathcal{C}_{\{\eta_n\}}(a) = \left\{\eta_n \sum_{k=0}^n a_n\right\}_{n=0}^\infty.$$

The sequences $(\eta) = \{\eta_n\}$ for which $\mathcal{C}_{\{\eta_n\}}$ is bounded on ℓ^p $(1 < p < \infty)$ have been characterized recently in [18].

Just as above, the operator $\mathcal{C}_{\{\eta_n\}}$ can be viewed as an operator acting on spaces of analytic functions in the disc. If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$ $(z \in \mathbb{D})$, we set

$$\mathcal{C}_{\{\eta_n\}}(f)(z) = \sum_{n=0}^\infty \eta_n \left(\sum_{k=0}^n a_k\right) z^n, \quad z \in \mathbb{D}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

Bao, Guo, Sung, and Wang [3, Theorem 1.3] have recently proved the following result.

$$\mathcal{C}_{\{\eta_n\}} \text{ is bounded on the Dirichlet space } \mathcal{D}$$
$$\text{if and only if } \sum_{n=N}^\infty n|\eta_n|^2 = \mathrm{O}\left(\frac{1}{\log(N+1)}\right). \tag{21}$$

Since H^2 is identifiable with ℓ^2, Theorem 1 of [18] gives that

$$\mathcal{C}_{\{\eta_n\}} \text{ is bounded on } H^2 \text{ if and only if} \sum_{n=0}^{N}(n+1)^2|\eta_n|^2 = \mathrm{O}(N). \tag{22}$$

Let us remark that the condition $\sum_{n=0}^{N}(n+1)^2|\eta_n|^2 = \mathrm{O}(N)$ is equivalent to saying that the function F_η defined by $F_\eta(z) = \sum_{n=0}^{\infty}\eta_n z^n$ ($z \in \mathbb{D}$) belongs to the space $\Lambda^2_{1/2}$ (see [6, 12, 30] or [19, Theorem A]).

In this section we shall use our results for the operators $\mathcal{H}_g$ to characterize the sequences $\{\eta_n\}$ for which $\mathcal{C}_{\{\eta_n\}}$ is bounded on $\mathcal{D}^2_\alpha$ ($0 \le \alpha \le 1$). In particular, we shall obtain new proofs of (21) and (22). These characterizations will follow from the next result.

Theorem 4 *Let $\{\eta_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that the power series $\sum_{n=0}^{\infty}\eta_n z^n$ defines an analytic function in $\mathbb{D}$. Set*

$$F_\eta(z) = \sum_{n=0}^{\infty}\eta_n z^n, \quad z \in \mathbb{D}.$$

Suppose that $0 \le \alpha \le 1$. If the operator $\mathcal{H}_{F_\eta}$ is bounded on $\mathcal{D}^2_\alpha$, then the operator $\mathcal{C}_{\{\eta_n\}}$ is also bounded on $\mathcal{D}^2_\alpha$.

Before we start with the proof, let us recall that if $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty}a_n z^n$ ($z \in \mathbb{D}$), then $f \in \mathcal{D}^2_\alpha$ if and only if $\sum_{n=1}^{\infty}n^{1-\alpha}|a_n|^2 < \infty$, and that

$$\|f\|^2_{\mathcal{D}^2_\alpha} \asymp |a_0|^2 + \sum_{n=1}^{\infty}(n+1)^{1-\alpha}|a_n|^2. \tag{23}$$

Proof of Theorem 4 Assume that $\mathcal{H}_{F_\eta}$ is bounded on $\mathcal{D}^2_\alpha$. Take $f \in \mathcal{D}^2_\alpha$, $f(z) = \sum_{n=0}^{\infty}a_n z^n$ ($z \in \mathbb{D}$). Set $h(z) = \sum_{n=0}^{\infty}|a_n|z^n$ ($z \in \mathbb{D}$). Then $h \in \mathcal{D}^2_\alpha$ and $\|f\|^2_{\mathcal{D}^2_\alpha} = \|h\|^2_{\mathcal{D}^2_\alpha}$. Using (23), we see that

$$\begin{aligned}
\|\mathcal{C}_{(\eta)}(f)\|^2_{\mathcal{D}^2_\alpha} &\asymp \sum_{n=0}(n+1)^{1-\alpha}|\eta_n|^2\left|\sum_{k=0}^{n}\alpha_k\right|^2 \\
&\le \sum_{n=0}(n+1)^{1-\alpha}|\eta_n|^2\left(\sum_{k=0}^{n}|a_k|\right)^2 \\
&\le \sum_{n=0}(n+1)^{1-\alpha}(2n+1)^2|\eta_n|^2\left(\sum_{k=0}^{n}\frac{|a_k|}{n+k+1}\right)^2 \\
&\lesssim \sum_{n=0}(n+1)^{1-\alpha}(n+1)^2|\eta_n|^2\left(\sum_{k=0}^{\infty}\frac{|a_k|}{n+k+1}\right)^2 \\
&\lesssim \|\mathcal{H}_{F_\eta}(h)\|^2_{\mathcal{D}^2_\alpha}.
\end{aligned}$$

Since $\mathcal{H}_{F_\eta}$ is bounded on $\mathcal{D}^2_\alpha$ and $\|f\|^2_{\mathcal{D}^2_\alpha} = \|h\|^2_{\mathcal{D}^2_\alpha}$, we obtain that $\|\mathcal{C}_{(\eta)}(f)\|^2_{\mathcal{D}^2_\alpha} \lesssim \|f\|^2_{\mathcal{D}^2_\alpha}$. □

Now we can state the mentioned characterization of the sequences $\{\eta_n\}_{n=0}^{\infty}$ for which $\mathcal{C}_{\{\eta_n\}}$ is bounded on $\mathcal{D}^2_\alpha$ ($0 \le \alpha \le 1$).

Theorem 5 *Let* $(\eta) = \{\eta_n\}_{n=0}^{\infty}$ *be a sequence of complex numbers.*

(A) *If* $0 < \alpha \le 1$, *then the following conditions are equivalent:*

(i) *The operator* $\mathcal{C}_{(\eta)}$ *is bounded on* $\mathcal{D}^2_\alpha$.

(ii) $\sum_{n=1}^{N} n^2|\eta_n|^2 = \mathrm{O}(N)$.

(iii) *The power series* $\sum_{n=0}^{\infty} \eta_n z^n$ *defines a function* F_η *which is analytic analytic in* $\mathbb{D}$ *and belongs to the mean Lipschitz space* $\Lambda^2_{1/2}$.

(B) *The following conditions are equivalent:*

(i) *The operator* $\mathcal{C}_{(\eta)}$ *is bounded on* $\mathcal{D} = \mathcal{D}^2_0$.

(ii) $\sum_{n=N}^{\infty} n|\eta_n|^2 = \mathrm{O}\left(\frac{1}{\log(N+1)}\right)$.

Of course, (B) is (21), while (A) for $\alpha = 1$ is (22).

The equivalence (ii) $\Leftrightarrow$ (iii) in (A) is well known (see Theorem A of [19]). The implication (iii) $\Rightarrow$ (i) in (A) follows using Theorem A and Theorem 4.

The implication (ii) $\Rightarrow$ (i) in (B) follows using Theorem 1 and Theorem 4.

Proof of the implication (i) ⇒ (ii) in (A) Assume $0 < \alpha \le 1$ and (i). For $0 < b < 1$, set

$$f_b(z) = \frac{(1-b)^{\frac{1-\alpha}{2}}}{(1-bz)^{1/2}}, \quad z \in \mathbb{D}.$$

We have $f_b(z) = (1-b)^{\frac{1-\alpha}{2}} \sum_{n=0}^{\infty} a_{n,b} z^n$ $(z \in \mathbb{D})$ with $a_{n,b} \asymp (n+1)^{-1/2} b^n$ and then it follows readily that

$$\|f_b\|^2_{\mathcal{D}^2_\alpha} \asymp 1. \tag{24}$$

Now, for every $N \ge 2$,

$$\begin{aligned}
\|\mathcal{C}_{(\eta)}(f_b)\|^2_{\mathcal{D}^2_\alpha} &\gtrsim (1-b)^{1-\alpha} \sum_{n=1}^{\infty} (n+1)^{1-\alpha} |\eta_n|^2 \left(\sum_{k=1}^{n} \frac{b^k}{k^{1/2}} \right)^2 \\
&\gtrsim (1-b)^{1-\alpha} \sum_{n=1}^{N} (n+1)^{1-\alpha} |\eta_n|^2 \left(\sum_{k=1}^{n} \frac{1}{k^{1/2}} \right)^2 \\
&\gtrsim (1-b)^{1-\alpha} \sum_{n=1}^{N} (n+1)^{2-\alpha} |\eta_n|^2 \\
&\gtrsim (1-b)^{1-\alpha} N^{-\alpha} \sum_{n=1}^{N} (n+1)^2 |\eta_n|^2.
\end{aligned}$$

Taking $b = b_N = 1 - \frac{1}{N}$, we obtain $\|\mathcal{C}_{(\eta)}(f_{b_N})\|^2_{\mathcal{D}^2_\alpha} \gtrsim \frac{1}{N} \sum_{n=1}^{N} (n+1)^2 |\eta_n|^2$. This, (i), and (28) give (ii). □

Proof of the implication (i) ⇒ (ii) in (B) Assume that $\mathcal{C}_{(\eta)}$ is bounded on $\mathcal{D}$. As in the proof of Theorem 1, for $0 < a < 1$ set

$$f_a(z) = \left(\log \frac{1}{1-a}\right)^{-1/2} \log \frac{1}{1-az} = \left(\log \frac{1}{1-a}\right)^{-1/2} \sum_{k=1}^{\infty} \frac{a^k}{k} z^k, \quad z \in \mathbb{D}.$$

Then $f_a \in \mathcal{D}$, for all $a \in (0,1)$, and $\| f_a \|_{\mathcal{D}}^2 \asymp 1$. For every $N \geq 2$, we have

$$\|\mathcal{C}_{(\eta)}(f_a)\|_{\mathcal{D}}^2 \gtrsim \left(\log \frac{1}{1-a}\right)^{-1} \sum_{n=1}^{\infty} (n+1)|\eta_n|^2 \left(\sum_{k=1}^{n} \frac{a^k}{k}\right)^2$$
$$\gtrsim \left(\log \frac{1}{1-a}\right)^{-1} \sum_{n=N}^{\infty} (n+1)|\eta_n|^2 \left(\sum_{k=1}^{N} \frac{a^k}{k}\right)^2 .$$

Taking $a = a_N = 1 - \frac{1}{N}$, we obtain

$$\|\mathcal{C}_{(\eta)}(f_{a_N})\|_{\mathcal{D}}^2 \gtrsim (\log N)^{-1} \sum_{n=N}^{\infty} (n+1)|\eta_n|^2 \left(\sum_{k=1}^{N} \frac{1}{k}\right)^2 \asymp (\log N) \sum_{n=N}^{\infty} (n+1)|\eta_n|^2.$$

Using (i) and the fact that $\| f_{a_N} \|_{\mathcal{D}}^2 \asymp 1$, (ii) follows. □

Using arguments similar to those used to prove Theorem 4 and Theorem 5 we can prove the corresponding results regarding compactness.

Theorem 6 *Let $\{\eta_n\}_{n=0}^{\infty}$ be a sequence of complex numbers such that the power series $\sum_{n=0}^{\infty} \eta_n z^n$ defines an analytic function in $\mathbb{D}$. Set*

$$F_\eta(z) = \sum_{n=0}^{\infty} \eta_n z^n, \quad z \in \mathbb{D}.$$

Suppose that $0 \leq \alpha \leq 1$. If the operator $\mathcal{H}_{F_\eta}$ is compact from $\mathcal{D}_\alpha^2$ into itself, then the operator $\mathcal{C}_{\{\eta_n\}}$ is also compact from $\mathcal{D}_\alpha^2$ into itself.

Theorem 7 *Let $(\eta) = \{\eta_n\}_{n=0}^{\infty}$ be a sequence of complex numbers.*

(A) *If $0 < \alpha \leq 1$, then the following conditions are equivalent:*

- (i) *The operator $\mathcal{C}_{(\eta)}$ is compact from $\mathcal{D}_\alpha^2$ into itself.*
- (ii) $\sum_{n=1}^{N} n^2 |\eta_n|^2 = \mathrm{o}(N)$.
- (iii) *The power series $\sum_{n=0}^{\infty} \eta_n z^n$ defines a function F_η which is analytic analytic in $\mathbb{D}$ and belongs to the space $\lambda_{1/2}^2$.*

(B) *The following conditions are equivalent:*

- (i) *The operator $\mathcal{C}_{(\eta)}$ is bounded from $\mathcal{D} = \mathcal{D}_0^2$ into itself.*
- (ii) $\sum_{n=N}^{\infty} n|\eta_n|^2 = \mathrm{o}\left(\frac{1}{\log(N+1)}\right)$.

We omit the details of the proofs.

Next we obtain a characterization of the sequences $(\eta) = \{\eta_n\}_{n=0}^{\infty}$ such that the operator $\mathcal{C}_{(\eta)}$ is a Hilbert-Schmidt operator on $\mathcal{D}_\alpha^2$ $(0 \leq \alpha \leq 1)$.

Theorem 8 *Let $(\eta) = \{\eta_n\}_{n=0}^{\infty}$ be a sequence of complex numbers.*

- (i) *If $0 < \alpha \leq 1$, then the operator $\mathcal{C}_{(\eta)}$ is Hilbert-Schmidt on $\mathcal{D}_\alpha^2$ if and only if $\sum_{n=1}^{\infty} n|\eta_n|^2 < \infty$.*
- (ii) *The operator $\mathcal{C}_{(\eta)}$ is Hilbert-Schmidt on $\mathcal{D}$ if and only if $\sum_{n=1}^{\infty} n|\eta_n|^2 \log n < \infty$.*

Proof Suppose that $0 \le \alpha \le 1$. Set $e_0(z) = 1$ $(z \in \mathbb{D})$ and

$$e_n(z) = d_n z^n, \quad z \in \mathbb{D}, \quad n = 1, 2, \ldots,$$

where,

$$d_n = \frac{1}{n}\sqrt{\frac{\Gamma(n-1+2+\alpha)}{(n-1)!\Gamma(2+\alpha)}}, \quad n = 1, 2, \ldots.$$

Then $\{e_n : n = 0, 1, 2, \ldots\}$ is an orthonormal basis of $\mathcal{D}^2_\alpha$. Notice that

$$d_n \asymp (n+1)^{\frac{\alpha-1}{2}}. \tag{25}$$

We have

$$\mathcal{C}_{(\eta)}(e_n) = d_n \sum_{k=n}^{\infty} \eta_k z^k, \quad z \in \mathbb{D}.$$

Then, using (25), we obtain

$$\|\mathcal{C}_{(\eta)}(e_n)\|^2_{\mathcal{D}^2_\alpha} \asymp (n+1)^{\alpha-1} \sum_{k=n}^{\infty} (k+1)^{1-\alpha} |\eta_k|^2.$$

This implies that

$$\sum_{n=0}^{\infty} \|\mathcal{C}_{(\eta)}(e_n)\|^2_{\mathcal{D}^2_\alpha} \asymp \sum_{n=0}^{\infty} (n+1)^{\alpha-1} \sum_{k=n}^{\infty} (k+1)^{1-\alpha} |\eta_k|^2$$
$$= \sum_{k=0}^{\infty} (k+1)^{1-\alpha} |\eta_k|^2 \sum_{n=0}^{k} (n+1)^{\alpha-1}.$$

If $0 < \alpha \le 1$, then $\sum_{n=0}^{k} (n+1)^{\alpha-1} \asymp (k+1)^\alpha$ and then

$$\sum_{n=0}^{\infty} \|\mathcal{C}_{(\eta)}(e_n)\|^2_{\mathcal{D}^2_\alpha} \asymp \sum_{k=0}^{\infty} (k+1)|\eta_k|^2.$$

This proves (i). If $\alpha = 0$, then $\sum_{n=0}^{k} (n+1)^{\alpha-1} \asymp \log(k+1)$ and then

$$\sum_{n=0}^{\infty} \|\mathcal{C}_{(\eta)}(e_n)\|^2_{\mathcal{D}^2_0} \asymp \sum_{k=0}^{\infty} (k+1)|\eta_k|^2 \log(k+1).$$

This proves (ii). □

4 The Cesàro type operators $\mathcal{C}_\mu$ acting on the Bergman spaces A^1_α $(\alpha > -1)$.

As we mentioned above, the work [15] includes a complete characterization of the finite positive Borel measure concentrated in [0, 1) for which $\mathcal{C}_\mu$ is bounded on A^p_α $(1 < p < \infty$, $\alpha > -1)$. Indeed, the following is Theorem 6 of [15].

Theorem C *Suppose that* $1 < p < \infty$ *and* $\alpha > -1$. *Let* μ *be a positive finite Borel measure on* $[0, 1)$. *Then the following conditions are equivalent.*

(i) *The measure μ is a Carleson measure, that is, there exists a positive constant C such that*

$$\mu([t,1)) \le C(1-t), \quad 0<t<1.$$

(ii) *The operator $\mathcal{C}_\mu$ is bounded from A^p_α into itself.*

Next we extend this result characterizing the positive finite Borel measures μ on $[0,1)$ for which $\mathcal{C}_\mu$ is bounded on A^1_α ($\alpha > -1$).

Theorem 9 *Suppose that $\alpha > -1$. Let μ be a positive finite Borel measure on* $[0,1)$. *Then the following conditions are equivalent.*

(i) *The measure μ is a Carleson measure, that is, there exists a positive constant C such that*

$$\mu([t,1)) \le C(1-t), \quad 0<t<1.$$

(ii) *The operator $\mathcal{C}_\mu$ is bounded from A^1_α into itself.*

Proof Suppose (i). Take $\beta > \alpha$. Using Theorem 4.24 of [43], we see that, for $f \in A^1_\alpha$, we have

$$f(z) = (\beta+1)\int_{\mathbb{D}} \frac{(1-|w|^2)^\beta f(w)}{(1-z\overline{w})^{2+\beta}}\, dA(w), \quad z\in\mathbb{D} \tag{26}$$

and then

$$\begin{aligned}
\|\mathcal{C}_\mu(f)\|_{A^1_\alpha} &\asymp \int_{\mathbb{D}} \left|\int_{[0,1)} \frac{f(tz)}{1-tz}\, d\mu(t)\right| (1-|z|^2)^\alpha\, dA(z)\\
&\le \int_{\mathbb{D}} (1-|z|^2)^\alpha \left[\int_{[0,1)} \left(\int_{\mathbb{D}} \frac{|f(w)|(1-|w|^2)^\beta}{|1-\overline{w}tz|^{2+\beta}|1-tz|}\, dA(w)\right) d\mu(t)\right] dA(z)\\
&\le \int_{\mathbb{D}} (1-|w|^2)^\beta |f(w)| \left[\int_{[0,1)} \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-\overline{w}tz|^{2+\beta}|1-tz|}\, dA(z)\right) d\mu(t)\right] dA(w).
\end{aligned}$$

Using Lemma 4.1 (vi) of [42] we deduce that

$$\begin{aligned}
\|\mathcal{C}_\mu(f)\|_{A^1_\alpha} &\lesssim \int_{\mathbb{D}} (1-|w|^2)^\beta |f(w)| \left(\int_{[0,1)} \frac{d\mu(t)}{(1-|w|t)^{\beta-\alpha}|1-tw|}\right) dA(w)\\
&\le \int_{\mathbb{D}} (1-|w|^2)^\beta |f(w)| \left(\int_{[0,1)} \frac{d\mu(t)}{(1-|w|t)^{\beta-\alpha+1}}\right) dA(w).
\end{aligned} \tag{27}$$

Integrating by parts, using the fact that μ is a 1-Carleson measure, and the facts that $|w|t \le t$ and $|w|t \le |w|$, we obtain

$$\begin{aligned}
\int_{[0,1)} \frac{d\mu(t)}{(1-|w|t)^{\beta-\alpha+1}} &\lesssim \int_0^1 \frac{1-t}{(1-|w|t)^{\beta-\alpha+2}} dt\\
&= \int_0^{|w|} \frac{1-t}{(1-|w|t)^{\beta-\alpha+2}} dt + \int_{|w|}^1 \frac{1-t}{(1-|w|t)^{\beta-\alpha+2}} dt\\
&\le \int_0^{|w|} \frac{1}{(1-t)^{\beta-\alpha+1}} dt + \frac{1}{(1-|w|)^{\beta-\alpha+2}} \int_{|w|}^1 (1-t) dt\\
&\lesssim \frac{1}{(1-|w|)^{\beta-\alpha}}.
\end{aligned}$$

This and (27) give that $\|\mathcal{C}_\mu(f)\|_{A^1_\alpha} \lesssim \|f\|_{A^1_\alpha}$. Thus, (ii) follows.

Suppose now that $\mathcal{C}_\mu$ is bounded on A^1_α. Take $A > 2$. For $0 < b < 1$, set

$$f_b(z) = \frac{(1-b)^{A-2}}{(1-bz)^{A+\alpha}} = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad z \in \mathbb{D}.$$

Using Lemma 3. 10 of [43], we see that $f_b \in A^1_\alpha$ for all $b \in (0, 1)$ and that

$$\|f_b\|_{A^1_\alpha} \asymp 1. \tag{28}$$

Also,

$$a_{k,b} \asymp (1-b)^{A-2} k^{A+\alpha-1} b^k. \tag{29}$$

Now, since

$$\mathcal{C}_\mu(f_b)(z) = \sum_{n=0}^{\infty} \mu_n \left(\sum_{k=0}^{n} a_{k,b} \right) z^n, \quad z \in \mathbb{D},$$

using the definitions, the fact that the sequence $\{\mu_n\}$ is decreasing, Theorem 2 of [32] (see also Theorem 8. 4. 1 (iii) of [26]), (28), and the fact that $\mathcal{C}_\mu$ is bounded on A^1_α, we deduce that, for every $N \geq 1$, we have

$$\begin{aligned} (1-b)^{A-2} \mu_N \sum_{n=1}^{N} n^{-\alpha-2} \left(\sum_{k=0}^{n} k^{A+\alpha-1} b^k \right) &\lesssim \sum_{n=1}^{N} n^{-\alpha-2} \mu_n \left(\sum_{k=0}^{n} a_{k,b} \right) \\ \lesssim \sum_{n=1}^{\infty} n^{-\alpha-2} \mu_n \left(\sum_{k=0}^{n} a_{k,b} \right) \lesssim \|\mathcal{C}_\mu(f_b)\|_{A^1_\alpha} &\lesssim \|f_b\|_{A^1_\alpha} \lesssim 1. \end{aligned}$$

Taking $b = 1 - \frac{1}{N}$, we obtain $\mu_N \lesssim \frac{1}{N}$ and it is well known that this is equivalent to saying that μ is a Carleson measure (see Proposition 8 of [7]). Thus, (i) holds. □

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