International Journal of Approximate Reasoning 172 (2024) 109243

Contents lists available at ScienceDirect

International Journal of Approximate Reasoning

journal homepage: www.elsevier.com/locate/ijar

Aggregation of fuzzy graphs

Francisco Javier Talavera [a,b], Carlos Bejines [c], Sergio Ardanza-Trevijano [a,b], Jorge Elorza [a,b,*]

[a] Universidad de Navarra, Departamento de Física y Matemática Aplicada, Facultad de Ciencias, C. Irunlarrea 1, 31008, Pamplona, Spain
[b] Instituto de Ciencia de Datos e Inteligencia Artificial, Universidad de Navarra, Spain
[c] Universidad de Málaga, Andalucía Tech, 29071, Málaga, Spain

ARTICLE INFO

Keywords:
Aggregation operator
Fuzzy graph
Algebraic structure
Aggregation of fuzzy graphs
Complete fuzzy graphs

ABSTRACT

Our study is centered on the aggregation of fuzzy graphs, looking for conditions under which the aggregation process yields another fuzzy graph. We conduct an in-depth analysis of the preservation of several important properties and structures inherent to fuzzy graphs, like paths, cycles, or bridges. In addition we obtain appropriate criteria for when the aggregation of complete fuzzy graphs is again a complete fuzzy graph.

1. Introduction

In many fields such as decision-making, image processing, classification, robotics, and control, the significance of aggregation functions is widely acknowledged. In these applications, it is important to preserve the inherent structure of the data when combining it using these operators. Consequently, in recent years, there has been a notable research trend focused on investigating the preservation of properties within fuzzy algebraic structures under aggregation operators. This framework has gathered wide attention and is continuously evolving, as indicated by the increasing number of papers addressing various aspects. Noteworthy topics include the aggregation of fuzzy subgroups [1–4], fuzzy relations [5], particularly indistinguishability operators [6], fuzzy implications [7], fuzzy metrics [8,9], fuzzy vector subspaces [10] and other types of algebraic structures [11–14].

Our research focuses on the aggregation of fuzzy graphs. This concept was introduced as an extension of the conventional graph notion to be used in contexts where there is vagueness or uncertainty in the objects and relations between the objects described by a graph. Fuzzy graph theory was first described in 1975 independently by Rosenfeld [15] and Yeh and Bah [16] where a first application to clustering was given. Since then a large number of publications have dealt with fuzzy graphs. A non-comprehensive list of foundational works by Bhattacharya, Bhutani, Mathew, Mordeson, Sunitha and Vijayakumar can be found in [17–34]. Recently, fuzzy graphs have been successfully used in a wide range of applications such as image processing, decision making or traffic light control (see for example [35–38]). In addition, Khamenehand and Kilicman defined in [39] the aggregation of m-polar fuzzy graphs and gave an application to modeling product manufacturing. All these antecedents, provide great perspectives on the applicability of the results obtained in the following study.

Furthermore, different types of graphs have been used to expand the field of Formal Concept Analysis (FCA) introduced by Wille in [40]. A formal context is a tuple (G, M, I) where G and M are sets whose elements are called objects and attributes of the formal context, respectively. $I : G \times M \longrightarrow \{0, 1\}$ is a relation where $I(g, m) = 1$ indicates that the object g possesses the attribute m. This

* Corresponding author.
E-mail address: jelorza@unav.es (J. Elorza).

https://doi.org/10.1016/j.ijar.2024.109243
Received 6 March 2024; Received in revised form 23 May 2024; Accepted 18 June 2024

Available online 26 June 2024

notion can be interpreted as a bipartite graph where the vertices are objects and attributes. An edge between an object vertex g and an attribute vertex m means that *the object g satisfies the attribute m*. Several researchers have adopted this perspective (see [41–45]). For example, [46] establishes a correlation between formal concepts and maximal bi-cliques. Notice also that the adjacency matrix of a graph can be interpreted as a formal context [47]. Algorithms for extracting adjacency matrix based on formal concept analysis can be found in [48]. Moreover, the recent article [49] provides a graph-based extension of FCA, and some real cases of study and quantitative experiments are depicted.

FCA was translated to a more general fuzzy framework with the emergence of fuzzy concept lattices in the seminal article [50]. The development of fuzzy concept lattices has extended the possibilities of FCA to new applications (an extensive survey can be found in [51] and a recent computational example in medicine in [52]). In [53], the authors obtained a fuzzy graph representation of fuzzy concept lattices. This representation provided a new efficient and easy-to-handle algorithm for computing fuzzy concepts from a fuzzy context. Additionally, more representations of fuzzy concept lattices have been developed based on more complex structures [54–58]. In particular, fuzzy graphs (see for example [59–61]). In some of the aforementioned works, the connectivity of the graphs is an important feature. Therefore, the study of the preservation of different properties regarding connectivity when aggregating different fuzzy graphs is useful. Also, graph theory has proven to be a valuable resource in various areas of logic programming. In [62], an alternative representation of multi-adjoint logic programs utilizing hypergraphs is introduced.

Our goal is to give a general definition for the aggregation of fuzzy graphs and to study whether the various properties of these fuzzy graphs are preserved or not. Furthermore, we analyze the conditions that an aggregation operator must satisfy to ensure such property preservation. Similar to other algebraic structures (fuzzy subgroups in [63], for instance), each non-empty level set of a fuzzy graph corresponds to a crisp graph, which plays a key role in understanding the nature of fuzzy graphs.

The approach of our article begins with a foundational section that outlines key concepts. In Section 3, we define the aggregation of fuzzy graphs and provide a powerful result: every aggregation operator preserves the structure of a fuzzy graph. Section 4 examines how the intricate structures of a graph persist through the aggregation process. Notably, some conditions for an aggregation operator to preserve common paths, cycles, or fuzzy bridges are given. In Section 5, our focus shifts to the aggregation of complete fuzzy graphs. Theorem 5.6 describes the conditions under which complete fuzzy graphs transform into another complete fuzzy graph during the aggregation process. Additionally, Theorem 5.8 outlines the criteria for a fixed aggregation operator to maintain the completeness of a set of fuzzy graphs when they are aggregated. The paper ends with a series of concluding remarks.

2. Preliminaries

As a foundational reading for the context of fuzzy graphs we suggest the book [64]. In this section, we present basic concepts that are essential for understanding the article. Let us consider a fixed set of crisp vertices denoted by V. We can define a binary equivalence relation $\sim$ between elements $(x_1, y_1), (x_2, y_2) \in V \times V$ as follows:

$$(x_1, y_1) \sim (x_2, y_2) \text{ if and only if } (x_1, y_1) = (x_2, y_2) \text{ or } (x_1, y_1) = (y_2, x_2).$$

This implies that the equivalence class of $(x, y) \in V \times V$ with respect to $\sim$ would be:

$$[(x, y)] = \{(x, y), (y, x)\}$$

and consequently, we define $\mathcal{E}_V = \{[(x, y)] \mid x, y \in V\}$. Thus, any pair (V, E) where $E \subseteq \mathcal{E}_V$ and $x \neq y$ for all $[(x, y)] \in E$ represents a crisp graph.

Remark 2.1. For each $x, y \in V$ we use xy to denote $[(x, y)]$. Hence $xy = yx$.

Fuzzy graphs were introduced by Rosenfeld making use of the concept of fuzzy set established by Zadeh in [65]. We recall that $x \wedge y$ denotes the minimum of x and y.

Definition 2.2 *([15])*. A fuzzy graph is a triple $G = (V, \sigma, \mu)$ where $V \neq \emptyset$, $\sigma : V \to [0, 1]$ and $\mu : \mathcal{E}_V \to [0, 1]$ such that for all $x, y \in V$:

$$\mu(xy) \leq \sigma(x) \wedge \sigma(y)$$

and $\mu(xy) = 0$ if $x = y$. The fuzzy set σ is called the fuzzy vertex set and μ is the fuzzy edge set of G.

In this paper, we only consider V as a finite set. When the context is clear and there is no possibility of confusion, we will use the simplified notation (σ, μ) to denote a fuzzy graph $G = (V, \sigma, \mu)$. There are some important concepts in fuzzy graph theory that we need to keep in mind.

Definition 2.3 *([64])*. Let $G = (V, \sigma, \mu)$ be a fuzzy graph. The fuzzy graph $H = (V, \tau, \nu)$ is called a partial fuzzy subgraph if $\tau \subseteq \sigma$ and $\nu \subseteq \mu$, that is, if for all $x, y \in V$,

$$\tau(x) \leq \sigma(x) \text{ and } \nu(xy) \leq \mu(xy).$$

Definition 2.4 *([64]).* Let $G = (\sigma, \mu)$ be a fuzzy graph. For each $1 \leq t \leq 1$, we define the level set of vertices $\sigma^t = \{x \in V \mid \sigma(x) \geq t\}$, and the level set of edges $\mu^t = \{e \in \mathcal{E}_V \mid \mu(e) \geq t\}$.

Definition 2.5 *([64]).* Given a fuzzy graph $G = (\sigma, \mu)$, we say that a set of distinct vertices $P = \{v_0, v_1, \dots, v_k\} \subseteq V$ (except possibly v_0 and v_k) is a path in G if $\mu(v_{j-1}v_j) > 0$ for all $1 \leq j \leq k$. It can be denoted as $v_0 v_1 \cdots v_k$ for simplicity. The number k is the length of the path and the expression:

$$\bigwedge_{j=1}^{n} \mu(v_{j-1}v_j)$$

is its strength. We call P a cycle if $v_0 = v_k$ with $k \geq 3$.

Given the strength of a path in a graph (σ, μ), the term $\mu^k(x, y)$ will denote the strength of connectedness between two vertices $x, y \in V$ by paths of length $k \geq 2$. This concept is defined as the maximum of the strengths of all paths of length k connecting x and y. More precisely,

$$\mu^k(x, y) = \bigvee \left\{ \mu(xv_1) \wedge \cdots \wedge \mu(v_{k-1}y) \mid (x, v_1, \dots, v_{k-1}, y) \in V_k^* \right\},$$

where

$$V_k^* = \left\{ (v_0, \dots, v_k) \in V^{k+1} \mid v_i \neq v_j \text{ for all } 0 \leq i < j \leq k \text{ except } v_0 \text{ and } v_k \right\}.$$

Since there is only one possible edge connecting x and y we define $\mu^1(x, y) = \mu(xy)$. As a consequence of the definition, if there is no path of a certain length $k \geq 2$ between x and y, it is clear that $\mu^k(x, y) = 0$. Moreover, if $k > |V|$ we take $\mu^k(x, y) = 0$ for all $x, y \in V$. Hence, the overall strength of connectedness between two vertices $x, y \in V$ can be defined as the maximum strength of all paths between x and y. More precisely, we denote this term as:

$$\mu^\infty(x, y) = \bigvee \left\{ \mu^k(x, y) \mid k \in \mathbb{N} \right\}.$$

Note that $\mu^\infty(x, y)$ reaches the maximum since V is a finite set and $\mu^k(x, y) = 0$ for all $k > |V|$.

Definition 2.6 *([64]).* A fuzzy graph is called a forest if the crisp graph consisting of its nonzero edges and vertices is a forest, i.e. it does not contain cycles.

We can define an important concept regarding connectivity in fuzzy graphs.

Definition 2.7 *([64]).* Let $G = (\sigma, \mu)$ be a fuzzy graph, let $u, v \in V$ be two distinct vertices and let $G' = (\sigma, \mu')$ be the partial fuzzy subgraph of G obtained by deleting the edge uv. That is $\mu'(uv) = 0$ and $\mu'(e) = \mu(e)$ for all $e \in \mathcal{E}_V \setminus \{uv\}$. We call uv a fuzzy bridge if ${\mu'}^\infty(x, y) < \mu^\infty(x, y)$ for some $x, y \in V$.

The next results will be important in the following section.

Theorem 2.8 *([15]). Let (σ, μ) be a fuzzy graph. Then the following statements are equivalent:*

1. *uv is a fuzzy bridge.*
2. *${\mu'}^\infty(u, v) < \mu(uv)$.*
3. *uv is not the weakest edge of any cycle, that is, for each cycle which contains uv, there is an edge ab such that $\mu(uv) > \mu(ab)$.*

Theorem 2.9 *([64]). If uv is a fuzzy bridge of a fuzzy graph (σ, μ), then $\mu^\infty(u, v) = \mu(uv)$.*

The next generalization of the concept of complete crisp graph was first introduced by Bhutani in 1989.

Definition 2.10 *([66]).* A complete fuzzy graph is a fuzzy graph $G = (\sigma, \mu)$ such that

$$\mu(xy) = \sigma(x) \wedge \sigma(y) \text{ for all } x, y \in V.$$

3. Aggregation of fuzzy graphs

Aggregation operators are crucial for combining data. As mentioned in the introductory chapter, they are used in several areas of science and technology. For this reason, the preservation of different properties under this type of functions has been an intense area of research in recent times.

Aggregation functions are defined as follows.

Definition 3.1 *([67]).* Let $A : [0,1]^n \to [0,1]$ be a function. A is called an aggregation operator or aggregation function if:

$A1$. $A(0,\dots,0) = 0$ and $A(1,\dots,1) = 1$. (Boundary conditions)
$A2$. For all $(x_1,\dots,x_n),(y_1,\dots,y_n) \in [0,1]^n$ such that $x_i \le y_i$ for each $1 \le i \le n$, $A(x_1,\dots,x_n) \le A(y_1,\dots,y_n)$ (Monotonicity)

The following families of aggregation operators play an important role in our article. We recall that $(x_1,\dots,x_n) \lneq (y_1,\dots,y_n)$ means that there exists i_0 such that $x_{i_0} \ne y_{i_0}$ and that $x_i \le y_i$ for all $1 \le i \le n$.

Definition 3.2 *([68]).* Let $A : [0,1]^n \to [0,1]$ be an aggregation function. We say that A is strictly monotone if assuming $(x_1,\dots,x_n) \lneq (y_1,\dots,y_n)$, then $A(x_1,\dots,x_n) < A(y_1,\dots,y_n)$. A is jointly strictly monotone if assuming $x_i < y_i$ for all $i \in \{1,\dots,n\}$, then $A(x_1,\dots,x_n) < A(y_1,\dots,y_n)$.

Aggregation operators can be used to merge multiple fuzzy graphs into a single one, as defined below.

Definition 3.3. Let $A : [0,1]^n \to [0,1]$ be an aggregation operator and $V \ne \emptyset$. Given n fuzzy graphs $\{(\sigma_i,\mu_i) \mid 1 \le i \le n\}$ on V, we denote by $\boldsymbol{\sigma}$ the map $\boldsymbol{\sigma} : V \to [0,1]^n$ such that:

$$\boldsymbol{\sigma}(x) = (\sigma_1(x),\dots,\sigma_n(x))$$

for each $x \in V$ and by $\boldsymbol{\mu}$ the map $\boldsymbol{\mu} : \mathcal{E}_V \to [0,1]^n$ such that:

$$\boldsymbol{\mu}(xy) = (\mu_1(xy),\dots,\mu_n(xy))$$

for each $xy \in \mathcal{E}_V$. We define the aggregation of fuzzy graphs as the pair $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ where:

$$(A\circ\boldsymbol{\sigma})(x) = A(\sigma_1(x),\dots,\sigma_n(x)) \text{ for each } x \in V \text{ and}$$

$$(A\circ\boldsymbol{\mu})(xy) = A(\mu_1(xy),\dots,\mu_n(xy)) \text{ for each } xy \in \mathcal{E}_V.$$

We say that A preserves fuzzy graphs if $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ is a fuzzy graph for any $\boldsymbol{\sigma},\boldsymbol{\mu}$ as above. That is:

$$(A\circ\boldsymbol{\mu})(xy) \le (A\circ\boldsymbol{\sigma})(x) \wedge (A\circ\boldsymbol{\sigma})(y)$$

for all $x, y \in V$.

The domination relation between aggregation operators is a fundamental concept in some aspects of fuzzy set theory. It characterizes the preservation of some fuzzy structures under aggregation as, for example, T-indistinguishability operators [6] or T-subgroups [2,4].

Definition 3.4 *([69]).* Consider two aggregation operators $A : [0,1]^n \to [0,1]$ and $B : [0,1]^m \to [0,1]$. We say that A dominates B if for all $x_{i,j} \in [0,1]$ with $i \in \{1,\dots,m\}$ and $j \in \{1,\dots,n\}$, the following property holds:

$$A(B(x_{1,1},\dots,x_{m,1}),\dots,B(x_{1,n},\dots,x_{m,n})) \\ \ge B(A(x_{1,1},\dots,x_{1,n}),\dots,A(x_{m,1},\dots,x_{m,n})). \quad (1)$$

We denote it by $A \gg B$.

If we substitute A for the minimum $\wedge$ in the previous definition, we obtain an interesting fact: any aggregation operator satisfies inequality (1). This is shown in the next proposition.

Proposition 3.5. *For any aggregation operator $A : [0,1]^n \to [0,1]$ we have that $\wedge \gg A$.*

Proof. Given $(x_1,\dots,x_n),(y_1,\dots,y_n) \in [0,1]^n$, by the monotonicity of A:

$$A(x_1,\dots,x_n) \ge A(x_1 \wedge y_1,\dots,x_n \wedge y_n)$$

and

$$A(y_1,\dots,y_n) \ge A(x_1 \wedge y_1,\dots,x_n \wedge y_n).$$

Hence $A(x_1,\dots,x_n) \wedge A(y_1,\dots,y_n) \ge A(x_1 \wedge y_1,\dots,x_n \wedge y_n)$. □

Using the property obtained in the previous result we can ensure that any pair $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ defined as in Definition 3.3, is a fuzzy graph.

Theorem 3.6. *Any aggregation operator* $A : [0,1]^n \to [0,1]$ *preserves fuzzy graphs.*

Proof. Let $\{(\sigma_i, \mu_i) \mid 1 \le i \le n\}$ be a set of fuzzy graphs on V. Thus, $\mu_i(xy) \le \sigma_i(x) \wedge \sigma_i(y)$ for all $i \in \{1, \ldots, n\}$ and $x, y \in V$. This fact, together with Proposition 3.5 lead us to the following chain of inequalities for each $x, y \in V$:

$$\begin{aligned}(A\circ\boldsymbol{\mu})(xy) &= A(\mu_1(xy), \ldots, \mu_n(xy))\\ &\le A(\sigma_1(x) \wedge \sigma_1(y), \ldots, \sigma_n(x) \wedge \sigma_n(y))\\ &\le A(\sigma_1(x), \ldots, \sigma_n(x)) \wedge A(\sigma_1(y), \ldots, \sigma_n(y))\\ &= (A\circ\boldsymbol{\sigma})(x) \wedge (A\circ\boldsymbol{\sigma})(y).\end{aligned}$$

Consequently, $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ is a fuzzy graph. □

There are specific types of graphs that can be utilized in different situations. We introduce bipartite fuzzy graphs whose preservation under aggregation is straightforward.

Definition 3.7 *([70]).* A bipartite fuzzy graph is a fuzzy graph $G = (V, \sigma, \mu)$ where V can be partitioned into two nonempty sets V_1 and V_2 such that $\mu(xy) = 0$ if $x, y \in V_1$ or $x, y \in V_2$.

From Theorem 3.6 and due to the boundary conditions of aggregation operators, the next result is clear.

Corollary 3.8. *Any aggregation operator preserves bipartite fuzzy graphs defined over the same set* $V = V_1 \cup V_2$ *with* $V_1 \cap V_2 = \emptyset$.

Since this kind of graph can help to visualize and manage fuzzy contexts, we will use them to state a possible application for the aggregation of fuzzy graphs. The following example shows how to aggregate different fuzzy contexts when they are seen as bipartite fuzzy graphs.

Example 3.9. A fuzzy $[0,1]$-context is formed by a triple (X, Y, R) where X is a set of objects, Y is a set of attributes, and $R : X \times Y \to [0,1]$ is a fuzzy relation between the objects and the attributes. Notice that we can identify a fuzzy context with a bipartite fuzzy graph defining $G = (X \cup Y, \sigma, \mu)$ where σ is the membership function of $X \cup Y$ and:

$$\mu(xy) = \begin{cases} R(x,y) & \text{if } x \in X \text{ and } y \in Y,\\ R(y,x) & \text{if } y \in X \text{ and } x \in Y,\\ 0 & \text{otherwise.}\end{cases}$$

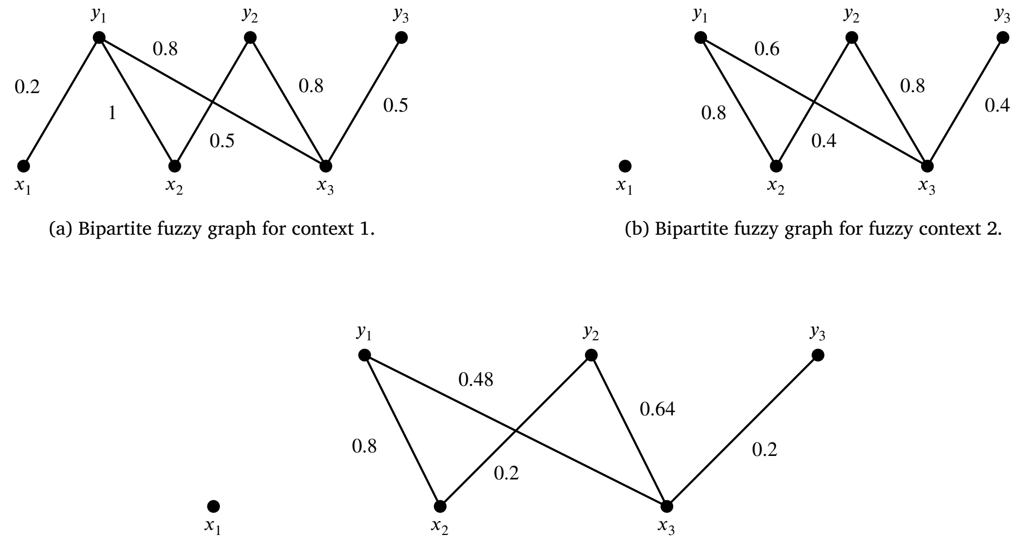
Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and R_1 and R_2 two fuzzy relations conforming two fuzzy contexts. Let us suppose that for each fuzzy relation we have the fuzzy graphs whose fuzzy edge sets are, respectively:

$$\mu_1(e) = \begin{cases} 1 & \text{if } e = x_2y_1,\\ 0.8 & \text{if } e \in \{x_3y_1, x_3y_2\},\\ 0.5 & \text{if } e \in \{x_2y_2, x_3y_3\},\\ 0.2 & \text{if } e \in \{x_1y_1\},\\ 0 & \text{otherwise.}\end{cases} \qquad \mu_2(e) = \begin{cases} 0.8 & \text{if } e \in \{x_2y_1, x_3y_2\}\\ 0.6 & \text{if } e = x_3y_1,\\ 0.4 & \text{if } e \in \{x_2y_2, x_3y_3\},\\ 0 & \text{otherwise.}\end{cases}$$

These graphs (σ, μ_1) and (σ, μ_2) are depicted in Figs. 1a and 1b. If we use the product of two elements T_P to fuse both graphs we obtain the fuzzy graph of Fig. 1c whose edge set is:

$$(T_P\circ\boldsymbol{\mu})(e) = \begin{cases} 0.8 & \text{if } e = x_2y_1,\\ 0.64 & \text{if } e = x_3y_2,\\ 0.48 & \text{if } e = x_3y_1,\\ 0.2 & \text{if } e \in \{x_2y_2, x_3y_3\},\\ 0 & \text{otherwise.}\end{cases}$$

Therefore, when different sources provide different fuzzy contexts of the same situation, the aggregation can be done considering that they are fuzzy graphs. Consequently, it is interesting to study which properties of these graphs are preserved when they are combined by arbitrary aggregation operators since the connections of the graphs provide information about the relation between objects and attributes.



(a) Bipartite fuzzy graph for context 1.

(b) Bipartite fuzzy graph for fuzzy context 2.

(c) Aggregated bipartite fuzzy graph.

Fig. 1. Example of the aggregation of two fuzzy contexts.

4. Preserving fuzzy graphs properties and structures under aggregation

In this section, we analyze the preservation of some important objects associated with fuzzy graphs during the aggregation process. We identify conditions that an aggregation operator must satisfy to ensure the preservation of a particular property. When discussing the preservation of a property, we refer to a property that is satisfied by all graphs $(\sigma_1, \mu_1), \dots, (\sigma_n, \mu_n)$ to be aggregated without exception. An aggregation operator is said to preserve such a property if $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ satisfies this same property. For instance, we say that A preserves paths if for any collection of n arbitrary fuzzy graphs $(\sigma_1, \mu_1), \dots, (\sigma_n, \mu_n)$ on a set V, all their common paths are also paths in $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$.

Proposition 4.1. *Let $A : [0,1]^n \to [0,1]$ be an aggregation operator. The next statements are equivalent:*

1. *$A(x_1, \dots, x_n) > 0$ for all $x_1, \dots, x_n \in (0,1]$.*
2. *A preserves paths.*
3. *A preserves cycles.*

Proof. 1. ⇒ 2. If $P = \{v_0, \dots, v_k\}$ is a common path to each one of the fuzzy graphs $(\sigma_1, \mu_1), \dots, (\sigma_n, \mu_n)$, then $\mu_i(v_{j-1}v_j) > 0$ for all $1 \le i \le n$ and $0 \le j \le k$. By hypothesis $(A\circ\boldsymbol{\mu})(v_{j-1}v_j) > 0$ for all $j \in \{0, \dots, k\}$ and P is a path in $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$.

2. ⇒ 3. It is straightforward since each cycle is a path by definition.

3. ⇒ 1. Let us set $C = \{v_0, v_1, v_2, v_0\} \subseteq V$ where all the elements are different. For any arbitrary $x_1, \dots, x_n \in (0,1]$, we take some fuzzy graphs $\{(\sigma_i, \mu_i) \mid 1 \le i \le n\}$ such that: $\sigma_i(v_0) = \sigma_i(v_1) = \sigma_i(v_2) = \mu_i(v_0v_1) = \mu_i(v_1v_2) = \mu_i(v_2v_0) = x_i > 0$ for all $i \in \{1, \dots, n\}$. Therefore, C is a cycle in each one of the fuzzy graphs. By hypothesis C is also a cycle in $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$, then:

$$A(x_1, \dots, x_n) = (A\circ\boldsymbol{\mu})(v_0v_1) > 0. \quad \square$$

Remark 4.2. Note that cycles only exist when $|V| \ge 3$. For the case where $|V| = 2$, it is easy to check that the equivalence between 1 and 2 still holds.

Now, let us provide a condition to ensure that all the existing paths in an aggregated graph are paths in each one of the graphs being aggregated.

Theorem 4.3. *Let $A : [0,1]^n \to [0,1]$ be an aggregation operator such that $A\left(x_1, \dots, x_n\right) > 0$ for all $x_1, \dots, x_n \in (0,1]$ and $A\left(x_1, \dots, x_n\right) = 0$ otherwise. For a given set of vertices $P = \left\{v_0, \dots, v_k\right\}$, the following statements are equivalent for a set of fuzzy graphs $\left\{\left(\sigma_i, \mu_i\right) \mid 1 \le i \le n\right\}$:*

1. *P is a path in the graph $\left(\sigma_i, \mu_i\right)$ for all $i \in \{1, \dots, n\}$.*
2. *P is a path in $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$.*

Proof. 1. ⇒ 2. This is a consequence of Proposition 4.1 since $A\left(x_1, \dots, x_n\right) > 0$ for all $x_1, \dots, x_n \in (0,1]$.

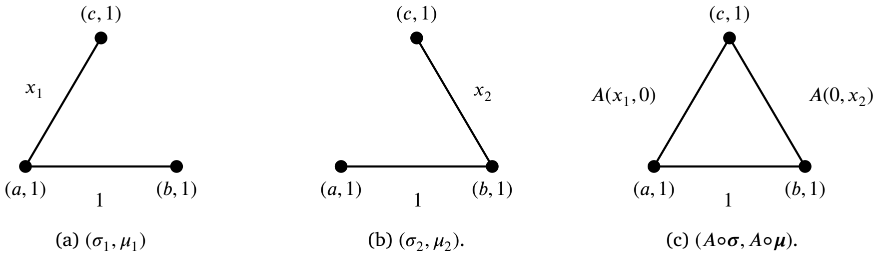


Fig. 2. Representation of (σ_1, μ_1), (σ_2, μ_2) and $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ in the proof of Theorem 4.4.

2. ⇒ 1. Let us suppose that $(A\circ\boldsymbol{\mu})\left(v_{j-1}v_j\right) > 0$ for all $j \in \{1,\dots,k\}$. Hence $A\left(\mu_1\left(v_{j-1}v_j\right),\dots,\mu_n\left(v_{j-1}v_j\right)\right) > 0$ and by hypothesis $\mu_i\left(v_{j-1}v_j\right) > 0$ for each $i \in \{1,\dots,n\}$ and $j \in \{1,\dots,k\}$. In this situation, P is a path in $\left(\sigma_i,\mu_i\right)$. □

The following theorem characterizes the aggregation operators with two arguments that preserve forests.

Theorem 4.4. *Let $A : [0,1]^2 \to [0,1]$ be an aggregation operator that preserves fuzzy graphs on V satisfying $|V| \geq 3$. Then, A preserves forests if and only if either $A(x,0) = 0$ for all $x \in (0,1]$ or $A(0,x) = 0$ for all $x \in (0,1]$.*

Proof. Without loss of generality, we suppose that $A(x,0) = 0$ for all $x \in (0,1]$. Given the forests $(\sigma_1,\mu_1),(\sigma_2,\mu_2)$, by reductio ad absurdum, assume that $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ is not a forest. Therefore, it contains a cycle $C = \{v_0,\dots,v_k\}$ with $v_0 = v_k$. In this case:

$$A(\mu_1(v_{j-1}v_j),\mu_2(v_{j-1}v_j)) = (A\circ\boldsymbol{\mu})(v_{j-1}v_j) > 0 \text{ for all } 1 \leq j \leq k.$$

Now, by hypothesis $\mu_2(v_{j-1}v_j) > 0$ for $1 \leq j \leq k$. Hence, C is a cycle in the fuzzy graph (σ_2,μ_2) which is a contradiction with the fact that it is a forest.

Conversely, if there exist $x_1, x_2 \in (0,1]$ such that $A(x_1,0) \neq 0$ and $A(0,x_2) \neq 0$, we define the following forests $(\sigma_1,\mu_1),(\sigma_2,\mu_2)$ whose support is a set of three different vertices $\{a,b,c\}$:

$$\sigma_1(v) = \sigma_2(v) = \begin{cases} 1 & \text{if } v \in \{a,b,c\}, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mu_1(e) = \begin{cases} 1 & \text{if } e = ab, \\ x_1 & \text{if } e = ac, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad \mu_2(e) = \begin{cases} 1 & \text{if } e = ab, \\ x_2 & \text{if } e = bc, \\ 0 & \text{otherwise.} \end{cases}$$

The aggregation of these two fuzzy graphs is given by $A\circ\boldsymbol{\sigma} = \sigma_1 = \sigma_2$ and

$$(A\circ\boldsymbol{\mu})(e) = \begin{cases} 1 & \text{if } e = ab, \\ A(x_1,0) & \text{if } e = ac, \\ A(0,x_2) & \text{if } e = bc, \\ 0 & \text{otherwise.} \end{cases}$$

This is straightforward checking the graphs described in Fig. 2. Note that $\{a,b,c\}$ is a cycle on $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$. Therefore, A does not preserve forests and we obtain a contradiction. □

We now extend the previous result to aggregations of an arbitrary number of fuzzy graphs.

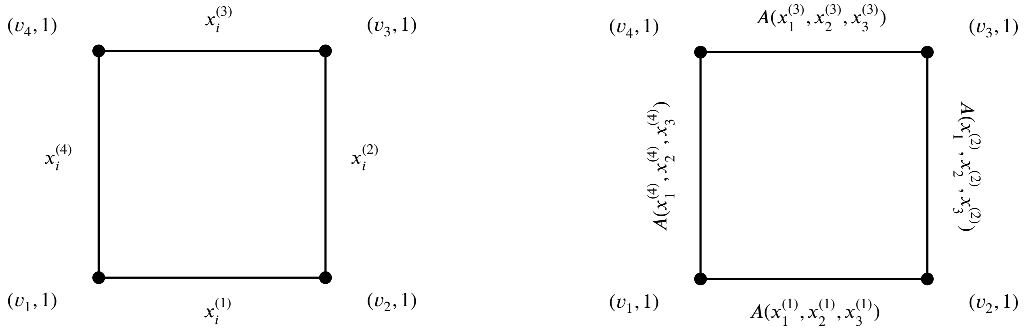
Theorem 4.5. *Let $A : [0,1]^n \to [0,1]$ be an aggregation operator that preserves fuzzy graphs on V. If there exists $l \in \{1,\dots,n\}$ such that $A(x_1,\dots,x_n) = 0$ for all $(x_1,\dots,x_n) \in [0,1]^n$ with $x_l = 0$, then A preserves forests. Moreover, when $|V| \geq n \geq 3$ we have an equivalence.*

Proof. Notice that the case $n = 2$ is exactly the previous theorem. Therefore, we study the general case $n > 2$. Let us assume that there is $l \in \{1,\dots,n\}$ such that $A(x_1,\dots,x_n) = 0$ for all $(x_1,\dots,x_n) \in [0,1]^n$ satisfying the condition $x_l = 0$. Given the forests $(\sigma_1,\mu_1),\dots,(\sigma_n,\mu_n)$, by reductio ad absurdum, assume that $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ is not a forest. Therefore, it contains a cycle $C = \{v_0,\dots,v_k\}$ with $v_0 = v_k$ for $k \geq 3$. In that case:

$$A(\mu_1(v_{j-1}v_j),\dots,\mu_n(v_{j-1}v_j)) = (A\circ\boldsymbol{\mu})(v_{j-1}v_j) > 0 \text{ for all } 1 \leq j \leq k.$$

Now, by hypothesis $\mu_l(v_{j-1}v_j) > 0$ for $1 \leq j \leq k$. Hence, C is a cycle in the fuzzy graph (σ_l,μ_l) which is a contradiction.

Conversely, if for each $l \in \{1,\dots,n\}$ we can find a n-tuple denoted by $(x_1^{(l)},\dots,x_n^{(l)}) \in [0,1]^n$ such that $x_l^{(l)} = 0$ and $A(x_1^{(l)},\dots,x_n^{(l)}) > 0$, then we can define the forests $(\sigma_1,\mu_1),\dots,(\sigma_n,\mu_n)$ as follows:



(a) (σ_i, μ_i) (b) $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$.

Fig. 3. Representation of the fuzzy graphs (σ_i, μ_i) and the aggregated graph for $n = 4$.

(a) (σ_1, μ_1).   (b) (σ_2, μ_2).

Fig. 4. Fuzzy graphs (σ_i, μ_i) in Example 4.7.

$$\sigma_i(v) = \begin{cases} 1 & \text{if } v \in \{v_1, \ldots, v_n\}, \\ 0 & \text{otherwise,} \end{cases}$$

and:

$$\mu_i(e) = \begin{cases} x_i^{(j)} & \text{if } e = v_j v_{j+1} \text{ with } 1 \le j \le n-1, \\ x_i^{(n)} & \text{if } e = v_n v_1, \\ 0 & \text{otherwise,} \end{cases}$$

for all $1 \le i \le n$ where $\{v_1, \ldots, v_n\}$ are distinct elements of V (recall that $|V| \ge n \ge 3$). Therefore:

$$(A\circ\boldsymbol{\mu})(e) = \begin{cases} A(x_1^{(j)}, \ldots x_n^{(j)}) & \text{if } e = v_j v_{j+1} \text{ with } 1 \le j \le n-1, \\ A(x_1^{(n)}, \ldots, x_n^{(n)}) & \text{if } e = v_n v_1, \\ 0 & \text{otherwise,} \end{cases}$$

and $A\circ\boldsymbol{\sigma} = \sigma_i$ for all $1 \le i \le n$. Since $\{v_1, \ldots, v_n\}$ is a cycle on $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$, we have a contradiction (see Fig. 3). □

Remark 4.6. Note that each fuzzy graph is a forest when $|V| \le 2$. Thus, every aggregation operator that preserves fuzzy graphs also preserves fuzzy forests.

Another significant feature of fuzzy graphs is the strength of connectedness between vertices since it is involved in the definition of other useful concepts of fuzzy graphs such as fuzzy bridges and fuzzy cutvertices. Unfortunately, the strength of connectedness for two vertices by paths of length k given by $(A\circ\boldsymbol{\mu})^k(u, v)$ in an aggregated fuzzy graph $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ can not be always described in terms of $A(\mu_1^k(u, v), \ldots, \mu_n^k(u, v))$ for arbitrary graphs (σ_i, μ_i) with $1 \le i \le n$ and $u, v \in V$. The next example shows that these two expressions are indeed different.

Example 4.7. Given the Łukasiewicz t-norm $T_L(x, y) = \max\{0, x + y - 1\}$ and the fuzzy graphs $(\sigma_1, \mu_1), (\sigma_2, \mu_2)$ on $V = \{a, b, c\}$ where $\sigma_1(v) = \sigma_2(v) = 1$ for all $v \in V$, $\mu_1(ab) = 0.5$, $\mu_1(bc) = \mu_2(ab) = 0.7$, $\mu_2(bc) = 0.4$ and $\mu_1(ac) = \mu_2(ac) = 0$ (see Fig. 4). It is easy to check that:

$$\mu_1^2(a, c) = \mu_1(ab) \wedge \mu_1(bc) = 0.5,$$
$$\mu_2^2(a, c) = \mu_2(ab) \wedge \mu_2(bc) = 0.4.$$

Consequently:

$$\begin{aligned} \left(T_L\circ\boldsymbol{\mu}\right)^2(a, c) &= \left(T_L\circ\boldsymbol{\mu}\right)(ab) \wedge \left(T_L\circ\boldsymbol{\mu}\right)(bc) \\ &= \max\{0, \mu_1(ab) + \mu_2(ab) - 1\} \wedge \max\{0, \mu_1(bc) + \mu_2(bc) - 1\} \\ &= 0.2 \wedge 0.1 > 0 = T_L\left(\mu_1^2(a, c), \mu_2^2(a, c)\right). \end{aligned}$$

Nevertheless, if we assume that the aggregation operator dominates the minimum then we can ensure one inequality between $(A\circ\boldsymbol{\mu})^k(u,v)$ and $A\left(\mu_1^k(u,v),\dots,\mu_n^k(u,v)\right)$ as we will show in Theorem 4.9. However, we must first recall that in [69], Saminger, Mesiar, and Bodenhofer showed that the inequality employed in defining the domination property always becomes an equality when the operator being dominated is the minimum operator.

Proposition 4.8 *([69]). Let* $A:[0,1]^n\to[0,1]$ *be an aggregation operator.* $A\gg\wedge$ *if and only if for any arbitrary* $m\in\mathbb{N}$*:*

$$A(x_{11},\dots,x_{1n})\wedge\cdots\wedge A(x_{m1},\dots,x_{mn})=A(x_{11}\wedge\cdots\wedge x_{m1},\dots,x_{1n}\wedge\cdots\wedge x_{mn})$$

for all $x_{ij}\in[0,1]$ *with* $i\in\{1,\dots,n\}$ *and* $j\in\{1,\dots,m\}$.

Now, we can proof the next result.

Theorem 4.9. *Let* $A:[0,1]^n\to[0,1]$ *be an aggregation operator.* $A\gg\wedge$ *if and only if for any collection of fuzzy graphs* $\{(V,\sigma_i,\mu_i)\mid 1\le i\le n\}$ *where* $|V|\ge 3$ *and for each* $k\in\mathbb{N}$*, we have:*

$$A\left(\mu_1^k(u,v),\dots,\mu_n^k(u,v)\right)\quad\ge\quad(A\circ\boldsymbol{\mu})^k(u,v)$$

for all $u,v\in V$*. As a consequence:*

$$A\left(\mu_1^\infty(u,v),\dots,\mu_n^\infty(u,v)\right)\quad\ge\quad(A\circ\boldsymbol{\mu})^\infty(u,v),$$

for all $u,v\in V$.

Proof. By definition, we have for each $u,v\in V$ that:

$$\begin{aligned}
&(A\circ\boldsymbol{\mu})^k(u,v)=\\
&\quad=\bigvee\{(A\circ\boldsymbol{\mu})(uv_1)\wedge\cdots\wedge(A\circ\boldsymbol{\mu})(v_{k-1}v)\mid(u,v_1,\dots,v_{k-1},v)\in V_k^*\}\\
&\quad=\bigvee\{A(\mu_1(uv_1),\dots,\mu_n(uv_1))\wedge\cdots\wedge A(\mu_1(v_{k-1}v),\dots,\mu_n(v_{k-1}v)) \qquad (2)\\
&\qquad\qquad\mid(u,v_1,\dots,v_{k-1},v)\in V_k^*\}\\
&\quad=A(\mu_1(uw_1),\dots,\mu_n(uw_1))\wedge\cdots\wedge A(\mu_1(w_{k-1}v),\dots,\mu_n(w_{k-1}v)) \qquad (3)\\
&\quad=A(\mu_1(uw_1)\wedge\cdots\wedge\mu_1(w_{k-1}v),\dots,\mu_n(uw_1)\wedge\cdots\wedge\mu_n(w_{k-1}v)). \qquad (4)
\end{aligned}$$

The expression (3) is derived from the fact that the supremum in (2) is attained for some $w_1,\dots,w_{k-1}\in V$ such that $(u,w_1,\dots,w_{k-1},v)\in V_k^*$. The equality between (3) and (4) is a consequence of Proposition 4.8 since $A\gg\wedge$. Additionally, A is a monotone operator and thus:

$$\begin{aligned}
&A(\mu_1^k(u,v),\dots,\mu_n^k(u,v))=\\
&\quad=A\Big(\bigvee\{\mu_1(uv_1)\wedge\cdots\wedge\mu_1(v_{k-1}v)\mid(u,v_1,\dots,v_{k-1},v)\in V_k^*\},\ \dots\\
&\qquad\dots,\bigvee\left\{\mu_n(uv_1)\wedge\cdots\wedge\mu_n(v_{k-1}v)\mid(u,v_1,\dots,v_{k-1},v)\in V_k^*\right\}\Big)\\
&\quad\ge A(\mu_1(uw_1)\wedge\cdots\wedge\mu_1(w_{k-1}v),\dots,\mu_n(uw_1)\wedge\cdots\wedge\mu_n(w_{k-1}v)).
\end{aligned}$$

Note that from this discussion we can deduce the inequality:

$$A\left(\mu_1^k(u,v),\dots,\mu_n^k(u,v)\right)\ge(A\circ\boldsymbol{\mu})^k(u,v)$$

as we wanted to prove.

Let us suppose now that $A\not\gg\wedge$. That is, there exist $(x_1,\dots,x_n),(y_1,\dots,y_n)\in[0,1]^n$ such that:

$$A(x_1,\dots,x_n)\wedge A(y_1,\dots,y_n)>A(x_1\wedge y_1,\dots,x_n\wedge y_n).$$

Therefore, we can take three distinct vertices $a,b,c\in V$ and the fuzzy graphs $(\sigma_1,\mu_1),\dots,(\sigma_n,\mu_n)$ where for each $i\in\{1,\dots,n\}$, $\sigma_i(v)=1$ for all $v\in V$, $\mu_i(ab)=x_i$, $\mu_i(bc)=y_i$ and $\mu_i(e)=0$ for all $e\in\mathcal{E}_V\setminus\{ab,bc\}$ (see Fig. 5). At this point, it is easy to check that:

$$\mu_i^2(a,c)=\mu_i(ab)\wedge\mu_i(bc)=x_i\wedge y_i.$$

Consequently:

$$\begin{aligned}
(A\circ\boldsymbol{\mu})^2(a,c)&=(A\circ\boldsymbol{\mu})(ab)\wedge(A\circ\boldsymbol{\mu})(bc)\\
&=A(\mu_1(ab),\dots,\mu_n(ab))\wedge A(\mu_1(bc),\dots,\mu_n(bc))
\end{aligned}$$

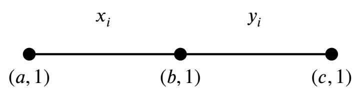


Fig. 5. Fuzzy graphs (σ_i, μ_i) in the proof of Theorem 4.9.

$$
\begin{aligned}
&= A(x_1, \dots, x_n) \wedge A(y_1, \dots, y_n) \\
&> A(x_1 \wedge y_1, \dots, x_n \wedge y_n) \\
&= A\left(\mu_1(ab) \wedge \mu_1(bc), \dots, \mu_n(ab) \wedge \mu_n(bc)\right) \\
&= A\left(\mu_1^2(a,c), \dots, \mu_n^2(a,c)\right)
\end{aligned}
$$

which is a contradiction with the fact that $A\left(\mu_1^k(u,v), \dots, \mu_n^k(u,v)\right) \geq (A \circ \boldsymbol{\mu})^k(u,v)$ for all $u, v \in V$ and $k \in \mathbb{N}$. □

Remark 4.10. When $|V| = 2$, we have the trivial case where $\mu^k(u,v) = 0$ for all $k \geq 2$ and the identity $A\left(\mu_1^k(u,v), \dots, \mu_n^k(u,v)\right) = (A \circ \boldsymbol{\mu})^k(u,v)$ holds for all $u, v \in V$ and for any aggregation operator.

We are now prepared to analyze the preservation of fuzzy bridges. It is worth noting that the definition of this concept depends on the strength of the connection between the vertices, which makes Theorem 4.9 crucial in the upcoming proofs.

Theorem 4.11. *Let $A : [0,1]^n \to [0,1]$ be an aggregation operator and $uv \in \mathcal{E}_V$ a common fuzzy bridge to all the fuzzy graphs $(\sigma_1, \mu_1), \dots, (\sigma_n, \mu_n)$. If A is jointly strictly monotone and $A \gg \wedge$, then uv is a fuzzy bridge in $(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$.*

Proof. Let us suppose that uv is not a fuzzy bridge in $(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$. Therefore, for all $x, y \in V$ we have:

$$(A \circ \boldsymbol{\mu})'^{\infty}(x,y) = (A \circ \boldsymbol{\mu})^{\infty}(x,y).$$

In particular:

$$(A \circ \boldsymbol{\mu})'^{\infty}(u,v) = (A \circ \boldsymbol{\mu})^{\infty}(u,v). \tag{5}$$

Recall that uv is a fuzzy bridge of all the fuzzy graphs in $\{(\sigma_i, \mu_i) \mid 1 \leq i \leq n\}$. Therefore, by Theorem 2.8 and Theorem 2.9:

$$\mu_i^{\infty}(u,v) = \mu_i(uv) > \mu_i'^{\infty}(u,v) \tag{6}$$

for all $1 \leq i \leq n$. With this, and making use of Theorem 4.9:

$$
\begin{aligned}
(A \circ \boldsymbol{\mu})(uv) &\leq \bigvee_{k \in \mathbb{N}} \{(A \circ \boldsymbol{\mu})^k(u,v)\} = (A \circ \boldsymbol{\mu})^{\infty}(u,v) \\
&\leq A\left(\mu_1^{\infty}(u,v), \dots, \mu_n^{\infty}(u,v)\right) \\
&= A(\mu_1(uv), \dots, \mu_n(uv)) = (A \circ \boldsymbol{\mu})(uv).
\end{aligned}
$$

As a direct consequence of this and equality (5):

$$
\begin{aligned}
(A \circ \boldsymbol{\mu})'^{\infty}(u,v) &= (A \circ \boldsymbol{\mu})^{\infty}(u,v) = A\left(\mu_1^{\infty}(u,v), \dots, \mu_n^{\infty}(u,v)\right) \\
&= (A \circ \boldsymbol{\mu})(uv).
\end{aligned}
\tag{7}
$$

Furthermore, we can define $\boldsymbol{\mu}' : \mathcal{E}_V \to [0,1]^n$ as $\boldsymbol{\mu}'(xy) = (\mu_1'(xy), \dots, \mu_n'(xy))$ where μ_i' is the fuzzy set such that $\mu_i'(uv) = 0$ and $\mu_i'(xy) = \mu_i(xy)$ for all $xy \neq uv$. Therefore, it is easy to check that:

$$(A \circ \boldsymbol{\mu})'(xy) = A\left(\mu_1'(xy), \dots, \mu_n'(xy)\right) = (A \circ \boldsymbol{\mu}')(xy)$$

where $(A \circ \boldsymbol{\mu})'(uv) = 0$ and $(A \circ \boldsymbol{\mu})'(xy) = (A \circ \boldsymbol{\mu})(xy)$ for all $xy \neq uv$. Using this, the Theorem 4.9, expressions (6)-(7) and the monotonicity of A:

$$
\begin{aligned}
A\left(\mu_1^{\infty}(u,v), \dots, \mu_n^{\infty}(u,v)\right) = (A \circ \boldsymbol{\mu})'^{\infty}(u,v) &= (A \circ \boldsymbol{\mu}')^{\infty}(u,v) \\
&\leq A\left(\mu_1'^{\infty}(u,v), \dots, \mu_n'^{\infty}(u,v)\right) \\
&\leq A\left(\mu_1^{\infty}(u,v), \dots, \mu_n^{\infty}(u,v)\right).
\end{aligned}
$$

Consequently, $A\left(\mu_1'^{\infty}(u,v), \dots, \mu_n'^{\infty}(u,v)\right) = A\left(\mu_1^{\infty}(u,v), \dots, \mu_n^{\infty}(u,v)\right)$. This is a contradiction with the fact that A is jointly strictly monotone since $\mu_i'^{\infty}(u,v) < \mu_i^{\infty}(u,v)$ for all $i \in \{1, \dots, n\}$. □

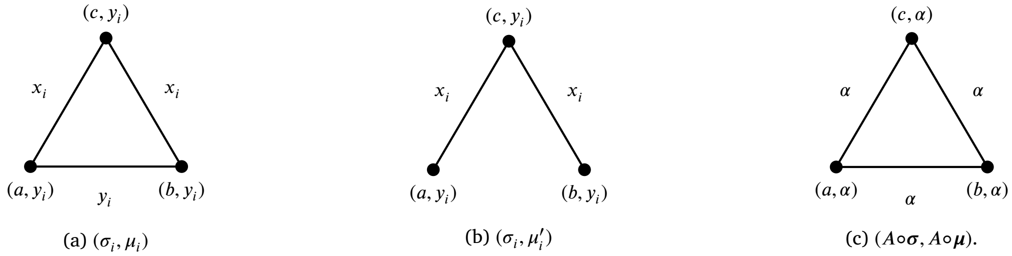


Fig. 6. Representation of (σ_i, μ_i), (σ_i, μ_i') and $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ in the proof of Theorem 4.12.

We can go even further from the previous theorem and prove that the only operators that preserve fuzzy bridges and such that $A \gg \wedge$ are those that are jointly strictly monotone.

Theorem 4.12. *Let $A : [0,1]^n \to [0,1]$ be an aggregation operator such that $A \gg \wedge$. The following statements are equivalent:*

1. *A is jointly strictly monotone.*
2. *A preserves fuzzy bridges. That is, for any collection of fuzzy graphs $\{(\sigma_i, \mu_i) \mid 1 \le i \le n\}$ where $uv \in \mathcal{E}_V$ is a common fuzzy bridge, uv is a fuzzy bridge in $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$.*

Proof. $1 \Rightarrow 2$. It is due to Theorem 4.11.

$2 \Rightarrow 1$. Let us suppose that A is not jointly strictly monotone. As a consequence, for each $i \in \{1,\dots,n\}$ there exist $x_i < y_i$ such that $A(x_1,\dots,x_n) = A(y_1,\dots,y_n)$. Let us define the fuzzy graphs $(\sigma_1, \mu_1), \dots, (\sigma_n, \mu_n)$ of Fig. 6a where for all $1 \le i \le n$:

$$\sigma_i(v) = \begin{cases} y_i & \text{if } v \in \{a,b,c\}, \\ 0 & \text{otherwise.} \end{cases} \qquad \text{and} \qquad \mu_i(e) = \begin{cases} y_i & \text{if } e = ab, \\ x_i & \text{if } e \in \{bc, ac\}, \\ 0 & \text{otherwise,} \end{cases}$$

with $\{a,b,c\} \subseteq V$. For all these fuzzy graphs, the edge ab is a fuzzy bridge since $\mu_i(ab) = y_i > x_i \ge {\mu_i'}^{\infty}(a,b)$ (see Fig. 6b). Hence, if we denote $A(x_1,\dots,x_n) = A(y_1,\dots,y_n) = \alpha$:

$$(A\circ\boldsymbol{\sigma})(v) = \begin{cases} \alpha & \text{if } v \in \{a,b,c\}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$(A\circ\boldsymbol{\mu})(e) = \begin{cases} \alpha & \text{if } e \in \{ab,bc,ac\}, \\ 0 & \text{otherwise.} \end{cases}$$

This aggregated graph is represented in Fig. 6c. Now, it is easy to check that $(A\circ\boldsymbol{\mu})^{\infty}(u,v) = (A\circ\boldsymbol{\mu})'^{\infty}(u,v)$ for all $u, v \in V$ where $(A\circ\boldsymbol{\mu})'(ab) = 0$ and $(A\circ\boldsymbol{\mu})'(e) = (A\circ\boldsymbol{\mu})(e)$ for all $e \ne ab$. Therefore, ab is not a fuzzy bridge in $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$. □

5. Aggregation of complete fuzzy graphs

This section is devoted to the study of the necessary and sufficient conditions for an aggregation operator to preserve complete fuzzy graphs. First, we recall a useful relation between two fuzzy sets.

Definition 5.1 *([63]).* We say that two fuzzy sets $\sigma : X \to [0,1]$ and $\tau : X \to [0,1]$ are similar, denoted by $\sigma \sim \tau$, if they have the same level sets, i.e.:

$$\sigma \sim \tau \qquad \Longleftrightarrow \qquad \left\{\sigma^t\right\}_{t \in Im\ \sigma} = \left\{\tau^t\right\}_{t \in Im\ \tau}$$

The next lemma is proved in [10] for fuzzy vector spaces. Nevertheless, the proof is the same for arbitrary fuzzy sets.

Lemma 5.2 *([10]). Let σ, τ be two fuzzy sets on X. The following assertions are equivalent:*

1. $\sigma(x) < \sigma(y) \iff \tau(x) < \tau(y)$.
2. $\sigma \sim \tau$.

We can check with a simple example that not every aggregation operator preserves complete fuzzy graphs.

Example 5.3. Let $V = \{a, b\}$ with $\sigma_1(a) = \sigma_2(b) = 1$ and $\sigma_2(a) = \sigma_1(b) = \mu_1(ab) = \mu_2(ab) = \alpha \in (0,1)$. It is clear that (σ_1, μ_1) and (σ_2, μ_2) are complete fuzzy graphs. However, fixing the product t-norm T_P as the aggregation operator:

$$(T_P \circ \boldsymbol{\sigma})(a) \wedge (T_P \circ \boldsymbol{\sigma})(b) = \alpha > \alpha^2 = (T_P \circ \boldsymbol{\mu})(ab).$$

Thus, the aggregated fuzzy graph is not complete.

Despite of the previous example, if we have n similar complete fuzzy graphs, then the aggregated graph is also complete.

Proposition 5.4. *Let $A : [0,1]^n \to [0,1]$ be an aggregation operator. If $(\sigma_1, \mu_1), \dots, (\sigma_n, \mu_n)$ are complete fuzzy graphs such that $\sigma_1 \sim \sigma_2 \sim \cdots \sim \sigma_n$, then $(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$ is a complete fuzzy graph.*

Proof. Let $a, b \in V$ and let us suppose without loss of generality that $\sigma_1(a) \leq \sigma_1(b)$. By Lemma 5.2, since $\sigma_i \sim \sigma_j$ for all $i, j \in \{1, \dots, n\}$, we have that $\sigma_i(a) \leq \sigma_i(b)$ for all $1 \leq i \leq n$. Moreover, all fuzzy graphs are complete, and A is a non-decreasing operator. With all this, we can establish the next chain of identities:

$$\begin{aligned}(A \circ \boldsymbol{\mu})(ab) &= A(\mu_1(ab), \dots, \mu_n(ab)) \\ &= A(\sigma_1(a) \wedge \sigma_1(b), \dots, \sigma_n(a) \wedge \sigma_n(b)) \\ &= A(\sigma_1(a), \dots \sigma_n(a)) \\ &= (A \circ \boldsymbol{\sigma})(a) \wedge (A \circ \boldsymbol{\sigma})(b).\end{aligned}$$

Hence, $(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$ is complete. □

This last result gives only a sufficient condition for the aggregation of a collection of complete fuzzy graphs to be complete again. The next example shows that it is not a necessary condition.

Example 5.5. Let $V = \{a, b\}$ with $\sigma_1(a) > \sigma_1(b) = \sigma_2(a) = \sigma_2(b) = \mu_1(ab) = \mu_2(ab) = \alpha \in [0,1]$. It is easy to check that (σ_1, μ_1) and (σ_2, μ_2) are complete fuzzy graphs. Additionally, using Lemma 5.2, we have that $\sigma_1 \nsim \sigma_2$. However, by taking A such that $A(\alpha, \alpha) = \alpha$ we get that:

$$(A \circ \boldsymbol{\sigma})(a) = A(\sigma_1(a), \sigma_2(a)) \geq A(\alpha, \alpha) = \alpha,$$

and therefore:

$$(A \circ \boldsymbol{\sigma})(a) \wedge (A \circ \boldsymbol{\sigma})(b) = (A \circ \boldsymbol{\sigma})(b) = \alpha = A(\mu_1(ab), \mu_2(ab)) = (A \circ \boldsymbol{\mu})(ab).$$

Consequently, $(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$ is complete.

We now extend a significant property from [10] in our context, which will enable us to establish a characterization of complete graphs. It will help us identify when the aggregation of complete fuzzy graphs is preserved.

Let X be a set and $\sigma_1, \dots, \sigma_n : X \to [0,1]$ fuzzy subsets. We denote the set $\mathcal{L}(\sigma_1, \dots, \sigma_n) = \{(x, y) \in X \times X \mid \exists i, j \in \{1, \dots, n\}$ with $\sigma_i(x) > \sigma_i(y)$ and $\sigma_j(y) > \sigma_j(x)\}$.

Theorem 5.6. *Let $(\sigma_1, \mu_1), \dots, (\sigma_n, \mu_n)$ be complete fuzzy graphs. The following statements are equivalent:*

1. $\mathcal{L}(\sigma_1, \dots, \sigma_n) = \emptyset$.
2. *$(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$ is a complete graph for any aggregation operator $A : [0,1]^n \to [0,1]$.*

Proof. $1 \Rightarrow 2$. Let $x, y \in V$. Let us distinguish two cases:

(a) For all $k \in \{1, \dots, n\}$, $\sigma_k(x) \leq \sigma_k(y)$. Since all fuzzy graphs are complete:

$$\begin{aligned}(A \circ \boldsymbol{\mu})(xy) &= A(\mu_1(xy), \dots, \mu_n(xy)) \\ &= A(\sigma_1(x) \wedge \sigma_1(y), \dots, \sigma_n(x) \wedge \sigma_n(y)) \\ &= A(\sigma_1(x), \dots, \sigma_n(x)) \\ &= (A \circ \boldsymbol{\sigma})(x) \wedge (A \circ \boldsymbol{\sigma})(y).\end{aligned}$$

Therefore, $(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$ is a complete fuzzy graph.

(b) There exists $i \in \{1, \dots, n\}$ such that $\sigma_i(x) > \sigma_i(y)$. By hypothesis, $\mathcal{L}(\sigma_1, \dots, \sigma_n) = \emptyset$ and hence $\sigma_k(x) \geq \sigma_k(y)$ for all $k \in \{1, \dots, n\}$. We can apply a similar reasoning as in the previous case to conclude that $(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$ is a complete fuzzy graph.

$2 \Rightarrow 1$. Let us assume $\mathcal{L}(\sigma_1,\dots,\sigma_n) \neq \emptyset$. As a consequence, we can find $x, y \in V$ and $i, j \in \{1,\dots,n\}$ satisfying $\sigma_i(x) < \sigma_i(y)$ and $\sigma_j(x) > \sigma_j(y)$. It is easy to check that the following function $A : [0,1]^n \longrightarrow [0,1]$ is an aggregation operator:

$$A(r_1,\dots,r_n) = \begin{cases} 1 & \text{if } r_i \geq \sigma_i(y) \text{ and } r_j \geq \sigma_j(y), \\ 1 & \text{if } r_j \geq \sigma_j(x) \text{ and } r_i \geq \sigma_i(x), \\ 0 & \text{otherwise.} \end{cases}$$

We show now that $(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ is not a complete graph, which would conclude the proof. Without loss of generality, we can set $i \leq j$ and hence:

$$\begin{aligned}(A\circ\boldsymbol{\mu})(xy) &= A(\sigma_1(x)\wedge\sigma_1(y),\dots,\sigma_n(x)\wedge\sigma_n(y)) = 0 \\ &< 1 = (A\circ\boldsymbol{\sigma})(x)\wedge(A\circ\boldsymbol{\sigma})(y). \quad \square\end{aligned}$$

As a consequence of the previous result, we give a new necessary and sufficient condition for a collection of complete fuzzy graphs to maintain the structure of a complete fuzzy graph when they are fused by any aggregation operator.

Corollary 5.7. *Let $B : [0,1]^n \to [0,1]$ be a strictly monotone aggregation operator and let $(\sigma_1,\mu_1),\dots,(\sigma_n,\mu_n)$ be complete fuzzy graphs. The following statements are equivalent:*

1. $\mathcal{L}(\sigma_1,\dots,\sigma_n) = \emptyset$.
2. *$(B\circ\boldsymbol{\sigma}, B\circ\boldsymbol{\mu})$ is a complete fuzzy graph.*
3. *$(A\circ\boldsymbol{\sigma}, A\circ\boldsymbol{\mu})$ is a complete fuzzy graph for each aggregation operator $A : [0,1]^n \to [0,1]$.*

Proof. $1 \Leftrightarrow 3$. It is a consequence of Theorem 5.6.

$3 \Rightarrow 2$. Straightforward.

$2 \Rightarrow 1$. Let $(B\circ\boldsymbol{\sigma}, B\circ\boldsymbol{\mu})$ be a complete fuzzy graph. Let us suppose that $\mathcal{L}(\sigma_1,\dots,\sigma_n) \neq \emptyset$ to get a contradiction. Therefore, there exist $x, y \in V$ and $i, j \in \{1,\dots,n\}$ such that $\sigma_i(x) < \sigma_i(y)$ and $\sigma_j(x) > \sigma_j(y)$. Hence:

$$\begin{aligned}(B\circ\boldsymbol{\mu})(xy) &= B(\mu_1(xy),\dots,\mu_n(xy)) \\ &\overset{(1)}{=} B(\sigma_1(x)\wedge\sigma_1(y),\dots,\sigma_n(x)\wedge\sigma_n(y)) \\ &\overset{(2)}{\leq} B(\sigma_1(x),\dots,\sigma_n(x))\wedge B(\sigma_1(y),\dots,\sigma_n(y)) \\ &= (B\circ\boldsymbol{\sigma})(x)\wedge(B\circ\boldsymbol{\sigma})(y)\end{aligned}$$

where (1) holds because the graphs are all complete and (2) is due to Proposition 3.5. It is important to note that, as a result of $\sigma_i(x) < \sigma_i(y)$, we observe:

$$(\sigma_1(x)\wedge\sigma_1(y),\dots,\sigma_n(x)\wedge\sigma_n(y)) \lneq (\sigma_1(y),\dots,\sigma_n(y))$$

Similarly, as $\sigma_j(x) > \sigma_j(y)$, we have:

$$(\sigma_1(x)\wedge\sigma_1(y),\dots,\sigma_n(x)\wedge\sigma_n(y)) \lneq (\sigma_1(x),\dots,\sigma_n(x))$$

Furthermore, since B is strictly increasing, the inequality (2) can never be an identity. Consequently, $(B\circ\boldsymbol{\mu})(xy) < (B\circ\boldsymbol{\sigma})(x) \wedge (B\circ\boldsymbol{\sigma})(y)$, and the pair $(B\circ\boldsymbol{\sigma}, B\circ\boldsymbol{\mu})$ would not form a complete set. $\square$

Finally, we provide a characterization of the aggregation operators that preserve any collection of complete fuzzy graphs. Thus, we get a different approach than the rest of the results in this section.

Theorem 5.8. *Let $A : [0,1]^n \to [0,1]$ be an aggregation operator. The following statements are equivalent:*

1. *A preserves complete fuzzy graphs.*
2. $A \gg \wedge$.

Proof. $1 \Rightarrow 2$. Let us suppose that A preserves complete fuzzy graphs. We aim to prove that $A \gg \wedge$. By Proposition 4.8, this is the same as proving:

$$A(x_1 \dots, x_n) \wedge A(y_1,\dots,y_n) = A(x_1\wedge y_1,\dots,x_n\wedge y_n),$$

for all $(x_1,\dots,x_n), (y_1,\dots,y_n) \in [0,1]^n$. Assume $a, b \in V$ with $a \neq b$. If we fix $(x_1,\dots,x_n), (y_1,\dots,y_n) \in [0,1]^n$, we can define the complete fuzzy graphs $(\sigma_1,\mu_1),\dots,(\sigma_n,\mu_n)$ such that:

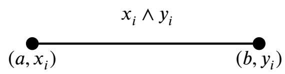


Fig. 7. Fuzzy graph (σ_i, μ_i) in the proof of Theorem 5.8.

$$\sigma_i(v) = \begin{cases} x_i & \text{if } v = a, \\ y_i & \text{if } v = b, \\ 0 & \text{otherwise,} \end{cases}$$

and $\mu_i(ab) = \sigma_i(a) \wedge \sigma_i(b) = x_i \wedge y_i$ for $1 \leq i \leq n$ (see Fig. 7). Therefore, since A preserves complete fuzzy graphs:

$$\begin{aligned} A(x_1 \ldots, x_n) \wedge A(y_1, \ldots, y_n) &= A(\sigma_1(a), \ldots \sigma_n(a)) \wedge A(\sigma_1(b), \ldots, \sigma_n(b)) \\ &= (A \circ \boldsymbol{\sigma})(a) \wedge (A \circ \boldsymbol{\sigma})(b) \\ &= (A \circ \boldsymbol{\mu})(ab) \\ &= A(\mu_1(ab), \ldots, \mu_n(ab)) \\ &= A(x_1 \wedge y_1, \ldots, x_n \wedge y_n). \end{aligned}$$

Hence $A \gg \wedge$.

$2 \Rightarrow 1$. Let $\{(\sigma_i, \mu_i) \mid 1 \leq i \leq n\}$ be a set of complete fuzzy graphs on V. Thus, $\mu_i(ab) = \sigma_i(a) \wedge \sigma_i(b)$ for all $i \in \{1, \ldots, n\}$ and $a, b \in V$. This fact, together with the hypothesis $A \gg \wedge$ leads us to the following chain of inequalities for each $a, b \in V$:

$$\begin{aligned} (A \circ \boldsymbol{\mu})(ab) &= A(\mu_1(ab), \ldots, \mu_n(ab)) \\ &= A(\sigma_1(a) \wedge \sigma_1(b), \ldots, \sigma_n(a) \wedge \sigma_n(b)) \\ &= A(\sigma_1(a), \ldots, \sigma_n(a)) \wedge A(\sigma_1(b), \ldots, \sigma_n(b)) \\ &= (A \circ \boldsymbol{\sigma})(a) \wedge (A \circ \boldsymbol{\sigma})(b). \end{aligned}$$

Consequently, $(A \circ \boldsymbol{\sigma}, A \circ \boldsymbol{\mu})$ is a complete fuzzy graph. □

6. Conclusions

We have explored the aggregation of fuzzy algebraic structures, with a focus on fuzzy graphs. We provided new insights into the preservation of properties within these structures that are influenced by aggregation operators. Some of the achievements of this work include:

1. Theoretical framework: we have established a theoretical framework for understanding aggregation in fuzzy graphs, clarifying the role of aggregation operators in the preservation of different properties.
2. Characterization of aggregation: our findings have outlined conditions for transforming fuzzy graphs while preserving their properties.
3. Insights into graph structures: we have uncovered important insights into preserving fuzzy graph structures during aggregation.
4. Preservation of completeness: additionally, we have elucidated criteria for preserving completeness in fuzzy graphs.

Our research contributes to the understanding of aggregation processes in fuzzy graph structures, which has implications for network analysis and decision-making. We expect that our findings will encourage further exploration in this area.

CRediT authorship contribution statement

Francisco Javier Talavera: Writing – original draft, Methodology, Investigation. **Carlos Bejines:** Writing – original draft, Methodology, Investigation, Conceptualization. **Sergio Ardanza-Trevijano:** Writing – review & editing, Methodology, Investigation. **Jorge Elorza:** Writing – review & editing, Writing – original draft, Supervision, Methodology, Investigation, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

C. Bejines acknowledges the financial support of the VALID research project (Grant PID2022-140630NB-I00) and the research project (Grant PID2021-127870OB-I00). The remaining authors acknowledge the Spanish Ministerio de Ciencia e Innovación/AEI/EU Government of Spain and EU FEDER Funds (Grant PID2021-122905NB-C22). The first author also thanks the support of the Asociación de Amigos of the University of Navarra.

References

[1] S. Ardanza-Trevijano, M.J. Chasco, J. Elorza, M. de Natividade, F.J. Talavera, Aggregation of t-subgroups, Fuzzy Sets Syst. 463 (2023) 108390.
[2] C. Bejines, M.J. Chasco, J. Elorza, Aggregation of fuzzy subgroups, Fuzzy Sets Syst. 418 (2021) 170–184.
[3] C. Bejines, S. Ardanza-Trevijano, J. Elorza, On self-aggregations of min-subgroups, Axioms 10 (3) (2021) 201.
[4] F.J. Talavera, S. Ardanza-Trevijano, J. Bragard, J. Elorza, Aggregation of t-subgroups of groups whose subgroup lattice is a chain, Fuzzy Sets Syst. 473 (2023) 108717.
[5] J. Drewniak, U. Dudziak, Preservation of properties of fuzzy relations during aggregation processes, Kybernetika 43 (2) (2007) 115–132.
[6] C. Bejines, S. Ardanza-Trevijano, M. Chasco, J. Elorza, Aggregation of indistinguishability operators, Fuzzy Sets Syst. 446 (2022) 53–67.
[7] R.H. Reiser, B. Bedregal, M. Baczyński, Aggregating fuzzy implications, Inf. Sci. 253 (2013) 126–146.
[8] T. Pedraza, J. Ramos-Canós, J. Rodríguez-López, Aggregation of weak fuzzy norms, Symmetry 13 (10) (2021) 1908.
[9] T. Pedraza, J. Rodríguez-López, Ó. Valero, Aggregation of fuzzy quasi-metrics, Inf. Sci. 581 (2021) 362–389.
[10] C. Bejines, Aggregation of fuzzy vector spaces, Kybernetika 59 (5) (2023) 752–767.
[11] L.T. Kóczy, M.E. Cornejo, J. Medina, Algebraic structure of fuzzy signatures, Fuzzy Sets Syst. 418 (2021) 25–50.
[12] J. Baz, I. Díaz, L. Garmendia, D. Gómez, L. Magdalena, S. Montes, Computable aggregations of random variables, Inf. Sci. 654 (2024) 119842.
[13] J. Baz, I. Díaz, S. Montes, Aggregation of random elements over bounded lattices, Int. J. Approx. Reason. 166 (2024) 109122.
[14] S. Massanet, J.V. Riera, J. Torrens, On the aggregation of Zadeh's z-numbers based on discrete fuzzy numbers, in: Aggregation Functions in Theory and in Practice, Springer, 2018, pp. 118–129.
[15] A. Rosenfeld, Fuzzy graphs, in: Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Elsevier, 1975, pp. 77–95.
[16] R.T. Yeh, S. Bang, Fuzzy relations, fuzzy graphs, and their applications to clustering analysis, in: Fuzzy Sets and Their Applications to Cognitive and Decision Processes, Elsevier, 1975, pp. 125–149.
[17] S. Banerjee, An optimal algorithm to find the degrees of connectedness in an undirected edge-weighted graph, Pattern Recognit. Lett. 12 (7) (1991) 421–424.
[18] P. Bhattacharya, Some remarks on fuzzy graphs, Pattern Recognit. Lett. 6 (5) (1987) 297–302.
[19] P. Bhattacharya, F. Suraweera, An algorithm to compute the supremum of max-min powers and a property of fuzzy graphs, Pattern Recognit. Lett. 12 (7) (1991) 413–420.
[20] K.R. Bhutani, On automorphisms of fuzzy graphs, Pattern Recognit. Lett. 9 (3) (1989) 159–162.
[21] K.R. Bhutani, A. Rosenfeld, Strong arcs in fuzzy graphs, Inf. Sci. 152 (2003) 319–322.
[22] M. Binu, S. Mathew, J.N. Mordeson, Connectivity status of fuzzy graphs, Inf. Sci. 573 (2021) 382–395.
[23] M. Delgado, J. Verdegay, M. Vila, On fuzzy tree definition, Eur. J. Oper. Res. 22 (2) (1985) 243–249.
[24] S. Mathew, M. Sunitha, Node connectivity and arc connectivity of a fuzzy graph, Inf. Sci. 180 (4) (2010) 519–531.
[25] S. Mathew, M. Sunitha, Strongest strong cycles and θ-fuzzy graphs, IEEE Trans. Fuzzy Syst. 21 (6) (2013) 1096–1104.
[26] S. Mathew, M. Sunitha, Menger's theorem for fuzzy graphs, Inf. Sci. 222 (2013) 717–726.
[27] S. Mathew, J.N. Mordeson, Connectivity concepts in fuzzy incidence graphs, Inf. Sci. 382 (2017) 326–333.
[28] J.N. Mordeson, Fuzzy line graphs, Pattern Recognit. Lett. 14 (5) (1993) 381–384.
[29] M. Sunitha, A. Vijayakumar, A characterization of fuzzy trees, Inf. Sci. 113 (1999) 293–300.
[30] M. Sunitha, A. Vijayakumar, Studies on fuzzy graphs, Ph.D. thesis, Department of Mathematics, 2001.
[31] M. Sunitha, S. Mathew, Fuzzy graph theory: a survey, Ann. Pure Appl. Math. 4 (1) (2013) 92–110.
[32] M. Sunitha, A. Vijayakumar, Some metric aspects of fuzzy graphs, in: Proceedings of the Conference on Graph Connections, CUSAT, Allied Publishers, 1999, pp. 111–114.
[33] M. Sunitha, A. Vijayakumar, Complement of a fuzzy graph, Indian J. Pure Appl. Math. 33 (9) (2002) 1451–1464.
[34] M. Sunitha, A. Vijayakumar, Blocks in fuzzy graphs, J. Fuzzy Math. 13 (1) (2005) 13–23.
[35] C. Karakuzu, O. Demirci, Fuzzy logic based smart traffic light simulator design and hardware implementation, Appl. Soft Comput. 10 (1) (2010) 66–73.
[36] G.K. Thakur, B. Priya, R. Mishra, An efficient coloring algorithm for time detraction of sign image segmentation based on fuzzy graph theory, J. Appl. Secur. Res. 14 (2) (2019) 210–226.
[37] A. Dey, A. Pal, Fuzzy graph coloring technique to classify the accidental zone of a traffic control, Ann. Pure Appl. Math. 3 (2) (2013) 169–178.
[38] M. Akram, A. Habib, B. Davvaz, Direct sum of n Pythagorean fuzzy graphs with application to group decision-making, J. Mult.-Valued Log. Soft Comput. 33 (1–2) (2019) 75–115.
[39] A.Z. Khameneh, A. Kilicman, m-polar fuzzy soft graphs in group decision making: a combining method by aggregation functions, in: Progress in Intelligent Decision Science: Proceeding of IDS 2020, Springer, 2021, pp. 425–455.
[40] R. Wille, Restructuring lattice theory: an approach based on hierarchies of concepts, in: I. Rival (Ed.), Ordered Sets: Proceedings, NATO Advanced Studies Institute, 83, Reidel, Dordrecht, 1982, pp. 445–470.
[41] H. Soldano, Extensional confluences and local closure operators, in: International Conference on Formal Concept Analysis, Springer, 2015, pp. 128–144.
[42] G. Cattaneo, G. Chiaselotti, D. Ciucci, T. Gentile, On the connection of hypergraph theory with formal concept analysis and rough set theory, Inf. Sci. 330 (2016) 342–357.
[43] H. Mao, S. Wang, C. Liu, G. Wang, Hypergraph-based attribute reduction of formal contexts in rough sets, Expert Syst. Appl. 234 (2023) 121062.
[44] T. Pattison, M. Enciso, Á. Mora, P. Cordero, D. Weber, M. Broughton, Scalable visual analytics in fca, in: Complex Data Analytics with Formal Concept Analysis, Springer, 2021, pp. 167–200.
[45] J.G. Stell, Formal concept analysis over graphs and hypergraphs, in: Graph Structures for Knowledge Representation and Reasoning: Third International Workshop, GKR 2013, Beijing, China, August 3, 2013, Springer, 2014, pp. 165–179, Revised Selected Papers.
[46] B. Gaume, E. Navarro, H. Prade, Clustering bipartite graphs in terms of approximate formal concepts and sub-contexts, Int. J. Comput. Intell. Syst. 6 (6) (2013) 1125–1142.
[47] G. Chiaselotti, T. Gentile, F.G. Infusino, P.A. Oliverio, The adjacency matrix of a graph as a data table: a geometric perspective, Ann. Mat. Pura Appl. (1923-) 196 (1917) 1073–1112.
[48] H. Hassan, M.I. Awang, M. Makhtar, A.H. Zakaria, R. Ismail, F. Ahmad, Algorithms for extracting adjacency matrix based on formal concept analysis (fca), in: Proceedings of the Second International Conference on Internet of Things, Data and Cloud Computing, 2017, pp. 1–6.

[49] S. Ferré, P. Cellier, Graph-fca: an extension of formal concept analysis to knowledge graphs, in: Advances in Formal Concept Analysis: Traces of CLA 2016, Discrete Appl. Math. 273 (2020) 81–102.
[50] A. Burusco Juandeaburre, R. Fuentes-González, The study of the l-fuzzy concept lattice, Mathw. Soft Comput. 1 (3) (1994) 209–218.
[51] J. Poelmans, D.I. Ignatov, S.O. Kuznetsov, G. Dedene, Fuzzy and rough formal concept analysis: a survey, Int. J. Gen. Syst. 43 (2) (2014) 105–134.
[52] I. Leceta, C. Alcalde, M. Urdanpilleta, P. Guerrero, K. de la Caba, A. Burusco, New algorithm for the elucidation of functional properties of gelatin-based materials, Comput. Chem. Eng. 153 (2021) 107410.
[53] P. Ghosh, K. Kundu, D. Sarkar, Fuzzy graph representation of a fuzzy concept lattice, Fuzzy Sets Syst. 161 (12) (2010) 1669–1675.
[54] F. Pérez-Gámez, P. Cordero, M. Enciso, Á. Mora, Simplification logic for the management of unknown information, Inf. Sci. 634 (2023) 505–519.
[55] O. Kridlo, M. Ojeda-Aciego, Extending formal concept analysis using intuitionistic l-fuzzy sets, in: 2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE), IEEE, 2017, pp. 1–6.
[56] M. Ojeda-Hernández, I.P. Cabrera, P. Cordero, E. Muñoz-Velasco, Fuzzy closure structures as formal concepts ii, Fuzzy Sets Syst. 473 (2023) 108734.
[57] M.E. Cornejo, J. Medina, F.J. Ocaña, Attribute implications in multi-adjoint concept lattices with hedges, Fuzzy Sets Syst. 479 (2024) 108854.
[58] R.G. Aragón, J. Medina, E. Ramírez-Poussa, Factorizing formal contexts from closures of necessity operators, Comput. Appl. Math. 43 (124) (2024), https://doi.org/10.1007/s40314-024-02590-0.
[59] P.K. Singh, C. Aswani Kumar, A note on bipolar fuzzy graph representation of concept lattice, Int. J. Comput. Sci. Math. 5 (4) (2014) 381–393.
[60] P.K. Singh, C.A. Kumar, Bipolar fuzzy graph representation of concept lattice, Inf. Sci. 288 (2014) 437–448.
[61] P.K. Singh, m-polar fuzzy graph representation of concept lattice, Eng. Appl. Artif. Intell. 67 (2018) 52–62.
[62] J.C. Díaz-Moreno, J. Medina, J.R. Portillo, Fuzzy logic programs as hypergraphs. Termination results, Fuzzy Sets Syst. 445 (2022) 22–42.
[63] P.S. Das, Fuzzy groups and level subgroups, J. Math. Anal. Appl. 84 (1) (1981) 264–269.
[64] S. Mathew, J.N. Mordeson, D.S. Malik, Fuzzy Graph Theory, vol. 363, Springer, 2018.
[65] L. Zadeh, Fuzzy sets, Inf. Control 8 (3) (1965) 338–353.
[66] K.R. Bhutani, On automorphisms of fuzzy graphs, Pattern Recognit. Lett. 9 (3) (1989) 159–162.
[67] T. Calvo, G. Mayor, R. Mesiar, Aggregation Operators: New Trends and Applications, vol. 97, Springer Science & Business Media, 2002.
[68] T. Calvo, A. Kolesárová, M. Komorníková, R. Mesiar, Aggregation operators: properties, classes and construction methods, in: Aggregation Operators, Springer, 2002, pp. 3–104.
[69] S. Saminger, R. Mesiar, U. Bodenhofer, Domination of aggregation operators and preservation of transitivity, Int. J. Uncertain. Fuzziness Knowl.-Based Syst. 10 (2002) 11–35.
[70] A. Somasundaram, S. Somasundaram, Domination in fuzzy graphs–i, Pattern Recognit. Lett. 19 (9) (1998) 787–791.