Hilbert-type Operators Acting on Bergman Spaces

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Abstract

If μ is a positive Borel measure on the interval $[0,1)$ we let $\mathcal{H}_\mu$ be the Hankel matrix $\mathcal{H}_\mu = (\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0,1,2,\dots$, μ_n denotes the moment of order n of μ. This matrix formally induces an operator, called also $\mathcal{H}_\mu$, on the space of all analytic functions in the unit disc $\mathbb{D}$ as follows: If f is an analytic function in $\mathbb{D}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \mathbb{D}$, $\mathcal{H}_\mu(f)$ is formally defined by

$$\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n+k}a_k\right)z^n, \quad z\in\mathbb{D}.$$

This is a natural generalization of the classical Hilbert operator. This paper is devoted to studying the operators H_μ acting on the Bergman spaces A^p, $1 \leq p < \infty$. Among other results, we give a complete characterization of those μ for which $\mathcal{H}_\mu$ is bounded or compact on the space A^p when p is either 1 or greater than 2. We also give a number of results concerning the boundedness and compactness of $\mathcal{H}_\mu$ on A^p for the other values of p, as well as on its membership in the Schatten classes $\mathcal{S}_p(A^2)$.

Keywords: The Hilbert matrix, Generalized Hilbert operator, Bergman spaces, Duality, Carleson measures, Bounded operator, Compact operator, Schatten classes

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1 Introduction and preliminary results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$. The space of all analytic functions in $\mathbb{D}$ will be denoted by $\mathrm{Hol}(\mathbb{D})$. We also let H^p $(0 < p \le \infty)$ be the classical Hardy space. We refer to [9] for notation and results regarding Hardy spaces.

For $0 < p < \infty$ the Bergman space A^p consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{A^p} \stackrel{\text{def}}{=} \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z)\right)^{1/p} < \infty.$$

Here, dA stands for the area measure on $\mathbb{D}$, normalized so that the total area of $\mathbb{D}$ is 1. Thus

$$dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta.$$

We refer to [10, 15, 25] for notation and results about Bergman spaces. Let us remark that if $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$ $z \in \mathbb{D}$, then $f \in A^2$ if and only if $\sum_{n=1}^\infty n^{-1}|a_n|^2 < \infty$ and, in fact,

$$\|f\|_{A^2} = \left(\sum_{n=0}^\infty \frac{|a_n|^2}{n+1}\right)^{1/2}.$$

If μ is a finite positive Borel measure on $[0,1)$ and $n = 0, 1, 2, \dots$, we let μ_n denote the moment of order n of μ, that is, $\mu_n = \int_{[0,1)} t^n \, d\mu(t)$, and we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\ge 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The matrix $\mathcal{H}_\mu$ formally induces an operator, which will be also called $\mathcal{H}_\mu$, on spaces of analytic functions by its action on the Taylor coefficients $a_n \mapsto \sum_{k=0}^\infty \mu_{n+k} a_k$, $n = 0, 1, 2, \dots$. To be precise:

$$\text{If } f(z) = \sum_{k=0}^\infty a_k z^k \in \mathrm{Hol}(\mathbb{D}) \text{ we define } \mathcal{H}_\mu(f)(z) = \sum_{n=0}^\infty \left(\sum_{k=0}^\infty \mu_{n+k} a_k\right) z^n,$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H} = \left((n+k+1)^{-1}\right)_{n,k\geq 0}$, which induces the classical Hilbert operator $\mathcal{H}$ which has recently been extensively studied (see [1, 5–7, 17, 18]).

Galanopoulos and Peláez [11] described the positive and finite Borel measures μ on $[0,1)$ so that the generalized Hilbert operator $\mathcal{H}_\mu$ becomes well defined and bounded on H^1. The corresponding question for H^2 had been solved by Widom in [23]. Chatzifountas, Girela and Peláez [4] extended this work describing those measures μ for which $\mathcal{H}_\mu$ is a bounded operator from H^p into H^q, $0 < p, q < \infty$.

Girela and Merchán obtained in [12] a characterization of those μ for which $\mathcal{H}_\mu$ is well defined and bounded on A^p for $2 < p < \infty$. Conditions on μ which are sufficient for $\mathcal{H}_\mu$ to be well defined and bounded on A^2 have been obtained in [11, 13]. The action of the operators $\mathcal{H}_\mu$ on A^p for $1 \leq p < 2$ has not been studied in the above mentioned articles.

This paper is devoted to studying the action of the operators $\mathcal{H}_\mu$ on all the Bergman spaces A^p $(1 \leq p < \infty)$. Among other results, we obtain a complete characterization of those μ for which $\mathcal{H}_\mu$ is either well defined, or bounded, or compact on A^p for $1 \leq p < \infty$ and $p \neq 2$. We also obtain a number of results for the case $p = 2$.

Obtaining an integral representation of $\mathcal{H}_\mu$ plays a basic role in this work. If μ is as above and $f \in \mathrm{Hol}(\mathbb{D})$, we shall write throughout the paper

$$\mathcal{I}_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t), \tag{1}$$

whenever the right hand side makes sense and defines an analytic function on $\mathbb{D}$. It turns out that the operators $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ are closely related. Indeed, $\mathcal{H}_\mu(f) = \mathcal{I}_\mu(f)$ for many functions $f \in \mathrm{Hol}(\mathbb{D})$. For instance, this is true if f is a polynomial.

Carleson measures play a basic role in our work. If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is the set defined as $S(I) = \{re^{it} : e^{it} \in I,\, 1 - |I|/(2\pi) \leq r < 1\}$.

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu\left(S(I)\right) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

If μ satisfies $\lim_{|I|\to 0} \mu(S(I))/|I|^s = 0$, then we say that μ is a *vanishing s-Carleson measure*.

A 1-Carleson measure, respectively, a vanishing 1-Carleson measure, will be simply called a Carleson measure, respectively, a vanishing Carleson measure.

Following [24], if μ is a positive Borel measure on $\mathbb{D}$, $0 \leq \alpha < \infty$, and $0 < s < \infty$ we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu\left(S(I)\right)\left(\log\frac{2\pi}{|I|}\right)^{\alpha}}{|I|^{s}} \leq C, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

If $\mu(S(I))(\log(2\pi/|I|)^{\alpha} = \mathrm{o}(|I|^{s})$, as $|I| \to 0$, we say that μ is a vanishing α-logarithmic s-Carleson measure.

A positive Borel measure μ on $[0,1)$ can be viewed as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \mu\left(A \cap [0,1)\right), \quad \text{for any Borel subset } A \text{ of } \mathbb{D}.$$

In this way a positive Borel measure μ on $[0,1)$ is an s-Carleson measure if and only if there exists a positive constant C such that

$$\mu\left([t,1)\right) \leq C(1-t)^{s}, \quad 0 \leq t < 1.$$

We have a similar statement for vanishing s-Carleson measures, for α-logarithmic s-Carleson measures, and for vanishing α-logarithmic s-Carleson measures.

If X is a Banach space of analytic functions in $\mathbb{D}$, such that convergence in X implies uniform convergence on compact subsets of $\mathbb{D}$, and $q > 0$, a positive Borel measure μ in $\mathbb{D}$ is said to be a "*q-Carleson measure for the space X*" or an "*(X,q)-Carleson measure*" if $X \subset L^{q}(d\mu)$. By the closed graph theorem, if μ is a q-Carleson measure for X then the embedding $X \to L^{p}(d\mu)$ is continuous. The q-Carleson measures for the spaces H^{p}, $0 < p, q < \infty$ are completely characterized. Carleson proved in [3] that the p-Carleson measures for H^{p} $(0 < p < \infty)$ are the (classical) Carleson measures. Duren [8] (see also [9, Thm. 9. 4]) extended this result showing that, for $0 < p < q$, the q-Carleson measures for H^{p} are precisely the q/p-Carleson measures. Luecking [19] solved the remaining case $0 < q < p$.

Hastings proved in [14] that, for all p, the p-Carleson measures for the Bergman space are the 2-Carleson measures. A complete characterization of the q-Carleson measures for A^{p} $(0 < p, q < \infty)$ was given in [20]. We shall be interested in the 1-Carleson measures for A^{p}, $p > 0$.

Following [20], for each $j = 1, 2, \ldots$, let us divide the annulus $\{z \in \mathbb{D} : 2^{-j} \leq 1 - |z| < 2^{1-j}\}$ into 2^{j+1} equal pieces by means of equally spaced radii. Number these sectors as $Q_{i,j}$, $i = 1, 2, \ldots, 2^{j+1}$. A particular case of [20, Thm. 2] asserts that a finite positive Borel measure ν on $\mathbb{D}$ is a 1-Carleson measure for A^{p} $(1 < p < \infty)$ if and only if the sequence $\left\{\mu\left(Q_{i,j}\right) 2^{2j/p}\right\}$ belongs to $\ell^{p'}$, where p' is the exponent conjugate to p, that is, $1/p + 1/p' = 1$.

For $k = 0, 1, 2, \ldots$, let I_k be the interval $[1 - 2^{-k}, 1 - 2^{-k-1})$. Since for any j at most two of the sectors $Q_{i,j}$, $i = 1, 2, \ldots, 2^{j+1}$, intersects the radius $[0,1)$, the following results follows.

Proposition 1 *Let μ be a positive finite Borel measure on $[0,1)$ and $1 < p < \infty$. Then μ is a 1-Carleson measure for A^p if and only if the sequence $\{2^{2k/p}\mu(I_k)\}_{k=0}^{\infty}$ belongs to $\ell^{p'}$.*

If μ is a positive finite Borel measure on $[0,1)$, $f \in \mathrm{Hol}(\mathbb{D})$ and $\int_{[0,1)} |f(t)|d\mu(t) < \infty$ it is clear that the integral

$$\int_{[0,1)} \left| \frac{f(t)}{1-tz} \right| d\mu < \infty, \quad \text{for every } z \in \mathbb{D}$$

and that $I_\mu(f)$ is a well defined analytic function in $\mathbb{D}$. Following the arguments in the proof of [12, Thm. 4], we have the following result.

Proposition 2 *Let μ be a positive finite Borel measure on $[0,1)$. Let h be a polynomial and let f be an analytic function in $\mathbb{D}$ such that $\int_{[0,1)} |f(t)|d\mu(t) < \infty$. Then*

$$\left| \int_{\mathbb{D}} h(z)\overline{I_\mu(f)}\, dA(z) \right| \leq 2 \int_{[0,1)} |f(t)|G(t)\, d\mu(t), \tag{2}$$

where $G(t) = \int_0^1 r|h(r^2 t)|dt$.

Throughout this paper the letter C denotes a positive constant that may change from one step to the next. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \leq CE_2$, respectively $E_1 \geq CE_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$.

Also, if $1 \leq p < \infty$, p' will stand for the exponent conjugate to p, that is, $1/p + 1/p' = 1$ if $1 < p < \infty$, while $1' = \infty$.

Let us remark that, just as in [12], a good number of the results that we are going to present could be extended to weighted Bergman spaces A^p_α for certain values of α. We have preferred to restrict ourselves to the unweighted case for simplicity.

2 The Hilbert-type operators $\mathcal{H}_\mu$ acting on the Bergman spaces A^p, with $1 \leq p < \infty$ and $p \neq 2$

We start considering the case $p = 1$. We have the following result.

Theorem 3 *Let μ be a positive finite Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) μ *is a* 2*-Carleson measure.*
(ii) μ *is a* 1*-Carleson measure for* A^1*.*
(iii) The operator I_μ *is well defined in* A^1*.*
(iv) The operator $\mathcal{H}_\mu$ *is well defined in* A^1*.*
(v) The operator I_μ *is well defined in* A^1 *and it is bounded from* A^1 *into itself.*
(vi) The operator $\mathcal{H}_\mu$ *is well defined in* A^1 *and it is bounded from* A^1 *into itself.*

Furthermore, if these conditions are satisfied then for any given $f \in A^1$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, *we have*

$$\mathcal{H}_\mu(f) = I_\mu(f), \tag{3}$$

$$\int_{[0,1)} \left| \frac{f(t)}{1-tz} \right| d\mu(t) < \infty, \quad z \in \mathbb{D}, \tag{4}$$

and

$$\sum_{k=1}^{\infty} \mu_{n+k}|a_k| < \infty, \quad n = 0, 1, 2, \dots. \tag{5}$$

The following result will be needed to prove Theorem 3.

Proposition 4 *There exists a positive constant* $K > 0$ *such that if* $f \in A^1$ *and* $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, *then there exists another function* $g \in A^1$ *of the form* $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $z \in \mathbb{D}$ *such that*

$$\|g\|_{A_1} \leq K\|f\|_{A_1}.$$
$$0 \leq |f(t)| \leq g(t), \ \ \textit{for all } t \in [0, 1).$$
$$0 \leq |a_n| \leq b_n, \ \ \textit{for } n = 0, 1, 2, \dots.$$

Proposition 4 will follow from the following result of Horowitz.

Theorem A. *There exists a constant* $K > 0$ *such that for any given* $f \in A^1$ *there exist two functions* $f_1, f_2 \in A^2$ *such that* $f = f_1 f_2$ *and* $\|f_1\|_{A^2}\|f_2\|_{A^2} \leq K\|f\|_{A^1}$.

Proof of Proposition 4. Let K be the constant in Theorem A. Take $f \in A^1$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, and $f_1, f_2 \in A^2$ such that $f = f_1 f_2$ and $\|f_1\|_{A^2}\|f_2\|_{A^2} \leq K\|f\|_{A^1}$.

Suppose that

$$f_1(z) = \sum_{n=0} \alpha_n z^n, \quad f_2(z) = \sum_{n=0} \beta_n z^n \quad z \in \mathbb{D}.$$

Set

$$g_1(z) = \sum_{n=0} |\alpha_n| z^n, \quad g_2(z) = \sum_{n=0} |\beta_n| z^n \quad z \in \mathbb{D}.$$

Then $g_1, g_2 \in A^2$ and $\|g_j\|_{A^2} = \|f_j\|_{A^2}$, $j = 1, 2$. Set $g(z) = g_1(z)g_2(z)$, $z \in \mathbb{D}$. Let $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $z \in \mathbb{D}$. We have

$$|f(t)| = |f_1(t)f_2(t)| \le g_1(t)g_2(t) = g(t), \quad 0 \le t < 1,$$

$$|a_n| = \left|\sum_{k=0}^{n} \alpha_k \beta_{n-k}\right| \le \sum_{k=0}^{n} |\alpha_k||\beta_{n-k}| = b_n, \quad n = 0, 1, 2, \ldots,$$
$$\|g\|_{A^1} \le \|g_1\|_{A^2}\|g_2\|_{A^2} = \|f_1\|_{A^2}\|f_2\|_{A^2} \le K\|f\|_{A^1}.$$

□

Proof of Theorem 3. The equivalence (i) ⇔ (ii) follows from the above mentioned result of Hastings.

The implication (ii) ⇒ (iii) is clear.

Nnow assume (iii). Take $f \in A^1$. Use Proposition 4 to take $g \in A^1$ with $0 \le |f(t)| \le g(t)$ for all $t \in [0, 1)$. Since $g \in A^1$, the integral defining $\mathcal{I}_\mu(g)(0)$ converges. Thus

$$\int_{[0,1)} |f(t)|\, d\mu(t) \le \int_{[0,1)} g(t)\, d\mu(t) = \mathcal{I}_\mu(g)(0) < \infty.$$

This gives that $f \in L^1(d\mu)$.

So far we have seen that (i), (ii), and (iii) are equivalent. Also, it is clear that these conditions imply (4).

Now assume (ii). Take $f \in A^1$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. Use Proposition 4 to take $g \in A^1$, $g(z) = \sum_{n=0}^{\infty} b_n z^n$, $z \in \mathbb{D}$, satisfying

$$\begin{aligned} \|g\|_{A_1} &\le K\|f\|_{A_1}, \\ 0 \le |f(t)| &\le g(t), \quad \text{for all } t \in [0, 1), \\ 0 \le |a_n| &\le b_n, \quad \text{for } n = 0, 1, 2, \ldots. \end{aligned}$$

Using (ii) we see that

$$\int_{[0,1)} g(t)\, d\mu(t) < \infty.$$

For every n, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \mu_{n+k}|a_k| \le \sum_{k=0}^{\infty} \mu_{n+k} b_k &= \sum_{k=0}^{\infty} \int_{[0,1)} t^{n+k} b_k d\mu(t) \\ &= \int_{[0,1)} t^n \sum_{k=0}^{\infty} b_k t^k\, d\mu(t) = \int_{[0,1)} t^n g(t)\, d\mu(t) \\ &\le \int_{[0,1)} g(t)\, d\mu(t) < \infty. \end{aligned}$$

Thus (5) holds. Also, Fubini's theorem yields that

$$\int_{[0,1)} t^n f(t)\, d\mu(t) = \sum_{k=0}^{\infty} \mu_{n+k} a_k, \quad n = 0, 1, 2, \ldots,$$

and

$$I_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n f(t)\, d\mu(t) \right) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n, \quad z \in \mathbb{D}.$$

Hence, $\mathcal{H}_\mu(f)$ is well defined and coincides with $\mathcal{I}_\mu(f)$. So, we have proved that (ii) implies (iv), (3), and (5).

Using Proposition 4 it follows readily that (iv) implies (5) and it is clear that this implies (ii).

We already know that the conditions (i), (ii), (iii), and (iv) are equivalent and that they imply (3), (4), and (5).

The implications (v) $\Rightarrow$ (iii) and (vi) $\Rightarrow$ (iv) are obvious. Hence it remains to prove that (i) implies (v) and (vi). To prove this we shall use duality.

Recall that a function $f \in \mathrm{Hol}(\mathbb{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions is denoted by $\mathcal{B}$, it is a Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ just defined. The little Bloch space $\mathcal{B}_0$ is the space of those $f \in \mathcal{B}$ such that

$$\lim_{|z| \to 1} (1 - |z|^2)|f'(z)| = 0.$$

The space $\mathcal{B}_0$ is a closed subspace of $\mathcal{B}$, in fact, it is the closure of the polynomials in the Bloch norm. We refer to [2] and to [25, Ch. 5] for the theory of Bloch functions. Let us remark that if $f \in \mathcal{B}$ then

$$|f(z)| \le \|f\|_{\mathcal{B}} \log \frac{2}{1 - |z|}, \quad z \in \mathbb{D}. \tag{6}$$

Also, the dual $(\mathcal{B}_0)^\star$ of $\mathcal{B}_0$ can be identified with A^1 under the integral pairing

$$\langle h, f \rangle = \int_{\mathbb{D}} h(z)\overline{f(z)}\, dA(z), \quad h \in \mathcal{B}_0, \;\; f \in A^1,$$

(see [25, Thm. 5.15]).

Assume (i) (or, equivalently (ii)). Take $f \in A^1$ and let h be a polynomial. Using Proposition 2, we see that

$$\left| \int_{\mathbb{D}} h(z)\overline{I_\mu(f)}\, dA(z) \right| \le 2 \int_{[0,1)} |f(t)| G(t)\, d\mu(t), \tag{7}$$

where $G(t) = \int_0^1 r|h(r^2t)|dr$. Using (6) we see that there exists $C > 0$ such

$$G(t) \le C\|h\|_{\mathcal{B}},$$

for every $t \in [0,1)$ and any polynomial h. This, (7), and the fact that the embedding $A^1 \to L^1(d\mu)$ is continuous give that

$$\begin{aligned}\left|\int_{\mathbb{D}} h(z)\overline{I_\mu(f)}\, dA(z)\right| &\le 2C\|h\|_{\mathcal{B}} \int_{[0,1)} |f(t)|\, d\mu(t) \\ &\le C'\|h\|_{\mathcal{B}}\|f\|_{A^1}, \quad \text{for any polynomial } h,\end{aligned}$$

for a certain positive constant C'. Since $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm, it follows that $I_\mu(f)$ defines a continuous linear functional in $\mathcal{B}_0$ with norm smaller than or equal to $C'\|f\|_{A^1}$. Thus, $\mathcal{I}_\mu(f) \in A^1$ and $\|\mathcal{I}_\mu(f)\|_{A^1} \le C'\|f\|_{A^1}$. Hence, we have proved that (i) implies (v).

Since we have seen that (i) implies that $\mathcal{H}_\mu$ and I_μ coincide in A^1 we also have that (i) implies (vi). □

Let us turn to the case $1 < p < 2$. We have the following result.

Theorem 5 *Suppose that $1 < p < 2$ and let μ be a positive finite Borel measure on $[0,1)$. Then:*

(i) The measure μ is a 1-Carleson measure for A^p if and only if the sequence $\left\{2^{2k/p}\mu(I_k)\right\}$ belongs to $\ell^{p'}$.
Furthermore, if these conditions are satisfied then μ is a vanishing $2/p$-Carleson measure and the operator I_μ is well defined in A^p and is bounded from A^p into itself.

(ii) If the operator $\mathcal{H}_\mu$ is well defined in A^p and is bounded from A^p into itself then μ is a $2/p$-Carleson measure.

(iii) If there exists a C^1-decreasing function $\varphi : [0,1) \to [0,\infty)$ for which

$$\int_0^1 \frac{\varphi(t)}{1-t}\, dt < \infty \quad \text{and} \quad \mu([t,1)) \lesssim (1-t)^{2/p}\varphi(t) \tag{8}$$

then $\mathcal{H}_\mu$ is well defined in A^p and is bounded from A^p into itself.

Proof The fact that μ is a 1-Carleson measure for A^p if and only if the sequence $\left\{2^{2k/p}\mu(I_k)\right\}$ belongs to $\ell^{p'}$ follows from Proposition 1.

Also, if μ is a 1-Carleson measure for A^p and $f \in A^p$, then

$$\int_{[0,1)} \left|\frac{f(t)}{1-tz}\right| d\mu(t) < \infty$$

and, hence, $\mathcal{I}_\mu(f)$ is well defined. Furthermore, using Minkowski's integral inequality, we see that

$$\left[\int_{\mathbb{D}} |I_\mu(f)(z)|^p \, dA(z)\right]^{1/p} \le \left[\int_{\mathbb{D}} \left(\int_{[0,1)} \left|\frac{f(t)}{1-tz}\right| d\mu(t)\right)^p\right]^{1/p}$$
$$\le \int_{[0,1)} |f(t)| \left(\int_{\mathbb{D}} \frac{dA(z)}{|1-tz|^p}\right)^{1/p} d\mu(t).$$

Since $1 < p < 2$,

$$\int_{\mathbb{D}} \frac{dA(z)}{|1-tz|^p} \lesssim \int_0^1 \frac{1}{(1-r)^{p-1}} \, dr < \infty.$$

Using the last two inequalities and the fact that μ is a 1-Carleson measure for A^p, we obtain

$$\left[\int_{\mathbb{D}} |I_\mu(f)(z)|^p \, dA(z)\right]^{1/p} \lesssim \int_{[0,1)} |f(t)| \, d\mu(t) \lesssim \|f\|_{A^p}.$$

Thus, I_μ is bounded from A^p into itself.

To finish the proof of (i), assume that

$$\sum_{k=0}^{\infty} 2^{2kp'/p} \mu(I_k)^{p'} < \infty.$$

Take $b \in (1/2, 1)$. Take $j \in \mathbb{N}$ such that $b \in I_j$. Then, we have $(1-b) \asymp 2^{-j}$ and

$$\left(\sum_{k=j}^{\infty} 2^{2kp'/p} \mu(I_k)^{p'}\right)^{1/p'} = o(1).$$

Using Hölder's inequality and these facts, we obtain

$$\mu([b,1)) \le \sum_{k=j}^{\infty} \mu(I_k) = \sum_{k=j}^{\infty} 2^{2k/p} \mu(I_k) 2^{-2k/p}$$
$$\le \left(\sum_{k=j}^{\infty} 2^{2kp'/p} \mu(I_k)^{p'}\right)^{1/p'} \left(\sum_{k=j}^{\infty} 2^{-2k}\right)^{1/p}$$
$$\asymp o(1) 2^{-2j/p} \asymp o(1)(1-b)^{2/p}.$$

Thus μ is a vanishing $2/p$-Carleson measure.

Let us turn to prove (ii).

Assume that $\mathcal{H}_\mu$ is well defined in A^p and that it is a bounded operator from A^p into itself. For $0 < b < 1$, set

$$f_b(z) = \frac{1-b}{(1-bz)^{2/p+1}}, \quad z \in \mathbb{D}.$$

Then

$$f_b \in A^p, \quad \text{and} \quad \|f_b\|_{A^p} \asymp 1.$$

Hence,

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{A^p}^p. \tag{9}$$

Now,

$$f_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k \quad z \in \mathbb{D}, \qquad \text{with } a_{k,b} \asymp (1-b) b^k k^{2/p}.$$

Since the $a_{k,b}$'s are positive, it follows that the sequence $\{\sum_{k=0}^{\infty}\mu_{n+k}a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers. Using this, [12, Prop. 1], and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}\|\mathcal{H}_\mu(f_b)\|_{A^p}^p &\gtrsim \sum_{n=1}^{\infty} n^{p-3}\left(\sum_{k=1}^{\infty}\mu_{n+k}a_{k,b}\right)^p \\ &= \sum_{n=1}^{\infty} n^{p-3}\left(\sum_{k=1}^{\infty}a_{k,b}\int_{[0,1)}t^{n+k}\,d\mu(t)\right)^p \\ &\gtrsim (1-b)^p\sum_{n=1}^{\infty} n^{p-3}\left(\sum_{k=1}^{\infty}k^{2/p}b^k\int_{[b,1)}t^{n+k}\,d\mu(t)\right)^p \\ &\ge (1-b)^p\mu([b,1))^p\sum_{n=1}^{\infty} n^{p-3}b^{np}\left(\sum_{k=1}^{\infty}k^{2/p}b^{2k}\right)^p \\ &\asymp (1-b)^p\mu([b,1))^p\frac{1}{(1-b)^{2+p}}\sum_{n=1}^{\infty} n^{p-3}b^{np} \\ &\asymp \frac{\mu([b,1))^p}{(1-b)^2}.\end{aligned} \tag{10}$$

To obtain the last relation we have used that $\sum_{n=1}^{\infty} n^{p-3}b^{np} \asymp 1$ because $p<2$. Using (10) and (9) we deduce that μ is a $2/p$-Carleson measure.

For future reference, let us remark that the argument we have just presented works also for $p=1$. We did not use it in the proof of Theorem 3 because the condition "$\mathcal{H}_\mu$ is well defined in A^1" is enough to conclude that μ is a 2-Carleson measure.

We shall use duality to prove (iii). Using [25, Thm. 4.25], we see that the dual of $A^{p'}$ can be identified with A^p under the pairing

$$\langle h,f\rangle = \int_{\mathbb{D}} h(z)\overline{f(z)}\,dA(z), \quad h\in A^{p'},\ \ f\in A^p.$$

Assume that μ and φ are as in (iii). Take h and P two polynomials. Clearly,

$$\int_{[0,1)}|P(t)|\,d\mu(t)<\infty, \text{ and } \mathcal{I}_\mu(P) \text{ and } \mathcal{H}_\mu(P) \text{ are well defined and } \mathcal{I}_\mu(P)=\mathcal{H}_\mu(P).$$

Then, using Proposition 2 we see that

$$\left|\int_{\mathbb{D}} h(z)\overline{\mathcal{H}_\mu(P)}\,dA(z)\right| \le 2\int_{[0,1)}|P(t)|G(t)\,d\mu(t), \tag{11}$$

where $G(t)=\int_0^1 r|h(r^2t)|dr$. Set $A=2/(pp')$ and let ν be the measure defined by

$$d\nu(t)=(1-t)^{Ap}d\mu(t)=(1-t)^{2/p'}\,d\mu(t).$$

It is clear that ν is a 2-Carleson measure and, hence, it is a p-Carleson measure for A^p. Thus,

$$\left(\int_{[0,1)}|P(t)|^p(1-t)^{Ap}\,d\mu(t)\right)^{1/p} \lesssim \|P\|_{A^p}.$$

Using Hölder's inequality and this in (11), we obtain

$$\left|\int_{\mathbb{D}} h(z)\overline{\mathcal{H}_\mu(P)}\,dA(z)\right| \lesssim \int_{[0,1)}|P(t)|(1-t)^{A}G(t)(1-t)^{-A}\,d\mu(t) \tag{12}$$

$$\lesssim \left(\int_{[0,1)} |P(t)|^p(1-t)^{Ap}d\mu(t)\right)^{1/p}\left(\int_{[0,1)} G(t)^{p'}(1-t)^{-Ap'}\,d\mu(t)\right)^{1/p'}$$
$$\lesssim \left(\int_{[0,1)} G(t)^{p'}(1-t)^{-Ap'}\,d\mu(t)\right)^{1/p'}\|P\|_{A^p}.$$

Arguing as in [12, pp. 814-816] we see that

$$\int_{[0,1/2)} G(t)^{p'}(1-t)^{-Ap'}\,d\mu(t) \lesssim \|h\|^{p'}_{A^{p'}} \tag{13}$$

and

$$\int_{[1/2,1)} G(t)^{p'}(1-t)^{-Ap'}\,d\mu(t) \lesssim \int_{[0,1)} F(t)d\mu(t), \tag{14}$$

where

$$F(t) = \left(\int_{1-t}^{1} M_\infty(1-s,h)ds\right)^{p'}(1-t)^{-Ap'} = \left(\int_0^t M_\infty(s,h)ds\right)^{p'}(1-t)^{-Ap'}.$$

Observe that F is an increasing function. Since $2/p' < 1$, we have

$$\begin{aligned} F(t)\mu([t,1)) &\lesssim (1-t)^{2/p}\varphi(t)F(t) = \varphi(t)\left(\int_0^t M_\infty(s,h)\,ds\right)^{p'} \\ &\lesssim \varphi(t)\left(\int_0^t \frac{\|h\|_{A_{p'}}}{(1-s)^{2/p'}}\,ds\right)^{p'} \lesssim \varphi(t)\|h\|^{p'}_{A_{p'}}. \end{aligned} \tag{15}$$

Integrating by parts and using (15) and the fact that $\varphi' \le 0$, we obtain

$$\begin{aligned} \int_{[0,1)} F(t)\,d\mu(t) &= F(0)\mu([0,1)) - \lim_{t\to 1}\mu([t,1))F(t) + \int_0^1 \mu([t,1))F'(t)dt \\ &= \int_0^1 \mu([t,1))F'(t)dt \\ &\lesssim \int_0^1 (1-t)^{2/p}\varphi(t)F'(t)dt \\ &\lesssim -\int_0^1 \frac{d}{dt}\left[(1-t)^{2/p}\varphi(t)\right]F(t)\,dt \\ &\lesssim \frac{2}{p}\int_0^1 F(t)(1-t)^{2/p-1}\varphi(t)\,dt \\ &\lesssim \left(\int_0^1 \frac{\varphi(t)}{1-t}dt\right)\|h\|^{p'}_{A_{p'}}. \\ &\lesssim \|h\|^{p'}_{A_{p'}}. \end{aligned}$$

Using this, (12), (13), and(14), we obtain that

$$\left|\int_{\mathbb{D}} h(z)\overline{\mathcal{H}_\mu(P)}\,dA(z)\right| \lesssim \|h\|_{A^{p'}}\|P\|_{A^p},$$

for any polynomial h. Since the polynomials are dense in $A^{p'}$, we deduce that $\mathcal{H}_\mu(P)$ gives a bounded linear functional in $A^{p'}$ with norm bounded by $\|P\|_{A^p}$, that is

$$\mathcal{H}_\mu(P) \in A^p \quad \text{and } \|\mathcal{H}_\mu(P)\|_{A^p} \lesssim \|P\|_{A^p}. \tag{16}$$

Take now $f \in A^p$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $z \in \mathbb{D}$. For $N = 0, 1, 2, \dots$, set

$$P_N(z) = \sum_{k=0}^{N} a_k z^k, \quad z \in \mathbb{D}.$$

Since (16) holds for any polynomial P, it is clear that the sequence $\{\mathcal{H}_\mu(P_N)\}_{N=0}^{\infty}$ converges in A^p to a certain function $g \in A^p$ with

$$\|g\|_{A^p} \lesssim \|f\|_{A^p}.$$

Let

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

For every N, we have

$$\mathcal{H}_\mu(P_N)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{N} \mu_{n+k} a_k \right) z^n, \quad z \in \mathbb{D}.$$

Since

$$\lim_{N\to\infty} \mathcal{H}_\mu(P_N) = g \quad \text{in } A^p,$$

it follows that, for every n,

$$\sum_{k=0}^{N} \mu_{n+k} a_k, \to b_n, \quad \text{as } N \to \infty,$$

that is, for every n, the series $\sum_{k=0}^{\infty} \mu_{n+k} a_k$ converges and

$$g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_k \right) z^n, \quad z \in \mathbb{D}.$$

Thus, $\mathcal{H}_\mu(f)$ is a well defined analytic function which belongs to A^p and $\|H_\mu(f)\|_{A^p} \lesssim \|f\|_{A^p}$. □

The finite positive Borel measures μ for which $\mathcal{H}_\mu$ is well defined and bounded from A^p into itself for $2 < p < \infty$ are characterized in [12, Thm. 4]. These results are complemented in the following theorem.

Theorem 6 *Suppose that $2 < p < \infty$ and let μ be a positive finite Borel measure on $[0, 1)$. Then:*

(i) The measure μ is a 1-Carleson measure for A^p if and only if the sequence $\left\{2^{2k/p}\mu(I_k)\right\}$ belongs to $\ell^{p'}$.
(ii) If μ is a 1-Carleson measure then it is a 1-Carleson measure for A^p.
(iii) If μ is a 1-Carleson measure and $f \in A^p$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, then (4) and (5) hold, the operators I_μ and $\mathcal{H}_\mu$ are well defined in A^p, and $\mathcal{H}_\mu(f) = I_\mu(f)$ for all $f \in A^p$.
(iv) The Hilbert-type operator $\mathcal{H}_\mu$ is well defined in A^p and it is bounded from A^p into itself if and only if μ is a 1-Carleson measure.

Part (i) follows from Proposition 1 and (ii) follows readily from (i). Let us remark that (ii) can be deduced also using that if $p > 2$ then $2/p < 1$ and that if $f \in A^p$ then $|f(z)| \lesssim (1-|z|)^{-2/p}$, arguing as in the proof of [12, Thm. 3].

Part (iii) is contained in [12, Thm. 3 and Cor. 1] and their proofs. Indeed, using (ii) we see that if $f \in A^p$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, then f satisfies condition [12, (12), p. 809] and $\sum_{k=0}^{\infty} |a_k|/(k+1) < \infty$ (see [12, p. 810]) and then (iii) follows.

Part (iv) is contained in [12, Thm. 4].

3 The Hilbert-type operators $\mathcal{H}_\mu$ acting on A^2

This section is devoted to considerin the operators $\mathcal{H}_\mu$ acting on A^2. The question of characterizing those μ for which I_μ and $\mathcal{H}_\mu$ are well defined in A^2 is solved in the following theorem where we give also a condition in terms of the moments of μ which is equivalent to μ being a 1-Carleson measure for A^2.

Theorem 7 *Let μ be a positive finite Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) μ is a 1-Carleson measure for A^2.
(ii) The sequence $\left\{2^k \mu(I_k)\right\}$ belongs to ℓ^2.
(iii) If $f \in A^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, then $\sum_{n=0}^{\infty} \mu_n |a_n| < \infty$.
(iv) $\sum_{n=0}^{\infty} n\mu_n^2 < \infty$.
(v) The operator I_μ is well defined on A^2.
(vi) The operator $\mathcal{H}_\mu$ is well defined on A^2.

Furthermore, when these conditions are satisfied then for any given $f \in A^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, we have

$$\mathcal{H}_\mu(f) = I_\mu(f), \tag{17}$$

$$\int_{[0,1)} \left| \frac{f(t)}{1-tz} \right| d\mu(t) < \infty, \quad z \in \mathbb{D}, \tag{18}$$

and

$$\sum_{k=1}^{\infty} \mu_{n+k} |a_k| < \infty, \quad n = 0, 1, 2, \dots. \tag{19}$$

Proof The equivalence (i) $\Leftrightarrow$ (ii) follows from Proposition 1.

To see that (i) $\Rightarrow$ (iii), take $f \in A^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. Set

$$g(z) = \sum_{n=0}^{\infty} |a_n| z^n, \quad z \in \mathbb{D}.$$

Then $g \in A^2$ and $\|g\|_{A^2} = \|f\|_{A^2}$. Since $g(t) = \sum_{n=0}^{\infty} |a_n| t^n$ for all $t \in [0,1)$, we have

$$\sum_{n=0}^{\infty} \mu_n |a_n| = \int_{[0,1)} g(t)\, d\mu(t) \lesssim \|g\|_{A^2} = \|f\|_{A^2} < \infty.$$

Thus (iii) holds.

Now assume (iii). Let f and g be as in the proof of the preceding implication. Then

$$\int_{[0,1)} |f(t)|\, d\mu(t) \le \int_{[0,1)} |g(t)| = \sum_{n=0}^{\infty} \mu_n |a_n| < \infty.$$

Thus, (ii) holds.

Condition (iii) if equivalent to saying that

$$\sum_{n=0}^{\infty} \left|\frac{a_n}{n+1}\right| (\sqrt{n}\,\mu_n) < \infty \text{ for any sequence } \{a_n\}_{n=0}^{\infty} \text{ such that } \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < \infty.$$

In other words (iii) is equivalent to saying that

$$\sum_{n=0}^{\infty} |b_n|(\sqrt{n}\,\mu_n) < \infty \text{ for any sequence } \{b_n\}_{n=0}^{\infty} \in \ell^2.$$

By duality, this is equivalent to saying that the sequence $\{\sqrt{n}\,\mu_n\}_{n=0}^{\infty}$ belongs to ℓ^2 which is (iv). Thus, we have proved that (iii) and (iv) are equivalent.

So far we have proved that the first four conditions are equivalent. The arguments used in the case $p = 1$ give that these conditions imply that (18) and (19) hold for any $f \in A^2$. This implies (v), (vi), and (17).

It remains to prove that (v) $\Rightarrow$ (i) and that (vi) $\Rightarrow$ (i).

For $f \in A^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, set $g(z) = \sum_{n=0}^{\infty} |a_n| z^n$, $z \in \mathbb{D}$, as above.

Assume (v). Since $g \in A^2$, $I_\mu(g)(0)$ is well defined, that is, $\int_{[0,1)} g(t)\, d\mu(t) < \infty$. Since $|f(t)| \le g(t)$, $0 \le t < 1$, we deduce that $\int_{[0,1)} |f(t)|\, d\mu(t) < \infty$. Thus (i) holds.

Now assume (vi). The fact that $g \in A^2$ implies that $\mathcal{H}_\mu(g)(0)$ is well defined, that is, $\sum_{n=0}^{\infty} \mu_n |a_n| < \infty$. Thus (iii) holds. Hence (i) holds.

□

Let us turn to deal with the boundednessof $\mathcal{H}_\mu$. A necessary condition for the boundedness of $\mathcal{H}_\mu$ was given in [13]. Indeed, the following result is proved in [13, Thm. 3].

Theorem B. *(i) Let μ be a positive and finite Borel measure on $[0,1)$. If μ is a 1-logarithmic 1-Carleson measure, then $\mathcal{H}_\mu$ is bounded from A^2 into itself.*

(ii) If $0 < \beta < 1$, then there exists a positive and finite Borel measure μ on $[0,1)$ which is a β-logarithmic 1-Carleson measure but such that $\mathcal{H}_\mu(A^2) \not\subset \mathcal{A}^2$.

In our following result we give a sufficient condition for the boundedness of $\mathcal{H}_\mu$ in A^2.

Theorem 8 *Let μ be a positive finite Borel measure on $[0,1)$. If the operator $\mathcal{H}_\mu$ is bounded from A^2 into itself then μ is a 1/2-logarithmic 1-Carleson measure.*

Proof For $0 < b < 1$, set

$$f_b(z) = \frac{1-b}{(1-bz)^2} = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad z \in \mathbb{D}.$$

Then

$$f_b \in A^2, \text{ and } \|f_b\|_{A^2} \asymp 1.$$

Hence

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|^2_{A^2}.$$

Bearing in mind that $a_{k,b} \asymp (1-b)kb^k$ and arguing as in the proof of part (ii) of Theorem 5 but taking $p=2$, we obtain

$$\begin{aligned}
1 \gtrsim \|\mathcal{H}_\mu(f_b)\|^2_{A^2} &\gtrsim \sum_{n=1}^{\infty} n^{-1}\left(\sum_{k=1}^{\infty}\mu_{n+k}a_{k,b}\right)^2 \\
&= \sum_{n=1}^{\infty} n^{-1}\left(\sum_{k=1}^{\infty}a_{k,b}\int_{[0,1)} t^{n+k}\,d\mu(t)\right)^2 \\
&\gtrsim (1-b)^2\sum_{n=1}^{\infty} n^{-1}\left(\sum_{k=1}^{\infty}kb^k\int_{[b,1)} t^{n+k}\,d\mu(t)\right)^2 \\
&\geq (1-b)^2\mu([b,1))^2\sum_{n=1}^{\infty} n^{-1}b^{2n}\left(\sum_{k=1}^{\infty}kb^{2k}\right)^2 \\
&\asymp (1-b)^2\mu([b,1))^2\frac{1}{(1-b)^4}\sum_{n=1}^{\infty} n^{-1}b^{2n} \\
&\asymp \frac{\mu([b,1))^2}{(1-b)^2}\log\frac{1}{1-b}.
\end{aligned}$$

Then it follows that μ is a 1/2-logarithmic 1-Carleson measure.

□

4 Compactness

In this section we deal with the question of characterizing the measures μ for which $\mathcal{H}_\mu$ is compact on A^p. For $p \neq 2$, we have the following result.

Theorem 9 *Let μ be a positive finite Borel measure on $[0,1)$.*

(i) The operator $\mathcal{H}_\mu$ is compact from A^1 into itself if and only if μ is a vanishing 2-Carleson measure.

(ii) If $1 < p < 2$ and $\mathcal{H}_\mu$ is compact from A^p into itself then μ is a vanishing $2/p$-Carleson measure.

(iii) If $2 < p < \infty$ then $\mathcal{H}_\mu$ is compact from A^p into itself if and only μ is a vanishing 1-Carleson measure.

Proof Let us start by proving that being a $2/p$-Carleson measure is a necessary condition for the compactness of $\mathcal{H}_\mu$ when $1 \leq p < 2$. So, assume that $1 \leq p < 2$ and that that $\mathcal{H}_\mu$ is a compact operator from A^p into itself. We argue as in the proof of the corresponding result regarding boundedness. For $0 < b < 1$, we set

$$f_b(z) = \frac{1-b^2}{(1-bz)^{2/p}}, \quad z \in \mathbb{D}.$$

We have $\|f_b\|_{A^p}^p \asymp 1$ and

$$\{f_b\} \to 0, \quad \text{as } b \to 1, \text{ uniformly on compact subsets of } \mathbb{D}.$$

Since $\mathcal{H}_\mu$ is compact from A^p into itself, we have

$$\|\mathcal{H}_\mu(f_b)\|_{A^p} \to 0, \quad \text{as } b \to 1. \tag{20}$$

Using (10) (see also the remark regarding A^1 at the end of the proof of part (ii) of Theorem 5), we see that

$$\mu([b,1))^p(1-b)^{-2} \lesssim \|\mathcal{H}_\mu(f_b)\|_{A^p}^p.$$

This and (20) give that μ is a vanishing $2/p$-Carleson measure.

A similar argument works to prove that being a vanishing 1 Carleson measure is a necessary condition for compactness in the case $2 < p < \infty$. Indeed, if f_b is as above and $2 < p < \infty$, the substitute for (10) is the inequality $\mu([b,1))(1-b)^{-1} \lesssim \|\mathcal{H}_\mu(f_b)\|_{A^p}$ (see [12, p. 813]). Using this and arguing as in the preceding paragraph we obtain the result. We omit the details.

Suppose now that $2 < p < \infty$ and that μ is a vanishing 1-Carleson measure. We recall that the dual of $A^{p'}$ can be identified with A^p with the pairing

$$\langle h, f\rangle = \int_{\mathbb{D}} h(z)\overline{f(z)}\, dA(z), \quad h \in H^{p'},\ f \in A^p.$$

Let $\{f_n\}$ be a sequence of functions in A^p which is bounded in A^p and tends to 0 uniformly on compact subsets of $\mathbb{D}$. Bearing in mind [22, Lem. 3.7], we have to prove that $\|\mathcal{H}_\mu(f_n)\|_{A^p} \to 0$. Looking at [12, pp. 813–814] we see that

$$|\langle h, \mathcal{H}_\mu(f_n)\rangle| \lesssim \|h\|_{A^{p'}}^{p'} \left(\int_{[0,1)} |f_n(t)|^p(1-t)\, d\mu(t)\right)^{1/p}, \quad h \in A^{p'},\ n \in \mathbb{N}.$$

This is equivalent to saying that, for every n, $\mathcal{H}_\mu(f_n) \in A^p$ and

$$\|\mathcal{H}_\mu(f_n)\|_{A^p} \lesssim \left(\int_{[0,1)} |f_n(t)|^p(1-t)\, d\mu(t)\right)^{1/p}. \tag{21}$$

Since μ is a vanishing 1-Carleson measure, the measure ν defined by

$$d\nu(t) = (1-t)d\mu(t)$$

is a vanishing 2-Carleson measure. Then, using [21, Thm. 4.3], we deduce that the embedding $A^p \hookrightarrow L^p(d\nu)$ is compact. We claim that

$$\{f_n\} \text{ tends to 0 in } L^p(d\nu). \tag{22}$$

Let us prove this by contradiction. Suppose (22) is not true. Then, using the fact that the embedding $A^p \hookrightarrow L^p(d\nu)$ is compact, we can find $\varepsilon > 0$, a subsequence $\{f_{n_j}\}$ of $\{f_n\}$, and $g \in L^p(d\nu)$ such that $\{f_{n_j}\} \to g$ in $L^p(d\nu)$ and $\|f_{n_j}\|_{L^p(d\nu)} \geq \varepsilon$, for all j. Now, using a standard result in measure theory we can assert that a subsequence of $\{f_{n_j}\}$ tends to g ν-almost everywhere. Since $\{f_n\} \to 0$ uniformly on compact subsets of $\mathbb{D}$, we deduce that

$$g = 0, \quad \nu\text{-almost everywhere.}$$

This yields that $\{f_{n_j}\} \to 0$ in $L^p(d\nu)$ which is a contradiction. Thus (22) is true. This is the same as saying that

$$\left(\int_{[0,1)} |f_n(t)|^p(1-t)\, d\mu(t)\right)^{1/p} \to 0.$$

This and (21) give that $\|\mathcal{H}_\mu(f_n)\|_{A^p} \to 0$, as desired.

Finally, let us prove that if μ is a vanishing 2-Carleson measure then $\mathcal{H}_\mu$ is compact on A^1. So suppose that μ is a vanishing 2-Carleson measure. We recall that the dual of $\mathcal{B}_0$ can be identified with A^1 with the pairing

$$\langle h, f\rangle = \int_{\mathbb{D}} h(z)\overline{f(z)}\, dA(z), \quad h \in \mathcal{B}_0,\ f \in A^1.$$

Let $\{f_n\}$ be a sequence of functions in A^1 which is bounded in A^1 and tends to 0 uniformly on compact subsets of $\mathbb{D}$. Just as in the preceding case, we have to prove that $\|\mathcal{H}_\mu(f_n)\|_{A^1} \to 0$. Looking at the proof of part (ii) of Theorem 3 we see that

$$|\langle h, \mathcal{H}_\mu(f_n)\rangle| \lesssim \|h\|_{\mathcal{B}} \int_{[0,1)} |f(t)|\, d\mu(t), \quad h \in \mathcal{B}_0,\ n \in \mathbb{N}. \tag{23}$$

Since μ is a vanishing 2-Carleson measure, using [21, Thm. 4.3], the embedding $A^1 \hookrightarrow L^1(d\nu)$ is compact. Then arguing as in the case $1 < p < 2$, we deduce that the sequence $\{f_n\}$ tends to to 0 in $L^1(d\mu)$. This is the same as saying that

$$\left(\int_{[0,1)} |f_n(t)|\, d\mu(t)\right)^{1/p} \to 0.$$

This and (23) give that $\|\mathcal{H}_\mu(f_n)\|_{A^1} \to 0$, as desired. □

Now, we turn to consider the case $p = 2$. We have the following result.

Theorem 10 *Let μ be a positive finite Borel measure on $[0, 1)$. If μ is a vanishing 1-logarithmic 1-Carleson measure, then $\mathcal{H}_\mu$ is a compact operator from A^2 into itself.*

Proof Assume that If μ is a vanishing 1-logarithmic 1-Carleson measure. Theorem B implies that $\mathcal{H}_\mu$ is bounded from A^2 onto itself.

For $N \in \mathbb{N}$ let $\mathcal{H}_{\mu,N}$ be defined as follows:

If $f \in A^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$, we set

$$\mathcal{H}_{\mu,N}(f)(z) = \sum_{n=0}^{N} \left(\sum_{k=0} \mu_{n+k} a_k\right) z^n, \quad z \in \mathbb{D}.$$

Clearly, the operators $\mathcal{H}_{\mu,N}$ are finite rank operators from A^2 into itself. Then the compactness of $\mathcal{H}_\mu$ will follow from the fact that

$$\mathcal{H}_{\mu,N} \to \mathcal{H}_\mu, \quad \text{in the operator norm.} \tag{24}$$

Let us prove (24).

Take $\varepsilon > 0$.

Take $f \in A^2$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z \in \mathbb{D}$. Set

$$g(z) = \sum_{n=0}^{\infty} |a_n| z^n, \quad z \in \mathbb{D}.$$

Then $g \in A^2$ and $\|g\|_{A^2} = \|f\|_{A^2}$. For every $N \in \mathbb{N}$, we have

$$\mathcal{H}_\mu(f)(z) - \mathcal{H}_{\mu,N}(f)(z) = \sum_{n=N+1}^{\infty} \left(\sum_{k=0} \mu_{n+k} a_k\right) z^n, \quad z \in \mathbb{D}.$$

Hence,

$$\|\mathcal{H}_\mu(f) - \mathcal{H}_{\mu,N}(f)\|_{A^2}^2 = \sum_{n=N+1}^{\infty} \frac{1}{n+1} \left| \sum_{k=0}^{\infty} \mu_{n+k} a_k \right|^2 \tag{25}$$

$$\leq \sum_{n=N+1}^{\infty} \frac{1}{n+1} \left(\sum_{k=0}^{\infty} \mu_{n+k} |a_k| \right)^2 = \|\mathcal{H}_\mu(g) - \mathcal{H}_{\mu,N}(g)\|_{A^2}^2.$$

Since μ is a vanishing 1-logarithmic 1-Carleson measure, $\mu_n = \mathrm{o}(1/(n \log n))$. Then take N_0 so that

$$\mu_n \leq \frac{\varepsilon}{(n+1)\log(n+1)}, \quad \text{for every } N \geq N_0. \tag{26}$$

Let ν be the Borel measure on $[0,1)$ defined by

$$d\nu(t) = \left(\log \frac{2}{1-t} \right)^{-1}.$$

Then ν is a 1-logarithmic 1-Carleson measure and

$$\nu_n \asymp \frac{1}{(n+1)\log(n+1)}.$$

This and (26) imply that, for $N \geq N_0$, we have

$$\|\mathcal{H}_\mu(f) - \mathcal{H}_{\mu,N}(f)\|_{A^2}^2 \leq \varepsilon^2 \sum_{n=N+1}^{\infty} \left(\sum_{k=0}^{\infty} \frac{|a_k|}{(n+k+1)\log(n+k+1)} \right)^2$$

$$\leq \varepsilon^2 \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{|a_k|}{(n+k+1)\log(n+k+1)} \right)^2 \asymp \varepsilon^2 \|\mathcal{H}_\nu(g)\|_{A^2}^2.$$

Using this and the fact that $\mathcal{H}_\nu$ is bounded from A^2 into itself, we deduce that

$$\|\mathcal{H}_\mu(f) - \mathcal{H}_{\mu,N}(f)\|_{A^2}^2 \lesssim \varepsilon^2 \|g\|_{A^2}^2 = \varepsilon^2 \|f\|_{A^2}^2.$$

Then (24) follows. This finishes the proof. □

Let us go one step forward by studying the membership of $\mathcal{H}_\mu$ in the Schatten classess $\mathcal{S}_p(A^2)$.

Given a separable Hilbert space X and $0 < p < \infty$, let $\mathcal{S}_p(X)$ denote the Schatten p-class of operators on X. The class $\mathcal{S}_p(X)$ consists of those compact operators T on X whose sequence of singular numbers $\{\lambda_n\}$ belongs to ℓ^p, the space of p-summable sequences. It is well known that, if λ_n are the singular numbers of an operator T, then

$$\lambda_n = \lambda_n(T) = \inf\{\|T - K\| : \text{ rank } K \leq n\}.$$

Thus finite rank operators belong to every $\mathcal{S}_p(X)$, and the membership of an operator in $\mathcal{S}_p(X)$ measures in some sense the size of the operator. The class $\mathcal{S}_2(X)$ is the class of Hilbert-Schmidt operators on X.

[11, Thm. 6] asserts that if μ is a positive finite Borel measure on $[0,1)$, then $\mathcal{H}_\mu$ is a Hilbert-Schmidt operator on A^2 if and only if

$$\int_{[0,1)} \frac{\mu([t,1))}{(1-t)^2} \log \frac{1}{1-t} \, d\mu(t) < \infty. \tag{27}$$

Let us remark that (27) is equivalent to

$$\sum_{n=0}^{\infty} n \int_{[0,1)} t^n \mu([t,1)) \log \frac{1}{1-t} \, d\mu(t) < \infty. \tag{28}$$

Let us turn to study the membership of $\mathcal{H}_\mu$ in $\mathcal{S}_p(A^2)$ for $p \neq 2$. We have the following result.

Theorem 11 *Let μ be a positive finite Borel measure on $[0,1)$ and $2 < p < \infty$.*
(i) If

$$\int_{[0,1)} \frac{1}{(1-t)^p} \left(\log \frac{1}{1-t}\right)^{p/2} d\mu(t) < \infty \tag{29}$$

then $\mathcal{H}_\mu \in \mathcal{S}_p(A^2)$.
(ii) If $\mathcal{H}_\mu \in \mathcal{S}_p(A^2)$ then

$$\sum_{n=0}^{\infty} n^{p/2} \left[\int_0^1 t^n \mu([t,1)) \log \frac{1}{1-t} d\mu(t)\right]^{p/2} < \infty. \tag{30}$$

Proof of Theorem 11 (i). Assume $2 < p < \infty$ and (29). Then, for $0 < b < 1$, we have

$$\infty > \int_{[0,1)} \frac{1}{(1-t)^p} \left(\log \frac{1}{1-t}\right)^{p/2} d\mu(t) \geq \int_{[b,1)} \frac{1}{(1-t)^p} \left(\log \frac{1}{1-t}\right)^{p/2} d\mu(t)$$
$$\geq \frac{\mu([b,1))}{(1-b)^p} \left(\log \frac{1}{1-b}\right)^{p/2} = \frac{\mu([b,1))}{(1-b)} \left(\log \frac{1}{1-b}\right) \frac{\left(\log \frac{1}{1-b}\right)^{p/2-1}}{(1-b)^{p-1}}.$$

Thus

$$\frac{\mu([b,1))}{(1-b)} \left(\log \frac{1}{1-b}\right) \lesssim \frac{(1-b)^{p-1}}{\left(\log \frac{1}{1-b}\right)^{p/2-1}}.$$

Since $p > 2$ this gives that μ is a vanishing 1-logarithmic 1-Carleson measure. Using Theorem 10, we see that this implies that $\mathcal{H}_\mu$ is a compact operator on A^2. Then, in order to prove that $\mathcal{H}_\mu \in \mathcal{S}_p(A^2)$, it suffices to prove that

$$\sum_{n=0}^{\infty} \|\mathcal{H}_\mu(e_n)\|_{A^2}^p < \infty, \tag{31}$$

for any orthonormal sequence $\{e_n\}_{n=0}^{\infty}$ in A^2 (see [25, Thm. 1.33]).

So, let $\{e_n\}_{n=0}^\infty$ be an orthogonal sequence in A^2. For any sequence $\{\lambda_n\}_{n=0}^\infty \in \ell^2$, we have that

$$\sum_{n=0}^\infty \lambda_n e_n \in A^2, \quad \text{and} \quad \|\sum_{n=0}^\infty \lambda_n e_n\|_{A^2} \le \left(\sum_{n=0}^\infty |\lambda_n|^2\right)^{1/2}.$$

Using this and [25, Thm. 4.14] we deduce that, for any $t \in [0,1)$,

$$\left|\sum_{n=0}^\infty \lambda_n e_n(t)\right| \le \frac{\left(\sum_{n=0}^\infty |\lambda_n|^2\right)^{1/2}}{1-t}, \quad \text{for any } \{\lambda_n\}_{n=0}^\infty \in \ell^2.$$

Using duality, we deduce that

$$\left(\sum_{n=}^\infty |e_n(t)|^2\right)^{1/2} \le \frac{1}{1-t}, \quad 0 \le t < 1. \tag{32}$$

Using Minkowski's integral inequality, the fact that

$$\int_{\mathbb{D}} \frac{dA(z)}{|1-tz|^2} \asymp \log \frac{1}{1-t},$$

Hölder's inequality and the fact that μ is a finite measure, the embedding of ℓ^2 into ℓ^p, and (31), we obtain

$$\begin{aligned}
\sum_{n=0}^\infty \|\mathcal{H}_\mu(e_n)\|_{A^2}^p &\le \left[\sum_{n=0}^\infty \int_{\mathbb{D}} \left(\int_{[0,1)} \frac{|e_n(t)|}{|1-tz|}\, d\mu(t)\right)^2 dA(z)\right]^{p/2} \\
&\le \sum_{n=0}^\infty \left[\int_{[0,1)} \left(\int_{\mathbb{D}} \frac{|e_n(t)|^2}{|1-tz|^2}\, dA(z)\right)^{1/2} d\mu(t)\right]^p \\
&= \sum_{n=0}^\infty \left[\int_{[0,1)} |e_n(t)| \left(\int_{\mathbb{D}} \frac{dA(z)}{|1-tz|^2}\right)^{1/2} d\mu(t)\right]^p \\
&\asymp \sum_{n=0}^\infty \left[\int_{[0,1)} |e_n(t)| \left(\log \frac{1}{1-t}\right)^{1/2} d\mu(t)\right]^p \\
&\lesssim \sum_{n=0}^\infty \int_{[0,1)} |e_n(t)|^p \left(\log \frac{1}{1-t}\right)^{p/2} d\mu(t) \\
&= \int_{[0,1)} \left(\log \frac{1}{1-t}\right)^{p/2} \left(\sum_{n=0}^\infty |e_n(t)|^p\right) d\mu(t) \\
&\le \int_{[0,1)} \left(\log \frac{1}{1-t}\right)^{p/2} \left(\sum_{n=0}^\infty |e_n(t)|^2\right)^{p/2} d\mu(t)
\end{aligned}$$

$$\leq \int_{[0,1)} \left(\log \frac{1}{1-t}\right)^{p/2} \frac{1}{(1-t)^p}\, d\mu(t) \, < \infty.$$

Thus (31) holds. This finishes the proof of (i). □

Proof of Theorem 11 (ii). Assume $p > 2$ and that $\mathcal{H}_\mu \in \mathcal{S}_p(A^2)$. Let $\{e_n\}_{n=0}^{\infty}$ be the orthonormal basis of A^2 defined by

$$e_n(z) \,=\, (n+1)^{1/2} z^n, \quad z \in \mathbb{D}, \quad n = 0, 1, 2, \dots.$$

Using again [25, Thm. 1.33] we see that

$$\sum_{n=0}^{\infty} \|H_\mu(e_n)\|_{A^2}^p \,<\, \infty. \tag{33}$$

We have

$$\mathcal{H}_\mu(e_n)(z) \,=\, \int_{[0,1)} \frac{e_n(t)}{1-tz} d\mu(t) = (n+1)^{1/2} \sum_{k=0}^{\infty} \left(\int_{[0,1)} t^{n+k} d\mu(t) \right) z^k.$$

Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} \|H_\mu(e_n)\|_{A^2}^p \,&=\, \sum_{n=0}^{\infty} (n+1)^{p/2} \left[\sum_{k=0}^{\infty} \frac{1}{k+1} \left(\int_{[0,1)} t^{n+k}\, d\mu(t) \right)^2 \right]^{p/2} \\
&= \sum_{n=0}^{\infty} (n+1)^{p/2} \left[\sum_{k=0}^{\infty} \frac{1}{k+1} \int_{[0,1)} \int_{[0,1)} (ts)^{n+k} d\mu(s) d\mu(t) \right]^{p/2} \\
&= 2 \sum_{n=0}^{\infty} (n+1)^{p/2} \left[\sum_{k=0}^{\infty} \frac{1}{k+1} \int_{[0,1)} \int_{[t,1)} (ts)^{n+k} d\mu(s) d\mu(t) \right]^{p/2} \\
&= 2 \sum_{n=0}^{\infty} (n+1)^{p/2} \left[\int_{[0,1)} t^n \left(\int_{[t,1)} s^n \sum_{k=0}^{\infty} \frac{(ts)^k}{k+1} d\mu(s) \right) d\mu(t) \right]^{p/2} \\
&\geq 2 \sum_{n=0}^{\infty} (n+1)^{p/2} \left[\int_{[0,1)} t^{2n} \left(\int_{[t,1)} \log \frac{1}{1-ts} d\mu(s) \right) d\mu(t) \right]^{p/2} \\
&\geq 2 \sum_{n=0}^{\infty} (n+1)^{p/2} \left[\int_{[0,1)} t^{2n} \left(\int_{[t,1)} \log \frac{1}{1-t^2} d\mu(s) \right) d\mu(t) \right]^{p/2} \\
&= 2 \sum_{n=0}^{\infty} (n+1)^{p/2} \left[\int_{[0,1)} t^{2n} \log \frac{1}{1-t^2}\, \mu([t,1)) d\mu(s) \right]^{p/2}.
\end{aligned}$$

This and (33) give (30). □

Regarding the membership of $\mathcal{H}_\mu$ in $\mathcal{S}_p(A^2)$ for $p<2$, we have the following result.

Theorem 12 *Let μ be a positive finite Borel measure on $[0,1)$ and $0<p<2$. If*

$$\sum_{n=0}^{\infty} n^{p/2}\left[\int_0^1 t^n \mu([t,1))\log\frac{1}{1-t}d\mu(t)\right]^{p/2} < \infty \tag{34}$$

then $\mathcal{H}_\mu \in \mathcal{S}_p(A^2)$.

Proof Assume $0<p<2$ and (34). Let us take again the orthonormal basis $\{e_n\}_{n=0}^\infty$ of A^2 defined by

$$e_n(z) = (n+1)^{1/2}z^n, \quad z\in\mathbb{D}, \quad n=0,1,2,\ldots.$$

Bearing in mind the embedding $\ell^p \hookrightarrow \ell^2$, we see that (34) implies (28). Hence, $\mathcal{H}_\mu\in\mathcal{S}_2(A^2)$ and, consequently, it is a compact operator in A^2. Using [25, Thm. 1.26], we see that it suffices to prove that $\mathcal{H}_\mu^*\mathcal{H}_\mu\in\mathcal{S}_{p/2}(A^2)$. Since $\mathcal{H}_\mu^*\mathcal{H}_\mu$ is a positive operator, using [25, Cor. 1.32] we see that $\mathcal{H}_\mu^*\mathcal{H}_\mu\in\mathcal{S}_{p/2}(A^2)$ will follow if we prove that

$$\sum_{n=0}^{\infty}\langle \mathcal{H}_\mu^*\mathcal{H}_\mu(e_n),e_n\rangle^{p/2} = \sum_{n=0}^{\infty}\|\mathcal{H}_\mu(e_n)\|_{A^2}^p < \infty. \tag{35}$$

In the proof of the preceding theorem we saw that

$$\sum_{n=0}^{\infty}\|\mathcal{H}_\mu(e_n)\|_{A^2}^p = 2\sum_{n=0}^{\infty}(n+1)^{p/2}\left[\int_{[0,1)} t^n\left(\int_{[t,1)} s^n\sum_{k=0}^{\infty}\frac{(ts)^k}{k+1}d\mu(s)\right)d\mu(t)\right]^{p/2}. \tag{36}$$

We have

$$\int_{[0,1)} t^n\left(\int_{[t,1)} s^n\sum_{k=0}^{\infty}\frac{(ts)^k}{k+1}d\mu(s)\right)d\mu(t) \le \int_{[0,1)} t^n\left(\int_{[t,1)}\sum_{k=0}^{\infty}\frac{t^k}{k+1}d\mu(s)\right)d\mu(t)$$
$$= \int_{[0,1)} t^n\left(\int_{[t,1)}\log\frac{1}{1-t}d\mu(s)\right)d\mu(t) = \int_{[0,1)} t^n\log\frac{1}{1-t}\,\mu([t,1))\,d\mu(t).$$

This and (36) show that (34) implies (35). This finishes the proof. □

Declarations

Conflict of Interests. The authors have no conflicts of interests to declare that are relevant to the content of this article.

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