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Special pure gradings on simple Lie algebras of types E_6, E_7, E_8

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ABSTRACT

A group grading on a semisimple Lie algebra over an algebraically closed field of characteristic zero is *special* if its identity component is zero; it is *pure* if at least one of its components, other than the identity component, contains a Cartan subalgebra. We classify special pure gradings on Lie algebras of types E_6, E_7, E_8 up to equivalence and up to isomorphism. To this end, we use quadratic forms over the field of two elements to show that there are exactly three equivalence classes for E_6, four for E_7, and five for E_8. The computation of the corresponding Weyl groups and their actions on the universal groups yields a set of invariants that allow us to distinguish the isomorphism classes.

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Weyl group
Quadratic form
Arf invariant

1. Introduction

Since the emergence of the theory of Lie algebras in the late 19th and early 20th centuries, gradings have been a key tool for studying the algebras themselves as well as their representations. After the classical root space decomposition of any semisimple Lie algebra—which provides a grading over the root lattice associated with a Cartan subalgebra—the study of gradings on Lie algebras has developed into a rich area of research. Over the past several decades, the most frequently occurring gradings in the literature have been those by the cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}$: $\mathbb{Z}_2$-gradings appear prominently in the theory of symmetric spaces in Differential Geometry (see the book [14]), while $\mathbb{Z}$-gradings naturally arise in filtered Lie algebras, in the context of Kac-Moody algebras, and in the study of the Virasoro and Heisenberg algebras. For other infinite-dimensional Lie algebras, gradings by $\mathbb{Z}^n$ as well as other finitely generated abelian groups help in constructing and classifying deformations as well as in the study of their representations.

Patera and Zassenhaus [19] initiated the task of systematically developing the theory of gradings on Lie algebras from an algebraic perspective. They formulated the problem of determining fine gradings in addition to the root space decomposition. For a group G, a G-grading on an algebra $\mathcal{A}$ is a decomposition, $\mathcal{A} = \oplus_{g\in G}\mathcal{A}_g$, into direct sum of subspaces, labeled by the elements of G and called *homogeneous components*, such that the product in the algebra agrees with the group operation: $\mathcal{A}_g\mathcal{A}_h \subset \mathcal{A}_{gh}$. A grading is called *fine* if its homogeneous components are as small as possible, i.e., it does not have a proper refinement.

The classification of fine gradings on simple Lie algebras over algebraically closed fields of characteristic zero became a major milestone, ultimately compiled in the monograph [10]. This book presents many classification results for gradings on simple Lie algebras, due to the contributions of many authors (see the references in [10] and, in addition, [23]). More precisely, one can consider two types of classification for gradings: up to equivalence and up to isomorphism. Two gradings on $\mathcal{A}$ are said to be *equivalent* (respectively, *isomorphic*) if there exists an automorphism of $\mathcal{A}$ that maps each component of the first grading onto some component (respectively, the component with the same label) of the second grading. The classification up to isomorphism is particularly relevant in applications where the specific labeling of components plays a role, as is the case when the group G carries some additional structure such as the commutation factor in the theory of Lie color algebras.

Since any grading can be obtained by joining some of the components of a fine grading through a process known as *coarsening*, the classification of fine gradings up to equivalence can serve as the starting point for a complete classification of gradings. However, implementing this in practice is far from straightforward, one difficulty being that the same grading may arise as a coarsening of several nonequivalent fine gradings. Indeed, the classification of all gradings up to isomorphism for classical simple Lie algebras was obtained in a different way, namely, by transferring the problem to certain associative algebras and then using the structure theory of graded associative algebras. The classification of all gradings up to isomorphism for the exceptional Lie algebras G_2 and F_4 was derived from fine gradings by ad hoc arguments, but the situation is far more complex for E_6, E_7 and E_8 (each having 14 nonequivalent fine gradings), so the problem remains open for these types, even over algebraically closed fields of characteristic zero. This is a significant problem that should be resolved, since the classification of graded-simple finite-dimensional Lie algebras would follow from the classification of gradings on all simple finite-dimensional Lie algebras via the so-called loop construction (see [3]). It should also be noted that the classification of all gradings on classical simple Lie superalgebras (with nonzero odd part) is now known [16,2,17], so completing the E types would also yield the classification of simple Lie color algebras via the so-called discoloration of Scheunert.

Due to the non-uniqueness issue mentioned above, the classification of gradings up to isomorphism has not yet been attempted for the E types. A new approach was recently suggested in [11], using what the authors called *almost fine gradings*. The definition is not immediately intuitive, but the idea is that a grading is almost fine if it cannot be refined while keeping the same toral rank (for instance, any fine grading is almost fine). The advantage of almost fine gradings is that any G-grading can be obtained by coarsening from an almost fine grading in an essentially unique way. To obtain a classification of G-gradings using this approach, one needs a list of the equivalence classes of almost fine gradings as well as their Weyl groups together with the action on their universal groups.

The purpose of this paper is to initiate the classification of gradings up to isomorphism on the simple Lie algebras of the E types by classifying, both up to isomorphism and up to equivalence, the special pure gradings on these algebras. A group grading on a semisimple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic zero is *pure* if at least one of its components, other than the identity component, contains a Cartan subalgebra of $\mathcal{L}$. This concept was introduced in [15], motivated by the so-called *Jordan gradings* (see [1,8]). The particular symmetry of these gradings makes them especially useful for constructing manageable models of exceptional Lie algebras (for instance, the one in [6]). A grading on $\mathcal{L}$ is *special* if its identity component is zero; these are precisely the almost fine gradings on $\mathcal{L}$ that have toral rank 0.

Partial information on almost fine gradings and their Weyl groups for simple Lie algebras of the E types appears in [22,23], where the objectives and methods are quite different from ours: the author classifies certain closed abelian subgroups of exceptional compact real simple Lie groups of adjoint type. The diagonalization of the action of these subgroups on the complexified tangent Lie algebra gives the almost fine gradings on it. We do not rely on these works, but we will point out some connections with our approach.

The structure of the paper is the following. Section 2 contains preliminaries on quadratic forms in characteristic 2, and how some of the orthogonal groups over the field of two elements, $\mathbb{F}_2$, appear in relation with the Weyl groups of the simple Lie algebras of the E types. These quadratic forms (as well as those induced by them on certain subquotients) will appear later in the explicit computations of the Weyl groups of the special pure gradings on these algebras. We also review the main concepts related to gradings (universal groups, Weyl groups, almost fine gradings) that are essential for achieving a classification up to isomorphism. Section 3 reviews the work [15], aiming to adapt its results to our notation and goals. It also addresses general properties of special pure gradings and their Weyl groups—specifically, the subgroup that preserves a homogeneous component which is a Cartan subalgebra. Section 4 is devoted to the classification, up to equivalence, of special pure gradings on the simple Lie algebras of the E types, based on the quadratic forms mentioned above. The main results here are Theorems 4.1, 4.2, and 4.3. Further information about the Weyl groups of the special pure gradings on these algebras is presented in Section 5. The goal is to determine what is preserved by these groups, so as to define a set of invariants for the gradings in Definition 6.2. This leads to a classification up to isomorphism, presented in Theorem 6.3.

2. Preliminaries

2.1. Quadratic forms in characteristic 2

In this work, we will need to understand the Weyl groups of certain gradings on the simple Lie algebras of the E-types, including their usual Weyl groups, which turn out to be closely related to certain orthogonal groups over the field of two elements, $\mathbb{F}_2$. Since quadratic forms in characteristic 2 are perhaps not so well-known, we start with reviewing them briefly. The material in this section is standard and has been extracted from the textbook [13].

If V is a (finite-dimensional) vector space over a field $\mathbb{F}$, recall that $Q\colon V \to \mathbb{F}$ is a *quadratic form* if $Q(cx) = c^2 Q(x)$ for any $c \in \mathbb{F}$ and $B_Q(x, y) := Q(x+y) - Q(x) - Q(y)$ is a bilinear form, called the *polarization* of Q. Observe that B_Q is necessarily symmetric (i.e., $B_Q(x, y) = B_Q(y, x)$) and, moreover, alternating in characteristic 2 (i.e., $B_Q(x, x) = 0$). In characteristic 2, Q cannot be recovered from B_Q.

If U and V are two vector spaces endowed with quadratic forms Q_U and Q_V, their orthogonal sum, denoted by $U \oplus V$, is the direct sum $U \oplus V$, endowed with the quadratic form $Q(u+v) := Q_U(u) + Q_V(v)$ for all $u \in U$ and $v \in V$. Then U and V are orthogonal to each other with respect to the polarization B_Q. We will also write $Q = Q_U \oplus Q_V$.

A symmetric bilinear form B is said to be *nondegenerate* if its radical $\operatorname{rad} B = \{x \in V \mid B(x, V) = 0\}$ vanishes. A quadratic form Q is nondegenerate if B_Q is nondegenerate. In characteristic 2, this concept is extended, and we will use the terms *regular* or *nonsingular* quadratic form to avoid possible confusion. Note that the restriction $Q|_{\operatorname{rad} B_Q}$ is semilinear (i.e., $Q(x + cy) = Q(x) + c^2 Q(y)$ for $x, y \in \operatorname{rad} B_Q$ and $c \in \mathbb{F}$), and the quadratic form is said to be *regular* if the radical of Q, defined as $\operatorname{rad} Q := \ker(Q|_{\operatorname{rad} B_Q})$ is trivial. For perfect fields, the regularity forces that $\dim \operatorname{rad} B_Q$ is 0 or 1. Moreover, the possibilities for a regular Q are not many. According to [13, Theorem 12.9],

(1) If $\dim V = 2m+1$, there is a basis $\{v_i\}$ of V such that $Q(\sum_i x_i v_i) = x_1x_2 + x_3x_4 + \cdots + x_{2m-1}x_{2m} + x_{2m+1}^2$.
(In this case $\operatorname{rad} B_Q = \mathbb{F}v_{2m+1}$ and $Q(v_{2m+1}) = 1$.)

(2) If $\dim V = 2m$, there is a basis $\{v_i\}$ of V such that either
 (a) $Q(\sum_i x_i v_i) = \sum_{i=1}^m x_{2i-1}x_{2i}$, or
 (b) $Q(\sum_i x_i v_i) = \sum_{i=1}^m x_{2i-1}x_{2i} + x_{2m-1}^2 + cx_{2m}^2$, for some $c \in \mathbb{F}$ such that x^2+x+c is irreducible in $\mathbb{F}[x]$.
 (In this case the polarization B_Q is nondegenerate.)

In characteristic 2, a *hyperbolic plane* H is spanned by $u, v \in V$ with $B_Q(u,v) = 1$ and $Q(u) = Q(v) = 0$. The maximum r such that $V = H_1 \oplus H_2 \oplus \cdots \oplus H_r \oplus W$, with each H_i a hyperbolic plane, is called the *Witt index*, and coincides with the dimension of a maximal totally isotropic subspace of V (in fact, any two of these are isometric, due to Witt's Extension Theorem). The Witt index equals m in case (1), and m and $m-1$, respectively, in cases (2a) and (2b).

The equivalence class of a regular quadratic form Q on V of even dimension is also determined by the so-called *Arf invariant*, $\operatorname{Arf}(Q)$, which has an especially simple definition for the field $\mathbb{F}_2$: it tells us whether there are more elements $v \in V$ with $Q(v) = 0$ (isotropic) or $Q(v) = 1$ (nonisotropic). In case (1) half of the elements are isotropic, so $\operatorname{Arf}(Q)$ is undefined, while in case (2a) there are $\binom{2^m+1}{2}$ isotropic elements and $\binom{2^m}{2}$ nonisotropic, so $\operatorname{Arf}(Q) = 0$ (or trivial), and in case (2b) there are $\binom{2^m}{2}$ isotropic elements and $\binom{2^m+1}{2}$ nonisotropic, so $\operatorname{Arf}(Q) = 1$.

We will later need the following property of the Arf invariant. If Q_1 and Q_2 are regular quadratic forms on spaces of even dimension, then $\operatorname{Arf}(Q_1 \oplus Q_2) = \operatorname{Arf}(Q_1) + \operatorname{Arf}(Q_2)$.

Since in characteristic 2 the polarization of a quadratic form is alternating, there is a close relationship between symplectic and orthogonal groups. For a regular quadratic form Q, we have $\mathrm{O}(V,Q) \leq \mathrm{Sp}(V, B_Q)$, where the latter denotes the stabilizer of B_Q (which is nondegenerate in even dimension). Besides, over a perfect field, if V has odd dimension and W is a complementary subspace of the one-dimensional radical of B_Q, then $B_Q|_W$ is nondegenerate and, by [13, Theorems 14.1 and 14.2], the map $\mathrm{O}(V,Q) \to \mathrm{Sp}(W, B_Q|_W)$, $\sigma \to \mathrm{pr}_W \circ \sigma|_W$, is an isomorphism.

2.2. Weyl groups of Lie algebras of the E types

We will now recall why the Weyl groups of the simple Lie algebras of types E_6, E_7 and E_8 are closely related to orthogonal groups over the field $\mathbb{F}_2$. Let $\mathcal{L}$ be one of these Lie algebras over an algebraically closed field $\mathbb{F}$ of characteristic 0, let $\mathcal{H}$ be a Cartan subalgebra of $\mathcal{L}$, and Φ the root system associated to $\mathcal{H}$. That is, we have the root space decomposition

$$\mathcal{L} = \mathcal{H} \oplus \big(\bigoplus_{\alpha\in\Phi} \mathcal{L}_\alpha\big), \tag{2.1}$$

where $\mathcal{L}_\alpha = \{x \in \mathcal{L} \mid [h,x] = \alpha(h)x \ \forall h \in \mathcal{H}\}$. Let $(.\,|\,.)$ be the bilinear form on $\mathcal{H}^*$, dual to the bilinear form obtained from the restriction of the Killing form to $\mathcal{H}$, normalized so that $(\alpha\,|\,\alpha) = 2$ for any $\alpha \in \Phi$. This bilinear form restricts to a $\mathbb{Z}$-bilinear form on the root lattice $R = \mathbb{Z}\Phi$. Consider the associated $\mathbb{Z}$-valued quadratic form $\frac{1}{2}(r\,|\,r)$, whose polarization is $(.\,|\,.)$. This quadratic form induces a quadratic form on the vector space $\overline{R} := R/2R$ over the field of two elements $\mathbb{F}_2$:

$$q\colon \overline{R} \longrightarrow \mathbb{F}_2,\ \bar{r} \mapsto \frac{1}{2}(r\,|\,r) \pmod 2, \tag{2.2}$$

with polarization b_q induced by the bilinear form $(.\,|\,.)$. Note that $q(\bar\alpha) = 1$ for any root $\alpha \in \Phi$. Also, in the cases at hand, if $\beta \neq \pm\alpha$ then $\bar\beta \neq \bar\alpha$.

By straightforward computations with the Cartan matrix, the quadratic form $q\colon \overline{R} \to \mathbb{F}_2$ is regular, and we get the following possibilities, depending on the type of the simple Lie algebra $\mathcal{L}$:

E_8: q is regular with trivial Arf invariant: $\mathrm{Arf}(q) = 0$. Moreover, in this case the nonisotropic vectors are exactly the vectors $\bar\alpha$ for $\alpha \in \Phi$.

E_7: q is regular. As the dimension is odd in this case, the radical of the polarization has dimension 1: $\mathrm{rad}\, b_q = \mathbb{F}_2 a$. In this case, the nonisotropic vectors are a and the elements $\bar\alpha$ for $\alpha \in \Phi$.

E_6: q is regular with $\mathrm{Arf}(q) = 1$. As for E_8, the nonisotropic vectors are exactly the vectors $\bar\alpha$ for $\alpha \in \Phi$.

The following description of the automorphism groups of the root systems, $\mathrm{Aut}(\Phi)$, is known (see, e.g., [21, p. 104]), but we will give a proof for completeness, and also because we are going to generalize this result in the next section. Any automorphism of Φ preserves $2R$, hence there is an induced group homomorphism

$$\rho\colon\ \mathrm{Aut}(\Phi) \longrightarrow \mathrm{GL}(\overline{R}). \tag{2.3}$$

Proposition 2.1. *Let $\mathcal{L}$ be a simple Lie algebra of type E_6, E_7, or E_8 over an algebraically closed field $\mathbb{F}$ of characteristic 0. Then the image of the homomorphism ρ is the orthogonal group $\mathrm{O}(q) := \mathrm{O}(\bar R, q)$, and the kernel is $\{\pm I\}$. Thus, the Weyl group is isomorphic to*

- $\mathrm{O}(q)$ *in type E_6;*

- $\mathrm{O}(q) \times \mathrm{C}_2$ *in type* E_7*;*
- *a central extension of* $\mathrm{O}(q)$ *by* C_2 *in type* E_8*.*

Proof. Any automorphism of the root system Φ preserves the quadratic form $\frac{1}{2}(.\,|\,.)$ on the root lattice $R = \mathbb{Z}\Phi$, and hence the image of ρ is contained in $\mathrm{O}(q)$. Now, the Weyl group is generated by the reflections relative to roots, and ρ maps them to the reflections (sometimes called orthogonal transvections in characteristic 2) relative to the nonisotropic vectors of $\overline{R}$. Recall that Cartan-Dieudonné Theorem [5] says that, with one exception, reflections generate the orthogonal group of a regular quadratic form. To be precise, for even-dimensional regular quadratic forms over a field of characteristic 2, the orthogonal group is generated by reflections unless the dimension is 4, the Arf invariant is 0 and the ground field is $\mathbb{F}_2$ (see, e.g., [13, Theorem 14.16]), and we have dimensions 6 and 8 in our cases. In odd dimension, the orthogonal group is isomorphic to a symplectic group, so the orthogonal group can be shown to be generated by reflections using the fact that the symplectic group is generated by transvections ([13, Theorem 3.4]). This completes the proof of the surjectivity of ρ.

If $\mu \in \mathrm{Aut}(\Phi)$ induces the identity map on $\overline{R}$, then it must map each root α to $\pm\alpha$, and it is clear that the sign must be the same for all simple roots, because the Dynkin diagram is connected. Hence $\ker \rho = \{\pm I\}$.

Finally, the Weyl group is $\mathrm{Aut}(\Phi)$ for types E_7 and E_8, whereas for type E_6 the Weyl group is a complement of $\{\pm I\}$ in $\mathrm{Aut}(\Phi)$. Also, for type E_7, $\mathrm{Aut}(\Phi)$ is the direct product of $\{\pm I\}$ and the subgroup of elements of determinant 1, as transformations of the euclidean vector space spanned by the roots. □

2.3. Gradings on Lie algebras

Though the concepts in this section can be considered in a more general setting, we will focus on group gradings on Lie algebras, in which case it is natural to consider only gradings by abelian groups. Actually, for simple Lie algebras there is no loss of generality in assuming the grading groups are abelian. All the algebras will be considered finite-dimensional over an algebraically closed field $\mathbb{F}$ of characteristic zero. Most of the material here is standard, and can be consulted, for instance, in [10, Chapter 1]. The notion and results on almost fine gradings are extracted from [11].

Definition 2.2. If G is an abelian group, and $\mathcal{L}$ is a Lie algebra, a *G-grading* Γ on $\mathcal{L}$ is a decomposition of $\mathcal{L}$ into a direct sum of subspaces indexed by G,

$$\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g, \tag{2.4}$$

such that $\mathcal{L}_g\mathcal{L}_h \subset \mathcal{L}_{g+h}$ for all $g, h \in G$. The subspace $\mathcal{L}_g$ is the *homogeneous component* of degree g, and its elements are the *homogeneous elements* of degree g.

The *support* of Γ is the set $\operatorname{Supp}\Gamma := \{g \in G \mid \mathcal{L}_g \neq 0\}$. For instance, for any G and any $\mathcal{L}$, we can consider the *trivial* G-grading on $\mathcal{L}$, given by $\mathcal{L}_e = \mathcal{L}$ and $\mathcal{L}_g = 0$ for all $g \neq e$, with support $\{e\}$. The *type* of Γ is the sequence $(n_1, \dots, n_m)$, where n_i is the number of homogeneous components of Γ of dimension i, for $i = 1, \dots, m$, with m the maximum of these dimensions. Note that $\sum_{1\leq i\leq m} n_i i = \dim \mathcal{L}$.

Two operations on gradings on $\mathcal{L}$ are quite natural: coarsening and refinement. In general, Γ' is said to be a *coarsening* of Γ if any homogeneous component of Γ is contained in some homogeneous component of Γ'. The coarsening is said to be proper if at least one of such inclusions is proper. For each group homomorphism $\alpha\colon G \to H$, we obtain a coarsening ${}^{\alpha}\Gamma$ of the grading Γ in (2.4) as follows:

$$ {}^{\alpha}\Gamma : \mathcal{L} = \bigoplus_{h\in H} {}^{\alpha}\mathcal{L}_h \quad \text{for} \quad {}^{\alpha}\mathcal{L}_h := \bigoplus_{g\in\alpha^{-1}(h)} \mathcal{L}_g. $$

Note that ${}^{\alpha}\Gamma$ is a proper coarsening of Γ if and only if the restriction of the map α to $\operatorname{Supp}\Gamma$ is not injective.

If Γ' is a coarsening (respectively, a proper coarsening) of Γ, it is also said that Γ is a *refinement* (respectively, a proper refinement) of Γ'. A grading is said to be *fine* if it admits no proper refinement.

Example 2.3. The simple Lie algebra $\mathcal{L} = \mathfrak{sl}(2, \mathbb{F})$ has the following grading by $G = \mathbb{Z}_2^2$:

$$ \Gamma_1 : \quad \mathcal{L}_{(\bar{0},\bar{1})} = \mathbb{F}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}_{(\bar{1},\bar{1})} = \mathbb{F}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{L}_{(\bar{1},\bar{0})} = \mathbb{F}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. $$

Since all the homogeneous components have dimension 1, Γ_1 is clearly fine.

Definition 2.4. Two gradings on $\mathcal{L}$, $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ and $\Gamma' : \mathcal{L} = \bigoplus_{h\in H} \mathcal{L}'_h$, are said to be *equivalent* if there is an automorphism $\varphi \in \operatorname{Aut}(\mathcal{L})$ and a bijection $\alpha_\varphi\colon \operatorname{Supp}\Gamma \to \operatorname{Supp}\Gamma'$ such that $\varphi(\mathcal{L}_g) = \mathcal{L}'_{\alpha_\varphi(g)}$ for all $g \in \operatorname{Supp}\Gamma$. In case $G = H$, Γ and Γ' are said to be *isomorphic* if there exists $\varphi \in \operatorname{Aut}(\mathcal{L})$ with $\alpha_\varphi = \mathrm{id}$, i.e., $\varphi(\mathcal{L}_g) = \mathcal{L}'_g$ for all $g \in G$.

Example 2.5. The following (fine) gradings on the Lie algebra $\mathcal{L} = \mathfrak{sl}(2, \mathbb{F})$ are equivalent: the $\mathbb{Z}$-grading

$$ \Gamma_2 : \quad \mathcal{L}_{-1} = \mathbb{F}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}_{0} = \mathbb{F}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{L}_{1} = \mathbb{F}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, $$

and the $\mathbb{Z}_3$-grading

$$ \Gamma_3 : \quad \mathcal{L}_{\bar{2}} = \mathbb{F}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathcal{L}_{\bar{0}} = \mathbb{F}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{L}_{\bar{1}} = \mathbb{F}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. $$

As we see from this example, the same set of subspaces of $\mathcal{L}$ can be regarded as a grading by different groups G, even if G is generated by the support. It turns out that $\mathbb{Z}$ is the “natural” group for this grading in the sense that it is not only generated by the support, but also the set of relations imposed on the generators is the minimum required to satisfy the definition of grading. More precisely, consider the grading (2.4) as a grading by the set $S := \operatorname{Supp}\Gamma$, $\mathcal{L} = \bigoplus_{s\in S}\mathcal{L}_s$, and let

$$S_* := \{(s_1, s_2) \in S \times S \mid [\mathcal{L}_{s_1}, \mathcal{L}_{s_2}] \neq 0\}.$$

For any $(s_1, s_2) \in S_*$, denote by $s_1 * s_2$ the unique element of S such that $[\mathcal{L}_{s_1}, \mathcal{L}_{s_2}] \subset \mathcal{L}_{s_1 * s_2}$.

Definition 2.6 *([19]).* The abelian group $U = U(\Gamma)$ generated by the set S subject only to the relations $s_1 + s_2 = s_1 * s_2$ for $(s_1, s_2) \in S_*$, is called the *universal group* of the grading $\mathcal{L} = \bigoplus_{s\in S}\mathcal{L}_s$.

By construction, whenever S is embedded in a group G to make $\mathcal{L} = \bigoplus_{s\in S}\mathcal{L}_s$ a G-grading, there is a unique group homomorphism $U \to G$ that extends the embedding $S \to G$. This universal property determines U up to isomorphism.

In Example 2.5, the universal group is $U(\Gamma_2) = U(\Gamma_3) = \mathbb{Z}$ (equivalent gradings have isomorphic universal groups). In Example 2.3, $U(\Gamma_1) = \mathbb{Z}_2^2$.

The universal group also has the following key property. For any coarsening $\Gamma' : \mathcal{L} = \bigoplus_{h\in H}\mathcal{L}'_h$ of Γ, there exists a unique group homomomorphism $\alpha\colon U \to H$ such that $\Gamma' = {}^{\alpha}\Gamma$. It is clear that every grading is a coarsening of at least one fine grading. As mentioned in the Introduction, this can be used to obtain all possible group gradings on a fixed Lie algebra starting from the classification of fine gradings up to equivalence, although there is a problem due to the lack of uniqueness.

Example 2.7. Consider again $\mathcal{L} = \mathfrak{sl}(2, \mathbb{F})$, endowed with the $\mathbb{Z}_2$-grading

$$\Gamma_4 : \quad \mathcal{L}_{\bar 0} = \mathbb{F}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathcal{L}_{\bar 1} = \mathbb{F}\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \oplus \mathbb{F}\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right).$$

Thus $\Gamma_4 = {}^{\alpha}\Gamma_1$ for $\alpha\colon \mathbb{Z}_2^2 \to \mathbb{Z}_2$, $(\bar a, \bar b) \mapsto \bar a + \bar b$, and also $\Gamma_4 = {}^{\beta}\Gamma_2$ for $\beta\colon \mathbb{Z} \to \mathbb{Z}_2$, $a \mapsto \bar a$.

In order to assign to each grading only one grading that is “fine” in some sense (see Definition 2.10) we need first to review some groups attached to a grading.

Definition 2.8 *([19]).* Given a grading Γ as in (2.4), consider the groups

- $\operatorname{Diag}(\Gamma) := \{\varphi \in \operatorname{Aut}(\mathcal{L}) \mid \forall g \in G\ \varphi|_{\mathcal{L}_g} \in \mathbb{F}^{\times}\mathrm{id}\}$;
- $\operatorname{Stab}(\Gamma) := \{\varphi \in \operatorname{Aut}(\mathcal{L}) \mid \forall g \in G\ \varphi(\mathcal{L}_g) \subseteq \mathcal{L}_g\}$;
- $\operatorname{Aut}(\Gamma) := \{\varphi \in \operatorname{Aut}(\mathcal{L}) \mid \forall g \in G\ \exists h \in G\ \varphi(\mathcal{L}_g) \subseteq \mathcal{L}_h\}$.

Note that $\operatorname{Diag}(\Gamma) \trianglelefteq \operatorname{Stab}(\Gamma) \trianglelefteq \operatorname{Aut}(\Gamma)$. The *Weyl group* of the grading Γ is defined by $W(\Gamma) := \operatorname{Aut}(\Gamma)/\operatorname{Stab}(\Gamma)$.

The Weyl group $W(\Gamma)$ provides a measure of the symmetry of the grading Γ. Note that, for the root space decomposition of a semisimple Lie algebra regarded as a grading by the root lattice, this definition gives $W(\Gamma) \cong \mathrm{Aut}(\Phi)$, which may be larger than the classical Weyl group.

The grading Γ (more precisely, the set of its homogeneous components) can be recovered as the simultaneous eigenspace decomposition with respect to $\mathrm{Diag}(\Gamma)$. This is a diagonalizable algebraic group, hence isomorphic to the direct product of a torus and a finite abelian group, usually called a *quasitorus*. Recall that the rank of an algebraic group is the dimension of any of its maximal tori (which are known to be conjugate over an algebraically closed field). Remarkably, this diagonal group is isomorphic to the group of characters $\widehat{U} = \mathrm{Hom}(U, \mathbb{F}^\times)$ of the universal group $U = U(\Gamma)$. In fact, the map which sends any $\chi \in \widehat{U}$ to the automorphism $\varphi_\chi \in \mathrm{Aut}(\mathcal{L})$ defined by $\varphi_\chi|_{\mathcal{L}_g} = \chi(g)\,\mathrm{id}$ for all $g \in U$, is an isomorphism. If $U \cong \mathbb{Z}^l \times H$ for H a finite abelian group, then $\mathrm{Diag}(\Gamma) \cong (\mathbb{F}^\times)^l \times H$ has dimension l, equal to the rank of the universal group.

Each $\varphi \in \mathrm{Aut}(\Gamma)$ produces a grading isomorphic to Γ, given by $\mathcal{L}'_g := \varphi(\mathcal{L}_g)$, and the bijection α_φ in Definition 2.4 is naturally extended to a group automorphism of the universal group $U = U(\Gamma)$. This gives a group homomorphism $\mathrm{Aut}(\Gamma) \to \mathrm{Aut}(U)$, whose kernel is precisely $\mathrm{Stab}(\Gamma)$, so the Weyl group of the grading can be seen as a subgroup of $\mathrm{Aut}(U)$.

Example 2.9. Consider, for $\mathcal{L} = \mathfrak{sl}(2, \mathbb{F})$ and each $c \in \mathbb{F}^\times$, the automorphism $\varphi_c \colon \mathcal{L} \to \mathcal{L}$ defined by $\varphi_c \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} x & cy \\ c^{-1}z & -x \end{pmatrix}$ and the one-dimensional torus $\mathcal{T} = \{\varphi_c \mid c \in \mathbb{F}^\times\}$. Take also the order two automorphism σ of $\mathcal{L}$ given by $\sigma \begin{pmatrix} x & y \\ z & -x \end{pmatrix} = \begin{pmatrix} -x & -z \\ -y & x \end{pmatrix}$.

For Γ_4 as in Example 2.7, it is easily computed that $\mathrm{Diag}(\Gamma_4) = \{\mathrm{id}, \varphi_{-1}\}$ while $\mathrm{Stab}(\Gamma_4) = \mathrm{Aut}(\Gamma_4) = \mathcal{T} \cup \mathcal{T}\sigma \cong \mathcal{T} \rtimes \mathrm{C}_2$, since $\sigma\varphi_c = \varphi_{c^{-1}}\sigma$. In particular, the Weyl group $W(\Gamma_4)$ is trivial.

For Γ_2 as in Example 2.5, the automorphism σ above permutes the homogeneous components. Note that $\mathrm{Diag}(\Gamma_2) = \mathrm{Stab}(\Gamma_2) = \mathcal{T}$, while $\mathrm{Aut}(\Gamma_2) \cong \mathcal{T} \rtimes \mathrm{C}_2$. Here $W(\Gamma_2) \cong \mathrm{C}_2$.

For Γ_1 as in Example 2.3, $\mathrm{Diag}(\Gamma_1) = \mathrm{Stab}(\Gamma_1) = \langle \varphi_{-1}, \sigma \rangle$ is isomorphic to the universal group $\mathbb{Z}_2^2$, while the Weyl group $W(\Gamma_1) = \mathrm{Aut}(\mathbb{Z}_2^2) \cong \mathrm{S}_3$, since we can easily find automorphisms interchanging any pair of homogeneous components.

The *toral rank* of Γ is defined as the rank of the algebraic group $\mathrm{Stab}(\Gamma)$, which is denoted by $\mathrm{tor.rank}(\Gamma)$. Thus the toral rank is at least the rank of the finitely generated abelian group $U(\Gamma)$. For instance, the toral rank of Γ_4 is 1, although the rank of the universal group is 0.

Definition 2.10. A grading Γ is *almost fine* if $\mathrm{rank}(U(\Gamma)) = \mathrm{tor.rank}(\Gamma)$ or, in other words, if the maximal torus contained in $\mathrm{Diag}(\Gamma)$ is also maximal in $\mathrm{Stab}(\Gamma)$.

If Γ is a fine grading, then $\mathrm{Diag}(\Gamma)$ is a maximal quasitorus of $\mathrm{Aut}(\mathcal{L})$ and hence Γ is almost fine, but the converse is not true. There are no examples of this fact in the case of $\mathcal{L} = \mathfrak{sl}(2, \mathbb{F})$, where Γ_1 and Γ_2 are the only almost fine gradings up to equivalence, but we will later see for $\mathcal{L}$ of the E types several examples of almost fine gradings that are not fine.

If Γ' is a coarsening of Γ, then the automorphisms of $\mathcal{L}$ preserving the homogeneous components of Γ will also preserve those of Γ', so $\mathrm{Stab}(\Gamma) \subseteq \mathrm{Stab}(\Gamma')$. This implies that refinements do not increase the toral rank. For almost fine gradings, any refinement preserves the toral rank.

For semisimple Lie algebras, an almost fine grading admits quite a simple characterization: $\mathrm{rank}(U(\Gamma)) = \dim \mathcal{L}_e$. We always have $\mathrm{rank}(U(\Gamma)) \leq \dim \mathcal{L}_e$ so, in particular, $\mathcal{L}_e = 0$ if and only if Γ is almost fine and has finite universal group. The gradings satisfying $\mathcal{L}_e = 0$ are called *special* gradings. One example that appeared earlier is Γ_1. In a special grading, every homogeneous element is semisimple [15, 3.6 Proposition], which gives another reason to study them: they provide nice bases of semisimple Lie algebras.

For our purposes, the reason to consider almost fine gradings in detail is that every grading admits a canonical almost fine refinement (thus avoiding duplications in the classification process). More precisely, if Γ is the G-grading in (2.4), and we pick a maximal torus $\mathcal{T}$ in $\mathrm{Stab}(\Gamma)$, consider the refinement Γ^* obtained by decomposing each homogeneous component $\mathcal{L}_g$ with respect to the action of $\mathcal{T}$:

$$\Gamma^* : \mathcal{L} = \bigoplus_{(g,\lambda)\in G\times\widehat{\mathcal{T}}} \mathcal{L}_{(g,\lambda)}, \text{ for } \mathcal{L}_{(g,\lambda)} := \{x \in \mathcal{L}_g \mid \tau(x) = \lambda(\tau)x \ \forall \tau \in \mathcal{T}\}. \tag{2.5}$$

The grading Γ^* is almost fine, with the same toral rank as Γ, and any other grading satisfying these properties is equivalent to Γ^*. In the above examples, Γ_4 is not almost fine and $\Gamma_4^* = \Gamma_2$. After this discussion, it becomes clear that the idea for achieving a classification consists in studying the coarsenings of almost fine gradings that preserve their toral rank. The suitable coarsenings are given by *admissible* homomorphisms $\alpha\colon U \to H$ as in [11, §4].

A grading is *pure* if there is a Cartan subalgebra contained in some homogeneous component distinct from the neutral homogeneous component. For instance, Γ_1 and Γ_4 are pure gradings, but only Γ_1 is special. In this paper, we will focus on special pure gradings, since they are almost fine and serve as the first step towards a classification of gradings on the simple Lie algebras of the E types.

There is a nice special pure grading on any semisimple $\mathcal{L}$, given by the following example.

Example 2.11. Let $\mathcal{L}$ be a semisimple Lie algebra of rank l. Consider a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$, the root system Φ associated to $\mathcal{H}$, and the root space decomposition as in (2.1). As in Example 2.9, there exists an order 2 automorphism σ of $\mathcal{L}$ which is $-\mathrm{id}$ on $\mathcal{H}$, and hence takes any root space $\mathcal{L}_\alpha$ to $\mathcal{L}_{-\alpha}$. Then σ commutes with the elements of order

2 in the torus $\mathcal{T}$ associated to $\mathcal{H}$. The eigenspaces of the quasitorus $\mathcal{T}_2 \times \langle\sigma\rangle$, where $\mathcal{T}_2 := \{\tau \in \mathcal{T} \mid \tau^2 = \mathrm{id}\}$, are the homogeneous components of a special pure grading on $\mathcal{L}$, whose universal group is isomorphic to $\mathbb{Z}_2^{l+1}$.

We are going to generalize this example in the next section, where it will correspond to the case $E = 2R$. The homogeneous components are given by (3.6). For a simple Lie algebra $\mathcal{L}$ with a simply laced root system, the nonzero homogeneous components of this grading have dimension 1, with the sole exception of the Cartan subalgebra $\mathcal{H}$, which implies that the grading is fine. For $\mathfrak{sl}(2,\mathbb{F})$, it is the grading in Example 2.3.

It turns out that all the special pure gradings are coarsenings of the grading in this example, as we will see in Section 3.

3. Weyl groups of special pure gradings

This section will be devoted to describing the special pure gradings of semisimple Lie algebras and to begin our study of the Weyl groups, following the work of Hesselink [15].

Let $\mathcal{L}$ be a finite-dimensional semisimple Lie algebra over an algebraically closed ground field $\mathbb{F}$ of characteristic 0, and let

$$\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$$

be a pure grading, i.e., there is an element $s \in G \setminus \{e\}$ such that $\mathcal{L}_s$ contains a Cartan subalgebra $\mathcal{H}$ of $\mathcal{L}$. Fix the element s in what follows.

Assume, from now on, that G is the universal group of Γ. Recall that its group of (multiplicative) characters $\widehat{G}$ is isomorphic to the diagonal group $\mathrm{Diag}(\Gamma)$, where $\chi \in \mathrm{Hom}(G, \mathbb{F}^\times)$ corresponds to the diagonal automorphism $\tau_\chi \colon x \in \mathcal{L}_g \mapsto \chi(g)x$ for any $g \in G$.

Consider the torus associated to the Cartan subalgebra $\mathcal{H}$:

$$\mathcal{T} = \{\tau \in \mathrm{Aut}(\mathcal{L}) \mid \tau|_{\mathcal{H}} = \mathrm{id}\}.$$

Then (see, e.g., [9, Proposition 2.2]), the diagonal group of Γ is

$$\mathrm{Diag}(\Gamma) = \big(\mathrm{Diag}(\Gamma) \cap \mathcal{T}_2\big) \times \langle\sigma\rangle \tag{3.1}$$

for some element $\sigma \in \mathrm{Aut}(\mathcal{L})$ such that $\sigma^2 = \mathrm{id}$ and $\sigma|_{\mathcal{H}} = -\mathrm{id}$, and where $\mathcal{T}_2$ is the 2-periodic part of $\mathcal{T}$: $\mathcal{T}_2 = \{\tau \in \mathcal{T} \mid \tau^2 = \mathrm{id}\}$. In particular, $\mathrm{Diag}(\Gamma)$ is an elementary abelian 2-group, and so is G.

For the sake of completeness, we include a sketch of the proof of (3.1). If χ is a character of G with $\chi(s) \neq 1$, the automorphism $\sigma = \tau_\chi$ restricts to $\chi(s)\mathrm{id}$ on $\mathcal{L}_s$ and hence on $\mathcal{H}$. This forces $\chi(s)^{-1}\alpha$ to be a root for any root α relative to $\mathcal{H}$. It follows that $\chi(s) = -1$, which forces $\sigma^2 = \mathrm{id}$ ([9, Lemma 2.1]). This same argument shows that

$\chi(s) = 1$ or -1 for any $\chi \in \widehat{G}$, and hence $\mathrm{Diag}(\Gamma)$ is contained in the centralizer of σ in $\{\varphi \in \mathrm{Aut}(\mathcal{L}) \mid \varphi|_{\mathcal{H}} = \pm\mathrm{id}\} = \mathcal{T} \rtimes \langle\sigma\rangle$, and this centralizer is $\mathcal{T}_2 \times \langle\sigma\rangle$ ([9, Lemma 2.1]). It follows that $\mathrm{Diag}(\Gamma) = \big(\mathrm{Diag}(\Gamma) \cap \mathcal{T}_2\big) \times \langle\sigma\rangle$.

Let Φ be the root system associated to $\mathcal{H}$:

$$\mathcal{L} = \mathcal{H} \oplus \big(\bigoplus_{\alpha\in\Phi} \mathcal{L}_\alpha\big),$$

and let $R = \mathbb{Z}\Phi$ be the root lattice. Write, as before, $\overline{R} := R/2R$. Then, identifying any character of $\overline{R}$ with a character of R trivial on $2R$, we have

$$\mathcal{T}_2 = \{\tau_\chi \mid \chi \in \mathrm{Hom}(\overline{R}, \{\pm 1\})\},$$

where $\tau_\chi \colon x \in \mathcal{L}_\alpha \mapsto \chi(\overline{\alpha})x$ for any $\alpha \in \Phi \cup \{0\}$.

Remark 3.1. Recall that, for $\mathcal{L}$ of types E_6, E_7, E_8, we have defined in (2.2) a regular quadratic form q on $\overline{R}$, regarded as a vector space over $\mathbb{F}_2$. Since $\mathcal{T}_2$ is identified with the character group of $\overline{R}$, which can be considered as the dual space of $\overline{R}$, we can transport the quadratic form q in the cases E_6 and E_8 along the isomorphism $\overline{R} \to \mathcal{T}_2$ defined by the nondegenerate bilinear form b_q. The resulting quadratic form $\hat{q}$ on $\mathcal{T}_2$ has the following meaning. For $\mathcal{L}$ of type E_8, there are two conjugacy classes of automorphisms of order 2, which are distinguished by the subalgebra of fixed points, and $\hat{q}$ takes value 0 on the nontrivial elements $\tau \in \mathcal{T}_2$ for which $\mathcal{L}^\tau$ is of type D_8 and value 1 on those for which $\mathcal{L}^\tau$ is of type $E_7 + A_1$. Moreover, the elements of $\mathcal{T}_2 \times \langle\sigma\rangle$ that are not in $\mathcal{T}_2$ give type D_8. For $\mathcal{L}$ of type E_6, there are two conjugacy classes of inner automorphisms of order 2, and $\hat{q}$ takes value 0 when $\mathcal{L}^\tau$ is of type $D_5 + Z$ (where Z stands for the 1-dimensional center) and value 1 when $\mathcal{L}^\tau$ is of type $A_5 + A_1$. Moreover, the elements of $\mathcal{T}_2 \times \langle\sigma\rangle$ that are not in $\mathcal{T}_2$ are outer automorphisms (with $\mathcal{L}^\tau$ of type C_4).

Now consider the quotient of G by the subgroup generated by the fixed element s: $\overline{G} = G/\langle s\rangle$, and the associated coarsening:

$$\overline{\Gamma} : \mathcal{L} = \bigoplus_{\bar{g}\in\overline{G}} \mathcal{L}_{\bar{g}},$$

where, for any $g \in G$, $\bar{g}$ denotes its coset $g\langle s\rangle$, and $\mathcal{L}_{\bar{g}} = \mathcal{L}_g \oplus \mathcal{L}_{gs}$. As the Cartan subalgebra $\mathcal{H}$ is contained in the neutral homogeneous component $\mathcal{L}_{\bar{s}} = \mathcal{L}_{\bar{e}}$, the grading $\overline{\Gamma}$ is a coarsening of the Cartan grading (2.1) and this gives a surjective group homomorphism

$$\pi \colon R \longrightarrow \overline{G} \tag{3.2}$$

such that for any root $\alpha \in \Phi$, $\pi(\alpha) = \bar{g}$ if $\mathcal{L}_\alpha \subseteq \mathcal{L}_{\bar{g}}$.

Let $E := \ker\pi$ be the kernel of π, which is called the *complementary lattice* in [15, 1.4]. As G is 2-elementary, $2R$ is contained in E. Also, π induces an isomorphism

$$\pi_E\colon R/E \longrightarrow \overline{G}. \tag{3.3}$$

Besides, G being the universal group of Γ, and $\overline{\Gamma}$ being a proper coarsening of Γ, there is a unique surjective map of G onto the universal group of $\overline{\Gamma}$, with s in its kernel, and hence, $\overline{G}$ is the universal group of $\overline{\Gamma}$. We therefore have:

$$E = 2R + \mathbb{Z}(\Phi^+ \cap E) + \mathbb{Z}\{\alpha - \beta \mid \alpha, \beta \in \Phi^+ \text{ and } \alpha - \beta \in E\}, \tag{3.4}$$

or, in the notation of [9, (2.4)], $E = E^\circ$. This is because the subgroup on the right hand side of (3.4) is contained in E, and the quotient of R by this subgroup gives the same coarsening $\overline{\Gamma}$, up to equivalence. (For details, see [9, §2].)

Moreover, the diagonal group of $\overline{\Gamma}$ is

$$\mathrm{Diag}(\overline{\Gamma}) = \mathrm{Diag}(\Gamma) \cap \mathcal{T}_2 = \{\tau_\chi \mid \chi \in \mathrm{Hom}(R/E, \{\pm 1\})\}.$$

A word of caution is needed here. Hesselink does not consider universal groups, although the grading groups appearing in [15] are universal in many cases.

The automorphism σ in (3.1) is not unique. If we fix one of them, and consider the corresponding character $\chi \in \widehat{G}$, so that $\chi(g) = \pm 1$ according to $\sigma|_{\mathcal{L}_g} = \pm \mathrm{id}$, we can identify our grading group G with $R/E \times \mathbb{F}_2$:

$$\begin{aligned} G &\simeq R/E \times \mathbb{F}_2 \\ g &\leftrightarrow \left(\pi_E^{-1}(\bar{g}), i\right), \end{aligned} \tag{3.5}$$

where $i = 0$ if $\chi(g) = 1$, and $i = 1$ if $\chi(g) = -1$. Under this identification, the element s corresponds to $(0, 1) \in R/E \times \mathbb{F}_2$ and the homogeneous components of Γ are the following ([9, (2.3)]):

$$\begin{aligned} \mathcal{L}_{(r+E,0)} &= \bigoplus_{\alpha \in \Phi^+ \cap (r+E)} \mathbb{F}\left(x_\alpha + \sigma(x_\alpha)\right), \\ \mathcal{L}_{(r+E,1)} &= \begin{cases} \bigoplus_{\alpha \in \Phi^+ \cap (r+E)} \mathbb{F}\left(x_\alpha - \sigma(x_\alpha)\right) & \text{if } r + E \neq E, \\ \mathcal{H} \oplus \left(\bigoplus_{\alpha \in \Phi^+ \cap (r+E)} \mathbb{F}\left(x_\alpha - \sigma(x_\alpha)\right)\right) & \text{if } r + E = E, \end{cases} \end{aligned} \tag{3.6}$$

while the homogeneous components of $\overline{\Gamma}$ are the following:

$$\begin{aligned} \overline{\mathcal{L}}_{r+E} &= \bigoplus_{\alpha \in \Phi^+ \cap (r+E)} \left(\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}\right), \qquad \text{if } r \notin E, \\ \overline{\mathcal{L}}_E &= \mathcal{H} \oplus \left(\bigoplus_{\alpha \in \Phi^+ \cap E} \left(\mathcal{L}_\alpha \oplus \mathcal{L}_{-\alpha}\right)\right). \end{aligned} \tag{3.7}$$

Here we have fixed a system of simple roots, which decomposes Φ into the set of positive roots Φ^+ and negative roots $\Phi^- = -\Phi^+$, and have picked nonzero elements $x_\alpha \in \mathcal{L}_\alpha$ for any $\alpha \in \Phi^+$.

Equation (3.6) gives at once the following result.

Proposition 3.2. *Let Γ be a pure grading on a semisimple Lie algebra $\mathcal{L}$ over an algebraically closed field of characteristic* 0*, with universal group G. Let $s \in G$ be an element such that $\mathcal{L}_s$ contains a Cartan subalgebra $\mathcal{H}$. Let Φ be the root system relative to $\mathcal{H}$, $R = \mathbb{Z}\Phi$ the root lattice, and E the complementary lattice of Γ. Then Γ is special if and only if $\Phi \cap E = \emptyset$.*

Our aim in this section is, assuming that Γ is also special, to compute the stabilizer $W_s(\Gamma)$ of s in the Weyl group of Γ:

$$W_s(\Gamma) = \{w \in W(\Gamma) \mid w(s) = s\}.$$

In many situations, there is a unique $s \in G$ with L_s containing a Cartan subalgebra, so $W_s(\Gamma)$ is the whole Weyl group $W(\Gamma)$.

Because of (3.5), we may identify $W(\Gamma)$ with a subgroup of $\mathrm{GL}(R/E \times \mathbb{F}_2) \simeq \mathrm{GL}_{l+1}(2)$, where l is the dimension of R/E, which is a 2-elementary group, and hence a vector space over $\mathbb{F}_2$. As s corresponds to $(0, 1)$, $W_s(\Gamma)$ is contained in the following subgroup:

$$\left(\begin{array}{c|c} \mathrm{GL}(R/E) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mathrm{Hom}(R/E, \mathbb{F}_2) & 1 \end{array}\right), \tag{3.8}$$

a semidirect product of $\mathrm{Hom}(R/E, \mathbb{F}_2)$ and $\mathrm{GL}(R/E)$.

Lemma 3.3. *Let Γ be a special pure grading on the semisimple Lie algebra $\mathcal{L}$. Under the identification above, the subgroup*

$$\left(\begin{array}{c|c} I_l & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \mathrm{Hom}(R/E, \mathbb{F}_2) & 1 \end{array}\right),$$

is contained in $W_s(\Gamma)$.

Proof. Let ξ be an arbitrary element of $\mathrm{Hom}(R/E, \mathbb{F}_2)$, and let $\tilde{\xi}\colon R \to \{\pm 1\}$ be the character of R given by $\tilde{\xi}(q) = (-1)^{\xi(r+E)}$. As $\mathbb{F}$ is algebraically closed, let $\chi \in \mathrm{Hom}(R, \mathbb{F}^\times)$ be a character with $\chi^2 = \tilde{\xi}$, and consider the corresponding automorphism $\tau_\chi \in \mathcal{T}$. For any $\alpha \in \Phi^+$, and $x_\alpha \in \mathcal{L}_\alpha$,

$$\tau_\chi(x_\alpha + \sigma(x_\alpha)) = \chi(\alpha)x_\alpha + \chi(\alpha)^{-1}\sigma(x_\alpha) \in \begin{cases} \mathcal{L}_{(\alpha+E,0)} & \text{if } \chi^2(\alpha) = \tilde{\xi}(\alpha) = 1, \\ \mathcal{L}_{(\alpha+E,1)} & \text{if } \chi^2(\alpha) = \tilde{\xi}(\alpha) = -1. \end{cases}$$

Moreover, if $\beta \in \Phi^+$ is another positive root with $\beta - \alpha \in E$, then $\chi^2(\alpha) = \chi^2(\beta)$, because $\tilde{\xi}(E) = 1$. Therefore, because of (3.6), τ_χ is an automorphism of Γ such that $\tau_\chi(\mathcal{L}_{(r+E,0)})$ is contained in $\mathcal{L}_{(r+E,\xi(r+E))}$ for any $r \in R$, so that its projection on $W(\Gamma)$ is

$$\left(\begin{array}{c|c} \begin{matrix} \\ I_l \\ \\ \end{matrix} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline \xi & 1 \end{array}\right),$$

thus proving our result. □

Any automorphism of the root system Φ preserves $2R$, and hence the induced group homomorphism in (2.3):

$$\rho\colon \operatorname{Aut}(\Phi) \longrightarrow \operatorname{GL}(\overline{R}),$$

gives an action of $\operatorname{Aut}(\Phi)$ on $\overline{R}$. Write $\overline{E} = E/2R$.

Any $\mu \in \operatorname{Stab}_{\operatorname{Aut}(\Phi)}(\overline{E})$ induces an automorphism of $R/E \simeq \overline{R}/\overline{E}$. Thus ρ induces a group homomorphism

$$\begin{aligned} \rho'\colon \operatorname{Stab}_{\operatorname{Aut}(\Phi)}(\overline{E}) &\longrightarrow \operatorname{GL}(R/E) \\ \mu &\mapsto \mu'. \end{aligned} \tag{3.9}$$

Consider the composition

$$\upsilon\colon G \longrightarrow \overline{G} \xrightarrow{\pi_E^{-1}} R/E, \tag{3.10}$$

where the first map is the natural homomorphism and the second is the inverse of the isomorphism π_E in (3.3). It induces a surjective group homomorphism

$$\tilde{\upsilon}\colon \operatorname{Stab}_{\operatorname{Aut}(G)}(s) \to \operatorname{GL}(R/E).$$

We get the diagram

$$\begin{array}{ccc} W_s(\Gamma) & \hookrightarrow & \operatorname{Stab}_{\operatorname{Aut}(G)}(s) \\ & & \downarrow \tilde{\upsilon} \\ \operatorname{Stab}_{\operatorname{Aut}(\Phi)}(\overline{E}) & \xrightarrow{\rho'} & \operatorname{GL}(R/E) \end{array}$$

The next result extends [15, 4.8].

Theorem 3.4. *Let Γ be a special pure grading on the semisimple Lie algebra $\mathcal{L}$, over an algebraically closed field of characteristic 0, with universal group G. Let $s \in G$ be an* *element such that $\mathcal{L}_s$ contains a Cartan subalgebra $\mathcal{H}$. Then, with the notations above, the stabilizer of s in the Weyl group $W(\Gamma)$ is obtained as follows:*

$$W_s(\Gamma) = \tilde{\upsilon}^{-1}(\mathrm{im}\ \rho').$$

In other words, $W_s(\Gamma)$ consists of the matrices in (3.8) with arbitrary lower left corner, and where the upper left corner (in $\mathrm{GL}(R/E)$) is induced by an automorphism of Φ that preserves $\overline{E}$.

Proof. Note that $\mathcal{L}_s = \mathcal{H}$ holds, because of Proposition 3.2 and Equation (3.6). We proceed in several steps.

1) Any $\varphi \in \mathrm{Aut}(\Gamma)$ preserving $\mathcal{L}_s = \mathcal{H}$ induces an automorphism $\mu_\varphi \in \mathrm{Aut}(\Phi)$ with $\varphi(\mathcal{L}_\alpha) = \mathcal{L}_{\mu_\varphi(\alpha)}$ for any $\alpha \in \Phi$.

Since Γ is special, we have $\Phi^+ \cap E = \emptyset$. For any $\alpha, \beta \in \Phi^+$ with $\alpha - \beta \in E$, the elements $x_\alpha + \sigma(x_\alpha)$ and $x_\beta + \sigma(x_\beta)$ are in the same homogeneous component of Γ, and hence so are their images under φ. Thus $\mu_\varphi(\alpha) - \mu_\varphi(\beta)$ is in E. Equation (3.4) shows $\mu_\varphi(E) \subseteq E$, which is equivalent to $\mu_\varphi \in \mathrm{Stab}_{\mathrm{Aut}(\Phi)}(\overline{E})$.

Note that the image under $\tilde{\upsilon}$ of the element of $W_s(\Gamma)$ determined by φ is the image under ρ' of μ_φ.

2) Conversely, for any $\mu \in \mathrm{Stab}_{\mathrm{Aut}(\Phi)}(\overline{E})$, take an element $\varphi \in \mathrm{Stab}_{\mathrm{Aut}(\mathcal{L})}(\mathcal{H})$ with $\mu_\varphi = \mu$ ([18, 14.2]). Note that we have $\mu(E) = E$ and $\varphi \in \mathrm{Aut}(\overline{\Gamma})$ (recall the homogeneous components of $\overline{\Gamma}$ in (3.7)).

For any $\tau \in \mathrm{Diag}(\Gamma) \cap \mathcal{T} = \mathrm{Diag}(\Gamma) \cap \mathcal{T}_2 = \mathrm{Diag}(\overline{\Gamma})$, we get $\varphi\tau\varphi^{-1} \in \mathrm{Diag}(\overline{\Gamma})$. Consider the automorphism $\sigma' = \varphi\sigma\varphi^{-1}$, which satisfies $\sigma'|_{\mathcal{H}} = -\mathrm{id}$, so there exists an element $\tau \in \mathcal{T}$ such that $\sigma' = \tau\sigma$. Take $\tilde{\tau} \in \mathcal{T}$ with $\tilde{\tau}^2 = \tau$. Then $\sigma' = \tau\sigma = \tilde{\tau}^2\sigma = \tilde{\tau}\sigma\tilde{\tau}^{-1}$. Therefore we have $\varphi\sigma\varphi^{-1} = \tilde{\tau}\sigma\tilde{\tau}^{-1}$ and $\tilde{\tau}^{-1}\varphi$ commutes with σ, stabilizes $\mathcal{H}$ and satisfies $\mu_{\tilde{\tau}^{-1}\varphi} = \mu_\varphi = \mu$.

Replacing φ by $\tilde{\tau}^{-1}\varphi$, we may assume $\varphi\sigma = \sigma\varphi$, and hence we get $\varphi \in \mathrm{Aut}(\Gamma)$ with $\varphi(\mathcal{L}_s) = \mathcal{L}_s$, because the homogeneous components of Γ (Equation (3.6)) are the eigenspaces for σ of the homogeneous components of $\overline{\Gamma}$.

3) For any $\varphi \in \mathrm{Aut}(\Gamma)$ preserving $\mathcal{L}_s = \mathcal{H}$ we obtain the following elements:

$$\mu_\varphi \in \mathrm{Stab}_{\mathrm{Aut}(\Phi)}(\overline{E}), \quad \rho'(\mu_\varphi) = \mu'_\varphi \in \mathrm{GL}(R/E),$$

and for any $\alpha \in \Phi^+$ we have

$$\varphi(\mathcal{L}_{(\alpha+E,0)}) = \mathcal{L}_{(\mu'_\varphi(\alpha+E),0 \text{ or } 1)}.$$

The associated element in $W_s(\Gamma)$ is then, considered as an element of $\mathrm{GL}(R/E \times \mathbb{F}_2)$, of the form

$$\left(\begin{array}{c|c} \mu'_\varphi & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline \xi & 1 \end{array}\right),$$

for some $\xi \in \operatorname{Hom}(R/E, \mathbb{F}_2)$. But step **2)** shows that given any element $\mu' = \rho'(\mu)$ with $\mu \in \operatorname{Stab}_{\operatorname{Aut}(\Phi)}(\overline{E})$, there is an element $w \in W_s(\Gamma)$ of the form

$$\left(\begin{array}{c|c} \mu' & \begin{matrix}0\\ \vdots\\ 0\end{matrix} \\ \hline \xi & 1 \end{array}\right),$$

for some $\xi \in \operatorname{Hom}(R/E, \mathbb{F}_2)$. Now Lemma 3.3 completes the proof. □

4. Special pure gradings on simple Lie algebras of types E_6, E_7, or E_8

The aim of this section is the classification, up to equivalence, of the special pure gradings on the simple Lie algebras of type E_6, E_7, and E_8. We will make use of Proposition 2.1 that relates the Weyl groups of these algebras with suitable orthogonal groups over the field of 2 elements.

We start with E_8.

Theorem 4.1. *Up to equivalence, there are exactly five special pure gradings on a simple Lie algebra of type E_8 over an algebraically closed field of characteristic 0. Their universal groups and types are given in the next table:*

	Universal group	*Type*
$\Gamma^9_{E_8}$	$\mathbb{Z}_2^9$	$(240, 0, 0, 0, 0, 0, 0, 1)$
$\Gamma^8_{E_8}$	$\mathbb{Z}_2^8$	$(128, 56, 0, 0, 0, 0, 0, 1)$
$\Gamma^7_{E_8}$	$\mathbb{Z}_2^7$	$(0, 96, 0, 12, 0, 0, 0, 1)$
$\Gamma^6_{E_8}$	$\mathbb{Z}_2^6$	$(0, 0, 0, 56, 0, 0, 0, 3)$
$\Gamma^5_{E_8}$	$\mathbb{Z}_2^5$	$(0, 0, 0, 0, 0, 0, 0, 31)$

Proof. The quadratic form $q\colon \overline{R} \to \mathbb{F}_2$ in (2.2) is regular with trivial Arf invariant. Moreover, the nonisotropic vectors of $\overline{R}$ are precisely the 120 vectors $\bar{\alpha} := \alpha + 2R$ for $\alpha \in \Phi^+$. By Proposition 3.2, the complementary lattice E satisfies that $\overline{E} = E/2R$ is a totally isotropic subspace of $\overline{R}$, and hence it has dimension at most 4. Moreover, for any dimension up to 4, any two totally isotropic subspaces of $\overline{R}$ are conjugate under the action of $\mathrm{O}(q)$. As a consequence, we get the following possibilities, according to (3.6):

$\dim \overline{E} = 0$**:** Here the universal group $R/E \times \mathbb{F}_2 \simeq \overline{R} \times \mathbb{F}_2$ (see (3.5)) is isomorphic to $\mathbb{Z}_2^9$, the homogeneous space $\mathcal{L}_{(0,1)}$ is a Cartan subalgebra, and all the other homogeneous components are one-dimensional. Hence, the type is $(240, 0, 0, 0, 0, 0, 0, 1)$.

$\dim \overline{E} = 1$**:** In this case the universal group is isomorphic to $\mathbb{Z}_2^8$. There is an isotropic vector u with $\overline{E} = \mathbb{F}_2 u$. If we fix an element v with $b_q(u, v) = 1$, then u and v span a hyperbolic plane H, and $\overline{R} = H \oplus H^\perp$, so $0 = \mathrm{Arf}(q) = \mathrm{Arf}(q|_H) + \mathrm{Arf}(q|_{H^\perp}) = \mathrm{Arf}(q|_{H^\perp})$. Moreover, q induces a quadratic form on the six-dimensional space $\overline{E}^\perp/\overline{E}$, which is isometric to $H^\perp$ and hence has trivial Arf invariant. Thus, $\overline{E}^\perp/\overline{E}$ contains 28 nonisotropic vectors. Besides, for any $v \in \overline{R}$ with $q(v) = 1$, we have $q(v + u) = 1$ if and only if $b_q(u, v) = 0$, that is, $v \in \overline{E}^\perp$, so there are 28 pairs of positive roots (α, β) such that $\alpha + E = \beta + E$ ($\alpha + 2R = v \in \overline{E}^\perp$, $\beta + 2R = u + v$). This gives 56 two-dimensional homogeneous components in (3.6). For the remaining positive roots, that is, those $\gamma \in \Phi^+$ such that $\bar{\gamma} \notin \overline{E}^\perp$, the corresponding homogeneous components in (3.6) are one-dimensional. Therefore, the type is $(128, 56, 0, 0, 0, 0, 0, 1)$.

$\dim \overline{E} = 2$**:** Here, if a positive root $\alpha \in \Phi^+$ satisfies $b_q(\bar{\alpha}, \overline{E}) = 0$, then $\bar{\alpha} + \overline{E}$ consists of 4 nonisotropic vectors. Since $\overline{E}^\perp/\overline{E}$ is a regular quadratic space with trivial Arf invariant of dimension 4, it contains 6 nonisotropic vectors, and thus $\overline{E}^\perp$ contains 6 left cosets of the form $\bar{\alpha} + \overline{E}$. This means that there are 12 homogeneous spaces of dimension 4 in (3.6).

On the other hand, there are $120 - 24 = 96$ positive roots $\alpha \in \Phi^+$ with $b_q(\bar{\alpha}, \overline{E}) \neq 0$, and then the left coset $\bar{\alpha} + \overline{E}$ contains just two nonisotropic vectors, thus obtaining 48 pairs of positive roots, which give 96 homogeneous components in (3.6) of dimension 2. Hence, the type is $(0, 96, 0, 12, 0, 0, 0, 1)$.

$\dim \overline{E} = 3$**:** In this case, the quadratic space $\overline{E}^\perp/\overline{E}$ is a regular quadratic space with trivial Arf invariant of dimension 2, so it contains a unique nonisotropic element. Thus, if $\alpha \in \Phi^+$ is in $\overline{E}^\perp$, the coset $\bar{\alpha} + \overline{E}$ consists of the 8 nonisotropic vectors in $\overline{E}^\perp$. This means that the homogeneous components $\mathcal{L}_{(\alpha+E,\bar{0})}$, $\mathcal{L}_{(\alpha+E,\bar{1})}$, and $\mathcal{L}_{(E,\bar{1})}$ are eight-dimensional, and actually Cartan subalgebras, because the homogeneous components in a special grading on a semisimple Lie algebra are always toral [15, 3.8 Theorem].

For any of the remaining 112 positive roots $\alpha \in \Phi^+$ with $b_q(\bar{\alpha}, \overline{E}) \neq 0$, the coset $\bar{\alpha} + \overline{E}$ contains 4 nonisotropic vectors. This forces all the other homogeneous components in (3.6) to be 4-dimensional, so that the type is $(0, 0, 0, 56, 0, 0, 0, 3)$.

$\dim \overline{E} = 4$**:** Here $\overline{E}^\perp = \overline{E}$, and all the cosets $\bar{\alpha} + \overline{E}$ contain 8 nonisotropic vectors. Thus, all homogeneous components have dimension 8 and the type is $(0, 0, 0, 0, 0, 0, 0, 31)$. □

Theorem 4.2. *Up to equivalence, there are exactly four special pure gradings on a simple Lie algebra of type* E_7 *over an algebraically closed field of characteristic* 0*. Their universal groups and types are given in the next table.*

Proof. The quadratic form $q\colon \overline{R} \to \mathbb{F}_2$ in (2.2) is regular, but odd-dimensional. The radical of b_q is one-dimensional: $\mathrm{rad}\, b_q = \mathbb{F}_2 a$, with $q(a) = 1$. Moreover, the nonisotropic

	Universal group	*Type*
$\Gamma^8_{E_7}$	$\mathbb{Z}_2^8$	$(126, 0, 0, 0, 0, 0, 1)$
$\Gamma^7_{E_7}$	$\mathbb{Z}_2^7$	$(66, 30, 0, 0, 0, 0, 1)$
$\Gamma^6_{E_7}$	$\mathbb{Z}_2^6$	$(0, 48, 2, 6, 0, 0, 1)$
$\Gamma^5_{E_7}$	$\mathbb{Z}_2^5$	$(0, 0, 0, 28, 0, 0, 3)$

vectors of $\overline{R}$ are precisely the 63 vectors $\bar{\alpha} := \alpha + 2R$ for $\alpha \in \Phi^+$ together with a. Any subspace of $\overline{R}$ containing strictly $\operatorname{rad} b_q$ contains other nonisotropic vectors besides a, so it is not of the form $\overline{E} = E/2R$ with E the complementary lattice of a special pure grading. Also, the lattice F with $\overline{F} = \operatorname{rad} b_q$ cannot be the complementary lattice of a special pure grading, graded by its universal group, because for any two different positive roots $\alpha, \beta \in \Phi^+$, $\bar{\alpha} - \bar{\beta} \neq a$, as otherwise we would have $1 = q(\bar{\beta}) = q(\bar{\alpha} + a) = q(\bar{\alpha}) + q(a) = 1 + 1 = 0$, and then (3.4) is not fulfilled. Therefore, the complementary lattice E of a special pure grading satisfies that $\overline{E}$ is a totally isotropic subspace of $\overline{R}$ and, in particular, it does not contain $\operatorname{rad} b_q$.

As a consequence, we get the following possibilities, according to (3.6):

$\dim \overline{E} = 0$: As for E_8, the universal group is isomorphic to $\mathbb{Z}_2^8$ and the type is $(126, 0, 0, 0, 0, 0, 1)$.

$\dim \overline{E} = 1$: Here the quadratic space $\overline{E}^{\perp}/\overline{E}$ is regular of dimension 5, so it contains 16 nonisotropic elements, one of them being $a + \overline{E}$. For each $\alpha \in \Phi^+$ with $\bar{\alpha} \in \overline{E}^{\perp}$, the coset $\bar{\alpha} + \overline{E}$ consists of two nonisotropic vectors, so either $\bar{\alpha} + \overline{E} = \{\bar{\alpha}, \bar{\beta}\}$ for another positive root β, or $\bar{\alpha} + \overline{E} = \{\bar{\alpha}, a\}$, which happens only for $\bar{\alpha} = a + u$, with $\overline{E} = \mathbb{F}_2 u$. Hence there are 15 pairs of positive roots that come together in (3.6). The other $35 = 63 - 30$ positive roots give rise to one-dimensional homogeneous components and the type is $(66, 30, 0, 0, 0, 0, 1)$.

$\dim \overline{E} = 2$: In this case the quadratic space $\overline{E}^{\perp}/\overline{E}$ is regular of dimension 3, so it contains 4 nonisotropic elements, one of them being $a + \overline{E}$. For each $\alpha \in \Phi^+$ with $\bar{\alpha} \in \overline{E}^{\perp}$, the coset $\bar{\alpha} + \overline{E}$ consists of four nonisotropic vectors, which come from four positive roots unless $\bar{\alpha} + \overline{E} = a + \overline{E}$ that contains three nonisotropic vectors obtained from positive roots. For the remaining $48 = 63 - 15$ positive roots α with $\bar{\alpha} \notin \overline{E}^{\perp}$, the coset $\bar{\alpha} + \overline{E}$ contains two nonisotropic vectors. The type is then $(0, 48, 2, 6, 0, 0, 1)$.

$\dim \overline{E} = 3$: Here the quadratic space $\overline{E}^{\perp}/\overline{E}$ is regular of dimension 1. The nonisotropic vectors of $\overline{E}^{\perp}$ form the left coset $a + \overline{E}$, that contains eight nonisotropic vectors of $\overline{R}$: the element a itself, and seven elements of the form $\bar{\alpha}$ with $\alpha \in \Phi^+$. Pick one of them, so that $a + \overline{E} = \bar{\alpha} + \overline{E}$, then the homogeneous components $\mathcal{L}_{(\alpha+E,\bar{0})}$, $\mathcal{L}_{(\alpha+E,\bar{1})}$, and $\mathcal{L}_{(E,\bar{1})}$ are seven-dimensional Cartan subalgebras.

For the remaining $56 = 63 - 7$ positive roots α with $\bar{\alpha} \notin \overline{E}^{\perp}$, the coset $\bar{\alpha} + \overline{E}$ contains four nonisotropic vectors. The type is then $(0, 0, 0, 28, 0, 0, 3)$. □

Theorem 4.3. *Up to equivalence, there are exactly three special pure gradings on a simple Lie algebra of type E_6 over an algebraically closed field of characteristic* 0*. Their universal groups and types are given in the next table:*

	Universal group	*Type*
$\Gamma^7_{E_6}$	$\mathbb{Z}_2^7$	$(72, 0, 0, 0, 0, 1)$
$\Gamma^6_{E_6}$	$\mathbb{Z}_2^6$	$(32, 20, 0, 0, 0, 1)$
$\Gamma^5_{E_6}$	$\mathbb{Z}_2^5$	$(0, 24, 0, 6, 0, 1)$

Proof. The quadratic form $q\colon \overline{R} \to \mathbb{F}_2$ in (2.2) is regular with Arf invariant 1, and the nonisotropic vectors are precisely the 36 vectors $\bar{\alpha}$ for $\alpha \in \Phi^+$. Therefore, because of Proposition 3.2, the complementary lattice E satisfies that $\overline{E} = E/2R$ is a totally isotropic subspace of $\bar{R}$, and hence it has dimension at most 2, as the Arf invariant is not trivial. The following possibilities appear:

$\dim \overline{E} = 0$**:** As for E_8 and E_7, the universal group is isomorphic to $\mathbb{Z}_2^7$ and the type is $(72, 0, 0, 0, 0, 1)$.

$\dim \overline{E} = 1$**:** Here the quadratic four-dimensional space $\overline{E}^{\perp}/\overline{E}$ is regular and its Arf invariant is 1, so it contains 10 nonisotropic elements, of the form $\bar{\alpha} + \overline{E}$, with $\alpha \in \Phi^+$ and $\bar{\alpha} \in \overline{E}^{\perp}$. Each such left coset contains two nonisotropic vectors. The remaining $16 = 36 - 20$ positive roots give rise to one-dimensional homogeneous components and the type is $(32, 20, 0, 0, 0, 1)$.

$\dim \overline{E} = 2$**:** In this case the quadratic two-dimensional space $\overline{E}^{\perp}/\overline{E}$ is regular and its Arf invariant is 1, so it contains 3 nonisotropic elements, of the form $\bar{\alpha} + \overline{E}$, with $\alpha \in \Phi^+$ and $\bar{\alpha} \in \overline{E}^{\perp}$. Each such left coset contains four nonisotropic vectors. The remaining $24 = 36 - 12$ positive roots α satisfy that the left coset $\bar{\alpha} + \overline{E}$ contains two nonisotropic vectors, and hence give rise to two-dimensional homogeneous components. The type is $(0, 24, 0, 6, 0, 1)$. □

5. Weyl groups of the special pure gradings on simple Lie algebras of types E_6, E_7, or E_8

This section is devoted to computing the Weyl group of each special pure grading on the simple Lie algebras of types E_6, E_7, E_8 and to describing its action on the corresponding universal group.

Let Γ be a special pure grading on a simple Lie algebra $\mathcal{L}$ of type E_6, E_7, or E_8, with universal group G. Let $s \in G$ be an element such that $\mathcal{L}_s$ contains a Cartan subalgebra $\mathcal{H}$. As the first result of this section, we are going to describe the structure of the group $W_s(\Gamma)$ in Theorem 3.4.

Let Φ be the root system relative to $\mathcal{H}$, let $R = \mathbb{Z}\Phi$ be the root lattice, and consider the quotient $\overline{R} = R/2R$. Let $\pi\colon R \to \overline{G}$ be the projection in (3.2) and let $q\colon \overline{R} \to \mathbb{F}_2$ be the quadratic form in (2.2). Let E be the complementary lattice of Γ and $\overline{E} = E/2R$. Denote by $\overline{\pi}\colon \overline{R} \to \overline{G}$ the projection induced from π (recall that G is an elementary 2-group) and let H be the subgroup of G such that $\overline{H} := H/\langle s\rangle = \overline{\pi}(\overline{E}^{\perp})$.

The quadratic form q restricts to a quadratic form $q_{\overline{E}}\colon \overline{E}^{\perp} \to \mathbb{F}_2$. As q is regular and $\overline{E}$ is totally isotropic (see Section 4), $\overline{E}$ is the radical of $q_{\overline{E}}$. Hence $q_{\overline{E}}$ is transferred by means of $\overline{\pi}$ to a regular quadratic form $q_{H,s}\colon \overline{H} \to \mathbb{F}_2$ (where we regard $\overline{H}$ as a vector space over $\mathbb{F}_2$).

There appears a natural flag $\mathcal{F}$ of subgroups of G:

$$\mathcal{F}: \quad 1 \le \langle s\rangle \le H \le G,$$

and we may consider the stabilizer of this flag, denoted by $\mathrm{Stab}_{\mathrm{Aut}(G)}(\mathcal{F})$. Any element $\varphi \in \mathrm{Stab}_{\mathrm{Aut}(G)}(\mathcal{F})$ induces an automorphism of $\overline{H} = H/\langle s\rangle$, denoted by $\varphi_{H,s}$.

Theorem 5.1. *Let Γ be a special pure grading on a simple Lie algebra $\mathcal{L}$ of type E_6, E_7, or E_8, over an algebraically closed field of characteristic 0, with universal group G. With the notations preceding the theorem, we have*

$$W_s(\Gamma) = \{\varphi \in \mathrm{Stab}_{\mathrm{Aut}(G)}(\mathcal{F}) \mid \varphi_{H,s} \in \mathrm{O}(q_{H,s})\}.$$

Proof. Since $\overline{E}$ is totally isotropic, we can choose a totally isotropic subspace $\overline{E}'$ of $\overline{R}$ such that $\overline{E}\cap\overline{E}' = 0$ and $b_q|_{\overline{E}\oplus\overline{E}'}$ is nondegenerate. To do this, it suffices to use induction on $\dim \overline{E}$ and the fact that any isotropic vector is contained in a hyperbolic plane (see, e.g., the proof of [12, Lemma 8.10]). Letting $\overline{W}$ be the orthogonal complement to $\overline{E}\oplus\overline{E}'$ relative to b_q, we get $\overline{R} = \overline{E} \oplus \overline{E}' \oplus \overline{W}$.

Identify $R/E \simeq \overline{R}/\overline{E}$ with $\overline{E}' \oplus \overline{W}$, and G with $(\overline{E}' \oplus \overline{W}) \times \mathbb{F}_2$ by means of (3.5). In this way, H corresponds to $\overline{W} \times \mathbb{F}_2$. Then we must prove that $W_s(\Gamma)$, seen as a group of transformations of $(\overline{E}' \oplus \overline{W}) \times \mathbb{F}_2$, consists of the linear maps

$$\left(\begin{array}{cc|c} f & 0 & 0 \\ \alpha & g & 0 \\ \hline \multicolumn{2}{c|}{\xi} & 1 \end{array}\right), \tag{5.1}$$

with $f \in \mathrm{GL}(\overline{E}')$, $g \in \mathrm{O}(q|_{\overline{W}})$, $\alpha \in \mathrm{Hom}(\overline{E}', \overline{W})$, and $\xi \in \mathrm{Hom}(\overline{E}' \oplus \overline{W}, \mathbb{F}_2)$.

Because of Proposition 2.1, the homomorphism

$$\rho'\colon \mathrm{Stab}_{\mathrm{Aut}(\Phi)}(\overline{E}) \to \mathrm{GL}(R/E)$$

in (3.9) induces a homomorphism

$$\begin{aligned} \bar{\rho}\colon \mathrm{Stab}_{\mathrm{O}(q)}(\overline{E}) &\longrightarrow \mathrm{GL}(R/E) \\ \mu &\mapsto \bar{\mu}, \end{aligned}$$

where $\bar{\mu}(r+E) = \mu(r+2R) + \overline{E} \in \overline{R}/\overline{E} \simeq R/E$ for any $r \in R$. Moreover, we have $\operatorname{im} \rho' = \operatorname{im} \bar{\rho}$.

Relative to the decomposition $\overline{R} = \overline{E} \oplus \overline{E}' \oplus \overline{W}$, the group $\operatorname{Stab}_{\mathrm{O}(q)}(\overline{E})$ consists of matrices of the form

$$A = \begin{pmatrix} h & \beta & \gamma \\ 0 & f & 0 \\ 0 & \alpha & g \end{pmatrix}$$

for $h \in \mathrm{GL}(\overline{E})$, $f \in \mathrm{GL}(\overline{E}')$, $g \in \mathrm{O}(q|_{\overline{W}})$, and linear maps (over $\mathbb{F}_2$) $\alpha\colon \overline{E}' \to \overline{W}$, $\beta\colon \overline{E}' \to \overline{E}$, $\gamma\colon \overline{W} \to \overline{E}$. Then A belongs to $\mathrm{O}(q)$ if and only if equations (5.2), (5.3), (5.4) hold.

Actually, we have:

- For any $u \in \overline{E}$ and $v \in \overline{E}'$,

$$b_q\big(h(u), f(v)\big) = b_q\big(h(u), f(v) + \beta(v)\big) = b_q(u, v), \tag{5.2}$$

  so that any $f \in \mathrm{GL}(\overline{E}')$ determines $h \in \mathrm{GL}(\overline{E})$ such that (5.2) holds, and conversely.
- For any $v \in \overline{E}'$ and $w \in \overline{W}$,

$$\begin{aligned} 0 &= b_q\big(\beta(v) + f(v) + \alpha(v), \gamma(w) + g(w)\big) \\ &= b_q\big(f(v), \gamma(w)\big) + b_q\big(\alpha(v), g(w)\big), \end{aligned}$$

  that is,

$$b_q\big(f(v), \gamma(w)\big) = b_q\big(\alpha(v), g(w)\big). \tag{5.3}$$

  Therefore, once f and g are fixed, any α determines γ such that (5.3) holds. (And if $\operatorname{rad} b_q = 0$, then any γ determines α too.)
- For any $v \in \overline{E}'$,

$$0 = q(v) = q\big(\beta(v) + f(v) + \alpha(v)\big) = q\big(\alpha(v)\big) + b_q\big(f(v), \beta(v)\big),$$

  that is,

$$q\big(\alpha(v)\big) = b_q\big(f(v), \beta(v)\big). \tag{5.4}$$

  For any $\alpha \in \operatorname{Hom}(\overline{E}', \overline{W})$ and $f \in \mathrm{GL}(\overline{E}')$, there is a linear map $\beta\colon \overline{E}' \to \overline{E}$ satisfying this equation.

The image under $\bar{\rho}$ of the matrix A above is, under our identification $R/E \simeq \overline{E}' \oplus \overline{W}$, the matrix

$$\begin{pmatrix} f & 0 \\ \alpha & g \end{pmatrix}.$$

For any $f \in \mathrm{GL}(\overline{E}')$, $g \in \mathrm{O}(q|_{\overline{W}})$, and $\alpha \in \mathrm{Hom}_{\mathbb{F}_2}(\overline{E}', \overline{W})$, there exist h, β, γ satisfying equations (5.2), (5.3), (5.4), and hence the result follows from Theorem 3.4. □

Corollary 5.2 *(of the proof). If $\mathcal{L}$ in Theorem 5.1 has type E_6 or E_8, then $W_s(\Gamma)$ is isomorphic to the affine orthogonal group of the restriction of q to $\overline{E}^{\perp}$.*

Proof. Note that $\overline{E}^{\perp} = \overline{E} \oplus \overline{W}$. Moreover, for E_6 and E_8, $\mathrm{rad}\, b_q$ is trivial, so the orthogonal group $\mathrm{O}(q|_{\overline{E}^{\perp}})$ is isomorphic to the image of $\bar{\rho}$ by means of the map

$$\begin{aligned} \Theta \colon \mathrm{O}(q|_{\overline{E}^{\perp}}) &\longrightarrow \mathrm{im}\ \bar{\rho} \\ \begin{pmatrix} h & \gamma \\ 0 & g \end{pmatrix} &\mapsto \begin{pmatrix} f & 0 \\ \alpha & g \end{pmatrix}, \end{aligned}$$

for $h \in \mathrm{GL}(\overline{E})$, $g \in \mathrm{O}(q|_{\overline{W}})$ and $\gamma \in \mathrm{Hom}(\overline{W}, \overline{E})$, where $f \in \mathrm{GL}(\overline{E}')$ is determined by (5.2) and α by (5.3).

On the other hand, $\overline{E}^{\perp}$ is naturally isomorphic to $\mathrm{Hom}(\overline{R}/\overline{E}, \mathbb{F}_2)$ by means of b_q. Denote this isomorphism by θ.

As we have seen, Theorem 3.4 shows that $W_s(\Gamma)$ is isomorphic to the semidirect product $\mathrm{Hom}(R/E, \mathbb{F}_2) \rtimes \mathrm{im}\ \bar{\rho} \leq \mathrm{Hom}(R/E, \mathbb{F}_2) \rtimes \mathrm{GL}(R/E)$, where the action of $\mathrm{GL}(R/E)$ on $\mathrm{Hom}(R/E, \mathbb{F}_2)$ is the natural one: $f.\xi = \xi \circ f^{-1}$. We claim that our isomorphisms Θ and θ are compatible with the action, i.e., $\theta(Ax) = \Theta(A).\theta(x)$ for all $A \in \mathrm{O}(q|_{\overline{E}^{\perp}})$ and $x \in \overline{E}^{\perp}$. Using the identification $R/E \simeq \overline{E}' \oplus \overline{W}$, this is equivalent to the equation $b_q(A(u+w), \Theta(A)(v'+w')) = b_q(u+w, v'+w')$ for all $u \in \overline{E}$, $v' \in \overline{E}'$, $w, w' \in \overline{W}$, and $A = \left(\begin{smallmatrix} h & \gamma \\ 0 & g \end{smallmatrix}\right) \in \mathrm{O}(q|_{\overline{E}^{\perp}})$, which is easy to check:

$$\begin{aligned} b_q\bigl(A(u+w), &\Theta(A)(v'+w')\bigr) \\ &= b_q\bigl(h(u)+\gamma(w)+g(w), f(v')+\alpha(v')+g(w')\bigr) \\ &= b_q\bigl(h(u)+\gamma(w), f(v')\bigr) + b_q\bigl(g(w), \alpha(v')+g(w')\bigr) \\ &= b_q(u, v') + b_q(w, w') = b_q(u+w, v'+w'), \end{aligned}$$

where we have used (5.2) and (5.3).

We conclude that $W_s(\Gamma)$ is isomorphic to the semidirect product $\overline{E}^{\perp} \rtimes \mathrm{O}(q|_{\overline{E}^{\perp}})$, which is the affine orthogonal group of the restriction of q to $\overline{E}^{\perp}$. □

Remark 5.3. In case the complementary lattice E is trivial: $E = 2R$, which is the case in Example 2.11, one has $\overline{R} = \overline{W}$ in the proof of Theorem 5.1, and hence $W_s(\Gamma)$ is the orthogonal group of the degenerate quadratic form q_G defined on G (a 2-elementary group, and hence a vector space over $\mathbb{F}_2$), by means of $q_G(g) := q\big(\upsilon(g)\big)$, with $\upsilon\colon G \to \overline{R}$ in (3.10).

According to Theorems 4.1, 4.2 and 4.3, the special pure gradings on a simple Lie algebra of type E_6, E_7, or E_8, contain a unique homogeneous component equal to a Cartan subalgebra, with the exceptions of the special pure gradings on E_8 with universal groups isomorphic to $\mathbb{Z}_2^6$ and $\mathbb{Z}_2^5$, and the special pure grading on E_7 with universal group isomorphic to $\mathbb{Z}_2^5$.

The Weyl group of the special pure grading on E_8 with universal group $\mathbb{Z}_2^5$ is the group of automorphisms of the universal group, hence isomorphic to $\mathrm{GL}_5(\mathbb{F}_2)$ (see [15, 6.2 Theorem]). This grading is an example of Jordan grading (see [1,20]).

For the remaining cases, $W_s(\Gamma) = W(\Gamma)$ by the uniqueness of the homogeneous component being equal to a Cartan subalgebra, so Theorem 5.1 determines the whole Weyl group.

Therefore, we must deal with the special pure grading on E_8 with universal group isomorphic to $\mathbb{Z}_2^6$, and with the special pure grading on E_7 with universal group isomorphic to $\mathbb{Z}_2^5$. The proofs of Theorems 4.1 and 4.2 show that, in both cases, there are exactly three homogeneous components that are Cartan subalgebras. The subgroup spanned by the degrees of these homogeneous components is isomorphic to $\mathbb{Z}_2^2$.

Proposition 5.4. *Let $\mathcal{L}$ be a simple Lie algebra of type E_7 or E_8 over an algebraically closed field of characteristic 0. Let $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ be the unique, up to equivalence, special pure grading with universal group G and with exactly three homogeneous components: $\mathcal{L}_s$, $\mathcal{L}_{s'}$ and $\mathcal{L}_{s''}$, being Cartan subalgebras. Let $W(\Gamma)$ be its Weyl group, and let $W'(\Gamma)$ be the pointwise stabilizer in $W(\Gamma)$ of $\{s, s', s''\}$:*

$$W'(\Gamma) = \{w \in W(\Gamma) \mid w(s) = s,\ w(s') = s',\ w(s'') = s''\}.$$

Then there is a short exact sequence

$$1 \to W'(\Gamma) \hookrightarrow W(\Gamma) \to S_3 \to 1,$$

where S_3 denotes the symmetric group of degree 3, and the homomorphism $W(\Gamma) \to S_3$ is induced from the action of $W(\Gamma)$ on the elements $s, s', s'' \in G$.

Proof. In the two cases we are dealing with, we can identify G with $R/E \times \mathbb{F}_2$ as in (3.5) with $\overline{E} = E/2R$ a totally isotropic subspace of $\overline{R} = R/2R$. Fix a decomposition $\overline{R} = \overline{E} \oplus \overline{E}' \oplus \overline{W}$ as in the proof of Theorem 5.1 and identify R/E with $\overline{E}' \oplus \overline{W}$. For E_8, $\overline{W}$ is a two-dimensional space and the Arf invariant of $q|_{\overline{W}}$ is trivial, so there is a unique vector $a \in \overline{W}$ with $q(a) = 1$. For E_7, $\overline{W} = \operatorname{rad} b_q = \mathbb{F}_2 a$, with $q(a) = 1$. The three homogeneous components that are Cartan subalgebras are $\mathcal{L}_{(0,1)}$, $\mathcal{L}_{(a,0)}$, and $\mathcal{L}_{(a,1)}$ (see (3.6)). Write $s = (0, 1)$, $s' = (a, 0)$ and $s'' = s + s' = (a, 1)$. Equation (5.1) shows that $W_s(\Gamma)$, as a group of transformations of $(\overline{E}' \oplus \overline{W}) \times \mathbb{F}_2$, consists of the linear maps

$$\left(\begin{array}{cc|c} f & 0 & 0 \\ \alpha & g & 0 \\ \hline \epsilon & \omega & 1 \end{array}\right), \tag{5.5}$$

with $g \in \mathrm{O}(q|_{\overline{W}})$, $f \in \mathrm{GL}(\overline{E}')$, $\alpha \in \mathrm{Hom}(\overline{E}', \overline{W})$, $\epsilon \in \mathrm{Hom}(\overline{E}', \mathbb{F}_2)$, and $\omega \in \mathrm{Hom}(\overline{W}, \mathbb{F}_2)$. Note that $\mathrm{O}(q|_{\overline{W}})$ is cyclic of order 2 and fixes a for E_8, while it is trivial for E_7. Therefore, $W'(\Gamma)$ consists of the linear maps in (5.5) with $\omega(a) = 0$.

The linear maps of the form

$$\left(\begin{array}{cc|c} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ \hline 0 & \omega & 1 \end{array}\right),$$

with $\omega(a) = 1$ have order 2 and permute the labels s' and s'', while fixing s.

However, the three Cartan subalgebras $\mathcal{L}_s$, $\mathcal{L}_{s'}$ and $\mathcal{L}_{s''}$ play the same role, so we could have started with the Cartan subalgebra $\mathcal{H} = \mathcal{L}_{s'}$ to conclude that there are elements in $W(\Gamma)$ permuting $\mathcal{L}_s$ and $\mathcal{L}_{s''}$. We conclude that the natural homomorphism $W(\Gamma) \to S_3$ is surjective, and its kernel is, by definition $W'(\Gamma)$. □

Corollary 5.5. *Let $\mathcal{L}$ be a simple Lie algebra of type E_7 or E_8 over an algebraically closed field of characteristic 0. Let $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ be the unique, up to equivalence, special pure grading with universal group G and with exactly three homogeneous components being Cartan subalgebras. Then the order of the Weyl group is given in the following table:*

Type of $\mathcal{L}$	$\lvert W(\Gamma)\rvert$
E_7	$2^{10} \times 3^2 \times 7$
E_8	$2^{15} \times 3^2 \times 7$

Proof. By Proposition 5.4, the order of $W(\Gamma)$ is $3 \times |W_s(\Gamma)|$. But (5.1) gives

$$\begin{aligned} |W_s(\Gamma)| &= |\mathrm{O}(q|_{\overline{W}})| \times \left|\mathrm{GL}(\overline{E}')\right| \times \left|\mathrm{Hom}(\overline{E}', \overline{W})\right| \times \left|\mathrm{Hom}(\overline{E}' \oplus \overline{W}, \mathbb{F}_2)\right| \\ &= \begin{cases} 1 \times (7 \times 6 \times 4) \times 2^3 \times 2^4 = 2^{10} \times 3 \times 7, & \text{for } E_7, \\ 2 \times (7 \times 6 \times 4) \times 2^6 \times 2^5 = 2^{15} \times 3 \times 7, & \text{for } E_8, \end{cases} \end{aligned}$$

as required. □

Our next goal is to show that the short exact sequence in Proposition 5.4 splits, and to give a simpler description of the Weyl groups. To do this, let us consider nice realizations of these gradings following ideas from [4] and the references therein.

For E_8, let H be the extended Hamming $[8,4,4]$ binary code, which is the four dimensional subspace of $\mathbb{F}_2^8$ that consists of the following words:

$$\begin{aligned}
&\mathbf{0} = (0,0,0,0,0,0,0,0),\ \mathbf{1} = (1,1,1,1,1,1,1,1),\\
&(1,1,1,1,0,0,0,0),\ (0,0,0,0,1,1,1,1),\\
&(1,1,0,0,1,1,0,0),\ (1,1,0,0,0,0,1,1),\ (0,0,1,1,1,1,0,0),\ (0,0,1,1,0,0,1,1),\\
&(1,0,1,0,1,0,1,0),\ (1,0,1,0,0,1,0,1),\ (0,1,0,1,1,0,1,0),\ (0,1,0,1,0,1,0,1),\\
&(1,0,0,1,1,0,0,1),\ (1,0,0,1,0,1,1,0),\ (0,1,1,0,1,0,0,1),\ (0,1,1,0,0,1,1,0).
\end{aligned}$$

There is an H-grading $\underline{\Gamma}$ of the simple Lie algebra $\mathcal{L}$ of type E_8 with

$$\mathcal{L}_{\mathbf{0}} = \mathfrak{sl}(V_1) \oplus \cdots \oplus \mathfrak{sl}(V_8), \qquad \mathcal{L}_{\mathbf{1}} = 0,$$

where V_i is a two-dimensional vector space for any $i = 1, \dots, 8$, and for any word $\mathbf{c} \neq \mathbf{0}, \mathbf{1}$, we have

$$\mathcal{L}_{\mathbf{c}} = V_{c_1} \otimes V_{c_2} \otimes V_{c_3} \otimes V_{c_4}$$

as a module for $\mathcal{L}_{\mathbf{0}}$, with $1 \leq c_1 < c_2 < c_3 < c_4 \leq 8$ being the slots of $\mathbf{c}$ that contain 1. For example, we have

$$\mathcal{L}_{(1,0,0,1,1,0,0,1)} = V_1 \otimes V_4 \otimes V_5 \otimes V_8.$$

The Lie bracket in $\mathcal{L}$ is shown in [7, Section 3.3].

There is a group homomorphism

$$\Psi\colon \mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_8) \longrightarrow \mathrm{Aut}(\mathcal{L}) \tag{5.6}$$

such that $\Psi\big((f_1, \dots, f_8)\big)$ acts on $\mathfrak{sl}(V_i) \leq \mathcal{L}_{\mathbf{0}}$ by $g \mapsto f_i g f_i^{-1}$, and on the component $V_{\mathbf{c}} = V_{c_1} \otimes V_{c_2} \otimes V_{c_3} \otimes V_{c_4}$ as $f_{c_1} \otimes f_{c_2} \otimes f_{c_3} \otimes f_{c_4}$.

Fix bases on the V_i's and consider the following automorphisms

$$\begin{aligned}
\sigma &= \Psi\left(\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\right),\\
\tau &= \Psi\left(\left(\begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}, \dots, \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}\right)\right),
\end{aligned} \tag{5.7}$$

with $\mathbf{i}^2 = -1$. From the conditions

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^2 = -\mathrm{id} = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}^2, \ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix} = -\begin{pmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

we get

$$\sigma^2 = \tau^2 = \mathrm{id}, \qquad \sigma\tau = \tau\sigma,$$

because we have $\Psi\big((-\mathrm{id}, \dots, -\mathrm{id})\big) = \mathrm{id}$.

Consider the grading by $H \times \mathbb{F}_2^2 \simeq \mathbb{Z}_2^6$:

$$\Gamma : \mathcal{L} = \bigoplus_{(\mathbf{c},i,j)\in H\times\mathbb{F}_2^2} \mathcal{L}_{(\mathbf{c},i,j)} \tag{5.8}$$

obtained by refining the previous H-grading $\underline{\Gamma}$ by means of σ and τ:

$$\begin{aligned}
\mathcal{L}_{(\mathbf{c},0,0)} &= \{x \in \mathcal{L}_{\mathbf{c}} \mid \sigma(x) = x = \tau(x)\},\\
\mathcal{L}_{(\mathbf{c},1,0)} &= \{x \in \mathcal{L}_{\mathbf{c}} \mid \sigma(x) = -x = -\tau(x)\},\\
\mathcal{L}_{(\mathbf{c},0,1)} &= \{x \in \mathcal{L}_{\mathbf{c}} \mid \sigma(x) = x = -\tau(x)\},\\
\mathcal{L}_{(\mathbf{c},1,1)} &= \{x \in \mathcal{L}_{\mathbf{c}} \mid \sigma(x) = -x = \tau(x)\}.
\end{aligned}$$

Lemma 5.6. *Up to equivalence, Γ is the special pure grading on E_8 with universal group $\mathbb{Z}_2^6$.*

Proof. It is enough to note that $\mathfrak{sl}_2(V_i)$ is the direct sum of the three Cartan subalgebras spanned by the linear maps with coordinate matrices in our fixed basis $\left(\begin{smallmatrix} 0 & 1\\ -1 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \mathbf{i} & 0\\ 0 & -\mathbf{i}\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix} 0 & \mathbf{i}\\ \mathbf{i} & 0\end{smallmatrix}\right)$. It follows that $\mathcal{L}_{(\mathbf{0},0,0)}$ is trivial, so that Γ is special, and $\mathcal{L}_{(\mathbf{0},1,0)}$, $\mathcal{L}_{(\mathbf{0},0,1)}$, and $\mathcal{L}_{(\mathbf{0},1,1)}$ are Cartan subalgebras whose sum is $\mathcal{L}_{\mathbf{0}} = \mathfrak{sl}(V_1) \oplus \cdots \oplus \mathfrak{sl}(V_8)$. In particular, Γ is pure.

Moreover, for $\mathbf{c} \neq \mathbf{0}, \mathbf{1}$ in H, $\mathcal{L}_{\mathbf{c}}$ splits into the direct sum of four homogeneous components of Γ, each of dimension 4. Hence, Γ has exactly three homogeneous components that are Cartan subalgebras. □

Now, we are in a position to prove that the short exact sequence in Proposition 5.4 splits for type E_8, and to provide a nice description of the Weyl group.

Theorem 5.7. *Let $\mathcal{L}$ be a simple Lie algebra of type E_8 over an algebraically closed field of characteristic 0. Let $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ be the unique, up to equivalence, special pure grading with universal group G and with exactly three homogeneous components: $\mathcal{L}_s$, $\mathcal{L}_{s'}$ and $\mathcal{L}_{s''}$, being Cartan subalgebras. Let $W(\Gamma)$ be its Weyl group, and let $W'(\Gamma)$ be the subgroup $W'(\Gamma) = \{w \in W(\Gamma) \mid w(s) = s,\ w(s') = s',\ w(s'') = s''\}$. Then the short exact sequence*

$$1 \to W'(\Gamma) \hookrightarrow W(\Gamma) \to S_3 \to 1,$$

in Proposition 5.4 splits.

Moreover, consider the flag

$$\mathcal{F}: \quad 1 \le S \le K \le G,$$

where $S = \langle s, s', s'' \rangle = \langle g \in G \mid \mathcal{L}_g$ *is a Cartan subalgebra*$\rangle$, *and* $K = \langle g \in G \mid \mathcal{L}_g = 0 \rangle$. *Then the Weyl group is the stabilizer of* $\mathcal{F}$*:*

$$W(\Gamma) = \mathrm{Stab}_{\mathrm{Aut}(G)}(\mathcal{F}).$$

Proof. We may identify G with $H \times \mathbb{F}_2^2$ as in (5.8). Then the subgroups S and K equal $\mathbb{F}_2^2$ and $\langle \mathbf{1} \rangle \times \mathbb{F}_2^2$ respectively, because the only elements g in G with $\mathcal{L}_g = 0$ are $(\mathbf{1}, i, j)$, for $i, j \in \mathbb{F}_2^2$.

The elements of the Weyl group clearly preserve the subgroups S and K. Therefore, the Weyl group, considered as a subgroup of $\mathrm{Aut}(H \times \mathbb{F}_2^2)$, consists of transformations that stabilize the flag $\mathcal{F}$ or, in other words, transformations of the form:

$$\left(\begin{array}{c|c} f & 0 \\ \hline \mu & g \end{array} \right), \tag{5.9}$$

with $f \in \mathrm{Stab}_{\mathrm{GL}(H)}(\mathbf{1})$, $g \in \mathrm{GL}_2(2)$, and $\mu \in \mathrm{Hom}(H, \mathbb{F}_2^2)$.

But an easy computation gives:

$$\begin{aligned} \left|\mathrm{Stab}_{\mathrm{GL}(H)}(\mathbf{1})\right| &= 2^3 \times |\mathrm{GL}_3(2)| = 2^3 \times 7 \times 6 \times 4 = 2^6 \times 3 \times 7, \\ \left|\mathrm{Hom}(H, \mathbb{F}_2^2)\right| &= 2^8, \\ |\mathrm{GL}_2(2)| &= 3 \times 2, \end{aligned}$$

so the order of the subgroup in (5.9) is $2^{15} \times 3^2 \times 7$, which coincides with the order of $W(\Gamma)$ by Corollary 5.5. Therefore, $W(\Gamma)$ is the group of all transformations in (5.9).

Moreover, $W'(\Gamma)$ is the subgroup of $W(\Gamma)$ consisting of the transformations that fix $(\mathbf{0}, i, j)$ for any $i, j \in \mathbb{F}_2^2$, and hence of the transformations in (5.9) with $g = \mathrm{id}$, which is complemented by the subgroup of transformations in (5.9) with $f = \mathrm{id}$, $\mu = 0$. This shows that the short exact sequence in Proposition 5.4 splits. □

For E_7 the situation is simpler. Let $\mathcal{L}$ be the simple Lie algebra of type E_7, and now let C be the simplex $[7, 4, 3]$ binary linear code, which is the three dimensional subspace of $\mathbb{F}_2^7$ consisting of the words:

$$\begin{aligned} &\mathbf{0} = (0,0,0,0,0,0,0),\ (1,1,0,0,1,1,0), \\ &(0,1,1,0,0,1,1),\ (1,0,1,0,1,0,1), \\ &(1,1,1,1,0,0,0),\ (0,0,1,1,1,1,0), \\ &(1,0,0,1,0,1,1),\ (0,1,0,1,1,0,1). \end{aligned}$$

There is a grading $\underline{\Gamma}$ by $C \simeq \mathbb{Z}_2^3$ with

$$\mathcal{L}_{\mathbf{0}} = \mathfrak{sl}(V_1) \oplus \cdots \oplus \mathfrak{sl}(V_7),$$

and

$$\mathcal{L}_{\mathbf{c}} = V_{c_1} \otimes V_{c_2} \otimes V_{c_3} \otimes V_{c_4}$$

as for E_8. Again, this is refined to a grading Γ by $C \times \mathbb{F}_2^2 \simeq \mathbb{Z}_2^5$ using the natural group homomorphism $\Psi\colon \mathrm{SL}(V_1) \times \cdots \times \mathrm{SL}(V_7) \longrightarrow \mathrm{Aut}(\mathcal{L})$ as in (5.6) and the automorphisms σ, τ as in (5.7), but with only seven components. The same arguments as for E_8 prove the following result.

Lemma 5.8. *Up to equivalence, Γ is the special pure grading on E_7 with universal group $\mathbb{Z}_2^5$.*

As in the case of E_8, we get that the short exact sequence in Proposition 5.4 splits, and the Weyl group gets a nice description.

Theorem 5.9. *Let $\mathcal{L}$ be a simple Lie algebra of type E_7 over an algebraically closed field of characteristic 0. Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be the unique, up to equivalence, special pure grading with universal group G and with exactly three homogeneous components: $\mathcal{L}_s$, $\mathcal{L}_{s'}$ and $\mathcal{L}_{s''}$, being Cartan subalgebras. Let $W(\Gamma)$ be its Weyl group, and let $W'(\Gamma)$ be the pointwise stabilizer in $W(\Gamma)$ of $\{s, s', s''\}$:*

$$W'(\Gamma) = \{w \in W(\Gamma) \mid w(s) = s,\, w(s') = s',\, w(s'') = s''\}.$$

Then the short exact sequence

$$1 \to W'(\Gamma) \hookrightarrow W(\Gamma) \to S_3 \to 1,$$

in Proposition 5.4 splits.

Moreover, consider the flag

$$\mathcal{F}: \quad 1 \leq S \leq G,$$

where $S = \langle s, s', s'' \rangle = \langle g \in G \mid \mathcal{L}_g \text{ is a Cartan subalgebra} \rangle$. Then the Weyl group is the stabilizer of $\mathcal{F}$:

$$W(\Gamma) = \mathrm{Stab}_{\mathrm{Aut}(G)}(\mathcal{F}).$$

Proof. Identify G with $C \times \mathbb{F}_2^2$. Then the subgroup S equals $\mathbb{F}_2^2$, and is preserved by $W(\Gamma)$, so $W(\Gamma)$ preserves the flag $\mathcal{F}$, and hence consists of transformations of the form

$$\left(\begin{array}{c|c} f & 0 \\ \hline \mu & g \end{array}\right), \tag{5.10}$$

with $f \in \mathrm{GL}(C)$, $g \in \mathrm{GL}_2(2)$, and $\mu \in \mathrm{Hom}(C, \mathbb{F}_2^2)$. Again, easy computations give:

$$\begin{aligned} |\mathrm{GL}(C)| &= |\mathrm{GL}_3(2)| = 7 \times 6 \times 4 = 2^3 \times 3 \times 7, \\ \left|\mathrm{Hom}(C, \mathbb{F}_2^2)\right| &= 2^6, \\ |\mathrm{GL}_2(2)| &= 3 \times 2, \end{aligned}$$

so the order of the subgroup in (5.10) is $2^{10} \times 3^2 \times 7$, which coincides with the order of $W(\Gamma)$ by Corollary 5.5. Therefore, $W(\Gamma)$ is the group of all transformations in (5.10), and the result follows. □

Remark 5.10. By duality, the quadratic form $q_{H,s}$ of Theorem 5.1 corresponds to the regular quadratic form induced by the restriction of $\hat{q}$ of Remark 3.1 to the subgroup $\mathrm{Diag}(\Gamma) \cap \mathcal{T}_2$ of $\mathcal{T}_2$. As a result, the Weyl groups $W(\Gamma)$ for $\mathcal{L}$ of types E_6 and E_8 could be computed using, respectively, Propositions 6.9 and 8.14 in [22].

6. Classification of special pure gradings up to isomorphism

Recall that any special grading $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ on a semisimple Lie algebra over an algebraically closed field of characteristic 0 is almost fine ([11, Proposition 3.8]). If U is the universal group of Γ, and we denote by Δ the grading Γ when considered as a grading over U, then there is a group homomorphism $\alpha \colon U \to G$ such that Γ is the coarsening ${}^{\alpha}\Delta$. This homomorphism α is bijective on the supports.

If now $\Gamma' : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}'_g$ is another special G-grading equivalent to Γ, then there is a group homomorphism $\beta \colon U \to G$, with $\Gamma' = {}^{\beta}\Delta$. Then the gradings Γ and Γ' are isomorphic if and only if there is an element ω in the Weyl group $W(\Delta)$ such that $\alpha = \beta \circ \omega$ ([11, Theorem 4.3]).

The goal of this section is the classification of the special pure gradings on the simple Lie algebras of types E_6, E_7, and E_8, up to isomorphism.

This will be achieved by defining suitable invariants for these gradings (Definition 6.2).

Let $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ be a special pure grading on the simple Lie algebra $\mathcal{L}$, of type E_6, E_7, or E_8, over an algebraically closed field of characteristic 0, but now we do not assume that G is the universal group of the grading. Pick an element s in the support of Γ such that $\mathcal{L}_s$ is a Cartan subalgebra. Denote by T the subgroup of G generated by the support, which is a 2-group, as it is a quotient of the universal group, and pick a character χ of T with $\chi(s) = -1$. Then $T = \ker \chi \times \langle s \rangle$. Write $\overline{T} = T/\langle s \rangle$, which is isomorphic to $\ker \chi$.

As in (3.2), there is a surjective group homomorphism

$$\pi \colon R \to \overline{T}.$$

The *complementary lattice* $E := \ker \pi$ contains $2R$. The grading Γ being special, we have $E \cap \Phi = \emptyset$ as in Proposition 3.2. As in (3.5) we conclude that T is isomorphic to $R/E \times \mathbb{F}_2$.

Write, as usual, $\overline{R} = R/2R$ and $\overline{E} = E/2R$, and consider the induced linear map $\overline{\pi}\colon \overline{R} \to \overline{T}$, and the quadratic form $q\colon \overline{R} \to \mathbb{F}_2$ in (2.2).

Lemma 6.1. *Unless $\mathcal{L}$ is of type E_7 and Γ is equivalent to the special pure grading $\Gamma^8_{E_7}$ (Theorem 4.2), the subgroup T is the universal group of Γ.*

Proof. From $E \cap \Phi = \emptyset$ we conclude, using the arguments previous to Proposition 2.1, that either $\overline{E}$ is a totally isotropic subspace of $\overline{R}$, or $\mathcal{L}$ is of type E_7 and $\overline{E} = \operatorname{rad} b_q$.

In the first case, T is the universal group, as so is $R/E \times \mathbb{F}_2$. In the second case, Γ is equivalent to $\Gamma^8_{E_7}$ in Theorem 4.2, and its universal group is $\overline{R}\times\mathbb{F}_2$, while T is isomorphic to its quotient $\overline{R}/\operatorname{rad}(b_q) \times \mathbb{F}_2$. $\square$

Definition 6.2. Let $\Gamma : \mathcal{L} = \bigoplus_{g\in G} \mathcal{L}_g$ be a special pure grading on the simple Lie algebra $\mathcal{L}$ of type E_6, E_7, or E_8, over an algebraically closed field of characteristic 0. Define the sequence $\operatorname{Inv}(\Gamma)$ as follows:

(i) If Γ is equivalent to $\Gamma^5_{E_8}$ in Theorem 4.1, then

$$\operatorname{Inv}(\Gamma) = (T),$$

with T the subgroup of G generated by the support of Γ.

(ii) If Γ is equivalent to $\Gamma^6_{E_8}$ in Theorem 4.1, then

$$\operatorname{Inv}(\Gamma) = (T, K, S),$$

with $T = \langle \operatorname{Supp}\Gamma\rangle$ as above, $S = \langle t \in T \mid \mathcal{L}_t \text{ is a Cartan subalgebra of } \mathcal{L}\rangle$, and $K = \langle t \in T \mid \mathcal{L}_t = 0\rangle$.

(iii) If Γ is equivalent to $\Gamma^5_{E_7}$ in Theorem 4.2, then

$$\operatorname{Inv}(\Gamma) = (T, S),$$

with $T = \langle \operatorname{Supp}\Gamma\rangle$ and $S = \langle t \in T \mid \mathcal{L}_t \text{ is a Cartan subalgebra of } \mathcal{L}\rangle$.

(iv) If Γ is equivalent to $\Gamma^8_{E_7}$ in Theorem 4.2, but the subgroup $T = \langle \operatorname{Supp}\Gamma\rangle$ is not the universal group (in other words, T is isomorphic to $\mathbb{Z}_2^7$), then

$$\operatorname{Inv}(\Gamma) = (T, S, b_{T,S}),$$

with $T = \langle \operatorname{Supp}\Gamma\rangle$, $S = \langle t \in T \mid \mathcal{L}_t \text{ is a Cartan subalgebra of } \mathcal{L}\rangle$ ($\simeq \mathbb{Z}_2$), $\overline{T} = T/S$, and $b_{T,S}\colon \overline{T}\times\overline{T} \to \mathbb{F}_2$ the nondegenerate alternating bilinear form defined as follows: the polar form b_q of the regular quadratic form $q\colon \overline{R} \to \mathbb{F}_2$ induces a nondegenerate alternating bilinear form $\bar{b}_q\colon \overline{R}/\operatorname{rad}(b_q) \times \overline{R}/\operatorname{rad}(b_q) \to \mathbb{F}_2$, and $b_{T,S}$ is the induced form on $\overline{T}$:

$$b_{T,S}\big(\overline{\pi}(x),\overline{\pi}(y)\big) := \bar{b}_q(x+\operatorname{rad} b_q, y+\operatorname{rad} b_q) = b_q(x,y),$$

for any $x, y \in \overline{R}$.

(v) In the remaining cases, so that Γ is equivalent to either $\Gamma^r_{E_8}$, $r=7,8,9$, in Theorem 4.1, or $\Gamma^r_{E_7}$, $r=6,7$, in Theorem 4.2, or $\Gamma^8_{E_7}$ with $\langle \operatorname{Supp}\Gamma\rangle \simeq \mathbb{Z}_2^8$, or $\Gamma^r_{E_6}$, $r=5,6,7$, in Theorem 4.3, then

$$\operatorname{Inv}(\Gamma) = (T, H, S, q_{H,S}),$$

with $T=\langle \operatorname{Supp}\Gamma\rangle$, $S=\langle t\in T \mid \mathcal{L}_t \text{ is a Cartan subalgebra of } \mathcal{L}\rangle$ ($\simeq \mathbb{Z}_2$), $\overline{T}=T/S$, H the subgroup of T such that $\overline{H}=H/S=\overline{\pi}(\overline{E}^\perp)$, where E is the complementary lattice, and $q_{H,S}\colon \overline{H}\to\mathbb{F}_2$ the regular quadratic form transferred from $q|_{\overline{E}^\perp}$ by means of $\overline{\pi}$: $q_{H,S}\big(\overline{\pi}(x)\big)=q(x)$, for any $x\in\overline{E}^\perp$.

The classification of special pure gradings up to isomorphism is now easy:

Theorem 6.3. *Let $\Gamma : \mathcal{L}=\bigoplus_{g\in G}\mathcal{L}_g$ and $\Gamma' : \mathcal{L}=\bigoplus_{g\in G}\mathcal{L}'_g$ be two special pure gradings on the simple Lie algebra $\mathcal{L}$ of type E_6, E_7, or E_8, over an algebraically closed field of characteristic 0. Then Γ is isomorphic to Γ' if and only if $\operatorname{Inv}(\Gamma)=\operatorname{Inv}(\Gamma')$.*

Proof. It is clear that if Γ and Γ' are isomorphic, then $\operatorname{Inv}(\Gamma)=\operatorname{Inv}(\Gamma')$.

For the converse, let us start with a simple case. Assume that Γ and Γ' are equivalent to $\Gamma^6_{E_8}$ and $\operatorname{Inv}(\Gamma)=\operatorname{Inv}(\Gamma')=(T,K,S)$. As in Theorem 5.7, the universal group of $\Gamma^6_{E_8}$ is $U=H\times\mathbb{F}_2^2$, and let $\alpha,\beta\colon U\to G$ be the corresponding group homomorphisms so that $\Gamma={}^\alpha(\Gamma^6_{E_8})$ and $\Gamma'={}^\beta(\Gamma^6_{E_8})$. Then α and β are isomorphisms when considered as maps $U\to T=\langle\operatorname{Supp}\Gamma\rangle=\langle\operatorname{Supp}\Gamma'\rangle$. Then the composition $\beta^{-1}\circ\alpha\colon U\to U$ makes sense and it is an automorphism that preserves the flag in Theorem 5.7. Therefore, $\beta^{-1}\circ\alpha$ belongs to the Weyl group $W(\Gamma^6_{E_8})$. It follows that $\alpha=\beta\circ\omega$ for an element $\omega\in W(\Gamma^6_{E_8})$, proving that Γ and Γ' are isomorphic.

Consider now the most problematic case, in which Γ and Γ' are equivalent to $\Gamma^8_{E_7}$, with $\operatorname{Inv}(\Gamma)=\operatorname{Inv}(\Gamma')=(T,S,b_{T,S})$ and $T\simeq\mathbb{Z}_2^7$. The universal group of $\Gamma^8_{E_7}$ is $\overline{R}\times\mathbb{F}_2$ and let $\alpha,\beta\colon\overline{R}\times\mathbb{F}_2\to G$ be the corresponding group homomorphisms. These induce surjective group homomorphisms $\overline{\alpha},\overline{\beta}\colon\overline{R}\to\overline{T}=T/S$.

Take a complementary space $\overline{W}$ of $\operatorname{rad}(b_q)$: $\overline{R}=\operatorname{rad}(b_q)\oplus\overline{W}$. Then $\overline{\alpha}$ and $\overline{\beta}$ give isomorphisms $\overline{W}\to\overline{T}$ and for any $w_1,w_2\in\overline{W}$, $b_q(w_1,w_2)=b_{T,S}\big(\overline{\alpha}(w_1),\overline{\alpha}(w_2)\big)=b_{T,S}\big(\overline{\beta}(w_1),\overline{\beta}(w_2)\big)$, so that $\overline{\beta}^{-1}\overline{\alpha}$ lies in the symplectic group of $b_q|_{\overline{W}}$.

But if V is an odd dimensional vector space over $\mathbb{F}_2$ endowed with a regular quadratic form Q, W is a complementary subspace of $\operatorname{rad}(b_Q)$ and $A\in\operatorname{Sp}(W,b_Q|_W)$, then there is a unique element $B\in\mathrm{O}(V,Q)$, such that $B(w)-A(w)\in\operatorname{rad}(b_Q)$ for any $w\in W$. (Actually, we have $\operatorname{Sp}(W,b_Q|_W)\simeq\mathrm{O}(V,Q)$.) Indeed, it is enough to define, with $\operatorname{rad}(b_Q)=\mathbb{F}_2a$, $B(a)=a$ and $B(w)=A(w)+\lambda(w)a$, for a linear form λ. Then B is an isometry if and only if $Q(w)=Q\big(B(w)\big)=Q\big(A(w)+\lambda(w)a\big)=Q\big(A(w)\big)+\lambda(w)^2=Q\big(A(w)\big)+\lambda(w)$

for any $w \in W$, if and only if $\lambda(w) = Q\big(A(w)\big) + q(w)$ for any $w \in W$ (this is linear because $A \in \mathrm{Sp}(W, b_Q|_W)$!).

Hence there is $\varphi \in \mathrm{O}(\overline{R}, q)$ such that $\varphi(x) + \overline{\beta}^{-1}\overline{\alpha}(x) \in \mathrm{rad}(b_q)$, and then $\alpha = \beta \circ \omega$ for an element $\omega \in W(\Gamma^8_{E_7})$ of the form

$$\left(\begin{array}{c|c} \varphi & 0 \\ \hline \epsilon & 1 \end{array}\right)$$

as in (5.5), and thus Γ and Γ' are isomorphic.

The remaining cases are easy because, as for $\Gamma^6_{E_8}$, the homomorphisms α, β are bijections from the universal group of the almost fine grading to the invariant T of Γ and Γ'. $\square$

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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