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STRUCTURE AND CHARACTERIZATIONS OF ALGEBRAS ASSOCIATED WITH GRAPHS

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Por la presente, yo, Cándido Martín González, Catedrático del Departamento de Álgebra, Geometría y Topología de la Universidad de Málaga, hago constar que he dirigido la Tesis Doctoral titulada “Structure and Characterisations of algebras associated with graphs” realizada por el doctorando Iván Ruiz Campos. La cual ha dado lugar a varios artículos en régimen de coautoría y que no han sido utilizados para ninguna otra tesis doctoral.

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Atentamente,

Cándido Martín González

A mi sobrina Paula

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INTRODUCTION

Algebras associated with graphs such as path, Leavitt path and Steinberg algebras, arise from different contexts independently, although they are related. For instance, Leavitt path algebras are quotient of path algebras and they also are a Steinberg algebra of a suitable groupoid. In addition, the three of them have in common that the link between the graph and the algebra is strong enough to obtain algebraic properties from geometric conditions in the graph and vice versa.

If we consider an algebra A, then it is possible to view its A-modules as a family of vector spaces connected by linear maps. These graphical representations took place in the late forties with A. Grothendieck [59] and P. Gabriel [48]. They became more popular in the seventies with the results of Gabriel [49, 50] where the notions of directed graph (or quiver) and their linear representations were formalized. A representation of a graph is a collection of vector spaces, one per each vertex, and a collection of linear maps, one per each edge. We will say that the graph is of finite type if it has a finite number of isomorphism classes of indecomposable representations. In this context, quiver (or path) algebras are important in representation theory because any algebra over an algebraically closed field is morita equivalent to a quotient of a quiver algebra. One of the most famous results is called Gabriel's Theorem [12, Theorem 5.10] which says that a quiver algebra has only finitely many indecomposable modules up to isomorphism if and only if the quiver is a disjoint union of Dynkin diagrams of types A, D and E. As a consequence of this result, the representation theory of graphs became more popular and led to several papers and books [12, 36, 68, 81, 94].

Although the popularity of the representations of graphs has been increasing during the years, there are not many references that deepen in the path algebra by itself. There are some notions in the mentioned citations, but more as a tool rather than a goal. In notes by W. Crawley-Boevery [32] and P. Smith [97] there are a few results about perfection and finiteness conditions on this algebra. However, they do not enter in full detail and even Crawley-Boevery defines them as "rather testing exercises".

On the other hand, we can set the origin of Leavitt path algebras in the late fifties and the early sixties with the investigations of W. G. Leavitt about rings that fail the Invariant Basis Number property [78, 79]. That is, Leavitt gave examples of a ring R such that $R^n \cong R^m$ with $n \neq m$. One of these examples is the Leavitt algebra $L_K(1,n)$ and more generally, the class $L_K(n,m)$. If $K = \mathbb{C}$, then $L_{\mathbb{C}}(1,n)$ is the unique unital nontrivial complex algebra generated by the elements $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ such that $\sum_{i=1}^n x_i y_i = 1$ and $y_i x_j = \delta_{i,j} 1$ for all $i, j \leqslant n$. Independent of Leavitt's work, Cuntz on the search for an explicit description of a separable simple infinite C^*-algebra developed the following Theorem [33].

Theorem (Cuntz's Theorem)**.** *Let $n \in \mathbb{N}$. Consider a Hilbert space $\mathcal{H}$, and a set $\{S_i\}_{i=1}^n$ of isometries (i.e., $S_i^* S_i = 1$) on $\mathcal{H}$. Assume that $\sum_{i=1}^n S_i^* S_i = 1$. Let $\mathcal{O}_n$ denote $C^*(S_1, S_2, \ldots, S_n)$ the C^*-algebra generated by $\{S_i\}_{i=1}^n$. Then the infinite separable C^*-algebra $\mathcal{O}_n$ is simple.*

After Cuntz's work, Cuntz and Krieger generalized the algebra $\mathcal{O}_n$ in [34] and nearly twenty years later, Kumjian, Pask and Raeburn developed a class of C^*-algebras from a graph [74]. Finally, in the early 2000's, Ara, Moreno and Pardo on one side [8]; and Abrams and Aranda Pino on the other [2], introduced the Leavitt path algebras. The first ones were inspired by the works of Cuntz and Krieger whereas the second ones were inspired by the results of Leavitt and their connection with the C^*-graph algebras. From that moment, many authors have been interested in this family of algebras. This has led to a plethora of works as well as a monograph that have helped to consolidate a solid research line in research years [1].

Leaving aside path and Leavitt path algebras, if we think of Steinberg algebras, we can set a similar origin to the second ones. Both of them arise from the study of C^*-algebras. In particular, Steinberg algebras first appeared in [100] and [46]. These $*$-algebras are built from a groupoid. A groupoid is just a small category where every arrow is an isomorphism. If we consider a Hausdorff topology in a groupoid $\mathcal{G}$ and ring R, we can take the family of locally constant, compactly supported functions $f\colon \mathcal{G} \to R$ which is a R-algebra together with the convolution product. This is named as the Steinberg algebra. If we consider a directed graph E and its corresponding graph groupoid, introduced in [74], then its Steinberg algebra is the Leavitt path algebra of the graph. In this way, Steinberg algebras generalize these last ones. Not only that, but it also includes: group algebras, semigroup algebras and Kumjian-Pask algebras. In this way, Steinberg algebras have been a strong tool to use [42, 85, 84, 101, 102, 104]

These three associative algebras, even though they have different origins, they are narrowly related. They are constructed from a combinatorial structure, a directed graph. In this way, algebraic properties in the algebra and geometric properties in the graph are linked. As we have already mentioned, Steinberg algebras generalize Leavitt path algebras. It is just enough to consider the Steinberg algebra of a path groupoid. On the other hand, if we consider the path algebra of the extended graph, then the Leavitt path algebra is a quotient of it. Not only that, but the path algebra is a subalgebra of the Leavitt path one.

Even though we will study these algebras independently, we want to obtain a common tool for them. This is what we will denote as algebraic entropy. The entropy is a physical magnitude. It is original from thermodynamics and it has been extended to fluids dynamics and statistical mechanics. In the early fifties, Shannon introduced the entropy in information theory. It became very popular since then and it gave place to many papers in information theory. During the late fifties entropy theory evolved again. Sinai constructed the concept of entropy in dynamical systems [96]. Kolmogorov, on the other hand, works with entropy in a similar way using Lebesgue spaces [72].

It was not until 1965 when Adler, Konheim and McAndrew established the topological entropy [4] and it became more popular among mathematicians. Since then, the production of articles has been fruitful in the different areas. In C^*-dynamics we can find several papers by Kerr and Voicolescu [70, 71, 105] as well as some others related to graphs and Exel-Laca C^*-algebras [66, 67]. More recently we can find [56] by Gonçalves, Royer and Tasca about entropy in shift spaces. The concept of entropy influenced the group theory too. We can find papers like [25] by Bowen and [63] by Hood. Not long ago we can find the works of Giordanno Bruno and Dikranjan [37, 38, 39, 40, 39].

In 1966, one year after the introduction of the topological entropy, Gelfand and Kirillov added a new dimension function in [51] which later would be called the Gelfand-Kirillov dimension. This new algebraic measure was based in the way that algebras grow.

Definition 0.1. [51] Let A be an algebra, which is generated by a finite-dimensional subspace V. Then, $g_V(n) := \dim V^n < \infty$ is the *growth function* of V. If $g_V(n)$ is polynomially bounded, then the *Gelfand-Kirillov dimension* of A is defined as

$$\text{GKdim}(A) := \limsup_{n\to\infty} \frac{\log g_V(n)}{\log(n)}.$$

This new dimension makes sense for algebras with polynomial growth. Naturally, there can be a measure for algebras with exponential growth. Inspired by this, in [87] M. F. Newman and C. Schneider construct the entropy for graded algebras. This last construction will be the inspiration for our notion of entropy in filtered algebras.

As a recap, we encourage the reader to keep in mind the following. During the course of this manuscript we will work mainly with path and Steinberg algebras, Leavitt path algebras will take a second place. We will focus on their structure and their algebraic properties, taking inspiration from other results in Leavitt path algebras. After this, we will develop the theory of algebraic entropy for filtered algebras and we will apply it to path and Leavitt path algebras. Finally, we will try to extend the independence of the base field property that these algebras present. In order to do so, we will make use of category theory and Hilbert's Nullstellensatz for a possibly infinite set of variables.

In the Preliminaries section (part i) we describe the basic concepts that we will need in future sections. In particular, in Chapter 1 we describe the path algebras. In order to do so, we study in Section 1.1 the definition of directed graphs which is, roughly speaking, a set of vertices and arrows. In addition, in Subsection 1.1.1, we briefly extend the concept of graph to k-graph. In Section 1.2, we focus in one of the topological properties of the graph, its connectivity. To be specific, we define connected and strongly connected. Consequently, they lead to the connected and strongly connected components of the graph. After that, in Section 1.3 we define the path and Leavitt path algebra of a directed graph being in Section 1.4 where we study some properties of the path algebra: ideal generated by a set of paths, collapse of a graph, etc. In addition, we see that the path algebra has local units and also a complete system of orthogonal idempotents, allowing the Peirce decomposition as a tool for the description of the algebra. In particular, we will focus on the path algebra of a cycle of n vertices, KC_n.

In Chapter 2, we focus our attention in the Steinberg algebras. To do so we first give the definition of groupoid in 2.1 and see some examples. Some groupoids have a topological structure compatible with the operations in the groupoid, we define then in Section 2.2 topological, étale and ample groupoid. Once that we have an ample groupoid, we can construct, as we do in Section 2.3, the Steinberg algebra.

Part ii is focused on studying the structure path algebras and Steinberg algebras. Although, both of them are interesting, we are going to pay more attention to the first ones. This part goes from Chapter 3 to Chapter 7. Through this text we explore finiteness and perfection conditions of path algebras together with the centroid, extended centroid and central closure of KC_n. In addition, we deepen the socle theory of the Steinberg algebras, taking inspiration from the papers [10, 11, 27].

In Chapter 3, we describe Artinianity, semiartinianity and left (resp. right) socle of a path algebra in terms of the geometric properties of the graph. In Section 3.1, we centre in describing the Artinianity and the left (resp. right) socle of the path algebra. Our study leads to Proposition 3.2 that states that a path algebra is Artinian if and only if the graph is finite and acyclic. Moreover, we characterize the minimal left (resp. right) ideals in Proposition 3.4 and consequently, we obtain a full description of the left (resp. right) socle of a path algebra in Corollary 3.7. We also include Section 3.2 for self-containment. It is completely based on the book by James Dugundji [41]. There, we just select the basic definitions and results about well-ordered sets and ordinals needed in Section 3.3. One of the key results is Theorem 3.20 that states all the properties of the class of ordinals. In the next Section, we define (Definition 3.21) the socle chain and, in general, the left (resp. right) socle of order α. That is why we need all the previous ordinal theory. In addition, we will say that an algebra is left (resp. right) Loewy if the left (resp. right) socle chain is stationary in the whole algebra. Since the left (resp. right) socle is related with the sources (resp. sinks), analogously we state in Definition 3.22 the concept of α-source (resp. α-sink) or source (resp. sink) of order α. These elements will be the ones that generate the socles of higher order. After that, we give in Definition 3.26 the notion of semiartinian algebra which is an algebra such that for every ideal I different from the whole algebra, the socle of the quotient of the algebra by the ideal has socle different from zero. As a tool to characterize semiartinian path algebras we prove Proposition 3.25 that says that if an ideal I verifies $\mathrm{soc}_l(KE/I) = \{0\}$, then I contains all the possible α-sources. Consequently, we declare Theorem 3.27 and prove that a path algebra KE is semiartinian if and only if all the vertices in E are α-sources.

In Chapter 4, we learn about the socle of the Steinberg algebra. In particular, we put our eyes on a type of Steinberg algebra that generalises the Leavitt path and Kumjian-Pask algebras. In Section 4.1, we study the minimal left ideals (the right ones are analogous) in our algebras. If we have a groupoid $\mathcal{G}$ such that there is an open compact bisection $\{x\} \in \mathcal{B}^{co}(\mathcal{G}^{(0)})$, with $x\mathcal{G}x$ a finite group, then it is possible to build a minimal left ideal as in Theorem 4.3. As we can observe, with less detail, the existence of open compact bisections with one element leads to minimal ideals. However, this condition is not only enough, it is necessary. In Proposition 4.6 we prove that the existence of minimal left ideals implies that there exits this type of open compact bisections. In Section 4.2, we finally describe the socle and the homogeneous components of the Steinberg algebras. All of this is compiled in Theorem 4.12. There, we prove that the socle is generated by the functions $1_{\{x\}}$ with $\{x\} \in \mathcal{B}^{co}(\mathcal{G}^{(0)})$.

Chapter 5 centres its attention in the perfection conditions of a path algebra. In particular, we study simple, prime, semiprime and primitive path algebras. These conditions are related to the type of connection that the graph presents. In Section 5.1, we characterize semiprime and prime path algebras. In [95] Mercedes Siles Molina characterises semiprime path algebras using Leavitt path algebras. In addition, in [32, 97] there are some results about prime path algebras by Crawley-Boevery and P. Smith. However, we include the proof of these results in Propositions 5.3 and 5.4 because we want to make use of the path algebras and add as much details as we can. For example, we verify that a path algebra KE is prime if the graph E is strongly connected. Next, in Section 5.2 we give details of the simple and primitive path algebras. The simple path algebras come from graphs formed by an isolated vertex. On the other hand, a path algebra KE is primitive if and only if E is strongly connected and every cycle has an exit (Theorem 5.7). In particular, a path algebra KE is primitive if and only if the graph E is strongly connected and is not a cycle. In Section 5.3 we focus on the structure of semiprimes path algebras. As we describe in Theorem 5.11 we see that the directed graph of semiprime path algebras consists of isolated vertices, isolated cycles and strongly connected components.

After this, we study, in Chapter 6, the centroid of a path algebra. This structure is non-trivial only when the graph is a cycle. This is the reason why we also study the extended centroid and central closure of KC_n. In Section 6.1, we characterize the centroid of a path algebra. This is a result that appears as an exercise in [32]. However, it does not contain any proof and we consider that this is not a trivial result. That is why we decided to include it. In Definition 6.1 we see that the centroid is the set of all the bimodule automorphisms of the algebra. This structure extends the center of the algebra. Since the centroid of a direct sum of algebra is the direct sum of the centroids, in the path algebra context we can consider that the graphs that we are taking are connected. That being said, in Theorem 6.13 we see that if $T \in \mathcal{C}(KE)$ (centroid) such that there is a vertex $u \in E^0$ with $T(u) = ku$ with $k \in K$, then $T = k\,\mathrm{Id}_{KE}$. This result, together with Lemmas 6.14 and 6.15, allow us to declare Theorem 6.16. In this sense, the centroid of a path algebra of a connected graph is either the ground field or the algebra of polynomials in one variable. In Section 6.2 we focus on the extended centroid and the central closure of KC_n. Since KC_n is a PI-algebra, then the extended centroid is the field of fractions of the centroid, in this case, $K(x)$. However, for the central closure we need the description given by Baxter and

Martindale [13]. The extended centroid is the set of all equivalence classes of admissible pairs. This is described with more detail in Definition 6.20. In order to prove the desired isomorphism, we make use of Lemma 6.21. After that, we construct in Proposition 6.22 the isomorphism which make easier to compute the central closure of KC_n as we do in Theorem 6.27.

Chapter 7 is dedicated to the study of left (resp. right) Noetherian and J-semisimple path algebras, that is, those with trivial Jacobson radical. In order to do so, we also characterise the radical. Both descriptions are "rather testing exercises" as we quote from Crawley-Boevery in [32]. However, due its complexity and the lack of a proof of both results in the literature, we decided to include it in this work. The main goal of Section 7.1 is describing left (resp. right) Noetherian path algebras. This are related with the cycles without entries (resp. exits). First, in Proposition 7.3 we study the case that the graph only has a cycle without entries (resp. exits). After that, we prove by induction in the number of cycles that a path algebra KE is left Noetherian if and only if E is a finite graph and every cycle does not have an entry (resp. exit) (Proposition 7.4). Using the Peirce decomposition we see in Proposition 7.5 that the Noetherian path algebras are formal upper triangular matrix rings. In Section 7.2 we check out the Jacobson radical of path algebras. In particular, in Proposition 7.10 we see that it is generated as a vector space by the path without return. Furthermore, in Theorem 7.11 we prove that the radical is the ideal generated by the edges without return. The main goal in Section 7.3 is studying the structure of J-semisimple path algebras, that is, with trivial radical. These path algebras are semiprime too, and we already know their structure from Chapter 5. In addition, we can consider that the algebra is left (resp. right) Noetherian. Consequently, we have Theorem 7.18 where we make use of a result from [65] and describe the whole structure.

Part iii is dedicated to the study of algebraic entropy. Most of the results that appear here are in the papers [21, 22]. In Chapter 8 we focus on the study of the algebraic entropy of filtered algebras. We apply this concept to the path and Leavitt path algebras. For this purpose, we take inspiration from the Gelfand-Kirillov dimension and the entropy of graded algebras. In Section 8.1 we construct the entropy for a filtered algebra. First, we define the notion of filtration in Definition 8.2 and later, define the algebraic entropy in 8.5. This new magnitude is conditioned by the chosen filtration. This fact manifests in Proposition 8.7. In Section 8.2 we focus on how the entropy behaves with the direct sum of algebras, monomorphisms and epimorphisms. In Proposition 8.10 we see how the entropy contracts in the presence of an epimorphism while in Proposition 8.11 we prove that the entropy expands with a monomorphism. Lastly, in Proposition 8.13 we check that the entropy of direct sums of algebras is the maximum of the entropies of each addend. Section 8.3 we centre our attention in the entropy applied to path and Leavitt path algebras of finite graphs. If we want to make use of our definition of entropy, we need to construct filtrations for each algebra. In Definition 8.14 we construct what we call the standard filtrations. By using Propositions 8.11 and 8.10 we prove Proposition 8.20 where we see that for a finite graph E, the entropy of $L_K(E)$ is lower bounded by the entropy of KE and upper bounded the one of $K\hat{E}$. Furthermore, by applying Proposition 8.13 we see that the entropy of a path and a Leavitt path algebra is the maximum of the entropies of its connected components (Proposition 8.22). We end this Section with Proposition 8.24 where we see once more the dependency of the entropy with chosen filtration. In this case, for a path and Leavitt path algebra we can obtain a value of entropy as small as we want. After this, we centre our attention in Section 8.4 in the computation of the algebraic entropy of path algebras of finite graphs. For this reason, we see in Lemma 8.28 that it is possible to obtain this value by using a norm of the adjacency matrix and its powers. Furthermore, in Proposition 8.30 we prove that eventually, the entropy is the logarithm of the spectral radius of the adjacency matrix of the graph. From this result, we induce Proposition 8.33 where the entropy of a path algebra is dictated by the strongly connected components of the graph. In Section 8.4.1 we experiment with the entropy of some Leavitt path algebras. Once again, the adjacency matrix of the graph is very useful, together with the software MATHEMATICA. In this way, we check numerically in Examples 8.37, 8.39, 8.40 and 8.41 that the algebraic entropy of the path algebra and the Leavitt path algebra coincide. The question that we ask ourselves is if this equality always holds. We explore this in Section 9.1, where we prove the truth of this assertion. For that, we use Lemma 8.46 where we study the entropy of the algebra $C_K(E)$ in terms of the adjacency matrix of the graph. To end this Section, taking into account some results of Banach algebras we get to Theorem 8.48 where we prove the equality of the entropy. Finally, we finish the chapter with Section 8.6 where we explore the relations of the algebraic entropy of path algebras and Part ii. There, we see in Proposition 8.49 how the entropy of Artinian or Noetherian path algebras is zero. Furthermore, the entropy allows us to distinguish primitive path algebras from simple algebras (Proposition 8.51). In addition, there are some ideals like the socle or the Jacobson radical that are superfluous for the computation on the entropy (Proposition 8.53 and Corollary 8.55). We finish this part with Chapter 9 where we explore a generalisation of the entropy of filtered algebras. In Section 9.1 we construct the direct limit of filtered algebras and prove that in general, the limit of finite filtered algebras is not a finite filtered algebra. In Section 9.2 we set the base of this entropy through Definitions 9.11, 9.12 and 9.13 where we establish the notions of strict monomorphism, strict system and entropy for a limit of a direct strict system. Once again, the entropy strongly depends on the chosen system of filtered algebras. We prove this in Propositions 9.15. Finally, we end this chapter with Proposition 9.16. There we prove that the new concept of entropy extends the notion of entropy in agreement with the one given in Chapter 8. In other words, this new method also allows the computation of the algebraic entropy of a finite filtered algebra as the entropy of a strict direct system of finite filtered algebras and this value agrees with the one obtained in Definition 8.5.

Part iv, in particular, Chapter 10 corresponds to the paper [52]. There, we study some properties of algebras associated with graphs in a categorical way. To give an example of the kind of result that we pursue, as a byproduct of the research herein we find for instance that if G_1, G_2 are finite groups, the group algebras $\mathbb{C}G_1$ and $\mathbb{C}G_2$ are isomorphic if and only if $KG_1 \cong KG_2$ for any algebraically closed field of characteristic 0. And we note that we do not need to consider only the group algebra: path algebras, Leavitt, Steinberg algebras can also be used. In Section 10.1 we first introduce the concept of algebra functor and, in Definition 10.3, we establish what is the K-algebra functor $\underline{A}$ for a certain K-algebra A. On the one hand, in Definition 10.5, we introduce the extension invariant functors, which will generalize the "nice" condition that the Leavitt path algebras satisfy. In fact, we will see that the extension invariant functors are precisely those of the form $\underline{A}$ for a certain K-algebra A. On the other hand, in Corollary 10.8 we remark that, for extension invariant functors $\mathcal{F}, \mathcal{G}$, once we have an isomorphism $\mathcal{F}(H) \cong \mathcal{G}(H)$ for some Hopf K-algebra H, this induces an isomorphism $\mathcal{F}(K) \cong \mathcal{G}(K)$. Hence $\mathcal{F}(S) \cong \mathcal{G}(S)$ for every commutative and unital K-algebra S. Besides, we will illustrate, in Example 10.10, an algebra functor which is not extension invariant thanks to derivations of an algebra. In Subsection 10.1.1, we include some well-known interesting examples of algebras which provide extension invariant algebra functors. It is worth to mention that they are the main motivation for the results presented in this work. As a first corollary we get that for any two graphs E_1, E_2 and any Hopf K-algebra

H, then $L_H(E_1) \cong L_H(E_2)$ if and only if $L_K(E_1) \cong L_K(E_2)$ (see Corollary 10.15). Next, in Section 10.2, we will present the key to develop our machinery for the isomorphism theorems. According to [77], the Hilbert's Nullstellensatz Theorem can be given in terms of infinite indeterminates. To achieve the hypotheses for Theorem 10.17, it is necessary that either the set of indeterminates is finite (in fact, the original Hilbert's Nullstellensatz Theorem) or that the transcendence degree of the ground field over its prime one has cardinality strictly higher than the number of variables. To unify those hypotheses in a single condition, we introduce Lemmas 10.18 and 10.20. This new environment consists of that the cardinal of the ground field is strictly higher that the cardinal of the set of indeterminates. The original paper [77] by S. Lang informs that I. Kaplansky obtained independently the result, but the equivalent formulation of Kaplansky was slightly different and did not use transcendence degree. Since Kaplansky never published his work, this encourages us to include the proofs of Lemmas 10.18 and 10.20. The remaining two sections are devoted, one for the finite-dimensional issue (Section 10.3) and the other for the infinite one (Section 10.4). The first case gives us a better understanding of the general one. The idea in both situations is analogous: we need to translate respectively the isomorphism problem into the existence of zeros of a certain ideal of polynomials. This corresponds to Proposition 10.23 and Corollary 10.24, for the finite-dimensional case, and Proposition 10.36, for the general one. For the finite dimension, as a consequence of Theorem 10.17 and Corollary 10.24, we prove Theorem 10.25 where for a field extension $K \subset F$ with K algebraically closed, the isomorphism between $A \otimes_K F$ and $B \otimes_K F$ (as F-algebras) is equivalent to the isomorphism between A and B (as K-algebras). From this main assertion, varying the hypotheses on the field extensions, it will be derived a series of Corollaries 10.27, 10.28, 10.29. We complete the finite-dimensional case by establishing the previous assertions in terms of extension invariant algebra functors giving a more global vision of the problem; these are Theorems 10.30 and 10.31. In particular, in order to highlight our motivation, we assert Corollary 10.33 and Corollary 10.34 in terms of relative Cohn path algebras. The same idea is pursued in the infinite-dimensional section by Theorem 10.37. However, we need the additional hypothesis that the cardinal of the ground field K is strictly higher than the dimension of the algebras. This hypothesis is essential in order to apply the Hilbert's Nullstellensatz Theorem for infinitely many variables. Along the lines of the finite-dimensional case, Corollaries 10.38, 10.39 and 10.40 are followed jointly with a functorial interpretation for those extension invariant ones (Theorem 10.41 and Theorem 10.42). Due to the fact that Leavitt path algebras become our major motivation, we include results that could be stated in the general setting of extension invariant functors, but that we expose in our particular favourite ambient. So, for instance, for countable graphs E_1 and E_2, an uncountable algebraically closed field K and $K \subseteq F_1, F_2$, then $L_{F_1}(E_1) \cong L_{F_1}(E_2)$ if and only if $L_{F_2}(E_1) \cong L_{F_2}(E_2)$ (Corollary 10.43).

RESUMEN

Las álgebras asociadas a grafos, como las álgebras de caminos, caminos de Leavitt y Steinberg, surgen de contextos diferentes y de forma independiente aunque están relacionadas. El álgebra de caminos de Leavitt es un cociente de un álgebra de caminos y además es un álgebra de Steinberg para un grupoide adecuado. Asimismo, el vínculo entre el grafo y el álgebra es lo suficientemente fuerte como para obtener ciertas propiedades algebraicas a partir de condiciones geométricas y viceversa.

Si consideramos un álgebra A, entonces es posible ver los A-módulos como una familia de espacios vectoriales conectados por aplicaciones lineales. Estas representaciones gráficas tuvieron lugar a finales de los años cuarenta gracias a A. Grothendieck [59] y P. Gabriel [48]. Se hizo más popular en los años setenta con los resultados de Gabriel [49, 50] donde se formalizaron las nociones de grafo dirigido (o quiver) y sus representaciones lineales. Una representación de un grafo es una colección de espacios vectoriales, uno por cada vértice, y una colección de aplicaciones lineales, uno por cada arista. Diremos que el grafo es de tipo finito si tiene un número finito de clases de isomorfía de representaciones

indescomponibles. Uno de los resultados conocidos es el llamado Teorema de Gabriel [12, Theorem 5.10].

Como consecuencia de estos trabajos, la Teoría de Representaciones de grafos se extendió convirtiéndose en un tópico relevante y dando lugar a varios artículos y libros etc[12, 36, 68, 81, 94].

Aunque el uso de la representación de grafos ha ido aumentando a lo largo de los años, no hay muchas referencias que profundicen en el álgebra de caminos per se. Hay algunas nociones en las citas antes mencionadas, pero más como herramienta que como objetivo. En las notas de W. Crawley-Boevery [32] y P. Smith [97] hay algunos resultados sobre condiciones de perfección y finitud en esta álgebra. Sin embargo, los autores no lo abordan con todo detalle e incluso Crawley-Boevery los define como "rather testing exercises".

Por otro lado, podemos situar el origen de las álgebras de caminos de Leavitt a finales de los cincuenta y principios de los sesenta con las investigaciones de W. G. Leavitt sobre anillos que no verifican la propiedad de mantener invariante el número de elementos de la base (IBN) [78, 79]. Es decir, Leavitt dio ejemplos de un anillo R tal que $R^n \cong R^m$ con $n \neq m$. Uno de estos ejemplos es el álgebra de Leavitt $L_K(1, n)$ y con mayor generalidad, la clase $L_K(n, m)$. Si $K = \mathbb{C}$, entonces $L_{\mathbb{C}}(1, n)$ es la única álgebra compleja unitaria no trivial generada por los elementos $x_1, \ldots, x_n$ y $y_1, \ldots, y_n$ tales que $\sum_{i=1}^{n} x_i y_i = 1$ y $y_i x_j = \delta_{i,j} 1$ para todo $i, j \leqslant n$. Independientemente del trabajo de Leavitt, Cuntz en la búsqueda de una descripción explícita de una C^*-álgebra simple infinita separable, obtuvo el siguiente Teorema [33].

Teorema (Teorema de Cuntz). Sea $n \in \mathbb{N}$. Consideremos un espacio de Hilbert $\mathcal{H}$, y un conjunto $\{S_i\}_{i=1}^{n}$ de isometrías (es decir, $S_i^* S_i = 1$) en $\mathcal{H}$. Supongamos que $\sum_{i=1}^{n} S_i^* S_i = 1$. Denotemos por $\mathcal{O}_n$ a $C^*(S_1, S_2, \ldots, S_n)$ la C^*-álgebra generada por $\{S_i\}_{i=1}^{n}$. Entonces la C^*-álgebra infinita separable $\mathcal{O}_n$ es simple.

Después del trabajo de Cuntz, él y Krieger generalizaron el álgebra $\mathcal{O}_n$ en [34] y casi veinte años después Kumjian, Pask y Raeburn desarrollaron una clase de C^*-álgebras a partir de un grafo [74]. Finalmente, a principios de los 2000, Ara, Moreno y Pardo por un lado [8]; y Abrams y Aranda Pino por otro [2], introdujeron las álgebras de caminos de Leavitt. Las primeras se inspiraron en los trabajos de Cuntz y Krieger mientras que los segundos se inspiraron en los resultados de Leavitt y su conexión con las C^*-álgebras de grafos. A partir de ese momento, muchos autores se han interesado por esta familia de álgebras. Esto ha dado lugar a un gran número de trabajos al igual que a una monografía que ha ayudado a consolidar una sólida línea de investigación en los años recientes [1].

Dejando a un lado las álgebras de caminos y de caminos de Leavitt, si pensamos en las álgebras de Steinberg, podemos establecer un origen similar a las segundas. Ambas surgen de las C^*-álgebras. En concreto, las álgebras de Steinberg aparecieron por primera vez en [100] y [46]. Estas C^*-álgebras se construyen a partir de un groupoide que es una categoría pequeña en la que cada flecha es un isomorfismo. Si consideramos un anillo R y una topología de Hausdorff en un grupoide $\mathcal{G}$, podemos tomar la familia de funciones $f\colon \mathcal{G} \to R$ localmente constantes y de soporte compacto. Esta es una R-álgebra con el producto de convolución, llamada álgebra de Steinberg. En el caso de que consideremos un grafo dirigido E y su correspondiente grupoide, introducido en [74], su álgebra de Steinberg es el álgebra de caminos de Leavitt del grafo. De este modo, las álgebras de Steinberg generalizan a estas últimas. No sólo eso, sino que también incluyen: álgebras de grupos, álgebras de semigrupos y álgebras de Kumjian-Pask. De esta manera, las álgebras de Steinberg han sido una fuerte herramienta a utilizar [42, 85, 84, 101, 102, 104].

Como ya hemos mencionado, estas tres álgebras asociativas, aunque tienen orígenes diferentes, están estrechamente relacionadas. Se construyen a partir de una estructura combinatoria, un grafo dirigido. De esta manera, las propiedades algebraicas en el álgebra y las propiedades geométricas en el grafo están enlazadas. Las álgebras de Steinberg generalizan las álgebras de caminos de Leavitt. Basta con considerar el álgebra de Steinberg del groupoide del grafo. Por otra parte, si consideramos el álgebra de caminos del grafo extendido, entonces el álgebra de caminos de Leavitt es un cociente de ella. No sólo eso, sino que el álgebra de caminos es una subálgebra de la de caminos de Leavitt.

Aunque estudiaremos estas álgebras de forma independiente, queremos obtener una herramienta común para ellas. Esto es lo que denotaremos como entropía algebraica. La entropía es una magnitud física. Es original de la termodinámica y se ha extendido a la dinámica de fluidos y a la mecánica estadística. A principios de los años cincuenta, Shannon introdujo la entropía en la Teoría de la Información. Desde entonces se hizo muy popular y dio lugar a muchos trabajos. A finales de los años cincuenta, la entropía volvió a evolucionar. Sinai construyó el concepto de entropía en sistemas dinámicos [96]. Kolmogorov, por su parte, trabajaba con la entropía de forma similar utilizando espacios de Lebesgue [72].

No fue hasta 1965 cuando Adler, Konheim y McAndrew establecieron la entropía topológica [4] y se hizo más popular entre los matemáticos. Desde entonces, la producción de artículos ha sido fructífera en las distintas áreas. En C^*-dinámica podemos encontrar varios artículos de Kerr y Voicolescu [70, 71, 105] así como algunos otros relacionados con grafos y Exel-Laca C^*-algebras [66, 67]. Recientemente podemos encontrar [56] de Gonçalves, Royer y Tasca sobre entropía en espacios de desplazamiento. El concepto de entropía también influyó en la teoría de grupos. Destacamos [25] de Bowen y [63] de Hood; además de los trabajos de Giordanno Bruno y Dikranjan [37, 38, 40, 39].

En 1966, un año después de la introducción de la entropía topológica, Gelfand y Kirillov añadieron una nueva dimensión en [51] que más tarde se llamaría dimensión de Gelfand-Kirillov. Esta nueva medida algebraica se basaba en la forma en que crecen las álgebras.

Definicion 0.1. [51] Sea A un álgebra generada por un subespacio de dimensión finita V. Entonces, $g_V(n) := \dim V^n < \infty$ es la función de crecimiento de V. Si $g_V(n)$ es polinómicamente acotada, entonces la dimensión de Gelfand-Kirillov de A se define como

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty} \frac{\log g_V(n)}{\log(n)}.$$

.

Esta nueva dimensión tiene sentido para álgebras con crecimiento polinómico. Naturalmente, puede haber una medida para álgebras con crecimiento exponencial. Inspirándose en esto, en [87] M.F Newman y C. Schneider construyen la entropía para álgebras graduadas. Esta última construcción será la inspiración para nuestra noción de entropía en álgebras filtradas.

Recapitulando, a lo largo de este trabajo trabajaremos principalmente con álgebras de caminos y algebras de Steinberg. Nos centraremos en su estructura y sus propiedades algebraicas, inspirándonos en otros resultados de las álgebras caminos de Leavitt. Estas últimas ocuparán un segundo lugar. A continuación, desarrollaremos la teoría sobre entropía algebraica para álgebras filtradas y la aplicaremos a las álgebras de caminos y de caminos de Leavitt. Por último, y aunque no se haya mencionado, trataremos de extender la propiedad de independencia del cuerpo base que presentan estas álgebras. Para ello haremos uso de la teoría de categorías y del Teorema de los ceros de Hilbert.

En la parte de Preliminaries describimos los conceptos básicos que necesitaremos en secciones futuras. En particular, en el Capítulo 1 describimos las álgebras de caminos. Para ello, estudiamos en la Sección 1.1 la definición de grafo dirigido que es, a grandes rasgos, un conjunto de vértices y flechas. Además, en la Subsección 1.1.1 extendemos el concepto de grafo a k-grafos con una breve definición. En la Sección 1.2 nos centramos en una de las propiedades topológicas del grafo, su conectividad. En concreto, definimos conexo y fuertemente conexo. Consecuentemente, se obtienen las componentes conexas y fuertemente conexas del grafo. Posteriormente, en la Sección 1.3 definimos el álgebra de caminos y caminos de Leavitt de un grafo dirigido siendo en la Sección 1.4 donde estudiamos algunas propiedades del álgebra de caminos: ideal generado por un conjunto de caminos, colapso de un grafo, etc. Además, vemos que el álgebra de caminos tiene unidades locales y un sistema completo de idempotentes ortogonales, permitiendo la descomposición de Peirce como herramienta para la descripción del álgebra. En particular, nos centraremos en el álgebra de caminos de un ciclo de n vértices, KC_n.

En el Capítulo 2 centramos nuestra atención en las álgebras de Steinberg. Para ello damos primero la definición de groupoide en 2.1 y vemos algunos ejemplos. Algunos groupoides tienen una estructura topológica compatible con las operaciones en el groupoide, definimos entonces en la Sección 2.2 groupoide topológico, étale y ample. Una vez que tenemos un grupoide ample, podemos construir, como hacemos en la Sección 2.3 el álgebra de Steinberg de un grupoide ample.

La parte ii se centra en el estudio de la estructura de las álgebras de caminos y de las álgebras de Steinberg. Aunque ambas son interesantes, vamos a prestar más atención a las primeras. Esta parte va desde el Capítulo 3 hasta el Capítulo 7. A través de este exploramos las condiciones de finitud y perfección de las álgebras de caminos junto con el centroide, centroide extendido y la clausura central de KC_n. Además, profundizamos en la teoría de zócalos de las álgebras de Steinberg, inspirándonos en los artículos [27, 10, 11].

En el Capítulo 3 describimos la Artinianidad, la semiartinianidad y el zócalo a izquierda (o derecha) de un álgebra de caminos en términos de las propiedades geométricas del grafo. En la Sección 3.1 nos centramos en describir la Artinianidad y el zócalo a izquierda (resp. derecha) del álgebra de caminos. Nuestro estudio conduce a la Proposición 3.2 que afirma que un álgebra de caminos es Artiniana si y sólo si el grafo es finito y acíclico. Además, caracterizamos los ideales mínimales a izquierda (resp. derecha) en la Proposición 3.4 y, en consecuencia, obtenemos una descripción completa del zócalo a izquierda (resp. derecha) de un álgebra de caminos en el Corolario 3.7. Incluimos la Sección 3.2 por autocontención. Se basa completamente en el libro de James Dugundji [41]. Allí solo seleccionamos las definiciones y resultados básicos sobre conjuntos bien ordenados y ordinales, necesarios en la Sección 3.3. Uno de los resultados clave es el Teorema 3.20 que establece todas las propiedades de la clase de ordinales. En la siguiente Sección definimos (Definición 3.21) la cadena de zócalos y en general el zócalo a izquierda (resp. derecha) de orden α. Por eso necesitamos toda la teoría de ordinales anterior. Además, diremos que un álgebra es de Loewy izquierda (resp. derecha) si la cadena de zócalos a izquierda (resp. derecha) es estacionaria en toda el álgebra. Como el zócalo a izquierda (resp. derecha) está relacionado con las fuentes (resp. sumideros), análogamente enunciamos en la Definición 3.22 el concepto de α-fuente (resp. α-sumidero) o fuente (resp. sumidero) de orden α. Estos elementos serán los que generen los zócalos de orden superior. A continuación, damos en la Definición 3.26 la noción de álgebra semiartiniana que es un álgebra tal que para cada ideal I distinto del total, el zócalo del cociente del álgebra por el ideal es distinto de cero. Como herramienta para caracterizar las álgebras semiartinianas probamos la Proposición 3.25 que dice que si un ideal I verifica $\mathrm{soc}_l(KE/I) = \{0\}$, entonces I contiene todas las posibles fuentes α. En consecuencia, declaramos el Teorema 3.27 y demostramos que un álgebra de caminos KE es semiartiniana si y sólo si todos los vértices de E son α-fuentes. Además, probamos en 3.30 que ambas son equivalentes.

En el Capítulo 4 investigamos sobre el zócalo del álgebra de Steinberg. En particular, ponemos el ojo en un tipo de álgebra de Steinberg que generaliza a la de caminos de Leavitt y las álgebras de Kumjian-Pask. En la Sección

4.1 estudiamos los ideales minimales a izquierda (a derecha, son análogos) en nuestras álgebras. Si tenemos un groupoide $\mathcal{G}$ tal que existe una bisección abierta compacta $\{x\} \in \mathcal{B}^{co}(\mathcal{G}^{(0)})$, con $x\mathcal{G}x$ un grupo finito, entonces es posible construir un ideal mínimal a izquierda como en el Teorema 4.3. Como podemos observar, con menos detalle, la existencia de bisecciones abiertas compactas con un elemento conducen a ideales mínimales. Sin embargo, esta condición no sólo es suficiente, sino necesaria. En la Proposición 4.6 demostramos que la existencia de ideales minimales a izquierda implica que existen este tipo de bisecciones abiertas compactas. En la Sección 4.2 describimos finalmente las componentes homogéneas de las álgebras de Steinberg. Todo ello se recopila en el Teorema 4.12. Allí demostramos que el zócalo está generado por las funciones $1_{\{x\}}$ con $\{x\} \in \mathcal{B}^{co}(\mathcal{G}^{(0)})$.

El Capítulo 5 centra su atención en las condiciones de perfección de un álgebra de caminos. En particular, se estudian las álgebras de caminos simples, primas, semiprimas y primitivas. Estas condiciones están relacionadas con el tipo de conexión que presenta el grafo. En la sección 5.1 caracterizamos las álgebras de caminos semiprimas y primas. En [95] Siles Molina caracteriza las álgebras de caminos semiprimas utilizando las álgebras de caminos de Leavitt. Además, en [32, 97] hay algunas nociones sobre álgebras de caminos primas de Crawley-Boevery y P. Smith. Sin embargo, incluimos la demostración de estos resultados en las Proposiciones 5.3 y 5.4 porque queremos hacer uso de las álgebras de caminos y añadir todos los detalles que podamos. Por ejemplo, verificamos que una álgebra de caminos KE es prima si el grafo E es fuertemente conexo. A continuación, en la Sección 5.2 damos detalles de las álgebras de caminos simples y primitivas. Las álgebras de caminos simples no son más que álgebras de grafos formados por un solo vértice aislado. Por otro lado, un álgebra de caminos KE es primitiva si y sólo si E es fuertemente conexo y todo ciclo tiene salida (Teorema 5.7). En particular, un álgebra de caminos KE es primitiva si y sólo si el grafo E es fuertemente conexo y no es un ciclo aislado. En la Sección 5.3 nos centramos en la estructura de las álgebras de caminos semiprimas. Como describimos en el Teorema 5.11 vemos que el grafo dirigido de las álgebras de caminos semiprimas consta de vértices aislados, ciclos aislados y componentes fuertemente conexas. Además, esta descomposición es única.

Después de esto, estudiamos en el Capítulo 6 el centroide de un álgebra de caminos. Esta estructura no es trivial sólo cuando el grafo es un ciclo. Esta es la razón por la que más adelante estudiamos el centroide extendido y la clausura central de KC_n. En la Sección 6.1 caracterizamos el centroide de un álgebra de caminos. Se trata de un resultado que aparece como ejercicio en [32]. Sin embargo, no contiene ninguna demostración y consideramos que no es un resultado trivial. Por eso decidimos incluirlo. En la Definición 6.1 vemos que el centroide es el conjunto de todos los automorfismos del álgebra vista como bimodulo. Esta estructura extiende el centro del álgebra. Como el centroide de una suma directa de álgebras es la suma directa de los centroides, en el contexto del álgebra de caminos podemos considerar que el grafo que estamos tomando es conexo. Dicho esto, en el Teorema 6.13 vemos que si $T \in \mathcal{C}(KE)$ (un elemento del centroide del álgebra de caminos) tal que hay un vértice $u \in E^0$ con $T(u) = ku$ con $k \in K$, entonces $T = k\,\mathrm{Id}_{KE}$. Este resultado, junto con los Lemas 6.14 y 6.15, nos permiten declarar el Teorema 6.16. En este sentido, el centroide de un álgebra de caminos de un grafo conexo es isomorfo al cuerpo base o al álgebra de polinomios en una variable. En la Sección 6.2 nos centramos en el centroide extendido y el cierre central de KC_n. Dado que KC_n es una PI-álgebra, entonces el centroide extendido es el cuerpo de fracciones del centroide, en este caso, $K(x)$. Sin embargo, para la clausura central necesitamos la descripción dada por Baxter y Martindale [13]. El centroide extendido es el conjunto de todas las clases de equivalencia de pares admisibles. Se describe con más detalle en la definición 6.20. Para encontrar el isomorfismo deseado, hacemos uso del Lemma 6.21. Después de eso, construimos en la Proposición 6.22 el isomorfismo. Así, calculamos el cierre central de KC_n como hacemos en el Teorema 6.27.

El Capítulo 7 está dedicado al estudio de las álgebras de caminos noetherianas a izquierda (resp. derecha) y J-semisimples, es decir, aquellas con radical de Jacobson trivial. Para ello, caracterizamos también el radical. Ambas descripciones son "rather testing exercises", tal y como dice Crawley-Boevery en [32]. Sin embargo, debido a su complejidad y a la falta de una demostración de ambos resultados en la literatura, decidimos incluirlo en este trabajo. El objetivo principal de la Sección 7.1 es describir las álgebras de caminos noetherianas a izquierda (resp. derecha). Estas están relacionadas con los ciclos sin entradas (resp. salidas). Primero, en la Proposición 7.3 estudiamos el caso de que el grafo sólo tenga un ciclo sin entradas (resp. salidas). Después, demostramos por inducción sobre el número de ciclos que un álgebra de caminos KE es noetheriana por la izquierda si y sólo si E es un grafo finito y cada ciclo no tiene una entrada (resp. salida) (Proposición 7.4). Utilizando la descomposición de Peirce vemos en la

Proposición 7.5 que las álgebras de caminos noetherianas son anillos matriciales triangulares superiores. En la Sección 7.2 caracterizamos el radical de Jacobson de las álgebras de caminos. En particular, en la Proposición 7.10 vemos que es generado como espacio vectorial por los caminos sin retorno. Además, en el Teorema 7.11 demostramos que el radical es el ideal generado por las aristas sin retorno. El objetivo principal de la Sección 7.3 es estudiar la estructura de las álgebras de caminos J-semisimples, es decir, con radical trivial. Estas álgebras de caminos también son semiprimas, y ya conocemos su estructura del Capítulo 5. Además, podemos considerar que el álgebra es noetheriana por la izquierda (resp. derecha). En consecuencia, tenemos el Teorema 7.18 donde hacemos uso de un resultado de [65] y describimos toda la estructura.

La parte iii está dedicada al estudio de la entropía algebraica. La mayoría de los resultados que aparecen aquí están en los trabajos [22, 21]. En el Capítulo 8 nos centramos en el estudio de la entropía algebraica de álgebras filtradas. Aplicamos este concepto a las álgebras de caminos y de caminos de Leavitt. Para ello, nos inspiramos en la dimensión de Gelfand-Kirillov y en la entropía de álgebras graduadas. En la sección 8.1 construimos la entropía para un álgebra filtrada. Primero, definimos la noción de filtración en la Definición 8.2 y después, definimos la entropía algebraica en 8.5. Esta nueva magnitud está condicionada por la filtración elegida. Este hecho se manifiesta en la Proposición 8.7. En la Sección 8.2 nos centramos en cómo se comporta la entropía con la suma directa de álgebras, monomorfismos y epimorfismos. En la Proposición 8.10 vemos cómo la entropía se contrae ante la presencia de un epimorfismo mientras que en la Proposición 8.11 probamos que la entropía se expande con un monomorfismo. Por último, en la Proposición 8.13 comprobamos que la entropía de sumas directas de álgebras es el máximo de las entropías de cada sumando. En la Sección 8.3 centramos nuestra atención en la entropía aplicada a álgebras de caminos y caminos de Leavitt de grafos finitos. Si queremos hacer uso de nuestra definición de entropía, necesitamos construir filtraciones para cada álgebra. En la Definición 8.14 construimos lo que llamamos las filtraciones estándar. Usando las Proposiciones 8.11 y 8.10 probamos la Proposición 8.20 donde vemos que para un grafo finito E, la entropía de $L_K(E)$ está acotada inferiormente por la entropía de KE y superiormente por la de $K\hat{E}$. Además, aplicando la Proposición 8.13 vemos que la entropía de un álgebra de caminos y de caminos de Leavitt es el máximo de las entropías de sus componentes conexas (Proposición 8.22). Terminamos esta Sección con la Proposición 8.24 donde vemos una vez más la dependencia de la entropía de la filtración elegida. En este caso, para el álgebra de caminos podemos obtener un valor de entropía tan pequeño como queramos. Después de esto, centramos nuestra atención en la Sección 8.4 en el cálculo de la entropía algebraica de álgebras de caminos de grafos finitos. Para ello, vemos en el Lemma 8.28 que es posible obtener este valor utilizando una norma de la matriz de adyacencia y sus potencias. Además, en la Proposición 8.30 demostramos que, finalmente, la entropía es el logaritmo del radio espectral de la matriz de adyacencia del grafo. A partir de este resultado, inducimos la Proposición 8.33 donde la entropía de un álgebra de caminos viene dictada por las componentes fuertemente conexas del grafo. En la sección 8.4.1 experimentamos con la entropía de algunas álgebras de caminos de Leavitt. Una vez más, la matriz de adyacencia del grafo es muy útil. Hacemos también uso del software Mathematica para realizar los cálculos. De esta forma, comprobamos numéricamente en los Ejemplos 8.37, 8.39, 8.40 y 8.41 que la entropía algebraica del álgebra de caminos y la del álgebra de caminos de Leavitt coinciden. La pregunta que nos hacemos es si esta igualdad se mantiene siempre. Exploramos esto en la Sección 8.5, donde demostramos la veracidad de esta afirmación. Para ello, utilizamos el Lemma 8.46 donde estudiamos la entropía del álgebra $C_K(E)$ en términos de la matriz de adyacencia del grafo. Para finalizar esta Sección, teniendo en cuenta algunos resultados de álgebras de Banach llegamos al Teorema 8.48 donde probamos la igualdad de ambas entropías. Finalmente, terminamos el Capítulo con la Sección 9.1 donde buscamos generalizar la entropía a álgebras que no son necesariamente filtradas, sino que son límite de álgebras filtradas. Sentamos las bases de este nuevo concepto en las Definiciones 9.11, 9.12 y 9.13 donde establecemos las nociones de monomorfismo estricto, sistema estricto y entropía para un límite de un sistema estricto directo. Una vez más, la entropía depende fuertemente del sistema elegido de álgebras filtradas. Demostramos esto en las Proposiciones 9.15. Además, probamos en la Proposición 9.16 que es posible calcular la entropía de un álgebra filtrada como la entropía de un sistema directo estricto de álgebras filtradas. No solo eso, sino ambos valores coinciden. Finalmente, acabamos el capítulo con la Sección 8.6 donde exploramos las relaciones entre la entropía algebraica y la Parte ii. Ahí, vemos en la Proposicion 8.49 cómo la entropia de las álgebras de caminos Artinianas o noetherianas es cero. Es más, la entropía nos permite distinguir álgebras de caminos primitivas de simples (Proposición 8.51). Además, existen ideales como el zócalo o el radical de Jacobson que son superfluos en el cálculo de la entropía (Proposición 8.53 y Corolario 8.55). Terminamos esta parte con el Capítulo 9 donde exploramos una generalización de la entropía. En la Sección 9.1 construimos el límite directo de álgebras filtradas y probamos que en general, el límite de álgebras finitamente filtradas no es un álgebra finitamente filtrada. En la Sección 9.2 sentamos las bases para esta nueva entropía a través de las Definiciones 9.11, 9.12 y 9.13 donde establecemos los conceptos de monomorphismo estricto, sistema dirigido estricto y entropía de un sistema dirigido estricto. De nuevo, la entropía depende fuertemente del sistema dirigido de álgebras filtradas. Probamos esto en la Proposición 9.15. Más aún probamos en la Proposición 9.16 que este nuevo concepto de entropía generaliza de manera compatible al dado en el Capítulo 8. Es decir, dada una álgebra finitamente filtrada, podemos calcular su entropía algebraica como el límite de un sistema dirigido estricto de álgebras finitamente filtradas y este valor coincide con el dado en la Definición 8.5.

Finalmente, la Parte iv y en particular el Capítulo 10, corresponde al artículo [52]. En él se estudian algunas propiedades de las álgebras asociadas a grafos de forma categórica. Como ejemplo de esta clase de resultados, encontramos por ejemplo que para grupos finitos G_1 y G_2, las algebras de grupo $\mathbb{C}G_1$ y $\mathbb{C}G_2$ son isomorfas si y solo si lo son KG_1 y KG_2 para cualquier cuerpo algebraicamente cerrado de característica 0. En la Sección 10.1 introducimos primero el concepto de functor de álgebra y, en la definición 10.3, establecemos qué es el functor de K-álgebras $\underline{A}$ para una cierta álgebra A. Por un lado, en la Definición 10.5, introducimos los extension invariant functors, que generalizarán la "buena" propiedad que satisfacen las álgebras de caminos de Leavitt. De hecho, veremos que los extension invariant functors son precisamente los de la forma $\underline{A}$ para una cierta K-álgebra A. Por otro lado, en el Corolario 10.8 observamos que, para los extension invariant functors $\mathcal{F}, \mathcal{G}$, una vez que tenemos un isomorfismo $\mathcal{F}(H) \cong \mathcal{G}(H)$ para alguna K-álgebra de Hopf H, este induce un isomorfismo $\mathcal{F}(K) \cong \mathcal{G}(K)$. Por lo tanto $\mathcal{F}(S) \cong \mathcal{G}(S)$ para toda K-álgebra conmutativa y unital S. Además, mostraremos, en el Ejemplo 10.10, un functor de álgebra que no es extension invariant gracias a las derivaciones de un álgebra perfecta. En la Subsección 10.1.1, incluimos algunos ejemplos interesantes bien conocidos de álgebras que proporcionan funtores extension invariant. Vale la pena mencionar que son la principal motivación para los resultados presentados en este trabajo. Como primer corolario obtenemos que para cualesquiera dos grafos E_1, E_2 y cualesquiera K-álgebras de Hopf H, entonces $L_H(E_1) \cong L_H(E_2)$ si y sólo si $L_K(E_1) \cong L_K(E_2)$ (ver Corolario 10.15). A continuación, en la Sección 10.2, presentamos las clave para el desarrollo del capítulo. Según [77], el Teorema de los ceros de Hilbert puede darse en términos de infinitas indeterminadas. Para alcanzar las hipótesis del Teorema 10.17, es necesario que o bien el conjunto de indeterminadas sea finito (de hecho, este es el Teorema de los ceros de Hilbert original) o bien que el grado de trascendencia del cuerpo base sobre su cuerpo primo tenga cardinalidad estrictamente mayor que el número de variables. Para unificar esas hipótesis en una sola condición, introducimos los Lemmas 10.18 y 10.20. Este nuevo entorno consiste en que el cardinal del cuerpo base es estrictamente mayor que el cardinal del conjunto de indeterminadas. El artículo original [77] de S. Lang informa de que I. Kaplansky obtuvo independientemente el resultado, pero la formulación equivalente de Kaplansky era ligeramente diferente y no utilizaba el grado de trascendencia. Dado que Kaplansky nunca publicó su trabajo, esto nos anima a incluir las pruebas de los Lemas 10.18 y 10.20. Las dos secciones restantes están dedicadas, una para la cuestión de dimensión finita (Sección 10.3) y la otra para la infinita (Sección 10.4). El primer caso nos permite comprender mejor el general. La idea en ambas situaciones es análoga: necesitamos traducir respectivamente el problema del isomorfismo en la existencia de ceros de un cierto ideal de polinomios. Esto corresponde a la Proposición 10.23 y al Corolario 10.24, para el caso de dimensión finita, y a la Proposición 10.36, para el general. Para la dimensión finita, como consecuencia del Teorema 10.17 y del Corolario 10.24, probamos el Teorema 10.25 donde para una extensión de cuerpos $K \subset F$ con K algebraicamente cerrado, el isomorfismo entre $A \otimes_K F$ y $B \otimes_K F$ (como F-álgebras) es equivalente al isomorfismo entre A y B (como K-álgebras). A partir de esta afirmación principal, variando las hipótesis sobre las extensiones de cuerpo, se derivarán una serie de Corolarios 10.27, 10.28, 10.29. Completamos el caso finito dimensional estableciendo las afirmaciones anteriores en términos de "extension invariant" functors, dando una visión más global del problema (Teoremas 10.30 y 10.31). En particular, para resaltar nuestra motivación, enunciamos el Corolario 10.33 y el Corolario 10.34 en términos de álgebras caminos de Cohn relativas. La misma idea se persigue en la sección de dimensión infinita mediante el Teorema 10.37. Sin embargo, necesitamos la hipótesis adicional de que el cardinal del cuerpo base K es estrictamente mayor que la dimensión de las álgebras. Esta hipótesis es esencial para aplicar el Teorema de los ceros de Hilbert en infinitas variables. En la línea del caso de dimensión finita, se siguen los Corolarios 10.38, 10.39 y 10.40 conjuntamente con una interpretación a través de funtores "extension invariant" (Teorema 10.41 y Teorema 10.42). Debido a que las álgebras de caminos de Leavitt se convierten en nuestra principal motivación, incluimos resultados que podrían enunciarse en el entorno general de los funtores "extension invariant", pero que exponemos en el entorno de las algebras de caminos de Leavitt. Así, por ejemplo, para grafos numerables E_1 y E_2, un cuerpo algebraicamente cerrado no numerable K y $K \subseteq F_1, F_2$, entonces $L_{F_1}(E_1) \cong L_{F_1}(E_2)$ si y sólo si $L_{F_2}(E_1) \cong L_{F_2}(E_2)$ (Corolario 10.43).

Part I

PRELIMINARIES

Chapter 1

ALGEBRAS ASSOCIATED TO A DIRECTED GRAPH

Directed graph theory is fundamental in mathematics, especially in discrete mathematics and computer science. Its importance lies in its ability to model asymmetric relationships, interactions or dependencies. For example: data flow, webpage link services, social networks, etc.

Algebraists also find interest in this object as a tool and aim, in representation theory [36, 94]. Not only that, since the introduction of Leavitt path algebra, there is a new field of study related to the graph theory [2, 11, 53]. Furthermore, if we have an evolution algebra and a natural basis of it, it is possible to define an associated graph. In this sense, it is possible to collect information of the algebra through the graph [29, 44].

1.1 DIRECTED GRAPHS

During the course of this work, we will develop a theory of algebras associated to directed graphs. As it is natural, in order to do so, we first need to know the definition of a directed graph and all the related concepts. When we first think of a directed graph we imagine it as a set of vertices and arrows that connect those vertices. Most of the times, we picture both sets as finite but the notion of directed graph is more general.

Definition 1.1. A *directed graph* consists of a 4-tuple $E = (E^0, E^1, s_E, r_E)$ with E^0 and E^1 two disjoint sets and $r_E, s_E : E^1 \to E^0$ two maps. The elements of E^0 are the *vertices* and the elements of E^1 are the *edges* of E. Further, for $e \in E^1$, $r_E(e)$ and $s_E(e)$ are called the *range* and the *source* of e, respectively. If there is no confusion with respect to the graph we are considering, we simply write $r(e)$ and $s(e)$.

The reader may observe that there is no restriction in the cardinality of the set of vertices and edges. If the number of vertices and edges is finite, then we will say that the directed graph is *finite* or that we have a finite graph.

It is possible to represent graphs, as its name says, graphically. For every vertex $v \in E^0$ we draw a dot in the plane and for every edge $e \in E^1$ we add an arrow from the vertex $s(e)$ pointing to the vertex $r(e)$. As an example, consider the graph E given by the tuple (E^0, E^1, r, s) with $E^0 = \{u, v, w\}$, $E^1 = \{f_{v,v}, f_{u,v}, f_{v,u}, f_{u,w}, g_{u,w}\}$ and the maps $s, r \colon E^1 \to E^0$ such that

$$s(f_{v,v}) = v, s(f_{u,v}) = u, s(f_{v,u}) = v, s(f_{u,w}) = u, s(g_{u,w}) = u,$$
$$r(f_{v,v}) = v, r(f_{u,v}) = v, r(f_{v,u}) = u, r(f_{u,w}) = w, r(g_{u,w}) = w.$$

Following the previous description of the representation of a graph, we can represent E as in Figure 1.

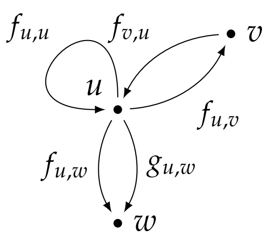

Figure 1: Example of a directed graph E.

It is possible to classify each vertex depending on the number of edges that emerge from it is zero, finite or infinite.

Definition 1.2. If E is a directed graph, a vertex $v \in E^0$ such that $s^{-1}(v) = \varnothing$ is called a *sink* and a vertex such that $r^{-1}(v) = \varnothing$ is a *source*. We will denote by sink(E) and source(E) the set of sinks and sources of E respectively. A vertex that is a sink and a source is *isolated*. The vertex v is called an *infinite emitter* if $s^{-1}(v)$ is an infinite set. Otherwise, a vertex that is neither a sink nor an infinite emitter is called a *regular vertex*. The set of regular vertices will be denoted by Reg(E) and the set of infinite emitters by Inf(E). If $s^{-1}(v)$ is finite for every vertex in E^0, then the graph is *row finite*.

In many occasions, we will not only be interested in the whole graph and we will centre our attention on just a part of it. In other words, we will be concerned with a subgraph.

Definition 1.3. Consider the directed graph $E = (E^0, E^1, s_E, r_E)$. We will say that $F = (F^0, F^1, s_F, r_F)$ is a *subgraph* of E, denoted as $F \subseteq E$, if and only if:

- $F^0 \subseteq E^0$ and $F^1 \subset E^1$.
- $s_E(F^1) \subseteq F^0$ and $r_E(F^1) \subseteq F^0$.
- $s_F = s_E|_{F^1}$ and $r_F = r_E|_{F^1}$.

As an example, the reader can check Figure 2 where we have two directed graphs $F \subset E$.

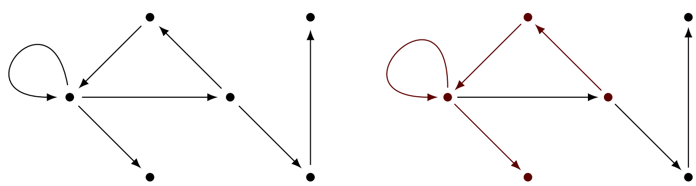

Figure 2: F (in red) subgraph of E (in black).

When we study directed graphs we do not just focus on vertices and edges. One of the main elements in directed graph theory are paths. As it is natural, a path is just a concatenation of edges. However, not every combination of edges forms a path.

Definition 1.4. Consider a directed graph E. A *path of length* $n \geqslant 1$ in E is a word $\mu = f_1 f_2 \cdots f_n$ with $f_i \in E^1$ for all $i = 1, \ldots, n$ and $r(f_i) = s(f_{i+1})$ for every $i = 1, \ldots, n-1$. The set of paths of length n is denoted by E^n, and we identify the set of vertices E^0 with the set of paths of length 0 (*trivial paths*) and the set of edges E^1 as the set of paths of length 1. The set of all paths of E is denoted by $\mathrm{Path}(E)$ and

$$\mathrm{Path}(E) = \coprod_{n=0}^{\infty} E^n.$$

Given a path $\mu \in \mathrm{Path}(E)$ we define the *length* map $\ell\colon \mathrm{Path}(E) \to \mathbb{N} \cup \{0\}$ such that $\ell(\mu) := 0$ if $\mu \in E^0$ and $\ell(\mu) := n$ if $\mu = f_1 f_2 \cdots f_n$ with $f_i \in E^1$ for $i = 1, \ldots, n$.

If $\lambda = e_1 \cdots e_n \in E^n$ and $\mu = f_1 \cdots f_m \in E^m$ with $r(\lambda) = s(\mu)$, then we denote $\lambda\mu = e_1 \cdots e_n f_1 \cdots f_m \in E^{n+m}$.

Coming back to the example of Figure 1 we can check that the word $f_{u,v} f_{u,u} g_{u,w}$ is not a path since $v = r(f_{u,v}) \neq s(f_{u,u}) = u$. However, $\mu = f_{u,u} f_{u,v} f_{v,u} g_{u,w}$ is a path. We can say that it starts in u and ends in w. In addition, the path μ is formed by the edges $\{f_{u,u}, f_{u,v}, f_{v,u}, g_{u,w}\}$ and it crosses the vertices $\{u, v, w\}$. All this information can be described for every path as follows.

Definition 1.5. Consider a directed graph $E = (E^0, E^1, s_E, r_E)$ and take a path $\mu \in \mathrm{Path}(E)$. If $\mu = f_1 \cdots f_n \in E^n$ we define the sets

$$\mu^{(0)} := \{u \in E^0 \colon \exists f_i \in s^{-1}(u) \cup r^{-1}(u)\},$$

$$\mu^{(1)} := \{f_1, f_2, \ldots, f_n\},$$

and if $\mu \in E^0$ then $\mu^{(0)} = \{\mu\}$ and $\mu^{(1)} = \varnothing$.

In addition, we define the *source* and *range* of a path as the maps $s, r\colon \mathrm{Path}(E) \to E^0$ with

$$s(\mu) := s_E(f_1),\ r(\mu) := r_E(f_n) \text{ if } \ell(\mu) \geqslant 1$$
$$s(\mu) = r(\mu) := \mu, \text{ if } \mu \in E^0.$$

In a directed graph there are different types of paths depending on whether they end at the same vertex they have started or if all the vertices that they go through are different.

Definition 1.6. Consider a directed graph $E = (E^0, E^1, r, s)$, we define the following:

1. A path $\mu = f_1 f_2 \cdots f_n \in \mathrm{Path}(E)$ with $n \geqslant 1$ and $s(\mu) = r(\mu) = v$ is called a *closed path based in v*. Furthermore, if $\mu \in E^1$ then μ is a *loop*.

2. A *simple path* is a path $\mu = f_1 f_2 \cdots f_n$ such that $s(f_i) \neq s(f_j)$ for $i \neq j$.

3. A *closed simple path based at v* is a closed path $\mu = f_1 f_2 \cdots f_n$ based at v such that $s(f_j) \neq v$ for every $j > 1$.

4. If $\mu = f_1 f_2 \cdots f_n$ is a closed path based at v with $s(f_i) \neq s(f_j)$ for $i \neq j$. Then μ is a *cycle based at v*. Notice that a cycle is a closed simple path based on any of its vertices, but a simple closed path based at a certain vertex v may not be a closed simple path based at another vertex.

5. Consider the path $\mu = f_1 f_2 \cdots f_n$, we say that e is an *exit* of μ if there is i $(1 \leqslant i \leqslant n)$ with $s(f_i) = s(e)$ and $e \neq f_i$ for every $i = 1, \dots, n$. On the other hand, it is possible to define the dual concept. We say that e is an *entry* of μ if there is a j $(1 \leqslant j \leqslant n)$ with $r(f_j) = r(e)$ and $f_j \neq e$ for every $j = 1, \dots, n$.

1.1.1 Higher rank graphs

When we represent a directed graph, we can observe that it is pretty similar to a category. Vertices would be the objects and edges would be the arrows or morphisms. All the paths would be in bijection with all the morphisms in the category. We highlight the fact that they are not equal because of the chosen definition of path. In other references, related to analysis and topology, paths are described as composition of functions instead of juxtaposition of edges. That inverts the order of the edges in compare to our convection.

This categorical vision makes possible the extension of the concept of a directed graph. By using concepts from category theory such us functors, we can describe what are called as k-graphs. In particular, directed graphs, as we have studied before, will just be 1-graphs.

Following Kumjian and Pask [73] we have:

Definition 1.7. For a positive integer k we consider the additive semigroup $(\mathbb{N} \cup \{0\})^k$ as a category with one object $(0, 0, \dots, 0)$ and composition given by the addition. A graph of rank k or a *k-graph* is a category $\Lambda = (\Lambda^0, \Lambda, s, r)$ together with a functor $d\colon \Lambda \to (\mathbb{N} \cup \{0\})^k$ called the *degree map* with the following property: If $\lambda \in \Lambda$ and $d(\lambda) = m + n$ with $m, n \in (\mathbb{N} \cup \{0\})^k$, then there are unique $\mu, \nu \in \Lambda$ with $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. This is referred as *the factorization property*.

The factorization property enables the identification of the objects in Λ^0 with the identity morphisms (with degree 0). In this way, we can represent the k-

graph as a directed graph with k colours. Consider $c_1, c_2, \dots, c_k$ colours and represent the objects Λ^0 as vertices of a directed graph. Then, the edges will be the morphisms $\lambda \in \Lambda$ with $d(\lambda) = e_i = (0, 0, \dots, 0, \overbrace{1}^{i}, 0, \dots, 0)$ for $i = 1, \dots, k$. The edge λ with degree e_i will be coloured with the colour c_i. This is called the *skeleton* of the k-graph. If $k = 1$ this is just a directed graph and it is enough to describe the whole 1-graph. However, when $k \geqslant 2$ then the skeleton is not enough to describe the k-graph, we will also need relations between the morphisms. In some cases, the relations will be implicit by the factorization property and in some others it will be necessary to specify.

Example 1.8. Have a look at the skeleton of the 2-graph Λ represented in Figure 3. We consider that $d(e) = (1, 0)$ and $d(f) = (0, 1)$. If we take the path ef, then we have that

$$d(ef) = d(e) + d(f) = (1, 0) + (0, 1) = (1, 1).$$

In addition,

$$d(ef) = d(e) + d(f) = d(f) + d(e) = (0, 1) + (1, 0),$$

so there must be two paths $\alpha, \beta \in \Lambda$ such that $d(\alpha) = (0, 1)$, $d(\beta) = (1, 0)$ and $ef = \alpha\beta$. The only possible elements that have degree $(0, 1)$ and $(1, 0)$ are f and e. In consequence, $ef = fe$ and the 2-graph is $\Lambda = (\Lambda^0, \Lambda, r, s)$ with $\Lambda^0 = \{v\}$, $\Lambda = \{e^n f^m : n, m \in \mathbb{N} \cup \{0\}\}$.

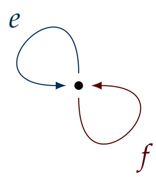

Figure 3: Skeleton of a 2-graph.

Although along the course of this work we will focus on the directed graph case, we want to highlight this generalisation since it is used in algebras such us Kumjian-Pask algebras, Steinberg algebras, etc. In addition, the more we answer questions related to the path algebra of a directed graph, the more questions related to the k-graphs will arise. However, answering these ones will not be the main goal of this thesis.

1.2 CONNECTIVITY OF A DIRECTED GRAPH

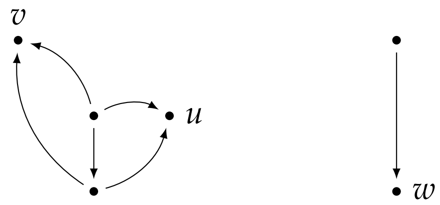

Figure 4: Graph F formed by two components.

If we consider the directed graph F as in Figure 4, we can differentiate at first sight two main parts or components. We could think that, for example, the vertices u and w do not belong to the same component because there is not a path $\mu \in \text{Path}(F)$ that connects u and w. However, this is not enough to describe the existent relation. Although there is not a path that connects v and u, they belong to the same component. This means that the relation is more general.

Definition 1.9. Consider a directed graph E and two vertices $u, v \in E^0$ we denote as $u \equiv v$ if there is an edge $e \in E$ with $s(e) = u$ and $r(e) = v$ or $s(e) = v$ and $r(e) = u$.

Two vertices u and v are related, $u \sim v$ if there is a collection of vertices $\{u_1, u_2, \dots, u_n\}$ such that

$$u \equiv u_1 \equiv u_2 \equiv \cdots \equiv u_n \equiv v.$$

We denote the equivalence class of a vertex v under this equivalence relation as $[v]$.

This last equivalence relation is the transitive closure of $\equiv$. The equivalence classes will be the ones that induce the connected components.

Definition 1.10. Consider a directed graph $E = (E^0, E^1, s, r)$ and a vertex $v \in E^0$. We construct the subgraph $F = (F^0, F^1, s_F, r_F)$ with

$$\begin{aligned} F^0 &:= [v], & r_F &:= r|_{F^1} \\ F^1 &:= s^{-1}([v]) \cup r^{-1}([v]), & s_F &:= s|_{F^1} \end{aligned}$$

denoted as the *connected component* of E that contains v.

The reader can wonder if there is a topology in the graph that agrees with the connected components. The answer is positive, the so called *DCC* topology implies that the elements of Definition 1.10 are precisely the connected components under this topology [29].

Once again, we refer to the graph F in Figure 4. We can observe that this graph is composed by two connected components. Perhaps, we could think that the graph F is a disjoint union of directed graphs. In fact, the disjoint union of its topological connected components.

Definition 1.11. Consider the directed graphs $E = (E^0, E^1, s_E, r_E)$ and (F^0, F^1, s_F, r_F). The *disjoint union* of the graphs E and F, denoted by $E \sqcup F$, is the directed graph given by the tuple $(E^0 \sqcup F^0, E^1 \sqcup F^1, r, s)$ with $r, s \colon E^1 \sqcup F^1 \to E^0 \sqcup F^0$ such that

$$\begin{aligned} s(e) = s_E(e),\ r(e) = r_E(e) \text{ if } e \in E^1 \\ s(f) = s_F(f),\ r(f) = r_F(e) \text{ if } f \in F^1. \end{aligned}$$

It is obvious that every directed graph E is a disjoint union of its connected components. When there is only one connected component, as it is natural, we will say that the graph is *connected*.

Even though the connection is not described using paths, as it is common in topology, there is a sense of connection in terms of paths known as path connection or strong connection.

Definition 1.12. If E is a directed graph, we say that two vertices $u, v \in E^0$ are *strongly related*, denoted by $u \leftrightarrow v$, if there are two paths $\lambda, \mu \in \text{Path}(E)$ such that $s(\lambda) = u$, $r(\lambda) = v$, $s(\mu) = v$ and $r(\mu) = u$.

The equivalence class of a vertex $v \in E^0$ under this relation is $\bar{v}$.

Once again, the reader can easily check that the relation given in the previous definition is an equivalence relation. This will allow us to define the strongly connected components of a graph.

Definition 1.13. Consider $E = (E^0, E^1, s, r)$ a directed graph. Given a vertex $v \in E^0$ we construct the subgraph $G = (G^0, G^1, s_G, r_G)$ with

$$\begin{aligned} G^0 &:= \bar{v}, & r_G &:= r|_{G^1} \\ G^1 &:= \{e \in E^1 : s(e), r(e) \in \bar{v}\}, & s_G &:= s|_{G^1}. \end{aligned}$$

denoted as the *strongly connected component* of E containing v.

If a directed graph E only has one strongly connected component, then we say that the graph is *strongly connected*. Equivalently, a graph is strongly connected if for every pair of vertices $u, v \in E^0$ there is a path $\lambda \in \text{Path}(E)$ such that $s(\lambda) = u$ and $r(\lambda) = v$.

Following the definitions of connected and strongly connected graph, it is obvious that strongly connected implies connected but connected does not always implies strongly connected. As an example we can have a look at Figure 5 where we have a connected graph with three different strongly connected components.

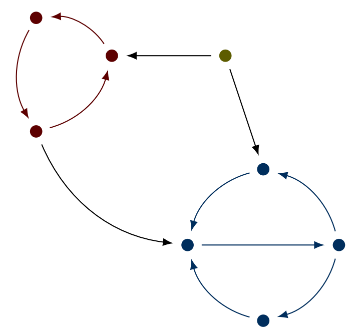

Figure 5: Directed graph G with one connected component and three strongly connected components (red, green and blue).

1.3 PATH ALGEBRAS AND LEAVITT PATH ALGEBRAS

On one hand, the study of representations of directed graphs, in particular of quivers (directed graphs with finite number of vertices and edges and without cycles), is strongly connected to the study of path algebras. In addition, the analysis of quantum groups is related to representations of quivers [82] and these are equivalent to the path algebra modules [12, 36, 94]. However, we are not going to put the spotlight on the representation theory of directed graphs, we are going to focus on the path algebra by itself. Even though the graph is used as a compact way to express all the relations in a free algebra, most of the algebraic properties, perfection, finiteness etc, are related to geometric qualities of the graph. That will be one of our main purposes in this work, creating connections between properties of the algebra and geometric configurations in the graph.

On the other hand, Leavitt path algebras arise as a generalisation of the Leavitt algebra $L_K(1, n)$ [2]. This last one is one of the main examples of an algebra that fails the so called Invariant Basis Number property. Leavitt path algebras led to a new research field that has been very fruitful. Algebraists have developed many results in relation to these algebras [1, 20, 31, 53, 95]. Furthermore, Leavitt path algebras have also been object of attraction to the analysis researchers since they are related to C^*-algebras and algebras of groupoids [46, 85, 102]. In many cases, results in Leavitt path algebra will be an inspiration to obtain some answers to questions in the path algebra context.

The path algebra of a directed graph is constructed as a free algebra with relations given by the graph.

Definition 1.14. Consider a directed graph E and R a commutative ring with unit. We call *path algebra*, denoted by RE to the free R-module generated by $\mathrm{Path}(E)$ together with the product $\cdot \colon RE \times RE \to RE$ given by

$$\begin{aligned} v \cdot \lambda &:= \delta_{v,s(\lambda)}\lambda && \text{if } v \in E^0, \lambda \in \mathrm{Path}(E), \\ \lambda \cdot v &:= \delta_{r(\lambda),v}\lambda && \text{if } v \in E^0, \lambda \in \mathrm{Path}(E) \\ \lambda \cdot \mu &:= \delta_{r(\lambda),s(\mu)}\lambda\mu && \text{if } \lambda, \mu \in \mathrm{Path}(E), \ell(\lambda), \ell(\mu) \geqslant 1. \end{aligned}$$

One can view this algebra as a quotient of the free algebra generated by the set $E^0 \cup E^1$ under a certain ideal generated by all the combinations that do not belong to $\mathrm{Path}(E)$.

Most of the times we will work with a field instead of a unital ring which we will denote as K. As a consequence, KE will be a vector space with basis $\mathrm{Path}(E)$. It is important to highlight that this set is linearly independent in the path algebra since several results in this work will be supported by this fact.

Example 1.15. Consider the directed graph E given in Figure 6 and compute the elements of KE.

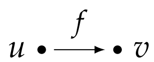

Figure 6: Directed graph E with just one arrow and two vertices.

Since $\mathrm{Path}(E) = \{u, v, f\}$ we have a three dimensional vector space $KE = Ku \oplus Kv \oplus Kf$. We can compute the multiplications of these elements and reflect them in Table 1.

	u	v	f
u	u	0	f
v	0	v	0
f	0	f	0

Table 1: Multiplication table of the path algebra of the directed graph of Figure 6.

The reader can check that the path algebra KE is isomorphic to

$$T_2(K) = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y, x \in K \right\}.$$

The length function $\ell\colon \operatorname{Path}(E) \to \mathbb{N} \cup \{0\}$ plays a key role in the path algebra of a directed graph. If $\lambda, \mu \in \operatorname{Path}(E)$ with $r(\lambda) = s(\mu)$, then $\ell(\lambda\mu) = \ell(\lambda) + \ell(\mu)$. This property allows us to define a $\mathbb{Z}$-grading in the path algebra. Consider a directed graph E and its corresponding path algebra KE with K a field. Define the vector spaces

$$KE_n := \begin{cases} \operatorname{span} \ell^{-1}(n) & \text{if } n \geqslant 0, \\ \{0\} & \text{if } n \leqslant 0. \end{cases}$$

In consequence, we have the $\mathbb{Z}$-grading

$$KE = \bigoplus_{n \geqslant 0}^{\infty} KE_n.$$

The fact that KE is a graded algebra will ease some of the future proofs since we will be able to work with the homogeneous components of an element in the algebra.

Once we know the construction of the path algebra we are in conditions to develop the Leavitt path algebra of a directed graph. In order to do so we first need to know the definition of the extended graph.

Definition 1.16. If $E = (E^0, E^1, s_E, r_E)$ is a directed graph, then we define the *ghost edges* $(E^1)^* = \{f^* \colon f \in E^1\}$ as a copy of E^1. The *extended graph* of E is the directed graph $\hat{E} = (\hat{E}^0, \hat{E}^1, s_{\hat{E}}, r_{\hat{E}})$ with $\hat{E}^0 = E^0$, $\hat{E}^1 = E^1 \cup (E^1)^*$ and $r_{\hat{E}}, s_{\hat{E}} \colon E^1 \cup (E^1)^* \to E^0$ such that

$$\begin{aligned} r_{\hat{E}}(f) &:= r_E(f), & s_{\hat{E}}(f) &= s_E(f) & &\text{if } f \in E^1, \\ r_{\hat{E}}(f^*) &:= s_E(f), & s_{\hat{E}}(f^*) &= r_E(f) & &\text{if } f^* \in (E^1)^*. \end{aligned}$$

The choice of $*$ as a way to differ the copy of the edges to the original ones is not random. The concept of ghost edge can be extended to a *ghost path*. If $u \in E^0$ then $u^* = u$ and if $\lambda \in \operatorname{Path}(E)$ with $\lambda = e_1 e_2 \cdots e_n$ then $\mu^* = e_n^* e_{n-1}^* \cdots e_1^*$. In addition, if μ^* is a ghost path with $\mu \in \operatorname{Path}(E)$, then $(\mu^*)^* = \mu$.

The Leavitt path algebra of a directed graph E will be the path algebra of the extended graph $\hat{E}$ with some extra relations.

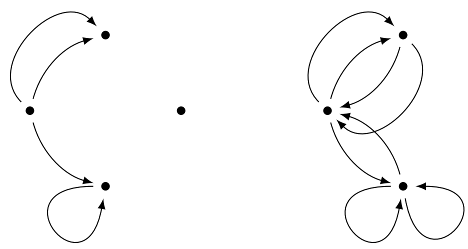

Figure 7: Directed graph E on the left and its corresponding extended graph $\hat{E}$ on the right.

Definition 1.17. For a directed graph E and a commutative ring R with identity, the *Leavitt path algebra* of E, denoted by $L_R(E)$, is the path algebra over the extended graph $\hat{E}$ with additional relations

(CK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$,

(CK2) $\sum_{\{e \in E^1 \,:\, s(e)=v\}} ee^* = v$ for every $v \in \mathrm{Reg}(E)$.

As it happened for the path algebras, most of the times we will work with a field K instead of with a ring. In general, it is not necessary to draw the extended graph to describe the Leavitt path algebra, but it is important to keep in mind that the ghost edges exist for the $(CK1)$ and $(CK2)$ conditions. Let us see an example of the computation of $L_K(E)$.

Example 1.18. If we consider once again the directed graph of Figure 6 we can this time compute $L_K(E)$ for a field K. In order to do so we first need to know that all the possible paths of $\mathrm{Path}(\hat{E})$ are

$$\mathrm{Path}(\hat{E}) = \{u, v\} \cup \{(ff^*)^n f, (f^*f)^n f^* : n \in \mathbb{N} \cup \{0\}\}.$$

However, we must take into account that now we have the $(CK1)$ and $(CK2)$ conditions. The first one implies that $f^*f = v$ whereas the second condition implies that $ff^* = u$. In consequence, the only elements that generate the algebra as a vector space are $\{u, v, f, f^*\}$ and $L_K(E) = Ku \oplus Kv \oplus Kf \oplus Kf^*$ as a vector space. We can reflect the multiplication in Table 2.

This algebra turns out to be isomorphic to $\mathcal{M}_2(K)$.

Unlike in the path algebra setting, in Leavitt theory the length map is not as important. However, there is an analogous map called the *degree map*. If E is a directed graph and $\hat{E}$ is its corresponding extended graph, then we define the

	u	v	f	f^*
u	u	0	f	0
v	0	v	0	f^*
f	0	f	0	u
f^*	f^*	0	v	0

Table 2: Multiplication table of the Leavitt path algebra of the directed graph in Figure 6.

degree map $d\colon \text{Path}(\hat{E}) \to \mathbb{Z}$ as $d(v) = 0$ if $v \in E^0$, $d(e) = 1$ if E^1, $d(e^*) = -1$ if $e^* \in (E^1)^*$ and if $x_1x_2\cdots x_n \in \text{Path}(\hat{E})$ with $x_i \in E^1 \cup (E^1)^*$ with $i = 1, \ldots, n$ then $d(x_1x_2\cdots x_n) = \sum_{i=1}^{n} d(x_i)$.

Analogously, the degree map induces a $\mathbb{Z}$-grading in $L_K(E)$ for a directed graph E and a field K. If we define the vector spaces

$$L_K(E)_n = \text{span}(d^{-1}(n)),$$

then we have the $\mathbb{Z}$-grading given by $L_K(E) = \bigoplus_{n\in\mathbb{Z}} L_K(E)_n$.

In the same way that directed graphs can be generalised through k-graphs, it is possible to extend the definition of path and Leavitt path algebra to these last ones.

By considering the *category algebra* [109] of a k-graph Λ we can define an analogous to the path algebra. The idea remains the same, if we can concatenate two paths the product is the concatenations whereas is zero if we can not. However, there is a slight difference since sometimes the category Λ would have relations between the paths which is something that does not happen in the path algebras.

The equivalent to Leavitt path algebras in k-graphs are known as Kumjian-Pask algebras. These ones have been object of study for different researchers, many of those related to the analysis and the C^*-algebras. Once again, this topic has given place to different articles [9, 27, 54, 61, 62].

1.4 BASIC PROPERTIES OF THE PATH ALGEBRAS

It is natural, since the construction of a path algebra comes from a directed graph, that some properties in the algebra are induced by the geometry of the graph. We can check, just by definition, that for a directed graph E the path algebra KE is an associative non-commutative algebra (the reader can check

Example 1.15). Furthermore, in this section we will explore some fundamental properties and we will obtain more specific results in future sections.

One of the first things that we can wonder from our naive intuition in path algebras is what happens when we have a disconnected graph. As we know, a product of paths in the algebra is different from zero if they can be concatenated. However, if we have two different connected components, there is no way such that elements in one component can multiply elements in another one and give a value different from zero. More precisely, if E is a directed graph and $\{E_i\}_{i\in I}$ are its corresponding connected components, then

$$KE = \bigoplus_{i\in I} KE_i$$

with $KE_i \cdot KE_j = \{0\}$ if $i \neq j$. That is, the path algebra is the direct sum of the path algebras of the connected components.

Another property that these algebras present is the existence of local. They are the vertices, and more precisely, the sums of those. Consider a set of elements $\{x_1, x_2, \ldots, x_n\} \subset KE$, there is an element $e \in KE$ such that $ex_i = x_i e = x_i$ for every $i = 1, \ldots, n$. In order to obtain such $e \in KE$ we need to observe each $x_i = \sum_{j=1}^{m_i} k_{ij}\lambda_{ij}$ with $k_{ij} \in K\backslash\{0\}$ and $\lambda_{ij} \in \mathrm{Path}(E)$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m_i$. To be more specific, we have to focus on $V = \{s(\lambda_{ij}), r(\lambda_{ij})\colon j = 1, \ldots, m_i,\ i = 1, \ldots, n\}$ and define $e = \sum_{v\in V} v$. Furthermore, the algebra KE is unital if and only if $|E^0| < +\infty$. In this case, $1 = \sum_{v\in E^0} v$.

The existence of local units in an associative algebra A implies that the smallest two-sided (resp. left or right) ideal generated by the elements $x_1, x_2, \ldots, x_n \in A$ is exactly $(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} Ax_iA$ (resp. $\sum_{i=1}^{n} Ax_i$ or $\sum_{i=1}^{n} x_iA$). In addition, this ideal contains the elements $x_1, \ldots, x_n$. If we put our attention in the path algebra $A = KE$ of a directed graph E and particularize to the case that $x_i \in \mathrm{Path}(E)$ for every $i = 1, \ldots, n$, then we have that

$$\begin{aligned}(x_1, x_2, \ldots, x_m) = \sum_{i=1}^{n} Ax_iA &= \mathrm{span}_K\{\lambda x_i \mu\colon \lambda, \mu \in \mathrm{Path}(E), i = 1, \ldots, n\},\\ \sum_{i=1}^{n} Ax_i &= \mathrm{span}_K\{\lambda x_i\colon \lambda \in \mathrm{Path}(E), i = 1, \ldots, n\},\\ \sum_{i=1}^{n} x_iA &= \mathrm{span}_K\{x_i\mu\colon \mu \in \mathrm{Path}(E), i = 1, \ldots, n\}.\end{aligned}$$

We can deepen even more and consider that all the elements are just vertices or just arrows. In consequence, we have that the two-sided (resp. left or right) ideal generated by those elements is the vector space generated by all the paths that go through (resp. start at or end at) one of those elements. This fact allows us to prove that certain quotients of a path algebras are also path algebras.

Definition 1.19. Let $E = (E^0, E^1, s, r)$ be a directed graph and K a field. Consider a subset $X \subset E^0$, *the collapse of E through the set of vertices X*, denoted by $E\backslash X$, is the graph given by the tuple $(E^0\backslash X, (E\backslash X)^1, s, r)$ with $(E\backslash X)^1 = E^1\backslash(s^{-1}(X) \cup r^{-1}(X))$.

If $Y \subset E^1$, then *the collapse of E through the set of edges Y*, denoted by $E\backslash Y$, is the graph given by the tuple $(E^0, E^1\backslash Y, s, r)$.

Proposition 1.20. *If (X) is the two-sided ideal in KE generated by the vertices in X, then $KE/(X) \cong K(E\backslash X)$ as K- algebras.*

If (Y) is the two-sided ideal in KE generated by the set of edges Y, then $KE/(Y) \cong K(E\backslash Y)$ as K-algebras.

The proof of this Proposition is direct and it can be considered general knowledge.

The reader can check Figure 8 as an example of a collapse through a set of vertices X and through a set of edges Y for the same graph E.

If we consider a directed graph E with a finite number of vertices, that is $|E^0| < \infty$, then, as we have already mentioned, the path algebra KE over the field K is unital. Not only that, but E^0 is a complete system of orthogonal idempotents. In other words, $1 = \sum_{v \in E^0} v$ and $uv = \delta_{uv} v$. Thus, we can perform the Peirce decomposition of $A = KE$. Ordering all the vertices as $E^0 = \{v_1, v_2, \ldots, v_n\}$, then

$$A = \bigoplus_{i,j=1}^{n} A_{i,j}$$

being $A_{i,j} := v_i A v_j$. Each of the $A_{i,j}$ is the vector space generated by all the paths that connect v_i with v_j. It is well known that each $A_{i,i}$ is a subalgebra of A and $A_{i,j}$ is an $(A_{i,i}, A_{j,j})$-bimodule. The algebra $A_{i,i}$ is called a *corner* of A through u_i. These structures are compatible with a matrix representation

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n} \\ A_{2,1} & A_{2,2} & \cdots & A_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n,1} & A_{n,2} & \cdots & A_{n,n} \end{pmatrix}$$

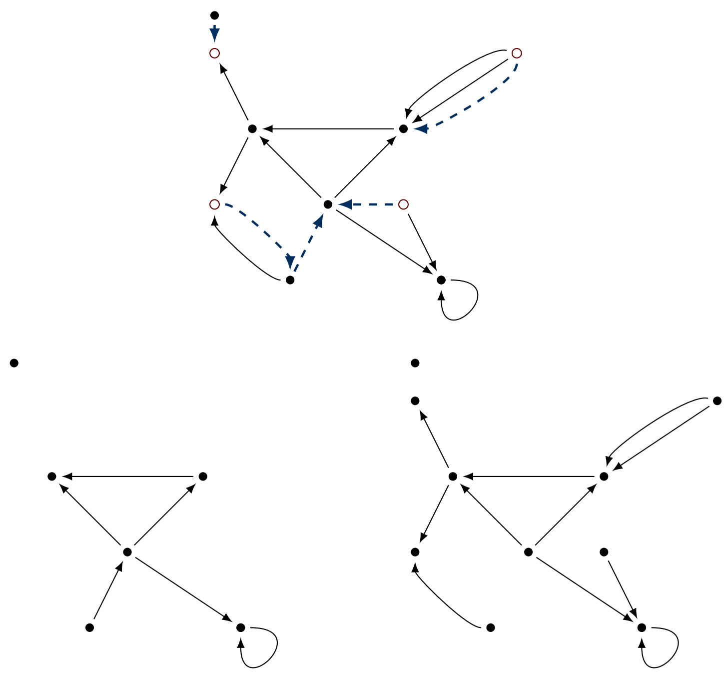

Figure 8: On the top the graph E with the set of vertices X (red circumferences) and the set of edges Y (blue dashed arrows). On the right the graph $E\backslash X$ and the graph $E\backslash Y$ on the right.

with its corresponding matrix product.

The set E^0 is not the only complete orthogonal set of idempotent elements. If we take a partition of the vertices $E^0 = V_1 \amalg V_2 \amalg \cdots \amalg V_m$ and define

$$e_1 = \sum_{v \in V_1} v, \cdots, e_m = \sum_{v \in V_m} v.$$

The set $\{e_1, e_2, \ldots, e_m\}$ corresponds to a complete orthogonal system of idempotents with m elements. That is, $e_i e_j = \delta_{i,j} e_i$ and $1 = \sum_{i=1}^m e_1$. In this context, we can write another Peirce decomposition

$$A = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,m} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ B_{m,1} & B_{m,2} & \cdots & B_{m,m} \end{pmatrix}$$

with $B_{i,j} := e_i A e_j$. This time $B_{i,j}$ is the vector space generated by all the paths that connect one vertex of V_i to a vertex of V_j.

However, these are just some possible Peirce decompositions using different combinations of vertices, but there are other complete systems of orthogonal idempotents possibly more general and with more complex description.

Let us see some examples of Peirce decompositions for some characteristic path algebras.

Example 1.21 (Straight line with n vertices.)**.** If we consider the directed graph E as in Figure 9, we can take the complete orthogonal system of idempotent elements $E^0 = \{v_1, v_2, \dots, v_n\}$ and find the Peirce decomposition of the algebra $A = KE$ over the field K. As we mentioned earlier, the set $A_{i,j} := v_i A v_j$ is the vector space

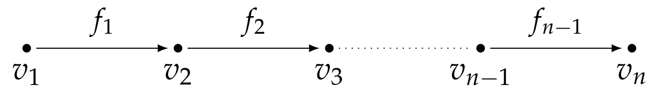

Figure 9: Arrow line with n vertices.

generated by all the paths from v_i to v_j. If we define $\mu_{i,j} := f_i f_{i+1} \cdots f_{j-1}$ for $i < j$ we have that

$$\begin{cases} A_{i,i} = Kv_i \\ A_{i,j} = K\mu_{i,j} \text{ for } i < j \\ A_{i,j} = 0 \text{ for } i > j \end{cases}$$

and then

$$A = \begin{pmatrix} Kv_1 & K\mu_{1,2} & \cdots & K\mu_{1,n} \\ 0 & Kv_2 & \cdots & K\mu_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & Kv_n \end{pmatrix}.$$

In this case, the map

$$\begin{pmatrix} k_{1,1}v_1 & k_{1,2}\mu_{1,2} & \cdots & k_{1,n}\mu_{1,n} \\ 0 & k_{2,2}v_2 & \cdots & k_{2,n}\mu_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{n,n}v_n \end{pmatrix} \mapsto \begin{pmatrix} k_{1,1} & k_{1,2} & \cdots & k_{1,n} \\ 0 & k_{2,2} & \cdots & k_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{n,n} \end{pmatrix}$$

is a K-algebra isomorphism between KE and the subalgebra of upper triangular matrices in $\mathcal{M}_n(K)$.

Example 1.22 (Cycle with n vertices)**.** For the purpose of this example all the subscripts will be modulo n. Consider the graph C_n as Figure 10, with the set of vertices as $C_n^0 = \{v_0, v_1, \dots, v_{n-1}\}$ and $C_n^1 = \{f_0, f_1, \dots, f_{n-1}\}$ where $s(f_i) = v_i$

and $r(f_i) = v_{i+1}$ for $i = 0, 1, \ldots, n-1$. Besides, define the simple paths $\mu_{i,i} := v_i$, $\mu_{i,j} := f_i f_{i+1} \cdots f_{j-1}$ and $c_i := f_i f_{i+1} \cdots f_{i-1}$ for $0 \leqslant i, j \leqslant n-1$.

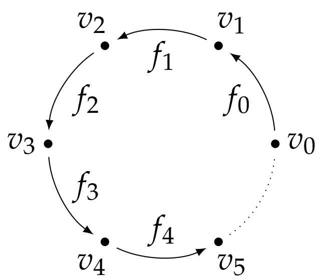

Figure 10: Directed graph C_n.

In this case, if we consider the K-algebra $A = KC_n$, we have that

$$A_{i,j} = K[c_i]\mu_{i,j}$$

with $K[c_i]$ the polynomial algebra over the variable c_i. This leads us to

$$A = \begin{pmatrix} K[c_0] & K[c_0]\mu_{0,1} & \cdots & K[c_0]\mu_{0,n-2} & K[c_0]\mu_{0,n-1} \\ K[c_1]\mu_{1,0} & K[c_1] & \cdots & K[c_1]\mu_{1,n-2} & K[c_1]\mu_{1,n-1} \\ K[c_2]\mu_{2,0} & K[c_2]\mu_{2,1} & \cdots & K[c_2]\mu_{2,n-2} & K[c_2]\mu_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K[c_{n-1}]\mu_{n-1,0} & K[c_{n-1}]\mu_{n-1,1} & \cdots & K[c_{n-1}]\mu_{n-1,n-2} & K[c_{n-1}] \end{pmatrix}.$$

Observe that we have the relation

$$c_i f_i = f_i c_{i+1}.$$

This, immediately implies that $p(c_i)f_i = f_i p(c_{i+1})$ for any polynomial p. Consequently,

$$\mu_{i,j} p(c_k) = \delta_{j,k} p(c_i)\mu_{i,j}$$

where again p is a polynomial.

Notice that for $i > 0$ we have $\mu_{0,i}\mu_{i,0} = c_0$ and $\mu_{i,0}\mu_{0,i} = c_i$. Furthermore:

$$\mu_{i,j}\mu_{j,k} = \begin{cases} \mu_{i,k} & \text{if } i \leqslant j \leqslant k \text{ or } j \leqslant k \leqslant i \text{ or } k \leqslant i \leqslant j, \\ c_i \mu_{i,k} & \text{cc.} \end{cases}$$

So we claim that $\mu_{i,j}\mu_{j,k} = c_i^{n(i,j,k)}\mu_{i,k}$ where $n(i,j,k) = 1$ if and only if $(i \leqslant k < j)$ or $(j < i \leqslant k)$ or $(k < j < i)$ (and $n(i,j,k) = 0$ otherwise).

The Peirce decomposition implies that $A = \bigoplus_{i,j=0}^{n-1} K[c_i]\mu_{i,j}$ and a general element is of the form $\sum_{i,j=0}^{n-1} p_{i,j}(c_i)\mu_{i,j}$. The product of two elements is then

$$\left(\sum_{i,j=0}^{n-1} p_{i,j}(c_i)\mu_{i,j}\right)\left(\sum_{i,j=0}^{n-1} q_{i,j}(c_i)\mu_{i,j}\right) = \sum_{i,j,k,l=0}^{n-1} p_{i,j}(c_i)\mu_{i,j}q_{k,l}(c_k)\mu_{k,l} =$$
$$= \sum_{i,j,k,l=0}^{n-1} \delta_{j,k}p_{i,j}(c_i)q_{k,l}(c_i)\mu_{i,j}\mu_{k,l} = \sum_{i,j,l=0}^{n-1} p_{i,j}(c_i)q_{j,l}(c_i)\mu_{i,j}\mu_{j,l} =$$
$$= \sum_{i,j,l=0}^{n-1} p_{i,j}(c_i)q_{j,l}(c_i)c_i^{n(i,j,l)}\mu_{i,l}.$$

Define next the numbers $m(i,j) = 1$ if $i > j$ and $m(i,j) = 0$ else. This enables us to define $\tau\colon A \to \mathcal{M}_n(K[x])$ such that $\tau\left(\sum_{ij} p_{ij}(c_i)\mu_{ij}\right) := \sum_{ij} p_{ij}(x)x^{m(i,j)}E_{ij}$ where E_{ij} is the usual elementary matrix. This map is a monomorphism of vector spaces. We reflect here in Table 3 the values of $n(i,jk)$ and various $m(i,j)$'s for the possible scenarios of the triplet (i,j,k):

	$n(i,j,k)$	$m(i,k)$	$m(i,j)$	$m(j,k)$
$i \leqslant j \leqslant k$	0	0	0	0
$i \leqslant k < j$	1	0	0	1
$j \leqslant i \leqslant k$	1	0	1	0
$j \leqslant k < i$	0	1	1	0
$k \leqslant i \leqslant j$	0	1	0	1
$k \leqslant j < i$	1	1	1	1

Table 3: Table of coefficients $n(i,j,k)$, $m(i,k)$, $m(i,j)$ and $m(j,k)$ for different relations of i,j and k.

Consequently, we have:

$$n(i,j,k) + m(i,k) = m(i,j) + m(j,k) \tag{1}$$

for any $i,j,k \in \{1,\ldots,n\}$. Then, it can be proved that τ is a K-algebra monomorphism by using Relation (1). Thus,

$$KC_n \cong \operatorname{Im}(\tau) = \begin{pmatrix} K[x] & K[x] & \cdots & K[x] & K[x] \\ xK[x] & K[x] & \cdots & K[x] & K[x] \\ xK[x] & xK[x] & \cdots & K[x] & K[x] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ xK[x] & xK[x] & \cdots & xK[x] & K[x] \end{pmatrix}.$$

Even though the behaviour of KC_n as a matrix algebra over $K[x]$ is well known, we have provided an extended proof because we have not been able to find any detailed reference to cite. Notice that the rotational symmetry of C_n can be visualized in the above representation. For instance, for KC_3 the rotation of the graph induced by $0 \to 1 \to 2 \to 0$ gives the automorphism of KC_3 such that

$$\begin{pmatrix} p_{0,0}(x) & p_{0,1}(x) & p_{0,2}(x) \\ xp_{1,0}(x) & p_{1,1}(x) & p_{1,2}(x) \\ xp_{2,0}(x) & xp_{2,1}(x) & p_{2,2}(x) \end{pmatrix} \mapsto \begin{pmatrix} p_{2,2}(x) & p_{2,0}(x) & p_{2,1}(x) \\ xp_{0,2}(x) & p_{0,0}(x) & p_{0,1}(x) \\ xp_{1,2}(x) & xp_{1,0}(x) & p_{1,1}(x) \end{pmatrix}.$$

This is nothing but the restriction of the inner automorphism of $\mathcal{M}_3(K(x))$ such that $M \mapsto PMP^{-1}$ with

$$P = \begin{pmatrix} 0 & 0 & 1 \\ x & 0 & 0 \\ 0 & x & 0 \end{pmatrix}.$$

Chapter 2

ALGEBRAS ASSOCIATED TO A GROUPOID

Groupoids generalize groups, in that, they are categories where every morphism is invertible. If we consider a “nice” topology, giving place to what we call as an ample groupoid, then we can construct an Steinberg algebra. These algebras were introduced, independently in [46] and [100]. They generalize the Leavitt path and Kumjian-Pask algebras, obtaining in this way a new approach and technique of study [46, 85].

2.1 GROUPOIDS

Continuing on the line of the algebras induced by directed graphs, in this section we are going to introduce the so called *Steinberg algebras*. This algebra first appeared in papers [46, 100] as a generalisation of the inverse semigroup algebras and the Leavitt path algebras. This algebra is constructed from a groupoid, which can be represented by a directed graph since a groupoid is a small category where every morphism is an inverse morphism. For those who are not familiar with the category theory, we will define it more precisely following [91].

Definition 2.1. A *groupoid* is a system $(\mathcal{G}^{(0)}, \mathcal{G}, s, r, m, i)$ such that:

1. $\mathcal{G}$ and $\mathcal{G}^{(0)}$ are nonempty sets called the *underlying set* and *unit space*;

2. $r, s\colon \mathcal{G} \to \mathcal{G}^{(0)}$ are maps called the *range* and the *source* map;

3. m is a partially defined operation on $\mathcal{G}$ called *composition*: in particular, it is a map from the set of *composable pairs*

$$\mathcal{G}^{(2)} = \{(g, h) \in \mathcal{G} \times \mathcal{G} \colon r(h) = s(g)\}$$

onto $\mathcal{G}$, denoted as $m(g, h) := gh$, with the following properties:

   - $s(gh) = s(h)$ and $r(gh) = r(g)$ for every $(g, h) \in \mathcal{G}^{(2)}$ and
   - $(gh)k = g(hk)$ whenever either side is defined;

4. For every $x \in \mathcal{G}^{(0)}$ there is a unique identity $id_x \in \mathcal{G}$ such that $id_x g = g$ whenever $r(g) = x$ and $h id_x = h$ whenever $s(h) = x$;

5. $i\colon \mathcal{G} \to \mathcal{G}$ is the *inversion map*, written as $i(g) := g^{-1}$ such that $s(g^{-1}) = r(g)$, $r(g^{-1}) = s(g)$, $g^{-1}g = id_{s(g)}$, $gg^{-1} = id_{r(g)}$ and $(g^{-1})^{-1} = g$.

It is important to highlight that we will always identify $x \in \mathcal{G}^{(0)}$ with $id_x \in \mathcal{G}$ in such a way that $\mathcal{G}^{(0)}$ is considered a subset of $\mathcal{G}$. The elements of $\mathcal{G}^{(0)}$ are called *units*.

Let us see some examples of groupoids:

Examples 2.2. 1. If G is a group with unit ε, then $(\{\varepsilon\}, G, s, r, m, i)$, with m and i the usual group multiplication and group inverse, is a groupoid.

2. Consider the sets

$$\mathcal{E}^{(0)} = \{x, y\}, \quad \mathcal{E} = \{x, y, \alpha, \alpha^{-1}, \beta, \beta^{-1}, \gamma, \delta\},$$

and the maps $s, r\colon \mathcal{E} \to \mathcal{E}^{(0)}$ given by $s(\gamma) = r(\gamma) = s(\alpha) = s(\beta) = x$, $r(\delta) = s(\delta) = r(\alpha) = r(\beta) = y$. We can define the multiplication by the relations

$$\gamma = \beta^{-1}\alpha, \delta = \beta\alpha^{-1}, \gamma^2 = x.$$

The first and third identities imply that $\beta^{-1}\alpha\beta^{-1}\alpha = x$ which is equivalent to $\alpha\beta^{-1} = \beta\alpha^{-1} = \delta$ and $\gamma = \beta^{-1}\alpha = \alpha^{-1}\beta$. Finally, we have that $\delta^2 = \beta\alpha^{-1}\beta\alpha^{-1} = \beta\beta^{-1}\alpha\alpha^{-1} = y$. Under these conditions, the tuple $(\mathcal{E}^{(0)}, \mathcal{E}, s, r)$ together with the respectively inverses and multiplication relations is a groupoid. The groupoid $\mathcal{E}$ can be represented by Figure 11.

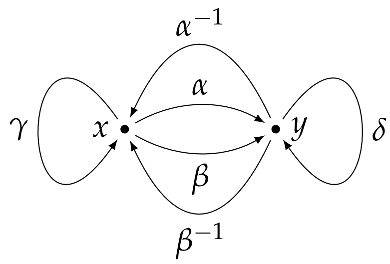

Figure 11: Finite groupoid $\mathcal{E}$.

3. Take a group G that acts on a set X on the left. That is, there is a map $\phi\colon G \times X \to X$ with $\phi(g, x) := g.x$ such that $\varepsilon.x = x$ with ε the identity in $\mathcal{G}$ and $h.(g.x) = (hg).x$ for every $g, h \in G$ and $x \in X$. Then, there is a groupoid structure for the set $G \times X$ with unit space $\{\varepsilon\} \times X$. The morphism (g, x) verifies $s(g, x) = (\varepsilon, g^{-1}.x)$ and $r(g, x) = (\varepsilon, x)$. The composition is defined by $(g, x)(h, g^{-1}.x) = (gh, x)$ and the inversion is given by $(g, x)^{-1} = (g^{-1}, g^{-1}.x)$ This is called the *transformation groupoid*.

When we work with groupoids, there are some elements or sets of elements that will be very useful. If $\mathcal{G}$ is a groupoid and $x, y \in \mathcal{G}^{(0)}$ we will denote $x\mathcal{G} = r^{-1}(x)$, $\mathcal{G}y = s^{-1}(y)$ and $x\mathcal{G}y = x\mathcal{G} \cap \mathcal{G}y = r^{-1}(x) \cap s^{-1}(y)$. In addition, the set $x\mathcal{G}x$ is a group called the *isotropy group* based at x. We call the *orbit* of x to the set $[x] = r(s^{-1}(x)) = s(r^{-1}(x))$. The set of orbits is a partition of $\mathcal{G}^{(0)}$. This is a consequence of the equivalence relation $x \sim y$ if and only if there is $\gamma \in \mathcal{G}$ such that $r(\gamma) = x$ and $s(\gamma) = y$. A subset $U \subset \mathcal{G}^{(0)}$ is *invariant* if for all $\gamma \in \mathcal{G}$ with $s(\gamma) \in U$ then $r(\gamma) \in U$ or equivalently if it is a union of orbits.

2.2 TOPOLOGICAL GROUPOIDS

Even though the anatomy of groupoids is interesting by itself, its flavour gets enriched when we endow the groupoid with a topology compatible with the structure. In this sense, we can define a *groupoid morphism* as a map that preserves its topological and algebraical structure similarly as a map between Lie groups. In particular, if $\mathcal{G}$ is a groupoid and R is a commutative unitary ring and consider it as an additive groupoid with the discrete topology, then we can take the maps $f\colon \mathcal{G} \to R$ that are continuous and preserve the groupoid structure. These maps, under certain conditions, will be the ones that define the so called Steinberg algebra.

Definition 2.3. A groupoid $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}, s, r, m, i)$ is a:

1. *topological groupoid* if $(\mathcal{G}, \mathcal{T})$ is a topological space and m and i are continuous maps with the topology of $\mathcal{G}^{(2)}$ inherited from $\mathcal{G} \times \mathcal{G}$.

2. *étale groupoid* if it is a topological groupoid and s is a local homeomorphism.

Remark 2.4. For any topological groupoid, since i is a continuous involution, then it is an homeomorphism. In addition, r and s are continous because $s = m \circ (i \times id_{\mathcal{G}})$ and $r = s \circ i$. Moreover, if the groupoid is étale, then r, s and m are local homeomorphism and the unit space $\mathcal{G}^{(0)}$ is open. If $\mathcal{G}$ is Hausdorff, then $\mathcal{G}^{(0)}$ is also closed.

When we consider étale groupoids we will be interested in the open sets that make r and s homeomorphism when we restrict to them. That is, if $U \subset \mathcal{G}$ is an open set such that $r|_U$ and $s|_U$ are homeomorphisms onto open sets of $\mathcal{G}^{(0)}$, then U is called an *open bisection*. Observe that if U is an open bisection, in particular $s|_U$ and $r|_U$ are injective. Actually, this is enough for an open set U to be a bisection. This new concept develops an equivalent definition of étale groupoid. A topological groupoid is étale if and only if there is a base of the topology formed by open bisections.

Even though we have already enriched our groupoids with very good properties, we still need more in order to define the Steinberg algebras. We will not only want our topological groupoid to have a base of open bisections, we will also want them to be compact.

Definition 2.5. An *ample groupoid* is a topological groupoid with a Hausdorff unit space and a base of compact open bisections.

If $\mathcal{G}$ is an ample groupoid we denote by $B^{co}(\mathcal{G})$ the set of all nonempty compact open bisections of $\mathcal{G}$ and $B^{co}(\mathcal{G}^{(0)})$ stands for the set of all nonemtpy compact bisections of $\mathcal{G}^{(0)}$.

Compact open bisections have several good properties compatible with the product and inversion in the groupoid. Many of those appear in [88]. For example, if $\mathcal{G}$ is an étale groupoid with $\mathcal{G}^{(0)}$ Hausdorff and $A, B \subset \mathcal{G}$ are compact open bisections then $A^{-1} = \{\alpha^{-1} \colon \alpha \in A\}$ and $AB = \{\alpha\beta \colon (\alpha, \beta) \in (A \times B) \cap \mathcal{G}^{(2)}\}$ are compact open bisections. In addition, if $\mathcal{G}$ is Hausdorff, then $A \cap B$ is an compact open bisection.

2.3 STEINBERG ALGEBRA OF AN AMPLE GROUPOID

In this section we will consider that $\mathcal{G}$ is an ample groupoid. As we have already mentioned, in order to define an Steinberg algebra we need to consider a commutative unitary ring R as a discrete topological space. If $U \subset \mathcal{G}$ we define the function $1_U \colon \mathcal{G} \to R$ as

$$1_U(\gamma) = \begin{cases} 1 & \text{if } \gamma \in U, \\ 0 & \text{if } \gamma \notin U. \end{cases}$$

The *support*, that is, the set of elements with image different from zero, of this function is supp $= U$. In addition, 1_U belongs to $R^{\mathcal{G}}$ the set of all the functions $f \colon \mathcal{G} \to R$. The set $R^{\mathcal{G}}$ has structure of R-module considering the pointwise addition and scalar multiplication. Finally, we can define the Steinberg algebra for an ample groupoid $\mathcal{G}$.

Definition 2.6. If $\mathcal{G}$ is an ample groupoid and R a commutative unitary ring, then $A_R(\mathcal{G})$ is the submodule of $R^{\mathcal{G}}$ defined by

$$A_R(\mathcal{G}) := \operatorname{span}_R\{1_U \colon U \text{ is a Hausdorff compact open subset of } \mathcal{G}\}. \tag{2}$$

The *convolution* of $f, g \in A_R(\mathcal{G})$ is defined as

$$f * g(\gamma) = \sum_{(\alpha,\beta) \in \mathcal{G}^{(2)}, \gamma = \alpha\beta} f(\alpha) g(\beta) \tag{3}$$

for all $\gamma \in \mathcal{G}$. The R-module $A_R(\mathcal{G})$, together with the convolution, is called the *Steinberg algebra* of $\mathcal{G}$ over R.

Following [100], the R-module $A_R(\mathcal{G})$ together with the convolution is an associative R-algebra which is unital if and only if the unit space is compact. In addition, the description of the algebra can be eased by using the compact open bisections. As [100, Proposition 4.3] states, the Steinberg algebra can be described as

$$A_R(\mathcal{G}) = \text{span}_R\{1_B : B \in B^{co}(\mathcal{G})\}. \tag{4}$$

Under this description, the convolution product is just a consequence of the property $1_A * 1_B = 1_{AB}$ for $A, B \in B^{co}(\mathcal{G})$.

One of the first examples that appear is the Steinberg algebra of a group. If we consider a group G with the discrete topology, then we are in front of an ample groupoid. Computing the Steinberg algebra of it we have that $A_R(G) \cong RG$ with RG the group algebra. In this sense, Steinberg algebras generalise group algebras. Furthermore, as it is proved in [46, Proposition 4.3] the Leavitt path algebra (in general the Kumjian-Pask algebra) is isomorphic to the Steinberg algebra of the so called path groupoid of a k-graph (a graph for $k = 1$). The path groupoid is described in [74] and even though it will be object of inspiration for future sections in this work, we will skip the details.

One last result that we would like to stand out is the possibility to decompose any element in a Steinberg algebra with more accessible elements. This translates into the following result (proved in [91, Proposition 2.5]).

Proposition 2.7. *If $\mathcal{G}$ is a Hausdorff ample groupoid, $B^{co}(\mathcal{G})$ is the set of compact open bisections and R is a commutative unitary ring, then every element $f \in A_R(\mathcal{G})$ is of the form $f = \sum_{i=1}^{n} r_i 1_{B_i}$ with $r_i \in R \backslash \{0\}$ and $B_1, B_2, \ldots, B_n \in B^{co}(\mathcal{G})$ are mutually disjoint.*

Part II

FINITENESS AND PERFECTION CONDITIONS

Chapter 3

ARTINIANITY, SEMIARTINIANITY AND SOCLE THEORY OF PATH ALGEBRAS

In the course of this chapter we are going to study the Artinianity and semiartinianity of the path algebra. Both of them belong to the so called finiteness conditions of an algebra and they are linked to the socle of the algebra itself. Not only to the socle but also to the socle chain of the algebra for which we will need to extend our knowledge in ordinal theory. Characterising the Artinianity in the path algebra context will be relatively easy. However, stepping outside and understanding the behaviour of the semiartinianity will need more complex tools. The theory of ordinals, or at least being aware of it, will be key for the following results. That is the reason why we have followed [41] as a reference. We will anyway include a section where we will show the basic definitions in ordinal theory to understand the concepts that involve semiartinianity.

3.1 SOCLE AND ARTINIANITY OF PATH ALGEBRAS

Definition 3.1. Let A be an algebra. We say that A is *left (resp. right, resp. two-sided) Artinian* if and only if for every descending chain of left (resp. right, resp. two-sided) ideals

$$I_1 \supset I_2 \supset \cdots I_n \supset \cdots$$

there is a number $n_0 \geqslant 1$ such that $I_{n_0} = I_n$ for every $n \geqslant n_0$.

Another way to state the last definition is saying that the algebra satisfies the descending chain condition (DCC). In other words, there is not any infinite strictly descending chain of ideals.

It is not difficult to check that if the algebra is finite-dimensional, then it is left Artinian (resp. right) because of the restrictions in the dimension. In the path algebra context the converse is true, although it is not true in general.

Proposition 3.2. *Let E be a directed graph and $A = KE$ the corresponding path algebra over the field K. The following statements are equivalent:*

1. *The algebra A is left (resp. right) Artinian.*
2. $\mathrm{Path}(E)$ *does not contain closed paths and* $|E^0 \cup E^1| < \infty$.
3. *The algebra A is finite-dimensional.*

Proof. It is well known that the second and third statements are equivalent. In addition, the third one implies the first one. We encourage the reader to try to prove these concepts by themselves. We only need to prove that the first one implies the second statement.

Let us consider that A is left Artinian, the proof is analogous for right and two-sided Artinian. If $|E^0 \cup E^1| = \infty$, then $|E^0| = \infty$ or $|E^1| = \infty$. Let us suppose that $|E^0| = \infty$ (the other case is analogous). We can consider the descending chain of ideals

$$\sum_{u \in E^0} Au \supset \sum_{u \in E^0 \setminus \{v_1\}} Au \supset \sum_{u \in E^0 \setminus \{v_1, v_2\}} Au \supset \cdots \supset \sum_{u \in E^0 \setminus \{v_1, \ldots, v_n\}} Au \supset \cdots$$

which is strictly descending. Consider that

$$\sum_{u \in E^0 \setminus \{v_1, \ldots, v_n\}} Au = \sum_{u \in E^0 \setminus \{v_1, \ldots, v_n, v_{n+1}\}} Au,$$

this means that $v_{n+1} \in \sum_{u \in E^0 \setminus \{v_1,\dots,v_n,v_{n+1}\}} Au$. In particular, $v_{n+1} = x_1u_1 + \cdots + x_mu_m$ with $x_i \in A$ and $u_i \neq v_{n+1}$ for $i = 1,\dots,m$. Multiplying to the right by v_{n+1} we obtain that $v_{n+1} = 0$ which is a contradiction.

On the other hand, if we consider that there is a closed path $\gamma \in A$. We construct the descending chain of ideals

$$A\gamma \supset A\gamma^2 \supset \cdots \supset A\gamma^n \supset \cdots .$$

Let us observe that the chain is strict. If it were not true, consider that $A\gamma^n = A\gamma^{n+1}$. Since we have local units, this implies that $\gamma^n \in A\gamma^{n+1}$. That is there is an element $x \in A$ such that $\gamma^n = x\gamma^{n+1}$. By using the base of paths in the algebra, we can write $x = \sum_{i=1}^m k_i\lambda_i$ with $k_i \in K\setminus\{0\}$ and $\lambda_i \in \mathrm{Path}(E)$. Then we have that $\gamma^n = \sum_{i=1}^m k_i\lambda_i\gamma^{n+1}$. However, this implies that $\sum_{i=1}^m k_i\lambda_i\gamma^{n+1} - \gamma^n = 0$ and this is a nonzero linear combination of different paths that equals to zero, which is not possible. □

In general, left and right Artinian are not equivalent, for example the algebra

$$\begin{pmatrix} \mathbb{Q} & \mathbb{R} \\ 0 & \mathbb{R} \end{pmatrix}$$

is right Artinian but it is not left Artinian [19, 4.2, Example 6]. However, due to the symmetry of the graph condition, it makes sense that both properties are equivalent in the path algebra context.

If we consider a left Artinian algebra, since it satisfies the descending chain condition, it implies the existence of minimal left ideals. That is, left ideals such that the only ideals that they contain are the trivial and themselves. It makes sense, in the search of a generalization of the Artinian path algebras, to study the minimal left (resp. right) ideals and try to characterize them in terms of the geometry of the graph. The study of these ideals is what give name to the socle theory of an algebra (or a module) and will be part of the definition of semiartinianity.

Definition 3.3. Let us consider a K-algebra A and M a left (resp. right) A-module. We define *the left (resp. right) socle of* M, denoted by $\mathrm{soc}(M)$ as the sum of all the minimal left (resp. right) submodules of M. If ${}_AA$ is the natural left A-module (resp. A_A the usual right module), we define the *left socle* (resp. *right socle*) of A as $\mathrm{soc}_l(A) := \mathrm{soc}({}_AA)$ (resp. $\mathrm{soc}_r(A) := \mathrm{soc}(A_A)$), that is, the sum of all the minimal left (resp. right) ideals of A.

As we will acknowledge in the following proposition, the minimal ideals are related to the sources and the sinks. When we try to build a minimal left (resp. right) ideal, the first we think of is building a one dimensional ideal. This ideal must take the form of $I = Kx$ with x a certain element in KE. If we do not choose wisely our generator of the ideal, there might be an element $y \in KE$ such that $yx \neq kx$ (resp. $xy \neq kx$) with $k \in K$ and I is not one dimensional. However, if we consider x a source (resp. a sink) there is not any possible path (or element) $y \in KE$ such that $yx \neq kx$ (resp. $xy \neq kx$). This means that sources and sinks will be related to the minimal ideals that we can construct in KE. We will illustrate this with more details in the next proposition.

Proposition 3.4. *Let us consider E a directed graph and KE its corresponding path algebra over a field K. The left ideal I is minimal if and only if $I = Kx$ with $x = \sum_{i=1}^{n} k_i\lambda_i$ ($k_i \in K\backslash\{0\}$, $\lambda_i \in \mathrm{Path}(E)$ and $\lambda_i \neq \lambda_j$ if $i \neq j$) and $s(\lambda_i) = v$ with $v \in \mathrm{source}(E)$, for every $i = 1, \ldots, n$. In addition, if $x = \sum_{i=1}^{n} k_i\lambda_i$ and $x' = \sum_{j=1}^{m} l_i\mu_j$, we have $Ix \cong Ix'$ as a left modules if and only if $s(\lambda_i) = s(\mu_j)$ for every $1 \leqslant i, j \leqslant n, m$.*

Proof. Let us consider $A = KE$ and I a minimal left ideal. By minimality, it can be expressed as $I = Ax$ with $x \in A$. If we write $x = \sum_{i=1}^{n} k_i\lambda_i$ with $k_i \in K\backslash\{0\}$, $\lambda_i \in \mathrm{Path}(E)$ and $\lambda_i \neq \lambda_j$ for $i \neq j$, there is a vertex $v \in E^0$ with $vx \neq 0$. Then, since the ideal is minimal, $I = Avx$ and without loss of generality we can consider $x = vx$, that is, all the paths start in v. If v is not a source this means that there is an edge $f \in E^1$ such that $fv = f$. We can take then $fx \in Ax\backslash\{0\}$. Again, this means that $I = Afx = Ax$, but this implies that there is $y \in A$ such that $yfx = x$. If we write $y = \sum_{j=1}^{m} l_j\mu_j$ with $l_j \in K\backslash\{0\}$, $\mu_j \in \mathrm{Path}(E)$ and $\mu_i \neq \mu_j$ if $i \neq j$, then we have that $\sum_{i,j=1}^{n,m} l_jk_i\mu_jf\lambda_i = \sum_{i=1}^{n} k_i\lambda_i$. This implies that $\sum_{i,j=1}^{n,m} l_jk_i\mu_jf\lambda_i - \sum_{i=1}^{n} k_i\lambda_i = 0$ which is a non trivial linear combination of different paths equal to zero and it can not be possible. Let us see this last assertion. If $\mu_jf\lambda_i = \mu_{j'}f\lambda_{i'}$, then by the definition of path, $\mu_j = \mu_{j'}$ and $\lambda_i = \lambda_{i'}$. If $\mu_jf\lambda_i = \lambda_s$, then $\lambda_i = \lambda_s$. However, using the length, we have that $\mu_jf\lambda_i \neq \lambda_i$ and all the paths are different. Thus, v must be a source and $Av = Kv$. Finally, $I = Ax = Avx = Kvx = Kx$.

Observe that the reverse is trivial because the ideal $I = Kx$ is one dimensional.

Let us now consider $\varphi\colon Kx \to Kx'$ an isomorphism of left modules (left ideals). If we have that $v = s(\lambda_i)$ for $i = 1, \ldots, n$, then $kx' = \varphi(x) = \varphi(vx) = v\varphi(x) = kvx'$ which implies that $v = s(\mu_j)$. For the converse, we just have to consider the isomorphism $\theta\colon Kx \to Kx'$ such that $\theta(x) = x'$. □

Remark 3.5. The dual of this result is also true. That is, I is a nonempty minimal right ideal of KE if and only if $I = Kx$ with x a linear combination of paths that end in the same sink. In addition, if $x = \sum_{i=1}^{n} k_i\lambda_i$ and $x' = \sum_{j=1}^{m} l_j\mu_j$ we have that $Kx \cong Kx'$ if and only if $r(\lambda_i) = r(\mu_j) = v \in \mathrm{sink}(E)$ for every $1 \leqslant i, j \leqslant n, m$.

Once we know how the minimal ideals in KE behave we are in conditions to describe the left and right socle. The first step to do so will be to give details of the homogeneous components of the socle.

Definition 3.6. Consider a K-algebra A, a minimal left (resp. right) ideal I and $[I]$ the equivalence class of all the left (resp. right) ideals in A such that are isomorphic to I. The sum $S_I = \sum_{J\in[I]} J$ is the *homogeneous component of the socle* determined by I.

In the case of the path algebras, following Proposition 3.4 we have that the homogeneous components of the left socle are determined by the set $\mathrm{source}(E)$ and the ones corresponding to the right socle by $\mathrm{sink}(E)$. If we fix a vertex $v \in \mathrm{source}(E)$ (resp. $u \in \mathrm{sink}(E)$), then we have that the homogeneous component associated to v (resp. to u) is $S_v = \sum_{x\in N_v} Kx$ where $N_v = \{x \in KE\colon vx = x\}$ (resp. $S_u = \sum_{y\in M_u} Ky$ where $M_u = \{y \in KE\colon yu = y\}$). Then, the reader might observe that in this case $S_v = \mathrm{span}_K\{\lambda \in \mathrm{Path}(E)\colon s(\lambda) = v\}$ (resp. $S_u = \mathrm{span}_K\{\mu \in \mathrm{Path}(E)\colon r(\mu) = u\}$). Eventually, if $A = KE$ the homogeneous components are just vA (resp. Au) with $v \in \mathrm{source}(E)$ (resp. $u \in \mathrm{sink}(E)$). Since the socle is the direct sum of its homogeneous components [64]. As a consequence we obtain the following result.

Corollary 3.7. *Let us consider E a directed graph. Then left socle and right socle of $A = KE$ are respectively $\mathrm{soc}_l(A) = \bigoplus_{v\in\mathrm{source}(E)} vA$ and $\mathrm{soc}_r(A) = \bigoplus_{v\in\mathrm{sink}(E)} Av$. That is, left socle is the ideal generated by $\mathrm{source}(E)$ and the right socle is the ideal generated by $\mathrm{sink}(E)$ (both of them two-sided ideals).*

Remark 3.8. If the graph that we are facing is a cycle with n vertices, then it is obvious that its left and right socle is trivial (Corollary 3.7).

We would like to emphasize that, generally, the left and right socle of a path algebra do not coincide. For example, consider the directed graph represented in Figure 12 and KE its corresponding path algebra over a field K. In this case, following Corollary 3.7 we have that $\mathrm{soc}_l(KE) = Ku \oplus Kf \oplus Kg$ and $\mathrm{soc}_r(KE) = Kv \oplus Kw$.

In the book [64] there is a well based socle theory for modules and rings. In particular, this theory is enriched once the nondegeneracy hypothesis is added. An associative algebra A is *nondegenerate* if $xAx = 0$ implies $x = 0$. In this

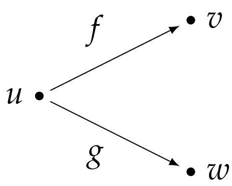

Figure 12: Example of a directed graph E which left and right socle of KE do not coincide.

case, left and right socle coincide (as it happens in Leavitt path algebras [11]). Furthermore, the homogeneous components are isomorphic to a matrix ring over a certain division algebra. If we put our attention in the left socle of the path algebra KE, then we can observe that there is only one homogeneous component and it can not be isomorphic as a matrix algebra over a division ring since its dimension is 3 which is not a perfect square. In this case, the algebra is degenerate since $fKEf = 0$ and $f \neq 0$.

Being capable of describing the left and right socle of the path algebras in terms of the directed graph with such a level of detail and without restrictions is a fact that we would like to highlight. Furthermore, we know the behaviour of the homogeneous components without the need of nondegeneracy.

3.2 THEORY OF ORDINALS

For the purpose of studying the semiartinianity of path algebras and since our intention is to make this work as self contained as we can, we are going to describe first the theory of ordinals. We will not describe the full theory because it will not be necessary. Hopefully, there will be just enough information to be able to work with ordinals confidently. As a main reference for this we will use the book [41]. This segment will be a summary of sections 1, 2, 3 and 5 of the second chapter. Once again, we want to highlight that these results are not new and we do not try to claim this theory as our own. That being said, we can start with a definition the reader might be familiar with.

Definition 3.9. A binary relation R in a set A is called a *preorder* if

1. for every $a \in A$, aRa (*reflexivity*),
2. if $a, b, c \in A$, aRb and bRc, then aRc (*transitivity*).

A set together with a preorder is called a *preordered set*. The preordered set A is a *chain* if every two elements in the set are related.

We will denote the preorder R by $<$ and $a < b$ will be read as 'a precedes b' or 'b follows a'.

We can add different requirements to a preorder to obtain other types of ordering relations.

Definition 3.10. A preordering in A with the *antisymmetry* property, that is, if $a, b \in A$ with $a < b$ and $b < a$, then $a = b$, is called a *partial ordering*. A set with a partial ordering is called a *partially ordered set*. A partially ordered set which is also a chain is called a *totally ordered set*.

One of the goals with this theory is generalize the induction. In order to do so, we will need a first or smallest element as it happens in $\mathbb{N}$.

Definition 3.11. A partially ordered set $(W, <)$ is called *well-ordered* or an *ordinal* if for each nonempty subset $B \subseteq W$ there is $b_0 \in B$ such that $b_0 < b$ for each $b \in B$. This element is called *first element*.

Example 3.12. The set $\mathbb{N}$ is a well-ordered set, whereas $\mathbb{Z}$ is not since $\mathbb{Z}^-$ has no first element.

As a direct consequence, if W is a well-ordered set, then it is a totally ordered set. It is enough to consider every subset $\{a, b\} \subset W$ to see that it is a chain.

Every element ω in a well-ordered set W that has a successor in the set has what is called an *immediate successor*. That is, an element γ in the set such that $\omega < \gamma$ and such that there is no $\delta \neq \gamma, \omega$ satisfying $\omega < \delta < \gamma$. In order to obtain such γ, we just need to consider the set $\{x \in W \backslash \{w\} \colon w < x\}$. However, an element ω may not have an *immediate predecessor* since there is the possibility that for each $\beta < \omega$ there may always be some $\gamma \neq \beta, \omega$ with $\beta < \gamma < \omega$. For example, consider the set $\{\{1, \ldots, n\} \colon n \in \mathbb{N}\} \cup \{\mathbb{N}\}$ with the preorder defined by the inclusion. This is a well-ordered set where every element $\{1, \ldots, n\}$ has $\{1, \ldots, n+1\}$ as a immediate successor. However, $\mathbb{N}$ does not have an immediate predecessor. An element without immediate predecessor is called a *limit element*.

Even though the theory of ordinals is very useful, there is some controversy as every area related with the affirmation of the axiom of choice. This is due to the next theorem.

Theorem 3.13. *[41, Theorem 2.1] The following three statements are equivalent:*

1. *The axiom of choice.*

2. *Zorn's lemma: Let X be a preoredered set. If each chain in X has an upper bound, then X has at least one maximal element.*

3. *Zermelo's theorem: Every set can be well ordered.*

The affirmation of the axiom of choice implies that there is a well order for every set. In particular, it means that there is a well order for the real numbers, even though there is no known construction of one. However, this does not infer that the well order is unique nor has any relation with the structure of the set.

Since every set can be well-ordered (if we assert the axiom of choice), we can study ordinals in greater detail and define the following.

Definition 3.14. Let $(W, <)$ be a well-ordered set. For every element $a \in W$, the set $W(a) = \{x \in W \backslash \{a\} \colon x < a\}$ is the *initial interval* determined by a.

The definition of initial interval will allow us to establish the Theorem of Transfinite Induction which we will make use of in Proposition 3.25.

Theorem 3.15. *[41, Transfinite induction] Let W be a well-ordered set and let $Q \subset W$. If $W(x) \subseteq Q$ implies that $x \in Q$ for every $x \in W$, then $Q = W$.*

This Theorem can be stated in the following form:

Theorem 3.16. *Let $\{P(x) \colon x \in W\}$ be a set of propositions. Assume both statements are true*

i $P(0)$ is true for 0, the first element in W.

ii For each x, the hypothesis that every $P(\alpha)$ with $\alpha \in W(x)$ is true implies that $P(x)$ is also true.

Then, every $P(x)$ is true.

To obtain the equivalent state, it is enough to consider $Q = \{x \colon P(x) \text{ is true}\}$.

A map φ between a well-ordered set $(X, <)$ and a well-ordered $(X', <')$ is a *monomorphism* if it preserves the ordering, that is, if $a < b$ then $\varphi(a) <' \varphi(b)$. We say that φ is an *isomorphism* if it is a monomorphism and it is bijective. In the class of all ordinals, we define $W = X$ if there is an isomorphism between both of them, that is, they are isomorphic. This is an equivalence relation that divides the class of all ordinals into mutually exclusive classes. We can select a representative element with good properties such that we will call the ordinal number of the subclass. In the end, what we will stay is that there is one well-ordered class $\mathcal{L}$

such that every well-ordered set is isomorphic to an initial interval of $\mathcal{L}$. This class $\mathcal{L}$ is constructed as follows: an element $\alpha \in \mathcal{L}$ will be simply an initial interval $\mathcal{L}(\alpha)$. The first five elements of $\mathcal{L}$ are the following:

$$\varnothing; \{\varnothing\}; \{\varnothing, \{\varnothing\}\}; \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}; \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}\}\}$$

In more detail, we have the following.

Definition 3.17. An *ordinal number* is a set α with the following properties:

1. If $x \in \alpha$ and $y \in \alpha$, then $x \in y$, $y \in x$ or $x = y$.
2. If $x \in y$ and $y \in \alpha$, then $x \in \alpha$.

Observe that $\in$ is not an ordering in the ordinal numbers since $x \in x$ is not true. However, we say that 'x precedes y' when $x \in y$. In addition, ordinal numbers have the following properties.

Proposition 3.18. *[41, 6.2] Let α be an ordinal number, then the next statements hold.*

1. *If $\varnothing \neq A \subset \alpha$, there is a unique $a \in A$, called the first element of A, such that $a \in x$ or $a = x$ for each $x \in A$.*
2. *The first element in α is $\varnothing$.*
3. *If $z \in \alpha$, then z is also an ordinal number.*

In this way, we can treat ordinal numbers indistinctly as sets or as elements of sets. There is also a sense of uniqueness in the ordinal numbers.

Proposition 3.19. *[41, 6.3] Consider α and β two ordinal numbers.*

1. *If $\alpha \neq \beta$, then $\alpha \subset \beta$ if and only if $\alpha \in \beta$.*
2. *Either $\alpha \subset \beta$ or $\beta \subset \alpha$.*

Finally, we have the Theorem that give sense to the theory of ordinals.

Theorem 3.20. *[41, 6.4] Let $\mathcal{L}$ be the class of all ordinal numbers. Define $\alpha \leqslant \beta$ if and only if $\alpha \subseteq \beta$. Then:*

1. *$(\mathcal{L}, \leqslant)$ is well-ordered.*
2. *$\mathcal{L}$ is not a set.*

3. *Each $\alpha \in \mathcal{L}$ is a set and $\mathcal{L}(\alpha) = \alpha$.*

4. *If E is any set of ordinal numbers, there is an ordinal number greater than all the elements in E.*

5. *If there is a nonincreasing sequence of ordinal numbers $\alpha_0 \geqslant \alpha_1 \geqslant \cdots$, then there is a number n such that $\alpha_i = \alpha_n$ for every $i \geqslant n$.*

6. *Each well-ordered set W is isomorphic to a suitable $\mathcal{L}(\alpha)$. The element α is called the ordinal number of W and it is denoted by* $\operatorname{ord} W$*. Finally,*

Following Proposition 3.18, the first ordinal number is $\varnothing$ which can be denoted as $\underline{0}$. The next ordinal number is $\{\varnothing\}$ and it can be denoted by $\underline{1}$. As it seems natural, the next ordinal number $\{\underline{0}, \underline{1}\}$ is denoted by $\underline{2}$. In general, the ordinal number with n elements $\{\underline{0}, \underline{1}, \ldots, \underline{n-1}\}$ is denoted by $\underline{n}$. If we have an ordinal number α we can construct the successor as $\alpha \cup \{\alpha\}$ written as $\alpha + \underline{1}$. Nevertheless, the axiom of infinity states the existence of ordinal numbers without an immediate predecessor, these are called the *limit ordinal* numbers. Finally, Theorem 3.20 allow us to consider every well-ordered set as a set of ordinal numbers.

In general, if there is no confusion we will denote the ordinal numbers without underlying. We also want to clarify that in general, ordinal numbers are associated with counting whereas cardinal numbers are related to the size.

3.3 SEMIARTINIANITY

As we know, the left (resp. right) socle of an algebra A is a two-sided ideal [64]. If we consider the quotient $A/\operatorname{soc}_l(A)$ we obtain a new algebra and we can try to compute the left socle of it. Once again, $\operatorname{soc}_l(A/\operatorname{soc}_l(A))$ is a two-sided ideal in $A/\operatorname{soc}_l(A)$ and the ideals of $A/\operatorname{soc}_l(A)$ come from ideals in A that contain $\operatorname{soc}_l(A)$. That is, $\operatorname{soc}_l(A/\operatorname{soc}_l(A)) = S/\operatorname{soc}_l(A)$ with S an ideal in A such that $\operatorname{soc}_l(A) \subseteq S$. We can continue this process of computing left socles of different quotients until we obtain an ascending chain of ideals. In this sense, we can define the following.

Definition 3.21. Let A be a K-algebra. We define the *left (resp. right) ascending socle chain of ideals* as the well-ordered chain of two-sided ideals

$$0 \subset S_1 \subset S_2 \subset \cdots \subset S_\alpha \subset \cdots$$

with $S_{\alpha+1}/S_\alpha = \operatorname{soc}_l(A/S_\alpha)$ (resp. $S_{\alpha+1}/S_\alpha = \operatorname{soc}_r(A/S_\alpha)$) and $S_\omega = \cup_{\alpha<\omega} S_\alpha$ if ω is a limit ordinal. We say that the algebra A is left (resp. right) *Loewy* if there is an ordinal number α such that $S_\alpha = A$.

In the path algebras context, Corollary 3.7 lets us compute the left and right chain of socles. If we consider a directed graph E and its corresponding path algebra KE, we know that $\operatorname{soc}_l(KE)$ is the ideal generated by $\operatorname{source}(E)$. As we know, the quotient $KE/\operatorname{soc}_l(KE)$ is isomorphic to the path algebra $K(E\backslash \operatorname{source}(E))$ and if we want to compute the socle of this new path algebra we need the sources of the graph $E\backslash \operatorname{source}(E)$. The reader might not find difficult to prove that, following the notation in Definition 3.21, S_2 is the ideal generated by the vertices $\operatorname{source}(E) \cup \operatorname{source}(E\backslash \operatorname{source}(E))$. We can keep this reasoning and obtain the whole ascending socle chain. Although all of these constructions are given in terms of the left socles and the sources of a graph, it is possible to replicate all this process in terms of the right socles and the sinks. In fact, during the rest of this section, we will focus on the left socles but all the results provided here also hold for the right socles.

In order to ease the notation we establish the next definition.

Definition 3.22. Consider a directed graph E, we define $\mathfrak{f}(E) := \operatorname{source}(E)$. In general, $\mathfrak{f}^0(E) := \varnothing$, $\mathfrak{f}^1(E) := \mathfrak{f}(E)$ and $\mathfrak{f}^\alpha(E) := \mathfrak{f}\left(E\backslash \bigcup_{\beta<\alpha} \mathfrak{f}^\beta(E)\right)$ for every ordinal $\alpha > 0$. We denote *the set of α-sources* by $\mathfrak{f}^\alpha(E) \subset E^0$.

Remark 3.23. With this new notation it is not difficult to check that S_α is the two-sided ideal generated by $\bigcup_{\beta\leqslant\alpha} \mathfrak{f}^\beta(E)$. Besides, it will ease the characterization of the semiartinian path algebras. As we will check, left semiartinian path algebras are highly connected to the α-sources. We first need these two key results.

Lemma 3.24. *If $u \in \mathfrak{f}^\alpha(E)$ with $\alpha > 1$, then there is an edge $e \in E^1$ such that $r(e) = u$ and $s(e) \in \mathfrak{f}^\beta(E)$ with $\beta < \alpha$. In addition, if α is not a limit ordinal, we can consider $\alpha = \beta + 1$.*

Proof. Let us consider $u \in \mathfrak{f}^\alpha(E)$ with $\alpha > 1$, and denote by $F_\alpha = E\backslash \bigcup_{\beta<\alpha} \mathfrak{f}^\beta(E)$. If $r(e) \neq u$ for every $e \in E^1$ then $u \in \operatorname{source}(E) = \mathfrak{f}^1(E)$ and this enter in contradiction with our hypothesis. Thus, there is an edge $e \in r^{-1}(u)$. Furthermore, we have that $e \notin F_\alpha^1$. Indeed, if $e \in F_\alpha^1$, then $u \notin \mathfrak{f}(F_\alpha) = \mathfrak{f}^\alpha(E)$ which contradicts our hypothesis. This leads to $s(e) \notin F_\alpha^0$ which implies that $s(e) \in \bigcup_{\beta<\alpha} \mathfrak{f}^\beta(E)$.

Let us also consider that α is not a limit ordinal and suppose that $s(e) \notin \mathfrak{f}^{\alpha-1}(E)$ for every $e \in r^{-1}(u)$. Thus, we have that $\{s(e) \colon e \in r^1(u)\} \subseteq \bigcup_{\beta<\alpha-1} \mathfrak{f}^\beta(E)$ and $u \in \mathfrak{f}\left(E\backslash \bigcup_{\beta<\alpha-1} \mathfrak{f}^\beta(E)\right) = \mathfrak{f}^{\alpha-1}(E)$ which is not true. □

Proposition 3.25. *Let E be a directed graph and KE its corresponding path algebra. Consider a left ideal I such that $\mathrm{soc}_l(KE/I) = \{0\}$, then $\{\lambda \in \mathrm{Path}(E) \colon s(\lambda) \in \mathfrak{f}^{\alpha}(E)\} \subset I$ for every ordinal α. In particular, $\mathfrak{f}^{\alpha}(E) \subset I$ for every ordinal α.*

Proof. We will approach this proof by transfinite induction in the order of the sources. Consider $A = KE$ and take a path λ with $s(\lambda) \in f^1(E)$ and consider $I \cap A\lambda$. We have two possibilities

- If $I \cap A\lambda = \{0\}$, then $(I + A\lambda)/I \cong A\lambda/\{0\} \cong K\lambda$ and this means that $\mathrm{soc}_l(A/I) \neq \{0\}$ which is not possible.

- If $I \cap A\lambda \neq \{0\}$, then $\lambda \in I$ by the minimality of $A\lambda$.

Consider that I contains all the paths that start in a i-source. Let us take a path λ such that $s(\lambda) \in \mathfrak{f}^{\alpha+1}(E)$ and consider $I \cap A\lambda$.

- If $I \cap A\lambda = \{0\}$, then $(I + A\lambda)/I \cong A\lambda$ is minimal and $\mathrm{soc}_l(A/I) \neq \{0\}$ which is not possible.

- If $I \cap A\lambda \neq \{0\}$, then we have two possibilities. If there is $0 \neq x \in I \cap A\lambda$ with $x = \sum_{j=1}^{n} k_j\lambda_j\lambda + k\lambda$ with $\lambda_j \neq \lambda_{j'}$ if $j \neq j'$, $k, k_j \in K$ and $k \neq 0$, then $\lambda = k^{-1}(x - \sum_{j=1} k_j\lambda_j\lambda) \in I$ because $\lambda_j\lambda \in I$ for every $j = 1, \ldots, n$ by our induction hypothesis. If there is no such element, this means that

$$K\lambda \cong \frac{A\lambda}{I \cap A\lambda} \cong \frac{I + As(\lambda)}{I}$$

  and $\mathrm{soc}_l(A/I) \neq \{0\}$ which is not possible

Finally, for $v \in \mathfrak{f}^{\omega}(E)$ with ω a limit ordinal, the proof is analogous because Lemma 3.24 is for sources of every order. □

Definition 3.26. We say that the K-algebra A is *left (resp. right) semiartinian* if for every proper left (resp. right) ideal I we have that $\mathrm{soc}(A/I) \neq \{0\}$.

Now we have the tools to characterize the semiartinianity of a path algebra in terms of geometric conditions of its graph.

Theorem 3.27. *Let E be a directed graph and $A = KE$ its corresponding path algebra over the field K. The algebra A is left semiartinian if and only if for every $u \in E^0$ there is an ordinal number α such that $u \in \mathfrak{f}^{\alpha}(E)$.*

Proof. Consider that KE is left semiartinian and suppose that there is a set of vertices $X \subset E^0$ such that for every $u \in X$ there is not any ordinal number α such that $u \in f^\alpha(E)$. If we consider I the ideal generated by the vertices $Y = E^0 \backslash X$ we have that $KE/I \cong K(E \backslash Y)$. There are not any sources in the graph $E \backslash Y$ and by Corollary 3.7 we have that $\mathrm{soc}_l(KE/I) \cong \mathrm{soc}_l(K(E\backslash Y)) = \{0\}$ but this is only possible if $X = \emptyset$.

For the converse, consider a left ideal I such that $\mathrm{soc}_l(KE/I) = \{0\}$. By Proposition 3.25 the ideal I contains all the α-sources, in particular, $E^0 \subset I$ by our hypothesis which implies that $I = KE$. Proving that KE is left semiartinian. □

Remark 3.28. If the algebra KE is left semiartinian, we will see that this implies that the graph E has no cycles: if we consider the ideal I generated by all the vertices that do not belong to a cycle, denoted by X, we have that $KE/I \cong K(E\backslash X)$ has a trivial socle since $E \backslash X$ is a graph without sources (each connected component in the graph is strongly connected). The only possible way for this to happen is that $I = KE$, that is, $X = E^0$. In addition, this result, together with Theorem 3.27, implies that the vertices that are basis of a cycle are not sources of any order. Besides, if the graph is finite, the algebra KE is not only left semiartinian, Proposition 3.2 implies that KE is a finite-dimensional, thus, we have that E is a tree.

Corollary 3.29. *Consider E a finite directed graph and $A = KE$ the corresponding path algebra over the field K. The following assertions are equivalent:*

1. *The algebra A is finite-dimensional.*
2. *The algebra A is left Artinian.*
3. *The algebra A is left semiartinian.*
4. *The graph E is acyclic.*

Proposition 3.30. *If E is a directed graph, then the path algebra KE is left (resp. right) semiartinian if and only if it is left (resp. right) Loewy.*

Proof. As it has been usual, we will prove just the left case. The right one is analogous. Consider that KE is left semiartinian. By Theorem 3.27, this means that every vertex $u \in E^0$ is a α-source for a certain ordinal number α. Following the theory of ordinals, in particular Theorem 3.20, there is an ordinal ω which is an upper bound of all the numbers α. This implies that S_ω is the ideal generated by $\bigcup_{\alpha \leqslant \omega} f^\alpha(E) = E^0$ which is the whole algebra and KE is Loewy.

For the converse assume that KE is left Loewy, then there is an ordinal ω such that $S_\omega = KE$. In particular, this means that every vertex is an α-source. Indeed, if there is a vertex $u \in E^0$ such that it is not an α-source for every ordinal number, then u is not in the ideal generated by all the α-sources, which contradicts that $KE = S_\omega$. □

Example 3.31. Consider the directed graph L which set of vertices is $L^0 = \{v\} \cup \{u_{i,j} \colon i \in \mathbb{N}\backslash\{0\}, 1 \leqslant j \leqslant i\}$ and set of edges is $L^1 = \{f_{i,j} \colon i \in \mathbb{N}\backslash\{0\}, 1 \leqslant j \leqslant i\}$ with $s(f_{i,j}) = u_{i,j}$ for every pair (i,j), $r(f_{i,j}) = u_{i,j+1}$ if $j < i$ and $r(f_{i,i}) = v$ for every $i \in \mathbb{N}$. We denote this as the *Levi* graph. One can think of the Levi graph as an countable infinite union of increasing finite lines connected through the same vertex v. The reader can find a graphic representation of the Levi graph in Figure 13. For a finite ordinal α we have that $\mathfrak{f}^\alpha(L) = \{u_{i,\alpha} \colon i \geqslant \alpha\}$ and for the limit ordinal $\omega = \mathbb{N}$ we have that $\mathfrak{f}^\omega(L) = \{v\}$. In particular, $\mathfrak{f}^\omega(L) \cup \left(\bigcup_{\alpha=1}^{\infty} \mathfrak{f}^\alpha(L)\right) = L^0$, that is, every vertex is a i-source and the path algebra KL is semiartinian.

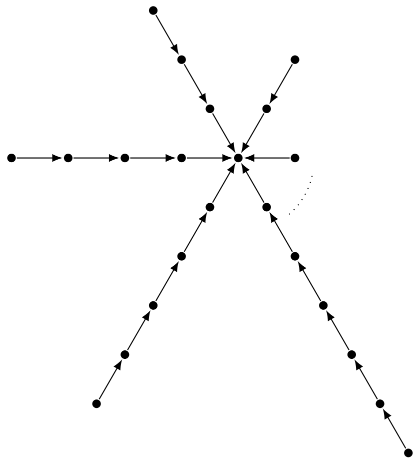

Figure 13: Levi graph.

In general, being acyclic is not enough to imply that the path algebra is semiartinian (if the graph is not finite). In Figure 14, we have an example of a directed graph which its corresponding path algebra is not semiartinian. It

Figure 14: Directed graph E such that there is no sources and no cycles.

is very important to highlight the fact that all these previous result have their corresponding dual. Instead of talking about left semiartinian, right socle and sources of order α, we refer to right semiartinian, right socle and sinks of order

α. All of the proofs above are analogous for the dual results just by taking the proper considerations.

If the directed graph E and KE is left semiartinian, then we can consider a total ordering in the set E^0 such that the adjacency matrix A_E is upper triangular. We just need to order each set $\mathfrak{f}^\alpha(E)$ (Zermelo's Theorem) and consider if $u \in \mathfrak{f}^\beta(E)$ and $v \in \mathfrak{f}^\alpha(E)$ such that $\beta < \alpha$, then $u < v$ and if $u, v \in \mathfrak{f}^\beta(E)$ we just consider the previous established order. Following this order of the vertices and by Lemma 3.24 the adjacency matrix A_E is upper triangular.

Chapter 4

SOCLE THEORY OF STEINBERG ALGEBRAS

The main purpose of this chapter is obtaining a socle theory for Steinberg algebras that can generalise the socle theory of the Leavitt path algebras [11] and of the Kumjian-Pask algebras [27]. We take inspiration for this from previous papers and also from the group algebra of a finite group, since the three algebras involved can be studied as Steinberg algebras. Similarly as we did in Chapter 3, the first step will be to study minimal left and right ideals. Following [11, Theorem 3.4], the existence of minimal left (resp. right) ideals, corresponds to the presence of special elements in the graph. For the Steinberg algebra case, this will be related to particular elements in a groupoid with notable properties. They will be elements in the unit space whose isotropy group is finite (and trivial in most cases), such that a singleton set containing one of them forms a compact open bisection

4.1 MINIMAL LEFT IDEALS OF A STEINBERG ALGEBRA

We first need a well known Lemma before giving a description of our canonical example of a minimal left ideal in a Steinberg algebra. Since we want this work to be as self-contained as possible and it is just general groupoid knowledge, we will include the proof.

Lemma 4.1. *Let us consider a groupoid $\mathcal{G}$ and $x, y \in \mathcal{G}^{(0)}$. If the set $y\mathcal{G}x$ is not empty, then $|x\mathcal{G}x| = |y\mathcal{G}x|$. As a consequence, $|z\mathcal{G}x| = |y\mathcal{G}x|$ for every $z, y \in r(s^{-1}(x))$.*

Proof. Since $y\mathcal{G}x \neq \emptyset$, there is $\beta \in y\mathcal{G}x$, consider the map $\phi\colon x\mathcal{G}x \to y\mathcal{G}x$ such that $\gamma \mapsto \phi(\gamma) := \beta\gamma$. This map is bijective since $\psi\colon y\mathcal{G}x \to x\mathcal{G}x$ with $\delta \mapsto \psi(\delta) := \beta^{-1}\delta$ is its inverse. Then $|y\mathcal{G}x| = |x\mathcal{G}x|$. □

Observation 4.2. The reader should remember that in this Chapter we are working with groupoids and in this case the range and the source of the arrows are inverted. For the composition of $\alpha\beta$ we have that $s(\alpha) = r(\beta)$.

Due to the symmetry on the results and the proofs, we will focus on the minimal left ideals and the left socle. The right minimal ideals and socle will be characterised in an analogous way.

In general if G is a finite group of cardinal n we can consider the n-dimensional group algebra KG for K a field. This algebra has nontrivial socle, that is, it possesses minimal left (resp. right) ideals. In fact, the ideal $K(\sum_{g\in G} g)$ is minimal. Defining $e = \sum_{g\in G} g$ we have that $e^2 = ne$. Then left minimal ideals are present in two flavours:

(i) Those generated by division idempotent. This case corresponds to the characteristic of K not dividing n because $e' = \frac{1}{n}e$ is a division idempotent.

(ii) And those generated by absolute zero divisors. Concretely, if the characteristic of K divides n, then e is an absolute zero divisor, that is, $e(KG)e = 0$. In particular, e is nilpotent of nilindex 2.

In case KG is a semiprime algebra, notice that the left minimal ideals of the second type (ii) do not exist. This dichotomy appears also in our study, as the following theorem shows.

Theorem 4.3. *Let $\mathcal{G}$ be an ample groupoid and $x \in \mathcal{G}^{(0)}$. If $\{x\}$ is a compact open bisection such that $|x\mathcal{G}x| = n < \infty$, then the left ideal $I = A_K(\mathcal{G})\sum_{\alpha\in x\mathcal{G}x} 1_{\{\alpha\}}$ is minimal. Furthermore:*

1. *If the characteristic of K does not divide n, then $e := \frac{1}{n}\sum_{\alpha\in x\mathcal{G}x} 1_{\{\alpha\}}$ is an idempotent.*

2. *If the characteristic of K divides n, then $\sum_{\alpha\in x\mathcal{G}x} 1_{\{\alpha\}}$ is a nilpotent element of nilindex* 2.

Proof. Consider $x\mathcal{G}x = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Since $\{x\}$ is a compact open bisection, then $\{\alpha_i\}$ is a compact open bisection for every $i = 1, \dots, n$ and the previous ideal I is well defined. In order to prove that I is minimal, we have to verify that for every $f \in A_K(\mathcal{G})$ such that $f * \sum_{i=1}^n 1_{\{\alpha_i\}} \neq 0$ we can find $g \in A_K(\mathcal{G})$ such that $g * f * \sum_{i=1}^n 1_{\{\alpha_i\}} = \sum_{i=1}^n 1_{\{\alpha_i\}}$. Take $f = \sum_{l=1}^t b_l 1_{B_l}$ with B_l a compact open bisection and $b_l \in K$ for every $l = 1, \dots, t$. Since $f * \sum_{i=1}^n 1_{\{\alpha_i\}} = f * 1_{\{x\}} * \left(\sum_{i=1}^n 1_{\{\alpha_i\}}\right) \neq 0$, then $f * 1_{\{x\}} \neq 0$ and without loss of generality we can assume that $f * 1_{\{x\}} = f$, that is, $f = \sum_{l=1}^t b_l 1_{\{\delta_l\}}$ with $s(\delta_l) = x$.

After a reordering, we can divide this sum in groups such that

$$f = \sum_{l=1}^{t_1} b_l 1_{\{\delta_l\}} + \sum_{l=t_1+1}^{t_2} b_l 1_{\{\delta_l\}} + \cdots + \sum_{l=t_{m-1}+1}^{t_m} b_l 1_{\{\delta_l\}}$$

with $r(\delta_l) = x_k$ for every $t_k + 1 \leqslant l \leqslant t_{k+1}$ and $x_i \neq x_j$ if $i \neq j$. For instance the δ_l in the first addend ($l = 1, \dots, t_1$) have $r(\delta_l) = x_1$ so that $\delta_l = \gamma_1\alpha_k$ for some $1 \leqslant k \leqslant n$ and a fixed γ_1 whose source is x and range x_1 (take into account Lemma 4.1). In virtue of the mentioned Lemma, we can take $\{\gamma_1, \dots, \gamma_m\} \subset \mathcal{G}$ such that $r(\gamma_j) = x_j$ with $x_i \neq x_j$ for $i \neq j$ and $s(\gamma_j) = x$ for every $j = 1, \dots, m$ and write $f = \sum_{j=1}^m \sum_{k=1}^n a_{jk} 1_{\{\gamma_j\alpha_k\}}$ with $a_{jk} \in K$ for every $j = 1, \dots, m$ and $k = 1, \dots, n$. If we multiply $f * \sum_{i=1}^n 1_{\{\alpha_i\}}$, then we have that

$$f * \sum_{i=1}^n 1_{\{\alpha_i\}} = \sum_{j=1}^m \sum_{i,k=1}^n a_{jk} 1_{\{\gamma_j\alpha_k\alpha_i\}} = \sum_{j=1}^m \sum_{k=1}^n a_{jk} 1_{\{\gamma_j\}} \sum_{i=1}^n 1_{\{\alpha_k\alpha_i\}}.$$

The sum $\sum_{i=1}^n 1_{\{\alpha_k\alpha_i\}}$ depends of $k = 1, \dots, n$. However, it is just a reordering of the original sum $\sum_{i=1}^n 1_{\{\alpha_i\}}$ (since we are multiplying all the elements of the isotropy for a fixed element of the group) and we have that

$$f * \sum_{i=1}^n 1_{\{\alpha_i\}} = \left(\sum_{j=1}^m \left(\sum_{k=1}^n a_{jk}\right) 1_{\{\gamma_j\}}\right)\left(\sum_{i=1}^n 1_{\{\alpha_i\}}\right).$$

Since $f * \sum_{i=1}^{n} 1_{\{\alpha_i\}} \neq 0$ there is at least one j such that $\sum_{k=1}^{n} a_{jk} \neq 0$. Without loss of generality we can fix $j = 1$ and take $g = \frac{1}{\Omega} 1_{\{\gamma_1^{-1}\}}$ with $\Omega = \sum_{k=1}^{n} a_{1k} \neq 0$. Then we have that

$$g * f * \sum_{i=1}^{n} 1_{\{\alpha_i\}} = \frac{1}{\Omega}\Omega 1_{\{x\}} \left(\sum_{i=1}^{n} 1_{\{\alpha_i\}} \right) = \sum_{i=1}^{n} 1_{\{\alpha_i\}}.$$

If $\mathrm{char}(K) \nmid n$ the element e defined in the statement of the theorem is an idempotent

$$e^2 = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} 1_{\{\alpha_i \alpha_j\}} = \frac{1}{n^2} n \sum_{i=1}^{n} 1_{\{\alpha_i\}} = e,$$

otherwise $\left(\sum_{i=1}^{n} 1_{\{\alpha_i\}}\right)^2 = 0$. □

Corollary 4.4. *Consider $\mathcal{G}$ an ample groupoid and $x \in \mathcal{G}^{(0)}$ with $\{x\}$ a compact open bisection such that $x\mathcal{G}x = \{x\}$, then $A_K(\mathcal{G})1_{\{x\}}$ is a minimal left ideal.*

In an associative algebra with local units, like a Steinberg algebra, minimal left (resp. right) ideals are of the form $A_K(\mathcal{G})a$ (resp. $aA_K(\mathcal{G})$) with $a \in A_K(\mathcal{G})$. As a first thought, the minimality of the ideal make us think that the element a has to be 'very small'. Our intuition says that if the compact open bisections implied in the decomposition of a are very small (very few elements in them), then our element is small and possibly the ideal that it generates is minimal. However, there is the possibility that the ideal is not generated by a 'small' element but it is isomorphic to one. In this sense, we describe a result analogous to [27, Lemma 3.1].

Lemma 4.5. *Consider $\mathcal{G}$ a Hausdorff ample groupoid and $A_K(\mathcal{G})a$ is a minimal left ideal of $A_K(\mathcal{G})$. There is a set $C \in B^{co}(\mathcal{G}^{(0)})$ and an element $b \in 1_C A_K(\mathcal{G}) 1_C$ such that $A_K(\mathcal{G})a \cong A_K(\mathcal{G})b$ are isomorphic as left-modules.*

Proof. Consider $a = \sum_{i=0}^{n} a_i 1_{B_i}$ with B_i non-empty compact open bisections and $B_i \cap B_j = \varnothing$ for every $i \neq j$. If we multiply a by $1_{B_0^{-1}}$, then we have that $1_{B_0^{-1}} a = a_0 1_{s(B_0)} + \sum_{i=1}^{n} a_i 1_{B_0^{-1} B_i}$. Let us see that all the compact open bisection implied in this new element are disjoint, which will imply that $1_{B_0^{-1}} a \neq 0$. Suppose that $B_0^{-1} B_i \cap B_0^{-1} B_j \neq \varnothing$, then there exist $\gamma, \delta \in B_0$, $\alpha \in B_i$ and $\beta \in B_j$ such that $\gamma^{-1}\alpha = \delta^{-1}\beta$. Hence, $r(\gamma^{-1}) = r(\delta^{-1})$, that is, $s(\gamma) = s(\delta)$ and since both of them are elements of the same compact open bisection we have that $\gamma = \delta$. In consequence, $\alpha = \beta$ but $B_i \cap B_j = \varnothing$ and this is not possible. Furthermore, we have that $r(B_0^{-1} B_i) \subseteq r(B_0^{-1}) = s(B_0) =: C$, so we can consider without loss of generality that $a = a_0 1_C + \sum_{i=1} a_i 1_{B_i}$ with $C \in B^{co}(\mathcal{G}^{(0)})$, $C \cap B_i = \emptyset$, $B_i \cap B_j = \emptyset$ for every $i \neq j$ and $r(B_i) \subseteq C$. Hence, we have that $1_C a = a$. If we compute $1_C a 1_C$, then $1_C a 1_C = a_0 1_C + \sum_{i=1}^{n} a_1 1_{B_i C} \neq 0$ because all of the compact open bisections involved in this sum are disjoint (using the same argument as we did before). If we take $b = 1_C a 1_C \in 1_C A_K(\mathcal{G}) 1_C$, we have that the map $R_{1_C} \colon A_K(\mathcal{G}) 1_C a \to A_K(\mathcal{G}) b$ such that $R_{1_C}(f) = f 1_C$ is an isomorphism. It is injective because $A_K(\mathcal{G}) 1_C a$ is minimal. In addition, it is surjective by construction. □

Under the conditions of the previous Lemma, we can prove what we were expecting. Minimal left (resp. right) ideals are related to compact open singletons.

Proposition 4.6. *Let us consider $\mathcal{G}$ a Hausdorff ample groupoid and $I = A_K(\mathcal{G})a$ a minimal left ideal, then there is an element $u \in \mathcal{G}^{(0)}$ such that $\{u\}$ is a compact open bisection with $u \in \mathrm{supp}(a)$.*

Proof. Since I is a minimal left ideal we have that $I = A_K(\mathcal{G})a$ with $a \in I$. Without loss of generality, we can consider a as in the proof of Lemma 4.5, that is, $a = a_0 1_C + \sum_{i=1}^{l} a_i 1_{B_i}$ with $\neq C \in B^{co}(\mathcal{G}^{(0)})$ and $a_i \neq 0$ for $i = 0, 1, \dots, l$. In other words, there is a unit $u \in C \subset \mathrm{supp}(a)$. We will prove that $\{u\}$ is open, thus it will be a compact open bisection. Suppose that $\{u\}$ is not open, then $\mathcal{G}^{(0)} \backslash \{u\}$ is not closed. Since $\mathcal{G}$ is Hausdorff (and $\mathcal{G}^{(0)}$ is closed) we can consider that there is a sequence $(x_n)_{n \geqslant 1} \subset \mathcal{G}^{(0)} \backslash \{u\}$ such that $x_n \to u$ and $x_i \neq x_j$ for every $i \neq j$. Taking into account that $\mathrm{supp}(a) \cap \mathcal{G}^{(0)}$ is an open neighbourhood of u and $x_n \to u$ we can assume that $(x_n)_{n \geqslant 1} \subset \mathcal{G}^{(0)} \cap \mathrm{supp}(a) \backslash \{u\}$. Take $x_1 \neq u$, since $\mathcal{G}$ is Hausdorff there exist V_1 and U_1 two compact open bisections such that $x_1 \in V_1 \subset \mathcal{G}^{(0)} \cap \mathrm{supp}(a)$, $u \in U_1 \subset \mathcal{G}^{(0)} \cap \mathrm{supp}(a)$ and $V_1 \cap U_1 = \emptyset$. Then, we have that $1_{V_1} a \neq 0$ and $1_{U_1} a \neq 0$. Since the ideal I is minimal, this implies that $I = A_K(\mathcal{G}) 1_{V_1} a = A_K(\mathcal{G}) 1_{U_1} a$. In particular, there is a function $g \in A_K(\mathcal{G})$ such that $g 1_{V_1} a = 1_{U_1} a$. If we evaluate both sides in u we have that $1_{U_1} a(u) = a(u) \neq 0$, then $g 1_{V_1} a(u) = \sum_{u = \gamma\beta\delta} g(\gamma) 1_{V_1}(\beta) a(\delta) \neq 0$. Taking into account that the sum is different from zero, this means that there is some $\gamma, \beta, \delta \in G$ with $u = \gamma\beta\delta$ and $g(\gamma) 1_{V_1}(\beta) a(\delta) \neq 0$. This necessarily means that $\beta \in V_1 \subset G^{(0)} \backslash U_1$ and we have that $\alpha_1 := \delta$ and $\alpha^{-1} := \gamma$. In summary, we have found an element $\alpha_1 \in \mathcal{G}$ such that $s(\alpha_1) = u$, $r(\alpha_1) \in V_1$ and $g(\alpha_1^{-1}) a(\alpha_1) \neq 0$, that is, $\alpha_1 \in \mathrm{supp}(a)$. In addition, there is an element n_1 such that $x_n \in U_1$ for every $n \geqslant n_1$ and we can repeat the same process as we did before. In this way, we construct $V_2, U_2 \in B^{co}(\mathcal{G}^{(0)})$ such that $x_{n_1} \in V_2$, $u \in U_2$, $V_2 \cap U_2 = \emptyset$ and $U_2 \cup V_2 \subset U_1$. Again, by the minimality of the ideal I this implies that there is some $\alpha_2 \in \mathcal{G}$ such that $s(\alpha_2) = u$ and $r(\alpha_2) \in V_2$

and $a(\alpha_2) \neq 0$. If we iterate this process infinitely many times, we can construct a sequence $(\alpha_n)_{n\geqslant 1} \subseteq \mathrm{supp}(a)$ with $\alpha_n \neq \alpha_m$ for $n \neq m$, $s(\alpha_n) = u$ and $r(\alpha_n) \in V_n$ for every $n \in V_n$. However, $a = a_0 1_C + \sum_{i=1}^{l} a_i 1_{B_i}$ and its support contains at most $l+1$ different elements with source u which is a contradiction. □

Corollary 4.7. *If $\mathcal{G}$ is a Hausdorff ample groupoid and I is a minimal left ideal of $A_K(\mathcal{G})$, then there is a compact open singleton $\{x\} \subset \mathcal{G}^{(0)}$ such that $I \cong A_K(\mathcal{G})a$ with $a \in 1_{\{x\}} A_K(\mathcal{G}) 1_{\{x\}}$.*

Proof. Consider $I = A_K(\mathcal{G})b$ a minimal left ideal where $b = \sum_{i=1}^{n} r_i 1_{B_i} \neq 0$ with $r_i \in K$, B_i compact open bisection such that $B_i \cap B_j = \varnothing$. By Proposition 4.6 there is a unit $x \in \mathrm{supp}(a)$ such that $\{x\}$ is a compact open bisection. Thus, $a = 1_{\{x\}} b 1_{\{x\}} \neq 0$ since $a = \sum_{i=1}^{n} r_i 1_{xB_i x}$ with $xB_i x \cap xB_j x = \varnothing$. Taking into account the minimality of the ideal we have that

$$I = A_K(\mathcal{G})b = A_K(\mathcal{G})1_{\{x\}}b \cong A_K(\mathcal{G})1_{\{x\}}b1_{\{x\}} = A_K(\mathcal{G})a.$$

□

On the one hand, Proposition 4.6 implies that the existence of minimal left ideals is related to the presence of compact open singletons. On the other hand, Theorem 4.3 implies that under certain conditions of the isotropy group, we can form minimal left ideals. However, not every compact open singleton allows the existence of a minimal left ideal. As an example, we will see that if $\{x\} \subset \mathcal{G}^{(0)}$ is a compact open bisection such that $x\mathcal{G}x \cong \mathbb{Z}$, then $A_K(\mathcal{G})1_{\{x\}}$ is not a minimal left ideal. This result and its corresponding proof, is a translation of [11, Proposition 2.5] into the Steinberg context. In order to prove that, we first need the next Lemma.

Lemma 4.8. *If $\mathcal{G}$ is an ample groupoid and $\{x\} \in B^{co}(\mathcal{G}^{(0)})$ with $x\mathcal{G}x \cong \mathbb{Z}$, then $1_{\{x\}} A_K(\mathcal{G}) 1_{\{x\}} \cong K[y, y^{-1}]$ as K-algebras.*

Proof. If $x\mathcal{G}x \cong \mathbb{Z}$, then there is an element $\gamma \in x\mathcal{G}x$ such that generates all the group. That is, for every $a \in x\mathcal{G}x$, $a = \gamma^k$ with $k \in \mathbb{Z}$. Finally, it is enough to just consider the K-algebra isomorphism $\phi\colon 1_x A_K(\mathcal{G}) 1_x \to K[y, y^{-1}]$ with $\phi(1_{\{\gamma\}}) = y$ and $\phi(1_{\{\gamma^{-1}\}}) = y^{-1}$. □

Proposition 4.9. *Let $\mathcal{G}$ be an ample Hausdorff groupoid. If we consider that $\{x\}$ is a compact open bisection with $x\mathcal{G}x \cong \mathbb{Z}$, then there is not any element $a \in 1_{\{x\}} A_K(\mathcal{G}) 1_{\{x\}}$ such $A_K(\mathcal{G})a$ is a minimal left ideal.*

Proof. Let us consider $\gamma \in x\mathcal{G}x \cong \mathbb{Z}$ the generator of the isotropy group. The element a can be written as $a = a_n 1_{\{\gamma^n\}} + a_{n-1}1_{\{\gamma^{n-1}\}} + \cdots + a_0 1_{\{x\}} + \cdots + a_{-m}1_{\{\gamma^{-m}\}}$ with $a_i \in K$ and $a_n \neq 0$. If $A_K(\mathcal{G})a$ is a minimal left ideal, then it coincides with $A_K(\mathcal{G})1_{\{\gamma^m\}}a$, so we can consider that a is a combination of elements of positive degree, that is, $a = \sum_{i=0}^n a_i 1_{\{\gamma^i\}}$. If the ideal $I = A_K(\mathcal{G})a$ is minimal, then every nonzero element $f \in A_K(\mathcal{G})$ such that $fa \neq 0$ generates the ideal I. In other words, there is an element $g \in A_K(\mathcal{G})$ such that $gfa = a$. If we express a as a polynomial in the same way as Lemma 4.8 using the morphism ϕ, then we have that $q(y) := \phi(a) = \sum_{i=0}^n a_i y^i$ and we can consider $y - b$ a polynomial that does not divide $\phi(a)$. We then multiply a times $f := 1_{\{\gamma\}} - b1_{\{x\}}$ and $fa \neq 0$. In consequence, we need to find $g \in A_K(\mathcal{G})$ such that $gfa = a$. If this element g exists then $1_{\{x\}}g1_{\{x\}}ga = a$ and we can consider that $g \in 1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$. Using once again Lemma 4.8 we have the equality, $p(y, y^{-1})(y - b)q(y) = q(y)$. Multiplying both elements by an appropriate power of y, we have that $y^s p(y, y^{-1}) = r(y)$ and $r(y)(y - b)q(y) = y^s q(y)$ which is not possible (no matter the g chosen). $\square$

As a summary of what we have already proven, we know that minimal left ideals are related to open compact singleton. If we have a minimal left ideal, then it will be isomorphic to one in a family of minimal left ideals that we can construct under some conditions in the isotropy groups. However, one of the question that we have is if this is enough. The answer is negative, we still need some properties in the groupoid to be able to characterise all minimal left ideals. This is because Steinberg algebras generalise group algebras, and characterising the socle of these algebras, in general, is tough.

Remark 4.10. Consider $\mathcal{G}$ a Hausdorff ample groupoid, $\{x\} \in B^{co}(\mathcal{G}^{(0)})$ and $I = A_K(\mathcal{G})a$ a minimal left ideal with $a \in 1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$. Take

$$J = 1_{\{x\}}I1_{\{x\}} = 1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}a \neq \{0\},$$

then J is a minimal left ideal of the algebra $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$. The fact that J is a left ideal of $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$ is a direct consequence of

$$1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}J = 1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}1_{\{x\}}I1_{\{x\}} = 1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}I1_{\{x\}} \subset 1_{\{x\}}I1_{\{x\}} = J.$$

Let us verify that J is minimal. Consider $f \in 1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$ such that $fa \neq 0$, since I is a minimal ideal, there is an element $g \in A_K(\mathcal{G})$ such that $gfa = a$ and since $1_{\{x\}}f = f$ and $1_{\{x\}}a = a$, this means that $1_{\{x\}}g1_{\{x\}}fa = a$. Thus, we can take $h = 1_{\{x\}}g1_{\{x\}} \in 1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$ such that $hga = a$. In addition, $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$ is an unital algebra with unit $1_{\{x\}}$. This fact, together with the previous equality imply that J is a minimal left ideal of $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$. As a summary, if the algebra $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$ does not contain minimal left ideals, then I can not be minimal. In this way, we have a necessary condition. The next we ask ourselves is if there is any characterisation of the minimal left ideals of the algebra $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$. The answer of that depends on the structure of the isotropy group $x\mathcal{G}x$.

Remark 4.11. Taking into account [3, Proposition 2.3], we have that if $\mathcal{G}$ is an ample groupoid and $x \in \mathcal{G}^{(0)}$ with $\{x\}$ a compact open bisection, then the algebra $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$ is isomorphic to $A_K(x\mathcal{G}x)$, that is, the Steinberg algebra of the isotropy group $x\mathcal{G}x$.

Remark 4.10 and Remark 4.11 imply that the minimal ideals of a Steinberg algebra are related to the minimal ideals of a group algebra. In general, finding the minimal ideals of a group algebra is a problem that has not been resolved yet. There are partial answers to that but there is not any complete characterisation. The reader can find some of them in [16, 45, 90]. In this sense, and with the objective of generalise the socle of the Leavitt path and Kumjian-Pask algebras, we are going to focus on the case that $\mathcal{G}$ is a Hausdorff ample groupoid such that for every $x \in \mathcal{G}^{(0)}$, $\operatorname{soc}(A_K(x\mathcal{G}x)) \neq \{0\}$ if and only if $x\mathcal{G}x = \{x\}$.

4.2 THE SOCLE OF A STEINBERG ALGEBRA

In this section we are going to give the final characterisation of the socle of a Steinberg algebra. Remember that we consider groupoids such that for every $x \in \mathcal{G}^{(0)}$, $\operatorname{soc}(A_K(x\mathcal{G}x)) \neq \{0\}$ if and only if $x\mathcal{G}x = \{x\}$. In case of a Leavitt path algebra of a directed graph E, if we consider the path groupoid $\mathcal{G}_E$, for every element $x \in \mathcal{G}^{(0)}$ we have that the isotropy group is $\{x\}$ or isomorphic to $\mathbb{Z}$. However, when $x\mathcal{G}x \cong \mathbb{Z}$, we have that $\operatorname{soc}(K\mathbb{Z}) = \{0\}$ and we are under the claimed condition. Similarly happens to the Kumjian-Pask algebras.

Theorem 4.12. *Let $\mathcal{G}$ be a Hausdorff ample groupoid such that* $\operatorname{soc}(A_K(x\mathcal{G}x)) \neq 0$ *if and only if $x\mathcal{G}x = \{x\}$. Then the left socle of the algebra $A_K(\mathcal{G})$ is the ideal*

$$\operatorname{soc}(A_K(\mathcal{G})) = \left(1_{\{x\}} \colon \{x\} \in B^{co}(\mathcal{G}^{(0)}), x\mathcal{G}x = \{x\}\right). \tag{5}$$

If $I_x = A_K(\mathcal{G})1_{\{x\}}$ is a minimal left ideal, then its corresponding homogenous component is

$$M = \sum_{I \cong I_x} I = (1_{\{x\}}) \tag{6}$$

where $(1_{\{x\}})$ is the two-sided ideal generated by $1_{\{x\}}$.

Proof. Let I be a minimal left ideal. We can consider that $I = A_K(\mathcal{G})a$ and by Proposition 4.6 there is a compact open bisection $\{x\}$ with $x \in \operatorname{supp}(a)$. This means that $1_{\{x\}}a \neq 0$ and by minimality, we have that $I = A_K(\mathcal{G})1_{\{x\}}a$. Not only that but $I \cong A_K(\mathcal{G})1_{\{x\}}a1_{\{x\}} = J$ as $A_K(\mathcal{G})$-modules. Since J is a minimal left $A_K(\mathcal{G})$-ideal, $1_{\{x\}}J1_{\{x\}}$ it is also a minimal left $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}$-ideal. In consequence, by Remark 4.11, $\operatorname{soc}(A_K(x\mathcal{G}x)) = \operatorname{soc}(1_{\{x\}}A_K(\mathcal{G})1_{\{x\}}) \neq \{0\}$ which is only possible if and only if $x\mathcal{G}x = \{x\}$. To sum up, we have that $I = A_K(\mathcal{G})1_{\{x\}}a$ with $\{x\} \in B^{co}(\mathcal{G}^{(0)})$ and $x\mathcal{G}x = \{x\}$, then $I \subseteq \left(1_{\{x\}} \colon \{x\} \in B^{co}(\mathcal{G}^{(0)}), x\mathcal{G}x = \{x\}\right)$ and $\operatorname{soc}(A_K(\mathcal{G})) \subseteq \left(1_{\{x\}} \colon x\mathcal{G}x = \{x\}\right)$.

Now consider $x \in \mathcal{G}^{(0)}$ with $\{x\} \in B^{co}(\mathcal{G}^{(0)})$ and $x\mathcal{G}x = \{x\}$. Then, by Corollary 4.4 we have that $I_x = A_K(\mathcal{G})1_{\{x\}}$ is a minimal left ideal and $I_x \subset \operatorname{soc}(A_K(\mathcal{G}))$. Since $1_{\{x\}} \in A_K(\mathcal{G})1_{\{x\}} \subseteq \operatorname{soc}(A_K(\mathcal{G}))$ and the socle is a two-sided ideal, we have that $\left(1_{\{x\}} \colon x\mathcal{G}x = \{x\}\right) \subseteq \operatorname{soc}(A_K(\mathcal{G}))$.

In order to prove the second part take an ideal I such that $I \cong I_x$. As we have mentioned before, we can consider $I = A_K(\mathcal{G})a$ with $a \in I$. If $\phi \colon I \to I_x$ is a $A_K(\mathcal{G})$-module isomorphism, then we have that $\phi(a) = f1_{\{x\}}$ and $\phi^{-1}(1_{\{x\}}) = ga$. Making use of this we get $a = \phi^{-1}(\phi(a)) = \phi^{-1}(f1_{\{x\}}) = \phi^{-1}(f1_{\{x\}}1_{\{x\}}) = f1_{\{x\}}\phi^{-1}(1_{\{x\}}) = f1_{\{x\}}ga \in (1_{\{x\}})$ and $I \subset (1_{\{x\}})$. As a consequence, $M \subset (1_{\{x\}})$.

For the counterpart, since I_x is a minimal ideal we have that $I_x \subset M$. And because M is a two-sided ideal we have that

$$(1_{\{x\}}) = A_K(\mathcal{G})1_{\{x\}}A_K(\mathcal{G}) = I_xA_K(\mathcal{G}) \subset M.$$

□

Remark 4.13. It is important to highlight that we can be more precise with the description of the socle. If we define the equivalence relation $\mathcal{R}$ as $x\mathcal{R}y$ if and only if there is $\gamma \in \mathcal{G}$ such that $r(\gamma) = x$ and $s(\gamma) = y$. In this case, we have that if $\{x\} \in B^{co}(\mathcal{G}^{(0)})$ and $x\mathcal{R}y$ then $(1_{\{x\}}) = (1_{\{y\}})$. If $x\mathcal{R}y$, then there is γ with $r(\gamma) = x$

and $s(\gamma) = y$. The rest is immediate from the fact that $1_{\{x\}} = 1_{\{\gamma\}}1_{\{y\}}1_{\{\gamma^{-1}\}}$ and $1_{\{y\}} = 1_{\{\gamma^{-1}\}}1_{\{x\}}1_{\{\gamma\}}$. As a consequence, we can describe the socle as

$$\mathrm{soc}(A_K(\mathcal{G})) = (1_{\{x\}} \colon [x] \in \mathcal{G}^{(0)}/\mathcal{R}, x\mathcal{G}x = \{x\}).$$

In the description of the homogeneous components, there are elements that generates the ideal that are redundant. We can give a more precise result as follows.

Proposition 4.14. *Let $\mathcal{G}$ be a Haussdorff ample groupoid such that for every $x \in \mathcal{G}^{(0)}$ we have that $\mathrm{soc}(A_K(x\mathcal{G}x)) \neq 0$ if and only if $x\mathcal{G}x = \{x\}$. If M is the homogeneous component of the socle corresponding to $[x]$ then*

$$M = (1_{\{x\}}) = \bigoplus_{y \in [x]} A_K(\mathcal{G})1_{\{y\}}.$$

Proof. First we check that it is a direct sum:

$$(A_K(\mathcal{G})1_{\{y\}}) \cap \left(\sum_{y_i \neq y} A_K(\mathcal{G})1_{\{y\}} \right) = \{0\}$$

for every $y, y_i \in [x]$ with $y \neq y_i$ for each i. If there is an element in the intersection, this means that there are $f, g_i \in A_K(\mathcal{G})$ such that $f1_{\{y\}} = \sum_{y \neq y_i} g_i 1_{\{y_i\}}$. Multiplying both sides to the right by $1_{\{y\}}$ we get that $f1_{\{y\}} = \sum_{y \neq y_i} g_i 1_{\{y_i\}}1_{\{y\}} = 0$ as we claimed. Define $J = \bigoplus_{y \in [x]} A_K(\mathcal{G})1_{\{y\}}$. Now, we will prove $M = J$. Consider $f1_{\{y\}} \in J$ and $\gamma \in y\mathcal{G}x$ ($x\mathcal{G}y \neq \emptyset$ because y belongs to the orbit of x), we have that $f1_{\{y\}} = f1_{\{\gamma\}}1_{\{x\}}1_{\{\gamma^{-1}\}} \in M$. As a consequence, $J \subseteq M$.

Take $0 \neq f1_{\{x\}}g \in M$, in particular $1_{\{x\}}g \neq 0$ and without loss of generality we can assume $1_{\{x\}}g = g = \sum_{i=1}^{n} a_i 1_{\{\delta_i\}}$ with $r(\delta_i) = x$ and $s(\delta_i) = y_i \in [x]$. Thus, $f1_{\{x\}}g = \sum_{i=1}^{n} a_i f 1_{\{\delta_i\}}1_{\{y_i\}} \in J$. Then $M \subseteq J$. □

Remark 4.15. Following [91, Proposition 2.8] the map $(\sum_i r_i 1_{B_i})^* := \sum_i r_i 1_{B_i^{-1}}$ is an involution on $A_K(\mathcal{G})$, hence taking into account Proposition 4.14 the homogeneous component $M = (1_{\{x\}})$ satisfies $M^* = M$, so the left socle is self-adjoint. Since the involution applied to the left socle is the right socle, then we have the coincidence of both socles.

Assume that $\mathcal{G}$ is a Hausdorff ample groupoid with $\{x\} \in B^{\mathrm{co}}(\mathcal{G}^{(0)})$ such that $x\mathcal{G}x = \{x\}$. Then we know that $1_{\{x\}}A_K(\mathcal{G})1_{\{x\}} = K1_{\{x\}}$. In addition it is easy to check that $\mathrm{End}_{A_K(\mathcal{G})}(A_K(\mathcal{G})1_{\{x\}}) \cong K1_{\{x\}}$ with isomorphism Ω : $\mathrm{End}_{A_K(\mathcal{G})}(A_K(\mathcal{G})1_{\{x\}}) \to K1_{\{x\}}$ given by $\Omega(\phi) = \phi(1_{\{x\}})$.

By using [65, Theorem 1, Chapter IV, §3] the homogeneous components of the socle of a semiprime algebra A (with minimal left ideals), are simple algebras with nonzero socle. In fact, the minimal left ideals are of the form $I = Ae$ for a division idempotent. Then $\Delta := eAe$ is a division algebra e. If A is simple then $A \cong \mathrm{End}_\Delta(I)$. In the particular case of a homogeneous component $M = (1_{\{x\}})$ of a semiprime Steinberg algebra with nonzero socle we have seen that $\mathrm{End}_{A_K(\mathcal{G})}(A_K(\mathcal{G})1_{\{x\}}) = \Delta \cong K$. So $M \cong \mathrm{End}_K(I)$ is of the form $\mathcal{M}_n(K)$ where n is finite or infinite. We know that $n = \dim_K(I)$.

Corollary 4.16. *Let $\mathcal{G}$ be a Hausdorff ample groupoid such that $A_K(\mathcal{G})$ is semiprime and* $\mathrm{soc}(A_K(x\mathcal{G}x)) \neq 0$ *if and only if $x\mathcal{G}x = \{x\}$. Then*

$$\mathrm{soc}(A_K(\mathcal{G})) \cong \bigoplus_{[x]\in\mathcal{G}^{(0)}/\mathcal{R}:\, x\mathcal{G}x=\{x\}} \mathcal{M}_{|[x]|}(K). \tag{7}$$

Proof. By [65, Theorem 1, Chapter IV, §3] the homogeneous components of the socle are simple algebras. The socle is a completely reducible module in the terminology of Jacobson, hence [65, Corollary, Chapter IV, §2] gives that the socle is the direct sum of its homogeneous components which are of the form $\mathrm{End}_K(A_K(\mathcal{G})1_{\{x\}}) \cong \mathcal{M}_n(K)$ where $n = \dim(A_K(\mathcal{G})1_{\{x\}})$ (possibly infinite). In order to compute n, note that the set $C = \{1_{\{\gamma\}} \colon \gamma \in s^{-1}(x)\}$ is a system of generators of $A_K(\mathcal{G})1_{\{x\}}$. Indeed, for any $B \in B^{\mathrm{co}}(\mathcal{G})$ we have $1_B 1_{\{x\}} = 1_{B\{x\}} = 1_{\{\gamma\}}$ for some $\gamma \in s^{-1}(x)$ or $1_B 1_{\{}x\} = 0$. So let us prove that C is a linearly independent set. Assume $\sum_{i=1}^m r_i 1_{\{\gamma_i\}} = 0$ where the $\gamma_i \in s^{-1}(x)$ for every $i = 1, \ldots, m$. Take one γ_k of those in the previous sum and compute $1_{\{\gamma_k^{-1}\}} \sum_{i=1}^m r_i 1_{\{\gamma_i\}} = 0$. We have $0 = \sum_i r_i 1_{\{\gamma_k^{-1}\}} 1_{\{\gamma_i\}} = \sum_i r_i 1_{\{\gamma_k^{-1}\gamma_i\}} = r_k 1_{\{x\}}$ implying $r_k = 0$. Thus, C is a basis of the ideal $A_K(\mathcal{G})1_{\{x\}}$. Then the homogeneous component

$$\mathrm{End}_K(A_K(\mathcal{G})1_{\{x\}}) \cong \mathcal{M}_n(K).$$

□

Chapter 5

PERFECTION CONDITIONS

Once we have characterized certain finiteness conditions on KE, such as Artinianity and semiartinianity, directly from the geometry of the graphs, we now pursue to describe perfection algebraic properties of KE. In this chapter, as a main goal we will study simple, primitive, prime and semiprime path algebras. And again, we will do it through conditions on their corresponding graphs. In general, perfection conditions are related, being each one of them stronger/weaker than the others. In general, the relation is as follows:

$$\text{Simple} \implies \text{Primitive} \implies \text{Prime} \implies \text{Semiprime}.$$

These properties will be translated into how strongly is the graph connected. From being just one vertex to having different strongly connected components.

5.1 PRIME AND SEMIPRIME PATH ALGEBRAS

Definition 5.1. An associative algebra A is said *semiprime* if and only if for $a \in A$ such that $aAa = 0$, then $a = 0$. In other words, if $a \neq 0$, then $aAa \neq 0$. It is said *prime* if and only if for every $a, b \in A$ with $aAb = 0$, then $a = 0$ or $b = 0$. That is, if $a \neq 0$ and $b \neq 0$, then $aAb \neq 0$.

These conditions are well known to be equivalent to:

1. For any ideal I of A, $I^2 = 0$ implies $I = 0$ (semiprimeness).
2. For any two ideals I and J of A, $IJ = 0$ implies $I = 0$ or $J = 0$ (primeness).

Remark 5.2. In case of the algebra A is graded, there is another characterisation of primeness and semiprimeness in terms of the homogeneous components. Consider $A = \bigoplus_{m \in \mathbb{Z}} A_m$ a $\mathbb{Z}$-graded associative algebra. The algebra A is semiprime if and only if $a \in A_m$ with $aAa = 0$ implies $a = 0$ for every $m \in \mathbb{Z}$. In addition, A is prime if and only if $a \in A_n, b \in A_m$ with $aAb = 0$ implies $a = 0$ or $b = 0$ for every $n, m \in \mathbb{Z}$.

As we have already mentioned, the next result is proved by M. Siles Molina in [95] in the Leavitt path algebras context. In this case, we prove it just making use of the path algebra.

Proposition 5.3. *Let E be a directed graph and K a field. The path algebra KE is semiprime if and only if for every path μ in E there is another path λ with $s(\lambda) = r(\mu)$ and $r(\lambda) = s(\mu)$. In other words, if there is a path μ from u to v, with $u, v \in E^0$, then there is another path λ from v to u.*

Proof. First, consider $A = KE$ semiprime and take any path μ. By definition, this path verifies that $\mu A \mu \neq 0$. This is only possible if there is a path λ with $\mu\lambda\mu \neq 0$ ($s(\lambda) = r(\mu)$ and $r(\lambda) = s(\mu)$).

On the other hand, consider that for every path μ from u to v there is another path λ from v to u and let us take an homogeneous element $a \in A_m$ different from zero. We can consider $a = \sum_{i=1}^{n} k_i \mu_i$ with $k_i \in K \backslash \{0\}$, $\ell(\mu_i) = m$ for every $i = 1, \ldots, n$ and $\mu_i \neq \mu_j$ for $i \neq j$. Fix a certain path μ_l with $1 \leqslant l \leqslant n$. As our hypothesis says, there is another path λ with $s(\lambda) = r(\mu_l)$ and $r(\lambda) = s(\mu_l)$. If we compute $a\lambda a$, then we have that

$$a\lambda a = \left(\sum_{i=1}^{n} k_i \mu_i \right) \lambda \left(\sum_{j=1}^{n} k_j \mu_j \right) = \sum_{i,j=1}^{n} k_i k_j \mu_i \lambda \mu_j = \sum_{(i,j) \in J} k_i k_j \mu_i \lambda \mu_j,$$

where $J = \{(i, j) \colon r(\mu_i) = r(\mu_l), s(\mu_j) = s(\mu_l)\}$. The set $J \neq \emptyset$ because $(l, l) \in J$. Also, $k_i k_j \neq 0$ and all the paths $\mu_i \lambda \mu_j$ are different. Suppose that $\mu_i \lambda \mu_j = \mu_{i'} \lambda \mu_{j'}$. Since all the paths have the same length, this implies that $\mu_i = \mu_{i'}$ and $\mu_j = \mu_{j'}$ which is only possible if $i = i'$ and $j = j'$. We have then a non trivial linear combination of different paths (linearly independents) and so $a\lambda a = \sum_{(i,j)\in J} k_i k_j \mu_i \lambda \mu_j \neq 0$. This implies that $aAa \neq 0$ and A is semiprime following Remark 5.2. □

Now let us focus on primeness of KE. Again, there is an approach (in a specific ambient) in [95, Theorem 3.5]. On the other hand, the reference [32] mentions, but not proves, that primeness of KE is equivalent to the strongly connectedness of the graph E. Also P. Smith in [97] proves the equivalence of primeness of KE with the strongly connectedness of E. Though the ambient graph in the aforementioned references is supposed to be finite, the statement in [97, Lemma 2.11] is also true for arbitrary graphs. This reference is not easily accessible and as far as we know has not been published in a regular journal or book. That is why we consider suitable to include an ample proof of the result.

Proposition 5.4. *Let E be a directed graph and K a field. The path algebra KE is prime if and only if for every pair of vertices $u, v \in E^0$ there is a path λ with $s(\lambda) = u$ and $r(\lambda) = v$. That is KE is prime if and only if E is strongly connected.*

Proof. Let us consider that $A = KE$ is a prime algebra and take $u, v \in E^0$, then we have $uAv \neq 0$ and necessarily there is a path μ with $u\lambda v \neq 0$. If not, $uAv = 0$ which enters in contradiction with the algebra being prime. As a consequence, $s(\lambda) = u$ and $r(\lambda) = v$.

On the other hand, take into account that for every pair of vertices there is a path that connect them. Consider $a \in A_n$ and $b \in A_m$ with $a, b \neq 0$. These two elements can be described as $a = \sum_{i=1}^{l} k_i \mu_i$ and $b = \sum_{j=1}^{l'} k'_j \nu_j$ with $k_i, k'_j \in K \setminus \{0\}$ for all $i = 1, \ldots, l$ and $j = 1, \ldots, l'$. Also, all the paths in a are different but with the same length n and all the paths in b are different with the same length m. Take two paths μ_{i_0} and ν_{j_0} in the respective linear combinations. Consider $u = r(\mu_{i_0})$ and $v = s(\nu_{j_0})$. As our hypothesis says, there is a path λ with $s(\lambda) = u$ and $r(\lambda) = v$. Then,

$$a\lambda b = \left(\sum_{i=1}^{l} k_i \mu_i\right) \lambda \left(\sum_{j=1}^{l'} k'_j \nu_j\right) = \sum_{i,j=1}^{l,l'} k_i k'_j \mu_i \lambda \nu_j = \sum_{(i,j)\in J} k_i k'_j \mu_i \lambda \nu_j.$$

where $J = \{(i,j) : r(\mu_i) = s(\lambda), s(\nu_j) = r(\lambda)\}$. We have that $J \neq \emptyset$ because $(i_0, j_0) \in J$. Besides, all the paths $\mu_i\lambda\nu_j$ are different for the different pairs (i,j). If $\mu_i\lambda\nu_j = \mu_{i'}\lambda\nu_{j'}$, since $\ell(\mu_i) = n$ for every $i = 1, \ldots, n$, this implies that, $\mu_i = \mu_{i'}$ and $\nu_j = \nu'_j$ which is only possible if $i = i'$ and $j = j'$. In consequence, we have a nontrivial linear combination of different paths, then $a\lambda b \neq 0$ and $aAb \neq 0$. Following Remark 5.2, this means that A is prime. □

5.2 SIMPLE AND PRIMITIVE PATH ALGEBRAS

Let E be a connected directed graph and KE its corresponding path algebra over the field K. If we consider that E is formed by just one vertex without edges, that is, $E^0 = \{u\}$ and $E^1 = \emptyset$, it is easy to see that $KE = Ku \cong K$ and then, A is simple. Conversely, if E is a directed graph with $A = KE$ being simple. Then, take $u \in E^0$ and define the two-sided ideal $I = AuA$. Under our hypothesis, this means that $I = 0$ or $I = A$. It is clear that $I \neq 0$ because $Ku \subset I$. If there is another vertex $v \in E^0$ with $v \neq u$, then we have that $v \notin I$ and $I \neq A$. In addition, if there is an edge $f \in E^1$, necessarily $r(f) = s(f) = u$ and the ideal $J = AfA \neq 0$ is different from A since $u \notin J$. Finally, $E^0 = \{u\}$ and $E^1 = \emptyset$. In this way, we have that the algebra KE is simple if and only if the graph E is just a vertex.

Once we have characterized prime, semiprime and simple path algebras, the next logic step is to study primitive path algebras. The associative context of the path algebras allow us to use two different characterizations of primitive algebras. One in terms of representations (modules) and the other one in terms of ideals.

Definition 5.5. Consider R a ring, a left R-module M is said to be *simple* or *irreducible* if for every left R-module N such that $\{0\} \subseteq N \subseteq M$ then $N = \{0\}$ or $N = M$. In addition, we say that M is *faithful* if the only $r \in R$ such that $rM = \{0\}$ is $r = 0$. A ring R is said *left (resp. right) primitive* if there is a faithful simple left (resp. right) R-module.

In the Leavitt path algebras context, the primitive algebras have already been characterized. This is stated in Theorem [1, 4.4.10]. The algebra $L_K(E)$ is primitive if and only if the graph E is downward directed and satisfies Condition (L). The graph E is *downward directed* if we consider two vertices $u, v \in E^0$ there is a third vertex $w \in E^0$ and two paths $\lambda, \mu \in \text{Path}(E)$ with $s(\lambda) = u$, $s(\mu) = v$ and $r(\lambda) = r(\mu) = w$. This is also equivalent to $L_K(E)$ being a prime algebra. On the other hand, condition (L) means that every cycle has an exit. In the path algebras context, characterization of primitive algebras is amusingly similar to the Leavitt path algebra case, but adapting Condition (i) in [1, Theorem 4.4.10] to the equivalent in path algebras. It is also important to highlight that the study of primitive Leavitt path algebras is done row finite directed graphs. However, for the path algebras context we do not need any conditions of finitude.

In order to give place for the main theorem of this section, we first need some previous results. If X is a set, we denote by $\mathrm{Ass}_K(X)$ to the unital associative K-algebra freely generated by the set X. This set is generated as a K-vector space by the words of X. It can be proven that the corners of a path algebra are isomorphic to this type of algebras.

Lemma 5.6. *Let E be a directed graph and K a field. Consider $u \in E^0$ and $X_u \subseteq \mathrm{Path}(E)$ the set of all paths $\lambda = f_1 f_2 \cdots f_n$ with $\ell(\lambda) \geqslant 1$, $s(\lambda) = r(\lambda) = u$ and $r(f_i) \neq u$ for all $i = 1, \ldots, n-1$. If $A = KE$ then $uAu \cong \mathrm{Ass}_K(X_u)$.*

Proof. If we consider the morphism

$$\begin{array}{rccc} \sigma\colon & \mathrm{Ass}_K(X_u) & \to & uAu \\ & 1 & \mapsto & u \\ & \lambda & \mapsto & \lambda \end{array}$$

It is obvious that this morphism is surjective. In order to prove that it is injective, we consider any polynomial $p \in \mathrm{Ass}_K(X_u)$ with $\sigma(p) = 0$. To check that $p = 0$, we need to verify that if $q = \lambda_1\lambda_2\cdots\lambda_n$ and $q' = \lambda'_1\lambda'_2\cdots\lambda'_m$ are two monomials such that $\sigma(q) = \sigma(q')$ then $q = q'$. If $\alpha = \sigma(q) = \lambda_1\lambda_2\cdots\lambda_n$ and $\beta = \sigma(q') = \lambda'_1\lambda'_2\cdots\lambda'_m$ (as paths), then these paths go through u the same amount of times because $\alpha = \beta$. The path α crosses $n+1$ times, while β crosses $m+1$ times, then $n = m$. Suppose without loss of generality that $\ell(\lambda_1) \geqslant \ell(\lambda'_1)$. As $\alpha = \beta$, this means that $\lambda_1 = \lambda'_1\gamma$, but λ_1 goes through u twice while $\lambda'_1\gamma$ three times, then $\gamma = u$ and $\lambda_1 = \lambda'_1$. Then, we have that $\lambda_2\lambda_3\cdots\lambda_n = \lambda'_2\lambda'_3\cdots\lambda'_n$. Repeating this argument we have that $\lambda_i = \lambda'_i$ for $i = 1, \ldots, n$ and $q = q'$. The previous polynomial p can be described as $p = \sum_{i=1}^{l} a_i q_i$ with q_i monomials in $\mathrm{Ass}_K(X_u)$, $a_i \in K\backslash\{0\}$ for every $i = 1, \ldots, l$ and $q_i \neq q_j$ for $i \neq j$. Thus, we have that $\sigma(p) = \sum_{i=1} a_i\sigma(q_i) = 0$ with $\sigma(q_i) \in \mathrm{Path}(E)$ and $\sigma(q_i) \neq \sigma(q_j)$ by the previous result. Finally, we have a linear combination of linear independent elements (different paths) equal to 0 which is only possible if $a_i = 0$ for every $i = 1, \ldots, n$ and this implies that $p = 0$. □

In addition to this Lemma, we need a result that ease the characterisation of the primitive algebras. The idea behind this is trying to prove what conditions make the algebra primitive at least in a local way. Once this is true, we can generalise for the whole algebra. This was stated in the next result by J. A. Anquela and T. Cortés.

Corollary. *[7, 1.2] Let R be a prime associative algebra such that eRe is left (resp. right) primitive for some idempotent $0 \neq e \in R$. Then R is left (resp. right) primitive*

Using Lemma 5.6 and Corollary [7, 1.2], together with the characterisation of the primitive Leavitt path algebras [1, 4.4.10] we can state the following Theorem.

Theorem 5.7. *Let E be a directed graph and K any field. The path algebra KE is left primitive if and only if:*

(i) E is strongly connected, and

(ii) E satisfies Condition (L).

Proof. Let E be a directed graph and K any field. Consider that the path algebra $A = KE$ is left primitive, in particular, this implies that A is prime [75, Propostion 11.6]. Following Proposition 5.4, the graph E must be strongly connected. On the other hand, suppose that A is a primitive algebra and there is a cycle without exit, this mean that there is a vertex $u \in E^0$ with the corner $uAu \cong K[x]$ which is not a left primitive algebra by [76, Proposition 11.8], this enters in contradiction with A being left primitive because all the corners must be left primitive.

For the converse, let us suppose that E satisfies conditions (i) and (ii). Once again, condition (i) implies that A is prime (Proposition 5.4) and this allow us to use [7, Corollary 1.2]. If we prove that there is an idempotent element $e \in A$ with eAe primitive, then A is primitive. Consider a vertex $u \in E^0$, we have that $uAu \cong \mathrm{Ass}_K(X_u)$ with X_u the set of closed paths $\lambda \in \mathrm{Path}(E)$ with $s(\lambda) = r(\lambda) = u$ and go through u just these two times (see Lemma 5.6). Conditions (i) and (ii) imply that $|X_u| \geqslant 2$ and this leads to uAu being primitive. This last result was proved by E. Formanek and appears in [75, p. 185] before [75, Theorem 11.27]. □

Corollary 5.8. *Let E be a directed graph with KE primitive, then KE is simple if and only if* $\mathrm{soc}_l(KE) = \mathrm{soc}_r(KE) \cong K$.

Remark 5.9. In general, left primitivity and right primitivity are not equivalent. There are examples of left primitive algebras that are not right primitive [17]. In the Leavitt path algebras context, the existence of an involution makes left primitivy and right primitivity equivalent notions. Despite the fact that in the path algebras context we do not have an involution, the symmetry in the conditions of Theorem 5.7 makes left and right primitivity equivalent. That is to say, the dual condition in Theorem 5.7 would be that every cycle has an entry. However, if the graph E is strongly connected, every cycle having and entry is equivalent to every cycle having an exit, which makes left and right primitivity equivalent.

5.3 STRUCTURE OF SEMIPRIME PATH ALGEBRAS

Primeness and semiprimenss of algebras are related, prime implies semiprime, but they are also linked to simpleness and primitiveness of algebras. As we know, simple implies primitive and primitive implies prime. In the path algebras context, they are even more related. As we will see, the basic blocks of what semiprime path algebras are constructed are: simple, primitive and prime algebras. To be more precise we will see that any semiprime algebra is a direct sum of simple path algebras, path algebras of cycles and primitive path algebras.

Let us consider a directed graph E such that its path algebra $A = KE$ is semiprime. We can consider each connected component in the graph E and describe A as a direct sum of path algebras. That is, if $\{E_m\}_{m\in\Lambda}$ are the connected components of E, then $A = \bigoplus_{i\in I} A_i$ with $A_m = KE_m$. If A is semiprime, Proposition 5.3 implies that each E_m is strongly connected, that is, each A_m is a prime algebra (Proposition 5.4). In this case, the algebra A_m can be simple, primitive (but non-simple) or prime (but non-primitive). If E_m is just a vertex, then the corresponding algebra is simple; if E_m has just one cycle, then $E_m = C_n$ and A_m is prime; if E_m has more than just one cycle, then every cycle has exit and A_m is a primitive algebra. In conclusion, we have the following decomposition:

$$KE = \left(\bigoplus_{i\in I} Ku_i\right) \oplus \left(\bigoplus_{j\in J} KC_{n_j}\right) \oplus \left(\bigoplus_{l\in L} KE_l\right)$$

with $\{u_i\}_{i\in I}$ all the isolated vertices in E, $\{C_{n_j}\}_{j\in J}$ all the isolated cycles and $\{E_l\}_{l\in L}$ all the strongly connected components that are not an isolated vertex or a cycle, that is, KE_l is primitive but not simple. The next question that we ask ourselves is if this decompositions is unique. In order to do so we need to add a result of Jacobson that appears in [65, p. 204] and extend it to the case that we have an algebra with local units and a decomposition with infinite ideals.

Let A be a K-algebra with local units and assume that A splits as $A = \oplus_{i\in\mathcal{A}} R_i$ where the R_is are indecomposable ideals (so they are also indecomposable as algebras on its own). The meaning of the term "indecomposable ideal"of an algebra A is that it is not a direct sum of nonzero ideals of A. Consider any ideal nonzero $B \subseteq A$ and take an arbitrary $b \in B$. Then there is a local unit for b, that is, an element $e \in A$ such that $b = be$. We have $e = \sum_{i \in \mathcal{A}} r_i$ with $r_i \in R_i$ (only finitely many of them being nonzero). Thus $b \in \sum_{i \in \mathcal{A}} BR_i$. Consequently we have $B = \sum_{i \in \mathcal{A}} BR_i$. Furthermore the sum is direct because for any i we have $BR_i \cap \sum_{j \neq i} BR_j \subset R_i \cap \sum_{j \neq i} R_j = 0$. So we have a splitting of B as a direct sum of ideals $B = \oplus_{i \in \mathcal{A}} BR_i$. Thus if B is an indecomposable ideal of A there is a unique $i \in \mathcal{A}$ such that $B = BR_i$. Moreover, with a similar argument there is a unique $j \in \mathcal{A}$ such that $B = R_j B$. Next we prove that $i = j$: we have $B = BR_i = R_j BR_i \subset R_j \cap R_i$ hence if $j \neq i$ we have $B = 0$ contrary to our assumption. So far we have:

Proposition 5.10. *If an algebra A with local units is a direct sum of indecomposable ideals, $A = \oplus_{i \in \mathcal{A}} R_i$, then any indecomposable ideal of A is one of the R_is. Furthermore, the decomposition is unique, that is, if there is another decomposition $A = \oplus_{j \in \mathcal{B}} R'_j$ in indecomposable ideals, then there is a bijection $\sigma \colon \mathcal{B} \cong \mathcal{A}$ such that $R'_j = R_{\sigma(j)}$.*

Proof. We have already proved the first part prior to our statement. We just need to prove the unity. If we have two decompositions as direct sum of indecomposable ideals $A = \oplus_{i \in \mathcal{A}} R_i = \oplus_{j \in \mathcal{B}} R'_j$, applying Proposition 5.10, for each $j \in B$ there is a unique $i \in A$ such that $R'_j = R_i$. So we have an injective map $\sigma \colon \mathcal{B} \hookrightarrow \mathcal{A}$ such that $R'_j = R_{\sigma(j)}$. Now given any $i \in \mathcal{A}$ there is a unique $k \in \mathcal{B}$ such that $R_i = R'_k$. But then $R'_k = R_{\sigma(k)}$ so $i = \sigma(k)$ which proves that σ is also surjective. Summarizing there is a bijection $\sigma \colon \mathcal{B} \cong \mathcal{A}$ such that $R'_j = R_{\sigma(j)}$ for any $j \in \mathcal{B}$. □

Finally, we are in conditions to state the following Theorem.

Theorem 5.11 (Structure Theorem for semiprime path algebras). *Let E be a directed graph and K a field. Every semiprime path algebra KE can be decomposed as*

$$KE = \bigoplus_{i \in I} Ku_i \oplus \bigoplus_{j \in J} KC_{n_j} \oplus \bigoplus_{l \in L} KE_l$$

with I the set of indices such that $E_i^0 = \{u_i\}$ for every $i \in I$, J the set of indices such that $E_j = C_{n_j}$ for every $j \in J$, L the set of indices such that KE_l is primitive (non-simple) and $|I| + |J| + |L|$ the number of connected components of E.

Chapter 6

CENTROID, EXTENDED CENTROID AND CENTRAL CLOSURE OF A PATH ALGEBRA

This chapter is dedicated to the commutative structures associated to an algebra such as the center. In particular, we will put the spotlight on the centroid and extended centroid of a path algebra, and the central closure of the path algebra of a cycle. Previously, the center of a path algebra has already been characterized in [31]. Following the results of that paper, we study the structure of the extended centroid, that generalises the centre of a path algebra. A characterization of this set is given by W. Crawley [97] as a "rather testing exercise". In our case, it has been a bit more challenging and since we have not found any reference that proves this statement, we have decided to include it in Section 6.1. However, we will not just leave it there. Since the centroid of a path algebra is linked to the existence of cycles without exits and entries, we study the extended centroid and central closure of the path algebra of a cycle. This will make an appearance in Section 6.2.

6.1 CENTROID OF A PATH ALGEBRA

In this section we will focus our attention on the centroid of a path algebra. Not only will westudy its elements, but also we will describe its structure as an algebra. In relation to this, we will work with the center of an algebra. Both concepts are really close and even are isomorphic when the algebra is unitary.

Definition 6.1. Consider an associative K-algebra A. The *centroid* of A, denoted by $\mathcal{C}(A)$, is the K-vector space of all linear maps $T\colon A \to A$ such that

$$T(xy) = T(x)y = xT(y)$$

for any $x, y \in A$.

The *center* of A, denoted by $\mathrm{Z}(A)$, is the K-vector space of the elements that commute with all of the elements of the algebra, that is,

$$\mathrm{Z}(A) := \{x \in A \mid xy = yx \ \forall y \in A\}$$

Observation 6.2. The centroid is also a K-algebra under composition. In particular if $A^2 = A$, then $\mathcal{C}(A)$ is commutative. Indeed, if $x = ab \in A^2 = A$ and $T, T' \in \mathcal{C}(A)$, then

$$TT'(x) = TT'(ab) = T(aT'(b)) = T(a)T'(b) = T'(T(a)b) = T'T(ab) = T'T(x).$$

The elements of $\mathcal{C}(A)$ will be called *centralizers*. For any centralizer $T \in \mathcal{C}(A)$ its kernel and image are both ideals of A. Thus, in the case of a simple algebra, each $T \in \mathcal{C}(A)$ is invertible and the centroid $\mathcal{C}(A)$ is a field. If A is prime and $T, T' \in \mathcal{C}(A)$ with $TT' = 0$, then for any $x, y \in A$ one has $T(x)T'(y) = T(xT'(y)) = TT'(xy) = 0$ so $\mathrm{Im}(T)\mathrm{Im}(T') = 0$ and since $\mathrm{Im}(T), \mathrm{Im}(T') \lhd A$, we have $\mathrm{Im}(T) = 0$ or $\mathrm{Im}(T') = 0$ by primeness of A. Thus, in this case $\mathcal{C}(A)$ is a domain.

Instead of speaking of general K-algebras, we can now focus on the path algebra context. We consider KE the path algebra of the directed graph E over the field K and choose $T \in \mathcal{C}(A)$. For a path $\mu \in \mathrm{Path}(E)$ we have that

$$T(\mu) = T(s(\mu)\mu) = T(s(\mu))\,\mu = T(\mu r(\mu)) = \mu T(r(\mu))\,.$$

As we can notice, the image of any path is characterised by the image of its starting vertex (thanks to the duality, it can also be characterised by its ending vertex). Even though this condition seems enough, this is not the case. Let us consider $u, v \in E^0$ with $u \neq v$. The element T verifies

$$T(u) = T(u^2) = T(u)u = uT(u),\ 0 = T(0) = T(uv) = uT(v) = T(u)v.$$

Lastly, if $f \in E^1$, $u = s(f)$ and $v = r(f)$ we have that

$$T(f) = T(uf) = T(u)f = T(fv) = fT(v).$$

If we collect all these equalities we have that

$$\begin{cases} T(u) = T(u)u = uT(u) & \forall u \in E^0, \\ 0 = uT(v) = T(u)v & \forall u, v \in E^0,\ u \neq v, \\ T(u)f = T(f) = fT(v) & \forall f \in E^1 \text{ with } u = s(f), v = r(f). \end{cases} \tag{8}$$

Definition 6.3. Let $E = (E^0, E^1, s, r)$ be a directed graph and A an associative algebra. A map $\phi\colon E \to A$ is a pair (ϕ_0, ϕ_1) where $\phi_0\colon E^0 \to A$ and $\phi_1\colon E^1 \to A$. In general, we will use the notation ϕ for both maps and we will know which one we are using by context. If $A = KE$ then we say that a map $\phi\colon E^0 \to KE$ is *centralizable* if it verifies the conditions

$$\begin{cases} \phi_0(u) = \phi_0(u)u = u\phi_0(u) & \forall u \in E^0, \\ 0 = u\phi_0(v) = \phi_0(u)v & \forall u, v \in E^0,\ u \neq v, \\ \phi_0(u)f = \phi_1(f) = f\phi_0(v) & \forall f \in E^1 \text{ with } u = s(f), v = r(f). \end{cases}$$

Remark 6.4. It is not difficult to notice that if $T \in \mathcal{C}(A)$, then $\phi = (\phi_0, \phi_1)$ where $\phi_0 := T|_{E^0}$ and $\phi_1 := T|_{E^1}$ is centralisable.

Before entering in the main result of this section, we first need some Lemmas that will ease the description of it. One of them corresponds to an easy way of describing the elements of the centroid just by knowing how the centralisers behave on E^0 and E^1.

Lemma 6.5. *Let E be a directed graph and $\phi\colon E \to KE$ a centralisable map. Define the linear map $T\colon KE \to KE$ as the map given by the image of paths $\mu \in \mathrm{Path}(E)$*

$$T(\mu) := \phi_0(s(\mu))\mu. \tag{9}$$

Then $T \in \mathcal{C}(KE)$.

Proof. First of all, it is easy to notice that $T|_{E^0} = \phi_0$ and by the definition of T and the equations that ϕ verifies, we have that

$$T(u) := \phi_0(u)u = u\phi_0(u) = \phi_0(u)\ \forall u \in E^0,$$

$$T(f) = T(s(f))f = \phi_0(s(f))f = \phi_1(f) = f\phi_0(r(f)) = fT(r(f))\ \forall f \in E^1$$

Let us now consider $\alpha = f_1 f_2 \cdots f_n, \beta = e_1 e_2 \cdots e_m \in \mathrm{Path}(E)$, then

$$T(\alpha\beta) = T(s(f_1))\alpha\beta = T(\alpha)\beta.$$

On the other hand,

$$\begin{aligned} T(s(f_1))\alpha\beta &= T(s(f_1))f_1 f_2 \cdots f_n e_1 e_2 \cdots e_m = f_1 T(r(f_1)) f_2 \cdots f_n e_1 e_2 \cdots e_m = \\ &= f_1 T(s(f_2)) f_2 \cdots f_n e_1 e_2 \cdots e_m = \cdots = \\ &= f_1 f_2 \cdots f_n T(s(e_1)) e_1 e_2 \cdots e_m = \alpha T(\beta) \end{aligned}$$

and so $T(\alpha\beta) = \alpha T(\beta) = T(\alpha)\beta$ for $\alpha, \beta \in \mathrm{Path}(E)$. Observe that these conditions together with linearity imply that for any $a, b \in KE$ we have $T(ab) = aT(b) = T(a)b$. □

Corollary 6.6. *There is a one to one relation between the set of elements of $\mathcal{C}(A)$ and the set of centralisable maps.*

As we have seen in Section 1.2 it is possible to decompose a graph in its connected components. If E is a directed graph with $\{E_i\}_{i\in I}$ its corresponding connected components we have that

$$KE = \bigoplus_{i\in I} KE_i.$$

As a direct consequence,

$$\mathcal{C}(KE) \cong \bigoplus_{i\in I} \mathcal{C}(KE_i).$$

In conclusion, if we characterise the centroid of a path algebra whose directed graph E is connected, we will be able to describe the centroid of any path algebra known just by studying its connected components.

Lemma 6.7. *Let E be a connected directed graph and $A = KE$ the path algebra over the field K. If we consider an edge $f \in E^1$ with $r(f) = v$,then* $\mathrm{Rann}_A(f) \cap vA = 0$*, where*

$$\mathrm{Rann}_A(f) = \{x \in A : fx = 0\}.$$

Proof. Let $x \in \mathrm{Rann}_A(f) \cap vA$. First of all $x \in vA$, so we can describe it as $x = \sum_{i=1}^{n} k_i \mu_i$ with $k_i \in K \backslash \{0\}$, $\mu_i \neq \mu_j$ if $i \neq j$ and $s(\mu_i) = v$ for every $i = 1, \ldots, n$. On the other hand, $x \in \mathrm{Rann}_A(f)$ verifying that $fx = 0$. Thus, $\sum_{i=1}^{n} k_i f \mu_i = 0$. From the fact that $s(\mu_i) = r(f) = v$ for every $i \in I$ we have that $f\mu_i \neq 0$ for every $i \in I$. This leads us to $fx = \sum_{i=1}^{n} k_i f \mu_i = 0$ which is a linear combination of linear independent elements (if $f\mu_i \neq 0$ and $f\mu_j \neq 0$, then $f\mu_i \neq f\mu_j$ if $i \neq j$) equal to zero. The only way this could be possible is by taking $k_i = 0$ for every $i = 1, \ldots, n$. In other words, $x = 0$. □

Proposition 6.8. *Let E be a connected directed graph and $A = KE$ the path algebra over the field K. Let us take $u \in E^0$ and $T \in \mathcal{C}(A)$ verifying $T(u) = ku$ for some $k \in K$. If there is an edge $f \in E^1$ with $s(f) = u$ and $r(f) = v \in E^0$, then $T(v) = kv$.*

Proof. Under the previous conditions, taking into account that $T \in \mathcal{C}(A)$, we can verify that $T(f) = T(uf) = T(u)f = kuf = kf = f(kv)$ and $T(f) = T(fv) = fT(v)$. As a consequence $f(kv - T(v)) = 0$, then $kv - T(v) \in \mathrm{Rann}_A(f)$. It is true that $T(v) = T(vv) = vT(v) \in vA$ and $v = vv \in vA$, then $kv - T(v) \in vA$. Finally $kv - T(v) \in \mathrm{Rann}_A(f) \cap vA$ and considering Lemma 6.7 $kv - T(v) = 0$ and $T(v) = kv$. □

Corollary 6.9. *Let E be a connected directed graph and $A = KE$ the path algebra over the field K. Let us take $u \in E^0$ and $T \in \mathcal{C}(A)$ verifying $T(u) = ku$ for some $k \in K$. If there is a path μ with $s(\mu) = u$ and $r(\mu) = v \in E^0$, then $T(v) = kv$.*

As a direct consequence of the dual property of the path algebras, we have the dual of Lemma 6.7, Proposition 6.8 and Corollary 6.9.

Lemma 6.10. *Let E be a connected directed graph and $A = KE$ the path algebra over the field K. If we consider an edge $f \in E^1$ with $s(f) = v$. Then, $\mathrm{Lann}_A(f) \cap Av = 0$ where*

$$\mathrm{Lann}_A(f) = \{x \in A \colon xf = 0\}.$$

Proposition 6.11. *Let E be a connected directed graph and $A = KE$ the path algebra over the field K. Let take $u \in E^0$ and $T \in \mathcal{C}(A)$ verifying $T(u) = ku$ for some $k \in K$. If there is an edge $f \in E^1$ with $r(f) = u$ and $s(f) = v \in E^0$, then $T(v) = kv$.*

Corollary 6.12. *Let E be a connected directed graph and $A = KE$ the path algebra over the field K. Let us take $u \in E^0$ and $T \in \mathcal{C}(A)$ verifying $T(u) = ku$ for some $k \in K$. If there is a path μ with $r(\mu) = u$ and $s(\mu) = v \in E^0$, then $T(v) = kv$.*

Corollaries 6.9 and 6.12 imply the first theorem that will be necessary to characterise the centroid.

Theorem 6.13. *Let E be a connected directed graph and KE the path algebra over the field K. Consider $T \in \mathcal{C}(KE)$, if there is some $u \in E^0$ that verifies $T(u) = ku$ for a $k \in K$, then $T = k\,\mathrm{Id}_{KE}$.*

Proof. First of all, as a direct consequence of Corollaries 6.9 and 6.12, we can prove that $T(u) = ku$ for every $u \in E^0$. Let us consider an edge $f \in E^1$ with $s(f) = u$, then $T(f) = T(uf) = T(u)f = kuf = kf$. Next, take a path $\mu = f_1 f_2 \cdots f_n$, it is easy to verify that $T(\mu) = T(f_1 f_2 \cdots f_n) = T(f_1) f_2 \cdots f_n = k f_1 f_2 \cdots f_n = k\mu$. We have proven that $T(\mu) = k\mu$ for every element of the basis of KE, by linearity it is direct that $T = k\,\mathrm{Id}_{KE}$. □

As we have done previously in Section 5.2, we are going to study the centroid of the path algebra from local to global. That is, we will study this structure on the corners of the algebra and we will try to generalise from there. In this way, we first need some 'local results'.

Lemma 6.14. *Let E be a connected directed graph and $A = KE$ the path algebra over the field K. Consider $T : A \to A$ with $T \in \mathcal{C}(A)$. If $u \in E^0$, then $T(u) \in \mathrm{Z}(uAu)$.*

Proof. Under the previous conditions, in order to prove $T(u) \in \mathrm{Z}(uAu)$ we have to demonstrate $T(u) \in uAu$ and $T(u)a = aT(u)$ for every $a \in uAu$.

1. $T(u) = T(uuu) = uT(uu) = uT(u)u \in uAu$.
2. Let $a \in uAu$, then $a = ua = au = uau$. Since $T \in \mathcal{C}(A)$ we have that $aT(u) = T(au) = T(ua) = T(u)a$ proving that $T(u) \in \mathrm{Z}(uAu)$.

□

Lemma 6.15. *If X is a set with $|X| > 1$, then $\mathrm{Z}(\mathrm{Ass}_K(X)) = K$.*

Proof. In order to prove this result, for an element $z \in \mathrm{Ass}_K(X)$ we define the set $\mathrm{C}(z) := \{a \in \mathrm{Ass}_K(X) |\ az = za\}$. It is not difficult to observe that

$$\mathrm{Z}(\mathrm{Ass}_K(X)) = \bigcap_{z \in \mathrm{Ass}_K(X)} \mathrm{C}(z).$$

Following Theorem [18, 5.3] we can affirm that $\mathrm{C}(z) = K[z]$ for every $z \in \mathrm{Ass}_K(X)$. As a consequence, $\mathrm{Z}(\mathrm{Ass}_K(X)) = \bigcap_{z \in \mathrm{Ass}_K(X)} K[z] = K$. □

Theorem 6.16. *Let us consider E a connected directed graph and $A = KE$ the corresponding path algebra over a field K. If $E = C_n$ (a cycle with n vertices), then $\mathcal{C}(A) \cong K[x]$, in other case $\mathcal{C}(A) = K\,\mathrm{Id}_A$.*

Proof. For every $u \in E^0$ we denote by X_u the set of all closed paths based at u such that they go through u just twice (in the same way as Lemma 5.6). We have two possibilities:

1. There is a vertex $u \in E^0$ such as $|X_u| \neq 1$. In this case we have two possibilities.

   a) If $|X_u| = 0$, that is $X_u = \emptyset$, then we have by Lemma 5.6 that $uAu \cong K$. In particular, $uAu = Ku$. Consider $T \in \mathcal{C}(A)$, following Lemma 6.14, $T(u) \in \mathrm{Z}(uAu) = \mathrm{Z}(Ku) = Ku$. Then, $T(u) = ku$. Theorem 6.13 implies that $T = k\,\mathrm{Id}$ and $\mathcal{C}(A) = K\,\mathrm{Id}_A$ as a consequence.

   b) In case that $|X_u| > 1$, by Lemma 6.15, we have that $\mathrm{Z}(\mathrm{Ass}_K(X_u)) = K$. In the same way as before, if $T \in \mathcal{C}(A)$, then $T(u) \in \mathrm{Z}(uAu) = Ku$ and $T(u) = ku$. We apply Theorem 6.13 once more and we obtain $T = k\,\mathrm{Id}_A$. This leads us to $\mathcal{C}(A) = K\,\mathrm{Id}_A$.

2. For all $u \in E^0$ we have that $|X_u| = 1$. Under this condition $uAu \cong \mathrm{Ass}_K(X_u) \cong K[x]$. In particular, $uAu = K[c_u]$ being c_u the only closed path based u. Observe that, c_u is also a cycle. If not, this will imply that $|X_u| \geqslant 2$. Under this hypothesis, we have two more possibilities:

   a) Suppose that E is not a cycle. Take $u \in E^0$ and c_u the only cycle based at u. Without loss of generality, we can suppose that there is an edge f such that $s(f) = u$ and $r(f) = v \notin c_u^0$. Furthermore, by hypothesis of the case 2, there is another cycle c_v based at v. Otherwise, it would not be true that $|X_u| = 1$ for every $u \in E^0$. If $c_u^0 \cap c_v^0 \neq \emptyset$, once again we enter in contradiction with $|X_u| = 1$ for every $u \in E^0$ because $w \in c_u^0 \cap c_v^0$ would verify $|X_w| \geqslant 2$. As a consequence $c_u^0 \cap c_v^0 = \emptyset$. An example of this situation is represented in Figure 15.

   Consider $T \in \mathcal{C}(A)$. From Lemma 6.14 we have that $T(u)$ is an element of $T(u) \in \mathrm{Z}(uAu) = K[c_u]$, that is $T(u) = k_0 u + \sum_{i=1}^{n} k_i c_u^i$ with $k_i \in K$. In a similar way, we also have that $T(v) = l_0 v + \sum_{j=1}^{m} l_j c_v^j$ with $l_j \in K$. If we are familiar with the properties of the elements of $\mathcal{C}(A)$, on the one

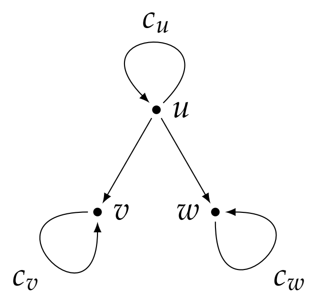

Figure 15: Example of a graph with $|X_u| = 1$ for every $u \in E^0$.

hand we have $T(f) = T(uf) = T(u)f = k_0 f + \sum_{i=1}^{n} k_i c_u^i f$. On the other hand $T(f) = T(fv) = fT(v) = l_0 f + \sum_{j=1}^{m} l_i f c_v^j$. In this way,

$$(k_0 - l_0)f + \sum_{i=1}^{n} k_i c_u^i f - \sum_{j=1}^{m} l_j f c_v^j = 0.$$

From our previous hypothesis we have that all paths in this last linear combination are different, and therefore $k_0 = l_0 = k$, $k_i = 0$ for every $i = 1, \ldots, n$ and $l_j = 0$ for all $j = 1, \ldots, m$. This allows us to prove that $T(u) = ku$ and using again Theorem 6.13, we have that $T = k\,\mathrm{Id}_A$. This give us $\mathcal{C}(A) = K\,\mathrm{Id}_A$.

b) The last possibility is that E is a cycle c with a finite number of vertices. This means that the algebra is unital and in consequence, $\mathcal{C}(A) \cong \mathrm{Z}(A)$. Following [31, Theorem 1] we have that $\mathcal{C}(A) \cong \mathrm{Z}(A) \cong K[x]$.

□

Corollary 6.17. *Let E be a directed graph with $\{E_i\}_{i \in I}$ its corresponding connected components. If KE is the path algebra associated to E over the field K then we have that*

$$\mathcal{C}(KE) \cong K[x]^J \oplus K^{I \setminus J}$$

where $J \subseteq I$ is $J = \{j \in I \colon E_j \text{ is a cycle}\}$.

6.2 EXTENDED CENTROID OF KC_n

Once we have studied the centroid of the path algebras, the next step lead us to identify the extended centroid and the central closure. Theorem 6.16 and

Corollary 6.17 state that the existence of non-trivial components in the centroid of a path algebra depends on the existence of isolated cycles in the directed graph. In any other case, the extended centroid is trivial (isomorphic to copies of the ground field). Since the path algebra of a cycle is prime (Proposition 5.4) the extended centroid is a field containing the centroid.

Following [26, Theorem 4.1] the extended centroid of PI-algebras (algebras satisfying a polynomial identity) is isomorphic to the field of fractions of its center. In Example 1.22 we prove that the algebra KC_n is isomorphic to a subalgebra of $\mathcal{M}_n(K[x])$, which means that KC_n is a PI-algebra because $\mathcal{M}_n(K[x])$ is also a PI-algebra by Amitsur-Levitzki Theorem [6]. Following Section 6.1, this implies that the extended centroid of the path algebra of a cycle, denoted by $\Gamma(KC_n)$, is isomorphic to $K(x)$, the field of polynomial fractions. However, knowing this will not be enough. In order to use the construction of the central closure given in [13] we will need the explicit isomorphism $\Gamma(KC_n) \to K(x)$.

The construction of $\Gamma(KC_n)$ will be in line with [13] so we will skip some meaningless details. One of the hypotheses of this process is considering a semiprime algebra. In our case, by Proposition 5.3 we have that KC_n is a semiprime algebra. Indeed, it is prime by 5.4. In addition, we will need to work with the essential ideals of the algebra.

Definition 6.18. Consider A an algebra and $I \neq \{0\}$ an ideal of A. We say that I is *essential* if for every ideal $J \neq \{0\}$ of A it satisfies that $I \cap J \neq \{0\}$.

Remark 6.19. Observe that since KC_n is a prime algebra, every ideal is essential. If $I \neq \{0\}$ is an ideal and consider any ideal $J \neq \{0\}$, then we have $\{0\} \neq IJ \subseteq I \cap J$. This means that we do not have to worry when we work with the essential ideals of KC_n since every ideal is essential. In addition, it is important to remember that KC_n is a unital algebra because $|C_n^0| = n < +\infty$.

Definition 6.20. [13] Let A be a semiprime K-algebra. An *admissible pair* is a pair (f, I) with I an essential ideal of A and $f : I \to A$ is a morphism of A-bimodules. In the set of all admissible pairs of we define the equivalence relation given by

$$(f, I) \sim (g, J) \iff f|_{I\cap J} = g|_{I\cap J}.$$

We denote by $\overline{(f, I)}$ to the equivalence class of the admissible pair (f, I). The *extended centroid* of A is the K-algebra given by the set

$$\Gamma(A) = \{\overline{(f, I)} : (f, I) \text{ is an admissible pair}\}$$

together with the operations:

$$k \cdot (\overline{f, I}) := (\overline{kf, I})$$
$$(\overline{f, I}) + (\overline{g, J}) := (\overline{f + g, I \cap J})$$
$$(\overline{f, I}) \cdot (\overline{g, J}) := (\overline{f \circ g, g^{-1}(I)}).$$

Before building the isomorphism $\Gamma(KC_n) \to K(x)$ we would like to remark that the notation that we will be using for all the computations is the one used in Example 1.22.

Lemma 6.21. *Let us consider KC_n the path algebra over K of C_n the oriented cycle of n indexed vertices. If I is an ideal of KC_n, then there is a polynomial $q(x) \in K[x]$ such that $J = \oplus_{i,j=0}^{n-1} J_{i,j}$, with $J_{i,j} := (q(c_i))\mu_{i,j}$, is a nontrivial ideal $J \subseteq I$. Here $(q(c_i))$ denotes the ideal generated by $q(c_i)$ in $K[c_i]$. In addition, this polynomial verifies that if there is an ideal $J' \subseteq I$ with $J'_{i,j} = (q'(c_i))\mu_{i,j}$, then $J' \subseteq J$.*

Proof. Denote $A = KC_n$ and observe that $A_{i,i} = v_i A v_i = K[c_i]$ is an algebra and $I_{i,i} = v_i I v_i$ is an $A_{i,i}$-ideal. That is

$$A_{i,i} I_{i,i} = v_i A v_i v_i I v_i \subseteq v_i I v_i = I_{i,i}$$

and

$$I_{i,i} A_{i,i} = v_i I v_i v_i A v_i \subseteq v_i I v_i = I_{i,i}.$$

The algebra $K[c_i]$ is an euclidean domain, thus, $I_{i,i}$ is a proper ideal generated by a polynomial $q_i(c_i)$. Take $q(x) = \operatorname{lcm}\{q_i(x), i = 0, \ldots, n-1\}$ and consider $J = \oplus_{i,j=0}^{n-1} J_{i,j}$ with $J_{i,j} := (q(c_i))\mu_{i,j}$ for $0 \leqslant i, j \leqslant n-1$. We can observe that, by definition, $J_{i,i}$ is an ideal of $A_{i,i}$ contained in $I_{i,i}$ since $q_i(c_i)$ divides $q(c_i)$. Since we are working with the Peirce decomposition of the path algebra of the cycle, in order to prove that J is an ideal of KC_n contained in I, it will only be necessary to verify:

1. $J_{i,j} \subseteq I_{i,j}$ for every $0 \leqslant i, j \leqslant n-1$.

2. $A_{i,k} J_{k,j} \subseteq J_{i,j}$ and $J_{i,k} A_{k,j} \subseteq J_{i,j}$ for every $0 \leqslant i, j, k \leqslant n-1$.

For the second part we have that $J_{i,j} = (q(c_i))\mu_{i,j} \subseteq (q_i(c_i))\mu_{i,j} = I_{i,i}\mu_{i,j} = v_i I v_i \mu_{i,j} v_j \subseteq v_i I v_j = I_{i,j}$. On the other hand, we have that

$$A_{i,k} J_{k,j} = K[c_i]\mu_{i,k}(q(c_k))\mu_{k,j} = K[c_i]\mu_{i,k}\mu_{k,j}(q(c_j)).$$

Depending if $l(\mu_{i,k}) + l(\mu_{k,j}) \geqslant n$ or not, the product $\mu_{i,k}\mu_{k,j}$ can be either $\mu_{i,k}\mu_{k,j} = c_i\mu_{i,j}$ or $\mu_{i,k}\mu_{k,j} = \mu_{i,j}$. In both cases $A_{i,k}J_{k,j} \subseteq K[c_i](q(c_i))\mu_{i,j} = (q(c_i))\mu_{i,j} = J_{i,j}$. The contention $J_{i,k}A_{k,j} \subseteq J_{i,j}$ is proven analogously.

The first result implies that $J \subseteq I$ while the second one implies that J is an ideal of KC_n.

For the last part consider $J' \subseteq I$ with $J'_{i,j} = (q'(c_i))\mu_{i,j}$. We have that $J'_{i,i} = v_iJ'v_i \subseteq v_iIv_i = I_{i,i} = (q_i(c_i))$ for every $i = 1, \ldots, n$. This implies that $q_i(x)$ divides $q'(x)$ for every $i = 1, \ldots, n$. Thus, $q(x) = \text{lcm}\{q_i(x), i = 1, \ldots, n\}$ divides $q'(x)$ and $J'_{i,j} \subseteq J_{i,j}$ for every $i, j = 0, \ldots, n-1$. Thus, $J' \subseteq J$. $\square$

Let us consider $A = KC_n$ the path algebra of a cycle with n indexed vertices over the field K. If we consider an admissible pair (f, I) and take $J \subseteq I$ and $q(x) \in K[x]$ as in Lemma 6.21. It is obvious that $(f, I) \sim (f, J)$. Furthermore, $f(q(c_i)) = f(v_iq(c_i)v_i) = v_if(q(c_i))v_i \in v_iAv_i = K[c_i]$. This means that for every $i = 0, \ldots, n-1$ there is a polynomial $p_i(x) \in K[x]$ such that $f(q(c_i)) = p_i(c_i)$. On one side, we have that $f(q(c_i)\mu_{i,j}) = f(q(c_i))\mu_{i,j} = p_i(c_i)\mu_{i,j}$, whereas on the other, $f(q(c_i)\mu_{i,j}) = f(\mu_{i,j}q(c_j)) = \mu_{i,j}f(q(c_j)) = \mu_{i,j}p_j(c_j) = p_j(c_i)\mu_{i,j}$. Thus, we have that $p_i(c_i) = p_j(c_i) = p(c_i)$ for every $0 \leqslant i, j \leqslant n-1$ and as a consequence $f(q(c_i)) = p(c_i)$ for every $i = 0, \ldots, n-1$. This will allow us to consider, for an admissible pair (f, I), a quotient of polynomials $\frac{p(x)}{q(x)}$.

Proposition 6.22. *If we consider KC_n the path K-algebra of a cycle of n indexed vertices. The map*

$$\begin{array}{rcl} \Omega\colon \Gamma(KC_n) & \to & K(x) \\ \overline{(f, I)} & \mapsto & \Omega\left(\overline{(f, I)}\right) := \frac{p(x)}{q(x)} \end{array}$$

where $q(x) \in K[x]$ is the one from Lemma 6.21 and $p(x) \in K[x]$ with $f(q(c_i)) = p(c_i)$ for every $i = 0, \ldots, n-1$, is a K-algebra isomorphism.

Proof. The first thing the readers might ask themselves is if the map is well defined. In order to do so we just need to prove that the image of $\overline{(f, I)}$ is the same independently of the element of the equivalence class. If we consider $(f|_{I'}, I')$ with $I' \subseteq I$ we have that $(f|_{I'}, I') \sim (f, I)$. Following Lemma 6.21 there is an ideal $J' \subseteq I' \subseteq I$ with good properties. However, the maximality of the ideal J corresponding to I implies that $J' \subseteq J$. In particular, this means that $J'_{i,i} = (q'(c_i)) \subseteq (q(c_i)) = J_{i,i}$ for certain $q(x), q'(x) \in K[x]$. This implies that $q(x)$ divides $q'(x)$, that is, there is some $r(x) \in K[x]$ such that $q'(x) = r(x)q(x)$. Consequently, $f(q'(c_i)) = f(r(c_i)q(c_i)) = r(c_i)p(c_i)$ for every $i = 0, \ldots, n-1$. Finally, this admissible pair is in correspondence with $\frac{r(x)p(x)}{r(x)q(x)} = \frac{p(x)}{q(x)}$. For the second part, we need to prove that this map is a morphism of K-algebras. In other words, it is compatible with the sum, the product and the product of scalars. Consider an admissible pair $(\overline{f,I})$ with $f(q_i(c_i)) = p(c_i)$ and $k \in K$, the map $k \cdot f \colon I \to KC_n$ verifies $kf(q(c_i)) = kp(c_i)$ for every $i = 0, \ldots, n-1$, then

$$\Omega\left(k \cdot (\overline{f,I})\right) = \Omega\left((\overline{kf,I})\right) = \frac{kp(x)}{q(x)} = k\frac{p(x)}{q(x)} = k \cdot \Omega\left((\overline{f,I})\right).$$

Now take $(\overline{f,I}), (\overline{g,I'}) \in \Gamma(KC_n)$ with $f(q(c_i)) = p(c_i)$ and $g(q'(c_i)) = p'(c_i)$ for $i = 0, \ldots, n-1$ with $q(x), q'(x) \in K[x]$ as in Lemma 6.21. If we consider $q_0(x) = \operatorname{lcm}\{q(x), q'(x)\}$ and J_0 an ideal with $(J_0)_{i,j} = (q_0(c_i))\mu_{i,j}$ for $i, j = 0, \ldots, n-1$. It is not difficult to check that, by construction, $J_0 \subseteq J \subseteq I$ and $J_0 \subseteq J' \subseteq I'$, thus, $J_0 \subseteq I \cap I'$ ($J \subseteq I$ and $J' \subseteq I'$ as in Lemma 6.21) . Take $Q(x) = \gcd\{q(x), q'(x)\}$, it verifies that $q_0(x)Q(x) = q(x)q'(x)$. Observe that, by the properties of the great common divisor and the lower common multiple, $\frac{q'(x)}{Q(x)}, \frac{q(x)}{Q(x)} \in K[x]$. In consequence,

$$\begin{aligned}
&(f+g)(q_0(c_i)) = f(q_0(c_i)) + g(q_0(c_i)) = \\
&= f\left(\frac{q'(x)}{Q(x)}q(c_i)\right) + g\left(\frac{q(x)}{Q(x)}q'(c_i)\right) = \frac{q'(x)}{Q(x)}f(q(c_i)) + \frac{q(x)}{Q(x)}g(q'(c_i)) = \\
&= \frac{q'(x)}{Q(x)}p(c_i) + \frac{q(x)}{Q(x)}p'(c_i)
\end{aligned}$$

for every $i = 0, \ldots, n-1$. If we do some computations, then

$$\begin{aligned}
\Omega\left((\overline{f,I}) + (\overline{g,I'})\right) &= \Omega\left((\overline{f+g, I \cap I'})\right) = \Omega\left((\overline{f+g, J_0})\right) = \\
&= \frac{p(x)\frac{q'(x)}{Q(x)} + p'(x)\frac{q(x)}{Q(x)}}{q_0(x)} = \frac{p(x)q'(x) + p'(x)q(x)}{Q(x)q_0(x)} = \\
&= \frac{p(x)q'(x) + p'(x)q(x)}{q(x)q'(x)} = \frac{p(x)}{q(x)} + \frac{p'(x)}{q'(x)} = \\
&= \Omega\left((\overline{f,I})\right) + \Omega\left((\overline{g,I'})\right).
\end{aligned}$$

Now if we consider L an ideal in KC_n with $L_{i,j} = (q(c_i)q'(c_i))\mu_{i,j}$, then we have that $L \subseteq g^{-1}(I)$ because $g(q(c_i)q'(c_i)\mu_{i,j}) = q(c_i)g(q'(c_i))\mu_{i,j} = q(c_i)p'(c_i)\mu_{i,j} \in J_{i,j} \subseteq I_{i,j}$

for every $i,j = 0,\dots,n-1$. In this sense, $f(g(q(c_i)q'(c_i))) = f(q(c_i)p'(c_i)) = f(q(c_i))p'(c_i) = p(c_i)p'(c_i)$ for every $i = 0,\dots,n-1$ and

$$\Omega\left((\overline{f,I})\cdot(\overline{g,I'})\right) = \Omega\left((\overline{f\circ g, g^{-1}(I)})\right) = \Omega\left((\overline{f\circ g, L})\right) = \\ = \frac{p(x)p'(x)}{q(x)q'(x)} = \Omega\left((\overline{f,I})\right)\cdot\Omega\left((\overline{g,I'})\right).$$

In this way, we have already proved that Ω is a well defined K-algebra morphism. Finally, we need to check it is bijective. It is not difficult to check that Ω is surjective. For an element $\frac{p(x)}{q(x)} \in K(x)$, we just need to consider the class $(\overline{f,J})$ with J an ideal verifying $J_{i,j} = (q(c_i))\mu_{i,j}$ and $f\colon J \to KC_n$ defined as $f(q(c_i)) = p(c_i)$ for every $i,j = 0,\dots,n-1$. Thus, $\Omega\left((\overline{f,J})\right) = \frac{p(x)}{q(x)}$. For the injectivity, if there is $(\overline{f,I})$ with $\Omega\left((\overline{f,I})\right) = 0$ this means that there is an ideal $J \subseteq I$ with $J_{i,j} = (q(c_i))\mu_{i,j}$ and $f(q(c_i)) = 0$ for $i = 0,\dots,n-1$. As a direct consequence, $f(J) = 0$ and $(\overline{f,I}) = (\overline{0,J}) = 0$. □

Now that we know how the isomorphism $\Omega\colon \Gamma(KC_n) \to K(x)$ behaves, we are in conditions to construct the central closure of KC_n. We will define and construct the central closure of the algebra following [13].

Definition 6.23. Consider a K-algebra A. For every $a \in A$ we define the *left and right multiplications morphisms* as

$$\begin{array}{rcl} L_a\colon A & \to & A \\ x & \mapsto & L_a(x) := ax \end{array} \qquad \begin{array}{rcl} R_a\colon A & \to & A \\ x & \mapsto & R_a(x) := xa \end{array}.$$

If $\mathrm{End}_K(A)$ is the *algebra of endomorphisms of A*, then $\mathrm{M}(A)$ is the subalgebra of $\mathrm{End}_A(A)$ generated by the left and right multiplications. This is called the *multiplication algebra* and in our case

$$\mathrm{M}(A) := \mathrm{Ass}_K(\{L_a, R_b\colon a,b \in A\}).$$

The readers can introduce themselves deeper in the multiplication algebras following the article by D.R Finston [47].

Remark 6.24. If A is a unitary associative K-algebra, then L_a and R_b commutes for every $a,b \in A$, and $L_1 = R_1 = \mathrm{Id}_A$. As a consequence,

$$\mathrm{M}(A) = \left\{\sum_{i=1}^{n} L_{a_i}R_{b_i}\colon a_i, b_i \in A\ \forall i = 1,\dots,n\right\}.$$

Definition 6.25. Let A be a semiprime K-algebra and $\Gamma(A)$ its extended centroid as in Definition 6.20. If $A \otimes_K \Gamma(A)$ is the usual tensor product of K-algebras, then an element $x \in A \otimes_K \Gamma(A)$ is *vanishing* if there is a representation $\sum_{k=1}^{m} a_k \otimes \lambda_k$ with $a_k \in A$ and $\lambda_k = \overline{(f_k, I)}$ for every $k = 1, \ldots, m$, verifying

$$\sum_{k=1}^{m} P(a_k) f_k(z) = 0$$

for every $z \in I$ and $P \in \mathrm{M}(A)$.

The vanishing elements will play a key role in the construction of the central closure of the algebra. In particular, if A is a semiprime K-algebra with $\Gamma(A)$ its extended centroid and M is the set of vanishing elements of $A \otimes_K \Gamma(A)$, then we have the next Lemma.

Lemma. *[13, 2.6] M is an ideal.*

Finally, we have all the necessary to build the central closure of an algebra

Definition 6.26. Let A be a semiprime K-algebra. The *central closure* of A is the quotient given by

$$\widehat{A} := \frac{A \otimes_K \Gamma(A)}{M}$$

where $\Gamma(A)$ is the extended centroid of A and M is the set of vanishing elements.

Even though we have a construction of the central closure of KC_n, the description of the algebra is a bit confusing and lacking of interest. However, Proposition 6.22 will let us characterise the central closure of KC_n as a better understanding algebra.

Theorem 6.27. *Denote by $A = KC_n$ the path algebra of a cycle of n vertices, then the central closure $\widehat{A}$ is isomorphic to $\mathcal{M}_n(K(x))$.*

Proof. Let us consider the morphism of K-algebras

$$\begin{array}{rcl} \theta \colon KC_n \otimes \Gamma(KC_n) & \to & \mathcal{M}_n(K(x)) \\ \sum_{k=1}^{m} a_k \otimes \lambda_k & \mapsto & \theta\left(\sum_{k=1}^{m} a_k \otimes \lambda_k\right) := \sum_{k=1}^{m} \tau(a_k)\Omega(\lambda_k), \end{array}$$

with $\tau \colon KC_n \to \mathcal{M}_n(K[x])$ the injective morphism given in Example 1.22 and $\Omega \colon \Gamma(KC_n) \to K(x)$ the isomorphism given in Proposition 6.22. The reader can check that $\theta = (\tau, \Omega)$ the coproduct morphism. For this reason, it will only be necessary to prove that $\ker(\theta) = M$ and that θ is surjective.

Take a vanishing element $\sum_{k=1}^{m} a_k \otimes \lambda_k$ with $a_k \in KC_n$ and $\lambda_k = \overline{(f_k, I)}$ for every $k = 1, \ldots, m$ such that $\sum_{k=1}^{m} P(a_k) f_k(z) = 0$ for all $P \in \mathrm{M}(KC_n)$ and $z \in I$. Consider the ideal $J \subset I$ with $J_{i,j} = (q(c_i))\mu_{i,j}$ for $i, j = 0, \ldots, n-1$ as in Lemma 6.21 and $f_k(q(c_i)) = p_k(c_i)$ for every $k = 1, \ldots, m$. If we take $z = \sum_{i=0}^{n-1} q(c_i) \in J \subseteq I$, then by hypothesis we have

$$\sum_{k=1}^{m} a_k f_k \left(\sum_{i=0}^{n-1} q(c_i) \right) = \sum_{k=1}^{m} a_k \left(\sum_{i=0}^{n-1} p_k(c_i) \right) = 0.$$

The morphism $\tau \colon KC_n \to \mathcal{M}_n(K[x])$ is injective so

$$\tau \left(\sum_{k=1}^{m} a_k \left(\sum_{i=0}^{n-1} p_k(c_i) \right) \right) = \sum_{k=1}^{m} \tau(a_k) \tau \left(\sum_{i=0}^{n-1} p_k(c_i) \right) = \sum_{k=1}^{m} \tau(a_k) p_k(x) = 0.$$

Dividing by $q(x)$ we have

$$0 = \sum_{k=1}^{m} \tau(a_k) \frac{p_k(x)}{q(x)} = \sum_{k=1}^{m} \tau(a_k) \Omega(\lambda_k) = \theta \left(\sum_{k=1}^{m} a_k \otimes \lambda_k \right).$$

This proves that $M \subseteq \ker\theta$. In order to prove the other contention we will make use of the maximality of M given in Lemma [13, 2.11]. This Lemma establish that M is the unique maximal ideal of $KC_n \otimes \Gamma(KC_n)$ with $I_0 \subseteq M$ and $M \cap (KC_n \otimes \mathbb{1}) = 0$. Here I_0 denotes the ideal generated by the elements of the form $z \otimes \lambda - f(z) \otimes \mathbb{1}$ where $\lambda = \overline{(f, I)} \in \Gamma(KC_n)$, $z \in I$ and $\mathbb{1} = \overline{(\mathrm{Id}_{KC_n}, KC_n)}$. Then, the only thing we need to verify is that $\ker\theta \cap (KC_n \otimes \mathbb{1}) = 0$ because this will mean that $\ker\theta \subseteq M$. If $a \otimes \mathbb{1} \in \ker\theta$, then $\theta(a \otimes \mathbb{1}) = \tau(a)\Omega(\mathbb{1}) = \tau(a) = 0$ and because τ is injective it is true that $a = 0$ and $a \otimes \mathbb{1} = 0$.

Finally, we need to prove that this morphism is surjective. Let us consider a matrix

$$\begin{pmatrix} \frac{p_{1,1}(x)}{q_{1,2}(x)} & \frac{p_{1,2}(x)}{q_{1,2}(x)} & \cdots & \frac{p_{1,n}(x)}{q_{1,n}(x)} \\ \frac{p_{2,1}(x)}{q_{2,1}(x)} & \frac{p_{2,2}(x)}{q_{2,2}(x)} & \cdots & \frac{p_{2,n}(x)}{q_{2,n}(x)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{p_{n,1}(x)}{q_{n,1}(x)} & \frac{p_{n,2}(x)}{q_{n,2}(x)} & \cdots & \frac{p_{n,n}(x)}{q_{n,n}(x)} \end{pmatrix} \in \mathcal{M}_n(K(x)).$$

We can multiply and divide by x each element below the diagonal and express the matrix as

$$\begin{pmatrix} \frac{p_{1,1}(x)}{q_{1,1}(x)} & \frac{p_{1,2}(x)}{q_{1,2}(x)} & \cdots & \frac{p_{1,n}(x)}{q_{1,n}(x)} \\ \frac{xp_{2,1}(x)}{xq_{2,1}(x)} & \frac{p_{2,2}(x)}{q_{2,2}(x)} & \cdots & \frac{p_{2,n}(x)}{q_{2,n}(x)} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{xp_{n,1}(x)}{xq_{n,1}(x)} & \frac{xp_{n,2}(x)}{xq_{n,2}(x)} & \cdots & \frac{p_{n,n}(x)}{q_{n,n}(x)} \end{pmatrix}.$$

Lastly, by calculating common denominator of all the entries of the matrix, we can rewrite it as

$$\frac{1}{q(x)}\begin{pmatrix} r_{1,1}(x) & r_{1,2}(x) & \cdots & r_{1,n}(x) \\ xr_{2,1}(x) & r_{2,2}(x) & \cdots & r_{2,n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ xr_{n,1}(x) & xr_{n,2}(x) & \cdots & r_{n,n}(x) \end{pmatrix}.$$

In consequence, this matrix is the image by Ω of the element $a \otimes \lambda$, where $a = \sum_{i,j=0}^{n-1} r_{i,j}(c_i)\mu_{i,j}$ and $\lambda = (\overline{f,I})$ with $I_{i,j} = (q(c_i))\mu_{i,j}$ and $f(q(c_i)) = v_i \in C_n^0$ for $0 \leqslant i,j \leqslant n-1$. □

Describing the central closure of the algebra of a cycle as an algebra of matrices, with coefficients in the field of fractions of polynomials, instead of as a quotient of a tensor product, eases the computations and the understanding of the behaviour of this algebra. Also, working with algebras of matrices over a field allow us to use, under certain conditions, well known theorems such that Wedderburn-Artin Theorem.

Before using those theorems, we need some results about central closures. It will be useful to know how the central closure behaves under isomorphism and direct sums. This is general knowledge, however due to the lack of a reference to appeal we find it a good opportunity to include it here. The extended centroid and central clousure behaves as expected and the reader can skip this part if they want and continue with the next chapter.

Proposition 6.28. *Let us consider A and B two prime associative K-algebras with unity. If A and B are isomorphic, then $\Gamma(A) \cong \Gamma(B)$ and $\widehat{A} \cong \widehat{B}$*

Proof. First of all define the map

$$\begin{aligned} \Gamma(\sigma)\colon \Gamma(A) &\to \Gamma(B) \\ (\overline{f,I}) &\mapsto \Gamma(\sigma)\left((\overline{f,I})\right) := (\overline{\sigma \circ f \circ \sigma^{-1}, \sigma(I)}) \end{aligned}$$

where $\sigma\colon A \to B$ is an isomorphism. This map is well defined. If we consider $J \subseteq I$ and $(f|_J, J) \sim (f,I)$ we have that $\sigma(J) \subseteq \sigma(I)$ and $\sigma \circ f|_J \circ \sigma^{-1} = \left(\sigma \circ f \circ \sigma^{-1}\right)|_{\sigma(J)}$ and then $(\sigma \circ f|_J \circ \sigma^{-1}, \sigma(J)) \sim (\sigma \circ f \circ \sigma^{-1}, \sigma(I))$. Also, this map is an isomorphism of K-algebras. If we consider $(\overline{f,I}), (\overline{g,J}) \in \Gamma(A)$ we have that

$$\sigma(kf(\sigma^{-1}(y))) = k \cdot (y)(\sigma \circ f \circ \sigma^{-1})$$

and then

$$\Gamma(\sigma)\left(k\overline{(f,I)}\right) = \Gamma(\sigma)\left(\overline{(kf,I)}\right) = \overline{(k(\sigma\circ f\circ\sigma^{-1}),\sigma(I))} =$$
$$= k\overline{(\sigma\circ f\circ\sigma^{-1},\sigma(I))} = k\Gamma(\sigma)\left(\overline{(f,I)}\right).$$

For the sum we have that

$$\sigma\circ(f+g)\circ\sigma^{-1} = \sigma\circ f\circ\sigma^{-1}+\sigma\circ g\circ\sigma^{-1}$$

and $\sigma(I\cap J) = \sigma(I)\cap\sigma(J)$ because σ is an isomorphism. As a consequence

$$\Gamma(\sigma)\left(\overline{(f,I)}+\overline{(g,J)}\right) = \Gamma(\sigma)\left(\overline{(f+g,I\cap J}\right) =$$
$$= \left(\overline{\sigma\circ f\circ\sigma^{-1}+\sigma\circ g\circ\sigma^{-1},\sigma(I)\cap\sigma(J)}\right) =$$
$$= \overline{(\sigma\circ f\circ\sigma^{-1},\sigma(I))}+\overline{(\sigma\circ g\circ\sigma^{-1},\sigma(J))} =$$
$$= \Gamma(\sigma)\left(\overline{(f,I)}\right)+\Gamma(\sigma)\left(\overline{(g,J)}\right)$$

For the product we can check that

$$\sigma\circ f\circ g\circ\sigma^{-1} = \left(\sigma\circ f\circ\sigma^{-1}\right)\circ\left(\sigma\circ g\circ\sigma^{-1}\right)$$

and also $\sigma(g^{-1}(I)) = \left(\sigma\circ g^{-1}\circ\sigma^{-1}\right)(\sigma(I))$. This implies that

$$\Gamma(\sigma)\left(\overline{(f,I)}\cdot\overline{(g,J)}\right) = \Gamma(\sigma)\left(\overline{(f\circ g,g^{-1}(I))}\right) =$$
$$= \left(\overline{\sigma\circ f\circ\sigma^{-1}\circ\sigma\circ g\circ\sigma^{-1},\left(\sigma\circ g^{-1}\circ\sigma^{-1}\right)(\sigma(I))}\right) =$$
$$= \left(\overline{\sigma\circ f\circ\sigma^{-1},\sigma(I)}\right)\cdot\left(\overline{\sigma\circ g\circ\sigma^{-1},\sigma(J)}\right) =$$
$$= \Gamma(\sigma)\left(\overline{(f,I)}\right)\cdot\Gamma(\sigma)\left(\overline{(g,J)}\right)$$

Finally, we need to find the inverse of $\Gamma(\sigma)$. The reader may find that $\Gamma(\sigma^{-1})\colon\Gamma(B)\to\Gamma(A)$ given by

$$\overline{(g,J)}\mapsto\left(\overline{\sigma^{-1}\circ g\circ\sigma,\sigma^{-1}(J)}\right)$$

is the inverse of $\Gamma(\sigma)$.

Once we have proven that $\Gamma(A) \cong \Gamma(B)$ we are in conditions to prove that their corresponding central closures are isomorphic too. The properties of the tensor product imply that $A \otimes \Gamma(A) \cong B \otimes \Gamma(B)$ through the algebra isomorphism

$$\begin{aligned} \sigma \otimes \Gamma(\sigma)\colon\ & A \otimes \Gamma(A) \to B \otimes \Gamma(B) \\ & \sum_{k=1}^{m} a_k \otimes \lambda_k \mapsto \sum_{k=1}^{m} \sigma(a_k) \otimes \Gamma(\sigma)(\lambda_k). \end{aligned}$$

If we consider M_A and M_B the set of vanishing elements in $A \otimes \Gamma(A)$ and $B \otimes \Gamma(B)$ and prove that $\sigma \otimes \Gamma(\sigma)(M_A) = M_B$ we have that the central closures are isomorphic. It will be only necessary to check that $\sigma \otimes \Gamma(\sigma)(M_A) \subseteq M_B$, the other contention will be a direct consequence of $\sigma^{-1} \otimes \Gamma(\sigma^{-1})(M_B) \subseteq M_A$ which is proven in the same way. Let us take a vanishing $x \in M$, that is, there is a representation of $x = \sum_{k=1}^{m} a_k \otimes \lambda_k$ with $\lambda_k = (f_k, I)$ that is I-vanishing. What we are going to verify is that $\sigma \otimes \Gamma(\sigma)(x)$ is $\sigma(I)$-vanishing. But before that, the reader might check that the multiplicative algebras $\mathrm{M}(A)$ and $\mathrm{M}(B)$ are isomorphic. This isomorphism is induced, again, by sigma

$$\begin{aligned} \mathrm{M}(\sigma)\colon\ & \mathrm{M}(A) \to \mathrm{M}(B) \\ & T_{a,b} \mapsto \mathrm{M}(\sigma)(T_{a,b}) := T_{\sigma(a),\sigma(b)} \end{aligned}$$

with $T_{a,b} = L_a R_b$ and $a, b \in A$. It is easy to check that

$$\begin{aligned} \mathrm{M}(\sigma)(T_{a,b}T_{c,d}) &= \mathrm{M}(\sigma)(T_{ac,bd}) = T_{\sigma(ac),\sigma(bd)} = \\ = T_{\sigma(a)\sigma(c),\sigma(b)\sigma(d)} &= T_{\sigma(a),\sigma(b)} T_{\sigma(c),\sigma(d)} = \mathrm{M}(\sigma)(T_{a,b})\,\mathrm{M}(\sigma)(T_{c,d}). \end{aligned}$$

Also, it is true that

$$\sigma \circ T_{a,b} = T_{\sigma(a),\sigma(b)} \circ \sigma$$

and in consequence

$$T_{\sigma(a),\sigma(b)} = \sigma \circ T_{a,b} \circ \sigma^{-1}.$$

Let us come back to the proof and check that $\sum_{k=1}^m \sigma(a_k) \otimes \Gamma(\sigma)(\lambda_k)$ is $\sigma(I)$-vanishing. Consider $P \in \mathrm{M}(B)$ and $\sigma(z) \in \sigma(I)$ with $z \in I$. We have that

$$\sum_{k=1}^m P(\sigma(a_k)) \cdot \left(\sigma \circ f_k \circ \sigma^{-1}(\sigma(z))\right) = \sum_{k=1}^m \left(\sum_{a,b} T_{\sigma(a),\sigma(b)}(\sigma(a_k))\right) \cdot \sigma(f_k(z)) =$$

$$= \sum_{k=1}^m \left(\sum_{a,b} \sigma \circ T_{a,b} \circ \sigma^{-1}(\sigma(a_k))\right) \cdot \sigma(f_k(z)) = \sigma\left(\sum_{k=1}^m \left(\sum_{a,b} T_{a,b}(a_k)\right) f_k(z)\right).$$

Following that $\sum_{k=1}^m a_k \otimes \lambda_k$ is I-vanishing, this means that

$$\sum_{k=1}^m \left(\sum_{a,b} T_{a,b}(a_k)\right) f_k(z) = 0$$

for every $z \in I$, and finally

$$\sum_{k=1}^m P(\sigma(a_k)) \cdot \left(\sigma \circ f_k \circ \sigma^{-1}(\sigma(z))\right) = \sigma(0) = 0$$

as we hoped to obtain, finishing here the proof. □

We can say that the central closure and the extended centroid are good definitions because they keep being isomorphic for isomorphic algebras. Checking this fact was not difficult but needed of some computations. Proving that the extended centroid and the central closure commutes with direct sums will not be less. We will need some previous computations that will ease the proof.

Lemma 6.29. *Let us consider A_1 and A_2 two prime associative K-algebras with unity. Then $I \subseteq A = A_1 \oplus A_2$ is an ideal if and only if $I = I_1 \oplus I_2$ with $I_i \subseteq A_i$ an ideal of A_i for $i = 1, 2$. The ideal $I_1 \oplus I_2$ is essential if and only if $I_i \neq \{0\}$ for $i = 1, 2$.*

Proof. Because of $A_iA_j = 0$ for $i \neq j$, it is immediate that $AI = (A_1 \oplus A_2)(I_1 \oplus I_2) = A_1I_1 \oplus A_2I_2$ and $IA = I_1A_1 \oplus I_2A_2$. If I_1 and I_2 are ideals of A_1 and A_2 respectively, we have that $I = I_1 \oplus I_2$ is an ideal of A. Let us consider $I \subseteq A$ an ideal of A and define $e_i = 1_{A_i}$ and $I_i = e_iI = Ie_i$. If we multiply $A_iI_i = Ae_ie_iI = Ae_iI = AIe_i \subseteq Ie_i = I_i$ and $I_iA_i \subseteq I_i$ analogously. This means that $I_i \subseteq A_i$ is an ideal for $i = 1, 2$. We just need to prove that $I = I_1 \oplus I_2$. Because $e_i \in A$ it is obvious that $I_i \subseteq I$ for $i = 1, 2$ and $I_1 \oplus I_2 \subseteq I$. If $x \in I$ we have that $x = x \cdot 1 = x(e_1 + e_2) = xe_1 + xe_2 \in I_1 \oplus I_2$.

Let us consider I an essential ideal of A and J a nonzero ideal. It is not difficult to check that $I \cap J = (I_1 \cap J_1) \oplus (I_2 \cap J_2)$. We have that $(I \cap J)\, e_i \subseteq Ie_i \cap Je_i = I_i \cap J_i$ and if $x \in I_i \cap J_i$, $x \in I \cap J$ and $x = xe_i$, so $x \in (I \cap J)e_i$. The intersection $I \cap J = (I_1 \cap J_1) \oplus (I_2 \oplus I_2)$ is different from zero if and only if $I_1 \cap J_1 \neq \{0\}$ or $I_2 \cap J_2 \neq \{0\}$. And because A_i is a prime algebra (the essential ideals are the nonzero ideals), this last assertion is true if and only if $I_i = 0$ for $i = 1, 2$. □

Lemma 6.30. *Let us consider A_1 and A_2 two prime associative K-algebras with unity. The extended centroid $\Gamma(A_1 \oplus A_2)$ is isomorphic to $\Gamma(A_1) \oplus \Gamma(A_2)$.*

Proof. Let us consider $f\colon I \to A$ a bimodule morphism and define $f_i = f|_{I_i}$. We have that $f_i(I_i) = f(e_i I) = e_i f(I) \subseteq e_i A = A_i$, and we can consider $f_i\colon I_i \to A_i$ a morphism of A_i-bimodules for $i = 1, 2$. The morphism f also verifies that

$$f(x) = f(xe_1 + xe_2) = f(xe_1) + f(xe_2) = f_1(xe_1) + f_2(xe_2).$$

Once we know this, we are in conditions to define the morphism

$$\begin{aligned} \phi\colon \Gamma(A_1 \oplus A_2) &\to \Gamma(A_1) \oplus \Gamma(A_2) \\ (\overline{f, I}) &\mapsto \phi\left((\overline{f, I})\right) := (\overline{f_1, I_1}) + (\overline{f_2, I_2}). \end{aligned}$$

This map, as the reader might have thought, is the isomorphism we were looking for.

If $J \subseteq I$, it is also true that $J_i = Je_i \subseteq Ie_i = I_i$ for $i = 1, 2$. If we consider $g\colon J \to A$ a morphism of A-bimodules with $g = f|_J$, we have that $g_i = g|_{J_i} = f|_{J_i} = f_i|_{J_i}$. Under this conditions, $(f, I) \sim (g, J)$ and $(f_i, I_i) \sim (g_i, J_i)$ for $i = 1, 2$. This implies that ϕ is well defined. Now we need to prove that it is a morphism. Take $k \in K$ and $(\overline{f, I}), (\overline{g, J}) \in \Gamma(A)$. First, we can observe that $(kf)_i = (kf)|_{I_i} = k\, f|_{I_i} = kf_i$ and this implies that

$$\phi(k(\overline{f, I})) = \phi((\overline{kf, I})) = (k\overline{f_1, I_1}) + (k\overline{f_2, I_2}) = k\left((\overline{f_1, I_1}) + (\overline{f_2, I_2})\right) = k\phi((\overline{f, I})).$$

On the other side, we have $(f+g)|_{(I_i \cap J_i)} = f|_{(I_i \cap J_i)} + g|_{(I_i \cap J_i)} = f_i|_{(I_i \cap J_i)} + g_i|_{(I_i \cap J_i)}$ for $i = 1, 2$. This implies

$$\begin{aligned} \phi((\overline{f, I}) + (\overline{g, J})) = \phi((\overline{f+g, I \cap J})) &= (\overline{f_1 + g_1, I_1 \cap J_1}) + (\overline{f_2 + g_2, I_2 \cap J_2}) = \\ (\overline{f_1, I_1}) + (\overline{g_1, J_1}) + (\overline{f_2, I_2}) + (\overline{g_2, J_2}) &= (\overline{f_1, I_1}) + (\overline{f_2, I_2}) + (\overline{g_1, J_1}) + (\overline{g_2, J_2}) = \\ &= \phi((\overline{f, I})) + \phi((\overline{g, J})). \end{aligned}$$

For the last part, we have that $g^{-1}(I) = g^{-1}(I_1 \oplus I_2) = g^{-1}(I_1) \oplus g^{-1}(I_2) = (g_1)^{-1}(I_1) \oplus (g_2)^{-1}(I_2)$. Also, the composition verifies that $(f \circ g)|_{g^{-1}(I_i)} = f \circ g|_{(g_i)^{-1}(I_i)} = f \circ g_i|_{(g_i)^{-1}(I_i)} = f|_{I_i} \circ g_i|_{(g_i)^{-1}(I_i)} = f_i \circ g_i|_{(g_i)^{-1}(I_i)} = (f_i \circ g_i)|_{(g_i)^{-1}(I_i)}$ for $i = 1, 2$. This implies

$$\begin{aligned}
\phi(\overline{(f,I)}\overline{(g,J)}) = \phi(\overline{(f \circ g, g^{-1}(I))}) &= \overline{(f_1 \circ g_1, (g_1)^{-1}(I_1))} + \overline{(f_2 \circ g_2, (g_2)^{-1}(I_2))} = \\
&= \overline{(f_1, I_1)}\overline{(g_1, J_1)} + \overline{(f_2, I_2)}\overline{(g_2, J_2)} = \\
&= \left(\overline{(f_1, I_1)} + \overline{(f_2, I_2)}\right)\left(\overline{(g_1, J_1)} + \overline{(g_2, J_2)}\right) = \\
&= \phi(\overline{(f,I)})\phi(\overline{(g,J)}).
\end{aligned}$$

Now that we have proven that ϕ is an isomorphism, we just need to check that is bijective. If $\phi(\overline{(f,I)}) = 0$, this means that $\overline{(f_1, I_1)} = 0$ and $\overline{(f_2, I_2)} = 0$. Then $f_1 = 0$ and $f_2 = 0$. In consequence, $\overline{(f,I)} = 0$ and ϕ is injective. The morphism ϕ is also surjective because for a pair $\overline{(f_1, I_1)} + \overline{(f_2, I_2)}$ we can find $\overline{(f,I)}$ with $\phi(\overline{(f,I)}) = \overline{(f_1, I_1)} + \overline{(f_2, I_2)}$. It is enough to consider $I = I_1 \oplus I_2$ and $f \colon I \to A$ defined as $f(x+y) = f_1(x) + f_2(y)$ for $x \in I_1$ and $y \in I_2$. □

Remark 6.31. It is obvious that under the right conditions

$$\Gamma\left(\bigoplus_{i=1}^{n} A_i\right) \cong \bigoplus_{i=1}^{n} \Gamma(A_i).$$

Once we know the extended centroid of the sum of algebras, we can calculate its central closure by quotienting with the vanishing elements. The tensor product preserves isomorphism and this implies

$$\begin{aligned}
(A_1 \oplus A_2) \otimes \Gamma(A_1 \oplus A_2) &\cong (A_1 \oplus A_2) \otimes (\Gamma(A_1) \oplus \Gamma(A_2)) \cong \\
&\cong (A_1 \otimes \Gamma(A_1)) \oplus (A_1 \otimes \Gamma(A_2)) \oplus (A_2 \otimes \Gamma(A_1)) \oplus (A_2 \otimes \Gamma(A_2)).
\end{aligned}$$

In the second term appear algebras like $A_i \otimes \Gamma(A_i)$ for $i = 1, 2$ which makes us think that the central closure of the sum is the sum of the central closures, but there are also terms $A_i \otimes \Gamma(A_j)$ for $i \neq j$. These last terms, once we calculate the vanishing elements and make the quotient, will not be problematic. Before that, we need to know how the multiplicative algebra $\mathrm{M}(A_1 \oplus A_2)$ is related with $\mathrm{M}(A_1)$ and $\mathrm{M}(A_2)$, because it will help us to understand the vanishing elements in $(A_1 \oplus A_2) \otimes \Gamma(A_1 \oplus A_2)$.

Lemma 6.32. *Let us consider A_1 and A_2 two associative K-algebras with unity. If $\sum_{i=1}^{n} L_{a_i+b_i} R_{c_i+d_i} \in \mathrm{M}(A_1 \oplus A_2)$ with $a_i, c_i \in A_1$ and $b_i, d_i \in A_2$ for $i = 1, \ldots, n$, and consider $x_1 + x_2 \in A_1 \oplus A_2$ with $x_j \in A_j$ for $j = 1, 2$, then*

$$\sum_{i=1}^{n} L_{a_i+b_i} R_{c_i+d_i}(x_1 + x_2) = \sum_{i=1}^{n} L_{a_i} R_{c_i}(x_1) + \sum_{i=1}^{n} L_{b_i} R_{d_i}(x_2)$$

with $\sum_{i=1}^{n} L_{a_i} R_{c_i} \in \mathrm{M}(A_1)$ and $\sum_{i=1}^{n} L_{b_i} R_{d_i} \in \mathrm{M}(A_2)$.

Proof. This proof is pure computation. First, we need to check is that $L_a R_b = 0$ for $a \in A_i$ and $b \in A_j$ for $i \neq j$. If $x \in A$ and $e_i = 1_{A_i}$ for $i = 1, 2$, we have

$$L_a R_b(x) = L_a(xb) = axb = a(xe_j + e_i x)b = axe_1 b + ae_2 xb = 0 + 0 = 0.$$

This implies that

$$\begin{aligned}
\sum_{i=1}^{n} L_{a_i+b_i} R_{c_i+d_i}(x_1 + x_2) &= \sum_{i=1}^{n} (L_{a_i} + L_{b_i})(R_{c_i} + R_{d_i})(x_1 + x_2) = \\
&= \sum_{i=1}^{n} (L_{a_i} R_{c_i} + L_{a_i} R_{d_i} + L_{b_i} R_{c_i} + L_{b_i} R_{d_i})(x_1 + x_2) = \\
&= \sum_{i=1}^{n} (L_{a_i} R_{c_i} + L_{b_i} R_{d_i})(x_1 + x_2) = \\
&= \sum_{i=1}^{n} L_{a_i} R_{c_i}(x_1 + x_2) + \sum_{i=1}^{n} L_{b_i} R_{d_i}(x_1 + x_2) = \\
&= \sum_{i=1}^{n} L_{a_i} R_{c_i}(x_1) + \sum_{i=1}^{n} L_{a_i} R_{c_i}(x_2) + \sum_{i=1}^{n} L_{b_i} R_{d_i}(x_1) + \sum_{i=1}^{n} L_{b_i} R_{d_i}(x_2) = \\
&= \sum_{i=1}^{n} L_{a_i} R_{c_i}(x_1) + \sum_{i=1}^{n} L_{b_i} R_{d_i}(x_2).
\end{aligned}$$

□

Theorem 6.33. *Let us consider A_1 and A_2 two associative prime K-algebras and take $A = A_1 \oplus A_2$. If M_1 and M_2 denote the sets of vanishing elements in $A \otimes \Gamma(A)$, $A_1 \otimes \Gamma(A_1)$ and $A_2 \otimes \Gamma(A_2)$ respectively. Then*

$$\frac{A \otimes \Gamma(A)}{M} \cong \frac{A_1 \otimes \Gamma(A_1)}{M_1} \oplus \frac{A_2 \otimes \Gamma(A_2)}{M_2}.$$

Proof. As we have mentioned before, we have that

$$A \otimes \Gamma(A) \cong A_1 \otimes \Gamma(A_1) \oplus A_1 \otimes \Gamma(A_2) \oplus A_2 \otimes \Gamma(A_1) \oplus A_2 \otimes \Gamma(A_2).$$

Let us find the vanishing elements of $A \otimes \Gamma(A)$ in terms of the vanishing elements of $A_1 \otimes \Gamma(A_1)$ and $A_2 \otimes \Gamma(A_2)$. If $x \in A \otimes \Gamma(A)$ is a vanishing element, we can write

$$x = \sum_{k=1}^{n} a_k \otimes \lambda_k$$

with $\lambda_k = \overline{(f_k, I)}$ for $k = 1, \ldots, n$ and for every element $P \in \mathrm{M}(A)$ and $z \in I$ we have

$$\sum_{k=1}^{n} P(a_k) f_k(z) = 0.$$

In virtue of Lemmas 6.30 and 6.32, we can consider $a_k = a_{k,1} + a_{k,2} \in A_1 \oplus A_2$ with $a_{k,i} \in A_i$ for $k = 1, \ldots, n$ and $i = 1, 2$; $z = z_1 + z_2 \in I = I_1 \oplus I_2$ with $z_i \in I_i$ for $i = 1, 2$ and $f_k(z_1 + z_2) = f_{k,1}(z_1) + f_{k,2}(z_2)$ with $f_{k_i} \colon I_i \to A_i$ for $i = 1, 2$; $P = \sum_{j=1}^{m} L_{\alpha_j + \beta_j} R_{\gamma_j + \delta_j}$ with $\alpha_j, \gamma_j \in A_1$ and $\beta_j, \delta_j \in A_2$ for $j = 1, \ldots, n$; and obtain

$$\begin{aligned}
\sum_{k=1}^{n} P(a_k) f_k(z) &= \sum_{k=1}^{n} \left(\sum_{j=1}^{m} L_{\alpha_j+\beta_j} R_{\gamma_j+\delta_j}(a_{k,1} + a_{k,2}) \right) \big(f_{k,1}(z_1) + f_{k,2}(z_2)\big) = \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{m} L_{\alpha_j} R_{\gamma_j}(a_{k,1}) + \sum_{j=1}^{m} L_{\beta_j} R_{\delta_j}(a_{k,2}) \right) \big(f_{k,1}(z_1) + f_{k,2}(z_2)\big) = \\
&= \sum_{k=1}^{n} \left(\sum_{j=1}^{m} L_{\alpha_j} R_{\gamma_j}(a_{k,1}) \right) f_{k,1}(z_1) + \sum_{k=1}^{n} \left(\sum_{j=1}^{m} L_{\alpha_j} R_{\gamma_j}(a_{k,1}) \right) f_{k,2}(z_2) + \\
&\quad \sum_{k=1}^{n} \left(\sum_{j=1}^{m} L_{\beta_j} R_{\delta_j}(a_{k,2}) \right) f_{k,1}(z_1) + \sum_{k=1}^{n} \left(\sum_{j=1}^{m} L_{\beta_j} R_{\delta_j}(a_{k,2}) \right) f_{k,2}(z_2) = 0.
\end{aligned}$$

The reader should not forget that $A_1 A_2 = 0$ which means that the second and third term in the last expression are zero. This leads us to

$$\sum_{k=1}^{n} P(a_k) f_k(z) = \sum_{k=1}^{n} \left(\sum_{j=1}^{m} L_{\alpha_j} R_{\gamma_j}(a_{k,1}) \right) f_{k,1}(z_1) + \sum_{k=1}^{n} \left(\sum_{j=1}^{m} L_{\beta_j} R_{\delta_j}(a_{k,2}) \right) f_{k,2}(z_2) = 0.$$

This identity is true if and only if

$$\begin{aligned}\sum_{k=1}^{n}\left(\sum_{j=1}^{m} L_{\alpha_j} R_{\gamma_j}(a_{k,1})\right) f_{k,1}(z_1) = 0 \text{ for every } z_1 \in I_1, \alpha_j, \gamma_j \in A_1 \\ \sum_{k=1}^{n}\left(\sum_{j=1}^{m} L_{\beta_j} R_{\delta_j}(a_{k,2})\right) f_{k,2}(z_2) = 0 \text{ for every } z_2 \in I_2, \beta_j, \delta_j \in A_2.\end{aligned} \tag{10}$$

This equivalence translates into: $\sum_{k=1}^{n} a_k\lambda_k = \sum_{k=1}^{n}(a_{k,1}+a_{k,2})(\lambda_{k,1}+\lambda_{k,2})$ is $I = I_1 \oplus I_2$-vanishing if and only if $\sum_{k=1}^{n} a_{k,1}\lambda_{k,1}$ is I_1-vanishing and $\sum_{k=1}^{n} a_{k,2}\lambda_{k,2}$ is I_2-vanishing. Then, $M = M_1 \oplus (A_1 \otimes \Gamma(A_2)) \oplus (A_2 \otimes \Gamma(A_1)) \oplus M_2$. Finally, we have that

$$\begin{aligned}\frac{A \otimes \Gamma(A)}{M} &\cong \frac{(A_1 \otimes \Gamma(A_1)) \oplus (A_1 \otimes \Gamma(A_2)) \oplus (A_2 \otimes \Gamma(A_1)) \oplus (A_2 \otimes \Gamma(A_2))}{M_1 \oplus (A_1 \otimes \Gamma(A_2)) \oplus (A_2 \otimes \Gamma(A_1)) \oplus M_2} \cong \\ &\cong \frac{A_1 \otimes \Gamma(A_1)}{M_1} \oplus \frac{A_2 \otimes \Gamma(A_2)}{M_2}\end{aligned}$$

□

Chapter 7

NOETHERIAN PATH ALGEBRAS

Developing a condition for a path algebra to be Artinian has not been difficult. We have been able to work with the descending chain condition successfully. In the following, we will work with the ascending chain condition, that is, we will develop a characterisation of Noetherian path algebras. This time it will not be as easy, but the results are promising. As we have already mentioned previously, some of the ideas of this chapter are not new, for example Proposition 7.4 appears in [32]. However, we decided to include it since we have not found any proof in the literature and we consider that it is not a trivial result.

7.1 NOETHERIAN PATH ALGEBRAS

Definition 7.1. Let A be an algebra. We say that A is *left (resp. right, resp. two-sided) Noetherian* if and only if for every ascending chain of left (resp. right, resp. two-sided) ideals

$$I_1 \subset I_2 \subset \cdots I_n \subset \cdots$$

there is a number $n_0 \geqslant 1$ such that $I_{n_0} = I_n$ for every $n \geqslant n_0$.

The left Noetherianity of path algebras is mentioned in [32] to be equivalent to the graph being finite and every cycle having no entries. However, the proof of this result is not given. In order to prove so we first need to see that the algebra KC_n is left (resp. right) Noetherian. This is immediate following the Peirce decomposition that we described in Example 1.22 and Proposition [58, 1.7].

Proposition. *[58, 1.7] Let R be a left (resp. right) Noetherian ring and S a subring of the matrix ring $\mathcal{M}_n(R)$. If S contains the subring*

$$R' = \left\{ \begin{pmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{pmatrix} : r \in R \right\}$$

of all 'scalar matrices', then S is left (resp. right) Noetherian. In particular, $\mathcal{M}_n(R)$ is a left (resp. right) Noetherian ring.

Lemma 7.2. *The path algebra KC_n of a cycle with n vertices over a field K is left (resp. right) Noetherian.*

Proof. Since $K[x]$ is a left (resp. right) Noetherian algebra and by Example 1.22 KC_n is isomorphic to a subring of $\mathcal{M}_n(K[x])$ that contains the so called 'scalar matrices', by [58, Lemma 1.7] the path algebra KC_n is left (resp. right) Noetherian. □

Lemma 7.2 establishes that a cycle with n vertices, as in Example 1.22, is left and right Noetherian. The next question is: what happens when we build more complex graphs? For example, by adding exits or entries to the cycle? If we add an exit to a cycle with n vertices, the left ideals of the path algebra do not change substantially. However, right ideals get easily affected. This could lead, as a first impression, to the path algebra still being left Noetherian but not right Noetherian.

Proposition 7.3. *Consider a directed graph E with only one cycle without entries (resp. exits) and $|E^0 \cup E^1| < \infty$. Then, the path algebra $A = KE$, with K a field, is left (resp. right) Noetherian.*

Proof. We will prove this result for the left Noetherian case. The other one is analogous (or dual). The arguments of the proof will rely on several results of [58] and the Peirce decomposition of KE.

Consider c the cycle without entries and $F^0 = E^0 \backslash c^{(0)}$. Without any loss of generality, we can order the vertices of E^0 such that $u_i \in c^{(0)}$ for $i = 0, \ldots, n-1$ and $u_i \in F^0$ for $i = n, \ldots, n+m$. Let us take the complete ortogonal system of idempotent elements $e_1 = \sum_{i=0}^{n-1} u_i$ and $e_2 = \sum_{i=n}^{n+m} u_i$, that is, $e_1e_2 = e_2e_1 = 0$, $e_j^2 = e_j$ for $j = 1, 2$ and $e_1 + e_2 = 1$. As we know c has no entries, thus there is no path from a vertex in F^0 to a vertex in $c^{(0)}$ and $A_{21} := e_2Ae_1 = 0$. In consequence, we have that

$$A \cong \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$$

with $S := e_1Ae_1$, $T := e_2Ae_2$ and $B := e_1Ae_2$. Following [58, Proposition 1.8], if we prove that S and T are left Noetherian and B is a finitely generated left S-module, we will prove that A is left Noetherian. As we know from the Peirce decomposition, S and T are subalgebras of A and B is a (S, T)-bimodule. The algebra S is isomorphic to the path algebra of a cycle and following Lemma 7.2, S is Noetherian. If we consider the algebra T, we have that it is finite-dimensional because there is a finite number of paths connecting vertices in F^0 (there are no cycles and the number of vertices and edges is finite). As a consequence, T is left Noetherian.

For the last part, we need to prove that B is finitely generated as a left S-module. If $X = \left\{u \in c^{(0)} : \left|s^{-1}(u)\right| = 1\right\}$, then we consider $G = E \backslash X$ the subgraph of E obtained by removing the vertices in the cycle that do not have exits. The reader can check Figure 16 for an example of this.

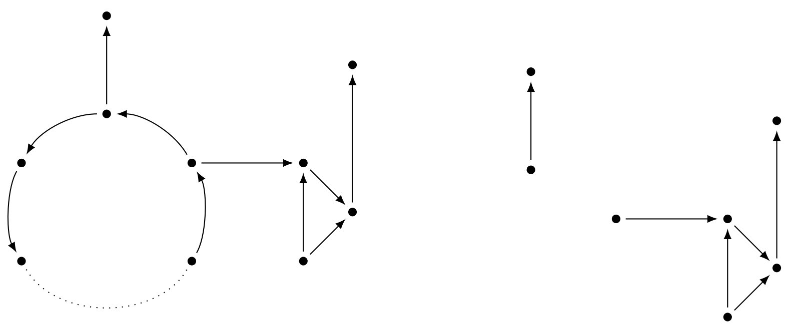

Figure 16: On the left side we have the graph E, a cycle without entries. On the right side we have the graph G, a product of eliminating in E the vertices in the cycle that are not part of an exit.

It is not difficult to check that the graph G, under our hypothesis, does not contain any cycle and $|G^0 \cup G^1| < +\infty$. This means that $|\operatorname{Path}(G)| < +\infty$. Let us now prove that $B = \sum_{\mu \in \operatorname{Path}(G)} S\mu$. As we know, B is the vector space generated by all the paths that start at some vertex in $c^{(0)}$ and end in a vertex in F^0. From that, it is easy to verify that $B' = \sum_{\mu \in \text{Path}(G)} S\mu \subseteq B$. On the other hand, if we consider a path $\lambda \in B$ such that $s(\lambda) \in c^{(0)}$, we have that $\lambda = \beta\mu$ with β a path in the cycle, that is $s(\beta), r(\beta) \in c^{(0)}$ and $\mu \in \text{Path}(G)$. Finally, for all the paths in B we have that $\lambda = \beta\mu \subseteq S\mu \subseteq B'$ and $B \subseteq B'$. This proves that B is finitely generated as a S-module and concludes the proof. □

Now that we know some examples of Noetherian path algebras, and we are aware of how the quotient of path algebras over ideals generated by vertices behave (Proposition 1.20), we are in conditions to prove the next characterisation.

Proposition 7.4. *Let E be a directed graph and K a field. The path algebra KE is left (resp. right) Noetherian if and only if $|E^0 \cup E^1| < +\infty$ and every cycle does not have entries (resp. exits).*

Proof. We will prove the left Noetherian case. The right one is analogous. In fact, it is the dual result.

First, suppose that $A = KE$ is left Noetherian. Consider that $E^0 \cup E^1$ is not finite. This means that E^0 or E^1 are infinite sets. If E^0 is not finite there is a set of vertices $\{u_i\}_{i=1}^{\infty} \subseteq E^0$ and then we can construct the ascending chain of left ideals

$$Au_1 \subset Au_1 + Au_2 \subset \cdots \subset \sum_{i=1}^{n} Au_i \subset \cdots$$

Let us verify that $u_{n+1} \notin \sum_{i=1}^{n} Au_i$ for every $i \geqslant 1$. If $u_{n+1} \in \sum_{i=1}^{n} Au_i$, then $u_{n+1} = \sum_{j=1}^{m} k_j \lambda_j$ with $k_j \in K \setminus \{0\}$ and $\lambda_j \in \text{Path}(E)$ with $r(\lambda_j) \in \{u_1, \ldots, u_n\}$ for every $j = 1, \ldots, m$. Multiplying both sides by u_{n+1} on the right we have $u_{n+1} = 0$, which is not true. As a consequence, we have constructed a strictly ascending chain of left ideals. This can not be possible because KE is Noetherian. That is, $|E^0| < +\infty$.

In a very similar way, if $|E^1| = \infty$, then there is a set of edges $\{f_i\}_{i=1}^{\infty} \subseteq E^1$ and we can construct the ascending chain of left ideals

$$Af_1 \subset Af_1 + Af_2 \subset \cdots \subset \sum_{i=1}^{n} Af_i \subset \cdots$$

To prove that it is a strict chain, let us verify that $f_{n+1} \notin \sum_{i=1}^{n} Af_i$. If $f_{n+1} \in \sum_{i=1}^{n} Af_i$, then $f_{n+1} = \sum_{j=1}^{m} k_j \lambda_j f_{i_j}$ with $k_j \in K \setminus \{0\}$, $\lambda_j f_{i_j} \in \text{Path}(E)$ and $\lambda_j f_{i_j} \neq \lambda_{j'} f_{i_{j'}}$ for every $j = 1, \ldots, m$ and $j \neq j'$. We can rewrite the expression as $\sum_{j=1}^{m} k_j \lambda_j f_{i_j} - f_{n+1} = 0$ which can not be possible since all the paths in the linear combination are different. In consequence, $|E^1| < +\infty$ and then, $|E^0 \cup E^1| < +\infty$.

Suppose that there is a cycle $c \in \text{Path}(E)$ with an entry $f \in E^1$, that is, $0 \neq fc \in \text{Path}(E)$. In the same way as we did before, we can construct the ascending chain of left ideals

$$Afc \subset Afc + Afc^2 \subset \cdots \subset \sum_{i=1}^{n} Afc^i \subset \cdots$$

Once again, and using similar arguments as before, this chain is strict that enters in contradiction with the fact that KE is left Noetherian. As a consequence, every cycle does not have entries.

Let us prove the reciprocal implication. Consider E a graph with $|E^0 \cup E^1| < +\infty$ and every cycle does not have entries. We will prove that KE is left Noetherian by induction on the number of cycles.

If E has no cycles, we have that KE is a finite-dimensional K-algebra and this means that KE is left Noetherian.

The case $n = 1$ is just a consequence of Proposition 7.3.

Suppose as our induction hypothesis that if the graph has n cycles, then KE is left Noetherian. Consider a finite graph E with $n+1$ cycles without entries. Let us take c any cycle and consider $I = (c^{(0)})$ the two-sided ideal generated by the vertices of $c^{(0)}$. Following Proposition [58, 1.2], if we prove that I and KE/I are KE-Noetherian, then KE will be Noetherian.

As we know from Proposition 1.20 it is true that $KE/I \cong KF$ where $F = E \backslash c^{(0)}$ (the collapse of E through the set of vertices $c^{(0)}$). Since all the cycles in E do not have entries, the graph F has n cycles without entries, and by our induction hypothesis, this implies that KF is KF-Noetherian. In addition, all the KE-left modules in KE/I are KE/I-left modules in KE/I and viceversa. This is a consequence of the surjectivity of the canonical projection. Then, we have that KE/I is KE-Noetherian.

Finally we just need to prove that I is KE-noetherian. Consider the subgraph $G = (G^0, G^1, s, r)$ with

$$G^0 = \left\{ u \in E^0 \colon \exists\, \mu \in I \cap \text{Path}(E) \text{ with } u \in \mu^{(0)} \right\}$$

$$G^1 = \left\{ f \in E^1 \colon \exists\, \mu \in I \cap \text{Path}(E) \text{ with } f \in \mu^{(1)} \right\}.$$

we can think of this subgraph as the graph formed by selecting all the vertices and edges in the paths that generate I. The reader can check Figure 17 for a better understanding.

It is important to notice some things.

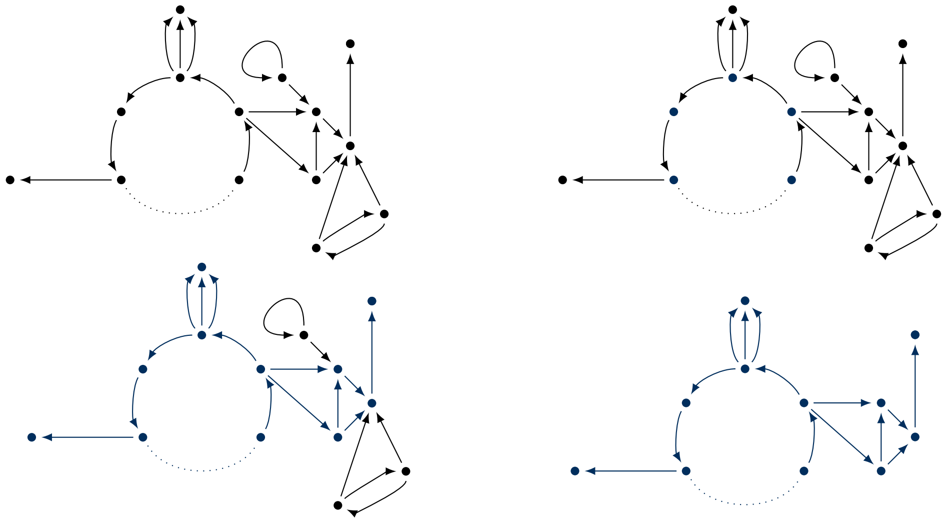

Figure 17: Process of building the graph G of the proof (in blue) from the graph E (in black).

1. The graph G has only one cycle. This fact is because every cycle in E has no entries. This means that there is no path $\mu \in I$ of the form $\mu = c\lambda c'$, with c' another cycle. That is, there is no path connecting two cycles.

2. The set I is contained in KG. As we know, the ideal I is generated by all the paths that go through some vertex $u \in c^{(0)}$ and all of them are in $\text{Path}(G)$ by construction. Then $I \subset KG$. Finally, because G is a subgraph of E, KG is a subalgebra of KE implying that I is an ideal of KG.

3. All the KE-modules contained in I are KG-modules contained in I and viceversa. If $\nu \in \text{Path}(E)\backslash\text{Path}(G)$, then $\nu \cdot (c^{(0)}) = 0$. Take J a left KG-module contained in I. Consider $a \in KE$, we can write $a = \sum_{i=1}^{n} a_i\mu_i + \sum_{j=1}^{m} b_j\lambda_i$ where $a_i, b_j \in K\backslash\{0\}$, $\mu_i \in \text{Path}(G)$ with $\mu_i \neq \mu_l$ for $i \neq l$ and $\lambda_j \in \text{Path}(E)\backslash\text{Path}(G)$ such that $\lambda_j \neq \lambda_k$ for $j \neq k$. Then $aJ = (\sum_{i=1}^{n} a_i\mu_i + \sum_{j=1}^{m} b_j\lambda_i)J = (\sum_{i=1}^{n} a_i\mu_i)J \subset J$ and J is a left KE-module. In addition, all the left KE-modules contained in I are left KG-modules because KG is a subalgebra of KE.

Finally, thanks to the case $n = 1$ we have that KG is left Noetherian and this implies that I is left Noetherian as a KG-module. However, all the left KE-submodules in I are left KG-submodules in I and vice-versa which means that I is left Noetherian. □

Once we know the geometric property of a graph for its path algebra to be left or right Noetherian, we would like to get to know better the structure of the Noetherian path algebras. The Peirce decomposition gives us a first clue o this.

Proposition 7.5. *Let us consider a directed graph E and $A = KE$ the corresponding path algebra over the field K. If the algebra A is left or right Noetherian, then A is isomorphic to a triangular matrix algebra. That is, A is isomorphic to*

$$\begin{pmatrix} S & B \\ 0 & T \end{pmatrix}$$

with S and T two K-algebras and B a (S,T)-bimodule.

Proof. As we know from Proposition 7.4, being left (resp. right) Noetherian means that $|E^0 \cup E^1| < +\infty$ and all the cycles in the graph do not have entries (resp. exits). Considering these facts, if we define $S^0 = \{v \in E^0 \colon v \text{ is in a cycle}\}$ (resp. $S^0 = \{v \in E^0 \colon v \text{ is not in a cycle}\}$) and $T^0 = E^0 \backslash S^0$. These two sets will allow us to construct a complete system of orthogonal idempotents $e_1 = \sum_{v \in S^0} v, \quad e_2 = \sum_{v \in T^0} v$. For the last part we just need to consider the Peirce decomposition with $S = e_1 A e_1$, $B = e_1 A e_2$ and $T = e_2 A e_2$. It is not difficult to check that under our hypothesis $e_2 A e_1 = 0$. □

Observation 7.6. The reader may not find very difficult to check that if A is left (resp. right) Noetherian the algebra S (resp. T) is of the form

$$\begin{pmatrix} S_1 & 0 & \cdots & 0 \\ 0 & S_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_m \end{pmatrix}$$

with m the number of cycles and S_i the algebra of a cycle for every $i = 1, \ldots, m$.

7.2 RADICAL OF A PATH ALGEBRA

Once we have a characterisation of the Noetherian path algebras, we would like to obtain a structure theorem for them. The Jacobson radical of an associative algebra, together with the central closure of the path algebra of a cycle, will be key to achieve our aim. The central closure is discussed in Section 6.1 and the characterization of the Jacobson radical of a path algebra is mentioned (without a proof) in [32] for finite acyclic graphs. However, we will prove the same result for arbitrary graphs.

Definition 7.7. The *Jacobson radical of* a ring R is the set

$$\operatorname{rad}(R) := \{r \in R \colon rM = \{0\} \text{ for all simple module } M\}.$$

Remark 7.8. It is well known that the Jacobson radical is also the intersection of all maximal left (resp. right) ideals of R (the reader can check [75, Chapter 2]). In case A is an associative algebra over a field K, then the Jacobson radical of A is defined analogously. It is important to remember that this set is an ideal.

Consider a ring R, not necessarily unital, and define the product

$$a \diamond b := a + b - ab.$$

An element $z \in R$ is said to be *quasi-regular* if there exists $z' \in R$ such that $z \diamond z' = z' \diamond z = 0$ (see [64, p. 8]). Then, in [64, Proposition 1, p. 9] it is proved that the radical is the set of all $z \in R$ such that azb is quasi-regular for any $a, b \in R$.

In the path algebra context, the Jacobson radical will be related to what will be called the path without return.

Definition 7.9. Consider a directed graph E. We say that $\lambda \in \operatorname{Path}(E)$ is *a path without return* if there is no $\mu \in \operatorname{Path}(E)$ such that $s(\mu) = r(\lambda)$ and $r(\mu) = s(\lambda)$.

Proposition 7.10. *If E is a directed graph and KE is the path algebra of E over the field K, then* $\operatorname{rad}(KE)$ *is the vector space generated by the set of paths without return, that is,*

$$\operatorname{rad}(KE) = \operatorname{span}_K \{\lambda \in \operatorname{Path}(E) \colon \ell(\lambda) \geqslant 1, \nexists\, \mu \in \operatorname{Path}(E), s(\mu) = r(\lambda), r(\mu) = s(\lambda)\}.$$

Proof. Consider λ a path without return. Following [64, Proposition 1, p. 9], we have to prove that $a\lambda b$ is quasi-regular for any $a, b \in KE$. First, observe that $z \diamond (-z) = (-z) \diamond z = z^2$. In this sense, if $z := a\lambda b$, then we are going to prove that $z^2 = 0$. This being said, $(a\lambda b) \diamond (-a\lambda b) = a\lambda ba\lambda b$ and if we write $ba = \sum_{i=1}^n k_i \tau_i$ with $k_i \in K$ and $\tau_i \in \operatorname{Path}(E)$, we have $\lambda ba\lambda = \sum_i k_i \lambda \tau_i \lambda$. However, since λ is a path without return, if $\lambda \tau_i \lambda \neq 0$, then $s(\tau_i) = r(\lambda)$ and $r(\tau_i) = s(\tau_i)$ which can not be possible. Then, $\lambda \tau_i \lambda = 0$ for every $i = 1, \ldots, n$ and $a\lambda ba\lambda b = a \left(\sum_i k_i \lambda \tau_i \lambda\right) b = 0$. Thus, $a\lambda b$ is quasi-regular for every $a, b \in KE$

implying $\lambda \in \mathrm{rad}(KE)$. Consequently, $\mathrm{rad}(KE)$ contains the linear span of all no-return paths because the Jacobson radical is an ideal.

Conversely, consider $z \in \mathrm{rad}(KE)$, we can express this element as $z = \sum_{i=1}^{n} k_i\lambda_i$ with $k_i \in K\setminus\{0\}$, $\lambda_i \in \mathrm{Path}(E)$ for every $i = 1\ldots, n$. If

$$J = \{i \in \{1, \ldots, n\} \colon \lambda_i \text{ is a path without return}\},$$

then we can decompose z as $z = \sum_{j\in J} k_j\lambda_j + \sum_{l\in\{1,\ldots,n\}\setminus J} k_l\lambda_l$. Following the previous implication, we have that $z' = \sum_{j\in J} k_j\lambda_j \in \mathrm{rad}(KE)$ and since $\mathrm{rad}(KE)$ is an ideal $z - z' \in \mathrm{rad}(KE)$. As a consequence, we can assume that z is a linear combination of paths with return. Consider $z := \sum_{i=1}^{n} k_i\lambda_i$, since $\mathrm{rad}(KE)$ is an ideal, $uzv \in \mathrm{rad}(KE)$ for every $u, v \in E^0$ and we can assume that $z := \sum_{i=1}^{n} k_i\lambda_i \in \mathrm{rad}(KE)$ where $s(\lambda_i) = u$, $r(\lambda_i) = v$ for every $i = 1, \ldots, n$ and $\lambda_i \neq \lambda_j$ if $i \neq j$. Furthermore, given that the λ_i are paths with return such that $s(\lambda_i) = u$ and $r(\lambda_i) = v$ for every $i = 1, \ldots, n$, there is a path $\mu \in \mathrm{Path}(E)$ such that $s(\mu) = v$ and $r(\mu) = u$. Then, $z\mu \in \mathrm{rad}(KE)$, and once again we may assume that $z := \sum_i k_i\lambda_i \in \mathrm{rad}(KE)$ with $k_i \in K\setminus\{0\}$, $\lambda_i \in \mathrm{Path}(E)$, $s(\lambda_i) = r(\lambda_i) = u$ for each $i = 1, \ldots, n$ and $\lambda_i \neq \lambda_j$. But then, taking into account [64, Proposition 1, p. 48], $z \in uKEu \cap \mathrm{rad}(KE) = \mathrm{rad}(uKEu) = 0$. We have that $\mathrm{rad}(uKEu) = 0$ because $uKEu$ is a free associative algebra by Lemma 5.6 and there are two possibilities. If $uKEu \cong K$ or $uKEu \cong K[x]$, then it is well known that their radical is null. If $uKEu$ is generated by more than one element, the by [75, Proposition 11.23], the algebra is left primitive. Since being primitive means that there is a faithful irreducible module, hence its radical is zero. Finally, this means that every nontrivial element in the radical is a linear combination of paths without return, which ends the proof. □

Although we have already obtained a characterisation of the Jacobson radical of a path algebra in terms of its generators as a vector space, we can go deeper and describe it in terms of its generators as an ideal. This is shown in the following theorem.

Theorem 7.11. *Consider a directed graph E and a field K. If KE is the path algebra of the graph E over K, then $\mathrm{rad}(KE)$ is the two-sided ideal generated by the set of edges without return, that is,*

$$\mathrm{rad}(KE) = \left(e \in E^1 \colon \nexists \lambda \in \mathrm{Path}(E) \text{ with } s(\lambda) = r(e), r(\lambda) = s(e)\right).$$

Proof. Define $I = \big(e \in E^1 \colon \nexists \lambda \in \mathrm{Path}(E) \text{ with } s(\lambda) = r(e), r(\lambda) = s(e)\big)$ the ideal generated by the set of edges without return. First of all, Proposition 7.10 directly implies that $I \subseteq \mathrm{rad}(KE)$. Second of all, if we prove that every path without return is in I, then $\mathrm{rad}(KE) \subseteq I$ by Proposition 7.10. Consider λ a path without return, that is, $\lambda = f_1 f_2 \cdots f_n \in \mathrm{Path}(E)$ such that there is no path $\mu \in \mathrm{Path}(E)$ with $s(\mu) = r(\lambda)$ and $r(\mu) = s(\lambda)$. Then, there is an edge f_i with $i \in \{1, \ldots, n\}$ such that f_i is an edge without return. If this were not true, then for every $i = 1, 2, \ldots, n$ there would exist a path $\mu_i \in \mathrm{Path}(E)$ such that $s(\mu_i) = r(f_i)$ and $r(\mu_i) = s(f_i)$. In this way, we could consider the path $\mu = \mu_n \mu_{n-1} \cdots \mu_1 \in \mathrm{Path}(E)$ which is a return path for λ and this can not be possible. In particular, $\lambda \in I$. □

Remark 7.12. If E is a finite graph without cycles, known as a *quiver*, then we denote KE_+ as the ideal generated by E^1. In this case, following Theorem 7.11 we have that $KE_+ = \mathrm{rad}(KE)$.

7.2.1 Semiprime radical

We have fully characterised the Jacobson radical of a path algebra. In fact, as we will see in this subsection, it coincides with the Semiprime radical of the path algebra.

Definition 7.13. An ideal I of an algebra A is said to be semiprime if the quotient A/I is a semiprime algebra. Equivalently I is a semiprime ideal when $xAx \subset I$ implies $x \in I$.

Note that if $\{I_\alpha\}_{\alpha \in \Lambda}$ is a collection of semiprime ideals, the intersection $\bigcap_{\alpha \in \Lambda} I_\alpha$ is also a semiprime ideal.

Definition 7.14. If $S \subset A$ is a subset of the algebra A, the intersection of all semiprime ideals containing S will be called *the semiprime ideal generated* by S.

If we compute then the intersection of all the semiprime ideals, then we have the minimal semiprime ideal.

Definition 7.15. The *semiprime radical* of an algebra A is defined as the intersection of all the semiprime ideals of the algebra. We will denote this ideal $\mathrm{Srad}(A)$.

Note that $\bar{A} := A/\mathrm{Srad}(A)$ is a semiprime algebra. The semiprime radical could be defined as the least ideal I such that U/I is semiprime.

We say that an element $a \in A$ is an *absolute zero divisor* if and only if $a \neq 0$ and $aAa = \{0\}$. Following Definition 5.1 an associative algebra A is semiprime if and only if there are not absolute zero divisors. Observe that, for any semiprime ideal I, if a is an absolute zero divisor $aAa = \{0\} \subseteq I$, then $a \in I$. This means that $\mathrm{Srad}(A)$ contains all the absolute zero divisors. Actually, by minimality, Srad is the semiprime ideal generated by the set of absolute zero divisors.

Proposition 7.16. *Consider E a directed graph and K a field, then*

$$\mathrm{Srad}(KE) = \mathrm{rad}(KE).$$

Proof. If Y is the set of edges without return, then by Theorem 7.11, $\mathrm{rad}(KE)$ is the ideal generated by Y. Consider $f \in E^1$ an edge without return, then $s(f) \neq r(f)$ and there is not any path $\lambda \in \mathrm{Path}(E)$ such that $s(\lambda) = r(f)$ and $r(\lambda) = s(f)$. This means that $fKEf = 0$ and f is an absolute zero divisor. Since $\mathrm{Srad}(KE)$ is an ideal, we have that $\mathrm{rad}(KE) \subseteq \mathrm{Srad}(KE)$.

As we have already mentioned, the Jacobson radical of a path algebra is an ideal generated by a set of edges, this means that $KE/\mathrm{rad}(KE)$ is isomorphic to the path algebra of the collapse $E\backslash Y$. The graph $E\backslash Y$ is a disjoint union of all the strongly connected components of E. Following Proposition 5.3, $K(E\backslash Y)$ is semiprime. In other words, $\mathrm{rad}(KE)$ is a semiprime ideal and $\mathrm{Srad}(KE) \subseteq \mathrm{rad}(KE)$. □

7.3 STRUCTURE OF NOETHERIAN PATH ALGEBRAS

The fact that the Jacobson radical of a path algebra is an ideal generated by edges is really interesting. This means that the algebra $KE/\mathrm{rad}(KE)$ is isomorphic to a path algebra of a new graph. In particular, by Proposition 1.20 we have that it is isomorphic to the path algebra of the graph $F = E\backslash Y$ with Y the set of edges without return. Furthermore, $KE/\mathrm{rad}(KE)$ is a semiprime algebra. By eliminating all the edges without return in the graph E all that we have left are paths with return and that is the condition needed for the path algebra to be semiprime (Proposition 5.3). In consequence, following Theorem 5.11,

$$KE/\mathrm{rad}(KE) \cong \bigoplus_{i\in I} Kv_i \oplus \bigoplus_{j\in J} KC_{n_j} \oplus \bigoplus_{l\in L} KF_l \tag{11}$$

where $\{v_i\}_{i\in I} \cup \{C_{n_j}\}_{j\in J} \cup \{F_l\}_{l\in L}$ are the strongly connected components of the graph E. The set $\{v_i\}_{i\in I}$ corresponds to isolated vertices, $\{C_{n_j}\}_{j\in J}$ isolated cycles and $\{F_l\}_{l\in L}$ strongly connected cycles with entries (exits). In particular, KF_l is primitive for every $l \in L$, KC_{n_j} is prime for every $j \in J$ and Kv_i is simple for every $i \in I$.

As we know from Proposition 7.5, the Noetherian path algebras (left or right) are isomorphic to upper triangular matrix algebras. In order to study their corresponding Jacobson radical we obtain the following result.

Proposition 7.17. *Let us consider a directed graph E with $A = KE$ its corresponding path algebra over the field K. If A is left Noetherian, following the description given in Proposition 7.5, the Jacobson radical of A is*

$$\mathrm{rad}(A) = \begin{pmatrix} 0 & B \\ 0 & \mathrm{rad}(T) \end{pmatrix}.$$

Analogously, if A is right Noetherian, then

$$\mathrm{rad}(A) = \begin{pmatrix} \mathrm{rad}(S) & B \\ 0 & 0 \end{pmatrix}.$$

Proof. These results are immediate to prove following Proposition 7.5 and Corollary [60, II.1]. Since A is Noetherian, by Proposition 7.4 is an upper triangular matrix ring with

$$A = \begin{pmatrix} S & B \\ 0 & T \end{pmatrix}.$$

In addition, by [60, Corollary II.1]

$$\mathrm{rad}\begin{pmatrix} S & B \\ 0 & T \end{pmatrix} = \begin{pmatrix} \mathrm{rad}(S) & B \\ 0 & \mathrm{rad}(T) \end{pmatrix}.$$

If A is left Noetherian and S is the component in the Peirce decomposition related to the vertices that belong to cycles, then by Theorem 7.11 $\mathrm{rad}(S) = \{0\}$. On the other hand, if A is right Noetherian, in this case, T is the component related to cycles and $\mathrm{rad}(T) = \{0\}$. □

Finally, after all the results obtained above, we are in conditions to develop an structure theorem for the Noetherian (left and right) path algebras. We will make use of the concepts of socle of a path algebra (Proposition 3.4) the radical of a path algebra (Theorem 7.11), together with Proposition 7.5.

Theorem 7.18 (Structure Theorem for Noetherian path algebras). *If E is a directed graph with KE left (resp. right) Noetherian, then*

$$KE/\operatorname{rad}(KE) \cong \left(\bigoplus_{i=1}^{n_0} Kv_i\right) \oplus \left(\bigoplus_{j=1}^{s} KC_{n_j}\right).$$

Consider two directed graphs E and F, and denote by s and t the number of cycles in the graph E and F respectively and n_i, m_j (for $i = 1, \ldots, s$ and $j = 1, \ldots, t$) the number of vertices in the corresponding cycle. If KE and KF are left (resp. right) Noetherian and $KE \cong KF$, then $|E^0| = |F^0|$, $s = t$ and there is a permutation $\sigma \in S_s$ such that $n_i = m_{\sigma(i)}$ for $i = 1, \ldots, s$.

Proof. If E is a directed graph such that KE is left Noetherian (resp. right Noetherian), then by Proposition 7.4 E is a finite directed graph such that every cycle does not have entries. In addition, the Jacobson radical of KE is the ideal generated by the edges without return (Theorem 7.11). In our case, these are the edges outside the cycles. Finally, by Proposition 1.20, $KE/\operatorname{rad}(KE) \cong K(E\backslash Y)$ where Y the set of edges without return which coincides with the set of edges outside the cycles. This means that

$$KE/\operatorname{rad}(KE) \cong \bigoplus_{i=1}^{n_0} Kv_i \oplus \bigoplus_{j=1}^{s} KC_{n_j}$$

where n_0 is the number of vertices outside the cycles, s is the number of cycles and n_j is the number of vertices in a cycle for every $j = 1, \ldots, s$.

For the second part of the result, consider $A_1 = KE$ and $A_2 = KF$ left Noetherian algebras (for the right Noetherian algebras is analogous). By Proposition 7.5 we have

$$A_1 \cong \begin{pmatrix} \bigoplus_{i=1}^{s} KC_{n_i} & B_1 \\ 0 & T_1 \end{pmatrix} \cong \begin{pmatrix} \bigoplus_{j=1}^{t} KC_{m_j} & B_2 \\ 0 & T_2 \end{pmatrix} \cong A_2.$$

In addition, $\operatorname{rad}(A_1) \cong \operatorname{rad}(A_2)$ by Theorem [69, Theorem 0.1.4]. Following Proposition 7.17, we can express this as

$$\operatorname{rad}(A_1) \cong \begin{pmatrix} 0 & B_1 \\ 0 & \operatorname{rad}(T_1) \end{pmatrix} \cong \begin{pmatrix} 0 & B_2 \\ 0 & \operatorname{rad}(T_2) \end{pmatrix} \cong \operatorname{rad}(A_2).$$

Define $\overline{A_1} = A_1/\operatorname{rad}(A_1)$ and $\overline{A_2} = A_2/\operatorname{rad}(A_2)$. Since the path algebras and their radicals are isomorphic, we have that $\overline{A_1} \cong \overline{A_2}$ which means

$$\overline{A_1} \cong \begin{pmatrix} \bigoplus_{i=1}^{s} KC_{n_i} & 0 \\ 0 & K^{n_0} \end{pmatrix} \cong \begin{pmatrix} \bigoplus_{j=1}^{t} KC_{m_j} & 0 \\ 0 & K^{m_0} \end{pmatrix} \cong \overline{A_2}$$

with n_0 the number of vertices outside the cycles of E and m_0 the number of vertices outside the cycles of F. Now, by Theorem [69, 9.1.4], we have that the socles are isomorphic, $\operatorname{soc}(\overline{A_1}) \cong \operatorname{soc}(\overline{A_2})$. By Corollary 3.7, $\operatorname{soc}(\overline{A_1}) = K^{n_0}$ and $\operatorname{soc}(\overline{A_2}) = K^{m_0}$, and since they are isomorphic this means that $n_0 = m_0$. Finally, $\bigoplus_{i=1}^{s} KC_{n_i} \cong \bigoplus_{j=1}^{t} KC_{m_j}$ because $\overline{A_1} \cong \bigoplus_{i=1}^{s} KC_{n_i} \oplus K^{n_0} \cong \bigoplus_{j=1}^{t} KC_{m_j} \oplus K^{n_0} \cong \overline{A_2}$. Since the path algebra of a cycle is prime (Proposition 5.4) this is a decomposition in indecomposable ideals. Consequently, we can apply Proposition 5.10 and we have that $s = t$ and there is a permutation $\sigma \in S_t$ such that $KC_{n_i} \cong KC_{m_{\sigma(i)}}$. Consequently, $n_i = m_{\sigma(i)}$. Indeed, consider the commutator of KC_n denoted by $[KC_n, KC_n]$. If $f \in C_n^1$, then $f = [f, r(f)] \in [KC_n, KC_n]$, which means that $KC_n/[KC_n, KC_n] \cong K^n$. If $KC_{n_i} \cong KC_{m_{\sigma(i)}}$, then

$$K^{n_i} \cong KC_{n_i}/[KC_{n_i}, KC_{n_i}] \cong KC_{m_{\sigma(i)}}/[KC_{m_{\sigma(i)}}, KC_{m_{\sigma(i)}}] \cong K^{m_{\sigma(i)}}$$

□

In order to describe the structure of left Noetherian path algebras KE, we first approach its quotient $\overline{KE} = KE/\operatorname{rad}(KE)$. Equivalently, we consider a left Noetherian Jacobson semisimple path algebra KE. Then $KE = \operatorname{soc}(KE) \oplus \operatorname{Ann}(\operatorname{soc}(KE))$ and the socle is a finite direct sum K^n (with componentwise product). A first invariant under isomorphism is the pair $(\operatorname{soc}(KE), \operatorname{Ann}(\operatorname{soc}(KE)))$. We have $\operatorname{Ann}(\operatorname{soc}(KE)) = \oplus_i KC_{n_i}$ (finite direct sum). Since the central closure of each KC_{n_i} is $\widehat{KC_{n_i}} \cong M_{n_i}(K(x))$ we have that the central closure of $\operatorname{Ann}(\operatorname{soc}(KE))$ is an Artinian semisimple $K(x)$-algebra and, in fact, is isomorphic to the (finite) direct sum $\oplus_i M_{n_i}(K(x))$. Therefore, an isomorphism of left Noetherian J-semisimple path algebras $KE \cong KF$ directly leads to an isomorphism of Artinian semisimple $K(x)$-algebras. Consequently the tuples $(n_1, \ldots, n_s)$, $(m_1, \ldots, m_t)$ of both algebras have the same length and up to a permutation the n_i's and the m_j's coincide. Summarizing: the tuple $(n, n_1, \ldots, n_k)$ determine completely KE and $KE = K^n \oplus KC_{n_1} \oplus \cdots \oplus KC_{n_k}$. Furthermore, $n = \dim_K(\operatorname{soc}(KE))$ and each n_i^2 is the dimension (as $K(x)$-vector space) of the simple components of the Artinian semisimple part $\widehat{\operatorname{Ann}(\operatorname{soc}(KE))}$. Alternatively, one can see that each n_i is the uniform dimension of KC_{n_i}.

Part III

ALGEBRAIC ENTROPY

Chapter 8

ENTROPY OF PATH AND LEAVITT PATH ALGEBRAS OF FINITE GRAPHS

In the study of algebras, the notion of dimension plays an important role when trying to classify them. It is, in most cases, the first clue if two algebras are isomorphic or not. However, when we are working with infinite-dimensional algebras it does not make much sense. But, instead of the dimension, one can categorize an algebra by the growth of the dimension of finitely generated subalgebras, if the algebra is finitely generated. For commutative algebras, this results in the Krull dimension, see e.g. [43] or in the Gelfand-Kirillov dimension in the general case, see e.g. [51, 98]. Even so, this new concept does not deal with the problem that we had before. There are many examples of algebras with infinite GK-dimension. For this purpose, in the case of groups and graded algebras, the notion of algebraic entropy is introduced, see e.g. [37, 87]. The concept of entropy itself was sketched for the first time by Adler, Konheim and McAndrew for compact topological spaces [4]. They define the topological entropy via covers and apply it to dynamical systems and endomorphisms of abelian groups. A generalisation of this notion is given by D. Gonçalves, D. Royer and F. A. Tasca in [56] which is used in infinite alphabet shift spaces. The algebraic entropy for groups was later studied by Weiss [108]. A different definition was given by Peters [89] for automorphisms of abelian groups. It coincides with the one by Weiss on endomorphisms of torsion abelian groups. Recently, the algebraic entropy was used and generalized in group theory in e.g. [38, 55]. For graded algebras the entropy was studied in [87]. This is where we will find inspiration to generalise this to the filtered algebras. The reader might be familiar with the concept of entropy in physics. In statistical mechanics the entropy consists of counting the number of possible states that a system can be in. The algebraic entropy can be considered similarly. The microstates that Boltzmann [24] used in his definition of the entropy are replaced by the span of finite-dimensional subspaces generating the infinite-dimensional algebra. The growth is then defined as the dimension of consecutive factor spaces.

8.1 ENTROPY OF FILTERED ALGEBRAS

We say that a K-algebra A is a *finitely generated algebra* if there is a finite set $S = \{a_1, \dots, a_n\} \subset A$ such that every element in A can be obtained by sums, linear combinations and products of elements in S. In particular, for every element $a \in A$ there is a polynomial $p \in K\langle x_1, \dots, x_n\rangle$ (the polynomial ring of non-commutative variables) such that $a = p(a_1, a_2, \dots, a_n)$. If we consider the vector space V generated by the set S, then we say that A is generated by a finite-dimensional vector space. Defining $V^n = \text{span}\{v_1 \cdots v_n \colon v_i \in V,\ i = 1, \dots, n\}$ and $V_n = V + V^2 + \cdots + V^n$, the algebra A is generated by the space V if and only if $A = \bigcup_{n=1}^{\infty} V_n$. In this sense, the Gelfand-Kirillov dimension can be defined.

Definition 8.1. [51] Let A be an algebra, which is generated by a finite-dimensional subspace V. Then, $g_V(n) := \dim V^n < \infty$ is the *growth function* of V. If $g_V(n)$ is polynomially bounded, then the *Gelfand-Kirillov dimension* of A is defined as

$$\text{GKdim}(A) := \limsup_{n\to\infty} \frac{\log g_V(n)}{\log(n)}.$$

The GK-dimension does not depend on a choice of the generating space V as long as $\dim(V) < \infty$. If the growth of A is not polynomially bounded, then $\text{GKdim}(A) = \infty$.

Note that if $g_V(n) = n^k$, then $\text{GKdim}(A) = k$ and if the algebra A is finite-dimensional, then $\text{GKdim}(A) = 0$.

For a graded algebra $A = \oplus_{n=0}^{\infty} A_n$ with $\dim(A_n) < +\infty$, the algebraic entropy of A has been defined in [87, Equation (1), p. 85] as

$$H(A) := \limsup_{n\to\infty} \sqrt[n]{\dim(A_n)}.$$

However, this definition is only restricted to a certain family of algebras. That is why we want to give a step forward and try to obtain a more general definition of algebraic entropy by using filtrations.

Definition 8.2. A K-algebra A is said to be *filtered* if it is endowed with a collection of subspaces $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ such that

1. $0 \subseteq V_1 \subseteq \cdots \subseteq V_n \subseteq V_{n+1} \subseteq \cdots \subseteq A$,
2. $A = \bigcup_{n\geqslant 1} V_n$,
3. $V_n V_m \subset V_{n+m}$.

The collection of vector spaces $\mathcal{F}$ is called a *filtration* of A. We say that the filtration is *finite* if it also verifies

4. $\dim(V_n/V_{n-1}) < +\infty$ for every $n \geqslant 2$.

Observation 8.3. We really want to highlight that during the course of this work all the filtrations that we are considering are finite. The reason for this will be explained in Section 9.1.

Remark 8.4. If the K-algebra A is finitely generated by a vector space V, then $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ with $V_n = V + V^2 + \cdots + V^n$ is a finite filtration for A. The first and second items are easy to check, let us verify the third one. Consider $x \in V_n$ and $y \in V_m$, in particular, we can express $x = \sum_{i=1}^{n} x_i$ and $y = \sum_{j=1}^{m} y_j$ with $x_i \in V^i$ and $y_j \in V^j$ for every $i = 1, \ldots, n$ and $j = 1, \ldots, m$. Then, $xy = \sum_{i,j=1}^{n,m} x_i y_j$. Each element $x_i y_j$ in the sum belongs to $V^i V^j = V^{i+j}$. In particular, we have that $xy \in V + V^2 + \cdots + V^{n+m} = V_{n+m}$. Finally, the fourth one is immediate since every vector space V_n is finite-dimensional as V^n is too.

If A is filtered K-algebra with (finite) filtration $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$, then we can define $\hat{A}_n := V_n/V_{n-1}$ for $n \geqslant 2$ and $\hat{A}_1 := V_1$. In addition, we can take $\hat{A} := \bigoplus_{n=1}^{\infty} \hat{A}_n$ and define the product

$$(x + V_{n-1})(y + V_{m-1}) := xy + V_{n+m-1}$$

where $x + V_{n-1} \in V_n/V_{n-1}$, $y + V_{m-1} \in V_m/V_{m-1}$. This product is well defined. Indeed, consider $x, x' \in V_n$ such that $x + V_{n-1} = x' + V_{n-1}$. This means that $x - x' \in V_{n-1}$. If we take $y \in V_m$, then $xy - x'y \in V_{n+m-1}$ which means that $xy + V_{n+m-1} = x'y + V_{n+m-1}$. Analogously, if $y + V_{m-1} = y' + V_{m-1}$ we have that $xy + V_{n+m-1} = xy' + V_{n+m-1}$. Finally,

$$(x' + V_{n-1})(y' + V_{m-1}) = x'y' + V_{n+m-1} =$$
$$= xy' + V_{n+m-1} = xy + V_{n+m-1} = (x + V_{n-1})(y + V_{m-1})$$

and the product does not depend on the element of the class chosen. In this way, $\hat{A}$, with the defined product and together the usual scalar multiplication and sum, is a K-algebra. Furthermore, it is a graded algebra. This allows us to define a notion of algebraic entropy for filtered algebras.

Definition 8.5. Let A be a filtered K-algebra with finite filtration $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$. Hence, we define the *algebraic entropy of* A for the filtration $\mathcal{F}$ as

$$\mathrm{h_{alg}}(A, \mathcal{F}) := \begin{cases} 0 & \text{if } \dim A < +\infty, \\ \limsup\limits_{n\to\infty} \dfrac{\log(\dim(V_n/V_{n-1}))}{n} & \text{otherwise.} \end{cases} \tag{12}$$

Remark 8.6. The only care we should have with this definition is that in case there is some step in the filtration for which $V_{n-1} = V_n$, then we have a $-\infty$ in the sequence. So the sequence $\frac{\log(\dim(V_n/V_{n-1}))}{n}$ takes values in $\mathbb{R} \cup \{-\infty\}$. Thus, the above limit could be $-\infty$ if the filtration stabilizes at some point. But this is only possible when A is finite-dimensional and we have defined $\mathrm{h_{alg}}(A, \mathcal{F}) = 0$ in this case.

As it is natural, this new definition of entropy extends the one given for graded algebras. Consider an infinite-dimensional graded algebra $A = \bigoplus_{n=0}^{\infty} A_n$ where $\dim(A_n) < +\infty$ for every $n \geqslant 0$. We can define the family of vector spaces $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ given by $V_n := \bigoplus_{i=0}^{n} A_i$. Indeed, $\mathcal{F}$ is a finite filtration. By construction, it is natural that

$$0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq V_{n+1} \subseteq \cdots \subseteq A$$

and $A = \bigcup_{n=1}^{\infty} V_n$. In addition,

$$V_n V_m = \left(\bigoplus_{i=0}^{n} A_i\right)\left(\bigoplus_{i=0}^{m} A_i\right) \subset \sum_{i,j=0}^{n,m} A_i A_j \subseteq \bigoplus_{k=0}^{n+m} A_k = V_{n+m}.$$

Finally, $\dim(V_n/V_{n-1}) = \dim(A_n) < +\infty$. If we compute the entropy of A for the filtration $\mathcal{F}$, we have that

$$\mathrm{h_{alg}}(A,\mathcal{F}) = \limsup_{n\to\infty} \frac{\log\left(\dim(V_n/V_{n-1})\right)}{n} = \limsup_{n\to\infty} \frac{\log\left(\dim(A_n)\right)}{n} =$$
$$= \limsup_{n\to\infty} \log\left(\sqrt[n]{\dim(A_n)}\right) = \log\left(\limsup_{n\to\infty} \sqrt[n]{\dim(A_n)}\right) = \log(H(A)).$$

This means that for a suitable filtration both entropies 'coincide'.

Whereas the Gelfand-Kirillov dimension is independent of the finite-dimensional system of generators chosen, the concept of entropy is more subtle. Actually, it is strongly tied to the filtration selected.

Proposition 8.7. *Let A be a filtered algebra with finite filtration $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$. Consider that the sequence $\left\{\frac{\log(\dim(V_n/V_{n-1}))}{n}\right\}_{n\geqslant 2}$ is convergent. If we define the family of vector spaces $\mathcal{G} = \{W_n\}_{n=1}^{\infty}$ such that $W_n := V_{nk}$ for any $n \geqslant 1$ and a fixed $k \in \mathbb{N}^*$, then*

1. *$\mathcal{G}$ is a finite filtration for A and*
2. *$\mathrm{h_{alg}}(A,\mathcal{G}) = k\,\mathrm{h_{alg}}(A,\mathcal{F})$.*

Proof. First of all, it is obvious, by the properties of the filtration that

$$0 \subseteq V_k \subseteq V_{2k} \subseteq V_{3k} \subseteq \cdots \subseteq V_{nk} \subseteq V_{(n+1)k} \subseteq \cdots$$

and taking into account that $W_n = V_{nk}$ we have that $\mathcal{G}$ forms a chain. Secondly, consider $x \in A$. By the properties of the filtration, there is a number $m \geqslant 1$ such that $x \in V_m$. Considering a proper number $n \in \mathbb{N}$ such that $m \leqslant nk$ we have that $x \in V_m \subseteq V_{nk} = W_n$ and $x \in \bigcup_{n=1}^{\infty} W_n$ proving that $A = \bigcup_{n=1}^{\infty} W_n$. In addition $W_n W_m = V_{nk}V_{mk} \subseteq V_{(n+m)k} = W_{n+m}$. Finally, we have that

$$\dim\left(W_n/W_{n-1}\right) = \dim\left(V_{nk}/V_{(n-1)k}\right) = \dim(V_{nk}) - \dim(V_{(n-1)k}) =$$
$$= \dim(V_{nk}) - \dim(V_{nk-1}) + \dim(V_{nk-1}) - \cdots + \dim(V_{(n-1)k+1}) - \dim(V_{(n-1)k}) =$$
$$= \sum_{j=0}^{k-1} \dim(V_{nk-j}) - \dim(V_{nk-j-1}) = \sum_{j=0}^{k-1} \dim\left(V_{nk-j}/V_{nk-j-1}\right) < +\infty$$

and $\mathcal{G}$ is a finite filtration for A.

Let us now focus on the second item. For every $n \geqslant 1$ we have seen that

$$\dim(V_{nk}/V_{nk-k}) = \sum_{j=0}^{k-1} \dim(V_{nk-j}/V_{nk-j-1}).$$

Then, we have that

$$\mathrm{h}_{\mathrm{alg}}(A,\mathcal{G}) = \limsup_{n\to+\infty} \frac{\log\left(\dim(W_n/W_{n-1})\right)}{n} =$$
$$= \limsup_{n\to+\infty} \frac{\log\left(\dim\left(V_{nk}/V_{(n-1)k}\right)\right)}{n} = \limsup_{n\to+\infty} \frac{1}{n}\log\left(\sum_{j=0}^{k-1}\dim(V_{kn-j}/V_{kn-j-1})\right).$$

Take into account that for two sequences $\{a_n\}_{n\geqslant 0}$ and $\{b_n\}_{n\geqslant 0}$ with $a_n,\ b_n > 0$ the following equality holds

$$\limsup_{n\to+\infty} \frac{\log(a_n+b_n)}{n} = \max\left\{\limsup_{n\to+\infty}\frac{\log(a_n)}{n}, \limsup_{n\to+\infty}\frac{\log(b_n)}{n}\right\}.$$

Consequently, we have that

$$\mathrm{h}_{\mathrm{alg}}(A,\mathcal{G}) = \max_{0\leqslant j\leqslant k-1}\left\{\limsup_{n\to+\infty}\frac{\log\left(\dim\left(V_{nk-j}/V_{nk-j-1}\right)\right)}{n}\right\}.$$

Since the sequence $\left\{\frac{\log(\dim(V_n/V_{n-1}))}{n}\right\}_{n\geqslant 2}$ is convergent, then

$$\mathrm{h}_{\mathrm{alg}}(A,\mathcal{G}) = \max_{0\leqslant j\leqslant k-1}\left\{\lim_{n\to+\infty}\frac{nk-j}{n}\frac{\log\left(\dim\left(V_{nk-j}/V_{nk-j-1}\right)\right)}{nk-j}\right\} =$$
$$= \max_{0\leqslant j\leqslant k-1}\left\{k\,\mathrm{h}_{\mathrm{alg}}(A,\mathcal{F})\right\} = k\,\mathrm{h}_{\mathrm{alg}}(A,\mathcal{F}).$$

□

Proposition 8.7 is the perfect way to picture the dependence of the entropy of the algebra with the filtration chosen. Not only that, if we would try to avoid this problem by taking the supremum between all the possible entropies, Proposition 8.7 will make this impossible.

Corollary 8.8. *If A is a filtered algebra with $\mathcal{F}$ a finite filtration verifying the conditions of Proposition 8.7 and $\mathrm{h}_{\mathrm{alg}}(A,\mathcal{F}) \neq 0$, then there is a family of finite filtrations $\{\mathcal{G}_k\}_{k=1}^{\infty}$ such that*

$$\lim_{k\to+\infty}\mathrm{h}_{\mathrm{alg}}(A,\mathcal{G}_k) = +\infty.$$

In this way, if we define the entropy of a filtered algebra as

$$\mathrm{h}_{\mathrm{alg}}(A) = \sup\{\mathrm{h}_{\mathrm{alg}}(A,\mathcal{F})\colon \mathcal{F} \text{ finite filtration}\},$$

in most of the cases, $h_{alg}(A) = +\infty$ and we are facing an useless way of measuring algebras. The other possible way of avoiding the dependence of the filtration is using the infimum instead of the supremum. However, as we will see in Section 8.3, it will give us trouble too. For this reason, we will focus on this chapter on computing the entropy for specific filtrations.

8.2 ENTROPY UNDER DIFFERENT CONSTRUCTIONS

In this Section we study how the entropy of an algebra behaves under epimorphisms, monomorphisms and direct sums. In order to do so we think of (finite) filtered algebras as elements in a category.

Definition 8.9. We consider the *category of filtered (resp. finite filtered)* K-algebras, denoted by $\mathbf{Falg_K}$ (resp. $\mathbf{falg_K}$). Its objects are the couples $(A, \mathcal{F})$ where $\mathcal{F}$ is a filtration (resp. finite filtration) in A and its morphisms $(A, \mathcal{F})$ to $(B, \mathcal{G})$ are the K-algebra morphisms $f\colon A \to B$ such that $f(V_n) \subseteq W_n$, for all $V_n \in \mathcal{F}$ and $W_n \in \mathcal{G}$.

Note that an isomorphism $f\colon (A, \mathcal{F}) \to (B, \mathcal{G})$ implies $\dim(V_n) = \dim(W_n)$ for any n. Moreover, if the filtrations consist of finite-dimensional subspaces, then $f(V_n) = W_n$ for any n.

8.2.1 Epimorphisms and entropy

Consider an epimorphism of algebras $f\colon A \to B$. If $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ is a finite filtration on A, then we can construct on B a family of vector spaces simply applying f to the subspaces of $\mathcal{F}$. Furthermore, the family $\mathcal{G} = \{f(V_n)\}_{n=1}^{\infty}$ is a finite filtration on B.

1. Since $V_n \subseteq V_{n+1}$ then $f(V_n) \subseteq f(V_{n+1})$ and the family of sets $\mathcal{G}$ forms a chain.
2. Consider $y \in B$, f is an epimorphism and this means that there is some $x \in A$ such that $f(x) = y$. In particular, $x \in V_n$ for some $n \geqslant 1$ and $f(x) = y \in f(V_n)$. In consequence, $B = \bigcup_{n=1}^{\infty} f(V_n)$.
3. Take $y \in f(V_n)$ and $y' \in f(V_m)$, then there are $x \in V_n$ and $x' \in V_m$ such that $f(x) = y$ and $f(x') = y$. Thus, $yy' = f(x)f(x') = f(xx') \in f(V_n V_m) \subseteq f(V_{n+m})$ which means that $f(V_n)f(V_m) \subseteq f(V_{n+m})$.

4. For every $n \geqslant 2$ it is possible to define a map $\bar{f}\colon V_n/V_{n-1} \to f(V_n)/f(V_{n-1})$ such that $\bar{f}(x+V_{n-1}) := f(x)+f(V_{n-1})$. This map is well defined and is surjective. Take $x, x' \in V_n$ such that $x+V_{n-1} = x'+V_{n-1}$ which means that $x-x' \in V_{n-1}$. Since f is a linear map $f(x-x') = f(x)-f(x') \in f(V_{n-1})$, thus $f(x)+f(V_{n-1}) = f(x')+f(V_{n-1})$ and the map is well defined. To see that it is surjective consider $y+f(V_{n-1})$ with $y \in f(V_n)$, in particular there is an element $x \in V_n$ such that $f(x) = y$. Then,

$$\bar{f}(x+V_{n-1}) = f(x)+f(V_{n-1}) = y+f(V_{n-1}).$$

In addition, the reader can easily check that the map $\bar{f}$ is linear. In consequence, $\bar{f}$ is an epimorphism of vector spaces which leads to

$$\dim\left(f(V_n)/f(V_{n-1})\right) \leqslant \dim\left(V_n/V_{n-1}\right) < +\infty.$$

Ultimately, we have an epimorphism in the category of filtered algebras which we denote as $f\colon (A,\mathcal{F}) \to (B,\mathcal{G})$. If there is no danger of confusion, we use the same symbol f for both the epimorphism of algebras and of filtered algebras. As the epimorphisms contract dimensions, not only in the subspaces but also in the quotients, then we have that

$$\mathrm{h_{alg}}(B,\mathcal{G}) = \limsup_{n\to\infty} \frac{\log\left(\dim\left(f(V_n)/f(V_{n-1})\right)\right)}{n} \leqslant$$
$$\leqslant \limsup_{n\to\infty} \frac{\log\left(\dim(V_n/V_{n-1})\right)}{n} = \mathrm{h_{alg}}(A,\mathcal{F}).$$

Eventually, we can summarise this in the following result.

Proposition 8.10. *If A is an algebra with finite filtration $\mathcal{F}$ and $f\colon A \to B$ is an algebra epimorphism, then the family of vector spaces $\mathcal{G} = f(\mathcal{F})$ is a finite filtration for B such that $f\colon (A,\mathcal{F}) \to (B,\mathcal{G})$ is an algebra epimorphism and*

$$\mathrm{h_{alg}}(B,\mathcal{G}) \leqslant \mathrm{h_{alg}}(A,\mathcal{F}).$$

8.2.2 Monomorphisms and entropy

Take a monomorphism of algebras $g\colon B \hookrightarrow A$. If A is a filtered algebra with finite filtration $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$, then we can construct a family of vector subspaces of B by considering the preimages of the elements in $\mathcal{F}$ by g. Not only that but $\mathcal{G} = \{g^{-1}(V_n)\}_{n=1}^{\infty}$ is a finite filtration on B.

1. The vector spaces in $\mathcal{F}$ form a chain, that is $V_n \subseteq V_{n+1}$ for every $n \geqslant 1$, then $g^{-1}(V_n) \subseteq g^{-1}(V_{n+1})$ for every $n \geqslant 1$.

2. Take $y \in B$, then $g(y) \in A$. Since $A = \bigcup_{n=1}^{\infty} V_n$, there exists $n \geqslant 1$ such that $g(y) \in V_n$, that is, $y \in g^{-1}(V_n) \subseteq \bigcup_{n=1}^{\infty} g^{-1}(V_n)$. Then, $B = \bigcup_{n=1}^{\infty} g^{-1}(V_n)$.

3. Let $x \in g^{-1}(V_n)$ and $y \in g^{-1}(V_m)$, we have that $g(xy) = g(x)g(y) \in V_n V_m \subseteq V_{n+m}$. Then, $xy \in g^{-1}(V_{n+m})$ and $g^{-1}(V_n)g^{-1}(V_m) \subseteq g^{-1}(V_{n+m})$.

4. For every $n \geqslant 2$ we define the linear map $\bar{g}\colon g^{-1}(V_n)/g^{-1}(V_{n-1}) \to V_n/V_{n-1}$ such that $\bar{g}(y + g^{-1}(V_{n-1}) := g(y) + V_{n-1}$. In addition, this linear map is injective. If $\bar{g}\left(y + g^{-1}(V_{n-1}))\right) = 0$, then $g(y) + V_{n-1} = 0 + V_{n-1}$ which implies that $g(y) \in V_{n-1}$ and $y \in g^{-1}(V_{n-1})$. In particular, $y + g^{-1}(V_{n-1}) = 0 + g^{-1}(V_{n-1})$. Since the map $\bar{g}$ is injective, we have that

$$\dim\left(g^{-1}(V_n)/g^{-1}(V_{n-1})\right) \leqslant \dim\left(V_n/V_{n-1}\right) < +\infty.$$

As a summary, we have constructed a monomorphism in the category of filtered algebras which we denote as $g\colon (B,\mathcal{G}) \to (A,\mathcal{F})$. We will use the same symbol g for both the monomorphism of algebras and of filtered algebras. As the monomorphism 'expands' dimensions, then we have that

$$\begin{aligned}\mathrm{h_{alg}}(B,\mathcal{G}) &= \limsup_{n\to\infty} \frac{\log\left(\dim\left(g^{-1}(V_n)/g^{-1}(V_{n-1})\right)\right)}{n} \leqslant \\ &\leqslant \limsup_{n\to\infty} \frac{\log\left(\dim(V_n/V_{n-1})\right)}{n} = \mathrm{h_{alg}}(A,\mathcal{F}).\end{aligned}$$

Eventually, we can summarise this in the following result.

Proposition 8.11. *If $(A,\mathcal{F})$ is a filtered algebra and $g\colon B \to A$ is an algebra monomorphism, then the family of vector spaces $\mathcal{G} = g^{-1}(\mathcal{F})$ is a finite filtration for B such that $g\colon (B,\mathcal{G}) \to (A,\mathcal{F})$ is an algebra monomorphism and*

$$\mathrm{h_{alg}}(B,\mathcal{G}) \leqslant \mathrm{h_{alg}}(A,\mathcal{F}).$$

8.2.3 Direct sums and entropy

Take two K-algebras A and B with filtrations $\{V_n\}_{n=1}^{\infty}$ and $\{W_n\}_{n=1}^{\infty}$ respectively, we can consider the algebra $A \oplus B$ and check that the set $\{V_n \oplus W_n\}_{n=1}^{\infty}$ is a filtration. First of all, by contruction the elements of $\{V_n \oplus W_n\}_{n=1}^{\infty}$ form a chain, that is $V_n \oplus W_n \subseteq V_{n+1} \oplus W_{n+1}$ for every $n \geqslant 1$. Secondly, if $x + y \in A \oplus B$, then since $A = \bigcup_{n=1}^{\infty} V_n$ and $B = \bigcup_{n=1}^{\infty} W_n$, we have that there are $n, m \geqslant 1$ such that $x \in V_n$ and $y \in W_m$. Without loss of generality we can consider $n \geqslant m$ and then $x + y \in V_n \oplus W_m \subseteq V_n \oplus W_n$ which means that $A \oplus B = \bigcup_{n=1}^{\infty} (V_n \oplus W_n)$. In addition, if we consider $A, B \subset A \oplus B$ as subalgebras of the direct sum, we have that $A \cdot B = 0$, which implies that $(V_n \oplus W_n)(V_m \oplus W_m) = V_n V_m \oplus W_n W_m \subseteq V_{n+m} \oplus W_{n+m}$. Finally, if both of the filtrations are finite

$$\begin{aligned}\dim\left((V_n \oplus W_n)/(V_{n-1} \oplus W_{n-1})\right) &= \dim(V_n \oplus W_n) - \dim(V_{n-1} \oplus W_{n-1}) = \\ &= \dim(V_n) - \dim(V_{n-1}) + \dim(W_n) - \dim(W_{n-1}) = \\ &= \dim(V_n/V_{n-1}) + \dim(W_n/W_{n-1}) < +\infty\end{aligned}$$

Definition 8.12. In the previous settings, we denote by *sum filtration* to the family of sets

$$\mathcal{F} \oplus \mathcal{G} = \{V_n \oplus W_n\}_{n=1}^{\infty}.$$

Proposition 8.13. *Consider $(A, \mathcal{F})$ and $(B, \mathcal{G})$ two filtered algebras with finite filtrations $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ and $\mathcal{G} = \{W_n\}_{n=1}^{\infty}$. If $A \oplus B$ is a filtered algebra with the sum filtration $\mathcal{H} = \mathcal{F} \oplus \mathcal{G}$, then*

$$\mathrm{h}_{\mathrm{alg}}(A \oplus B, \mathcal{H}) = \max\left\{\mathrm{h}_{\mathrm{alg}}(A, \mathcal{F}), \mathrm{h}_{\mathrm{alg}}(B, \mathcal{G})\right\}.$$

Proof. Define $H_n := V_n \oplus W_n$ for $n \geqslant 1$. First observe that for every $n \geqslant 2$,

$$\dim\left(\frac{H_n}{H_{n-1}}\right) = \dim\left(\frac{V_n \oplus W_n}{V_{n-1} \oplus W_{n-1}}\right) = \dim\left(\frac{V_n}{V_{n-1}}\right) + \dim\left(\frac{W_n}{W_{n-1}}\right).$$

Secondly, as we did in the proof of Proposition 8.7, we take into account that for two sequences $\{a_n\}_{n \geqslant 0}$ and $\{b_n\}_{n \geqslant 0}$ with $a_n,\ b_n > 0$ the following equality holds:

$$\limsup_{n \to +\infty} \frac{\log(a_n + b_n)}{n} = \max\left\{\limsup_{n \to +\infty} \frac{\log(a_n)}{n}, \limsup_{n \to +\infty} \frac{\log(b_n)}{n}\right\}.$$

Consequently, we have that

$$
\begin{aligned}
\mathrm{h}_{\mathrm{alg}}(A \oplus B, \mathcal{H}) &= \limsup_{n \to +\infty} \frac{\log\left(\dim\left(H_{n+1}/H_n\right)\right)}{n} = \\
&= \limsup_{n \to +\infty} \frac{\log\left(\dim\left(V_{n+1}/V_n\right) + \dim\left(W_{n+1}/W_n\right)\right)}{n} = \\
&= \max\left\{\limsup_{n \to +\infty} \frac{\log\left(V_{n+1}/V_n\right)}{n}, \limsup_{n \to +\infty} \frac{\log\left(W_{n+1}/W_n\right)}{n}\right\} = \\
&= \max\left\{\mathrm{h}_{\mathrm{alg}}(A\mathcal{F}), \mathrm{h}_{\mathrm{alg}}(B, \mathcal{G})\right\}.
\end{aligned}
$$

□

8.3 ENTROPY OF PATH AND LEAVITT PATH ALGEBRAS

On the study of path algebras and Leavitt path algebras, the Gelfand-Kirillov dimension has been a new tool to consider. This new invariant has been analysed in both cases on the references [5, 86]. The next natural step, at least for us, consists of using the entropy. There is an approach of this in [56] where they compute it via covers and via metric spaces. Our point of view will be different, although we will eventually agree, as we will compute the algebraic entropy of path and Leavitt path algebras as filtered algebras. First, we will focus on the finite graph case and later we will generalise it to locally finite graphs via direct limits of algebras.

In order to compute the entropy by our definition, it is necessary to define a finite filtration in KE and $L_K(E)$. If the directed graph E is finite, then both algebras are finitely generated and we can construct a filtration as in the Gelfand-Kirillov dimension. For the path algebra, the generators are $E^0 \cup E^1$, so we consider $V = \mathrm{span}_K(E^0 \cup E^1)$. If we compute the next V_n as in the Gelfand-Kirillov dimension, then

$$
\begin{aligned}
&V_1 = V; \quad V_2 = V + V^2 = \mathrm{span}_K(E^0 \cup E^1 \cup E^2); \\
&V_3 = V + V^2 + V^3 = \mathrm{span}_K(E^0 \cup E^1 \cup E^2 \cup E^3)
\end{aligned}
$$

and in general $V_n = \mathrm{span}_K\left(\bigcup_{k=0}^{n} E^k\right)$.

In the Leavitt path algebra case, the generators are $E^0 \cup E^1 \cup \left(E^1\right)^*$. Consequently, considering $W = \text{span}_K(E^0 \cup E^1 \cup (E^1)^*$, we have that

$$W_1 = W,\ \ W_2 = W + W^2 = \text{span}_K \left(E^0 \cup E^1 \cup (E^1)^* \cup E^2 \cup (E^2)^* \cup E^1(E^1)^*\right);$$

$$W_3 = W + W^2 + W^3 =$$

$$= \text{span}_K \left(E^0 \cup E^1 \cup (E^1)^* \cup E^2 \cup E^1(E^1)^* \cup (E^2)^* \cup E^3 \cup E^2(E^1)^* \cup E^1(E^2)^* \cup (E^3)^*\right)$$

and in general $W_n = \text{span}_K \left(\bigcup_{k=0}^{n} \bigcup_{i=0}^{k} E^{k-i}(E^i)^*\right)$. Keep in mind that in order to compute these products it is necessary to take into account the relations $(CK1)$ and $(CK2)$.

Definition 8.14. Consider E a finite graph and K a field. We define the *standard filtration for the path algebra KE* as the family of vector spaces $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ with

$$\begin{aligned} V_1 &= \text{span}_K \{\mu \in \text{Path}(E) \colon \ell(\mu) \leqslant 1\}, \\ V_2 &= \text{span}_K \{\mu \in \text{Path}(E) \colon \ell(\mu) \leqslant 2\}, \\ &\vdots \\ V_n &= \text{span}_K \{\mu \in \text{Path}(E) \colon \ell(\mu) \leqslant n\}, \\ &\vdots \end{aligned} \tag{13}$$

In addition, the *standard filtration of the Leavitt path algebra $L_K(E)$* is the family of vector spaces $\mathcal{G} = \{W_n\}_{n=1}^{\infty}$ with

$$\begin{aligned} W_1 &= \text{span}_K\{\lambda\mu^* \colon \lambda, \mu \in \text{Path}(E),\ \ell(\lambda) + \ell(\mu) \leqslant 1\}, \\ W_2 &= \text{span}_K\{\lambda\mu^* \colon \lambda, \mu \in \text{Path}(E),\ \ell(\lambda) + \ell(\mu) \leqslant 2\}, \\ &\vdots \\ W_n &= \text{span}_K\{\lambda\mu^* \colon \lambda, \mu \in \text{Path}(E),\ \ell(\lambda) + \ell(\mu) \leqslant n\}, \\ &\vdots \end{aligned} \tag{14}$$

In general, during the course of future sections, we will omit the filtration in the computation of the algebraic entropy of path algebras and Leavitt path algebras of finite graphs since we will always be using the corresponding standard filtrations. That is, we will refer to

$$\text{h}_{\text{alg}}(KE) = \text{h}_{\text{alg}}(KE, \mathcal{F}),$$
$$\text{h}_{\text{alg}}(L_K(E)) = \text{h}_{\text{alg}}(L_K(E), \mathcal{G}).$$

Example 8.15. Consider the m-petals rose R_m as in Figure 18,

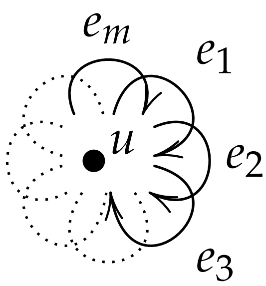

Figure 18: Graph R_m of the m-petals rose

where $R_m^0 = \{u\}$ and $R_m^1 = \{e_1, \dots, e_m\}$ with $s(e_i) = r(e_i) = u$ for every $i = 1, \dots, m$. Let us obtain the algebraic entropy of the path algebra KR_m. Remember that in order to do so we have to compute the dimension of the quotients. Since the vector spaces in the standard filtrations are generated by the path of length less or equal a certain n, and these are linearly independent, by counting the number of paths of length less or equal than n we can compute the dimension of the quotients. Define Δ_n the number of paths of length n. Since every combination of edges is possible, we have that $\Delta_0 = 1$, $\Delta_1 = m$, $\Delta_2 = m^2$ and in general $\Delta_n = m^n$. In the end, we have

$$\dim(V_n/V_{n-1}) = \dim(V_n) - \dim(V_{n-1}) = \sum_{k=0}^{n} \Delta_k - \sum_{k=0}^{n-1} \Delta_k = \Delta_n = m^n$$

and

$$\mathrm{h}_{\mathrm{alg}}(KR_m) = \limsup_{n\to+\infty} \frac{\log(\dim(V_n/V_{n-1}))}{n} = \limsup_{n\to+\infty} \frac{\log(m^n)}{n} =$$
$$= \limsup_{n\to+\infty} \log(m) = \log(m).$$

Now that we have computed the entropy for the path algebra, let us obtain it for the Leavitt path algebra, this case will be harder since we have to keep in mind the ghost edges and the (CK_1) and (CK_2) conditions. To be more specific, we have that $u = \sum_{k=1}^{m} e_i e_i^*$ which means that $\{u, e_1e_1^*, \dots, e_me_m^*\}$ are linearly dependent and we have to be careful with the computations. A system of generators of V_n is that of V_{n-1} plus the elements of $\bigcup_{i=0}^{n} E^{n-i}(E^i)^*$. To get a linearly independent set we must remove from each $E^{n-i}(E^i)^*$ with $i = 1, \dots, n-1$ the elements of $E^{n-i-1}e_1e_1^*(E^{i-1})^*$ (apply [1, Corollary 1.5.12]). That is, we are removing $m^{n-i-1}m^{i-1} = m^{n-2}$ elements. Finally, for $n \geqslant 2$ we have that

$$\dim(W_n/W_{n-1}) = \left|\bigcup_{i=0}^{n} E^{n-i}(E^i)^* \setminus \bigcup_{i=1}^{n-1} E^{n-i-1}e_1e_1^*(E^{i-1})^*\right| =$$
$$= m^n + (n-1)(m^n - m^{n-2}) + m^n = (n+1)m^n - (n-1)m^{n-2},$$

and we can compute

$$\begin{aligned} \mathrm{h}_{\mathrm{alg}}(L_K(R_m)) &= \limsup_{n\to+\infty} \frac{\log\left[(n+1)m^n - (n-1)m^{n-2}\right]}{n} = \\ &= \limsup_{n\to+\infty} \frac{\log\left[m^{n-2}\left((n+1)m^2 - n + 1\right)\right]}{n} = \\ &= \limsup_{n\to+\infty} \frac{(n-2)\log(m) + \log\left[(n+1)m^2 - n + 1\right]}{n} = \\ &= \log(m). \end{aligned}$$

Remark 8.16. Following [5, 86] the Gelfand-Kirillov dimension is infinite for both the path and the Leavitt path algebra of the graph R_m. However, the algebraic entropy is finite, having a first example of a way of measuring algebras that is finite when the dimension and the Gelfand-Kirillov is not.

The entropy of the algebra of the m-petal rose is a good first example since it allow us to observe the following.

Remark 8.17. If E is a finite directed graph, then the path algebra KE is generated by the vector space $V = \mathrm{span}_K(E^0 \cup E^1)$. The reader can easily check that $V^n = \mathrm{span}_K(E^n)$. In addition, $V^n \cap V^m = \{0\}$. Indeed, each V^n is an homogeneous component of the usual grading in KE. Then, $V_n = V \oplus V^2 \oplus \cdots \oplus V^n$ and $\dim(V_n/V_{n-1}) = \dim(V^n) = |E^n|$. This fact eases significantly the computation of the algebraic entropy. However, this is not the case for the Leavitt path algebra $L_K(E)$ because of the $(CK2)$ condition. In particular, if we consider the graph R_m we have that $L_K(R_m)$ is generated by the vector space $W = \mathrm{span}_K\{u, e_1, e_2, \ldots, e_m, e_1^*, e_2^*, \ldots, e_m^*\}$ and the square of it is

$$W^2 = \mathrm{span}_K\{e_ie_j, e_ie_j^*, e_i^*e_j^* : 1 \leqslant i, j \leqslant m\},$$

but $W \cap W^2 \neq \{0\}$ since $u = \sum_{i=1}^{m} e_i e_i^*$. This means that we are not facing a direct sum as it happened in the path algebra case and the computations get more difficult.

Even though there will be cases where the algebraic entropy of a Leavitt path algebra will be complex to obtain, we can still find certain bounds of its value by using the properties described in Section 8.2. If E is finite directed graph and $\hat{E}$ is its extended graph, there is a natural epimorphism of algebras $p\colon K\hat{E} \to L_K(E)$ since $L_K(E)$ is a quotient of the path algebra $K\hat{E}$ by the ideal generated by the relations $(CK1)$ and $(CK2)$. Take $\hat{\mathcal{F}} = \{\hat{V}_n\}_{n=1}^{\infty}$ the standard filtration of $K\hat{E}$ and $\mathcal{G} = \{W_n\}_{n=1}^{\infty}$ the standard filtration of $L_K(E)$, then $p(\hat{V}_n) = W_n$ and we have a filtered algebras epimorphism $p\colon (K\hat{E}, \hat{\mathcal{F}}) \to (L_K(E), \mathcal{G})$. In consequence, by Proposition 8.10 we have the next Lemma:

Lemma 8.18. *If E is a finite directed graph and K is a field, then*

$$\mathrm{h}_{\mathrm{alg}}(L_K(E)) \leqslant \mathrm{h}_{\mathrm{alg}}(K\hat{E}). \tag{15}$$

Not only that, there is also a natural monomorphism $i\colon KE \hookrightarrow L_K(E)$ since the subalgebra generated by $E^0 \cup E^1$ in $L_K(E)$ is isomorphic to KE. If $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ is the standard filtration of the path algebra KE, then $V_n = i^{-1}(W_n)$ and $i\colon (KE, \mathcal{F}) \hookrightarrow (L_K(E), \mathcal{G})$ is a monomorphism of filtered algebras. Applying Proposition 8.11 we have the following Lemma.

Lemma 8.19. *If E is a finite directed graph and K is a field, then*

$$\mathrm{h}_{\mathrm{alg}}(KE) \leqslant \mathrm{h}_{\mathrm{alg}}(L_K(E)). \tag{16}$$

That is, the entropy of the Leavitt path algebras is bounded by the entropy of two path algebras

Proposition 8.20. *The entropy of the algebra $L_K(E)$ of a finite graph is bounded as*

$$\mathrm{h}_{\mathrm{alg}}(KE) \leqslant \mathrm{h}_{\mathrm{alg}}(L_K(E)) \leqslant \mathrm{h}_{\mathrm{alg}}(K\hat{E}).$$

Example 8.21. As another example, consider the cycle with m vertices C_m and the extended graph $\hat{C}_m$ as in Figure 19. Let us compute the algebraic entropy of KC_m and $K\hat{C}_m$ in order to bound the possible values of $L_K(E)$. As we have already mentioned, in order to evaluate the entropy of a path algebra we need to count

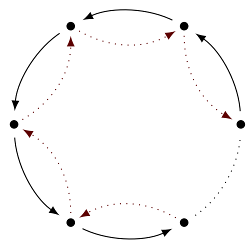

Figure 19: Extended graph $\hat{C}_m$.

the number of paths of a certain length. For each vertex $u \in C_m^0$, there is only a path of length n that starts from u, which means that $\dim V^n = m$. Consequently,

$$\mathrm{h_{alg}}(KC_m) = \limsup_{n \to +\infty} \frac{\log m}{n} = 0.$$

Counting the number of paths of length n in $\hat{C}_m$ is trickier. If we start at a certain vertex $u \in \hat{C}_m^0$, then we can go to another one by an edge or a ghost edge. In consequence, the number of paths of length n that start from a vertex u coincide with the number of words in the alphabet $\{f, b\}$ where f means 'go forward' and b means 'go backward'. Thus, there are 2^n possible words. Since there are m vertices, we have $2^n m$ paths of length n in total. Finally,

$$\mathrm{h_{alg}}(K\hat{C}_m) = \limsup_{n \to +\infty} \frac{\log(2^n m)}{n} = \limsup_{n \to +\infty} \frac{n \log(2) + \log(m)}{n} = \log(2).$$

By Proposition 8.20 we have that $0 \leqslant \mathrm{h_{alg}}(L_K(E)) \leqslant \log(2)$.

Another property of the entropy of filtered algebras that we can apply to the path and Leavitt path algebra is the one corresponding to direct sums of filtered algebras. If E is a finite directed graph with $\{E_i\}_{i=1}^m$ its connected components, then $KE = \bigoplus_{i=1}^m KE_i$ and $L_K(E) = \bigoplus_{i=1}^m L_K(E_i)$. Not only that, if $\{\mathcal{F}_i\}_{i=1}^m$ and $\{\mathcal{G}_i\}_{i=1}^m$ are families of standard filtrations with $\mathcal{F}_i$ and $\mathcal{G}_i$ corresponding to each of the components KE_i and $L_K(E_i)$, then $\mathcal{F} = \bigoplus_{i=1}^m \mathcal{F}_i$ and $\mathcal{G} = \bigoplus_{i=1}^m \mathcal{G}_i$ are standard filtrations for KE and $L_K(E)$ respectively. As a direct consequence of this and Proposition 8.13 the next result holds:

Proposition 8.22. *Consider E a finite directed graph with $\{E_i\}_{i=1}^m$ its corresponding connected components, then*

$$\mathrm{h_{alg}}(KE) = \max\left\{\mathrm{h_{alg}}(KE_i) \colon i = 1, \dots, m\right\},$$
$$\mathrm{h_{alg}}(L_K(E)) = \max\left\{\mathrm{h_{alg}}(L_K(E_i)) \colon i = 1, \dots, m\right\}.$$

We have computed the entropy for different examples. If we observe them, we can see that if there is a vertex which is base of two different cycles, the value of the entropy is different from zero. Indeed, this is always true.

Proposition 8.23. *Let E be a directed graph, if there is a vertex $u \in E^0$ which is a base of two different cycles, then*

$$\mathrm{h_{alg}}(KE) > 0.$$

Proof. If $u \in E^0$ is a base of two different cycles λ and μ, then take $n_\lambda := \ell(\lambda) \geqslant 1$ and $n_\mu := \ell(\mu) \geqslant 1$ and consider the ordered sequence $S = \{(k_1, k_2)_n\}_{n=1}^{\infty} = \{(1,1), (1,2), (2,2), (2,3), (3,3), \dots, (k_1, k_2), \dots\}$. Observe that this sequence verifies the following:

- $(k_1, k_2) \leqslant (k'_1, k'_2)$ if and only if $k_1 \leqslant k'_1$ and $k_2 \leqslant k'_2$,
- $(k_1, k_2) < (k'_1, k'_2)$ if and only if $k_1 < k'_1$ and $k_2 \leqslant k'_2$, or $k_1 \leqslant k'_1$ and $k_2 < k'_2$,
- if $(k_1, k_2) \leqslant (k'_1, k'_2)$, then $n_\lambda k_1 + n_\mu k_2 \leqslant n_\lambda k'_1 + n_\mu k'_2$,
- if $(k_1, k_2) < (k'_1, k'_2)$, then $n_\lambda k_1 + n_\mu k_2 < n_\lambda k'_1 + n_\mu k'_2$,
- if $n \to \infty$ then $k_1/k_2 \to 1$.

This means that $\{n_\lambda k_1 + n_\mu k_2\}_{(k_1,k_2)\in S}$ forms a strictly ascending sequence of natural numbers. Consequently, we have that, following Remark 8.17

$$\mathrm{h_{alg}}(KE) = \limsup_{n\to\infty} \frac{\log |E^n|}{n} \geqslant \limsup_{n_\lambda k_1 + n_\mu k_2 \to \infty} \frac{\log |E^{n_\lambda k_1 + n_\mu k_2}|}{n_\lambda k_1 + n_\mu k_2}.$$

The set of paths of length $n_\lambda k_1 + n_\mu k_2$ contain all the paths product of combinations of k_1 times the path λ and k_2 times the path μ. For example, if we consider $(1, 2)$, then

$$\{\lambda\mu^2, \mu\lambda\mu, \mu^2\lambda\} \subseteq E^{n_\lambda + 2n_\mu}.$$

Furthermore, we have that

$$|E^{n_\lambda k_1 + n_\mu k_2}| \geqslant \binom{k_1 + k_2}{k_1} = \frac{(k_1 + k_2)!}{k_1! k_2!}$$

which leads to

$$\limsup_{n_\lambda k_1 + n_\mu k_2 \to \infty} \frac{\log |E^{n_\lambda k_1 + n_\mu k_2}|}{n_\lambda k_1 + n_\mu k_2} \geqslant \limsup_{n_\lambda k_1 + n_\mu k_2 \to \infty} \frac{\log\left(\frac{(k_1+k_2)!}{k_1! k_2!}\right)}{n_\lambda k_1 + n_\mu k_2}.$$

Stirling's formula states that $\log(n!) = n\log n - n + \mathcal{O}(\log(n))$ where $|\mathcal{O}(\log n)| \leqslant C\log n$ with $C > 0$ for a big enough n. Using this formula we will prove that the last sequence is convergent and the value of its limit is positive.

$$\lim_{n_\lambda k_1+n_\mu k_2\to\infty} \frac{\log\left(\frac{(k_1+k_2)!}{k_1!k_2!}\right)}{n_\lambda k_1 + n_\mu k_2} = \lim_{n_\lambda k_1+n_\mu k_2\to\infty} \frac{\log[(k_1+k_2)!] - \log(k_1!) - \log(k_2!)}{n_\lambda k_1 + n_\mu k_2}$$

$$= \lim_{n_\lambda k_1+n_\mu k_2\to\infty} \frac{k_1\log\left(1+\frac{k_2}{k_1}\right) + k_2\log\left(1+\frac{k_1}{k_2}\right)}{n_\lambda k_1 + n_\mu k_2}$$

$$+ \lim_{n_\lambda k_1+n_\mu k_2\to\infty} \frac{\mathcal{O}(\log(k_1+k_2)) - \mathcal{O}(\log k_1) - \mathcal{O}(\log k_2)}{n_\lambda k_1 + n_\mu k_2} =$$

$$\lim_{n_\lambda k_1+n_\mu k_2\to\infty} \frac{\log\left(1+\frac{k_2}{k_1}\right) + \frac{k_2}{k_1}\log\left(1+\frac{k_1}{k_2}\right)}{n_\lambda + n_\mu \frac{k_2}{k_1}} = \frac{2}{n_\lambda + n_\mu}\log 2 > 0.$$

□

When we defined the concept of algebraic entropy we emphasized the dependency of this value with the chosen filtration. A possible way to avoid this was considering the supremum of the possible values of the entropy. However, in many cases, this value would be infinity. The other possible solution to this would be taking the infimum. However, this gives us some trouble too. Consider a finite directed graph E and define for a natural number $k \geqslant 2$ the family of vector spaces $\mathcal{H}_k = \{H_n^k\}_{n=1}^{\infty}$ in KE as

$$\begin{aligned} H_1^k &:= \operatorname{span}_K(E^0), \\ H_2^k &:= \operatorname{span}_K(E^0), \\ &\vdots \\ H_{k-1}^k &:= \operatorname{span}_K(E^0), \\ H_k^k &:= \operatorname{span}_K(E^0 \cup E^1), \\ H_{k+1}^k &:= \operatorname{span}_K(E^0 \cup E^1), \\ &\vdots \end{aligned}$$

and in general $H_n^k := \operatorname{span}_K\left(\bigcup_{m=0}^{\lfloor \frac{n}{k} \rfloor} E^m\right)$. This family of vector spaces it is pretty similar to the family of vector spaces given by the standard filtration. The main difference is that we start with the vector space generated by just the vertices and we do not increase the length of the paths until we get to a multiple of k. Indeed, if $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ is the standard filtration, then

$$H_n^k = V_{\lfloor \frac{n}{k} \rfloor} \text{ if } n \geqslant k.$$

The family $\mathcal{H}_k$ is also a finite filtration of KE. By construction it is immediate that $0 \subseteq H_1^k \subseteq H_2^k \subseteq \cdots \subseteq H_n^k \subseteq$ and $\bigcup_{n=1}^{\infty} H_n^k = \bigcup_{n=1}^{\infty} V_n = KE$ and since all of the vector spaces in the family are finite-dimensional then $\dim\left(H_n^k/H_{n-1}^k\right) < +\infty$ for every $n \geqslant 2$. Finally for every $n, m \geqslant k$

$$H_n^k H_m^k = V_{\lfloor \frac{n}{k} \rfloor} V_{\lfloor \frac{m}{k} \rfloor} \subseteq V_{\lfloor \frac{n}{k} \rfloor + \lfloor \frac{m}{k} \rfloor}$$

and since $\lfloor x \rfloor + \lfloor y \rfloor \leqslant \lfloor x + y \rfloor$ for every real number $x, y \in \mathbb{R}$ we have that

$$H_n^k H_m^k \subseteq V_{\lfloor \frac{n}{k} \rfloor + \lfloor \frac{m}{k} \rfloor} \subseteq V_{\lfloor \frac{n+m}{k} \rfloor} = H_{n+m}^k.$$

If $n < k$, then $H_n = \operatorname{span}_K(E^0)$ which is the space generated by the local units in the path algebra. As a consequence $H_n V = V$ for every vector space V in KE. Then,

$$H_n^k H_m^k = H_m^k \subseteq H_{n+m}^k.$$

Let us compute the entropy of KE under the filtration $\mathcal{H}_k$. First of all, observe that for $n \geqslant k$

$$\dim(H_n^k/H_{n-1}^k) = \dim\left(V_{\lfloor \frac{n}{k} \rfloor}/V_{\lfloor \frac{n-1}{k} \rfloor}\right)$$

which is only different from 0 when $k|n$, that is, when $n = qk$ with $q \in \mathbb{N}$. Thus, we have that

$$\begin{aligned} \mathrm{h_{alg}}(KE, \mathcal{H}_k) &= \limsup_{n \to \infty} \frac{\log(\dim(H_n^k/H_{n-1}^k))}{n} = \limsup_{q \to \infty} \frac{\log(\dim V_q/V_{q-1})}{qk} = \\ &= \limsup_{q \to \infty} \frac{1}{k} \frac{\log(\dim(V_q/V_{q-1}))}{q} = \frac{1}{k} \mathrm{h_{alg}}(KE). \end{aligned}$$

By considering different values of k, we have decreasing values of entropy.

Proposition 8.24. *Consider E a finite directed graph and K a field*

$$\inf\{\mathrm{h_{alg}}(KE, \mathcal{F}) \colon \mathcal{F} \text{ finite filtration } \} = 0.$$

We are now in a dead end as we can not make use of the supremum or the infimum and the entropy strongly depends with the chosen filtration. However, the family of filtrations that do not make possible the use of the infimum is very artificial. That is, this new filtration does not preserve certain properties that the standard filtration has. That is why, there is still certain hope as we could restrict the infimum to a certain family of filtrations with 'good' properties.

8.4 ENTROPY AND THE ADJACENCY MATRIX

One of the advantages of working with finite graphs is that we can make use of the adjacency matrix. In this matrix we have all of the information of the geometry of the graph and it even allow us to obtain information of the paths just by computing powers of it.

Definition 8.25. Let E be a finite directed graph with vertices $E^0 = \{v_1, v_2, \dots, v_m\}$. We define the *adjacency matrix* of E as the matrix

$$A_E := (a_{i,j})_{m\times m} \text{ where } a_{i,j} = \left|\{e \in E^1 : s(e) = v_i, r(e) = v_j\}\right|.$$

The size of the matrix indicates the number of vertices and each entry codes the number of edges from a vertex to another one. Not only that, if A_E is the adjacency matrix associated to a graph E, then $(A_E^n)_{i,j}$ is the number of paths of length n from the vertex v_i to v_j. Since what we need compute the entropy of the algebra KE is the number of paths of length n, just by adding all of the entries in A_E^n we have what we wanted.

Definition 8.26. Consider a square matrix $A = (a_{i,j}) \in \mathcal{M}_n(\mathbb{C})$. We denote by $\|\cdot\|_{1,1} \colon \mathcal{M}_n(\mathbb{C}) \to [0, +\infty)$ to the norm defined as

$$\|A\|_{1,1} := \sum_{i,j=1}^{n} |a_{i,j}|.$$

Lemma 8.27. *The norm $\|\cdot\|_{1,1}$ is submultiplicative.*

Proof. Take two matrices $A, B \in \mathcal{M}_n(\mathbb{C})$, we have that $(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$ and $\|AB\|_{1,1} = \sum_{i,j=1}^{n} \left|\sum_{k=1}^{n} a_{i,k}b_{k,j}\right|$. By the properties of the module of complex numbers

$$\|AB\|_{1,1} = \sum_{i,j=1}^{n} \left|\sum_{k=1}^{n} a_{i,k}b_{k,j}\right| \leqslant \sum_{i,j=1}^{n} \sum_{k=1}^{n} |a_{i,k}||b_{k,j}| =$$
$$= \sum_{i,k,l,j=1}^{n} |a_{i,k}||b_{l,j}|\delta_{k,l} \leqslant \sum_{i,k,l,j=1}^{n} |a_{i,k}||b_{l,j}| =$$
$$= \left(\sum_{i,k=1}^{n} |a_{i,k}|\right)\left(\sum_{l,j=1}^{n} |b_{l,j}|\right) = \|A\|_{1,1}\|B\|_{1,1}$$

□

Using the matrix norm that we have defined we have the next result.

Lemma 8.28. *Let A_E be the adjacency matrix associated to a finite directed graph E. Then*

$$\mathrm{h_{alg}}(KE) = \limsup_{n\to\infty} \frac{\log(\|A_E^n\|_{1,1})}{n}.$$

Proof. If $A_E^n = (a_{i,j})$, then $a_{i,j} \geqslant 0$ is the number of paths of length n from the vertex v_i to the vertex v_j in the graph. This implies that

$$\|A_E^n\|_{1,1} = \sum_{i,j=1}^{m} |a_{i,j}| = \sum_{i,j=1}^{m} a_{i,j} = |E^n| = \dim\left(V_n/V_{n-1}\right)$$

with $\{V_n\}_{n=1}^{\infty}$ the standard filtration of KE. □

Example 8.29. Let E be the finite directed graph given in Figure 20.

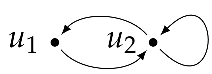

Figure 20: Directed graph E.

The corresponding adjacency matrix is $A_E = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. It is easy to prove by induction that

$$A_E^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix},$$

where $f_0 = 0$, $f_1 = 1$, $f_2 = 1$, and $f_n = f_{n-1} + f_{n-2}$ for $n \geqslant 3$. Denote

$$\varphi_+ := \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \varphi_- := \frac{1-\sqrt{5}}{2}.$$

Diagonalizing we have that $A_E^n = PD^nP^{-1}$ where

$$D = \begin{pmatrix} \varphi_+ & 0 \\ 0 & \varphi_- \end{pmatrix} \quad \text{and } P = \begin{pmatrix} -\varphi_- & -\varphi_+ \\ 1 & 1 \end{pmatrix},$$

that is,

$$A_E^n = \frac{1}{\sqrt{5}} \begin{pmatrix} \varphi_+^{n-1} - \varphi_-^{n-1} & \varphi_+^n - \varphi_-^n \\ \varphi_+^n - \varphi_-^n & \varphi_+^{n+1} - \varphi_-^{n+1} \end{pmatrix}.$$

Finally,

$$\|A_E^n\| = \frac{1}{\sqrt{5}} \left(\varphi_+^n - \varphi_-^n + \varphi_+^{n+1} - \varphi_-^{n+1} \right).$$

In order to compute

$$\mathrm{h_{alg}}(KE) = \limsup_{n\to\infty} \frac{\log(\|A_E^n\|)}{n}$$

we apply Stolz criterion. This criterion says that if $\{a_n\}_{n\geqslant 1}$ and $\{b_n\}_{n\geqslant 1}$ are two real numbers sequences such that $\{b_n\}_{n\geqslant 1}$ is strictly positive, increasing and not bounded such that $\lim_{n\to\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = L$, then $\lim_{n\to\infty} \frac{a_n}{b_n} = L$. This results in

$$\lim_{n\to\infty} \frac{\log(\|A_E^{n+1}\|) - \log\left(\|A_E^n\|\right)}{n+1-n} = \lim_{n\to\infty} \log\left(\frac{\|A_E^{n+1}\|}{\|A_E^n\|} \right) =$$

$$= \lim_{n\to\infty} \log\left(\frac{\varphi_+^{n+1} - \varphi_-^{n+1} + \varphi_+^n - \varphi_-^n}{\varphi_+^n - \varphi_-^n + \varphi_+^{n-1} - \varphi_-^{n-1}} \right) = \lim_{n\to\infty} \log\left(\frac{\varphi_+ - \varphi_- \frac{\varphi_-^n}{\varphi_+^n} + 1 - \frac{\varphi_-^n}{\varphi_+^n}}{1 - \frac{\varphi_-^n}{\varphi_+^n} + \varphi_+^{-1} - \varphi_-^{-1}\frac{\varphi_-^n}{\varphi_+^n}} \right).$$

Since $|\varphi_+| > |\varphi_-|$ we have that

$$\lim_{n\to\infty} \log\left(\frac{\varphi_+ - \varphi_- \frac{\varphi_-^n}{\varphi_+^n} + 1 - \frac{\varphi_-^n}{\varphi_+^n}}{1 - \frac{\varphi_-^n}{\varphi_+^n} + \varphi_+^{-1} - \varphi_-^{-1}\frac{\varphi_-^n}{\varphi_+^n}} \right) =$$

$$= \lim_{n\to\infty} \log\left(\frac{\varphi_+ + 1}{1 + \varphi_+^{-1}} \right) = \lim_{n\to\infty} \log\left(\varphi_+ \frac{\varphi_+ + 1}{\varphi_+ + 1} \right) = \log(\varphi_+).$$

Consequently,

$$\mathrm{h_{alg}}(KE) = \log(\varphi_+).$$

Observe in detail Example 8.29, after all the computations done, the value of the entropy coincides with the logarithm of the biggest eigenvalue. This is not a coincidence. The *spectral radius* of a square matrix A, denoted by $\rho(A)$, is the maximum of the modules of its eigenvalues, see [92, 18.8 Definition, p. 355]. Then, the formula for the spectral radius in a Banach algebra [92, 18.9 Theorem, p. 355] states that for a matrix norm $\|\cdot\|$

$$\lim_{n\to\infty} \sqrt[n]{\|A^n\|} = \rho(A).$$

As a direct consequence of this formula and Lemma 8.28 we have the next Proposition.

Proposition 8.30. *Let E be a finite directed graph and A_E its adjacency matrix. If KE is the corresponding path algebra we have that the entropy of KE is finite and*

$$\mathrm{h_{alg}}(KE) = \log(\rho(A_E)) \tag{17}$$

with $\rho(A_E)$ the spectral radius of A_E. In particular,

$$\mathrm{h_{alg}}(KE) = \limsup_{n\to\infty} \frac{\log\dim(V_n/V_{n-1})}{n} = \lim_{n\to\infty} \frac{\log\dim(V_n/V_{n-1})}{n}$$

for the standard filtration $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$.

Remark 8.31. As we have proved in Example 8.21, the entropy for the path algebra of a cycle C_n is $\mathrm{h_{alg}}(KC_n) = 0$. This means, by Proposition 8.30, that the spectral radius of the adjacency matrix of a cycle is 1.

We want to emphasize that Proposition 8.30 is not a new result. In the book by Lind and Marcus [80] they prove this ([80, Theorem 4.3.1]). Actually, using Perron-Frobenius they prove that there is a unique positive eigenvalue such that the entropy of a subshift space associated to a graph is the logarithm of it. The coincidence of both results is immediate even though the contexts are different. Once we translate both definitions of entropy to the adjacency matrix of a graph, the computations remain the same. We decided to add our prove because it just makes use of the properties of the matrix norm which will be handy in Section8.5.

As a direct consequence of Proposition 8.30 it is possible do certain transformations in the graph that do not modify the value of the entropy of the algebra. For example, eliminating sources and sinks does not change the algebraic entropy.

Corollary 8.32. *Consider E a finite directed graph and $X \subseteq \text{sink}(E) \cup \text{source}(E)$. Then*

$$\text{h}_{\text{alg}}(KE) = \text{h}_{\text{alg}}(KE \backslash X).$$

Proof. We will just need to prove this for the case that $X = \{v\} \subseteq E^0$ with v being a sink and iterate this argument. When v is a source the proof is analogous. If we consider the adjacency matrix A_E, then we have that

$$A_E = \begin{pmatrix} & & * & & \\ & & \vdots & & \\ & & * & & \\ 0 & \cdots & 0 & \cdots & 0 \\ & & * & & \\ & & \vdots & & \\ & & * & & \end{pmatrix}$$

with the i-th row of zeros corresponding to the vertex v. If we eliminate the i-th row and column, then we get the adjacency matrix of $E\backslash\{v\}$. If $p_E, p_{E\backslash\{v\}} \in K[x]$ are the characteristic polynomials of A_E and $A_{E\backslash\{v\}}$, then $p_E(x) = x p_{E\backslash\{x\}}(x)$. As a consequence, the spectral radius is

$$\rho(A_E) = \max\{0, \rho(A_{E\backslash v})\} = \rho(A_{E\backslash\{v\}}).$$

By Proposition 8.30 we get that $\text{h}_{\text{alg}}(KE) = \text{h}_{\text{alg}}(KE\backslash v)$. □

One direct consequence of the properties of the entropy with direct sums, is that for a finite directed graph, the entropy of the path algebra is the maximum of the algebraic entropies of each component. Actually, the result is stronger, to prove so we just need to make use of the adjacency matrix.

Proposition 8.33. *Let us consider E a finite graph and $\{E_i\}_{i=1}^m$ the strongly connected components of E, then*

$$\text{h}_{\text{alg}}(KE) = \max\{\text{h}_{\text{alg}}(KE_1), \text{h}_{\text{alg}}(KE_2), \ldots, \text{h}_{\text{alg}}(KE_m)\}.$$

Proof. For every finite directed graph, there is an enumeration of the vertices $E^0 = \{v_1, v_2, \ldots, v_n\}$ in such a way that the adjacency matrix A_E is upper triangular by blocks. In particular,

$$A_E = \begin{pmatrix} A_{E_1} & * & * & \cdots & * \\ 0 & A_{E_2} & * & \cdots & * \\ 0 & 0 & A_{E_3} & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A_{E_m} \end{pmatrix}.$$

where A_{E_i} is the adjacency matrix of the strongly connected component of E_i for $i = 1, \ldots, n$. In this case,

$$\rho(A_E) = \max\{\rho(A_{E_1}), \rho(A_{E_2}), \ldots, \rho(A_{E_m})\}$$

and by Proposition 8.30 we have the desired result. □

It is also possible to relate the algebraic entropy of a path algebra with its Gelfand-Kirillov dimension. In order to do so we make use of a result in [86].

Theorem. *[86, 3.12] Let E be a finite graph*

(i) The path algebra KE has finite Gelfand-Kirillov dimension if and only if E satisfies condition (EXC).

(ii) Assume that E satisfies condition (EXC). If t denotes the maximal length of chains of cycles in E, then $\mathrm{GKdim}(KE) = t$.

Cycles C and D are said to be *disjoint* if there is no common vertex. We will say that a cycle C is an *exclusive cycle* if it is disjoint with every other cycle; equivalently, no vertex on C is the base of a different cycle other than a rotated of C. Otherwise, we will say that C is a *non-exclusive cycle*. We say that E satisfies *Condition (EXC)* if every cycle of E is an exclusive cycle. In other words, a graph with Condition (EXC) is one without non-disjoint cycles. A *chain of cycles* of length n is a sequence of cycles $C_1, C_2, \cdots, C_n$ such that there is a path from C_i to C_{i+1} for each $i < n$. This chain has an *exit* if the last cycle C_n has an exit.

Lemma 8.34. *Let E be a finite directed graph satisfying Condition (EXC) and without sources and sinks. Then* $\mathrm{h_{alg}}(KE) = 0$.

Proof. It will only be necessary to prove this result for connected graphs, the case of non-connected graphs is a direct consequence of Proposition 8.22. We will proceed to make our proof by induction on the number of cycles. If E has one cycle without sinks and sources, then $E = C_n$ and we have proven in Example 8.21 that $\mathrm{h_{alg}}(KE) = \mathrm{h_{alg}}(KC_n) = 0$. Let us assume that this result is true for k cycles. Let us consider a graph E with $k+1$ cycles without sources or sinks. This implies that there is at least one cycle without entries or a cycle without exits. Without loss of generality, suppose there is a cycle C_n without entries. For a suitable order of the vertices, the adjacency matrix of E is:

$$A_E = \begin{pmatrix} A_{C_n} & * \\ 0 & A_F \end{pmatrix}$$

with $F = E \backslash C_n^0$. Accordingly, $\rho(A_E) = \max\{\rho(C_n), \rho(A_F)\} = \max\{1, \rho(A_F)\}$. The graph F has k cycles, but we may have sinks or sources. However, Corollary 8.32 implies that we get the same entropy even if we eliminate all the sources and sinks. Now we are under our induction hypothesis so $\mathrm{h_{alg}}(KF) = 0$ and this means by Proposition 8.30 that $\rho(A_F) = 1$. Finally, $\rho(A_E) = 1$ and $\mathrm{h_{alg}}(KE) = 0$. □

Summarizing we obtain the following trichotomy:

Theorem 8.35. *Let E be a finite directed graph and KE its associated path algebra over a field K, then*

i) $\dim KE < +\infty$, $\mathrm{GKdim}(KE) = 0$ *and* $\mathrm{h_{alg}}(KE) = 0$*;*

ii) $\dim KE = +\infty$, $0 < \mathrm{GKdim}(KE) < +\infty$ *and* $\mathrm{h_{alg}}(KE) = 0$*;*

(iii) $\dim KE = +\infty$, $\mathrm{GKdim}(KE) = +\infty$ *and* $0 < \mathrm{h_{alg}}(KE) < +\infty$*.*

Proof. For the first statement apply [86, Theorem 3.12] and the definition of entropy. Now we prove item *(ii)*. If $0 < \mathrm{GKdim}(KE) = k < \infty$, then by [86, Theorem 3.12] the graph satisfies Condition (EXC) and the maximal length of chains of cycles in E is k. By removing subsequently all sinks and all sources we get a graph with the same entropy (Corollary 8.32) and by Lemma 8.34, each connected component has null entropy. By Corollary 8.22, we obtain $\mathrm{h_{alg}}(KE) = 0$. Moreover, the dimension of KE is infinite since the graph E has at least one cycle. Finally, for item *(iii)*, if $\mathrm{GKdim}(KE) = +\infty$, then by [86, Theorem 3.12] there is a vertex which is base of two different cycles. This means that $\dim KE = +\infty$ and

$\mathrm{h_{alg}}(KE) > 0$ by Proposition 8.23. For a finite matrix the spectral radius is finite, consequently $\mathrm{h_{alg}}(KE) < +\infty$. □

The recent theorem is highly significant in the path algebra context. The Gelfand-Kirillov dimension enlightens the study of infinite-dimensional algebras, however it is not enough since there is a family of algebras with infinite Gelfand-Kirillov dimension. In this sense the entropy goes one step further since it allows us to differentiate them.

8.4.1 Computing the entropy for a Leavitt path algebra

The adjacency matrix of a graph has been very handy for the computation of the entropy of paths algebras. However, it has not been very useful for the Leavitt path algebras yet. Remember that the computations of the dimension are harder since we have to take into account the $(CK1)$ and $(CK2)$ conditions. If E is a finite directed graph and $\mathcal{G} = \{W_n\}_{n=1}^{\infty}$ is the standard filtration for the Leavitt path algebra $L_K(E)$ we have that:

$$\dim(W_n/W_{n-1}) = \dim(W_n) - \dim(W_{n-1}).$$

Since W_n is generated by the elements of the form $\lambda\mu^*$ with $\ell(\lambda) + \ell(\mu) \leqslant n$, then the dimension of W_n/W_{n-1} coincides with the number of linearly independent elements of the form $\lambda\mu^*$ with $\ell(\lambda) + \ell(\mu) = n$. The authors W. Bock and A. Sebandal prove in [23, Theorem 40] the following:

Theorem 8.36. *[23, Theorem 40] Let E be a finite graph, A_E its corresponding adjacency matrix of size m. In $L_K(E)$ the number of linearly independent elements of the form $\lambda\mu^*$ such that $\ell(\lambda) + \ell(\mu) = n \geqslant 2$, $\lambda, \mu \in \mathrm{Path}(E)$, that is $\dim(V_n/V_{n-1})$, is equal to:*

$$\sum_{s=0,j=1}^{n,m} \left(\sum_{i=1}^{m}(A_E^s)_{i,j}\right)\left(\sum_{i=1}^{m}(A_E^{n-s})_{i,j}\right) - \sum_{s=1,j=1}^{n-1,m}\left(\sum_{i=1}^{m}(A_E^{s-1})_{i,j}\right)\left(\sum_{i=1}^{m}(A_E^{n-s-1})_{i,j}\right)\gamma_j \tag{18}$$

where $\gamma_j = 0$ if $\sum_{k=1}^{m}(A_E)_{j,k} = 0$, else $\gamma_j = 1$.

The reader can check that the notation chosen is different to the one provided in [23], this is because the publication of the article in a journal was after the development of the results that appear in this section. We have decided to use this notation because it is consistent with the previous theory.

Even though we have a formula, computing each element in the sequence is not easy and the limit is even harder. However, we can make use of the Mathematica software (*Mathematica-Wolfram Research, Inc.*) to approximate the value of the entropy numerically. This is the code that we used:

```
  p[A_, n_, a_, b_] := Module[{m}, m = Length[A];
 If[n == 0, IdentityMatrix[m][[a, b]], (MatrixPower[A, n])[[a, b]]]];
```

The above command p[A,n,a,b] computes the (a, b)-entry of the n-th power of the matrix A. The next one, cond[A,j], codifies the γ_j function defined in (18):

```
cond[A_, j_] := Module[{m}, m = Length[A];
Sign[Sum[p[A, 1, j, n], {n, m}]]];
```

The function aux[A,j,s,m] computes the second part in (18) when $\gamma_j = 1$, that is, when cond[A,J] evaluates to "TRUE".

```
aux[A_, j_, s_, k_] :=
 Module[{m}, m = Length[A];
  Sum[p[A, s - 1, i, j], {i, m}] Sum[p[A, k - s - 1, i, j], {i, m}]];
```

And finally h[A,n] computes the element $h_n = \frac{\log(\dim(W_n/W_{n-1})}{n}$ of the Leavitt path algebra which graph has adjacency matrix A, based on the computations of linearly independent elements $\lambda\mu^*$ given by equation (18).

```
h[A_, n_] := N[Log[Module[{m}, m = Length[A]; Sum[
       Sum[
        Sum[p[A, s, i, j], {i, m}] Sum[p[A, k - s, i, j], {i, m}], {j,
          m}], {s, 1, n - 1}] -
      Sum[Sum[aux[A, j, s, n] cond[A, j], {j, m}], {s, 1, n - 1}] +
      2 Sum[p[A, n, i, j], {i, m}, {j, m}]]]/n];
```

By using the above Mathematica code, all the examples considered below seem to suggest the coincidence of both numbers $\mathrm{h_{alg}}(L_K(E))$ and $\mathrm{h_{alg}}(KE)$.

Example 8.37. Consider the graph E from Figure 21:

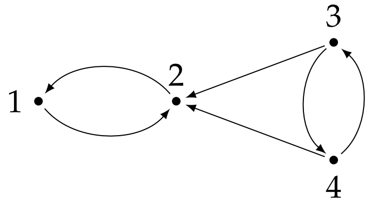

Figure 21: Directed graph E in Example 8.37.

Taking into account formula (18), computing $\dfrac{\log(\dim(W_n/W_{n-1}))}{n}$ is the best numerical approximation to $\mathrm{h_{alg}}(L_K(E))$. For instance, we get with *Mathematica* that

$$\frac{\log(\dim(W_{1000}/W_{999}))}{1000} = 0.0145107.$$

On the other hand, the logarithm of the spectral radius (the maximum of the absolute values of the eigenvalues of A_E) equals 0, having $\mathrm{h_{alg}}(KE) = 0$ by Proposition 8.30. Increasing the value of n we approximate more closely to 0.

Example 8.38. Next we take the graphs F_1 and F_2 as in Figure 22:

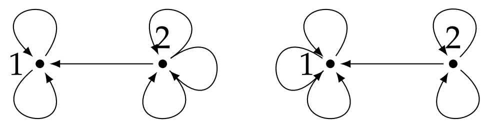

Figure 22: Directed graphs F_1 and F_2 in Example 8.38.

Again by formula (18), according to the calculations made with *Mathematica* we have that $\dfrac{\log(\dim(W_{1000}/W_{999}))}{1000} = 1.1061$ (for both graphs) and

$$\mathrm{h_{alg}}(KF_1) = \mathrm{h_{alg}}(KF_2) = \log(3) \simeq 1.09861.$$

We observe again the same phenomenon. In addition, notice that it seems to be independent of the orientation of the edges (as it happens in the path algebra case as the entropy is just the maximum of the entropies of the strongly connected components).

Example 8.39. Now suppose the graph G is the one given in Figure 23.

In this case, for $n = 1000$ we obtain $\dfrac{\log(\dim(W_n/W_{n-1}))}{n} = 0.00352636$ and $\mathrm{h_{alg}}(KG) = 0$ having convergence once more.

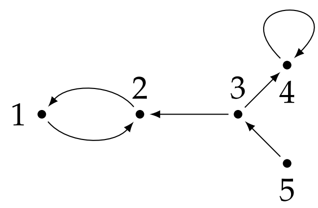

Figure 23: Directed graph G in Example 8.39

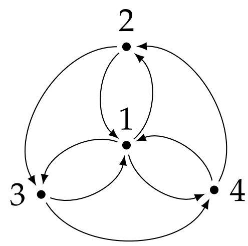

Figure 24: Directed graph D in Example 8.40

Example 8.40. Let D be the following graph:

For $n = 1000$, *Mathematica* gives us $\dfrac{\log(\dim(W_n/W_{n-1}))}{n} = 0.842187$ and $\mathrm{h}_{\mathrm{alg}}(KD) = \log\left(\frac{1+\sqrt{13}}{2}\right) \approx 0.834115$.

Example 8.41. Consider the graph of Figure 20. Computing $\mathrm{h}_{\mathrm{alg}}(L_K(E))$ with *Mathematica* we have numerical convergence to $\log(\varphi_+)$ where $\varphi_+ = \frac{1+\sqrt{5}}{2}$.

At first glance, since the number of generators in the Leavitt path algebra grows with relation to the path algebra, we might think that the entropy would be larger. This agree with the bound given in Lemma 8.19 where for a finite graph E, $\mathrm{h}_{\mathrm{alg}}(KE) \leqslant \mathrm{h}_{\mathrm{alg}}(L_K(E)) \leqslant \mathrm{h}_{\mathrm{alg}}(K\hat{E})$. However, after doing several numerical examples, the experience suggests that both values of the entropy coincide. Actually, this is the case in general. Both entropies, the one of the path algebra and that of the Leavitt path algebra are the same.

8.5 AGREEMENT OF THE ENTROPY

To prove that $\mathrm{h}_{\mathrm{alg}}(KE) = \mathrm{h}_{\mathrm{alg}}(L_K(E))$ we are going to use all the tools available so far and even more. We will need the adjacency matrix of the graph, the matrix norm, some results in Banach algebras and more. The first step will consist of defining the relative Cohn path algebras.

Definition 8.42. Let E be a directed graph, K a field and $X \subseteq \mathrm{Reg}(E)$. The *Cohn path algebra relative to the set* X, denoted by $C_K^X(E)$ is the algebra $K\hat{E}$ together with the relations

(CK1) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$,

(CK2) $\sum_{\{e\in E^1 :\, s(e)=v\}} ee^* = v$ for every $v \in X$.

In particular, if $X = \varnothing$ then we have the *Cohn path algebra* denoted by $C_K(E)$.

In general, since the algebra $C_K^X(E)$ is finitely generated by $E^0 \cup E^1 \cup (E^1)^*$ we can construct in a analogous way a standard filtration. This filtration will be generated by the vector space $V = \mathrm{span}_K(E^0 \cup E^1 \cup (E^1)^*)$ as $V_n := V + V^2 + \cdots + V^n$.

Definition 8.43. Let E be a finite directed graph, $X \subseteq \mathrm{Reg}(E)$ and K a field. We define the *standard filtration for* $C_K^X(E)$ as the family $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$

$$\begin{aligned} H_1 &= \mathrm{span}_K\{\lambda\mu^* \colon \lambda, \mu \in \mathrm{Path}(E), \ell(\lambda) + \ell(\mu) \leqslant 1\}, \\ H_2 &= \mathrm{span}_K\{\lambda\mu^* \colon \lambda, \mu \in \mathrm{Path}(E), \ell(\lambda) + \ell(\mu) \leqslant 2\}, \\ &\vdots \\ H_n &= \mathrm{span}_K\{\lambda\mu^* \colon \lambda, \mu \in \mathrm{Path}(E), \ell(\lambda) + \ell(\mu) \leqslant n\}, \\ &\vdots \end{aligned} \tag{19}$$

Observe that the description of the family of vector spaces in the filtration is the same as in the Leavitt path algebra case. This makes sense since the reader can observe that if $X = \mathrm{Reg}(E)$ then $C_K^X(E) = L_K(E)$. In addition, even though the generators are the same ones, the relations are different. In consequence the vector spaces and the dimensions are different.

In general, the algebraic entropy of $C_K^X(E)$ over the standard filtration will be denoted by $\mathrm{h_{alg}}(C_K^X(E))$.

Proposition 8.44. *Let E be a finite directed graph and $X \subseteq \mathrm{Reg}(E)$. Then*

$$\mathrm{h_{alg}}(KE) \leqslant \mathrm{h_{alg}}(C_K^X(E)).$$

Proof. As it is natural, the path algebra can be embedded in the relative Cohn path algebra $i\colon KE \hookrightarrow C_K^X(E)$. If $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ and $\mathcal{H} = \{H_n\}_{n=1}^{\infty}$, then we have that $i^{-1}(H_n) = V_n$ for every $n \geqslant 1$. Then, by Proposition 8.11 we have that

$$\mathrm{h_{alg}}(KE, \mathcal{F}) \leqslant \mathrm{h_{alg}}(C_K^X(E), \mathcal{H}).$$

□

In addition, if E is a finite directed graph and $X \subseteq Y \subseteq \mathrm{Reg}(E)$, then $C_K^Y(E) \simeq C_K^X(E)/I$ for I the ideal generated by the elements $v - \sum_{e\in E^1\colon e\in s^{-1}(v)}$ with $v \in Y \backslash X$. In particular, there is a projection $p\colon C_K^X(E) \to C_K^Y(E)$ which preserves the standard filtration in each of them. Once again, we can apply Proposition 8.10 and obtain the following bound.

Proposition 8.45. *If E is a finite directed graph and $X \subseteq Y \subseteq \mathrm{Reg}(E)$, then*

$$\mathrm{h_{alg}}(C_K^Y(E)) \leqslant \mathrm{h_{alg}}(C_K^X(E)). \tag{20}$$

In particular,

$$\mathrm{h_{alg}}(L_K(E)) \leqslant \mathrm{h_{alg}}(C_K(E)). \tag{21}$$

Putting all the bounds that we have together,

$$\mathrm{h_{alg}}(KE) \leqslant \mathrm{h_{alg}}(L_K(E)) \leqslant \mathrm{h_{alg}}(C_K(E)).$$

If we were able to prove that $\mathrm{h_{alg}}(C_K(E)) = \mathrm{h_{alg}}(KE)$, then we would have that $\mathrm{h_{alg}}(L_K(E)) = \mathrm{h_{alg}}(KE)$. Not only that, but $\mathrm{h_{alg}}(C_K^X(E)) = \mathrm{h_{alg}}(KE)$. As a first Lemma, we need a way to compute the algebraic entropy of $\mathrm{h_{alg}}(C_K(E))$ using the adjacency matrix.

Lemma 8.46. *If E is a finite directed graph with A_E its corresponding adjacency matrix and A_E^* its transposed matrix, then*

$$\mathrm{h_{alg}}(C_K(E)) = \limsup_{n\to\infty} \frac{\log\left(\sum_{s=0}^{n} \|A_E^s (A_E^*)^{n-s}\|_{1,1}\right)}{n}. \tag{22}$$

Proof. By the definition of the standard filtration of $C_K(E)$, the dimension of H_n/H_{n-1} is equal to the number of linearly independent elements of the form

$\lambda\mu^*$ with $\lambda, \mu \in \mathrm{Path}(E)$ and $\ell(\lambda) + \ell(\mu) = n$. Following the formula in Theorem 8.36, if m is the size of the matrix A_E, then

$$\dim(H_n/H_{n-1}) = \sum_{s=0,j=1}^{n,m} \left(\sum_{i=1}^{m} (A_E^s)_{i,j}\right)\left(\sum_{i=1}^{m} (A_E^{n-s})_{i,j}\right). \tag{23}$$

Observe that, on the contrary of the formula in Theorem 8.36, the negative part it is omitted since it was necessary to consider the $(CK2)$ condition and that is not necessary in $C_K(E)$. Then

$$\begin{aligned}\dim(H_n/H_{n-1}) &= \sum_{s=0,j=1}^{n,m} \left(\sum_{i=1}^{m} (A_E^s)_{i,j}\right)\left(\sum_{i=1}^{m} (A_E^{n-s})_{i,j}\right) = \\ &= \sum_{s=0;i,j,l=1}^{n,m} (A_E^s)_{i,j}(A_E^{n-s})_{l,j} = \sum_{s=0;i,l=1}^{n,m} \sum_{j=1}^{m} (A_E^s)_{i,j}((A_E^*)^{n-s})_{j,l} = \\ &= \sum_{s=0}^{n} \sum_{i,l=1}^{m} (A_E^s(A_E^*)^{n-s})_{i,l} = \sum_{s=0}^{n} \|A_E^s(A_E^*)^{n-s}\|_{1,1}.\end{aligned}$$

Finally,

$$\mathrm{h_{alg}}(C_K(E)) = \limsup_{n\to\infty} \frac{\log\left(\dim(H_n/H_{n-1})\right)}{n} = \limsup_{n\to\infty} \frac{\log\left(\sum_{s=0}^{n} \|A_E^s(A_E^*)^{n-s}\|_{1,1}\right)}{n}.$$

□

Before proving that the entropies of a Leavitt path algebra and of a path algebra agree, we recall the following well-known result about holomorphic functional calculus (see [93, Definition 10.26, formula (2) and Theorem 10.27]).

Theorem 8.47. *[93, Theorem 10.27] If $A \in \mathcal{M}_m(\mathbb{C})$, then for every $n \geqslant 1$*

$$A^n = \frac{r^{n+1}}{2\pi}\int_0^{2\pi} e^{it(n+1)}(re^{it}I - A)^{-1}\,dt,$$

where r is larger than the spectral radius $\rho(A)$.

Theorem 8.48. *If E is a finite directed graph, then $\mathrm{h_{alg}}(C_K(E)) \leqslant \mathrm{h_{alg}}(KE)$. Consequently,*

$$\mathrm{h_{alg}}(KE) = \mathrm{h_{alg}}(L_K(E)) = \mathrm{h_{alg}}(C_K(E)).$$

Proof. Write $A := A_E$ the adjacency matrix of E with size m, A^* the corresponding transposed matrix of A and suppose $r > \rho(A)$. First we compute $\left\|(re^{it}I - A)^{-1}\right\|_{1,1}$. On the one hand, $(re^{it}I - A)^{-1} = \frac{1}{r}e^{-it}\left(I + \frac{A}{re^{it}} + \frac{A^2}{(re^{it})^2} + \cdots\right)$ if $\|A\|_{1,1} < r$. So

$$\begin{aligned}\left\|(re^{it}I - A)^{-1}\right\|_{1,1} &= \frac{1}{r}\left\|I + \frac{A}{re^{it}} + \frac{A^2}{(re^{it})^2} + \cdots\right\|_{1,1} \leqslant \\ &\leqslant \frac{1}{r}\left(\|I\|_{1,1} + \frac{\|A\|_{1,1}}{r} + \frac{\|A\|_{1,1}^2}{r^2} + \cdots\right) = \\ &= \frac{1}{r}\left(m + \frac{\frac{\|A\|_{1,1}}{r}}{1 - \frac{\|A\|_{1,1}}{r}}\right) = \frac{1}{r}\left(m + \frac{\|A\|_{1,1}}{r - \|A\|_{1,1}}\right).\end{aligned}$$

On the other hand, we bound $\left\|\int_0^{2\pi} e^{it(n+1)}(re^{it}I - A)^{-1}\,dt\right\|_{1,1}$. In fact, we have

$$\begin{aligned}\left\|\int_0^{2\pi} e^{it(n+1)}(re^{it}I - A)^{-1}\,dt\right\|_{1,1} &\leqslant \int_0^{2\pi}\left\|(re^{it}I - A)^{-1}\right\|_{1,1} dt \leqslant \\ &\leqslant \frac{1}{r}\left(m + \frac{\|A\|_{1,1}}{r - \|A\|_{1,1}}\right)2\pi,\end{aligned}$$

taking into account the previous computation. Hence by Theorem 8.47, for every $n \geqslant 1$, we obtain

$$\|A^n\|_{1,1} \leqslant \frac{r^{n+1}}{2\pi}\frac{1}{r}\left(m + \frac{\|A\|_{1,1}}{r - \|A\|_{1,1}}\right)2\pi = r^n\left(m + \frac{\|A\|_{1,1}}{r - \|A\|_{1,1}}\right).$$

At this point recall that by Lemma 8.46

$$\mathrm{h_{alg}}(C_K(E)) = \limsup_{n\to\infty}\frac{1}{n}\log\left(\sum_{s=0}^{n}\|A^s(A^*)^{n-s}\|_{1,1}\right).$$

Consider $\|A^s(A^*)^{n-s}\|_{1,1}$. Since the eigenvalues of A coincide with the eigenvalues of A^*, then $\rho(A) = \rho(A^*)$ and we have that

$$\begin{aligned}&\|A^s(A^*)^{n-s}\|_{1,1} \leqslant \|A^s\|_{1,1}\|(A^*)^{n-s}\|_{1,1} \leqslant \\ &\leqslant r^s\left(m + \frac{\|A\|_{1,1}}{r - \|A\|_{1,1}}\right)r^{n-s}\left(m + \frac{\|A^*\|_{1,1}}{r - \|A^*\|_{1,1}}\right) = r^n\left(m + \frac{\|A\|_{1,1}}{r - \|A\|_{1,1}}\right)^2.\end{aligned}$$

Therefore, $\sum_{s=0}^{n} \|A^s(A^*)^{n-s}\|_{1,1} \leqslant r^n \left(m + \frac{\|A\|_{1,1}}{r - \|A\|_{1,1}}\right)^2 (n+1)$. Finally,

$$\frac{1}{n}\log\left(\sum_{s=0}^{n} \|A^s(A^*)^{n-s}\|_{1,1}\right) \leqslant \frac{\log(n+1)}{n} + \log(r) + \frac{2}{n}\log\left(m + \frac{\|A\|_{1,1}}{r-\|A\|_{1,1}}\right),$$

which means, $\mathrm{h}_{\mathrm{alg}}(C_K(E)) \leqslant \log(r)$ whenever $r > \rho(A)$. In particular,

$$\mathrm{h}_{\mathrm{alg}}(C_K(E)) \leqslant \log(\rho(A)) = \mathrm{h}_{\mathrm{alg}}(KE)$$

by Proposition 8.30. Finally, taking also into account Lemma 8.19 and Proposition 8.45 we have that for every finite graph E,

$$\mathrm{h}_{\mathrm{alg}}(KE) \leqslant \mathrm{h}_{\mathrm{alg}}(L_K(E)) \leqslant \mathrm{h}_{\mathrm{alg}}(C_K(E)) \leqslant \mathrm{h}_{\mathrm{alg}}(KE).$$

□

8.6 ENTROPY AND PATH ALGEBRA ALGEBRAIC PROPERTIES

The goal of this section is to relate Parts ii and iii. In this sense, we search for an algebraic motivation of the use of the entropy.

Proposition 8.49. *If E is a graph with KE Artinian or (left or right) Noetherian, then* $\mathrm{h}_{\mathrm{alg}}(KE) = 0$.

Proof. If KE is Artinian, then $\dim KE < +\infty$ and $\mathrm{h}_{\mathrm{alg}}(KE) = 0$. In case that KE is Noetherian, then by Proposition 7.4, the graph E satisfies Condition (EXC) and this means that $\mathrm{GKdim}(KE) = 0$ ([86, 3.12]). Finally, by Theorem 8.35 the equality holds. □

Proposition 8.50. *Take E a finite graph with KE semiprime, $\mathrm{h}_{\mathrm{alg}}(KE) = 0$ if and only if KE is left (and right) Noetherian.*

Proof. We have already proved in Proposition 8.49 that if KE is noetherian, then $\mathrm{h}_{\mathrm{alg}}(KE) = 0$.

Conversely, consider KE semiprime with $\mathrm{h}_{\mathrm{alg}}(KE) = 0$. By Proposition 5.11, if $\{E_i\}_{i=1}^n$ are the connected components of E, each E_i is strongly connected. Following Proposition 8.33, $\mathrm{h}_{\mathrm{alg}}(KE) = \max_{i=1,\ldots,n}\{\mathrm{h}_{\mathrm{alg}}(KE_i)\}$. Let us focus on each E_i. If E_i is an isolated vertex (resp. cycle), then $h_{alg}(KE_i) = 0$. However, if there is $v \in E_i^0$ which is a base of two different cycles, that is, E_i has a cycle with entries and exits, then by Proposition 8.23 we have that $h_{alg}(KE_i) > 0$. This enters in contradiction with the fact that $h_{alg}(KE) = 0$. Consequently, E is a disjoint union of isolated vertices and cycles. Finally, KE is left and right Noetherian because of Proposition 7.4. □

Proposition 8.51. *Consider a finite directed graph E with KE primitive. The path algebra KE is simple if and only if* $h_{alg}(KE) = 0$.

Proof. If the path algebra is simple, then the graph $|E^0| = 1$ and $|E^1| = 0$, which means we have finite dimension and $h_{alg}(KE) = 0$.

If the path algebra is primitive but not simple, then by Theorem 5.7 the graph E is strongly connected and every cycle has an exit which means that there is a vertex $u \in E^0$ base of two cycles. Consequently, following Proposition 8.33 we have that $h_{alg}(KE) > 0$. □

We have seen in previous sections that if we consider a finite graph E and erase sinks or sources then the entropy of the path algebra does not change. Furthermore, since all the sources generate the left socle and the sinks generate the right socle, we have.

Proposition 8.52. *Consider E a finite graph, then*

$$h_{alg}(KE) = h_{alg}(KE/\operatorname{soc}_l(KE)) = h_{alg}(KE/\operatorname{soc}_r(KE)).$$

Proof. Since the left (resp. right) socle is the ideal generated by the sources (resp. sinks), the rest is given by Corollary 8.32. □

Taking into account the socle chain of a path algebra, we have that.

Corollary 8.53. *Consider a finite graph E and S_α the left (resp. right) socle of KE of order α. Then*

$$h_{alg}(KE) = h_{alg}(KE/S_\alpha).$$

Furthermore, if we eliminate edges without return the entropy remains the same.

Proposition 8.54. *If E is a finite directed graph and $e \in E^1$ is an edge without return, then*

$$h_{alg}(KE) = h_{alg}(K(E\backslash\{e\}).$$

Proof. If $e \in E^1$ is an edge without return, then $s(e) \neq r(e)$. Consider $u := s(e)$ and $v := r(e)$, then u and v belong to different strongly connected components of the graph. On the contrary, there is a path $\lambda \in \text{Path}(E)$ such that $s(\lambda) = v$ and $r(\lambda) = u$ and the edge e has a return. Consequently, the edge e does not belong to the graph of any strongly connected component and this means that if we erase it, they do not change. Since the strongly connected components remain the same, by Proposition 8.33

$$\text{h}_{\text{alg}}(KE) = \text{h}_{\text{alg}}(K(E \backslash \{e\})).$$

□

If we eliminate all the edges without return, then we get a new graph where each connected component is strongly connected and the entropy of the new path algebra remains the same.

Corollary 8.55. *Consider E a finite graph, then*

$$\text{h}_{\text{alg}}(KE) = \text{h}_{\text{alg}}(KE/\operatorname{rad}(KE)).$$

Since the Jacobson radical of a path algebra coincides with its semiprime radical and it is generated by a set of edges, the quotient $KE/\operatorname{rad}(KE)$ is a semiprime path algebra of a new graph. In this sense, we are able to construct a graph which its associated path algebra is semiprime and its entropy remains the same.

Chapter 9

ENTROPY OF LIMITS OF FILTERED ALGEBRAS

Denote the graph Ch_∞ given by vertices $Ch^0_\infty = \{v_n\}_{n=1}^\infty$ and edges $Ch^1_\infty = \{e_n, f_n\}_{n=1}^\infty$ with $s(e_n) = r(f_n) = v_n$ and $r(e_n) = s(f_n) = v_{n+1}$. We call it the *infinity chain* and represent it as in Figure 25. Consider the path algebra KCh_∞. If we

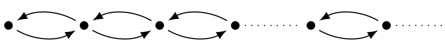

Figure 25: Infinity chain.

think of its standard filtration $\mathcal{F} = \{V_n\}_{n=1}^\infty$, then we can observe that

$$V_n = \operatorname{span}_K\{\lambda \in \operatorname{Path}(E) \colon \ell(\lambda) \leqslant n\},$$

and

$$\dim(V_n/V_{n-1}) = |\{\lambda \in \operatorname{Path}(E) \colon \ell(\lambda) = n\}| = +\infty$$

which means that the filtration is not finite. If we forgot that we have defined the entropy for finite filtrations, we would have

$$\mathrm{h_{alg}}(KCh_\infty) = \limsup_{n\to\infty} \frac{\dim\left(V_n/V_{n-1}\right)}{n} = +\infty.$$

On the other hand, consider the graph Ch_2 given in Figure 26,

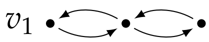

Figure 26: Chain of two links Ch_2.

its adjacency matrix is $A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ and by Proposition 8.30 we have that $\mathrm{h_{alg}}(KCh_2) = \frac{1}{2}\log(2) \simeq 0.34657$.

If we now take the graph Ch_3 as in Figure 27, then the adjacency matrix is

Figure 27: Chain of three links Ch_3.

$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ and $\mathrm{h_{alg}}(KCh_3) = \log\left(\frac{1+\sqrt{5}}{2}\right) \simeq 0.48121$.

The reader can observe that, in general, we can consider the families of graphs $\{Ch_n\}_{n=2}^{\infty}$ with adjacency matrices

$$A_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathcal{M}_{n+1}(\mathbb{R}).$$

Using the software *Mathematica* we have computed numerically the values $\mathrm{h_{alg}}(Kh_n)$ for several n:

THe reader can check the numbers in Table 4 and observe that there is a certain tendency to the value $\log(2) \simeq 0.693147$ which is quite opposite to the $+\infty$ that we have obtained before. Our intuition says that if we want to extend our definition

n	$h_{alg}(\mathbf{KCh_n})$
5	0.549306
15	0.673745
105	0.692708
203	0.693029
275	0.693082

Table 4: Algebraic entropy of the family of graphs Ch_n for different values of n.

of entropy to algebras of infinite graphs, it has to be compatible with the one that we already have. For this purpose we restrict ourselves to finite filtrations.

9.1 LIMIT OF FILTERED ALGEBRAS

In this section, we explore the concept of direct limit of a family of filtered algebras. In the category $\mathbf{Falg_K}$ this limit exists, however in general it is not in $\mathbf{falg_K}$ (see Definition 8.9). That is why we need to be cautious when we want to define a new notion of entropy.

First, we need to know what we are talking about when we refer to a "family" of objects.

Definition 9.1. Consider a category $\mathcal{C}$ and I a directed set. A *direct system in* $\mathcal{C}$ is a pair $\left(\{X_i\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j}\right)$ where X_i is an object of $\mathcal{C}$ for every $i \in I$ and $\varphi_{i,j}\colon X_i \to X_j$ is a morphism in $\mathcal{C}$ for every $i \leqslant j$ such that

1. $\varphi_{i,i}$ is the identity,

2. $\varphi_{i,k} = \varphi_{j,k} \circ \varphi_{i,j}$ for every $i \leqslant j \leqslant k$.

A pair $(X, \{\varphi_i\}_{i\in I})$, where X is an object in $\mathcal{C}$ and $\varphi_i : X_i \to X$ is a morphism for every $i \in I$, is a *target of* $(\{X_i\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j})$ if $\varphi_i = \varphi_j \circ \varphi_{i,j}$. We say that the target $(X, \{\varphi_i\}_{i\in I})$ is a *direct limit* if for any other target $(Y, \{\psi_i\}_{i\in I})$ there is an unique morphism $\theta\colon X \to Y$ such that $\psi_i = \theta \circ \varphi_i$ for every $i \in I$. The object X usually is denoted as $X = \varinjlim X_i$. This can be represented as the commutative diagram of Figure 28.

In general, not every direct system has a direct limit. However, there are some categories where every direct system has. For example the category of groups, algebras, rings, etc. We include some examples for self-containment, however these are not new and can be checked in [83].

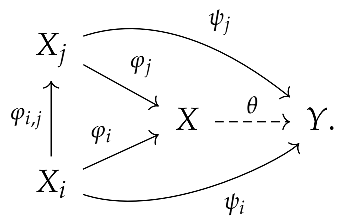

Figure 28: Commutative diagram of the direct limit of a direct system.

Example 9.2. 1. ***Limit of vector spaces.*** In the category of K-vector spaces, the direct limit is analogous to the limit of K-algebras. If $(\{V_i\}_{i\in I}, \{\varphi_{i,j}\}_{i,j})$ is a direct system, then the direct limit is the vector space

$$V = \varinjlim V_i = \bigsqcup_{i\in I} V_i / \sim$$

where $x_i \in V_i$ and $x_j \in V_j$ are related if there is some $k \geqslant i, j$ such that

$$\varphi_{i,k}(x_i) = \varphi_{j,k}(x_j).$$

The morphisms $\varphi_i \colon V_i \to V$ are the maps that send each element to its equivalent class. In this set, the elements of the equivalence class are denoted by $[x] \in \varinjlim V_i$ and the structure of vector space is given by the well-defined operations

$$\begin{aligned} \lambda \cdot [x_i] &:= [\lambda x_i] \ \forall \lambda \in K, \\ [x_i] + [x_j] &:= [\varphi_{i,k}(x_i) + \varphi_{j,k}(x_j)] \text{ with } i, j \leqslant k. \end{aligned}$$

2. ***Limit of algebras.*** It is well known that if $(\{A_i\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j})$ is a direct system of K-algebras, then

$$A = \varinjlim A_i = \bigsqcup_{i\in I} A_i / \sim .$$

The morphism $\varphi_i \colon A_i \to A$ is the one that sends each element to its equivalent class. The product in this algebra is given by

$$[x_i][x_j] = [\varphi_{i,k}(x_i)\varphi_{j,k}(x_j)] \text{ with } i, j \leqslant k.$$

Consider a direct system of filtered algebras $\{(A_i, \mathcal{F}_i)\}_{i\in I}$ where $\mathcal{F}_i = \{V_n^i\}_{n=1}^{\infty}$ is a filtration and $\varphi_{i,j}\colon (A_i, \mathcal{F}_i) \to (A_j, \mathcal{F}_j)$ is a filtered algebra morphism for every $i \leqslant j$.

Lemma 9.3. *The pairs* $\left(\{A_i\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j}\right)$ *and* $\left(\{V_n^i\}_{i\in I}, \{\varphi_{i,j}|_{V_n^i}\}_{i\leqslant j}\right)$ *for any* $n \geqslant 1$*, are direct systems .*

Proof. It is trivial to check that the first pair is directed. The second one is also directed because $\varphi_{i,j}(V_n^i) \subseteq V_n^j$ for every $n \geqslant 1$ and $i \leqslant j$. Consequently:

1. $\varphi_{i,i}|_{V_n^i} = \mathrm{Id}|_{V_n^i}$,
2. $\varphi_{i,k}|_{V_n^i} = (\varphi_{j,k}\varphi_{i,j})|_{V_n^i} = \varphi_{j,k}|_{V_n^j}\ \varphi_{i,j}|_{V_n^i}$.

□

Following the previous lemma, it makes sense to consider the direct limit of the family of algebras once we forget that they are also filtered. This leads to the construction of the limit of filtered algebras.

Proposition 9.4. *Let* $S = \left(\{(A_i, \mathcal{F}_i)\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j}\right)$ *be a direct system of filtered algebras with* $\mathcal{F}_i = \{V_n^i\}_{n=1}^{\infty}$*. If we consider* $A = \varinjlim A_i$ *with* $\varphi_i\colon A_i \to A$ *given by* $x_i \mapsto [x_i]$ *and* $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ *with* $V_n = \bigcup_{i\in I} \varphi_i(V_n^i)$*, then* $((A, \mathcal{F}), \{\varphi_i\}_{i\in I})$ *is the direct limit of the system* S*.*

Proof. We need to check that $(A, \mathcal{F}) \in \mathbf{Falg_K}$, φ_i is a morphism of filtered algebras for every $i \in I$, $((A, \mathcal{F}), \{\varphi_i\}_{i\in I})$ is a target and it verifies the universal property.

To see that $\mathcal{F}$ is a filtration we follow Definition 8.2.

1. By construction $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n \subseteq V_{n+1} \subseteq \cdots$.
2. If $[x] \in A$, then there is $i \in I$ such that $x \in A_i$. Since $\mathcal{F}_i$ is a filtration in A_i, there is some n such that $x \in V_n^i$ and $\varphi_i(x) = [x] \in V_n$.
3. Take $[x] \in V_n$ and $[y] \in V_m$, then there are some $i, j \in I$ such that $x \in V_n^i$ and $y \in V_m^j$. If we take $k \geqslant i, j$, then $\varphi_{i,k}(x) \in V_n^k$, $\varphi_{j,k}(y) \in V_m^k$ and $\varphi_{i,k}(x)\varphi_{j,k}(y) \in V_{n+m}^k$ which means that $[x][y] = [\varphi_{i,k}\varphi_{j,k}(y)] \in V_{n+m}$.

We already know that φ_i is a morphism of algebras and by definition it verifies that $\varphi_i(V_n^i) \subseteq V_n$. Consequently, $\varphi_i\colon (A_i, \mathcal{F}_i) \to (A, \mathcal{F})$ is a filtered algebra morphism. In addition, since $(A, \{\varphi_i\}_{i\in I})$ is already a direct limit for $(\{A_i\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j})$, it satisfies that $\varphi_i = \varphi_j\varphi_{i,j}$.

Finally, we just need to check that $((A,\mathcal{F}),\{\varphi_i\}_{i\in I})$ satisfies the universal property. Take a target $((B,\mathcal{G}),\{\psi_i\}_{i\in i})$, we need to find an unique $\theta\colon (A,\mathcal{F})\to(B,\mathcal{G})$ such that $\psi_i=\theta\circ\varphi_i$ for every $i\in I$. If we have $x_i\in A_i$, then θ has to verify that $\psi_i(x)=\theta(\varphi(x_i))=\theta([x_i])$. In this sense we define $\theta\colon A\to B$ such that $\theta([x]):=\psi_i(x_i)$ where $x_i\in A_i$ and $[x_i]=[x]$. This map is well-defined and is unique by construction. Indeed, take some $x_j\in A_j$ with $[x_j]=[x_i]$, then there is some $k\geqslant i,j$ such that $\varphi_{j,k}(x_j)=\varphi_{i,k}(x_i)$. Thus, if we compose with ψ_k we have that

$$\theta([x_j])=\psi_j(x_j)=\psi_k(\varphi_{j,k}(x_j))=\psi_k(\varphi_{i,k}(x_i))=\psi_i(x_i)=\theta([x_i]).$$

Lastly, it is a filtered algebra morphism. Consider $\mathcal{G}=\{W_n\}_{n=1}^{\infty}$, if $[x]\in V_n$, then there is some $x_i\in V_n^i$ such that $[x_i]=[x]$ and $\theta([x])=\psi_i(x_i)\in W_n$ because ψ_i is a filtered algebra morphism. In addition, if $\lambda\in K$, $x_i\in A_i$, $y_j\in A_j$ and $k\geqslant i,j$, then

$$\theta(\lambda[x_i]+[y_j])=\theta([\varphi_{i,k}(\lambda x_i)+\varphi_{j,k}(x_j)])=\psi_k(\varphi_{i,k}(\lambda x_i)+\varphi_{j,k}(x_j))=$$
$$\lambda\psi_k(\varphi_{i,k}(x_i)+\psi_k(\varphi_{j,k}(y_j))=\lambda\psi_i(x_i)+\psi_j(y_j)=\lambda\theta([x_i])+\theta([y_j]).$$

and

$$\theta([x_i][y_j])=\theta([\varphi_{i,k}(x_i)\varphi_{j,k}(x_j)])=\psi_k(\varphi_{i,k}(x_i)\varphi_{j,k}(x_j))=$$
$$\psi_k(\varphi_{i,k}(x_i))\psi_k(\varphi_{j,k}(y_j))=\psi_i(x_i)\psi_j(y_j)=\theta([x_i])\theta([y_j]).$$

□

Remark 9.5. The reader can check that in the context of the previous Proposition, $V_n=\varinjlim V_n^i$ since $(V_n,\{\varphi_i|_{V_n^i}\}_{i\in I})$ satisfies the universal property. To check so we only need to find an unique map $\theta\colon V_n\to\bigsqcup V_n^i/\sim$ such that $\phi_i=\theta\circ\varphi_i$ with $\phi_i\colon V_n^i\to V_n^i/\sim$ the map that sends x to $\overline{x}\in V_n^i/\sim$. We change the notation for the new equivalence class in order to distinguish the one in V_n to the one in $V_n^i/\sim$. The needed θ is just the map that satisfies $\theta([x_i])=\overline{x_i}$. It is well-defined since $[x_i]=[x_j]$ if and only if there is some $k\geqslant i,j$ such that $\varphi_{i,k}(x_i)=\varphi_{j,k}(x_j)$ which is equivalent to $\overline{x_i}=\overline{x_j}$. Finally, highlight that since $V_n=\varinjlim V_n^i$ we will not make distinction between them or their respective equivalence classes and use the $[\cdot]$ notation.

Even though the direct limit exists in the category of filtered algebras, this is not true in the categories of finite filtered algebras. More specifically, the direct limit of finite filtered algebras as in Proposition 9.16 in general is not a finite filtered algebra.

The path algebra KCh_∞ with Ch_∞ the infinity chain is a good example of this. Consider the direct system $(\{(KCh_i, \mathcal{F}_i)\}_{i=2}^{\infty}, \{\varphi_{i,j}\}_{i,j=2}^{\infty})$ with $\varphi_{i,j}\colon KCh_i \hookrightarrow KCh_j$ the natural inclusion for $j \geqslant i$ and $\mathcal{F}_i$ the standard filtration of KCh_i. In this case, the direct limit is $\big((KCh_\infty, \mathcal{F}), \{\varphi_i\}_{i=2}^{\infty}\big)$ with $\varphi_i\colon KCh_i \to KCh_\infty$ the inclusion and $\mathcal{F}$ the standard filtration (a way to prove this is using [57, Lemma 2.5]). However, the standard filtration is not finite for the path algebra of the infinity chain whereas the standard filtrations for the graphs in the direct system are.

Instead of a particular example, we can see more generally that the limit of finite filtered algebras is not (in general) a finite filtered algebra. We can see this by computing the dimension of quotient of the limit. We will need for that Lemmas 9.6, 9.7 and 9.9. Some of them can be considered general knowledge and a person who is familiar with category theory and direct limits might think of them as a triviality. If that is the case, the reader can skip the proofs of the next results until Section 9.2. However, since we want to be as self-contained as we can and we do not want to deviate the content from our problem, we include the explicit proofs of the desired lemmas.

Lemma 9.6. *If $S = (\{A_i\}_{i\in I}, \{\varphi_{i,j}\}_{i,j\in I})$ is a direct system where $\varphi_{i,j}\colon A_i \to A_j$ is a monomorphism for every $i \leqslant j$, then the morphisms in the direct limit $(A, \{\varphi_i\}_{i\in I})$ of S are injective for every $i \in I$.*

Proof. First of all, consider $\hat{A} = \bigsqcup A_i/\sim$, since $\hat{A}$ also is the direct limit of S, then $A \cong \hat{A}$. By the universal property, we just need to prove that the morphisms $A_i \to \hat{A}$ are injective. Remember that

$$\begin{array}{rccc} \psi_i\colon & A_i & \to & \hat{A} \\ & x & \mapsto & [x] \end{array}$$

and consider $x, y \in A_i$ with $[x] = [y]$. This means that there is some $k \geqslant i$ such that $\varphi_{i,k}(x) = \varphi_{i,k}(y)$ and since all the morphisms $\varphi_{i,j}$ are injective, $x = y$ as we expected. □

Consider $S = (\{(A_i, \mathcal{F}_i)\}_{i\in I}, \{\varphi_{i,j}\}_{i,j\in I})$ a direct system of filtered algebras with $\mathcal{F}_i = \{V_n^i\}_{n=1}^{\infty}$ for $i \in I$. The map $\varphi_{i,j}$ is a filtered algebra morphism, then $\varphi_{i,j}(V_{n-1}^i) \subseteq V_{n-1}^j$. In a similar way to what we did for Propositions 8.10 and 8.11, we can induce a vector space morphism $\bar{\varphi}_{i,j}\colon V_n^i/V_{n-1}^i \to V_n^j/V_{n-1}^j$ such that $\bar{\varphi}_{i,j}(x + V_{n-1}^i) = \varphi_{i,j}(x) + V_{n-1}^j$.

Lemma 9.7. *Let S be a direct system of filtered algebras as before. The family of vector spaces $\{V_n^i/V_{n-1}^i\}_{i\in I}$ together with the morphisms $\{\bar{\varphi}_{i,j}\}_{i\leqslant j}$ is a direct system and for every $n \geqslant 2$*

$$\frac{\bigsqcup_{i\in I} V_n^i/\sim}{\bigsqcup_{i\in I} V_{n-1}^i/\sim} \cong \frac{\bigsqcup_{i\in I} V_n^i/V_{n-1}^i}{\sim} \tag{24}$$

as K-vector spaces.

Proof. First we need to check that $(\{V_n^i/V_{n-1}^i\}_{i\in I}, \{\bar{\varphi}_{i,j}\}_{i\leqslant j})$ is a direct system. If $x_i + V_{n-1}^i \in V_n^i/V_{n-1}^i$ and $i \leqslant j \leqslant k$, then

$$\bar{\varphi}_{i,i}(x_i + V_{n-1}^i) = \bar{\mathrm{Id}}(x_i + V_{n-1}^i) = x_i + V_{n-1}^i,$$

and

$$\begin{aligned}\bar{\varphi}_{i,k}(x_i + V_{n-1}^i) = \varphi_{i,k}(x_i) + V_{n-1}^k = \varphi_{j,k}(\varphi_{i,j}(x_i)) + V_{n-1}^k = \\ \bar{\varphi}_{j,k}(\varphi_{i,j}(x_i) + V_{n-1}^j) = \bar{\varphi}_{j,k} \circ \bar{\varphi}_{i,j}(x_i + V_{n-1}^i).\end{aligned}$$

To prove (24) consider $V_n = \bigsqcup_{i\in I} V_n^i/\sim$ and $V_{n-1} = \bigsqcup_{i\in I} V_{n-1}^i/\sim$. We define the map

$$\begin{aligned}\Omega\colon \quad \frac{\bigsqcup_{i\in I} V_n^i/V_{n-1}^i}{\sim} &\to V_n/V_{n-1} \\ [x_i + V_{n-1}^i] &\mapsto [x_i] + V_{n-1}\end{aligned}$$

which is an isomorphism of K-vector spaces. In order to prove so we need to verify that it is well-defined, linear and bijective:

1. Consider two elements $x_i + V_{n-1}^i$ and $x_j + V_{n-1}^j$ related. This means that there is certain $k \in I$ with $k \geqslant i, j$ such that $\bar{\varphi}_{i,k}(x_i + V_{n-1}^i) = \bar{\varphi}_{j,k}(x_j + V_{n-1}^j)$. In particular, $\varphi_{i,k}(x_i) + V_{n-1}^k = \varphi_{j,k}(x_j) + V_{n-1}^k$ which implies that $\varphi_{i,k}(x_i) - \varphi_{j,k}(x_j) \in V_{n-1}^k$. Then, $[\varphi_{i,k}(x_i) - \varphi_{j,k}(x_j)] = [x_i] - [x_j] \in V_{n-1}$ and

$$\Omega\left([x_i + V_{n-1}^i]\right) = [x_i] + V_{n-1} = [x_j] + V_{n-1} = \Omega\left([x_j + V_{n-1}^j]\right).$$

2. Take $\lambda \in K$, then

$$\begin{aligned}\Omega(\lambda \cdot [x_i + V_{n-1}^i]) = \Omega([\lambda x_i + V_{n-1}^i]) = [\lambda x_i] + V_{n-1} = \\ \lambda[x_i] + V_{n-1} = \lambda \cdot ([x_i] + V_{n-1}) = \lambda\Omega([x_i + V_{n-1}^i]).\end{aligned}$$

Consider $[x_i + V^i_{n-1}], [x_j + V^j_{n-1}] \in \frac{\bigsqcup_{i\in I} V^i_n/V^i_{n-1}}{\sim}$ then we have

$$\Omega\left([x_i + V^i_{n-1}] + [x_j + V^j_{n-1}]\right) = \Omega\left([\bar\varphi_{i,k}(x_i + V^i_{n-1}) + \bar\varphi_{j,k}(x_j + V^j_{n-1})]\right) =$$
$$= \Omega\left([\varphi_{i,k}(x_i) + \varphi_{j,k}(x_j) + V^k_{n-1}]\right) = [\varphi_{i,k}(x_i) + \varphi_{j,k}(x_j)] + V_{n-1} =$$
$$= [x_i] + [x_j] + V_{n-1} = ([x_i] + V_{n-1}) + \left([x_j] + V_{n-1}\right) =$$
$$= \Omega\left([x_i + V^i_{n-1}]\right) + \Omega\left([x_j + V^j_{n-1}]\right).$$

3. To prove the injectivity, take $[x_i + V^i_{n-1}]$ with $\Omega([x_i + V^i_{n-1}]) = 0$. That is $[x_i] + V_{n-1} = 0$, which means that $[x_i] \in V_{n-1}$. In particular, there is a $x_j \in V^j_{n-1}$ such that $[x_j] = [x_i] \in V_{n-1}$, that is, there is $k \geqslant i, j$ with $\varphi_{i,k}(x_i) = \varphi_{j,k}(x_j)$. Since, $\varphi_{j,k}$ is a filtered algebra morphism and $x_j \in V^j_{n-1}$ then $\varphi_{i,k}(x_i) = \varphi_{j,k}(x_j) \in V^k_{n-1}$. Then, $\varphi_{i,k}(x_i) + V^k_{n-1} = 0$ and $\bar\varphi_{i,k}(x_i + V^i_{n-1}) = 0$. Consequently, by the relation $\sim$ in $\frac{\bigsqcup_{i\in I} V^i_n/V^i_{n-1}}{\sim}$, this means that $[x_i + V^i_{n-1}] = 0$. The surjectivity of the map is immediate by construction. If we take $[x_i] + V_{n-1} \in V_n/V_{n-1}$, then $\Omega([x_i + V^i_{n-1}]) = [x_i] + V_{n-1}$.

□

Remark 9.8. There is a smarter way to prove this using exact functors. If $\mathbf{Vect}_K$ denotes the category of K-vector spaces and I is directed set, then $\mathbf{Vect}_K^I$ is the category which objects are direct systems of vector spaces. Its objects are direct systems $(\{V_i\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j})$ and if $S, S' \in \mathbf{Vect}_K^I$ then a morphism $\theta\colon S \to S'$ is a family of maps $\{\theta_i\}_{i\in I}$ with $\theta_i\colon V_i \to V'_i$ such that $\varphi'_{i,j} \circ \theta_i = \theta_j \circ \varphi_{i,j}$ for every $i \leqslant j$. Under this context, we can define the functor $\mathcal{F}\colon \mathbf{Vect}_K^I \to \mathbf{Vect}_K$ such that if $S = (\{V_i\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j})$, $S' = (\{V'_i\}_{i\in I}, \{\varphi'_{i,j}\}_{i\leqslant j})$ and $\theta\colon S \to S'$, then $\mathcal{F}(S) := \varinjlim V_i$ and $\mathcal{F}(\theta)\colon \varinjlim V_i \to \varinjlim V'_i$ is the only morphism given by the universal property such that $\varphi'_i \circ \theta_i = \mathcal{F}(\theta) \circ \varphi_i$. This functor is exact ([107, Theorem 2.6.15]), that is, if

$$0 \to S_1 \hookrightarrow S_2 \twoheadrightarrow S_3 \to 0$$

is an exact sequence, then

$$0 \to \mathcal{F}(S_1) \hookrightarrow \mathcal{F}(S_2) \twoheadrightarrow \mathcal{F}(S_3) \to 0$$

is also an exact sequence. In particular, if $(\{(A_i, \mathcal{F}_i)\}_{i\in I}, \{\varphi_{i,j}\}_{i,j\in I})$ is a direct system of filter algebras with $\mathcal{F}_i = \{V_n^i\}_{n=1}^{\infty}$ then there is a exact sequence in $\mathbf{Vect}_K^I$ induced by

$$0 \to V_{n-1}^i \hookrightarrow V_n^i \twoheadrightarrow V_n^i/V_{n-1}^i \to 0.$$

Consequently,

$$0 \to \varinjlim V_{n-1}^i \hookrightarrow \varinjlim V_n^i \twoheadrightarrow \varinjlim V_n^i/V_{n-1}^i \to 0$$

is an exact sequence because $\mathcal{F}$ is an exact functor. Finally this implies that

$$\varinjlim V_n^i / \varinjlim V_{n-1}^i \cong \varinjlim V_n^i/V_{n-1}^i.$$

Lemma 9.9. *Consider $(\{V_i\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j})$ a direct system of vector spaces with $\varphi_{i,j}\colon V_i \to V_j$ injective for every $i \leqslant j$. If $V = \varinjlim V_i$, then*

$$\dim V = \sup_{i\in I}\{\dim V_i\} \tag{25}$$

Proof. Firstly, we consider the case that $\dim V = n < +\infty$. By Lemma 9.6 we have that $\dim V_i \leqslant \dim V$ for every $i \in I$. Take a basis $\mathcal{B} = \{[x_{i_1}], [x_{i_2}], \dots, [x_{i_n}]\}$. Since I is directed, there is some $k \in I$ such that $k \geqslant i_j$ for every $j = 1, \dots, n$. In addition, $[x_{i_j}] = [\varphi_{i_j,k}(x_{i_j})]$ for every $j = 1, \dots, n$ and without loss of generality we can consider a basis $\mathcal{B} = \{[x_1], [x_2], \dots, [x_n]\}$ of V, with $x_i \in V_k$ for every $i = 1, \dots, n$. Since B is a set of linearly independent vectors in V, it is easy to check that $\{x_1, \dots, x_n\}$ are linearly independent in V_k. Consequently $n \leqslant \dim V_k \leqslant n$ and $\dim V = \sup_{i\in I}\{\dim V_i\}$.

Secondly, if $\dim V = +\infty$, then there is a sequence of finite-dimensional vector spaces $\{W_n\}_{n=1}^{\infty}$ such that $W_n \subseteq V$ and $\lim_{n\to\infty} \dim W_n = +\infty$. Using the previous argument, we can construct a sequence of vector spaces $\{V_{i_n}\}_{n=1}^{\infty}$ such that $\dim W_n \leqslant \dim V_{i_n}$. Consequently,

$$\dim V = +\infty = \sup_{i\in I}\{\dim V_i\}.$$

□

Remark 9.10. By Lemmas 9.7 and 9.9 we have that

$$\dim V_n/V_{n-1} = \dim \varinjlim V_n^i/V_{n-1}^i = \sup_{i\in I}\left\{\dim V_n^i/V_{n-1}^i\right\}.$$

This makes possible that $\dim V_n/V_{n-1} = +\infty$ leading to a non-finite filtration.

9.2 ENTROPY OF STRICT DIRECT SYSTEMS

It is important not to lose the spotlight in our problem. We are trying to obtain a definition of the entropy that generalise the one that we have for filtered algebras and that is compatible with the previous one. To do so we take inspiration from the infinity chain. In this case, for a field K, we have that KCh_i is included in KCh_∞ by the usual inclusion $\mathrm{i}\colon KCh_i \hookrightarrow KCh_\infty$. In addition, if $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ is the standard filtration of KCh_∞ and $\mathcal{F}_i = \{V_n^i\}_{n=1}^{\infty}$ is the standard filtration of KCh_i, then $\mathrm{i}^{-1}(V_n) = V_n^i$. Following these ideas, we define:

Definition 9.11. Let $(A, \mathcal{F})$ and $(B, \mathcal{G})$ be two filtered algebras with $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ and $\mathcal{G} = \{W_n\}_{n=1}^{\infty}$. We say that a monomorphism of filtered algebras $f\colon (A, \mathcal{F}) \to (B, \mathcal{G})$ is *strict* if and only if $V_n = f^{-1}(W_n)$.

Definition 9.12. A direct system of filtered algebras $S = \big(\{(A_i, \mathcal{F}_i)\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j}\big)$ is *strict* if $\varphi_{i,j}$ is a strict monomorphism for every $i \leqslant j$.

Once we know the concepts of strict monomorphism and strict direct system, we can define the entropy of an algebra as the entropy of a strict direct system.

Definition 9.13. Let A be a K-algebra such that there is a strict direct system $S = \big(\{(A_i, \mathcal{F}_i)\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j}\big)$ of finite filtered algebras with $A = \varinjlim A_i$ as K-algebras. Then, we define the entropy of A for the system S, denoted by $\mathrm{h}_{\mathrm{alg}}(A, S)$, as

$$\mathrm{h}_{\mathrm{alg}}(A, S) := \sup_{i\in I}\{\mathrm{h}_{\mathrm{alg}}(A_i, \mathcal{F}_i)\}$$

Let us compute the entropy of the path algebra of the infinity chain.

Example 9.14. Consider $S = (\{(KCh_i, \mathcal{F}_i)\}_{i\geqslant 1}, \{\varphi_{i,j}\}_{i\leqslant j})$ a strict direct system of filtered algebras with $\mathcal{F}_i$ the standard filtration and $\varphi_{i,j}\colon KCh_i \hookrightarrow KCh_j$ the inclusion. The adjacency matrix of Ch_i is $A_i = \sum_{j=1}^{n-1}(E_{j,j+1} + E_{j+1,j})$ where E_{jk} is the elementary (j,k)-matrix (which unique nonzero entry is 1 in the (j,k) place). This matrix is symmetric. By Proposition 8.11 the sequence of entropies $h_i := \mathrm{h}_{\mathrm{alg}}(KCh_i, \mathcal{F}_i)$ is non-decreasing $h_i \leqslant h_{i+1}$ and bounded since each Ch_i is a subgraph of the extended graph of a cycle with $i+1$ vertices, denoted by $\hat{C}_{i+1}$ (see Figure 29).

The entropy of the $K\hat{C}_{i+1}$ is $\log 2$ (Example 8.21) and does not depend on the number of vertices, then we have an upper bound for $\{h_i\}_{i=2}^{\infty}$ which is $\log 2$. So $\{h_i\}_{i=1}^{\infty}$ is convergent and $\sup_{i\geqslant 2}\{h_i\} \leqslant \log 2$.

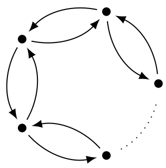

Figure 29: Extended directed cycle with i vertices.

Our next goal is to prove the equality $\sup_{i\geqslant 2}\{h_i\} = \log 2$. From Proposition 8.30 we have that $h_i = \log\rho(A_i)$. Computing the spectral radius of the different matrices is a complex computational problem. However, we can obtain an smart lower bound that will prove our assertion. For every symmetric matrix A of size m with $\mathrm{spec}(A)$ the set of its eigenvalues, we have that

$$(A)_{1,1} \leqslant \mathrm{tr}(A) = \sum_{\lambda\in\mathrm{spec}(A)} \lambda \leqslant \sum_{\lambda\in\mathrm{spec}(A)} |\lambda| \leqslant m\rho(A).$$

Thus, the following inequality holds

$$(A^n)_{1,1} \leqslant m\rho(A^n) \leqslant m\rho(A)^n.$$

In particular $\sqrt[n]{(A^n)_{1,1}} \leqslant \sqrt[n]{i+1}\rho(A)$. If we apply this to the adjacency matrix A_i of Ch_i we have that

$$\sqrt[n]{(A_i^n)_{1,1}} \leqslant \sqrt[n]{i+1}\rho(A_i)$$

for every $n \geqslant 1$. On the other hand, for every n, each entry $(A_i^{2n})_{1,1}$ counts the number of paths of length $2n$ from the vertex v_1 to v_1. It can be seen that this number is also the number of Dyck's words of length $2n$. By Corollary 6.2.3 (ii) or (iii) of [99], the number of Dyck's words of length $2n$ is given by the Catalan number $c_n = \binom{2n}{n}\frac{1}{n+1}$. Then $(A_i^{2n})_{1,1} = c_n$ where $\{c_n\}_{n\geqslant 1}$ denotes the Catalan numbers. Thus, $\sqrt[2n]{c_n} \leqslant \sqrt[2n]{i+1}\rho(A_i)$ for every n and every $i \geqslant 2$. Since the logarithm is an increasing function, we have that

$$\frac{1}{2n}\log c_n \leqslant \frac{\log(i+1)}{2n} + \log\rho(A_i) = \frac{\log(i+1)}{2n} + h_i$$

for every n and every $i \geqslant 2$. Using Stirling formula we have that $\lim_{n\to\infty}\frac{1}{2n}\log c_n = \log 2$. Consequently, if n tends to infinity, then $\log 2 \leqslant h_i$ for every $i \geqslant 2$. Finally,

$$\log 2 \leqslant \sup_{i\geqslant 2} h_i \leqslant \log 2$$

which means that the equality holds. In particular, $\mathrm{h_{alg}}(KCh_\infty, S) = \log 2$.

Similarly as it happens for the entropy of filtered algebras, the entropy of an algebra which is direct limit of a strict system S depends on the chosen system.

Proposition 9.15. *Consider $S = \big(\{(A_i, \mathcal{F}_i)\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j}\big)$ a strict direct system of finite filtered algebras with $A = \varinjlim A_i$ and $\mathcal{F}_i = \{V_n^i\}_{i\in I}$. If we take $\mathcal{G}_i := \{W_n^i\}_{n=1}^\infty$ for a fixed $k \geqslant 1$ such that $W_n^i := V_{kn}^i$, then we have the following:*

1. *$S' = \big(\{(A_i, \mathcal{G}_i)\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j}\big)$ is a strict direct system of finite filtered algebras,*
2. *$\mathrm{h_{alg}}(A, S') = k\,\mathrm{h_{alg}}(A, S)$.*

Proof. First of all, we have to prove that S' is a strict direct system of filtered algebras. By Proposition 8.7, the family $\mathcal{G}_i$ is a filtration. In addition, by construction, the system is strict. The morphisms $\varphi_{i,j}$ remain being injective and $\varphi_{i,j}^{-1}(W_n^j) = \varphi_{i,j}^{-1}(V_{kn}^j) = V_{kn}^i = W_n^i$.

For the second part, observe that, by Proposition 8.7

$$\mathrm{h_{alg}}(A_i, \mathcal{G}_i) = k\,\mathrm{h_{alg}}(A_i, \mathcal{F}_i).$$

Consequently,

$$\mathrm{h_{alg}}(A, S') = \sup_{i\in I}\{\mathrm{h_{alg}}(A_i, \mathcal{G}_i)\} =$$
$$= \sup_{i\in I}\{k\,\mathrm{h_{alg}}(A_i, \mathcal{F}_i)\} = k \sup_{i\in I}\{h(A_i \mathcal{F}_i)\} = k\,\mathrm{h_{alg}}(A, S).$$

□

Under this setting we ask ourselves if the entropy of filtered algebras and the entropy of direct limits are compatible. That is, if $(A, \mathcal{F})$ is a finite filtered algebra and $A = \varinjlim A_i$ with $(\{A_i\}_{i\in I}, \{\varphi_{i,j}\}_{i,j\in I})$ a direct system where $\varphi_{i,j}\colon A_i \to A_j$ is a monomorphism, is there a strict direct system S such that $\mathrm{h_{alg}}(A, \mathcal{F}) = \mathrm{h_{alg}}(A, S)$? If $\mathcal{F} = \{V_n\}_{n=1}^\infty$, then by Proposition 8.11 we can construct a finite filtration in each A_i of the form $\mathcal{F}_i = \{\varphi_i^{-1}(V_n)\}_{n=1}^\infty$.

Proposition 9.16. *Let $(A, \mathcal{F})$ be a finite filtered algebra with $A = \varinjlim A_i$ for a direct system $(\{A_i\}_{i\in I}, \{\varphi_{i,j}\}_{i,j\in I})$ where $\varphi_{i,j}\colon A_i \to A_j$ is a monomorphism for every $i \leqslant j$. The pair $S = \big(\{(A_i, \mathcal{F}_i)\}_{i\in I}, \{\varphi_{i,j}\}_{i\leqslant j}\big)$ is a strict direct system of finite filtered algebras such that*

$$\mathrm{h_{alg}}(A, \mathcal{F}) = \mathrm{h_{alg}}(A, S).$$

Proof. If $(A, \{\varphi_i\}_{i\in I})$ is the direct limit, then by Lemma 9.6, the map φ_i is a monomorphism for every $i \in I$. Following Proposition 8.11 each $\mathcal{F}_i$ is a filtration and

$$\mathrm{h_{alg}}(A_i, \mathcal{F}_i) \leqslant \mathrm{h_{alg}}(A, \mathcal{F})$$

for every $i \in I$. If we take the supremum, then

$$\mathrm{h_{alg}}(A, S) = \sup_{i\in I}\{\mathrm{h_{alg}}(A_i, \mathcal{F}_i)\} \leqslant \mathrm{h_{alg}}(A, \mathcal{F}).$$

In order to prove that $\mathrm{h_{alg}}(A, \mathcal{F}) \leqslant \mathrm{h_{alg}}(A, S)$ we need to check that $\varinjlim V_n^i = V_n$.

First observe that S is a direct system of finite filtered algebras. Indeed, we only have to check that $\varphi_{i,j}(V_n^i) \subseteq V_n^j$. Since $\varphi_i = \varphi_j \circ \varphi_{i,j}$, then

$$\begin{aligned}\varphi_{i,j}(V_n^i) = \varphi_{i,j}(\varphi_i^{-1}(V_n)) = \varphi_{i,j}\left((\varphi_j \circ \varphi_{i,j})^{-1}(V_n)\right) = \\ = \varphi_{i,j}\left(\varphi_{i,j}^{-1}\left(\varphi_j^{-1}(V_n)\right)\right) \subseteq V_n^j.\end{aligned}$$

Consequently, by Lemma 9.7 the pair $S_n = (\{V_n^i\}_{i\in I}, \{\varphi_{i,j}\}_{i,j})$ is a direct system too. Finally, to verify that $\varinjlim V_n^i = V_n$ we are going to see that it verifies its universal property. Consider a target $(W, \{\psi_i\}_{i\in I})$ for the direct system S_n. For a element $v \in V_n$, there exists a $i \in I$ such that $\varphi_i^{-1}(v) \in V_n^i$ (which is unique since φ_i is a monomorphism). Then, we define the map

$$\begin{aligned}\theta\colon \quad V_n &\to \quad W \\ v &\mapsto \psi_i(\varphi_i^{-1}(v)).\end{aligned}$$

By construction it is direct that $\psi_i = \theta \circ \varphi_i$. However, we need to prove that the map θ is well-defined, that is, it does not depend on the chosen $i \in I$. Suppose $j \in I$ with $\varphi_j^{-1}(v) \in V_n^j$, there is some $k \geqslant i, j$ with $\psi_i = \psi_k \circ \varphi_{i,k}$, $\psi_j = \psi_k \circ \varphi_{j,k}$, $\varphi_i = \varphi_k \circ \varphi_{i,k}$ and $\varphi_j = \varphi_k \circ \varphi_{j,k}$, then

$$\begin{aligned}\psi_i(\varphi_i^{-1}(v)) = \psi_k(\varphi_{i,k}((\varphi_k \circ \varphi_{i,k})^{-1}(v))) = \\ \psi_k(\varphi_{i,k}(\varphi_{i,k}^{-1}(\varphi_k^{-1}(v)))) = \psi_k(\varphi_k^{-1}(v)).\end{aligned}$$

Analogously we have that $\psi_i(\varphi_j^{-1}(v)) = \psi_k(\varphi_k^{-1}(v))$ and θ is well-defined. Consequently, $\varinjlim V_n^i = V_n$. Using Lemma 9.7 and since $\varinjlim V_n^i = V_n$, we have that

$$\dim(V_n/V_{n-1}) = \sup_{i\in I}\left\{\dim\left(V_n^i/V_{n-1}^i\right)\right\} \tag{26}$$

because the maps $\varphi_{i,j}$ are injective.

For the last part, check that

$$\frac{\log(\dim(V_n^i/V_{n-1}^i)}{n} \leqslant \limsup_{n\to\infty} \frac{\log(\dim(V_n^i/V_{n-1}^i)}{n}$$

for every $i \in I$ and every $n \geqslant 2$, then

$$\sup_{i\in I}\left\{\frac{\log(\dim(V_n^i/V_{n-1}^i))}{n}\right\} \leqslant \sup_{i\in I}\left\{\limsup_{n\to\infty}\frac{\log(\dim(V_n^i/V_{n-1}^i))}{n}\right\} \tag{27}$$

for every $n \geqslant 2$. Finally, we have that

$$\begin{aligned} \mathrm{h}_{\mathrm{alg}}(A,\mathcal{F}) &= \limsup_{n\to\infty}\frac{\log\left(\dim(V_n/V_{n-1})\right)}{n} = \\ &= \limsup_{n\to\infty}\frac{\log\left(\sup_{i\in I}\left\{\dim(V_n^i/V_{n-1}^i)\right\}\right)}{n} = \\ &= \limsup_{n\to\infty}\sup_{i\in I}\left\{\frac{\log(\dim(V_n^i/V_{n-1}^i)}{n}\right\} \leqslant \sup_{i\in I}\left\{\limsup_{n\to\infty}\frac{\log(\dim(V_n^i/V_{n-1}^i)}{n}\right\} = \\ &= \sup_{i\in I}\left\{\mathrm{h}_{\mathrm{alg}}(A_i,\mathcal{F}_i)\right\} = \mathrm{h}_{\mathrm{alg}}(A,S). \end{aligned}$$

□

Part IV

ALGEBRAS ASSOCIATED TO GRAPHS AS FUNCTORS

Chapter 10

ISOMORPHISMS INDEPENDENT OF THE FIELD

One of the isomorphism problems for Leavitt path algebras is the following: when E_1 and E_2 are graphs, K a field and $L_K(E_1) \cong L_K(E_2)$ as$K$-algebras, can we have a K'-algebra isomorphism $L_{K'}(E_1) \cong L_{K'}(E_2)$ for some other field K'? In the literature, there are positive answers to closely related problems. For example, in [28] it is proved that the graph groupoids of directed graphs E_1 and E_2 are topologically isomorphic if and only if there is a diagonal-preserving ring $*$-isomorphism between the Leavitt path algebras of E_1 and E_2. More generally, we can find questions dealing with the equivalence between the existence of graded isomorphism of groupoids and diagonal-preserving graded isomorphism of Steinberg algebras, like in the works [1, 30, 103]. Another isomorphism-type problem is the following: assume $K \subset K'$ is a field extension and $L_{K'}(E_1) \cong L_{K'}(E_2)$ as K'-algebras. We would like to know under what conditions this implies that $L_K(E_1) \cong L_K(E_2)$ as K-algebras. This kind of results could be termed as 'descending conditions for isomorphism'. In case such a condition exists for an extension $K \subset K'$, then $L_{K'}(E_1) \cong L_{K'}(E_2)$ implies $L_{K''}(E_1) \cong L_{K''}(E_2)$ for any superfield $K'' \supset K$. When trying to give partial solutions to these questions, we find that there is no need to focus on Leavitt path algebras. We can answer them using suitable functors. Such are the so-called extension invariant functors. One of the main tools that we use is the Hilbert's Nullstellensatz Theorem for polynomials with possibly infinitely many variables. This is a result of S. Lang [77] which generalizes the classical Nullstellensatz Theorem but imposes an additional condition on the cardinal of the ground field.

10.1 ALGEBRA FUNCTORS

For a commutative, associative and unital ring K we denote by $\mathbf{alg}_K$ the *category of associative and commutative K-algebras with non-zero unit* where morphisms are given by the unit preserving K-algebra morphisms. In addition, we will denote by $\mathbf{Alg}_K$ the *category of all K-algebras* with the morphisms given by the K-algebra morphisms.

Definition 10.1. A functor $\mathcal{F}\colon \mathbf{alg}_K \to \mathbf{Alg}_K$ such that $\mathcal{F}(R)$ is an R-algebra for any $R \in \mathbf{alg}_K$ will be called a *K-algebra functor*.

We will identify the category $\mathbf{alg}_{\mathbb{Z}}$ with the category of (associative commutative and unital) rings, usually denoted $\mathbf{rng}$. Some well-known examples are in order.

Example 10.2. 1. Take a positive integer n and consider the algebra functor $\mathcal{M}_n\colon \mathbf{rng} \to \mathbf{Alg}_{\mathbb{Z}}$ such that $\mathcal{M}_n(R)$ is the R-algebra of $n \times n$ matrices with coefficients in R with the usual matrix product.If $\varphi\colon R \to R'$ is a morphism of rings, then

$$\begin{array}{cccc} \mathcal{M}_n(\varphi)\colon & \mathcal{M}_n(R) & \to & \mathcal{M}_n(R') \\ & \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} & \mapsto & \begin{pmatrix} \varphi(a_{1,1}) & \varphi(a_{1,2}) & \cdots & \varphi(a_{1,n}) \\ \varphi(a_{2,1}) & \varphi(a_{2,2}) & \cdots & \varphi(a_{2,n}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi(a_{n,1}) & \varphi(a_{n,2}) & \cdots & \varphi(a_{n,n}) \end{pmatrix}. \end{array}$$

is a $\mathbb{Z}$-algebra homomorphism. This functor produces associative unitary algebras.

2. Again, for a positive integer n, consider $\mathrm{sl}_n\colon \mathbf{rng} \to \mathbf{Alg}_{\mathbb{Z}}$ such that

$$\mathrm{sl}_n(R) := \{X \in \mathcal{M}_n(R)\colon \ \mathrm{tr}(X) = 0\},$$

where $\mathrm{tr}(X)$ denotes the trace of the matrix X, with the Lie product

$$[X,Y] := XY - YX$$

for any $X, Y \in \mathrm{sl}_n(R)$.

Another algebra functor is Another source of algebra functors, that we will use in the sequel, is the following:

Definition 10.3. Let $A \in \mathbf{Alg}_K$, we are able to define the K-algebra functor $\underline{A}\colon \mathbf{alg}_K \to \mathbf{Alg}_K$ given by $\underline{A}(R) := A \otimes_K R = A_R$. Furthermore, for any $R, R' \in \mathbf{alg}_K$, if $\alpha : R \to R'$ is a morphism in $\mathbf{alg}_K$, then $\underline{A}(\alpha)\colon \underline{A}(R) \to \underline{A}(R')$ is defined as $\underline{A}(\alpha) = \mathrm{Id}_A \otimes_K \alpha\colon A \otimes_K R \to A \otimes_K R'$ such that $a \otimes_K r \mapsto a \otimes_K \alpha(r)$.

Observation 10.4. When it is understood by the context, the tensor product will be simply denoted by $\otimes$ instead of $\otimes_K$.

As we previously mentioned, this set of functors are well behaved. If K is a commutative and unital ring, $R \in \mathbf{alg}_K$, $S \in \mathbf{alg}_R$ and $A \in \mathbf{Alg}_K$, then it is straightforward to prove that $(A \otimes_K R) \otimes_R S \cong A \otimes_K (R \otimes_R S)$. As a consequence, $(A_R)_S \cong A_S$ as S-algebras. For more details, define $\omega_{RS}\colon (A_R)_S \to A_S$ the morphism (of S-algebras) such that $\omega_{RS}((a \otimes_K r) \otimes_R s) := a \otimes_R rs$.In fact, it is an isomorphism since $\phi\colon A_S \to (A_R)_S$ given by $\phi(a \otimes rs) = (a \otimes r) \otimes s$ with $a \in A$, $r \in R$ and $s \in S$. Not only that but ω_{RS} is natural in S, that is, for any $\beta \in \hom_{\mathbf{alg}_R}(S, S')$ we have the commutativity of the diagram:

$$\begin{array}{ccc} (A_R)_S & \xrightarrow{\omega_{RS}} & A_S \\ {\scriptstyle \mathrm{Id}_{A_R} \otimes_R \beta}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Id}_A \otimes_K \beta} \\ (A_R)_{S'} & \xrightarrow[\omega_{RS'}]{} & A_{S'} \end{array} \tag{28}$$

Indeed, if we consider $(a \otimes r) \otimes s \in (A_R)_S$, then

$$\omega_{RS'}(\mathrm{Id}_{A_R} \otimes \beta((a \otimes r) \otimes s)) = \omega_{RS'}((a \otimes r) \otimes \beta(s)) = a \otimes r\beta(s).$$

Since β is a R-algebra morphism, we have that

$$\begin{aligned} \omega_{RS'}(\mathrm{Id}_{A_R} \otimes \beta((a \otimes r) \otimes s)) &= a \otimes r\beta(s) = a \otimes \beta(rs) = \\ &= \mathrm{Id}_A \otimes \beta(a \otimes rs) = \mathrm{Id}_A \otimes \beta(\omega_{RS}((a \otimes r) \otimes s)). \end{aligned}$$

One important property associated to K-algebra functors that we will use along this work is introduced now.

Definition 10.5. Let K be a commutative and unital ring. A K-algebra functor $\mathcal{F}\colon \mathbf{alg}_K \to \mathbf{Alg}_K$ is said to be *extension invariant* if for every $R \in \mathbf{alg}_K$ there is an R-algebra isomorphism $\tau_R\colon \mathcal{F}(R) \to \mathcal{F}(K) \otimes_K R$ in $\mathbf{Alg}_K$ which is natural in R, that is, for any $\alpha\colon R \to R'$ homomorphism between $R, R' \in \mathbf{alg}_K$ the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}(R) & \xrightarrow{\tau_R} & \mathcal{F}(K) \otimes_K R \\ {\scriptstyle \mathcal{F}(\alpha)}\downarrow & & \downarrow{\scriptstyle 1_{\mathcal{F}(K)}\otimes_K \alpha} \\ \mathcal{F}(R') & \xrightarrow[\tau_{R'}]{} & \mathcal{F}(K) \otimes_K R' \end{array} \tag{29}$$

Proposition 10.6. *If $\mathcal{F}$ is an extension invariant functor, then $\mathcal{F}(S) \cong \mathcal{F}(R) \otimes_R S$ as S-algebras for any $R \in \mathbf{alg}_K$ and $S \in \mathbf{alg}_R$. Furthermore, the above isomorphism can be chosen to be natural in S.*

Proof. The isomorphism is the composition $\tau_S^{-1}\omega_{RS}(\tau_R \otimes \mathrm{Id}_S)\colon \mathcal{F}(R) \otimes_R S \to \mathcal{F}(S)$ and it is natural in S because for any homomorphism of R-algebras $\beta\colon S \to S'$, we have the commutativity of the three squares in the diagram:

$$\begin{array}{ccccccc} \mathcal{F}(R) \otimes_R S & \xrightarrow{\tau_R \otimes_R \mathrm{Id}_S} & (\mathcal{F}(K) \otimes_K R) \otimes_R S & \xrightarrow{\omega_{RS}} & \mathcal{F}(K) \otimes_K S & \xrightarrow{\tau_S^{-1}} & \mathcal{F}(S) \\ {\scriptstyle \mathrm{Id}_{\mathcal{F}(R)} \otimes_R \beta}\downarrow & & {\scriptstyle \mathrm{Id}_{\mathcal{F}(K)\otimes_K R} \otimes_R \beta}\downarrow & & {\scriptstyle \mathrm{Id}_{\mathcal{F}(K)} \otimes_K \beta}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(\beta)} \\ \mathcal{F}(R) \otimes_R S' & \xrightarrow{\tau_R \otimes_R \mathrm{Id}_{S'}} & (\mathcal{F}(K) \otimes_K R) \otimes_R S' & \xrightarrow{\omega_{RS'}} & \mathcal{F}(K) \otimes_K S' & \xrightarrow{\tau_{S'}^{-1}} & \mathcal{F}(S') \end{array}$$

The first square on the left commutes since

$$(\tau_R \otimes_R \mathrm{Id}_{S'}) \circ (\mathrm{Id}_{\mathcal{F}(R)} \otimes_R \beta) = \tau_R \otimes_R \beta = (\mathrm{Id}_{\mathcal{F}(K)\otimes_K R} \otimes \beta) \circ (\tau_R \otimes_R \mathrm{Id}_S).$$

The second and third squares commute because of (28) and (29). □

Remark 10.7. Observe that if $\mathcal{F}\colon \mathbf{alg}_K \to \mathbf{Alg}_K$ is a K-algebra functor which is extension invariant, we can consider $A := \mathcal{F}(K)$, then $\mathcal{F} \cong \underline{A}$. Indeed, take $R \in \mathbf{alg}_K$, since $\mathcal{F}$ is extension invariant there is a natural isomorphism $\tau_R\colon \mathcal{F}(R) \to \mathcal{F}(K) \otimes_K R$ such that $\mathcal{F}(R) \cong \mathcal{F}(K) \otimes_K R = \underline{A}(R)$. Conversely, if $A \in \mathbf{Alg}_K$, then the functor $\underline{A}$ is extension invariant. We need to find a natural isomorphism τ such that $\underline{A}(R) \cong \underline{A}(K) \otimes_K R$. We just need to define $\tau_R\colon \underline{A}(R) \to \underline{A}(K) \otimes_K R$ given by $\tau_R(a \otimes_R r) = (a \otimes_K 1) \otimes_K r$ with inverse given by $\tau_R^{-1}((a \otimes_K k) \otimes_K r) = a \otimes_R kr$. Finally, we need to see that it is natural. Consider $\alpha \in \hom_{\mathbf{alg}_k}(R, R')$, then

$$\tau_{R'} \circ \underline{A}(\alpha)(a \otimes_R r) = \tau_{R'}(a \otimes_K \alpha(r)) = (a \otimes_K 1) \otimes_K \alpha(r) =$$
$$= \mathrm{Id}_{\underline{A}(K) \otimes_K} \otimes \alpha((a \otimes_K 1) \otimes_K r) = (\mathrm{Id}_{\underline{A}(K) \otimes_K} \otimes \alpha) \circ \tau_R(a \otimes_K r).$$

As a consequence, $\underline{A}$ is extension invariant.

That is, the extension invariant functors are precisely those algebra functors which are (naturally) isomorphic to some $\underline{A}$.

Consider $R, S \in \mathbf{alg}_K$ and $\varphi \colon R \to S$ a morphism in their category. Then, it is possible to endow S with a structure of R-algebra by considering the product:

$$\begin{array}{ccc} R \times S & \to & S \\ (r, s) & \mapsto & \varphi(r)s. \end{array}$$

As a consequence of Proposition 29 we have the next result.

Corollary 10.8. *If $\mathcal{F}, \mathcal{G} \colon \mathbf{alg}_K \to \mathbf{Alg}_K$ are extension invariant functors and $\varphi \colon R \to S$ a homomorphism in $\mathbf{alg}_K$, then $\mathcal{F}(R) \cong \mathcal{G}(R)$ as R-algebras implies $\mathcal{F}(S) \cong \mathcal{G}(S)$ as S-algebras. In particular for $R \in \mathbf{alg}_K$, if we have an homomorphism $\psi \colon R \to K$, then $\mathcal{F}(R) \cong \mathcal{G}(R)$ as R-algebras implies that $\mathcal{F}(K) \cong \mathcal{G}(K)$, which means that $\mathcal{F}(S) \cong \mathcal{G}(S)$ for every $S \in \mathbf{alg}_K$.*

Note that Corollary 10.8 is true for any Hopf algebra since by definition a Hopf algebra is endowed with a homomorphism to the ground field called *augmentation* (see [106, Section 1.4 (page 8)]). In other words, we can verify that two extension invariant functors are isomorphic just by checking in a Hopf algebra. Any group algebra is a Hopf algebra, so observe that we are in the conditions of Corollary 10.8: we deduce that if for some group G and extension invariant functors $\mathcal{F}$ and $\mathcal{G}$ we have $\mathcal{F}(KG) \cong \mathcal{G}(KG)$, then $\mathcal{F}(S) \cong \mathcal{G}(S)$ for any $S \in \mathbf{alg}_K$.

Next, we recall an extension algebra functor from which we will derive some subfunctors (derivations, centroid).

Example 10.9. Let V be a vector space over a field K and denote by $\mathrm{End}_K(V)$ the K-algebra of all K-linear maps $V \to V$. If $R \in \mathbf{alg}_K$, we will also denote $\mathrm{End}_R(V_R)$ the R-algebra of all R-module endomorphisms $V_R \to V_R$. There is a functor $\mathfrak{G} \colon \mathbf{alg}_K \to \mathbf{Alg}_K$ such that $\mathfrak{G}(R) := \mathrm{End}_R(V_R)$ and for any $\alpha \colon R \to R'$ (in $\mathbf{alg}_K$) we have $\mathfrak{G}(\alpha) \colon \mathrm{End}_R(V_R) \to \mathrm{End}_{R'}(V_{R'})$ given by $\mathfrak{G}(\alpha)(T) = T'$ such that $T' \colon V_{R'} \to V_{R'}$ is defined by $T'(v \otimes 1_{R'}) := (\mathrm{Id}_V \otimes \alpha) T(v \otimes 1_R)$ whenever

$T \in \mathrm{End}_R(V_R)$ ($1_R \in R$ and $1_{R'} \in R'$ are the units in R and R' respectively). We will call $\mathfrak{G}$ the *endomorphism functor*. This algebra functor is well-known and the reader can find it in [35, p. 185]

In order to see that it is extension invariant, consider the map

$$\begin{array}{rccc} \Omega_R : & \mathrm{End}_K(V) \otimes_K R & \to & \mathrm{End}_R V_R \\ & f \otimes r & \mapsto & \Omega_R(f \otimes r) \end{array}$$

where

$$\begin{array}{rccc} \Omega_R(f \otimes r) & V_R & \to & V_R \\ & v \otimes s & \mapsto & f(v) \otimes rs \end{array}$$

is a R-lineal map. Not only that but, Ω_R is a natural isomorphism. Indeed, first observe that

$$\begin{aligned} \Omega_R((f \otimes r)(f' \otimes r'))(v \otimes s) &= \Omega_R(ff' \otimes rr')(v \otimes s) = f(f'(v)) \otimes rr's = \\ &= \Omega_R(f \otimes r)(f'(v) \otimes r's) = \Omega_R(f \otimes r)\left(\Omega_R(f' \otimes r')(v \otimes s)\right) = \\ &= (\Omega_R(f \otimes r) \circ \Omega_R(f' \otimes r'))(v \otimes s) \end{aligned}$$

which indicates that it is a morphisms of R-algebras (the R-linearity is immediate). To prove the surjectivity consider $\mathcal{B}_V = \{e_i\}_{i \in I}$ a basis of V and $\mathcal{B}_R = \{u_j\}_{j \in J}$ a basis of R as K-vector spaces. Consequently, $\{e_i \otimes u_j\}_{i \in I, j \in J}$ is a basis for V_R. Let $S \in \mathrm{End}_R(V_R)$ and suppose

$$S(e_i \otimes 1) = \sum_{p \in I, q \in J} \lambda_i^{p,q} e_p \otimes u_q$$

with $\lambda_i^{p,q} \in K$ and only a finite number of $\lambda_i^{p,q} \neq 0$. Now consider the K-linear endomorphism

$$\begin{array}{rccc} T^q : & V & \to & V \\ & e_i & \mapsto & T^q(e_i) := \sum_{p \in I} \lambda_i^{p,q} e_p. \end{array}$$

It is straightforward to check that

$$\Omega_R\left(\sum_{q \in J} T^q \otimes u_q\right)(e_i \otimes 1) = \sum_{q \in J} T^q(e_i) \otimes u_q = \sum_{p \in I, q \in J} \lambda_i^{p,q} e_p \otimes u_q = S(e_i \otimes 1),$$

that is, $\Omega_R\left(\sum_{q \in J} T^q \otimes u_q\right) = S$ and the map is surjective. Secondly, we have to see that Ω is injective. Assume that $\Omega(\sum_{p=1}^n T_p \otimes r_p) = 0$ for $T_p \in \mathrm{End}_K(V)$,

$r_p \in R$ with $T_p(e_i) = \sum_{k\in I} t^k_{p,i} e_k$ where $t^k_{p,i} \in K$ (a finite number of $t^k_{p,i} \neq 0$) and $r_p = \sum_{q\in J} \gamma^q_p u_q$ where $\gamma^q_p \in K$ (with a finite numer of $\gamma^q_p \neq 0$). In particular, we have that for every $i \in I$

$$0 = \Omega_R\left(\sum_{p=1}^{n} T_p \otimes r_p\right)(e_i \otimes 1) = \sum_{p=1}^{n} T_p(e_i) \otimes r_p = \sum_{p=1,k\in I}^{n} t^k_{p,i} e_k \otimes r_p =$$
$$= \sum_{p=1,k\in I,q\in J} t^k_{p,i}\gamma^q_p e_k \otimes u_q = \sum_{k\in I,q\in J}\left(\sum_{p=1}^{n} t^k_{p,i}\gamma^q_p\right) e_k \otimes u_q.$$

Since $\{e_k \otimes u_q\}_{k\in I,q\in J}$ is basis of V_R we get

$$\sum_{p=1}^{n} t^k_{p,i}\gamma^q_p = 0 \tag{30}$$

for every $i,k \in I$ and $q \in J$. If we consider the map

$$\begin{array}{cccc} E_{i,j} & V & \to & V \\ & e_p & \mapsto & E_{i,j}(e_p) := \delta_{i,p} e_j \end{array}$$

then we can express $T_p = \sum_{i,k\in I} t^k_{pi} E_{i,k}$ and

$$\sum_{p=1}^{n} T_p \otimes r_p = \sum_{p=1,i,k\in I,q\in J}^{n} t^k_{p,i}\gamma^q_p E_{i,k} \otimes u_q = \sum_{i,k\in I,q\in I}\left(\sum_{p=1}^{n} t^k_{p,i}\gamma^q_p\right) E_{i,k} \otimes u_q = 0$$

which is zero by 30.

$$\begin{array}{ccc} \mathrm{End}_K(V) \otimes_K R & \xrightarrow{\Omega_R} & \mathrm{End}_R(V_R) \\ {\scriptstyle \mathrm{Id}_{\mathrm{End}_K(V)} \otimes \alpha}\downarrow & & \downarrow{\scriptstyle \mathfrak{G}(\alpha)} \\ \mathrm{End}_K(V) \otimes_K R' & \xrightarrow[\Omega_{R'}]{} & \mathrm{End}_{R'}(V_{R'}). \end{array} \tag{31}$$

Finally, in order to prove that it is natural, take $\alpha\colon R \to R'$,$T \in \mathrm{End}_K(V)$ and $r \in R$, for every $v \in V$ we have that

$$\begin{gathered}\mathfrak{G}(\alpha)\left(\Omega_R(T\otimes r)\right)(v\otimes 1_{R'}) = (\mathrm{Id}_V \otimes \alpha)(\Omega_R(T\otimes r)(v\otimes 1_R)) = \\ = (\mathrm{Id}_V \otimes \alpha)(T(v)\otimes r) = T(v)\otimes \alpha(r) = \Omega_{R'}(T\otimes \alpha(r))(v\otimes 1_{'R}) = \\ = \Omega_{R'}\left((\mathrm{Id}_{\mathrm{End}_K(V)} \otimes \alpha)(T\otimes r)\right)(v\otimes 1_{r'}).\end{gathered}$$

Consequently

$$\mathfrak{G}(\alpha)\left(\Omega_R(T\otimes r)\right) = \Omega_{R'}\left((\mathrm{Id}_{\mathrm{End}_K(V)} \otimes \alpha)(T\otimes r)\right)$$

for every $T\otimes r \in \mathrm{End}_K(V)\otimes_K R$, which means that

$$\mathfrak{G}(\alpha)\circ \Omega_R = \Omega_{R'} \circ (\mathrm{Id}_{\mathrm{End}_K(V)} \otimes \alpha).$$

The following example gives an algebra functor which is not extension invariant.

Example 10.10. For a K-algebra U, denote by $\mathrm{Der}(U)$ the *derivations* of the corresponding algebra, that is to say, all the K-linear applications $d\colon U \to U$ satisfying $d(ab) = ad(b) + d(a)b$ for any elements $a, b \in U$. it can be proved that $R \mapsto \mathrm{Der}(U_R)$ is the object map of a functor

$$\mathcal{D}\colon \mathbf{alg}_K \to \mathbf{Alg}_K$$

which is a subfunctor of the endomorphism functor $\mathfrak{G}$. However, even though $\mathfrak{G}$ is extension invariant, $\mathcal{D}$ is not. Following [14, Theorem 1] or [15, Remark 2.31], if we consider a perfect finite-dimensional nontrivial K-algebra U, then we have that

$$\mathrm{Der}(U\otimes K[x]) \cong (\mathrm{Der}(U)\otimes K[x]) \oplus (\mathcal{C}(U)\otimes \mathrm{Der}(K[x])),$$

where $\mathcal{C}(U)$ is the centroid of U. Taking into account that $\mathrm{Der}(K[x]) \neq 0$ (since the well known diferential that we study in school is a derivation) and $\mathcal{C}(U) \neq 0$ (because the identity is always in the centroid of any algebra) we have that $\mathcal{D}(K[x]) \not\cong \mathcal{D}(K)\otimes_K K[x]$. In fact, this construction gives us an example of a non extension invariant subfunctor of an extension invariant functor.

10.1.1 Algebra functors associated to graphs

Even though we have worked most of the times during the course of this dissertation with a field K, the definition of path, Leavitt path, relative Cohn path, Clark and Steinberg algebra allow the use of a unitary commutative ring. In this way, for a certain algebra associated to a graph we can obtain an algebra functor. Moreover, a extension invariant functor.

Definition 10.11. Consider a directed graph E, we define the *path functor* $\_E\colon \mathbf{rng} \to \mathbf{Alg}_{\mathbb{Z}}$ such that sends a ring R into RE and a morphism $\alpha\colon R \to R'$ into the morphism

$$\begin{array}{rccc} \hat{\alpha}\colon & RE & \to & R'E \\ & \sum_{i=1}^{n} r_i\lambda_i & \mapsto & \sum_{i=1}^{n} \alpha(r_i)\lambda_i. \end{array}$$

Take $X \subseteq \operatorname{Reg}(E)$, we define the *relative Cohn path functor* $C^X(E)\colon \mathbf{rng} \to \mathbf{Alg}_{\mathbb{Z}}$ such that $R \mapsto C^X_R(E)$ and if $\alpha\colon R \to R'$ then

$$\begin{array}{rccc} C^X(E)(\alpha)\colon & C^X_R(E) & \to & C^X_{R'}(E) \\ & \sum_{i=1}^{n} r_i\lambda_i\mu_i^* & \mapsto & \sum_{i=1}^{n} \alpha(r_i)\lambda_i\mu_i^*. \end{array}$$

In particular, if $X = \operatorname{Reg}(X)$ then define the *Leavitt path functor* denoted by $L(E) := C^{\operatorname{Reg}(X)}(E)$.

Remark 10.12. As an important fact, by [1, Corollary 1.5.14], we have that

$$C^X_S(E) \cong C^X_R(E) \otimes_R S$$

for any unital commutative ring R and any unital commutative R-algebra S. It is easy to check that this isomorphism is natural, therefore the functor $C^X(E)$ is extension invariant.

Definition 10.13. Consider $\mathcal{G}$ an ample Hausdorf groupoid, then we define the *Steinberg functor*, denoted by $A(\mathcal{G})\colon \mathbf{rng} \to \mathbf{Alg}_{\mathbb{Z}}$, to the functor such that $R \mapsto A_R(G)$ and for an homomorphism $\alpha\colon R \to R'$ we define

$$\begin{array}{rccc} A(\mathcal{G})(\alpha)\colon & A_R(\mathcal{G}) & \to & A_{R'}(\mathcal{G}) \\ & \sum_{i=1}^{n} r_i 1_{B_i} & \mapsto & \sum_{i=1}^{n} \alpha(r_i) 1_{B_i}. \end{array}$$

In the particular case that the groupoid is a group G, we might consider the *group ring functor* $\_G\colon \mathbf{rng} \to \mathbf{Alg}_{\mathbb{Z}}$ such that $R \mapsto RG$. For any $\alpha\colon R \to R'$, we define

$$\begin{array}{rccc} \hat{\alpha}\colon & RG & \to & R'G \\ & \sum_{i=1}^{n} r_i g_i & \mapsto & \sum_{i=1}^{n} \alpha(r_i) g_i. \end{array}$$

Remark 10.14. All the functors defined in this section are extension invariant. We will omit the proof since it is analogous to previous ones.

Also, as a consequence of Corollary 10.8 and due to the fact that the Leavitt path algebra functor is extension invariant, we conclude our first isomorphism result in the following context:

Corollary 10.15. *If E_1 and E_2 are graphs and H any K-algebra endowed with an augmentation (in particular any group K-algebra), then $L_H(E_1) \cong L_H(E_2)$ if and only if $L_K(E_1) \cong L_K(E_2)$.*

10.2 ON THE HILBERT'S NULLSTELLENSATZ THEOREM

Definition 10.16. Let $\mathcal{A}$ be an indexing set. Consider $K[x_\alpha]$ the polynomial ring in the indeterminates $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ with coefficients in the field K and an ideal $\mathfrak{i} \lhd K[x_\alpha]$. Following [77], a *zero* of $\mathfrak{i}$ is an element $(\xi_\alpha)_{\alpha\in\mathcal{A}}$ where each ξ_α is in some extension field of K such that $p(\xi_\alpha) = 0$ for all $p \in \mathfrak{i}$. A zero of $\mathfrak{i}$ will be called *algebraic* if all ξ_α lie in K. The set of all algebraic zeros of an ideal $\mathfrak{i}$ will be called the *variety defined by* $\mathfrak{i}$ denoted by

$$V(\mathfrak{i}) = \{a \in K^{\mathcal{A}}\colon p(a) = 0 \text{ for any } p \in \mathfrak{i}\}.$$

The Hilbert's Nullstellensatz Theorem is in general not valid if the number of indeterminates is infinite. However, the main result of [77], that we recall in this section for self-containedness, holds.

Theorem 10.17. *[77] Let K be an algebraically closed field and $\{x_\alpha\}$ a set of indeterminates indexed by $\mathcal{A}$. The following three statements are equivalent:*

S1. If $\mathfrak{i}$ is an ideal of $K[x_\alpha]$ and $p \in K[x_\alpha]$ vanishes on the variety defined by $\mathfrak{i}$, then $p^m \in \mathfrak{i}$ for some integer m.

S2. If $\mathfrak{i}$ is an ideal of $K[x_\alpha]$ and $\mathfrak{i} \neq K[x_\alpha]$, then $\mathfrak{i}$ has an algebraic zero.

S3. A ring extension $K[\xi_\alpha]$ by elements ξ_α in some extension field of K is a field if and only if all ξ_α lie in K.

Furthermore, the three statements hold if one of the following two conditions is satisfied:

(i) $\mathcal{A}$ is a finite set.

(ii) Let $\mathcal{A}$ have cardinality a. Let the transcendence degree of K over the prime field Q have cardinality b. Then $a < b$.

As pointed out by S. Lang in his paper [77], Professor I. Kaplansky informed him that he had obtained the above theorem independently. It is also mentioned in [77] that the two conditions *(i)* and *(ii)* can be replaced by the single condition that the cardinality of the field K itself should be greater than the cardinality of $\mathcal{A}$. We include a proof for self-containment.

Lemma 10.18. *If Q is a finite field or the field of rationals and $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ an infinite set of variables, then the extension field $Q(x_\alpha)$ of Q containing all the indeterminates x_α has cardinal $|\mathcal{A}|$. As a consequence, if $\{b_i\}_{i\in I}$ is an infinite transcendence basis of an extension field $K \supset Q$, the cardinals of the fields $Q(b_i)$ and K are also $|I|$. In particular, if a field K has transcendence degree over its prime field greater than a given infinite cardinal b, then $|K| > b$ too.*

Proof. Since $\{x_\alpha\}_{\alpha\in\mathcal{A}} \subseteq Q(x_\alpha)$ we have $|\mathcal{A}| \leqslant |Q(x_\alpha)|$. The polynomial ring in all the indeterminates $Q[x_\alpha]$ has cardinal $|\mathcal{A}|$. Furthermore, the field of rational functions $Q(x_\alpha)$ has the same cardinal $|\mathcal{A}|$. Note that if we have an extension of fields such as the smallest field is infinite and the largest one is algebraic over the first, then it is well known that both fields have the same cardinal. Consequently, since K is algebraic over $Q(b_i)$ (and this is infinite) we have $|K| = |Q(b_i)| = |I|$. □

Remark 10.19. Consider a field Q with $|Q| = \aleph_0$. If $\{b_1, \dots, b_n\}$ is a finite transcendence basis of the extension $K \supset Q$, then the cardinal of $Q(b_1, \dots, b_n)$ is $\aleph_0$.

Next, we have a converse for the Lemma 10.18:

Lemma 10.20. *Let K be a field and $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ an infinite set of indeterminates. If $|K| > |\mathcal{A}|$, then the transcendence degree of K over its prime field is greater than $|\mathcal{A}|$.*

Proof. Assume $|K| > |\mathcal{A}|$ and $\mathcal{A}$ is not finite. Let $\{b_i\}_{i\in I}$ be a transcendence basis of K over Q and b the transcendence degree of K over Q, that is $b = |\{b_i\}_{i\in I}|$. Every element in K is a root of a unique minimal monic irreducible polynomial with coefficients in $Q(b_i)$ because K is algebraic over $Q(b_i)$. We say that two elements $k, k' \in K$ are related if and only if they have the same minimal polynomial. Consider the quotient set $K/\mathcal{R}$ with $\mathcal{R}$ the previous equivalence relation. After this, we can take the map that for each equivalence class corresponds its minimal polynomial. This map, from $K/\mathcal{R}$ to the set of irreducible polynomials with coefficients in $Q(b_i)$ is injective, which means that

$$|K/\mathcal{R}| \leqslant |Q(b_i)[x]| = \aleph_0 |Q(b_i)|.$$

In addition, since in every equivalence class there is a finite number of elements of K, we have that $|K| \leqslant \aleph_0 |K/\mathcal{R}| \leqslant \aleph_0^2 |Q(b_i)|$ which means that

$$|K| \leqslant \aleph_0 |Q(b_i)|. \tag{32}$$

If b is infinite we apply Lemma 10.18. From (32) we obtain $|K| \leqslant b$. Since $|\mathcal{A}| < |K|$ we get the result.

Next, we prove that b is necessarily infinite. If b is finite, $|Q(b_i)| = |Q| = \aleph_0$ (Remark 10.19) but $|K| > \aleph_0$ by hypothesis. Then equation (32) gives

$$\aleph_0 < |K| \leqslant \aleph_0^2 = \aleph_0$$

which is a contradiction. □

In view of the results obtained in the previous Lemmas 10.18 and 10.20 we can replace item *(ii)* of Theorem 10.17 for the next one:

(ii)′ Let $\mathcal{A}$ have cardinal a. Let b be the cardinal of the field K. Then $a < b$. (33)

10.3 THE FINITE-DIMENSIONAL CASE

In this section A will denote an algebra (associative or not) over a field K. The prime field of K will be denoted by Q.

Definition 10.21. For an K-algebra A and a basis $\mathcal{B} = \{e_i\}_{i \in \Lambda}$ of it as a K-vector space, the *structure constants* $\{\gamma_{ij}^k\}_{i,j,k \in \Lambda}$ relative to the basis $\mathcal{B}$ are those elements $\gamma_{ij}^k \in K$ such that

$$e_i e_j = \sum_{k \in \Lambda} \gamma_{ij}^k e_k$$

for every $i, j \in \Lambda$. For every pair $(i, j) \in \Lambda \times \Lambda$, denote the set

$$\Sigma_{i,j} := \{k \in \Lambda \mid \gamma_{i,j}^k \neq 0\}.$$

Observe that each $\Lambda_{i,j}$ is finite. In particular, $e_i e_j = \sum_{k \in \Sigma_{i,j}} \gamma_{i,j}^k e_k$.

Remark 10.22. Take a K-algebra A with basis $\mathcal{B}_K = \{e_i\}_{i\in\Lambda}$. If $K \subset F$ is a field extension, then A_F is a F-algebra with basis $\mathcal{B}_F = \{e_i \otimes 1\}_{i\in\Lambda}$ as a F-vector space. In particular, if $\{\gamma_{i,j}^k\}_{i,j,k\in\Lambda}$ are the structure constants of A for the base $\mathcal{B}_K$, it is also the set of structure constants of A_F for the base $\mathcal{B}_F$. That is,

$$(e_i \otimes 1) \otimes (e_j \otimes 1) = e_i e_j \otimes 1 = \sum_{k\in\Lambda} \gamma_{i,j}^k e_k \otimes 1.$$

Note that the isomorphism question for finite-dimensional algebras can be translated into a problem of existence of zeros for suitable systems of algebraic equations. This is formulated in the next proposition.

Proposition 10.23. *Let A and B be finite-dimensional K-algebras with bases $\mathcal{B}_A = \{e_i\}_{i\in\Lambda}$ and $\mathcal{B}_B = \{u_i\}_{i\in\Lambda}$, respectively. Let $\{\gamma_{i,j}^k\}_{i,j,k,\in\Lambda}$ and $\{\tau_{l,m}^n\}_{l,m,n\in\Lambda}$ be the structure constants of A and B relative to the bases $\mathcal{B}_A$ and $\mathcal{B}_B$, respectively. Then A and B are isomorphic K-algebras if and only if there exist $\{t_i^j\}_{i,j\in\Lambda} \cup \{s_i^j\}_{i,j\in\Lambda} \subseteq K$, such that the following equations are satisfied:*

$$\begin{aligned} \sum_{k\in\Lambda} \gamma_{i,j}^k t_k^n &= \sum_{l,m\in\Lambda} \tau_{l,m}^n t_i^l t_j^m \quad \text{for every } i, j, n \in \Lambda, \\ \sum_{j\in\Lambda} t_i^j s_j^k &= \delta_{i,k} \qquad\qquad\quad \text{for every } i, k \in \Lambda. \end{aligned} \tag{34}$$

Proof. Assume first that $f \colon A \to B$ is an isomorphism of K-algebras. Let $t_i^j \in K$ be such that $f(e_i) = \sum_{j\in\Lambda} t_i^j u_j$. The hypothesis of being f a homomorphism of algebras means $f(e_i e_j) = f(e_i) f(e_j)$ for every $i, j \in \Lambda$. Then

$$\begin{aligned} f\left(\sum_{k\in\Lambda} \gamma_{ij}^k e_k\right) &= \sum_{l\in\Lambda} t_i^l u_l \sum_{m\in\Lambda} t_j^m u_m, \\ \sum_{k\in\Lambda} \gamma_{ij}^k f(e_k) &= \sum_{l,m\in\Lambda} t_i^l t_j^m u_l u_m, \\ \sum_{k\in\Lambda} \gamma_{ij}^k \sum_{n\in\Lambda} t_k^n u_n &= \sum_{l,m\in\Lambda,} t_i^l t_j^m \sum_{n\in\Lambda} \tau_{lm}^n u_n. \end{aligned}$$

Then, taking into account the linear independence of the u_n's we have, for any $i, j, n, \in \Lambda$

$$\sum_{k\in\Lambda} \gamma_{i,j}^k t_k^n = \sum_{l,m\in\Lambda} t_i^l t_j^m \tau_{l,m}^n.$$

The following step is to impose that f is bijective. If this is the case, f^{-1} denotes the inverse of f and $f^{-1}(u_i) = \sum_{j\in\Lambda} s_i^j e_j$, where $s_i^j \in K$. Then

$$e_i = f^{-1}f(e_i) = f^{-1}\left(\sum_{j\in\Lambda} t_i^j u_j\right) = \sum_{j,k\in\Lambda} t_i^j s_j^k e_k,$$

hence

$$\sum_{j\in\Lambda} t_i^j s_j^k = \delta_{ik}.$$

The converse follows immediately. □

Let A and B be K-algebras having the same finite dimension as K-vector spaces, and let $\{\gamma_{ij}^k\}_{i,j,k,\in\Lambda}$ and $\{\tau_{ij}^k\}_{i,j,k\in\Lambda}$ be the structure constants of A and B, respectively, relative to certain bases indexed in a set Λ. Consider

$$X = \{T_i^j\}_{i,j\in\Lambda} \cup \{S_i^j\}_{i,j\in\Lambda}$$

a set of indeterminates and let $K[X]$ be the K-algebra of commuting polynomials in the indeterminates of X. For every $i, j, n \in \Lambda$, define

$$p_{i,j,n} := \sum_{k\in\Lambda} \gamma_{ij}^k T_k^n - \sum_{l,m\in\Lambda} \tau_{lm}^n T_i^l T_j^m \tag{35}$$

and for every $i, k \in \Lambda$,

$$q_{i,k} := \sum_{j\in\Lambda} T_i^j S_j^k - \delta_{i,k}. \tag{36}$$

Since A and B are finite-dimensional, then the sets of polynomials $\{p_{i,j,n}\}_{i,j,n\in\Lambda}$ and $\{q_{i,k}\}_{i,k\in\Lambda}$ are finite.

Corollary 10.24. *In the previous conditions, let $\mathfrak{i}_K$ be the ideal of $K[X]$ generated by the polynomials of the set $P_K := \{p_{ijn}\}_{i,j,n\in\Lambda} \cup \{q_{ik}\}_{i,k\in\Lambda}$. Then the set of isomorphisms $f : A \to B$ is in one-to-one correspondence with the points of the algebraic variety $V(\mathfrak{i}_K)$, that is, the common zeros of the polynomials in P_K.*

Now that we have translated the isomorphism question of finite-dimensional algebras to a problem of varieties, we can exploit the Hilbert's Nullstellensatz Theorem in the following result.

Theorem 10.25. *Let $K \subseteq F$ be a field extension, with K algebraically closed. Consider A and B two K-algebras having finite dimension as K-vector spaces. Then A_F and B_F are isomorphic as F-algebras if and only if A and B are isomorphic as K-algebras.*

Proof. Let $X = \{T_i^j\}_{i,j\in\Lambda} \cup \{S_i^j\}_{i,j\in\Lambda}$ be a set of commuting indeterminates, and consider the polynomial algebras $K[X]$ and $F[X]$. Take, as in Corollary 10.24, the ideals $\mathfrak{i}_K$ and $\mathfrak{i}_F$ of $K[X]$ and $F[X]$ generated by the polynomials in P_K and P_F respectively. Observe that since the structure constant remains the same after the field extension, then $P_K = P_F$. Consequently, the K-algebra monomorphism $i\colon K[X] \to F[X]$ maps $\mathfrak{i}_K$ into $\mathfrak{i}_F$ so that if $1 \in \mathfrak{i}_K$, then $1 \in \mathfrak{i}_F$. If A_F and B_F are isomorphic as F-algebras, then $V(\mathfrak{i}_F) \neq \emptyset$ by Corollary 10.24, which implies $1 \notin \mathfrak{i}_K$. Therefore $\mathfrak{i}_K \neq K[X]$ and by the Hilbert's Nullstellensatz Theorem we get $V(\mathfrak{i}_K) \neq \emptyset$. Again, by Corollary 10.24, this implies that A and B are isomorphic as K-algebras. □

Remark 10.26. The hypothesis of being algebraically closed field cannot be eliminated. For example, consider the quaternion division algebra $\mathbb{H}$ and $M_2(\mathbb{R})$, which are non isomorphic $\mathbb{R}$-algebras. However, their complexifications are isomorphic as $\mathbb{C}$-algebras.

Firstly, as an immediate consequence of the previous theorem, we contemplate the case when we have two fields containing another algebraically closed.

Corollary 10.27. *Let F_1 and F_2 be two fields containing an algebraically closed field K. Consider A and B be two K-algebras with finite dimension. Then A_{F_1} and B_{F_1} are isomorphic as F_1-algebras if and only if A_{F_2} and B_{F_2} are isomorphic as F_2-algebras.*

Proof. Let us consider A_{F_1} and B_{F_1} as F_1-algebras. By Theorem 10.25 these two algebras are isomorphic if and only if A and B are isomorphic as K-algebras. Again, by Theorem 10.25 A and B are isomorphic if and only if A_{F_2} and B_{F_2} are isomorphic as F_2-algebras. By transitivity, we have our claim. □

Secondly, we may have the situation when two algebraically closed fields contains another one which is not necessarily algebraically closed.

Corollary 10.28. *Let F_1 and F_2 be two algebraically closed fields containing a field K. Consider A and B two K-algebras with finite dimension. Then A_{F_1} and B_{F_1} are isomorphic as F_1-algebras if and only if A_{F_2} and B_{F_2} are isomorphic as F_2-algebras.*

Proof. First, we know that if A is a K-algebra and K_1, K_2 are two fields verifying $K \subseteq K_1 \subseteq K_2$, then $(A_{K_1})_{K_2} \cong A_{K_2}$. Due to the fact that F_1 and F_2 are algebraically closed, it is clear that $\bar{K} \subseteq F_1, F_2$, with $\bar{K}$ denoting the algebraic closure of K. Therefore, we can consider $A_{\bar{K}}$ and $B_{\bar{K}}$ two $\bar{K}$-algebras. We are now in the conditions of Corollary 10.27 implying $(A_{\bar{K}})_{F_1} \cong (B_{\bar{K}})_{F_1}$ if and only if $(A_{\bar{K}})_{F_2} \cong (B_{\bar{K}})_{F_2}$. □

On the other hand, we can establish a more general case but taking into account that the algebraic closure of the prime field of the smallest one is contained in one of the largest fields.

Corollary 10.29. *Let $K \subseteq F_i$ for $i = 1, 2$ field extensions. Consider Q the prime field of K and A and B two K-algebras of finite dimension which structure constants (relative to suitable bases) are in $\bar{Q}$. Suppose $\bar{Q} \subseteq F_2$, if $A_{F_1} \cong B_{F_1}$, then $A_{F_2} \cong B_{F_2}$.*

Proof. Since $A_{F_1} \cong B_{F_1}$, then $A_{\bar{F}_1} \cong B_{\bar{F}_1}$ because $K \subseteq \bar{F}_1, \bar{F}_2$. Applying Corollary 10.28 we have $A_{\bar{F}_2} \cong B_{\bar{F}_2}$. Suppose $\{a_i\}_{i\in\Lambda}$ and $\{b_i\}_{i\in\Lambda}$ are respectively the bases of A and B as K-algebras with structure constants in $\bar{Q}$. Write $X = \bigoplus_{i\in\Lambda} \bar{Q}a_i$ and $Y = \bigoplus_{i\in\Lambda} \bar{Q}b_i$ as $\bar{Q}$-algebras with the product given by the structure constants in A and B respectively. Therefore, we have that $X_{\bar{F}_2} \cong Y_{\bar{F}_2}$ because $A_{\bar{F}_2} \cong B_{\bar{F}_2}$. Since $\bar{Q} \subseteq \bar{F}_2$, by Theorem 10.25, $X \cong Y$. Finally, $A_{F_2} \cong B_{F_2}$ considering the fact that $\bar{Q} \subseteq F_2$. □

To end this section, we can restate all the results in terms of algebra functors which will give a more global vision. To begin with, we show that Theorem 10.25 could be rewritten in a functorial way giving Theorem 10.30. In other words, both statements are equivalent.

Theorem 10.30. *Let $K \subseteq F$ be a field extension with K algebraically closed. Let $\mathcal{F}, \mathcal{G} \colon \mathbf{alg}_K \to \mathbf{Alg}_K$ be two extension invariant algebra functors and assume that $\mathcal{F}(K)$ and $\mathcal{G}(K)$ are finite-dimensional as K-algebras. Then $\mathcal{F}(K) \cong \mathcal{G}(K)$ if and only if $\mathcal{F}(F) \cong \mathcal{G}(F)$.*

Proof. Note that it is straightforward considering, in Theorem 10.25, $A := \mathcal{F}(K)$ and $B := \mathcal{G}(K)$. □

Also, as a consequence of Theorem 10.25, we get the following results in terms of extension invariant functors.

Theorem 10.31. *Let $\mathcal{F}, \mathcal{G} \colon \mathbf{rng} \to \mathbf{Alg}_{\mathbb{Z}}$ be two extension invariant algebra functors and assume that for some field K we have $\mathcal{F}(K) \cong \mathcal{G}(K)$ isomorphic as finite-dimensional*

K-algebras. Let Q be the prime field of K and R any ring containing $\bar{Q}$ as a subring. Then $\mathcal{F}(R) \cong \mathcal{G}(R)$.

Proof. First we take $A := \mathcal{F}(\bar{Q})$ and $B := \mathcal{G}(\bar{Q})$. Since $\mathcal{F}$ is extension invariant, $\mathcal{F}(K) \cong \mathcal{G}(K)$ implies $\mathcal{F}(\bar{K}) \cong \mathcal{G}(\bar{K})$ as $\bar{K}$-algebras. Then, taking into account that A and B are finite-dimensional $\bar{Q}$-algebras, by Theorem 10.25, we know that $\mathcal{F}(\bar{K}) \cong \mathcal{G}(\bar{K})$ implies $\mathcal{F}(\bar{Q}) \cong \mathcal{G}(\bar{Q})$ as $\bar{Q}$-algebras. Finally $\mathcal{F}(R) \cong A \otimes_{\bar{Q}} R \cong B \otimes_{\bar{Q}} R \cong \mathcal{G}(R)$ as desired. □

Remark 10.32. Analogously to the equivalence between Theorem 10.25 and Theorem 10.30, we can rewrite Corollaries 10.27 and 10.28 in terms of functors. That is, under the corresponding hypotheses (K algebraically closed field and F_1, F_2 containing K) in each result, we have that for $\mathcal{F}, \mathcal{G} \colon \mathbf{alg}_K \to \mathbf{Alg}_K$ extension invariant functors it is straightforward that $\mathcal{F}(F_1) \cong \mathcal{G}(F_1)$ if and only if $\mathcal{F}(F_2) \cong \mathcal{G}(F_2)$. So we have freedom to replace a field with another one and still to have an isomorphism, as long as both contain an algebraically closed one. Respectively for Corollary 10.29 we could say $\mathcal{F}(F_1) \cong \mathcal{G}(F_1)$ implies $\mathcal{F}(F_2) \cong \mathcal{G}(F_2)$.

We can specialize Theorem 10.31 to any extension invariant functor. Since our main motivation comes from Leavitt path algebras and related structures (path, Cohn and Steinberg algebras), we claim:

Corollary 10.33. *Let $C_K^{X_{E_1}}(E_1)$ and $C_K^{X_{E_2}}(E_2)$ be isomorphic finite-dimensional relative Cohn path algebras. Let Q be the prime field of K, then for any field K' of the same characteristic that K such that $\bar{Q} \subseteq K'$, we have $C_{K'}^{X_{E_1}}(E_1) \cong C_{K'}^{X_{E_2}}(E_2)$ as K'-algebras.*

Also an immediate consequence of Corollary 10.27 for the relative Cohn path algebras is the following:

Corollary 10.34. *Let K and K' be algebraically closed fields of the same characteristic and assume that $C_K^{X_{E_1}}(E_1)$ is finite-dimensional. Then $C_K^{X_{E_1}}(E_1) \cong C_K^{X_{E_2}}(E_2)$ if and only if $C_{K'}^{X_{E_1}}(E_1) \cong C_{K'}^{X_{E_2}}(E_2)$.*

As a consequence of these results, take for instance an algebraically closed field K and extensions $K \subset F_i$ for $i = 1, 2$. Then, for finite-dimensional Steinberg algebras we have $A_{F_1}(G_1) \cong A_{F_1}(G_2)$ if and only if $A_{F_2}(\mathcal{G}_1) \cong A_{F_2}(\mathcal{G}_2)$ for groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$. In the same vein, being $\mathcal{G}_1$ and G_2 two finite groups and using the group ring functors suitably, we have $F_1G_1 \cong F_1G_2$ if and only if $F_2G_1 \cong F_2G_2$.

Being aware that, in the setting of relative Cohn path algebras, the finite-dimensionality imposes a strong restriction to the algebras under study, we will try to relax the hypothesis on finite-dimensionality of the algebras in the next section.

10.4 THE INFINITE-DIMENSIONAL CASE

Now we will extend some of the results of the previous section to the case in which the algebras are of arbitrary dimension.

Definition 10.35. Consider A and B two K-algebras with bases $\mathcal{B}_A = \{e_i\}_{i\in\Lambda}$ and $\mathcal{B}_B = \{u_i\}_{i\in\Lambda}$ respectively. Let $f : A \to B$ be an isomorphism of K-algebras. Assume $f(e_i) = \sum_{j\in\Lambda} t_i^j u_j$ and $f^{-1}(u_i) = \sum_{j\in\Lambda} s_i^j e_j$ for certain $\{t_i^j\}_{i,j\in\Lambda}, \{s_i^j\}_{i,j\in\Lambda} \subseteq K$. By $\mathrm{Isom}_f(A,B)$ we will understand the set of all isomorphisms g from A into B such that if $g(e_i) = \sum_{j\in\Lambda} \bar{t}_i^j u_j$ and $g^{-1}(u_i) = \sum_{j\in\Lambda} \bar{s}_i^j e_j$, then $t_i^j = 0$ implies $\bar{t}_i^j = 0$ and $s_i^j = 0$ implies $\bar{s}_i^j = 0$. Note that, in particular, $f \in \mathrm{Isom}_f(A,B)$.

As we did in Section 10.3, we can translate the isomorphism question for infinite-dimensional algebras in terms of the existence of zeros of a certain ideal of polynomials.

Proposition 10.36. *Consider A and B two K-algebras with bases $\mathcal{B}_A = \{e_i\}_{i\in\Lambda}$ and $\mathcal{B}_B = \{u_i\}_{i\in\Lambda}$. Let $\{\gamma_{i,j}^k\}_{i,j,k\in\Lambda}, \{\tau_{l,m}^n\}_{l,m,n\in\Lambda}$ be the structure constants of A and B relative to the bases $\mathcal{B}_A$ and $\mathcal{B}_B$ respectively. Assume that $f\colon A \to B$ is an isomorphism of K-algebras. Let $\{t_i^j\}_{i,j\in\Lambda}$ and $\{s_i^j\}_{i,j\in\Lambda}$ be subsets of K such that $f(e_i) = \sum_{j\in\Lambda} t_i^j u_j$ and $f^{-1}(u_i) = \sum_{j\in\Lambda} s_i^j e_j$. Define*

$$\Lambda_i^f := \{j \in \Lambda \mid t_i^j \neq 0\}, \quad \Lambda_i^{f^{-1}} := \{j \in \Lambda \mid s_i^j \neq 0\}, \quad \Sigma_{i,j} := \{k \in \Lambda \mid \gamma_{ij}^k \neq 0\}.$$

Now, consider a set of indeterminates $X = \{T_i^j\}_{i,j\in\Lambda} \cup \{S_i^j\}_{i,j\in\Lambda}$ and the polynomials

$$\begin{aligned} p_{i,j,n} &:= \sum_{k\in\Sigma_{i,j}} \gamma_{i,j}^k T_k^n - \sum_{l\in\Lambda_i^f, m\in\Lambda_j^f} \tau_{l,m}^n T_i^l T_j^m, \quad q_{i,k} := \sum_{j\in\Lambda_i^f} T_i^j S_j^k - \delta_{i,k}, \\ r_{i,k} &:= \sum_{j\in\Lambda_i^{f^{-1}}} S_i^j T_j^k - \delta_{i,k}, \quad T_i^j\ (j \notin \Lambda_i^f), \quad S_i^j\ (j \notin \Lambda_i^{f^{-1}}), \end{aligned} \tag{37}$$

for every $i, j, k, n \in \Lambda$. Denote by I_K^f the ideal of $K[X]$ generated by all the polynomials defined above. Then there is a bijective map from $V(I_K^f)$ into $\mathrm{Isom}_f(A,B)$.

Proof. Note that, by construction, Λ_i^f, $\Lambda_i^{f^{-1}}$ and $\Sigma_{i,j}$ are finite sets. In particular, this implies that the polynomials p_{ijn}, q_{ik} and r_{ik} are well-defined.

Let $z = \left((t_i'^j), (s_i'^j)\right)$ be a zero in $V(I_K^f)$; given any $i \in \Lambda$ we have that the number of nonzero elements in $\{t_i'^j\}_{j\in\Lambda}$ and $\{s_i'^j\}_{j\in\Lambda}$ is finite. This allows us to define

$g : A \to B$ as the linear map having (t'^j_i) as the associated matrix relative to the bases $\mathcal{B}_A$ and $\mathcal{B}_B$. Let $h : B \to A$ be the linear map whose associated matrix relative to the bases $\mathcal{B}_B$ and $\mathcal{B}_A$ is (s'^j_i). Since z is a zero of the polynomials in (37), it is clear that g is a homomorphism of K-algebras and that $hg = 1_A$ and $gh = 1_B$ (similarly to Proposition 10.23). Moreover, taking into account the construction of g we deduce that $g \in \mathrm{Isom}_f(A, B)$. This way, we can define a map $\varphi : V(I^f_K) \to \mathrm{Isom}_f(A, B)$ such that $z \mapsto g$.

Conversely, take $g \in \mathrm{Isom}_f(A, B)$. Denote by (a^j_i) and by (b^j_i) the matrices of g and g^{-1} relative to the bases $\mathcal{B}_A$ and $\mathcal{B}_B$. Notice that

$$\begin{cases} a^j_i = 0 & \text{if } j \notin \Lambda^f_i \\ b^j_i = 0 & \text{if } j \notin \Lambda^{f^{-1}}_i . \end{cases} \tag{38}$$

Since g is the inverse of g^{-1} (and conversely), we have that $z = ((a^j_i), (b^j_i))$ is a zero of $\{q_{i,k}\}_{i,k\in\Lambda} \cup \{r_{i,k}\}_{i,k\in\Lambda}$. Apply that g is a homomorphism of K-algebras to get that z is a zero of $\{p_{i,j,n}\}_{i,j,n\in\Lambda}$ and also of T^j_i and S^j_i because of (38). Thus we can define a map ψ: $\mathrm{Isom}_f(A, B) \to V(I^f_K)$ and it is not difficult to prove that ψ is the inverse of φ, again similarly to Proposition 10.23. □

Since the algebras we are dealing with are possibly infinite-dimensional, in order to apply Theorem 10.17, we need to consider additional hypothesis about the cardinal of the ground field to proceed similarly as in Section 10.3.

Theorem 10.37. *Let $K \subseteq F$ be a field extension with K algebraically closed. Let A and B be K-algebras of the same dimension ω. Assume that the cardinal of K is strictly higher than ω. Then A_F and B_F are isomorphic as F-algebras if and only if A and B are isomorphic as K-algebras.*

Proof. For the nontrivial part assume that $A_F \cong B_F$. Let $f : A_F \to B_F$ be an isomorphism. Let Λ be a set of cardinal ω. Fix bases $\mathcal{B}_A = \{e_i\}_{i\in\Lambda}$ of A and $\mathcal{B}_B = \{u_i\}_{i\in\Lambda}$ of B. Assume that the structure constants of A and B relative to the bases $\mathcal{B}_A$ and $\mathcal{B}_B$ are respectively given by:

$$e_i e_j = \sum_{k\in\Lambda} \gamma^k_{i,j} e_k, \qquad u_i u_j = \sum_{k\in\Lambda} \tau^k_{i,j} u_k. \tag{39}$$

Note that the structure constants of A_F and B_F relative to the bases $\{u_i \otimes 1\}_{i\in\Lambda}$ and $\{e_i \otimes 1\}_{i\in\Lambda}$ are in K. Assume that

$$f(e_i \otimes 1) = \sum_{j\in\Lambda} t_i^j (u_j \otimes 1) \text{ and } f^{-1}(u_i \otimes 1) = \sum_{j\in\Lambda} s_i^j (e_j \otimes 1)$$

for any $i \in \Lambda$. Define now the sets Λ_i^f, $\Lambda_i^{f^{-1}}$ and $\Sigma_{i,j}$ as in Proposition 10.36. Consider the set of indeterminates $X = \{T_i^j\}_{i,j\in\Lambda} \cup \{S_i^j\}_{i,j\in\Lambda}$. In the polynomial algebras $K[X]$ and $F[X]$, define respectively the ideals I_K^f and I_F^f generated by the polynomials $p_{i,j,n}$, $q_{i,k}$, $r_{i,k}$, T_i^j (with $j \notin \Lambda_i^f$) and S_i^j (with $j \notin \Lambda_i^{f^{-1}}$). Then $V(I_F^f) \neq \varnothing$ whence $1 \notin I_F^f$. Observe that $I_K^f \subseteq I_F^f$ because the polynomials defined as in (37) have their coefficients in K. Thus $1 \notin I_K^f$ implying $V(I_K^f) \neq \varnothing$ by Hilbert's Nullstellensatz Theorem 10.17 taking into account condition (33). So there is an isomorphism $A \to B$. □

Corollary 10.38. *Let F_1 and F_2 be two fields containing an algebraically closed field K. Consider A and B two K-algebras with the same dimension ω. Suppose that the cardinal of K is strictly higher than ω. Then A_{F_1} and B_{F_1} are isomorphic as F_1-algebras if and only if A_{F_2} and B_{F_2} are isomorphic as F_2-algebras.*

Proof. Let us consider A_{F_1} and B_{F_1} as F_1-algebras. By Theorem 10.37 these two algebras are isomorphic if and only if A and B are isomorphic as K-algebras. Again, by Theorem 10.37, we have that A and B are isomorphic if and only if A_{F_2} and B_{F_2} are isomorphic as F_2-algebras. By transitivity, we have our claim. □

Corollary 10.39. *Let F_1 and F_2 be two algebraically closed fields containing a field K. Suppose that A and B are two K-algebras with the same dimension ω and the cardinal of K is strictly higher than ω. Then A_{F_1} and B_{F_1} are isomorphic as F_1-algebras if and only if A_{F_2} and B_{F_2} are isomorphic as F_2-algebras.*

Proof. Due to the fact that F_1 and F_2 are algebraically closed, it is clear that $\bar{K} \subseteq F_1, F_2$. Therefore, we can consider $A_{\bar{K}}$ and $B_{\bar{K}}$ as two $\bar{K}$-algebras. Observe that the cardinal of $\bar{K}$ is bigger than ω. We are now in the conditions of Corollary 10.38 implying $(A_{\bar{K}})_{F_1} \cong (B_{\bar{K}})_{F_1}$ if and only if $(A_{\bar{K}})_{F_2} \cong (B_{\bar{K}})_{F_2}$. □

Corollary 10.40. *Let $K \subseteq F_i$ for $i = 1, 2$ be extension of fields. Consider A and B two K-algebras with the same dimension ω. Suppose that the cardinal of K is strictly higher than ω and $\bar{K} \subseteq F_2$. If $A_{F_1} \cong B_{F_1}$ then $A_{F_2} \cong B_{F_2}$.*

Proof. Since $A_{F_1} \cong B_{F_1}$, then $A_{\bar{F}_1} \cong B_{\bar{F}_1}$ taking into account $K \subseteq \bar{F}_1$. By Corollary 10.38 we have $A_{\bar{K}} \cong B_{\bar{K}}$. Because of $\bar{K} \subseteq F_2$, we get $A_{F_2} \cong B_{F_2}$. □

Finally, we can express the previous results in terms of extension invariant algebra functors. Concretely, Theorem 10.37 gives Theorem 10.41 and from Corollary 10.38 we obtain Theorem 10.42.

Theorem 10.41. *Let $K \subseteq F$ be a field extension with K algebraically closed. Let $\mathcal{F}, \mathcal{G}\colon \mathbf{alg}_K \to \mathbf{Alg}_K$ be two extension invariant algebra functors and assume that $|K| > \dim(\mathcal{F}(K)), \dim(\mathcal{G}(K))$. Then $\mathcal{F}(K) \cong \mathcal{G}(K)$ if and only if $\mathcal{F}(F) \cong \mathcal{G}(F)$.*

Theorem 10.42. *Let F_1 and F_2 be two fields containing an algebraically closed field K. Consider $\mathcal{F}$ and $\mathcal{G}$ two extension invariant algebra functors with $|K| > \dim(\mathcal{F}(K)), \dim(\mathcal{G}(K))$. Then $\mathcal{F}(F_1)$ and $\mathcal{G}(F_1)$ are isomorphic as F_1-algebras if and only if $\mathcal{F}(F_2)$ and $\mathcal{G}(F_2)$ are isomorphic as F_2-algebras.*

In fact, if A and B are two K-algebras such that $\dim(A) = \dim(B) = \aleph_0$, then for any uncountable algebraically closed field K we meet the hypothesis of Corollary 10.38. For the path algebras, if E_1 and E_2 are countable graphs and $K \subset F_i$ $i = 1, 2$, we have that $F_1E_1 \cong F_1E_2$ if and only if $F_2E_1 \cong F_2E_2$. Moreover, for Steinberg algebras of countable dimension one has $A_{F_1}(G_1) \cong A_{F_1}(G_2)$ if and only if $A_{F_2}(G_1) \cong A_{F_2}(G_2)$ for groupoids G_1 and G_2. In order to highlight the corresponding result, in the case of Leavitt path algebras, we establish the following corollary:

Corollary 10.43. *Suppose that E_1 and E_2 are countable graphs, K an uncountable algebraically closed field and $K \subseteq F_1, F_2$ two field extensions. Then $L_{F_1}(E_1) \cong L_{F_1}(E_2)$ if and only if $L_{F_2}(E_1) \cong L_{F_2}(E_2)$.*

Note that if K is an uncountable algebraically closed field, E_1 and E_2 countable graphs and R, S commutative and unital K-algebras, then $L_R(E_1) \cong L_R(E_2)$ as R-algebras if and only if $L_S(E_1) \cong L_S(E_2)$ as S-algebras. Indeed, assuming $L_R(E_1) \cong L_R(E_2)$ we consider the field $K' = R/\mathfrak{m}$ where $\mathfrak{m}$ is any maximal ideal of R. We have a field extension $K \subset K'$ and from the isomorphism $L_R(E_1) \cong L_R(E_2)$ we deduce an isomorphism $L_{K'}(E_1) \cong L_{K'}(E_2)$ so that Corollary 10.43 implies the existence of an isomorphism $L_K(E_1) \cong L_K(E_2)$ whence $L_S(E_1) \cong L_S(E_2)$.

Chapter A

FURTHER QUESTIONS

A.1 ORLOFF ALGEBRAS

The Steinberg algebra works as a generalisation of the Leavitt path algebras and the group algebras. However, that is not the case for the path algebras. The existence of an inverse in the groupoid is not compatible with the path category of a directed graph. That is why the next question is: Is there a generalisation of the Steinberg algebra without the use of inverses? In other words, is there a similar generalization of the path algebra?

A.1.1 Monoidoids

During the definition of a groupoid and the construction of a path algebra, we can omit the existence of an inverse and just consider that we are facing a small category or as we will call it: a monoidoid. Following the description of Definition 2.1 we have:

Definition A.1. A *monoidoid* is a system $(\mathcal{C}^{(0)}, \mathcal{C}, s, r, m)$ such that:

1. The nonempty sets $\mathcal{C}$ and $\mathcal{C}^{(0)}$ are called the *underlying set* and *unit space* respectively;
2. the maps $r, s \colon \mathcal{C} \to \mathcal{C}^{(0)}$ are called the *range* and the *source* map;
3. m is a partially defined operation on $\mathcal{C}$ called *composition*: in particular, it is a map from the set of *composable pairs*

$$\mathcal{C}^{(2)} := \{(g,h) \in \mathcal{C} \times \mathcal{C} \colon r(h) = s(g)\}$$

onto $\mathcal{C}$, denoted as $m(g,h) := gh$, with the following properties:

   - $s(gh) = s(h)$ and $r(gh) = r(g)$ for every $(g,h) \in \mathcal{C}^{(2)}$ and
   - $(gh)k = g(hk)$ whenever either side is defined.

4. For every $x \in \mathcal{C}^{(0)}$ there is a unique identity $id_x \in \mathcal{C}$ such that $id_x g = g$ whenever $r(g) = x$ and $h id_x = h$ whenever $s(h) = x$.

In a similar way as we did in the groupoid case, we will identify the elements $x \in \mathcal{C}^{(0)}$ with $id_x \in \mathcal{C}$ in such a way that the set $\mathcal{C}^{(0)}$ will be a subset of $\mathcal{C}$.

Observation A.2. Observe that the term groupoid generalizes the concept of group. Then, the natural expression to generalize a monoid would be monoidoid. Even thought it is quite amusing, we decided to use this term instead of small category. This is because in Subsection A.1.2 we add a topology to this structure and the name "topological category" is referred to something else.

Example A.3. There are many examples of monoidoids. Here are some interesting ones:

1. Take N a monoid with unit ε we can consider it as a monoidoid with one unit and morphisms the elements of N.
2. Let X be a set with a reflexive transitive relation $>$. We define the *monoidoid of pairs*

$$\mathcal{C}_X = \{(x,y) \in X \times X \colon x < y\}$$

with unit space X and view the morphisms (x,y) with source y and range x. The morphisms (x,y) and (w,z) are composable if and only if $y = w$, and composition is defined as

$$(x,y)(y,z) = (x,z).$$

3. If E is a directed graph, then we can consider the *path category* of the graph E as $\mathcal{C}_E = (E^0, \mathrm{Path}(E), s, r, m)$ with multiplication described by $m(\lambda, \mu) := \lambda\mu$ if $r(\lambda) = s(\mu)$. As it is expected, it is a monoidoid.

A.1.2 Topological monoidoids

If we intend to replicate the construction of the Steinberg algebras for a general monoidoid, then we will need to make use of a topology. Nevertheless, we have to be careful since we do not have an inversion map available. In this case, we will have to introduce additional assumptions and generalize our definitions.

Definition A.4. A monoidoid $\mathcal{C} = (\mathcal{C}^{(0)}, \mathcal{C}, s, r, m)$ is a:

1. *topological monoidoid* if $(\mathcal{C}, \mathcal{T})$ is a topological space and r, s and m are continuous maps.
2. *étale monoidoid* if it is a topological monoidoid and r, s and m are local homeomorphisms.

Remark A.5. On the one hand, if $\mathcal{C}$ is an étale monoidoid, then $\mathcal{C}^{(0)}$ is open in $\mathcal{C}$. On the other hand, if it is a Hausdorff topological monoidoid, then $\mathcal{C}^{(0)}$ is closed. In general, for a topological monoidoid the maps $r \times s \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}^{(0)} \times \mathcal{C}^{(0)}$ and $(r, s) \colon \mathcal{C} \to \mathcal{C}^{(0)} \times \mathcal{C}^{(0)}$ are continuous. In addition, if the monoidoid is also Hausdorff, then the *diagonal* $\Delta := \{(x, x) \colon x \in \mathcal{C}^{(0)}\}$ is closed and as a consequence $\mathcal{C}^{(2)} = (r \times s)^{-1}(\Delta)$ is closed too.

Since the definition of an open bisection does not make use of the inversion map, the reader might wonder if we can replicate this concept in this new context. And again, a monoidoid is étale if and only if there is a base of open bisections.

Definition A.6. An *ample monoidoid* is a topological monoidoid with a Hausdorff unit space and a base of compact open bisections.

We will denote by $B^{co}(\mathcal{C})$ to the set of all nonempty compact open bisections of $\mathcal{C}$ and $B^{co}(\mathcal{C}^{(0)})$ will denote the set of all compact open bisections in $\mathcal{C}^{(0)}$.

We are almost ready to construct an analogue of the Steinberg algebras for a general monoidoid, but we will need first to introduce a Lemma that will be handy later. The proof is similar to the one provided in [91, Lemma 2.1].

Lemma A.7. *Let $\mathcal{C}$ be an étale monoidoid with $\mathcal{C}^{(0)}$ a Hausdorff space. If $A, B \subset \mathcal{C}$ are compact open bisections, then*

1. *$AB = \{\alpha\beta\colon (\alpha, \beta) \in (A \times B) \cap \mathcal{C}^{(2)}\}$ is a compact open bisection*

2. *If $\mathcal{C}$ is Hausdorff then $A \cap B$ is a compact open bisection.*

Proof. 1. If AB is empty, then it is a compact open bisection. Otherwise, $AB = m((A \times B) \cap \mathcal{C}^{(2)})$. Since $\mathcal{C}^{(2)}$ is closed, then $(A \times B) \cap \mathcal{C}^{(2)}$ is closed in the relative topology of $A \times B$. In addition, $A \times B$ is compact, then $(A \times B) \cap \mathcal{C}^{(2)}$ is compact in $A \times B$ and in consequence $(A \times B) \cap \mathcal{C}^{(2)}$ is compact in $\mathcal{C}^{(2)}$. Finally, since m is continuous AB is compact.

To prove that AB is open, observe that $A \times B$ is open in $\mathcal{C} \times \mathcal{C}$, then $(A \times B) \cap \mathcal{C}^{(2)}$ is open in the relative topology of $\mathcal{C}^{(2)}$. Finally, since m is also a local homeomorphism we have that is open which means that the image AB is open.

Lastly, we need to prove that it is a bisection. As we have mentioned previously, it is only necessary to check that $s|_{AB}$ and $r|_{AB}$ are injective. Consider $\gamma, \gamma' \in AB$ with $s(\gamma) = s(\gamma')$. This means that there are $\alpha, \alpha' \in A$ and $\beta, \beta' \in B$ with $\gamma = \alpha\beta$, $\gamma' = \alpha'\beta'$ and $s(\alpha\beta) = s(\alpha'\beta')$. Thus, $s(\beta) = s(\beta')$ and since B is a bisection, we have that $\beta = \beta'$. As a consequence, $s(\alpha) = r(\beta) = r(\beta') = s(\alpha')$ and again this implies that $\alpha = \alpha'$. Finally, $\gamma = \gamma'$ and $s|_{AB}$ is injective. We work analogously to prove that $r|_{AB}$ is injective.

2. $A \cap B$ is an open bisection just by definition. Let us see that it is compact. Since $\mathcal{C}$ is Hausdorff, then A and B are closed and the finite intersection of closed is closed. Finally, since $A \cap B$ is a closed contained in a compact, then it is compact.

□

The importance of Lemma A.7 (1) shows that the structure of $B^{co}(\mathcal{C})$ for an ample monoidoid is compatible with a monoidal structure.

Under a little bit of work, by using Lemma A.7 (2) it is possible to prove that when an ample monoidoid $\mathcal{C}$ is Hausdorff, then $B^{co}(\mathcal{C})$ is closed under finite intersections.

A.1.3 The Orloff algebra of an ample monoidoid

As the reader might find obvious, now that we have all the base to construct the analogue of the Steinberg algebra, the next step is to give a proper definition of what we will call Orloff algebra. The reason why we chose this name to denote this algebra is because the researcher Lisa Orloff Clark has made important contributions to the study of Steinberg algebras, Leavitt path algebras and groupoids [46, 54, 85]. In addition, she has been the one who has introduced us in this field and gave the motivation to deepen into the generalisation of Steinberg algebras.

Definition A.8. If $\mathcal{C}$ is an ample monoidoid and R a commutative unitary ring, then $\mathcal{I}_R(\mathcal{C})$ is the submodule of $R^{\mathcal{C}}$ defined by

$$\mathcal{I}_R(\mathcal{C}) := \operatorname{span}_R\{1_U : U \text{ is a Hausdorff compact open subset of } \mathcal{C}\}.$$

The *convolution* of $f, g \in \mathcal{I}_R(\mathcal{C})$ is defined as

$$f * g(\gamma) = \sum_{(\alpha,\beta)\in\mathcal{C}^{(2)},\gamma=\alpha\beta} f(\alpha)g(\beta)$$

for all $\gamma \in \mathcal{C}$. The R-module $\mathcal{I}_R(\mathcal{C})$, together with the convolution, is called the *Orloff algebra* of $\mathcal{C}$ over R.

The name 'algebra' is not arbitrary. As we will prove later $\mathcal{I}_R(\mathcal{C})$ is an associative algebra. However, we will need to do some preparatory work: verifying that the sum is finite, the convolution is closed and the product associative. The results here shown will be analogous to the ones in [91] but we will have to be careful when the inverse map is used.

The following results, proper of the Steinberg algebras theory, appear in [91] and do not make any use of the inverse map. That is why we will just write them without a proof just changing the hypothesis.

Proposition A.9. *Let $\mathcal{B}$ be a base for an ample monoidoid $\mathcal{C}$ consisting on compact open sets, with the property:*

$$\left\{\bigcap_{i=1}^{n} B_i : B_i \in \mathcal{B}, \bigcup_{i=1}^{n} B_i \text{ is Hausdorff}\right\} \subseteq \mathcal{B} \cup \{\emptyset\}.$$

Then $\mathcal{I}_R(\mathcal{C}) = \operatorname{span}_R\{1_B : B \in \mathcal{B}\}$.

Corollary A.10. *If $\mathcal{C}$ is a Hausdorff ample monoidoid and $\mathcal{B}$ is a base of compact open sets that is closed under finite intersections, then* $\mathcal{I}_R(\mathcal{C}) = \operatorname{span}_R\{1_B : B \in \mathcal{B}\}$.

Corollary A.11. *If $\mathcal{C}$ is an ample monoidoid, then the Orloff algebra is finitely generated as an R-module by the characteristic functions of compact open bisections. That is*

$$\mathcal{I}_R(\mathcal{C}) = \text{span}_R\{1_B : B \in B^{co}(\mathcal{C})\}.$$

Now that we know that the $\mathcal{I}_R(\mathcal{C})$ has a system of generators, it makes sense to find what is the convolution product of these elements.

Lemma A.12. *Let $A, B \in B^{co}(\mathcal{C})$ then $1_A * 1_B = 1_{AB}$.*

Proof. By definition we have that $1_A * 1_B(\gamma) = \sum_{\gamma=\alpha\beta} 1_A(\alpha)1_B(\beta) = \sum_{\gamma=\alpha\beta,\alpha\in A,\beta\in B} 1$. Suppose that $\gamma \in AB$, since A and B are compact open bisections, then there exist unique elements $\alpha \in A$ and $\beta \in B$ such that $\gamma = \alpha\beta$. This means that $1_A * 1_B(\gamma) = 1$.

On the other side, if $\gamma \notin AB$ and $\gamma = \alpha\beta$, then $\alpha \notin A$ or $\beta \notin B$ because if that is not true, then $\gamma \in AB$. Thus, we have that $1_A * 1_B = 0$. □

Finally we are in condition to state the following result:

Proposition A.13. *If $\mathcal{C}$ is an ample monoidoid, then the R-module $\mathcal{I}_R(\mathcal{C})$, equipped with the convolution, is an associative R-algebra.*

Proof. The R-linearity can be easily proved by definition. In addition, Lemma A.12 and Corollary A.11 imply that the convolution is an internal operation. Furthermore, both of them, together with the fact that the multiplication in a monoidoid is associative, prove that convolution is associative. □

As a first example, take a directed graph E and its corresponding path category $\mathcal{C}_E$. We can endow this monoidoid with the discrete topology, becoming an ample Hausdorff monoidoid. If R is commutative unitary ring, then it is possible to prove that the path algebra RE is isomorphic to $\mathcal{I}_R(\mathcal{C}_E)$. Every finite set $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a Hausdorff compact open subset of $\mathcal{C}_E$. In addition, since the singletons (sets with just one element) are compact open bisections, we can decompose $1_B = \sum_{i=1}^n 1_{\{\alpha_i\}}$. In consequence, we can write every element $f \in \mathcal{I}_R(\mathcal{C}_E)$ as $f = \sum_{i=1} r_i 1_{\{\gamma_i\}}$ with $r_i \in R\backslash\{0\}$ and $\gamma_i \in \text{Path}(E)$ for every $i = 1, 2, \ldots, n$. Lastly, we consider the map

$$\begin{aligned} \varphi\colon RE &\to \mathcal{I}_R(\mathcal{C}_E) \\ \sum_{i=1}^n r_i\gamma_i &\mapsto \sum_{i=1}^n r_i 1_{\{\gamma_i\}} \end{aligned}$$

which is an surjective homomorphism of R-algebras. We have to prove the injectivity. Suppose that $\varphi\left(\sum_{i=1}^{n} r_i\gamma_i\right) = \sum_{i=1}^{n} r_i 1_{\{\gamma_i\}} = 0$. If we evaluate $\varphi\left(\sum_{i=1}^{n} r_i\gamma_i\right)$ in γ_i, then we have that $r_i = 0$ for every $i = 1, \ldots, n$. In consequence, $\sum_{i=1}^{n} r_i\gamma_i = 0$ and φ is injective. We want to highlight the importance of this last example since it was the motivation to try to find a generalisation of the Steinberg algebras.

After all these constructions the natural questions that we ask ourselves are:

1. Is it possible to prove different path algebra result using Orloff algebras?
2. How do these algebras behave in compare to the Steinberg algebras?
3. What are their finiteness and perfection conditions? What is the structure of the socle?

In general, many questions arise from this new structure due to its relation with the Steinberg and path algebras.

A.2 ENTROPY INDEPENDENT OF THE FILTRATION

During the course of Chapter 8 we have insisted in the fact that the entropy strongly depends on the chosen filtration. Proposition 8.7 and Proposition 8.24 prevent to use the supremum or the infimum respectively in the definition of entropy. If we look at Proposition 8.24 we should observe that the filtration is quite pathological. The selection is quite specific and it makes use of properties of the graph algebras, such as the existence of local units. Not only that, filtrations like the standard filtration have a very interesting property: if $\mathcal{F} = \{V_n\}_{n=1}^{\infty}$ and $V := V_1$, then $V_n = V + V^2 + \cdots + V^n$ for every $n \geqslant 1$. However, the filtration in 8.24 does not achieve that. We can say in this case that the filtration is not *multiplicative*

Lemma A.14. *Consider E a finite graph an the path algebra KE for a field K. The filtration $\mathcal{H}_k = \{H_n\}_{n=1}^{\infty}$ given right before Proposition 8.24 is not multiplicative.*

Proof. Since $H := H_1 = \operatorname{span}_K(E^0)$ we have that $H + H^2 + H^3 + \cdots + H^n = H$ for every $n \geqslant 1$. In particular, we have that $H_k \neq H + H^2 + \cdots H^k$. □

Although the filtration in 8.7 deny the possibility of using the supremum, the family of filtrations that do not allow this are multiplicative. The reader can check that. In this sense, the natural way of defining the entropy of a filtered algebra is the following:

Definition A.15 (Conjecture)**.** Let A be a filtered algebra and $\hat{F}$ is the family of all multiplicative filtrations of A. The *entropy* of A is the magnitude defined as

$$\mathrm{h}(A) := \inf \left\{ \mathrm{h}_{\mathrm{alg}}(A, \mathcal{F}) \colon \mathcal{F} \in \hat{F} \right\}.$$

This new definition is not a solution to our problem unless we find an example of an algebra with a value of entropy different from 0. The possible solution that we propose is the path algebra of a two-petals rose R_2 (Figure 30)

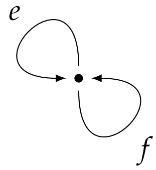

Figure 30: Two-petals rose

Conjecture A.16. Let R_2 be the two-petals rose and KR_2 its path algebra, then

$$\mathrm{h}(KR_2) = \log(2).$$

If Conjecture A.16 is true, then we can translate all of the results in Chapter 8 to the new definition of entropy. It would need a little bit of more work, but it is not impossible. Furthermore, the next question that arises is: in the case of the graph algebras, does the algebraic entropy of the standard filtration coincide with the entropy of the algebra defined with an infimum? We conjecture that this is true.

Conjecture A.17. Let E be a finite graph with A_E its corresponding adjacency matrix, then

$$\mathrm{h}(KE) = \log(\rho(A_E))$$

with $\rho(A_E)$ the spectral radius of A_E.

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