

ON TWO WEIGHT ESTIMATES FOR ITERATED COMMUTATORS

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ABSTRACT. In this paper we extend the bump conjecture and a particular case of the separated bump conjecture with logarithmic bumps to iterated commutators T_b^m . Our results are new even for the first order commutator T_b^1 . A new bump type necessary condition for the two-weighted boundedness of T_b^m is obtained as well. We also provide some results related to a converse to Bloom's theorem.

1. INTRODUCTION

Let T be a Calderón-Zygmund operator, and let b be a locally integrable function on \mathbb{R}^n . The commutator $[b, T]$ of T and b is defined by

$$[b, T]f(x) = b(x)T(f)(x) - T(bf)(x).$$

The iterated commutators T_b^m , $m \in \mathbb{N}$, are defined inductively by

$$T_b^m f = [b, T_b^{m-1}]f, \quad T_b^1 f = [b, T]f.$$

By a weight we mean a non-negative, locally integrable function. In this paper we study two weight estimates

$$\int_{\mathbb{R}^n} |T_b^m f|^p u \lesssim \int_{\mathbb{R}^n} |f|^p v \quad (p > 1)$$

with emphasis on A_p type conditions on a couple of weights (u, v) .

This subject has a long history, and in our brief exposition below we mention only several papers of specific interest to us. Consider first a Calderón-Zygmund operator T (which formally can be regarded as T_b^0). In the one-weighted case when $u = v = w$, the A_p condition,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

is the sufficient (and for a subclass of non-degenerate Calderón-Zygmund operators is also necessary) condition for T to be bounded on $L^p(w)$ (see, e.g., [35]).

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It is well known that the two weight problem is much more complicated, and, in particular, the A_p condition for a couple (u, v) ,

$$(1.1) \quad \sup_Q \left(\frac{1}{|Q|} \int_Q u \right) \left(\frac{1}{|Q|} \int_Q v^{-\frac{1}{p-1}} \right)^{p-1} < \infty,$$

is no longer sufficient for $T : L^p(v) \rightarrow L^p(u)$.

There was a great deal of effort to find slightly stronger bump conditions which are sufficient for $T : L^p(v) \rightarrow L^p(u)$. To formulate such conditions, define the normalized Luxemburg norm (for a Young function φ) by

$$\|f\|_{\varphi, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q \varphi(|f(y)|/\lambda) dy \leq 1 \right\}.$$

If $\varphi(t) = t^p \log^\alpha(e+t)$, $\alpha \geq 0$, we will use the notation $\|f\|_{L^p(\log L)^\alpha, Q}$. Observe that in this notation, (1.1) can be written in the form

$$\sup_Q \|u^{1/p}\|_{L^p, Q} \|v^{-1/p}\|_{L^{p'}, Q} < \infty,$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. The bump conditions strengthen this condition by replacing the L^p norms by slightly larger Luxemburg norms.

We say that a Young function A satisfies the B_p condition if $\int_1^\infty \frac{A(t)}{t^p} \frac{dt}{t} < \infty$. Let \bar{A} denote the Young function complementary to A . The bump conjecture of D. Cruz-Uribe and C. Pérez (see [7, p. 187]) says that if A and B are Young functions such that $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$, $p > 1$, and

$$(1.2) \quad \sup_Q \|u^{1/p}\|_{A, Q} \|v^{-1/p}\|_{B, Q} < \infty,$$

then $T : L^p(v) \rightarrow L^p(u)$. The bump conjecture was solved positively in [31] for $p = 2$ and in [23] for all $p > 1$.

Observe that typical examples of A and B satisfying $\bar{A} \in B_{p'}$ and $\bar{B} \in B_p$ are

$$(1.3) \quad A(t) = t^p \log^{p-1+\delta}(e+t) \quad \text{and} \quad B(t) = t^{p'} \log^{p'-1+\delta}(e+t),$$

where $\delta > 0$. Such functions are called the logarithmic bumps.

The separated bump conjecture (probably first formulated in [10]) asserts that T is bounded from $L^p(v)$ to $L^p(u)$ when (1.2) is replaced by

$$\sup_Q \|u^{1/p}\|_{L^p, Q} \|v^{-1/p}\|_{B, Q} < \infty \quad \text{and} \quad \sup_Q \|u^{1/p}\|_{A, Q} \|v^{-1/p}\|_{L^{p'}, Q} < \infty.$$

This conjecture is still open, in general. In [10], D. Cruz-Uribe, A. Reznikov and A. Volberg established that in the particular case of A and B in (1.3), this conjecture is true (see also [15] for a different proof of this result).

We also mention the works of M. Lacey [19] and K. Li [29] where some different variants of the separated bump conjecture were obtained (which in the particular case of the logarithmic bumps in (1.3) provide yet another approach to the result in [10]). In Section 4.2 below a more detailed exposition of the results in [19, 29] is given.

Turn now to the commutators T_b^m . Our first goal is to extend the bump conjectures to T_b^m . It was shown in our previous works [27, 28] (we recall the proof in Lemma 3.1) that T_b^m (for $b \in BMO$) is controlled by the $(m+1)$ -th iteration of A_S , denoted by A_S^{m+1} , where A_S is the standard sparse operator defined by

$$A_S f(x) = \sum_{Q \in \mathcal{S}} f_Q \chi_Q(x) \quad (f_Q = \frac{1}{|Q|} \int_Q f).$$

Observe that in the case $m = 0$ this result is well known, see [5, 16, 20, 24, 25, 26].

A domination of T by A_S is the standard tool in most of the works dealing with the bump conjectures for T . Therefore, it is not surprising that in our attempt of extending these results to T_b^m we deal with A_S^{m+1} . We will show (Lemma 3.2) that A_S^{m+1} is essentially equivalent to $T_m + T_m^*$, where T_m is a positive linear operator controlled by

$$A_{L(\log L)^m, \mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} \|f\|_{L(\log L)^m, Q} \chi_Q(x).$$

Thus, the operator $A_{L(\log L)^m, \mathcal{S}}$ is the key object in our analysis. The $L^p(v) \rightarrow L^p(u)$ bounds for T_b^m follow from the corresponding bounds for $A_{L(\log L)^m, \mathcal{S}}$ and their dual counterpart.

In what follows, it will be convenient to use the notation

$$[\lambda, \mu]_{A, B} = \sup_Q \|\lambda\|_{A, Q} \|\mu\|_{B, Q}.$$

Our extension of the bump conjecture to T_b^m is the following.

Theorem 1.1. *Let \mathcal{S} be a sparse family. Assume that $m \in \mathbb{Z}_+$ and $p > 1$. Let α_p be an arbitrary Young function such that $\bar{\alpha}_p \in B_{p'}$. Next, let $\beta_{p, m}$ be an arbitrary Young function such that $\beta_{p, m}^{-1}(t) \varphi^{-1}(t) \simeq \frac{t}{\log^m(e+t)}$, where $\varphi \in B_p$. Then*

$$\|A_{L(\log L)^m, \mathcal{S}}\|_{L^p(v) \rightarrow L^p(u)} \lesssim [u^{1/p}, v^{-1/p}]_{\alpha_p, \beta_{p, m}}.$$

If T is a Calderón-Zygmund operator with Dini-continuous kernel, then

$$(1.4) \quad \|T_b^m\|_{L^p(v) \rightarrow L^p(u)} \lesssim \|b\|_{BMO}^m ([u^{1/p}, v^{-1/p}]_{\alpha_p, \beta_{p, m}} + [u^{1/p}, v^{-1/p}]_{\beta_{p', m}, \alpha_{p'}}).$$

Observe that if $m = 0$, then $\beta_{p, 0} = \alpha_{p'}$. Hence in this case the first and the second terms in (1.4) coincide, and we obtain the bump conjecture for T stated above.

In the case $m = 1$, D. Cruz-Uribe and K. Moen [9] obtained different bump-type sufficient conditions. We give an example (in the Appendix) showing that the conditions in Theorem 1.1 provide a wider class of weights (u, v) for which $\|T_b^1\|_{L^p(v) \rightarrow L^p(u)} < \infty$.

The standard computations show that typical examples of α_p and $\beta_{p, m}$ in Theorem 1.1 are

$$(1.5) \quad \alpha_p(t) = t^p \log^{p-1+\delta}(e+t), \quad \beta_{p, m}(t) = t^{p'} \log^{(m+1)p'-1+\delta}(e+t).$$

Our next result is an extension of the separated bump conjecture with logarithmic bumps to T_b^m . Here we fix α_p and $\beta_{p, m}$ precisely as in (1.5).

Theorem 1.2. *Let \mathcal{S} be a sparse family. Assume that $m \in \mathbb{Z}_+$ and $p > 1$. Take α_p and $\beta_{p,m}$ as in (1.5). Define also*

$$\gamma_{p,m}(t) = t^{p'} \log^{m(p'+\delta)}(e+t).$$

Then

$$\|A_{L(\log L)^m, \mathcal{S}}\|_{L^p(v) \rightarrow L^p(u)} \lesssim [u^{1/p}, v^{-1/p}]_{t^p, \beta_{p,m}} + [u^{1/p}, v^{-1/p}]_{\alpha_p, \gamma_{p,m}}.$$

If T is a Calderón-Zygmund operator with Dini-continuous kernel, then

$$(1.6) \quad \begin{aligned} \|T_b^m\|_{L^p(v) \rightarrow L^p(u)} &\lesssim \|b\|_{BMO}^m \left([u^{1/p}, v^{-1/p}]_{t^p, \beta_{p,m}} + [u^{1/p}, v^{-1/p}]_{\alpha_p, \gamma_{p,m}} \right. \\ &\quad \left. + [u^{1/p}, v^{-1/p}]_{\beta_{p',m}, t^{p'}} + [u^{1/p}, v^{-1/p}]_{\gamma_{p',m}, \alpha_{p'}} \right). \end{aligned}$$

The proof of this result is much more involved than the proof of Theorem 1.1. It is based on some extension of the works [19, 29].

If $m = 0$, then $\beta_{p,0} = \alpha_{p'}$ and $\gamma_{p,0} = t^{p'}$. In this case the second line in (1.6) coincides with the first line and we obtain the above mentioned separated bump conjecture with logarithmic bumps from [10]. In the case $m \geq 1$ one can see that our estimates are not totally in the spirit of the separated bumps. Simple manipulations with terms involving $\gamma_{p,m}$ in (1.6) allow us to get the following.

Corollary 1.3. *For $m \in \mathbb{N}$ define*

$$\psi_{p,m}(t) = t^{p'} \log^{\max((m+1)p'-1, mp'+1)+\varepsilon}(e+t).$$

Then, with T as above,

$$\|T_b^m\|_{L^p(v) \rightarrow L^p(u)} \lesssim \|b\|_{BMO}^m \left([u^{1/p}, v^{-1/p}]_{t^p, \psi_{p,m}} + [u^{1/p}, v^{-1/p}]_{\psi_{p',m}, t^{p'}} \right).$$

Remark 1.4. At this point it is worth comparing Theorems 1.1, and 1.2. Note that the joint bump conditions in Theorem 1.1 hold for arbitrary bumps in contrast to the conditions in Theorem 1.2 that hold just for certain bumps but allow to derive separated log bump conditions as we show in Corollary 1.3. For the choices of bumps for which Theorem 1.2 holds, even though we have four conditions, those conditions are much better than the two conditions in Theorem 1.1, and hence they are new sufficient conditions.

Remark 1.5. Observe that the preceding Corollary in the case $p = 2$ yields

$$\|T_b^m\|_{L^2(v) \rightarrow L^2(u)} \lesssim \|b\|_{BMO}^m \left([u^{1/2}, v^{-1/2}]_{t^2, \phi(t)} + [u^{1/2}, v^{-1/2}]_{\phi(t), t^2} \right).$$

where $\phi(t) = t^2 \log^{2(m+1)-1+\varepsilon}(e+t)$. In this case this estimate generalizes the logarithmic separated bumps estimate from [10] to commutators of any order.

We conjecture that the terms with $\gamma_{p,m}$ in (1.6) can be fully avoided.

Conjecture 1.6. Let T be as above. Then

$$\|T_b^m\|_{L^p(v) \rightarrow L^p(u)} \lesssim \|b\|_{BMO}^m \left([u^{1/p}, v^{-1/p}]_{t^p, \beta_{p,m}} + [u^{1/p}, v^{-1/p}]_{\beta_{p',m}, t^{p'}} \right).$$

It is interesting that Corollary 1.3 coincides with Conjecture 1.6 in the case $p = 2$ but for every $p \neq 2$, Conjecture 1.6 provides a better result.

Turn to a necessary condition for the two-weighted boundedness of T_b^m .

Theorem 1.7. *Let T be a non-degenerate Calderón-Zygmund operator. Let $m \in \mathbb{Z}_+$ and $p > 1$. Assume that for every $b \in BMO$,*

$$\|T_b^m f\|_{L^{p,\infty}(u)} \lesssim \|b\|_{BMO}^m \|f\|_{L^p(v)}.$$

Assume additionally that u is a doubling weight. Then

$$(1.7) \quad \sup_Q \|u^{1/p}\|_{L^{p,Q}} \|v^{-1/p}\|_{L^{p'(\log L)^{mp'},Q}} < \infty.$$

Consider first the case $m = 0$. Then (1.7) is just the usual A_p condition for (u, v) . In this case, when T is the Hilbert transform, the necessity of the A_p condition was obtained by B. Muckenhoupt and R. Wheeden [30] without assuming the doubling condition on u . However, it is not clear to us whether this condition can be removed, in general. In a very recent work [6], a similar statement was obtained assuming a slightly weaker directionally doubling condition. In [21], it was shown that the doubling condition can be avoided but assuming $T : L^p(v) \rightarrow L^{p,\infty}(u)$ for a family of operators.

To the best of our knowledge, in the case $m \geq 1$, Theorem 1.7 is new. Its proof is based on a special construction of BMO functions which goes back to P. Jones [18].

By duality, it follows from Theorem 1.7 that if $T_b^{m+1} : L^p(v) \rightarrow L^p(u)$ and u and $v^{1-p'}$ are doubling, then, additionally to (1.7) (with $m + 1$), we also have that

$$\sup_Q \|u^{1/p}\|_{L^{p(\log L)^{(m+1)p},Q}} \|v^{-1/p}\|_{L^{p',Q}} < \infty.$$

Hence, assuming also that T is Dini-continuous, by Corollary 1.3 for $m \geq 1$ and by the separated bump conjecture with logarithmic bumps for $m = 0$, we obtain that $T_b^m : L^p(v) \rightarrow L^p(u)$. Observe that by a well-known intuition, T_b^1 is more “singular” than T . Therefore, it is natural to expect that $T_b^{m+1} : L^p(v) \rightarrow L^p(u)$ implies $T_b^m : L^p(v) \rightarrow L^p(u)$ even without assuming the doubling conditions on u and $v^{1-p'}$. However, we do not know any direct proof of this fact even in the case $m = 0$, and, in particular, it is not clear to us whether these doubling conditions can be removed. See also Remark 5.4 for some comments about this.

In this paper we also obtain several results related to Bloom type estimates. We recall the following result, see [14, 28] and the references therein. Its sufficiency part in the case $m = 1$ is due to S. Bloom [2].

Theorem 1.8. *Let T be a non-degenerate Calderón-Zygmund operator with Dini-continuous kernel. Let $m \in \mathbb{N}$ and $p > 1$. Assume that $\lambda, \mu \in A_p$. Further, let $\eta = \left(\frac{\mu}{\lambda}\right)^{1/pm}$. Then*

$$b \in BMO_\eta \Rightarrow \|T_b^m\|_{L^p(\lambda)} \lesssim \|b\|_{BMO_\eta}^m \|f\|_{L^p(\mu)}$$

and

$$\|T_b^m f\|_{L^p(\lambda)} \lesssim \|f\|_{L^p(\mu)} \Rightarrow b \in BMO_\eta.$$

It is natural to ask whether this result remains valid under more general assumptions on λ, μ and η . We conjecture that this is not the case.

Conjecture 1.9. Let $m \in \mathbb{N}$ and $p > 1$. Let λ, μ and η be arbitrary weights such that the following holds:

$$b \in BMO_\eta \Rightarrow \|T_b^m\|_{L^p(\lambda)} \lesssim \|b\|_{BMO_\eta}^m \|f\|_{L^p(\mu)}$$

and

$$\|T_b^m f\|_{L^p(\lambda)} \lesssim \|f\|_{L^p(\mu)} \Rightarrow b \in BMO_\eta.$$

Then $\lambda, \mu \in A_p$ and $\eta \simeq \left(\frac{\mu}{\lambda}\right)^{1/pm}$.

Unluckily, our current tools are still far away from enabling us to establish this conjecture in full generality. However we still manage to provide several partial results in Section 6.

The remainder of the paper is organized as follows. Section 2 contains some preliminary results and definitions. In Section 3 we gather some sparse bounds that will be crucial to establish our main results. In Section 4 we prove Theorems 1.1 and 1.2. Section 5 is devoted to prove Theorem 1.7. In Section 6 we discuss the partial results related to Conjecture 1.9. Finally, in the Appendix we provide an example comparing our sufficient condition in Theorem 1.1 with the condition from [9].

Throughout the paper we use the notation $A \lesssim B$ if $A \leq CB$ with some independent constant C . We write $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

2. PRELIMINARIES

2.1. Calderón-Zygmund operators. We say that K is an ω -Calderón-Zygmund kernel if $|K(x, y)| \leq \frac{C_K}{|x-y|^n}$, $x \neq y$, and

$$|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq \omega\left(\frac{|x-x'|}{|x-y|}\right) \frac{1}{|x-y|^n},$$

whenever $|x-y| > 2|x-x'|$, where $\omega : [0, 1] \rightarrow [0, \infty)$ is the modulus of continuity.

A linear, L^2 bounded operator T is called ω -Calderón-Zygmund if it has a representation

$$Tf(x) = \int_{\mathbb{R}^n} K(x, y)f(y)dy \quad \text{for all } x \notin \text{supp } f,$$

where K is an ω -Calderón-Zygmund kernel.

An ω -Calderón-Zygmund kernel is called Dini-continuous if $\int_0^1 \omega(t) \frac{dt}{t} < \infty$.

We say that an ω -Calderón-Zygmund kernel is non-degenerate if $\omega(t) \rightarrow 0$ as $t \rightarrow 0$, and for every $y \in \mathbb{R}^n$ and $r > 0$, there exists $x \notin B(y, r)$ with

$$|K(x, y)| \geq \frac{c_0}{r^n}.$$

This definition was given by T. Hytönen [14] (in the case when $K(x, y) = k(x - y)$, a similar notion was introduced by E. Stein [35, p. 210]).

We say that T is a non-degenerate Calderón-Zygmund operator if T and its adjoint T^* are associated with non-degenerate kernels. In other words, we require that if T is associated with kernel K , then $K(x, y)$ and $\tilde{K}(x, y) = K(y, x)$ are non-degenerate.

The following result is contained in the proof of [14, Prop 2.2].

Proposition 2.1. *Let K be a non-degenerate kernel. Then for every $A \geq 3$ and every ball $B(y_0, r)$, there is a disjoint ball $\tilde{B} = B(x_0, r)$ at distance $\text{dist}(B, \tilde{B}) \simeq Ar$ such that*

$$|K(x_0, y_0)| \simeq \frac{1}{A^n r^n},$$

and for all $y \in B$ and $x \in \tilde{B}$, we have

$$|K(x, y) - K(x_0, y_0)| \lesssim \frac{\varepsilon_A}{A^n r^n},$$

where $\varepsilon_A \rightarrow 0$ as $A \rightarrow \infty$.

2.2. Dyadic lattices and sparse families. By a cube in \mathbb{R}^n we mean a half-open cube $Q = \prod_{i=1}^n [a_i, a_i + h)$, $h > 0$. Denote by ℓ_Q the sidelength of Q . Given a cube $Q_0 \subset \mathbb{R}^n$, let $\mathcal{D}(Q_0)$ denote the set of all dyadic cubes with respect to Q_0 , that is, the cubes obtained by repeated subdivision of Q_0 and each of its descendants into 2^n congruent subcubes.

A dyadic lattice \mathcal{D} in \mathbb{R}^n is any collection of cubes such that

- (i) if $Q \in \mathcal{D}$, then each child of Q is in \mathcal{D} as well;
- (ii) every 2 cubes $Q', Q'' \in \mathcal{D}$ have a common ancestor, i.e., there exists $Q \in \mathcal{D}$ such that $Q', Q'' \in \mathcal{D}(Q)$;
- (iii) for every compact set $K \subset \mathbb{R}^n$, there exists a cube $Q \in \mathcal{D}$ containing K .

For this definition we refer to [25].

A family of cubes \mathcal{S} is called sparse if there exists $0 < \alpha < 1$ such that for every $Q \in \mathcal{S}$ one can find a measurable set $E_Q \subset Q$ with $|E_Q| \geq \alpha|Q|$, and the sets $\{E_Q\}$ are pairwise disjoint.

In our results the sparseness constant α will depend only on n , and its precise value will not be relevant.

2.3. Young functions and normalized Luxemburg norms. By a Young function we mean a continuous, convex, strictly increasing function $\varphi : [0, \infty) \rightarrow [0, \infty)$ with $\varphi(0) = 0$ and $\varphi(t)/t \rightarrow \infty$ as $t \rightarrow \infty$.

The Orlicz maximal operator M_φ is defined for a Young function φ by

$$M_\varphi f(x) = \sup_{Q \ni x} \|f\|_{\varphi, Q},$$

where the supremum is taken over all cubes Q containing the point x . If $\varphi(t) = t$, then $M_\varphi = M$ is the standard Hardy-Littlewood maximal operator. It was shown

by C. Pérez [34] that

$$(2.1) \quad \varphi \in B_p \Rightarrow \|M_\varphi f\|_{L^p} \lesssim \|f\|_{L^p} \quad (p > 1).$$

Given a Young function φ , its complementary function is defined by

$$\bar{\varphi}(t) = \sup_{s \geq 0} (st - \varphi(s)).$$

Then $\bar{\varphi}$ is also a Young function satisfying $t \leq \bar{\varphi}^{-1}(t)\varphi^{-1}(t) \leq 2t$. Also the following Hölder type estimate holds:

$$(2.2) \quad \frac{1}{|Q|} \int_Q |f(x)g(x)| dx \leq 2\|f\|_{\varphi,Q} \|g\|_{\bar{\varphi},Q}.$$

There is a more general version of (2.2).

Lemma 2.2. *Let A, B and C be non-negative, continuous, strictly increasing functions on $[0, \infty)$ satisfying $A^{-1}(t)B^{-1}(t) \leq C^{-1}(t)$ for all $t \geq 0$. Assume also that C is convex. Then*

$$\|fg\|_{C,Q} \leq 2\|f\|_{A,Q} \|g\|_{B,Q}.$$

This lemma was proved by R. O’Neil [33] under the assumption that A, B and C are Young functions but the same proof works under the above conditions (see [27] for the details).

Recall the well-known facts (see, e.g., [37, Ch. 10]) that

$$(2.3) \quad \frac{1}{|Q|} \int_Q M^k(f\chi_Q) \lesssim \|f\|_{L(\log L)^k, Q},$$

where M^k denotes the k -th iteration of the maximal operator, and

$$(2.4) \quad \|f\|_{L(\log L)^\alpha, Q} \simeq \frac{1}{|Q|} \int_Q |f(x)| \log^\alpha \left(\frac{|f(x)|}{|f|_Q} + e \right) dx.$$

Dealing with the logarithmic bumps of the form $t^p \log^\alpha(e+t)$, we will frequently use the following estimates. First, since $\| |f|^r \|_{\varphi, Q} = \|f\|_{\varphi^{(tr)}, Q}^r$ for any $r > 0$, we have

$$\| |f|^{1/p} \|_{L^p(\log L)^\alpha, Q} \simeq \|f\|_{L(\log L)^\alpha, Q}^{1/p}.$$

Second, it follows from (2.4) and Hölder’s inequality that for any $0 < \delta < 1$,

$$\|f\|_{L(\log L)^\alpha, Q} \lesssim (|f|_Q)^\delta \|f\|_{L(\log L)^{\frac{\alpha}{1-\delta}}, Q}^{1-\delta}.$$

3. SPARSE BOUNDS FOR ITERATED COMMUTATORS

Let T be a Calderón-Zygmund operator with Dini-continuous kernel, and assume that b is a locally integrable function. Consider the m -th iterated commutator T_b^m .

Recall the following pointwise sparse bound established for $m = 1$ in [27] and for $m \geq 1$ in [17]: for every bounded and compactly supported f , there exist 3^n sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that for a.e. $x \in \mathbb{R}^n$,

$$(3.1) \quad |T_b^m f(x)| \lesssim \sum_{j=1}^{3^n} \sum_{Q \in \mathcal{S}_j} \sum_{k=0}^m |b(x) - b_Q|^{m-k} \left(\frac{1}{|Q|} \int_Q |b - b_Q|^k |f| \right) \chi_Q(x).$$

Next, it was shown in [27, Lemma 5.1] that given a sparse family $\mathcal{S} \subset \mathcal{D}$, there exists a sparse family $\tilde{\mathcal{S}} \subset \mathcal{D}$ containing \mathcal{S} and such that if $Q \in \tilde{\mathcal{S}}$, then for a.e. $x \in Q$,

$$(3.2) \quad |b(x) - b_Q| \leq 2^{n+2} \sum_{P \in \tilde{\mathcal{S}}, P \subseteq Q} \left(\frac{1}{|P|} \int_P |b - b_P| \right) \chi_P(x).$$

Suppose now that η is a weight, and $b \in BMO_\eta$, namely,

$$\|b\|_{BMO_\eta} = \sup_Q \frac{1}{\eta(Q)} \int_Q |b(x) - b_Q| dx < \infty.$$

Then (3.2) allows to transform the right-hand side of (3.1) in the following way.

Given a sparse family \mathcal{S} , define the sparse operator $A_{\mathcal{S}}$ by

$$A_{\mathcal{S}} f(x) = \sum_{Q \in \mathcal{S}} f_Q \chi_Q(x).$$

Further, consider the operator $A_{\mathcal{S}, \eta}$ defined by

$$A_{\mathcal{S}, \eta} f(x) = \eta A_{\mathcal{S}} f(x).$$

Denote by $A_{\mathcal{S}, \eta}^m$ the m -th iteration of $A_{\mathcal{S}, \eta}$.

Lemma 3.1. *Let $m \in \mathbb{N}$. For every bounded and compactly supported f and for every $b \in BMO_\eta$, there exist 3^n sparse families $\mathcal{S}_j \subset \mathcal{D}_j$ such that for every $g \geq 0$,*

$$(3.3) \quad |\langle T_b^m f, g \rangle| \lesssim \|b\|_{BMO_\eta}^m \sum_{j=1}^{3^n} \langle A_{\mathcal{S}_j} (A_{\mathcal{S}_j, \eta}^m |f|), g \rangle.$$

This result is contained implicitly in [27] for $m = 1$ and in [28] for $m \geq 1$. We outline briefly its proof for the sake of the completeness.

Proof of Lemma 3.1. By (3.1),

$$(3.4) \quad |\langle T_b^m f, g \rangle| \lesssim \sum_{j=1}^{3^n} \sum_{Q \in \mathcal{S}_j} \sum_{k=0}^m (|b - b_Q|^k |f|)_Q (|b - b_Q|^{m-k} |g|)_Q |Q|.$$

Next, by (3.2), there exist extended families $\tilde{\mathcal{S}}_j$ such that for a.e. $x \in Q$,

$$|b(x) - b_Q| \lesssim \|b\|_{BMO_\eta} \sum_{P \in \tilde{\mathcal{S}}_j, P \subseteq Q} \eta_P \chi_P(x).$$

Since the cubes from $\tilde{\mathcal{S}}_j$ are dyadic, for every $l \in \mathbb{N}$,

$$\left(\sum_{P \in \tilde{\mathcal{S}}_j, P \subseteq Q} \eta_P \chi_P \right)^l \lesssim \sum_{P_1 \subseteq P_{-1} \subseteq \dots \subseteq P_1 \subseteq Q, P_i \in \tilde{\mathcal{S}}_j} \eta_{P_1} \eta_{P_2} \dots \eta_{P_l} \chi_{P_l}.$$

Combining this estimate with the previous yields

$$\int_Q |b - b_Q|^l |h| \lesssim \|b\|_{BMO_\eta}^l \int_Q A_{\tilde{\mathcal{S}}_j, \eta}^l |h|.$$

Therefore, replacing the right-hand side of (3.4) by a larger sum over $\tilde{\mathcal{S}}_j$ and redenoting $\tilde{\mathcal{S}}_j$ again by \mathcal{S}_j , we obtain that

$$|\langle T_b^m f, g \rangle| \lesssim \|b\|_{BMO_\eta}^m \sum_{j=1}^{3^n} \sum_{Q \in \mathcal{S}_j} \sum_{k=0}^m (A_{\mathcal{S}_j, \eta}^k |f|)_Q (A_{\mathcal{S}_j, \eta}^{m-k} g)_Q |Q|,$$

where $A_{\mathcal{S}_j, \eta}^0 |f| = |f|$. It remains to observe that, since $A_{\mathcal{S}}$ is self-adjoint,

$$\begin{aligned} \sum_{Q \in \mathcal{S}} (A_{\mathcal{S}, \eta}^k |f|)_Q (A_{\mathcal{S}, \eta}^{m-k} g)_Q |Q| &= \langle A_{\mathcal{S}}(A_{\mathcal{S}, \eta}^k |f|), A_{\mathcal{S}, \eta}^{m-k} g \rangle \\ &= \langle \eta A_{\mathcal{S}}(A_{\mathcal{S}, \eta}^k |f|), A_{\mathcal{S}}(A_{\mathcal{S}, \eta}^{m-k-1} g) \rangle = \langle A_{\mathcal{S}}(A_{\mathcal{S}, \eta}^{k+1} |f|), A_{\mathcal{S}, \eta}^{m-k-1} g \rangle \\ &= \dots = \langle A_{\mathcal{S}}(A_{\mathcal{S}, \eta}^m |f|), g \rangle. \end{aligned}$$

This along with the previous estimate completes the proof. \square

Lemma 3.1 says that if $b \in BMO$ (that is, $\eta = 1$), then T_b^m is controlled by $A_{\mathcal{S}}^{m+1}$. Let us show that in this case the right-hand side of (3.3) can be further transformed.

Given $m \in \mathbb{N}$, define the operator T_m by

$$T_m f(x) = \sum_{Q \in \mathcal{S}} \left(\frac{1}{|Q|} \sum_{P_1 \in \mathcal{S}: P_1 \subseteq Q} \dots \sum_{P_m \in \mathcal{S}: P_m \subseteq P_{m-1}} \int_{P_m} f \right) \chi_Q(x),$$

where we assume that $P_0 = Q$. Observe that the adjoint operator T_m^* is given by

$$T_m^* f(x) = \sum_{Q \in \mathcal{S}} \left(\sum_{P_1 \in \mathcal{S}: Q \subseteq P_1} \dots \sum_{P_m \in \mathcal{S}: P_{m-1} \subseteq P_m} f_{P_m} \right) \chi_Q(x)$$

(this can be easily checked by changing the summation and switching the indices).

Lemma 3.2. *For all $f, g \geq 0$ and for every $m \in \mathbb{N}$,*

$$\langle A_{\mathcal{S}}^{m+1} f, g \rangle \lesssim \langle T_m f, g \rangle + \langle T_m^* f, g \rangle.$$

Proof. The proof is by induction on m . In the case $m = 1$ we have

$$\begin{aligned} \langle A_S^2 f, g \rangle &= \sum_{Q \in \mathcal{S}} \sum_{P \in \mathcal{S}} \frac{|P \cap Q|}{|P|} g_Q \int_P f \\ &= \sum_{Q \in \mathcal{S}} \sum_{P \subseteq Q} g_Q \int_P f + \sum_{Q \in \mathcal{S}} \sum_{P: Q \subset P} f_P \int_Q g \leq \langle T_1 f, g \rangle + \langle T_1^* f, g \rangle. \end{aligned}$$

Suppose that the lemma is true for $m = k - 1$, and let us prove it for $m = k$, $k \geq 2$. Since A_S is self-adjoint and by the inductive assumption,

$$\begin{aligned} \langle A_S^{k+1} f, g \rangle &= \langle A_S^k f, A_S g \rangle \\ &\lesssim \langle T_{k-1} f, A_S g \rangle + \langle T_{k-1}^* f, A_S g \rangle \\ &= \langle f, T_{k-1}^* A_S g \rangle + \langle f, T_{k-1} A_S g \rangle. \end{aligned}$$

It follows from this that the proof will be completed if we show that

$$(3.5) \quad \max(T_{k-1} A_S g, T_{k-1}^* A_S g) \lesssim T_k g + T_k^* g.$$

Consider first $T_{k-1} A_S g$. We have

$$(3.6) \quad \begin{aligned} \int_{P_{k-1}} A_S g &= \sum_{P_k \in \mathcal{S}} \frac{|P_k \cap P_{k-1}|}{|P_k|} \int_{P_k} g \\ &= \sum_{P_k \in \mathcal{S}: P_k \subseteq P_{k-1}} \int_{P_k} g + \sum_{P_k \in \mathcal{S}: P_{k-1} \subset P_k} \frac{|P_{k-1}|}{|P_k|} \int_{P_k} g. \end{aligned}$$

Therefore,

$$(3.7) \quad T_{k-1} A_S g \leq T_k g + \sum_{Q \in \mathcal{S}} a_k(g, Q) \chi_Q,$$

where

$$(3.8) \quad a_k(g, Q) = \frac{1}{|Q|} \sum_{P_1 \in \mathcal{S}: P_1 \subseteq Q} \cdots \sum_{P_{k-1} \in \mathcal{S}: P_{k-1} \subseteq P_{k-2}} \sum_{P_k \in \mathcal{S}: P_{k-1} \subset P_k} \frac{|P_{k-1}|}{|P_k|} \int_{P_k} g.$$

Transform the last two sums in (3.8) as follows:

$$\begin{aligned} \sum_{P_{k-1} \subseteq P_{k-2}} \sum_{P_k: P_{k-1} \subset P_k} &= \sum_{P_{k-1} \subseteq P_{k-2}} \sum_{P_k: P_{k-1} \subset P_k \subseteq P_{k-2}} + \sum_{P_{k-1} \subseteq P_{k-2}} \sum_{P_k: P_{k-2} \subset P_k} \\ &= \sum_{P_k \subseteq P_{k-2}} \sum_{P_{k-1} \subset P_k} + \sum_{P_{k-1} \subseteq P_{k-2}} \sum_{P_k: P_{k-2} \subset P_k}. \end{aligned}$$

By the sparseness,

$$\sum_{P_k \subseteq P_{k-2}} \sum_{P_{k-1} \subset P_k} \frac{|P_{k-1}|}{|P_k|} \int_{P_k} g \lesssim \sum_{P_k \subseteq P_{k-2}} \int_{P_k} g$$

and

$$\sum_{P_{k-1} \subseteq P_{k-2}} \sum_{P_k: P_{k-2} \subset P_k} \frac{|P_{k-1}|}{|P_k|} \int_{P_k} g \lesssim \sum_{P_k: P_{k-2} \subset P_k} \frac{|P_{k-2}|}{|P_k|} \int_{P_k} g.$$

It follows from this that

$$(3.9) \quad \sum_{Q \in \mathcal{S}} a_k(g, Q) \chi_Q \lesssim T_{k-1}g + \sum_{Q \in \mathcal{S}} a_{k-1}(g, Q) \chi_Q$$

and

$$\sum_{Q \in \mathcal{S}} a_2(g, Q) \chi_Q \lesssim T_1g + T_1^*g.$$

Therefore, iterating (3.9) yields

$$\sum_{Q \in \mathcal{S}} a_k(g, Q) \chi_Q \lesssim \sum_{j=1}^{k-1} T_jg + T_1^*g \lesssim T_kg + T_k^*g,$$

which, along with (3.7), proves the first part of (3.5).

The proof of the second part of (3.5) is similar. Consider $T_{k-1}^*A_Sg$. Applying (3.6), we obtain

$$(3.10) \quad T_{k-1}^*A_Sg \leq T_k^*g + \sum_{Q \in \mathcal{S}} b_k(g, Q) \chi_Q,$$

where

$$(3.11) \quad b_k(g, Q) = \frac{1}{|Q|} \sum_{P_1 \in \mathcal{S}: Q \subseteq P_1} \cdots \sum_{P_{k-1} \in \mathcal{S}: P_{k-2} \subseteq P_{k-1}} \sum_{P_k \in \mathcal{S}: P_k \subseteq P_{k-1}} \frac{1}{|P_{k-1}|} \int_{P_k} g$$

Transform the last two sums in (3.11) as follows:

$$\begin{aligned} \sum_{P_{k-1}: P_{k-2} \subseteq P_{k-1}} \sum_{P_k \subseteq P_{k-1}} &= \sum_{P_{k-1}: P_{k-2} \subseteq P_{k-1}} \sum_{P_k \subseteq P_{k-2}} + \sum_{P_{k-1}: P_{k-2} \subseteq P_{k-1}} \sum_{P_{k-2} \subset P_k \subseteq P_{k-1}} \\ &= \sum_{P_{k-1}: P_{k-2} \subseteq P_{k-1}} \sum_{P_k \subseteq P_{k-2}} + \sum_{P_k: P_{k-2} \subset P_k} \sum_{P_{k-1}: P_k \subseteq P_{k-1}}. \end{aligned}$$

Using the standard fact that $\sum_{Q \in \mathcal{Q}: P \subseteq Q} \frac{1}{|Q|} \lesssim \frac{1}{|P|}$, we obtain that

$$\sum_{P_{k-1}: P_{k-2} \subseteq P_{k-1}} \sum_{P_k \subseteq P_{k-2}} \frac{1}{|P_{k-1}|} \int_{P_k} g \lesssim \sum_{P_k \subseteq P_{k-2}} \frac{1}{|P_{k-2}|} \int_{P_k} g$$

and

$$\sum_{P_k: P_{k-2} \subset P_k} \sum_{P_{k-1}: P_k \subseteq P_{k-1}} \frac{1}{|P_{k-1}|} \int_{P_k} g \lesssim \sum_{P_k: P_{k-2} \subset P_k} \frac{1}{|P_k|} \int_{P_k} g.$$

It follows from this that

$$(3.12) \quad \sum_{Q \in \mathcal{S}} b_k(g, Q) \chi_Q \lesssim T_{k-1}^*g + \sum_{Q \in \mathcal{S}} b_{k-1}(g, Q) \chi_Q$$

and

$$\sum_{Q \in \mathcal{S}} b_2(g, Q) \chi_Q \lesssim T_1 g + T_1^* g.$$

Therefore, iterating (3.12) yields

$$\sum_{Q \in \mathcal{S}} b_k(g, Q) \chi_Q \lesssim \sum_{j=1}^{k-1} T_j^* g + T_1 g \lesssim T_k g + T_k^* g,$$

which, along with (3.10), proves the second part of (3.5).

This completes the proof of (3.5), and therefore, the lemma is proved. \square

We end this section with a lemma which will be quite useful for our purposes.

Lemma 3.3. *Let T be a Calderón-Zygmund operator with Dini-continuous kernel. Let $m \in \mathbb{Z}_+$ and $p > 1$. Let u, v be weights. Then*

$$\|T_b^m\|_{L^p(v) \rightarrow L^p(u)} \lesssim \|A_{L(\log L)^m, \mathcal{S}}\|_{L^p(v) \rightarrow L^p(u)} + \|A_{L(\log L)^m, \mathcal{S}}\|_{L^{p'}(u^{1-p'}) \rightarrow L^{p'}(v^{1-p'})}.$$

Proof. First we note that by Lemmata 3.1, 3.2 and by duality,

$$\|T_b^m\|_{L^p(v) \rightarrow L^p(u)} \lesssim \|T_m\|_{L^p(v) \rightarrow L^p(u)} + \|T_m\|_{L^{p'}(u^{1-p'}) \rightarrow L^{p'}(v^{1-p'})}.$$

Consider the expression defining T_m . Since \mathcal{S} is sparse,

$$\sum_{P_m \in \mathcal{S}: P_m \subseteq P_{m-1}} \int_{P_m} f \lesssim \sum_{P_m \in \mathcal{S}: P_m \subseteq P_{m-1}} \int_{E_{P_m}} M(f \chi_{P_0}) \lesssim \int_{P_{m-1}} M(f \chi_{P_0}).$$

Applying subsequently this argument implies

$$\sum_{P_1 \in \mathcal{S}: P_1 \subseteq P_0} \cdots \sum_{P_m \in \mathcal{S}: P_m \subseteq P_{m-1}} \int_{P_m} f \lesssim \int_{P_0} M^m(f \chi_{P_0}).$$

Therefore, by (2.3),

$$T_m f(x) \lesssim \sum_{Q \in \mathcal{S}} \|f\|_{L(\log L)^m, Q} \chi_Q(x)$$

and we are done. \square

4. PROOFS OF THEOREMS 1.1 AND 1.2

4.1. Proof of Theorem 1.1. By Lemma 3.3, it suffices to prove the first part of Theorem 1.1 for $A_{L(\log L)^m, \mathcal{S}}$. Consider the bilinear form

$$\langle A_{L(\log L)^m, \mathcal{S}} f, g \rangle = \sum_{Q \in \mathcal{S}} \|f\|_{L(\log L)^m, Q} g_Q |Q|.$$

By Hölder's inequality (2.2),

$$g_Q \leq 2 \|u^{1/p}\|_{\alpha_p, Q} \|g u^{-1/p}\|_{\tilde{\alpha}_p, Q}.$$

Further, using the fact that if $C(t) = t \log^m(e+t)$, then $C^{-1}(t) \sim \frac{t}{\log^m(e+t)}$, and applying Lemma 2.2, we obtain

$$\|f\|_{L(\log L)^m, Q} \lesssim \|fv^{1/p}\|_{\varphi, Q} \|v^{-1/p}\|_{\beta_{p,m}, Q}.$$

Therefore,

$$(4.1) \quad \langle A_{L(\log L)^m, \mathcal{S}} f, g \rangle \lesssim [u^{1/p}, v^{-1/p}]_{\alpha_p, \beta_{p,m}} \sum_{Q \in \mathcal{S}} \|fv^{1/p}\|_{\varphi, Q} \|gu^{-1/p}\|_{\bar{\alpha}_p, Q} |Q|.$$

Using that \mathcal{S} is sparse and by Hölder's inequality along with (2.1),

$$\begin{aligned} \sum_{Q \in \mathcal{S}} \|fv^{1/p}\|_{\varphi, Q} \|gu^{-1/p}\|_{\bar{\alpha}_p, Q} |Q| &\lesssim \sum_{Q \in \mathcal{S}} \int_{E_Q} M_\varphi(fv^{1/p}) M_{\bar{\alpha}_p}(gu^{-1/p}) dx \\ &\lesssim \int_{\mathbb{R}^n} M_\varphi(fv^{1/p}) M_{\bar{\alpha}_p}(gu^{-1/p}) dx \lesssim \|M_\varphi(fv^{1/p})\|_{L^p} \|M_{\bar{\alpha}_p}(gu^{-1/p})\|_{L^{p'}} \\ &\lesssim \|f\|_{L^p(v)} \|g\|_{L^{p'}(u^{1-p'})}. \end{aligned}$$

This combined with (4.1) proves, by duality, the desired estimate for $A_{L(\log L)^m, \mathcal{S}}$, and therefore, Theorem 1.1 is proved.

The proof of Theorem 1.2 requires some preliminaries which we mention in the following subsection.

4.2. Auxiliary statements. Given a sparse family \mathcal{S} and a non-negative sequence $\{\tau_Q\}_{Q \in \mathcal{S}}$, consider the operator $T_{\mathcal{S}, \tau}$ defined by

$$T_{\mathcal{S}, \tau} f(x) = \sum_{Q \in \mathcal{S}} \tau_Q f_Q \chi_Q(x).$$

Given a cube R , denote $\mathcal{S}(R) = \{Q \in \mathcal{S} : Q \subseteq R\}$ and

$$T_{\mathcal{S}, \tau}^R f(x) = \sum_{Q \in \mathcal{S}(R)} \tau_Q f_Q \chi_Q(x).$$

The following result is due to M. Lacey, E. Sawyer and I. Uriarte-Tuero [22] (see also [13, 36] for different proofs).

Theorem 4.1. *Let $p > 1$. We have*

$$\|T_{\mathcal{S}, \tau}(\cdot\sigma)\|_{L^p(\sigma) \rightarrow L^p(u)} \sim \sup_{R \in \mathcal{S}} \frac{\|T_{\mathcal{S}, \tau}^R(\sigma)\|_{L^p(u)}}{\sigma(R)^{1/p}} + \sup_{R \in \mathcal{S}} \frac{\|T_{\mathcal{S}, \tau}^R(u)\|_{L^{p'}(\sigma)}}{u(R)^{1/p'}}.$$

Let $p > 1$. Suppose that $A \in B_p$ and φ is a decreasing function such that $\int_{1/2}^\infty \frac{1}{\varphi(t)^{p'}} \frac{dt}{t} < \infty$. In [19], M. Lacey established that the condition

$$(4.2) \quad \sup_Q (u_Q)^{1/p} \|\sigma^{1/p'}\|_{\bar{A}, Q} \varphi \left(\frac{\|\sigma^{1/p'}\|_{\bar{A}, Q}}{(\sigma_Q)^{1/p'}} \right) < \infty$$

implies that

$$\left\| \sum_{Q \in \mathcal{S}(R)} \sigma_Q \chi_Q \right\|_{L^p(u)} \lesssim \sigma(R)^{1/p}.$$

It was also shown in [19] that this result implies a particular case of the separated bump conjecture with logarithmic bumps proved in [10].

In [29], K. Li provided a different proof of a slightly stronger result where (4.2) is replaced by

$$(4.3) \quad \sup_Q (u_Q)^{1/p} \frac{\sigma_Q}{\|\sigma^{1/p}\|_{A,Q}} \varphi \left(\frac{(\sigma_Q)^{1/p}}{\|\sigma^{1/p}\|_{A,Q}} \right) < \infty.$$

Observe that, by Hölder's inequality,

$$\frac{(\sigma_Q)^{1/p}}{\|\sigma^{1/p}\|_{A,Q}} \leq 2 \frac{\|\sigma^{1/p'}\|_{\bar{A},Q}}{(\sigma_Q)^{1/p'}},$$

and therefore (4.2) is stronger than (4.3).

We will need the following extension of the above results.

Theorem 4.2. *Let $p > 1$, and let φ and ψ be increasing functions such that*

$$\int_{1/2}^{\infty} \left(\frac{1}{\varphi(t)^{p'}} + \frac{1}{\psi(t)^{p'}} \right) \frac{dt}{t} < \infty.$$

Let \mathcal{S} be a sparse family, and let $\{\lambda_Q\}_{Q \in \mathcal{S}}$ be a sequence such that $\lambda_Q \geq 1$ for every $Q \in \mathcal{S}$. If $A \in B_p$ and

$$K \equiv \sup_Q (u_Q)^{1/p} \lambda_Q \psi(\lambda_Q) \frac{\sigma_Q}{\|\sigma^{1/p}\|_{A,Q}} \varphi \left(\frac{(\sigma_Q)^{1/p}}{\|\sigma^{1/p}\|_{A,Q}} \right) < \infty,$$

then

$$\left\| \sum_{Q \in \mathcal{S}(R)} \lambda_Q \sigma_Q \chi_Q \right\|_{L^p(u)} \lesssim K \sigma(R)^{1/p}.$$

The proof of this result is a minor modification of the corresponding proof in [29]. In particular, as in [29], it is based on the two following statements.

Proposition 4.3. [3, Proposition 2.2] *let \mathcal{D} be a dyadic lattice, and let $p > 1$. For any non-negative sequence $\{a_Q\}_{Q \in \mathcal{D}}$ and for every weight w ,*

$$\left\| \sum_{Q \in \mathcal{D}} a_Q \chi_Q \right\|_{L^p(w)} \simeq \left(\sum_{Q \in \mathcal{D}} a_Q \left(\frac{1}{w(Q)} \sum_{Q' \in \mathcal{D}, Q' \subseteq Q} a_{Q'} w(Q') \right)^{p-1} w(Q) \right)^{1/p}.$$

Proposition 4.4. [13, Lemma 5.2] *Let \mathcal{S} be a sparse family, and let $0 < s < 1$. For every weight w ,*

$$\sum_{Q \in \mathcal{S}, Q \subseteq R} (w_Q)^s |Q| \lesssim (w_R)^s |R|.$$

Proof of Theorem 4.2. For $k, m \geq 0$ define the sets

$$\mathcal{S}_{k,m} = \left\{ Q \in \mathcal{S}(R) : 2^k \leq \lambda_Q \leq 2^{k+1}, 2^m \leq \frac{(\sigma_Q)^{1/p}}{\|\sigma^{1/p}\|_{A,Q}} \leq 2^{m+1} \right\}.$$

Then, applying Proposition 4.3 yields

$$(4.4) \quad \left\| \sum_{Q \in \mathcal{S}(R)} \lambda_Q \sigma_Q \chi_Q \right\|_{L^p(u)} \leq 2 \sum_{k,m \geq 0} 2^k \left\| \sum_{Q \in \mathcal{S}_{k,m}} \sigma_Q \chi_Q \right\|_{L^p(u)} \\ \lesssim \sum_{k,m \geq 0} 2^k \left(\sum_{Q \in \mathcal{S}_{k,m}} \sigma_Q \left(\frac{1}{u(Q)} \sum_{Q' \in \mathcal{S}_{k,m}, Q' \subseteq Q} \sigma_{Q'} u(Q') \right)^{p-1} u(Q) \right)^{1/p}.$$

Suppose first that $p \geq 2$. Then, by Proposition 4.4,

$$\sum_{Q' \in \mathcal{S}_{k,m}, Q' \subseteq Q} \sigma_{Q'} u(Q') = \sum_{Q' \in \mathcal{S}_{k,m}, Q' \subseteq Q} \sigma_{Q'} (u_{Q'})^{\frac{1}{p-1}} (u_{Q'})^{1-\frac{1}{p-1}} |Q'| \\ \lesssim \left(\frac{K}{2^k \psi(2^k) 2^m \varphi(2^m)} \right)^{p'} \sum_{Q' \in \mathcal{S}_{k,m}, Q' \subseteq Q} (u_{Q'})^{1-\frac{1}{p-1}} |Q'| \\ \lesssim \left(\frac{K}{2^k \psi(2^k) 2^m \varphi(2^m)} \right)^{p'} (u_Q)^{1-\frac{1}{p-1}} |Q|.$$

Therefore, by (4.4),

$$(4.5) \quad \left\| \sum_{Q \in \mathcal{S}(R)} \lambda_Q \sigma_Q \chi_Q \right\|_{L^p(u)} \lesssim K \sum_{k,m \geq 0} \frac{1}{\psi(2^k) 2^m \varphi(2^m)} \left(\sum_{Q \in \mathcal{S}_{k,m}} \sigma(Q) \right)^{1/p}.$$

From this, using that $\sigma(Q) \lesssim 2^{mp} \|\sigma^{1/p}\|_{A,Q}^p |Q|$ for $Q \in \mathcal{S}_{k,m}$, we obtain

$$\left\| \sum_{Q \in \mathcal{S}(R)} \lambda_Q \sigma_Q \chi_Q \right\|_{L^p(u)} \lesssim K \sum_{k,m \geq 0} \frac{1}{\psi(2^k) \varphi(2^m)} \left(\sum_{Q \in \mathcal{S}_{k,m}} \|\sigma^{1/p}\|_{A,Q}^p |Q| \right)^{1/p} \\ \lesssim K \left(\sum_{k,m \geq 0} \left(\frac{1}{\psi(2^k) \varphi(2^m)} \right)^{p'} \right)^{1/p'} \left(\sum_{k,m \geq 0} \sum_{Q \in \mathcal{S}_{k,m}} \|\sigma^{1/p}\|_{A,Q}^p |Q| \right)^{1/p} \\ \lesssim K \left(\int_{1/2}^{\infty} \frac{1}{\psi(t)^{p'}} dt \right)^{1/p'} \left(\int_{1/2}^{\infty} \frac{1}{\varphi(t)^{p'}} dt \right)^{1/p'} \left(\sum_{Q \in \mathcal{S}(R)} \|\sigma^{1/p}\|_{A,Q}^p |Q| \right)^{1/p} \\ \lesssim \left(\int_R M_A(\sigma^{1/p} \chi_R)^p \right)^{1/p} \lesssim \sigma(R)^{1/p}.$$

Consider now the case $1 < p < 2$. Then, by Proposition 4.4,

$$\begin{aligned} \sum_{Q' \in \mathcal{S}_{k,m}, Q' \subseteq Q} \sigma_{Q'} u(Q') &= \sum_{Q' \in \mathcal{S}_{k,m}, Q' \subseteq Q} (\sigma_{Q'})^{p-1} u_{Q'} (\sigma_{Q'})^{2-p} |Q'| \\ &\lesssim \left(\frac{K}{2^k \psi(2^k) 2^m \varphi(2^m)} \right)^p \sum_{Q' \in \mathcal{S}_{k,m}, Q' \subseteq Q} (\sigma_{Q'})^{2-p} |Q'| \\ &\lesssim \left(\frac{K}{2^k \psi(2^k) 2^m \varphi(2^m)} \right)^p (\sigma_Q)^{2-p} |Q|. \end{aligned}$$

Therefore, by (4.4),

$$\begin{aligned} &\left\| \sum_{Q \in \mathcal{S}(R)} \lambda_Q \sigma_Q \chi_Q \right\|_{L^p(u)} \\ &\lesssim \sum_{k,m \geq 0} 2^k \left(\frac{K}{2^k \psi(2^k) 2^m \varphi(2^m)} \right)^{p-1} \left(\sum_{Q \in \mathcal{S}_{k,m}} \sigma_Q ((\sigma_Q)^{p-1} u_Q)^{2-p} |Q| \right)^{1/p} \\ &\lesssim K \sum_{k,m \geq 0} \frac{1}{\psi(2^k) 2^m \varphi(2^m)} \left(\sum_{Q \in \mathcal{S}_{k,m}} \sigma(Q) \right)^{1/p}, \end{aligned}$$

and we again arrived at (4.5), which completes the proof. \square

4.3. Proof of Theorem 1.2. As before, by Lemma 3.3, it suffices to establish the first part of Theorem 1.2 for $A_{L(\log L)^m, \mathcal{S}}$.

Note that in [29], K. Li found a characterization of a similar inequality

$$\left\| \sum_{Q \in \mathcal{S}} \|f\|_{L^r, Q} \chi_Q \right\|_{L^p(u)} \lesssim \|f\|_{L^p(v)} \quad (1 < r < p).$$

We partially follow his approach.

It will be more convenient to deal with an equivalent form of the statement written as

$$(4.6) \quad \left\| \sum_{Q \in \mathcal{S}} \|f\sigma\|_{L(\log L)^m, Q} \chi_Q \right\|_{L^p(u)} \lesssim \|f\|_{L^p(\sigma)},$$

where $\sigma = v^{1-p'}$. Note that in terms of σ , the assumptions that

$$[u^{1/p}, v^{-1/p}]_{t^p, \beta_{p,m}} + [u^{1/p}, v^{-1/p}]_{\alpha_p, \gamma_{p,m}} < \infty$$

can be rewritten in the form

$$(4.7) \quad \sup_Q u_Q \|\sigma\|_{L(\log L)^{(m+1)p'-1+\delta}, Q}^{p-1} < \infty$$

and

$$(4.8) \quad \sup_Q \|u\|_{L(\log L)^{p-1+\delta}, Q} \|\sigma\|_{L(\log L)^{m(p'+\delta)}, Q}^{p-1} < \infty.$$

We will use the notation

$$\|f\|_{\varphi,Q}^{\sigma} = \inf \left\{ \lambda > 0 : \frac{1}{\sigma(Q)} \int_Q \varphi(|f(y)|/\lambda) \sigma(y) dy \right\}$$

and $f_{Q,\sigma} = \frac{1}{\sigma(Q)} \int_Q f \sigma$.

We start by observing that

$$\begin{aligned} (4.9) \quad \|f\sigma\|_{L(\log L)^m, Q} &\simeq \frac{1}{|Q|} \int_Q f \log^m \left(\frac{f\sigma}{(f\sigma)_Q} + e \right) \sigma \\ &= \frac{1}{|Q|} \int_Q f \log^m \left(\frac{f}{f_{Q,\sigma}} \frac{\sigma}{\sigma_Q} + e \right) \sigma \\ &\lesssim \frac{1}{|Q|} \int_Q f \log^m \left(\frac{f}{f_{Q,\sigma}} + e \right) \sigma + \frac{1}{|Q|} \int_Q f \log^m \left(\frac{\sigma}{\sigma_Q} + e \right) \sigma. \end{aligned}$$

Take $1 < r < p$, which will be specified later on. By Hölder's inequality,

$$\begin{aligned} \frac{1}{|Q|} \int_Q f \log^m \left(\frac{\sigma}{\sigma_Q} + e \right) \sigma &\leq \sigma_Q \|f\|_{L^r, Q}^{\sigma} \left(\frac{1}{\sigma(Q)} \int_Q \log^{mr'} \left(\frac{\sigma}{\sigma_Q} + e \right) \sigma \right)^{1/r'} \\ &\simeq \|\sigma\|_{L(\log L)^{mr'}, Q}^{1/r'} (\sigma_Q)^{1/r} \|f\|_{L^r, Q}^{\sigma}. \end{aligned}$$

Next,

$$\frac{1}{|Q|} \int_Q f \log^m \left(\frac{f}{f_{Q,\sigma}} + e \right) \sigma \simeq \sigma_Q \|f\|_{L(\log L)^m, Q}^{\sigma} \lesssim \|\sigma\|_{L(\log L)^{mr'}, Q}^{1/r'} (\sigma_Q)^{1/r} \|f\|_{L^r, Q}^{\sigma}.$$

Therefore, by (4.9),

$$\|f\sigma\|_{L(\log L)^m, Q} \lesssim \|\sigma\|_{L(\log L)^{mr'}, Q}^{1/r'} (\sigma_Q)^{1/r} \|f\|_{L^r, Q}^{\sigma}.$$

We obtain that (4.6) will follow from

$$(4.10) \quad \left\| \sum_{Q \in \mathcal{S}} \|\sigma\|_{L(\log L)^{mr'}, Q}^{1/r'} (\sigma_Q)^{1/r} \|f\|_{L^r, Q}^{\sigma} \chi_Q \right\|_{L^p(u)} \lesssim \|f\|_{L^p(\sigma)}.$$

Observe that (4.10) is equivalent to

$$(4.11) \quad \left\| \sum_{Q \in \mathcal{S}} \|\sigma\|_{L(\log L)^{mr'}, Q}^{1/r'} (\sigma_Q)^{1/r} \left(\frac{1}{\sigma(Q)} \int_Q f \sigma \right) \chi_Q \right\|_{L^p(u)} \lesssim \|f\|_{L^p(\sigma)}.$$

Indeed, on the one hand, (4.11) follows from (4.10) by Hölder's inequality. On the other hand, since

$$\|f\|_{L^r, Q}^{\sigma} \leq \frac{1}{\sigma(Q)} \int_Q (M_{r,\sigma}^{\mathcal{D}} f) \sigma,$$

and $M_{r,\sigma}^{\mathcal{D}}$ is bounded on $L^p(\sigma)$ (here is important that $r < p$), we obtain that (4.11) implies (4.10).

Denote

$$\lambda_Q = \left(\frac{\|\sigma\|_{L(\log L)^{mr'}, Q}}{\sigma_Q} \right)^{1/r'}.$$

By Theorems 4.1 and 4.2, in order to establish (4.11), it suffices to show that there exist $A \in B_p, B \in B_{p'}$ and functions $\varphi, \psi, \rho, \theta$ satisfying

$$\int_{1/2}^{\infty} \left(\frac{1}{\varphi(t)^{p'}} + \frac{1}{\psi(t)^{p'}} \right) \frac{dt}{t} < \infty \quad \text{and} \quad \int_{1/2}^{\infty} \left(\frac{1}{\rho(t)^p} + \frac{1}{\theta(t)^p} \right) \frac{dt}{t} < \infty$$

such that

$$(4.12) \quad \sup_Q (u_Q)^{1/p} \lambda_Q \psi(\lambda_Q) \|\sigma^{1/p'}\|_{\bar{A}, Q} \varphi \left(\frac{\|\sigma^{1/p'}\|_{\bar{A}, Q}}{(\sigma_Q)^{1/p'}} \right) < \infty$$

and

$$(4.13) \quad \sup_Q (\sigma_Q)^{1/p'} \lambda_Q \rho(\lambda_Q) \|u^{1/p}\|_{\bar{B}, Q} \theta \left(\frac{\|u^{1/p}\|_{\bar{B}, Q}}{(u_Q)^{1/p}} \right) < \infty.$$

We start by verifying (4.12). In what follows we introduce several parameters that will be fixed later on. Take $\varphi(t) = \psi(t) = \log(e+t)$. Next, let $A(t) = \frac{t^p}{\log^{1+\mu}(e+t)}$. Then $A \in B_p$ and $\bar{A}(t) \sim t^{p'} \log^{\frac{1+\mu}{p-1}}(e+t)$.

Take $0 < \nu < 1$ such that $\frac{1+\mu}{1-\nu} = 1 + 2\mu$. Then, by Hölder's inequality,

$$\|\sigma^{1/p'}\|_{\bar{A}, Q} \sim \|\sigma\|_{L(\log L)^{\frac{1+\mu}{p-1}}, Q}^{1/p'} \lesssim \left((\sigma_Q)^\nu \|\sigma\|_{L(\log L)^{\frac{1+2\mu}{p-1}}, Q}^{1-\nu} \right)^{1/p'}.$$

Hence, setting

$$t_Q = \frac{\|\sigma\|_{L(\log L)^{\frac{1+2\mu}{p-1}}, Q}}{\sigma_Q}$$

and using that $\sup_{t \geq 1} t^{-\frac{\nu}{p'}} \varphi(t^{\frac{1-\nu}{p'}}) < \infty$, we obtain

$$(4.14) \quad \begin{aligned} \|\sigma^{1/p'}\|_{\bar{A}, Q} \varphi \left(\frac{\|\sigma^{1/p'}\|_{\bar{A}, Q}}{(\sigma_Q)^{1/p'}} \right) &\lesssim \|\sigma\|_{L(\log L)^{\frac{1+2\mu}{p-1}}, Q}^{1/p'} t_Q^{-\frac{\nu}{p'}} \varphi(t_Q^{\frac{1-\nu}{p'}}) \\ &\lesssim \|\sigma\|_{L(\log L)^{\frac{1+2\mu}{p-1}}, Q}^{1/p'}. \end{aligned}$$

Similarly, by Hölder's inequality, if $s < r$, then

$$\left(\frac{\|\sigma\|_{L(\log L)^{mr'}, Q}}{\sigma_Q} \right)^{1/r'} \lesssim \left(\frac{\|\sigma\|_{L(\log L)^{ms'}, Q}}{\sigma_Q} \right)^{1/s'}.$$

Therefore, setting

$$\tau_Q = \frac{\|\sigma\|_{L(\log L)^{ms'}, Q}}{\sigma_Q}$$

and using that $\sup_{t \geq 1} t^{\frac{1}{s'} - \frac{1}{r'}} \psi(t^{1/s'}) < \infty$, we obtain

$$(4.15) \quad \begin{aligned} \lambda_Q \psi(\lambda_Q) &\lesssim \tau_Q^{1/s'} \psi(\tau_Q^{1/s'}) = \tau_Q^{1/r'} \tau_Q^{1/s' - 1/r'} \psi(\tau_Q^{1/s'}) \\ &\lesssim \left(\frac{\|\sigma\|_{L(\log L)^{ms'}, Q}}{\sigma_Q} \right)^{1/r'}. \end{aligned}$$

From this, and by (4.14), the left-hand side of (4.12) is controlled by

$$(4.16) \quad (u_Q)^{1/p} \|\sigma\|_{L(\log L)^{\frac{1+2\mu}{p-1}, Q}}^{1/p'} \left(\frac{\|\sigma\|_{L(\log L)^{ms'}, Q}}{\sigma_Q} \right)^{1/r'}.$$

Let $0 < q < 1$ and $s_0 < s$. Then, by Hölder's inequality, the expression in (4.16) is at most

$$(4.17) \quad (u_Q)^{1/p} \left((\sigma_Q)^q \|\sigma\|_{L(\log L)^{\frac{1+2\mu}{(p-1)(1-q)}, Q}}^{1-q} \right)^{1/p'} \left(\frac{\|\sigma\|_{L(\log L)^{ms'_0}, Q}}{\sigma_Q} \right)^{s'/s'_0 r'}.$$

We now fix the parameters in such a way that

$$\frac{q}{p'} = \frac{s'}{s'_0 r'} \quad \text{and} \quad \frac{1}{(p-1)(1-q)} = \frac{ms'_0 r'}{s'}.$$

It follows from this that

$$q = \frac{mp}{mp+1} \quad \text{and} \quad ms'_0 = \frac{s'}{r'} ((m+1)p' - 1).$$

Since $s_0 < s$, we have $s'_0 > s'$, and hence $r' < \frac{(m+1)p'-1}{m}$. Therefore, the additional assumption on r is that $\frac{mp+1}{m+1} < r$.

Let δ be a constant from condition (4.7). Take $s < r$ in such a way that

$$(4.18) \quad \frac{s'}{r'} ((m+1)p' - 1) \leq (m+1)p' - 1 + \delta.$$

Also fix $\mu > 0$ such that $\frac{2\mu}{(p-1)(1-q)} = \delta$. We obtain that the expression in (4.17) is at most

$$(u_Q)^{1/p} \|\sigma\|_{L(\log L)^{(m+1)p'-1+\delta}, Q}^{1/p'}$$

which, by (4.7), proves (4.12).

The proof of (4.13) is based on similar ideas. As before, set $\rho(t) = \theta(t) = \log(e+t)$. Next, let $B(t) = \frac{t^{p'}}{\log^{1+\mu}(e+t)}$. Then $B \in B_{p'}$ and $\bar{B}(t) \sim t^p \log^{(1+\mu)(p-1)}(e+t)$.

The same arguments as above show that

$$\|u^{1/p}\|_{\bar{B}, Q} \theta \left(\frac{\|u^{1/p}\|_{\bar{B}, Q}}{(u_Q)^{1/p}} \right) \lesssim \|u\|_{L(\log L)^{(1+2\mu)(p-1), Q}^{1/p}}.$$

From this and from (4.15) we obtain that the left-hand side of (4.13) is at most

$$(4.19) \quad \begin{aligned} & \|u\|_{L(\log L)^{(1+2\mu)(p-1),Q}}^{1/p} (\sigma_Q)^{1/p'} \left(\frac{\|\sigma\|_{L(\log L)^{ms'},Q}}{\sigma_Q} \right)^{1/r'} \\ & \lesssim \|u\|_{L(\log L)^{(1+2\mu)(p-1),Q}}^{1/p} \|\sigma\|_{L(\log L)^{ms'},Q}^{1/p'}. \end{aligned}$$

Observe that our current assumptions on s and r (guaranteeing that (4.12) holds) are $\frac{mp+1}{m+1} < r < p$ and $s < r$ such that (4.18) holds. We now assume additionally that s and r are so close to p that $s' \leq p' + \delta$. Fix also μ such that $2\mu(p-1) = \delta$. Then we obtain that the expression in (4.19) is controlled by condition (4.8), and therefore, Theorem 1.2 is proved.

Proof of Corollary 1.3. Recall that

$$\beta_{p,m}(t) = t^{p'} \log^{(m+1)p'-1+\delta}(e+t), \quad \gamma_{p,m}(t) = t^{p'} \log^{m(p'+\delta)}(e+t)$$

and

$$\psi_{p,m}(t) = t^{p'} \log^{\max((m+1)p'-1, mp'+1)+\varepsilon}(e+t).$$

It suffices to prove that, with suitable choice of δ ,

$$[u^{1/p}, v^{-1/p}]_{t^p, \beta_{p,m}} + [u^{1/p}, v^{-1/p}]_{\alpha_p, \gamma_{p,m}} \lesssim [u^{1/p}, v^{-1/p}]_{t^p, \psi_{p,m}} + [u^{1/p}, v^{-1/p}]_{\psi_{p',m}, t^{p'}}.$$

This would provide the estimate for $\|A_{L(\log L)^m, S}\|_{L^p(v) \rightarrow L^p(u)}$. Since the right-hand side here is self-dual, from this and from Lemma 3.3 we obtain the desired bound for T_b^m .

Observe that for $\delta \leq \varepsilon$ we have $\beta_{p,m} \leq \psi_{p,m}$. Therefore,

$$[u^{1/p}, v^{-1/p}]_{t^p, \beta_{p,m}} \leq [u^{1/p}, v^{-1/p}]_{t^p, \psi_{p,m}}.$$

Hence, the result will follow if we show that

$$(4.20) \quad [u^{1/p}, v^{-1/p}]_{\alpha_p, \gamma_{p,m}} \lesssim [u^{1/p}, v^{-1/p}]_{t^p, \psi_{p,m}} + [u^{1/p}, v^{-1/p}]_{\psi_{p',m}, t^{p'}}.$$

By Hölder's inequality, for $0 < \alpha < 1$,

$$\|u^{1/p}\|_{L^p(\log L)^{p-1+\delta}, Q} \lesssim \|u^{1/p}\|_{L^p, Q}^\alpha \|u^{1/p}\|_{L^p(\log L)^{\frac{p-1+\delta}{1-\alpha}, Q}^{1-\alpha}}$$

and

$$\|v^{-1/p}\|_{L^{p'}(\log L)^{m(p'+\delta)}, Q} \lesssim \|v^{-1/p}\|_{L^{p'}, Q}^{1-\alpha} \|v^{-1/p}\|_{L^{p'}(\log L)^{\frac{m(p'+\delta)}{\alpha}, Q}^\alpha.$$

Therefore, the left-hand side of (4.20) is at most

$$(4.21) \quad \left(\|u^{1/p}\|_{L^p, Q} \|v^{-1/p}\|_{L^{p'}(\log L)^{\frac{m(p'+\delta)}{\alpha}, Q}} \right)^\alpha \left(\|v^{-1/p}\|_{L^{p'}, Q} \|u^{1/p}\|_{L^p(\log L)^{\frac{p-1+\delta}{1-\alpha}, Q}} \right)^{1-\alpha}.$$

Fix α in such a way that $\frac{p-1}{1-\alpha} = (m+1)p-1$. Then $\alpha = \frac{mp}{mp+p-1}$ and $\frac{mp'}{\alpha} = mp'+1$. Hence, taking δ such that $\frac{\delta m}{\alpha} \leq \varepsilon$ and $\frac{\delta}{1-\alpha} \leq \varepsilon$, we obtain that the expression in (4.21) is bounded by the right-hand side of (4.20), and therefore, the proof is complete. \square

5. A NECESSARY CONDITION

5.1. On a theorem of P. Jones. In [18], P. Jones established a rather general result allowing to decide whether a function from $BMO(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is a connected open set, can be extended to a function from $BMO(\mathbb{R}^n)$.

We will need a particular version of this result when $\Omega = Q$ is a cube. Observe that the proof of a general result in [18] is long and involved. In the particular case we need, it is much simpler. Therefore we outline the proof below.

Theorem 5.1. *Assume that $f \in BMO$, and let R be a cube such that $f_R = 0$. Then there exists a function φ such that $\varphi = f$ on R , $\varphi = 0$ on $\mathbb{R}^n \setminus 2R$ and*

$$\|\varphi\|_{BMO} \lesssim \|f\|_{BMO}.$$

Remark 5.2. In the one-dimensional case this statement can be found in [12, Ex. 3.1.10, p. 167].

Remark 5.3. The proof we give is an adaptation of the method in [18]. In particular, as in [18], we shall make use of the Whitney covering theorem. We refer to [1, p. 348] for the statement and various properties of Whitney's cubes.

Proof of Theorem 5.1. Let $E = \{Q_j\}$ and $E' = \{Q'_j\}$ be the Whitney coverings of the interiors of $\mathbb{R}^n \setminus R$ and R , respectively.

Take $\alpha = \alpha_n$ with the following property: for every $Q_j \in E$ with $\ell_{Q_j} \leq \alpha \ell_R$ we have $Q_j \subset 2R$, and, moreover, there exists the nearest cube $P'_j \in E'$ such that $|P'_j| \geq |Q_j|$. Denote

$$F = \{Q_j \in E : \ell_{Q_j} \leq \alpha \ell_R\},$$

and define

$$\varphi = f\chi_R + \sum_{Q_j \in F} f_{P'_j}\chi_{Q_j}.$$

Observe that each cube $P'_j \in E'$ may appear in this sum not more than $k = k_n$ times.

Denote $\tilde{R} = R \cup (\cup_{Q_j \in F} Q_j)$. To prove that $\|\varphi\|_{BMO} \lesssim \|f\|_{BMO}$, it suffices to show that for every cube Q , there exists $c \in \mathbb{R}$ such that

$$(5.1) \quad \frac{1}{|Q|} \int_{Q \cap R} |f - c| + \sum_{Q_j \in F} \frac{|Q \cap Q_j|}{|Q||P'_j|} \left| \int_{P'_j} (f - c) \right| + \frac{|Q \setminus \tilde{R}|}{|Q|} |c| \lesssim \|f\|_{BMO}.$$

Denote $A = \{Q_j \in F : Q \cap Q_j \neq \emptyset\}$. If $A = \emptyset$, then either $Q \subset R$ or $Q \subset \mathbb{R}^n \setminus \tilde{R}$, and this case is trivial. Therefore, suppose that $A \neq \emptyset$. There are two main cases.

(i) Suppose that $\ell_{Q_j} \leq 4\ell_Q$ for every $Q_j \in A$.

If $|Q \setminus \tilde{R}| > 0$, then there exists $Q_j \in A$ such that $\text{dist}(Q_j, \partial R) \sim \text{diam} R$, and hence $|R| \lesssim |Q|$. In this case we take $c = f_R = 0$. Then the left-hand

side of (5.1) is bounded by

$$\frac{1}{|R|} \int_R |f - f_R| + \frac{1}{|R|} \sum_{Q_j \in F} \int_{P'_j} |f - f_R| \lesssim \frac{1}{|R|} \int_R |f - f_R| \lesssim \|f\|_{BMO}.$$

Suppose that $|Q \setminus \tilde{R}| = 0$. It follows from the definition of P'_j that for every $Q_j \in A$ with $\ell_{Q_j} \leq 4\ell_Q$ the corresponding P'_j is contained in βQ , where $\beta = \beta_n$. Therefore, taking $c = f_{\beta Q}$, we obtain that the left-hand side of (5.1) is bounded by

$$\frac{1}{|Q|} \int_{\beta Q} |f - f_{\beta Q}| + \frac{1}{|Q|} \sum_{Q_j \in F} \int_{P'_j} |f - f_{\beta Q}| \lesssim \frac{1}{|Q|} \int_{\beta Q} |f - f_{\beta Q}| \lesssim \|f\|_{BMO}.$$

- (ii) Suppose that there exists $Q_{j_0} \in A$ such that $\ell_Q < \frac{1}{4}\ell_{Q_{j_0}}$. Then $Q \subset \frac{3}{2}Q_{j_0}$, and hence $Q \cap R = \emptyset$. It follows from the properties of Whitney cubes that every other cube $Q_j \in A$ touches Q_{j_0} , and therefore $|Q_j| \sim |Q_{j_0}|$, and the corresponding cube P'_j is contained in γQ_{j_0} , where $\gamma = \gamma_n$.

Now, if $|Q \setminus \tilde{R}| = 0$, we take $c = f_{\gamma Q_{j_0}}$. Then the left-hand side of (5.1) is bounded by

$$\frac{1}{|Q_{j_0}|} \sum_{Q_j \in F} \int_{P'_j} |f - f_{\gamma Q_{j_0}}| \lesssim \frac{1}{|Q_{j_0}|} \int_{\gamma Q_{j_0}} |f - f_{\gamma Q_{j_0}}| \lesssim \|f\|_{BMO}.$$

If $|Q \setminus \tilde{R}| \neq 0$, then $|Q_{j_0}| \sim |R|$. In this case, taking $c = f_R$, we obtain that the left-hand side of (5.1) is bounded by

$$\frac{1}{|Q_{j_0}|} \int_{\gamma Q_{j_0}} |f - f_R| \lesssim \frac{1}{|R|} \int_{2\gamma R} |f - f_{2\gamma R}| \lesssim \|f\|_{BMO}.$$

This completes the proof of (5.1), and therefore, the theorem is proved. \square

5.2. Proof of Theorem 1.7. We start by observing that, by duality, the estimates

$$\|T_b^m f\|_{L^{p,\infty}(u)} \lesssim \|b\|_{BMO} \|f\|_{L^p(v)}$$

and

$$(5.2) \quad \|(T_b^m)^* f\|_{L^{p'(v^{1-p'})}} \lesssim \|b\|_{BMO} \|f/u\|_{L^{p',1}(u)}$$

are equivalent.

Note that

$$T_b^m f(x) = \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x, y) f(y) dy \quad (x \notin \text{supp } f).$$

Hence $(T_b^m)^*$ is essentially the same operator but associated with $\tilde{K}(x, y) = K(y, x)$. Since \tilde{K} is non-degenerate (by our definition of a non-degenerate Calderón-Zygmund operator), it suffices to prove the theorem assuming that (5.2) holds for T_b^m instead of $(T_b^m)^*$.

Next, by the reasons explained in Section 2.3, (1.7) is equivalent to

$$(5.3) \quad \sup_Q \left(\frac{1}{|Q|} \int_Q u \right) \left(\frac{1}{|Q|} \int_Q v^{1-p'} \log^{mp'} \left(\frac{v^{1-p'}}{(v^{1-p'})_Q} + e \right) \right)^{p-1} < \infty.$$

Let Q be an arbitrary cube. Define

$$g(x) = \log^+ \left(\frac{M(v^{1-p'} \chi_Q)(x)}{(v^{1-p'})_Q} \right).$$

It is well known [4] that $g \in BMO$ and $\|g\|_{BMO} \lesssim 1$. Also, using that (see, e.g., [11, Ex. 2.1.5, p. 100])

$$\int_Q (M(f \chi_Q))^\delta \lesssim \left(\frac{1}{|Q|} \int_Q |f| \right)^\delta |Q| \quad (0 < \delta < 1),$$

we obtain

$$(5.4) \quad g_Q \lesssim 1.$$

By Theorem 5.1, there exists a function φ such that $\varphi = g - g_Q$ on Q and $\varphi = 0$ outside $2Q$, and $\|\varphi\|_{BMO} \lesssim 1$. Let B be the ball concentric with Q or radius $r = \text{diam } Q$. In accordance with Proposition 2.1, take the corresponding ball \tilde{B} of the same radius at distance $\text{dist}(B, \tilde{B}) \simeq Ar$, where $A \geq 3$ will be chosen later.

Let f be a non-negative function supported in \tilde{B} . Set $b = \varphi$. Observe that b is supported in B , and hence $b = 0$ on \tilde{B} . Thus, for $x \in B$,

$$\begin{aligned} T_b^m f(x) &= \int_{\tilde{B}} (b(x) - b(y))^m K(x, y) f(y) dy \\ &= \varphi^m(x) \int_{\tilde{B}} K(x, y) f(y) dy. \end{aligned}$$

Therefore, by (5.2) (with T_b^m),

$$\int_B \left| \int_{\tilde{B}} K(x, y) f(y) dy \right|^{p'} |\varphi|^{mp'} v^{1-p'} dx \lesssim \|f \chi_{\tilde{B}}/u\|_{L^{p',1}(u)}^{p'}.$$

From this, and by Proposition 2.1,

$$\begin{aligned} & \frac{1}{A^n} \left(\int_B |\varphi|^{mp'} v^{1-p'} \right)^{1/p'} f_{\tilde{B}} \lesssim \left(\int_B \left| \int_{\tilde{B}} K(x_0, y_0) f(y) dy \right|^{p'} |\varphi|^{mp'} v^{1-p'} dx \right)^{1/p'} \\ & \leq \left(\int_B \left(\int_{\tilde{B}} |K(x, y) - K(x_0, y_0)| f(y) dy \right)^{p'} |\varphi|^{mp'} v^{1-p'} dx \right)^{1/p'} \\ & \quad + \left(\int_B \left| \int_{\tilde{B}} K(x, y) f(y) dy \right|^{p'} |\varphi|^{mp'} v^{1-p'} dx \right)^{1/p'} \\ & \lesssim \frac{\varepsilon_A}{A^n} \left(\int_B |\varphi|^{mp'} v^{1-p'} \right)^{1/p'} f_{\tilde{B}} + \|f \chi_{\tilde{B}}/u\|_{L^{p',1}(u)}. \end{aligned}$$

Therefore, taking A large enough, we obtain

$$(5.5) \quad \left(\int_B |\varphi|^{mp'} v^{1-p'} \right)^{1/p'} f_{\tilde{B}} \lesssim \|f \chi_{\tilde{B}}/u\|_{L^{p',1}(u)}.$$

Setting here $f = u$ and using that $\|\chi_{\tilde{B}}\|_{L^{p',1}(u)} \simeq (\int_{\tilde{B}} u)^{1/p'}$ yields

$$\left(\frac{1}{|\tilde{B}|} \int_{\tilde{B}} u \right) \left(\int_B |\varphi|^{mp'} v^{1-p'} \right)^{p-1} \lesssim 1.$$

Note that $|\tilde{B}| \simeq |Q|$ and $Q \subset \gamma\tilde{B}$, where γ depends only on A and n . Therefore, since u is doubling (recall that this means that there exists $c > 0$ such that $u(2Q) \leq cu(Q)$ for every cube Q), we obtain

$$(5.6) \quad \left(\frac{1}{|Q|} \int_Q u \right) \left(\int_Q |g - g_Q|^{mp'} v^{1-p'} \right)^{p-1} \lesssim 1.$$

We now observe that the same proof with the choice $b = \chi_B$ shows that (5.6) holds with $m = 0$. Combining this with (5.4) yields

$$\left(\frac{1}{|Q|} \int_Q u \right) \left(\int_Q g^{mp'} v^{1-p'} \right)^{p-1} \lesssim 1,$$

which, in turn, implies (5.3), and therefore, the theorem is proved.

Remark 5.4. As we have mentioned in the Introduction, if T is non-degenerate Calderón-Zygmund operator with Dini-continuous kernel, and if both weights u and $v^{1-p'}$ are doubling, then $T_b^{m+1} : L^p(v) \rightarrow L^p(u)$ implies $T_b^m : L^p(v) \rightarrow L^p(u)$. We add two remarks here. First, it is easy to see that under the above assumptions we actually obtain that $A_S^{m+1} : L^p(v) \rightarrow L^p(u)$, and therefore in the conclusion $T_b^m : L^p(v) \rightarrow L^p(u)$, T can be replaced by any Calderón-Zygmund operator with Dini-continuous kernel.

Second, the assumptions that both u and $v^{1-p'}$ are doubling can be replaced by that either u or $v^{1-p'}$ belongs to A_∞ . Indeed, assume, for example, that $u \in A_\infty$ and that $T_b^{m+1} : L^p(v) \rightarrow L^p(u)$. Then, by Theorem 1.7,

$$\sup_Q \|u^{1/p}\|_{L^p,Q} \|v^{-1/p}\|_{L^{p'(\log L)^{(m+1)p'},Q}} < \infty.$$

Since $u \in A_\infty$, it satisfies the reverse Hölder inequality, and therefore, the above condition can be self-improved to

$$\sup_Q \|u^{1/p}\|_{L^{rp},Q} \|v^{-1/p}\|_{L^{p'(\log L)^{(m+1)p'},Q}} < \infty$$

with some $r > 1$. It is easy to see that this condition is stronger than the assumptions of Theorem 1.2, and therefore $T_b^m : L^p(v) \rightarrow L^p(u)$.

6. ON A CONVERSE TO BLOOM'S THEOREM

Throughout this section we assume that T is a non-degenerate Calderón-Zygmund operator with Dini-continuous kernel, and $m \in \mathbb{N}$.

As we announced in the introduction, we obtain several partial results related to Conjecture 1.9, being the first of them the following theorem.

Theorem 6.1. *Let $\lambda, \mu \in A_p, p > 1$. Let η be an arbitrary weight such that*

$$(6.1) \quad b \in BMO_\eta \Rightarrow \|T_b^m\|_{L^p(\lambda)} \lesssim \|b\|_{BMO_\eta}^m \|f\|_{L^p(\mu)}$$

and

$$(6.2) \quad \|T_b^m f\|_{L^p(\lambda)} \lesssim \|f\|_{L^p(\mu)} \Rightarrow b \in BMO_\eta.$$

Then $\eta \simeq \left(\frac{\mu}{\lambda}\right)^{1/pm}$ almost everywhere.

In the following theorem we assume that $\eta = 1$.

Theorem 6.2. *Let $p > 1$. Let λ and μ be the weights satisfying either one of the following conditions:*

- (i) $\lambda \in A_p$ and μ is an arbitrary weight;
- (ii) $\lambda \in A_\infty$ and $\mu^{1-p'} \in A_\infty$.

Suppose also that

$$(6.3) \quad b \in BMO \Rightarrow \|T_b^m\|_{L^p(\lambda)} \lesssim \|b\|_{BMO}^m \|f\|_{L^p(\mu)}$$

and

$$(6.4) \quad \|T_b^m f\|_{L^p(\lambda)} \lesssim \|f\|_{L^p(\mu)} \Rightarrow b \in BMO.$$

Then $\lambda \simeq \mu$ almost everywhere and $\lambda, \mu \in A_p$.

As we will see below, Theorem 6.2 is more difficult than Theorem 6.1. In particular, in the simplest case when $\lambda = 1$, μ is an arbitrary weight and $m = 1$ this result says that the implication $b \in BMO \Leftrightarrow \|T_b^1 f\|_{L^p} \lesssim \|f\|_{L^p(\mu)}$ holds if and only if $\mu \sim 1$. Even in such a simple form this result seems to be new.

6.1. Auxiliary propositions. We first recall several standard properties of A_p weights (see, e.g., [11, Ch. 9]):

- (i) if $w \in A_p$, then w is a doubling weight;
- (ii) if $w \in A_p$, then $w \in A_\infty$, which means that there exist $c, \rho > 0$ such that for every cube Q and any subset $E \subset Q$,

$$(6.5) \quad w(E) \leq c \left(\frac{|E|}{|Q|}\right)^\rho w(Q);$$

- (iii) if $w \in A_p$, then for every cube Q and any subset $E \subset Q$,

$$(6.6) \quad |E| \leq [w]_{A_p}^{1/p} \left(\frac{w(E)}{w(Q)}\right)^{1/p} |Q|,$$

where

$$[w]_{A_p} = \sup_Q \left(\frac{1}{|Q|} \int_Q w \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}} \right)^{p-1};$$

(iv) if $\lambda \in A_p$, then there exists $\varepsilon > 0$ such that $\lambda^{1+\varepsilon} \in A_p$.

Proposition 6.3. *Let λ and μ be arbitrary weights such that (6.1) holds. Then for every ball $B = B(y_0, r)$, there is a disjoint ball $\tilde{B} = B(x_0, r)$ at distance $\text{dist}(B, \tilde{B}) \simeq r$ such that for any $f \geq 0$*

$$(6.7) \quad (f_B)^p (\eta^{pm} \lambda)(\tilde{B}) \lesssim \int_B f^p \mu.$$

Proof. Let B be an arbitrary ball, and let \tilde{B} be the corresponding ball from Proposition 2.1. Set $b = \eta \chi_B$. We trivially have that $\|b\|_{BMO_\eta} \leq 2$. Then, following exactly the same argument led to (5.5), we obtain (6.7). \square

Corollary 6.4. *Let λ and μ be arbitrary weights such that (6.1) holds. Then*

$$\lambda \eta^{pm} \lesssim \mu$$

almost everywhere.

Proof. This follows immediately by taking $f = 1$ in (6.7) and applying the Lebesgue differentiation theorem. \square

Corollary 6.5. *Assume that λ and μ satisfy the conditions of Theorem 6.2. Then $(\lambda, \mu) \in A_p$.*

Proof. This follows at once from (6.7) (with $\eta = 1$) by taking $f = \mu^{1-p'}$ and using that λ is doubling. \square

Proposition 6.6. *Let v be a weight such that $v \notin L^\infty$. Then there exists a sequence $\alpha_k \uparrow \infty$ with the following property: for every sequence $0 < \delta_k < 1$ there exist pairwise disjoint cubes Q_k and subsets $E_k \subset Q_k$ with $|E_k| \geq \delta_k |Q_k|$ such that $\alpha_k < v \leq 2\alpha_k$ on E_k .*

Proof. For $j \in \mathbb{N}$ denote

$$\Omega_j = \{x : 2^j < v \leq 2^{j+1}\}.$$

Since v is unbounded, there is a subsequence $j_k \rightarrow \infty$ for which the sets Ω_{j_k} have positive measure. Denote $\alpha_k = 2^{j_k}$.

By the standard density argument, there exist cubes Q_k such that $|Q_k \cap \Omega_{j_k}| \geq \delta_k |Q_k|$. Moreover, since the sets Ω_{j_k} are pairwise disjoint, the cubes Q_k can be taken pairwise disjoint as well (taking them as small as necessary). Setting $E_k = Q_k \cap \Omega_{j_k}$ completes the proof. \square

Lemma 6.7. *Let η_1, η_2 be the weights such that $\frac{\eta_1}{\eta_2} \notin L^\infty$. Then there exists $b \in BMO_{\eta_1} \setminus BMO_{\eta_2}$.*

Proof. Apply Proposition 6.6 to the weight $v = \frac{\eta_1}{\eta_2}$ with $\delta_k = \frac{1}{2}$. We obtain that there exists an unbounded sequence α_k and there are sequences of pairwise disjoint cubes Q_k and measurable subsets $E_k \subset Q_k$ such that $|E_k| \geq \frac{1}{2}|Q_k|$ and $\alpha_k < \frac{\eta_1}{\eta_2} \leq 2\alpha_k$ on E_k .

Next, there exists $i_k \in \mathbb{Z}$ such that the set

$$E'_k = \{x \in E_k : 2^{i_k} < \eta_2(x) \leq 2^{i_k+1}\}$$

has positive measure. Denote $\beta_k = 2^{i_k}$.

Let x_k be the density point of E'_k and the Lebesgue point of η_2 . Then there exists a cube R_k centered at x_k and containing in Q_k such that $\eta_2(R_k \cap E'_k) \geq \frac{1}{2}\eta_2(R_k)$ and $|R_k \cap E'_k| \geq \frac{1}{2}|R_k|$.

Choose $A_k \subset R_k \cap E'_k$ such that $|A_k| = \frac{1}{2}|R_k|$, and set

$$b = \sum_k \alpha_k \beta_k \chi_{A_k}.$$

Since $b \leq \eta_1$, we trivially obtain that $b \in BMO_{\eta_1}$.

Let us show that $b \notin BMO_{\eta_2}$. Note that $b_{R_k} = \frac{1}{2}\alpha_k \beta_k$. Hence

$$\begin{aligned} \frac{1}{\eta_2(R_k)} \int_{R_k} |b(x) - b_{R_k}| dx &\geq \frac{1}{\eta_2(R_k)} \int_{A_k} |\alpha_k \beta_k - \frac{1}{2}\alpha_k \beta_k| dx \\ &= \frac{1}{4} \alpha_k \beta_k \frac{|R_k|}{\eta_2(R_k)}. \end{aligned}$$

On the other hand,

$$\eta_2(R_k) \leq 2\eta_2(R_k \cap E'_k) \leq 4\beta_k |R_k|.$$

Therefore,

$$\frac{1}{\eta_2(R_k)} \int_{R_k} |b(x) - b_{R_k}| dx \geq \frac{1}{16} \alpha_k.$$

This completes the proof since α_k is unbounded. \square

Lemma 6.8. *Let $\lambda \in A_p$. Assume that μ is a weight such that $\lambda \leq \mu$ and $\frac{\mu}{\lambda} \notin L^\infty$. Then there exists a weight $u \notin L^\infty$ such that $(\lambda u, \mu) \in A_p$.*

Proof. Apply Proposition 6.6 to the weight $v = \frac{\mu}{\lambda}$. We obtain the corresponding sequences α_k, δ_k, Q_k and E_k . Denote $G_k = Q_k \setminus E_k$. Then $|G_k| \leq (1 - \delta_k)|Q_k|$. Set also $\sigma_\lambda = \lambda^{-\frac{1}{p-1}}$.

Let us show that taking suitable sequence δ_k one can choose the sets $A_k \subset \frac{1}{2}Q_k$ of positive measure and satisfying the following properties:

- (i) $|A_k| = \gamma_k |Q_k|$, where $\sum_k \alpha_k \gamma_k^\rho < \infty$, and ρ is the constant from the A_∞ property (6.5) with $w = \lambda$;
- (ii) if $Q \subset Q_k$ and $Q \cap A_k \neq \emptyset$, then

$$(6.8) \quad \sigma_\lambda(Q \cap G_k) \leq \alpha_k^{-\frac{1}{p-1}} \sigma_\lambda(Q).$$

Define the weighted local maximal operator

$$M_{\sigma_\lambda, Q_k} f(x) = \sup_{Q \ni x, Q \subset Q_k} \frac{1}{\sigma_\lambda(Q)} \int_Q |f| \sigma_\lambda,$$

and consider the sets

$$B_k = \{x \in Q_k : M_{\sigma_\lambda, Q_k} \chi_{G_k}(x) > \alpha_k^{-\frac{1}{p-1}}\}.$$

Observe that $\sigma_\lambda \in A_{p'}$, and therefore it is a doubling weight. Thus, by the weighted weak type estimate for M_{σ_λ} along with (6.5),

$$\sigma_\lambda(B_k) \leq c_{n,\lambda} \alpha_k^{\frac{1}{p-1}} \sigma_\lambda(G_k) \leq c'_{n,\lambda} \alpha_k^{\frac{1}{p-1}} (1 - \delta_k)^\varepsilon \sigma_\lambda(Q_k).$$

Hence, by (6.6),

$$\begin{aligned} |B_k| &\leq [\sigma_\lambda]_{A_{p'}}^{1/p'} \left(c'_{n,\lambda} \alpha_k^{\frac{1}{p-1}} (1 - \delta_k)^\varepsilon \right)^{1/p'} |Q_k| \\ &= (c'_{n,\lambda})^{1/p'} (\alpha_k [w]_{A_p})^{1/p} (1 - \delta_k)^{\varepsilon/p'} |Q_k|. \end{aligned}$$

Take now δ_k such that $(c'_{n,\lambda})^{1/p'} (\alpha_k [w]_{A_p})^{1/p} (1 - \delta_k)^{\varepsilon/p'} = \frac{1}{2^{n+1}}$.

We have that $|B_k| \leq \frac{1}{2^{n+1}} |Q_k|$. Take γ_k such that $\sum_k \alpha_k \gamma_k^\rho < \infty$ and $\gamma_k < \frac{1}{2^{n+1}}$ for all k . Then there exists $A_k \subset \frac{1}{2} Q_k$ such that $|A_k| = \gamma_k |Q_k|$ and $A_k \cap B_k = \emptyset$. We have that property (i) is satisfied, and property (ii) holds as well: if $Q \subset Q_k$ and $Q \cap A_k \neq \emptyset$, then $Q \cap (Q_k \setminus B_k) \neq \emptyset$, and hence, by the definition of B_k , (6.8) holds.

Taking the sets A_k that satisfy properties (i) and (ii), define

$$u = \sum_k \alpha_k \chi_{A_k} + \chi_{\mathbb{R}^n \setminus \cup_k A_k}.$$

Let us show that $(\lambda u, \mu) \in A_p$.

Denote

$$F(Q) = \left(\frac{1}{|Q|} \int_Q \lambda u \right) \left(\frac{1}{|Q|} \int_Q \mu^{-\frac{1}{p-1}} \right)^{p-1}.$$

Assume that Q is not contained in any Q_k and $Q \cap \frac{1}{2} Q_k = \emptyset$ for every k . Then

$$F(Q) = \left(\frac{1}{|Q|} \int_Q \lambda \right) \left(\frac{1}{|Q|} \int_Q \mu^{-\frac{1}{p-1}} \right)^{p-1} \leq [\lambda]_{A_p}.$$

Assume that Q is not contained in any Q_k and the set

$$\mathcal{K} = \{k : Q \not\subset Q_k, Q \cap \frac{1}{2} Q_k \neq \emptyset\}$$

is not empty. Then $Q_k \subset 7Q$ for every $k \in \mathcal{K}$. We obtain

$$(6.9) \quad \int_Q \lambda u \leq \sum_{k \in \mathcal{K}} \alpha_k \int_{A_k} \lambda + \int_Q \lambda.$$

Applying (6.5) along with the doubling property of λ and (i) yields

$$\begin{aligned} \sum_{k \in \mathcal{K}} \alpha_k \int_{A_k} \lambda &\leq c \sum_{k \in \mathcal{K}} \alpha_k \gamma_k^\rho \int_{Q_k} \lambda \\ &\leq c \left(\sum_k \alpha_k \gamma_k^\rho \right) \int_{7Q} \lambda \leq c' \int_Q \lambda. \end{aligned}$$

Combining this with (6.9), we obtain $\int_Q \lambda u \leq c \int_Q \lambda$, which implies

$$F(Q) \leq c[\lambda]_{A_p}.$$

It remains to consider the case when there exists k such that $Q \subset Q_k$. Observe that

$$(6.10) \quad \frac{1}{|Q|} \int_Q \lambda u \leq \frac{1}{|Q|} \int_Q \lambda + \left(\frac{1}{|Q|} \int_{Q \cap A_k} \lambda \right) \alpha_k$$

and

$$(6.11) \quad \frac{1}{|Q|} \int_Q \mu^{-\frac{1}{p-1}} \leq \alpha_k^{-\frac{1}{p-1}} \frac{1}{|Q|} \int_Q \lambda^{-\frac{1}{p-1}} + \frac{1}{|Q|} \int_{Q \cap G_k} \lambda^{-\frac{1}{p-1}}.$$

If

$$\left(\frac{1}{|Q|} \int_{Q \cap A_k} \lambda \right) \alpha_k \leq \frac{1}{|Q|} \int_Q \lambda,$$

then, by (6.10),

$$F(Q) \leq 2[\lambda]_{A_p}.$$

Assume that

$$\frac{1}{|Q|} \int_Q \lambda < \left(\frac{1}{|Q|} \int_{Q \cap A_k} \lambda \right) \alpha_k.$$

Then $Q \cap A_k \neq \emptyset$. Hence, by property (ii) along with (6.10) and (6.11),

$$\frac{1}{|Q|} \int_Q \lambda u \leq 2\alpha_k \frac{1}{|Q|} \int_Q \lambda \quad \text{and} \quad \frac{1}{|Q|} \int_Q \mu^{-\frac{1}{p-1}} \leq 2\alpha_k^{-\frac{1}{p-1}} \frac{1}{|Q|} \int_Q \lambda^{-\frac{1}{p-1}}.$$

From this,

$$F(Q) \leq 2^p[\lambda]_{A_p},$$

and therefore, the proof is complete. \square

Property (iv) of A_p weights mentioned in the beginning of this section along with Lemma 6.8 implies the following.

Corollary 6.9. *Let $\lambda \in A_p$. Assume that μ is a weight such that $\lambda \leq \mu$ and $\frac{\mu}{\lambda} \notin L^\infty$. Then there exist $\varepsilon > 0$ and a weight $u \notin L^\infty$ such that $((\lambda u)^{1+\varepsilon}, \mu^{1+\varepsilon}) \in A_p$.*

Finally, an important role in our proofs will be played by the following result of C. Neugebauer [32].

Theorem 6.10. *Let λ and μ be the weights such that $(\lambda^r, \mu^r) \in A_p, p > 1$, for some $r > 0$. Then there exists a weight $w \in A_p$ such that*

$$\lambda \lesssim w \lesssim \mu$$

almost everywhere.

6.2. Proofs of Theorems 6.1 and 6.2.

Proof of Theorem 6.1. By Corollary 6.4, it remains to prove that

$$\mu \lesssim \lambda \eta^{pm}$$

almost everywhere. Assume that this is not true. Define $\tilde{\eta} = (\mu/\lambda)^{1/mp}$. Then $\tilde{\eta}/\eta \notin L^\infty$.

In order to get a contradiction, it suffices to show that

$$(6.12) \quad \|A_S(A_{S,\tilde{\eta}}^m f)\|_{L^p(\lambda)} \lesssim \|f\|_{L^p(\mu)},$$

where $A_{S,\tilde{\eta}}^m$ is the m -th iteration of $A_{S,\tilde{\eta}} f = \tilde{\eta} A_S f$. Indeed, from this, by Lemma 3.1, we obtain that for $b \in BMO_{\tilde{\eta}}$,

$$\|T_b^m f\|_{L^p(\lambda)} \lesssim \|b\|_{BMO_{\tilde{\eta}}}^m \|f\|_{L^p(\mu)},$$

which, by Lemma 6.7, contradicts (6.2).

To show (6.12), we will use the well-known fact that A_S is bounded on $L^p(w)$ for $w \in A_p$ (see, e.g., [8]). Also, by Hölder's inequality,

$$\lambda \tilde{\eta}^{kp} = \lambda^{1-\frac{k}{m}} \mu^{\frac{k}{m}} \in A_p \quad (k = 0, \dots, m).$$

Hence,

$$\begin{aligned} \|A_S(A_{S,\tilde{\eta}}^m f)\|_{L^p(\lambda)} &\lesssim \|A_{S,\tilde{\eta}}^m f\|_{L^p(\lambda)} \lesssim \|A_{S,\tilde{\eta}}^{m-1} f\|_{L^p(\lambda \tilde{\eta}^p)} \\ &\lesssim \cdots \lesssim \|A_S f\|_{L^p(\mu)} \lesssim \|f\|_{L^p(\mu)}, \end{aligned}$$

proving (6.12). \square

The proof of Theorem 6.2 is similar but now Lemma 6.8 along with Theorem 6.10 will play the crucial role.

Proof of Theorem 6.2. By Corollary 6.4, $\lambda \lesssim \mu$ a.e., and therefore it remains to prove the converse estimate. Assume that this is not true. As in the previous proof, it suffices to show that there exists a weight $u \notin L^\infty$ such that

$$(6.13) \quad \|A_S(A_{S,u}^m f)\|_{L^p(\lambda)} \lesssim \|f\|_{L^p(\mu)}.$$

Assume first that $\lambda \in A_p$ and μ is an arbitrary weight. Corollary 6.9 along with Theorem 6.10 shows that there exist $u \notin L^\infty$ and $w \in A_p$ such that

$$\lambda u^{mp} \lesssim w \lesssim \mu.$$

It follows from this that

$$\lambda u^p \lesssim \lambda^{1-\frac{1}{m}} w^{\frac{1}{m}}.$$

Also, by Hölder's inequality, $\lambda^{1-\frac{1}{m}} w^{\frac{1}{m}} \in A_p$. Hence,

$$\begin{aligned} \|A_{\mathcal{S}}(A_{\mathcal{S},u}^m f)\|_{L^p(\lambda)} &\lesssim \|A_{\mathcal{S},u}^m f\|_{L^p(\lambda)} = \|A_{\mathcal{S}}(uA_{\mathcal{S},u}^{m-1} f)\|_{L^p(\lambda u^p)} \\ &\lesssim \|A_{\mathcal{S}}(uA_{\mathcal{S},u}^{m-1} f)\|_{L^p(\lambda^{1-\frac{1}{m}} w^{\frac{1}{m}})} \lesssim \|A_{\mathcal{S},u}^{m-1} f\|_{L^p(\lambda^{1-\frac{1}{m}} w^{\frac{1}{m}} u^p)}. \end{aligned}$$

Arguing similarly, we have that

$$\lambda^{1-\frac{k-1}{m}} w^{\frac{k-1}{m}} u^p \lesssim \lambda^{1-\frac{k}{m}} w^{\frac{k}{m}}$$

and $\lambda^{1-\frac{k}{m}} w^{\frac{k}{m}} \in A_p$ for all $2 \leq k \leq m$. Therefore, iterating this argument yields

$$\begin{aligned} \|A_{\mathcal{S},u}^{m-1} f\|_{L^p(\lambda^{1-\frac{1}{m}} w^{\frac{1}{m}} u^p)} &\lesssim \|A_{\mathcal{S}}(uA_{\mathcal{S},u}^{m-2} f)\|_{L^p(\lambda^{1-\frac{2}{m}} w^{\frac{2}{m}})} \\ &\lesssim \|A_{\mathcal{S},u}^{m-2} f\|_{L^p(\lambda^{1-\frac{2}{m}} w^{\frac{2}{m}} u^p)} \lesssim \cdots \lesssim \|A_{\mathcal{S}} f\|_{L^p(w)} \lesssim \|f\|_{L^p(w)} \lesssim \|f\|_{L^p(\mu)}, \end{aligned}$$

which proves (6.13).

Consider now the assumption $\lambda, \mu^{1-p'} \in A_{\infty}$. Observe that if $\mu \in A_p$, then $\mu^{1-p'} \in A_{p'}$, and then, by duality, the situation is reduced to the previously considered case (and we even do not need to use that $\lambda \in A_{\infty}$). Therefore, assume that $\mu \notin A_p$.

Combining Corollary 6.5 and the fact that λ and $\mu^{1-p'}$ satisfy the reverse Hölder inequality, we obtain that there exists $r > 1$ such that $(\lambda^r, \mu^r) \in A_p$. Therefore, by Theorem 6.10, there exists $\nu \in A_p$ such that $\lambda \lesssim \nu \lesssim \mu$ a.e. Since $\mu \notin A_p$, we have that $\frac{\mu}{\nu} \notin L^{\infty}$. Therefore, we are in position to repeat the previous argument with λ replaced by ν . This completes the proof. \square

7. APPENDIX

D. Cruz-Uribe and K. Moen [9] showed that the condition

$$(7.1) \quad \sup_Q \|u^{1/p}\|_{L^p(\log L)^{2p-1+\varepsilon}, Q} \|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\delta}, Q} < \infty$$

is sufficient for

$$(7.2) \quad \|T_b^1 f\|_{L^p(u)} \lesssim \|b\|_{BMO} \|f\|_{L^p(v)},$$

and also they showed that this result is not true for $\varepsilon = \delta = 0$.

On the other hand, by Theorem 1.1,

$$(7.3) \quad \sup_Q \|u^{1/p}\|_{L^p(\log L)^{p-1+\varepsilon}, Q} \|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1+\delta}, Q} < \infty$$

and

$$(7.4) \quad \sup_Q \|u^{1/p}\|_{L^p(\log L)^{2p-1+\varepsilon}, Q} \|v^{-1/p}\|_{L^{p'}(\log L)^{p'-1+\delta}, Q} < \infty$$

provide another sufficient condition for (7.2).

It is obvious that condition (7.1) implies (7.3) and (7.4). We give an example showing that (7.3) and (7.4) are weaker than (7.1), in general.

Theorem 7.1. *There exist weights u and v on \mathbb{R} such that*

$$(7.5) \quad \sup_I \|u^{1/p}\|_{L^p(\log L)^{p-\frac{1}{2}}, I} \|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-\frac{1}{2}}, I} < \infty$$

and

$$(7.6) \quad \sup_I \|u^{1/p}\|_{L^p(\log L)^{2p-\frac{1}{2}}, I} \|v^{-1/p}\|_{L^{p'}(\log L)^{p'-\frac{1}{2}}, I} < \infty,$$

while

$$(7.7) \quad \sup_I \|u^{1/p}\|_{L^p(\log L)^{2p-1}, I} \|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1}, I} = \infty.$$

7.1. Auxiliary propositions. We say that a Young function Φ is submultiplicative if there exists $\kappa \geq 1$ such that

$$(7.8) \quad \Phi(ab) \leq \kappa \Phi(a)\Phi(b)$$

for all $a, b \geq 0$. It is easy to see that the function

$$\varphi(t) = t \log^\alpha(t + e) \quad (\alpha \geq 0)$$

is submultiplicative.

In the following propositions we assume that Φ satisfies (7.8).

Proposition 7.2. *Let $I, J \subset \mathbb{R}$ be the intervals such that $J \subset I$. Then*

$$\|f\|_{\Phi, J} \leq \|f\|_{\Phi, I} \frac{1}{\Phi^{-1}\left(\frac{1}{\kappa} \frac{|J|}{|I|}\right)}.$$

Proof. By (7.8),

$$\Phi\left(\frac{|f|}{\lambda}\right) \leq \frac{|J|}{|I|} \Phi\left(\frac{|f|}{\lambda \Phi^{-1}\left(\frac{1}{\kappa} \frac{|J|}{|I|}\right)}\right).$$

Using also that $J \subset I$, we obtain

$$\frac{1}{|J|} \int_J \Phi\left(\frac{|f|}{\lambda}\right) \leq \frac{1}{|I|} \int_I \Phi\left(\frac{|f|}{\lambda \Phi^{-1}\left(\frac{1}{\kappa} \frac{|J|}{|I|}\right)}\right).$$

Hence if $\lambda = \|f\|_{\Phi, I} \frac{1}{\Phi^{-1}\left(\frac{1}{\kappa} \frac{|J|}{|I|}\right)}$, the latter is controlled by 1, and the desired conclusion follows. \square

Proposition 7.3. *Let $I, J \subset \mathbb{R}$ be the intervals such that $|J \cap I| \neq 0$. If $\text{supp } f \subset I$, then*

$$\|f\|_{\Phi, J} \leq \|f\|_{\Phi, J \cap I} \frac{1}{\Phi^{-1}\left(\frac{1}{\kappa} \frac{|J|}{|J \cap I|}\right)}.$$

Proof. The proof is similar to the previous. By (7.8),

$$\Phi\left(\frac{|f|}{\lambda}\right) \leq \frac{|J|}{|J \cap I|} \Phi\left(\frac{|f|}{\lambda \Phi^{-1}\left(\frac{1}{\kappa} \frac{|J|}{|J \cap I|}\right)}\right).$$

Therefore,

$$\frac{1}{|J|} \int_J \Phi\left(\frac{|f|}{\lambda}\right) \leq \frac{1}{|J \cap I|} \int_{J \cap I} \Phi\left(\frac{|f|}{\lambda \Phi^{-1}\left(\frac{1}{\kappa} \frac{|J|}{|J \cap I|}\right)}\right).$$

Hence if $\lambda = \|f\|_{\Phi, J \cap I} \frac{1}{\Phi^{-1}\left(\frac{1}{\kappa} \frac{|J|}{|J \cap I|}\right)}$, the latter is controlled by 1, and the desired conclusion follows. \square

7.2. Localized weights. Let $0 < a < b < 1/2$ be the numbers such that

$$b = \log^{-\frac{1}{2}}(1/a).$$

We also assume that a is small enough so that

$$(7.9) \quad a < (b/2)^{\max(p-1, 1)}.$$

Let m any real number. We define two functions on the interval $I = [m, m + b]$ as follows

$$(7.10) \quad u_I(x) = \frac{1}{a} \chi_{[m, m+a]}(x) + a \chi_{[m+b-a, m+b]}(x)$$

and

$$(7.11) \quad v_I(x) = \frac{1}{a} \chi_{[m, m+b-a^{\frac{1}{p-1}}]}(x) + a \log^{3p}(1/a) \chi_{[m+b-a^{\frac{1}{p-1}}, m+b]}(x).$$

Observe that, by (7.9),

$$m + a < \min(m + b - a, m + b - a^{\frac{1}{p-1}}).$$

Proposition 7.4. *We have*

$$(7.12) \quad \|u_I^{1/p}\|_{L^p(\log L)^{p-\frac{1}{2}}, I} \|v_I^{-1/p}\|_{L^{p'}(\log L)^{2p'-\frac{1}{2}}, I} \lesssim 1,$$

$$(7.13) \quad \|u_I^{1/p}\|_{L^p(\log L)^{2p-\frac{1}{2}}, I} \|v_I^{-1/p}\|_{L^{p'}(\log L)^{p'-\frac{1}{2}}, I} \lesssim 1$$

and

$$(7.14) \quad \|u_I^{1/p}\|_{L^p(\log L)^{2p-1}, I} \|v_I^{-1/p}\|_{L^{p'}(\log L)^{2p'-1}, I} \sim \frac{1}{b}.$$

Proof. The proof is based on the observation that for any $\alpha, \beta > 0$

$$\|u_I^{1/p}\|_{L^p(\log L)^\alpha, I} \sim \|u_I\|_{L(\log L)^\alpha, I}^{1/p} \quad \text{and} \quad \|v_I^{-1/p}\|_{L^{p'}(\log L)^\beta, I} \sim \|v_I^{1-p'}\|_{L(\log L)^\beta, I}^{1/p'}$$

and on a straightforward computation by (2.4).

We have

$$(u_I)_I = \frac{1}{b}(1+a^2) \quad \text{and} \quad (v_I^{1-p'})_I = \frac{1}{b}\left((b-a^{\frac{p'}{p}})a^{\frac{p'}{p}} + \log^{3p(1-p')}(1/a)\right).$$

Therefore, by (2.4),

$$\begin{aligned} \|u_I\|_{L(\log L)^\alpha, I} &\sim \frac{1}{|I|} \int_I u_I \log^\alpha \left(\frac{u_I}{(u_I)_I} + e \right) \\ &= \frac{1}{b} \log^\alpha \left(\frac{1}{(u_I)_I} \frac{1}{a} + e \right) + \frac{a^2}{b} \log^\alpha \left(\frac{a}{(u_I)_I} + e \right) \\ &\sim \frac{1}{b} \log^\alpha \left(\frac{b}{a} + e \right) \sim \frac{1}{b} \log^\alpha(1/a). \end{aligned}$$

Similarly,

$$\begin{aligned} \|v_I^{1-p'}\|_{L(\log L)^\beta, I} &\sim \frac{1}{|I|} \int_I v_I^{1-p'} \log^\beta \left(\frac{v_I^{1-p'}}{(v_I^{1-p'})_I} + e \right) \\ &= \frac{1}{b} (b - a^{\frac{1}{p-1}}) a^{\frac{1}{p-1}} \log^\beta \left(\frac{a^{\frac{1}{p-1}}}{(v_I^{1-p'})_I} + e \right) \\ &\quad + \frac{1}{b} \log^{\gamma_2(1-p')}(1/a) \log^\beta \left(\frac{a^{1-p'} \log^{3p(1-p')}(1/a)}{(v_I^{1-p'})_I} + e \right) \\ &\sim \frac{1}{b} \log^{3p(1-p')}(1/a) \log^\beta (ba^{1-p'} + e) \sim \frac{1}{b} \log^{\beta-3p(p'-1)}(1/a). \end{aligned}$$

From this,

$$\begin{aligned} \|u_I^{1/p}\|_{L^p(\log L)^\alpha, I} \|v_I^{-1/p}\|_{L^{p'}(\log L)^\beta, I} &\sim \frac{1}{b} \log^{\frac{\alpha}{p} + \frac{\beta-3p(p'-1)}{p'}}(1/a) \\ &= \frac{1}{b} \log^{\frac{\alpha}{p} + \frac{\beta}{p'} - 3}(1/a), \end{aligned}$$

which implies the proposition. \square

7.3. Proof of Theorem 7.1. For each $n \geq N$, where N is large enough, set $a_n = e^{-n}$, $b_n = n^{-1/2}$ and $m_n = e^n$. Set $I_n = [m_n, m_n + b_n]$, and define $u_n(x) = u_{I_n}(x)$ and $v_n(x) = v_{I_n}(x)$ as described in the previous section.

Finally, set

$$u(x) = \sum_{n=N}^{\infty} u_n(x)$$

and

$$v(x) = \chi_{(-\infty, e^N)} + \sum_{n=N}^{\infty} v_n(x) + \sum_{n=N}^{\infty} e^n \chi_{(e^n + b_n, e^{n+1})}(x).$$

By (7.14),

$$\|u^{1/p}\|_{L^p(\log L)^{2p-1}, I_n} \|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1}, I_n} \sim n^{1/2},$$

which proves (7.7).

It remains to show that (7.5) and (7.6) hold, namely that for every interval $J \subset \mathbb{R}$,

$$(7.15) \quad \|u^{1/p}\|_{L^p(\log L)^{p-\frac{1}{2}}, J} \|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-\frac{1}{2}}, J} \lesssim 1$$

and

$$(7.16) \quad \|u^{1/p}\|_{L^p(\log L)^{2p-\frac{1}{2}}, J} \|v^{-1/p}\|_{L^{p'}(\log L)^{p'-\frac{1}{2}}, J} \lesssim 1.$$

We shall consider several subcases.

Case 1: $J \subset I_n$ for some n . We split I_n as

$$I_n = I^1 \cup I^2 \cup I^3,$$

where

$$I^1 = [m, m + a_n), \quad I^2 = [m + a_n, m + b_n - a_n^{\frac{1}{p-1}}) \quad \text{and} \quad I^3 = [m + b_n - a_n^{\frac{1}{p-1}}, m + b_n].$$

There are four possible situations:

(i) $J \subset I^i$ for some i .

- If $J \subset I^1$, then, since u and v are constant on I^1 ,

$$\|u^{1/p}\|_{L^p(\log L)^{p-1/2}, J} = \|u^{1/p}\|_{L^p(\log L)^{2p-1/2}, J} = a_n^{-1/p}$$

and

$$(7.17) \quad \|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1/2}, J} = \|v^{-1/p}\|_{L^{p'}(\log L)^{p'-1/2}, J} = a_n^{1/p}.$$

If $J \subset I^3$, then, since $u \leq a_n$ on I^3 ,

$$(7.18) \quad \|u^{1/p}\|_{L^p(\log L)^{p-1/2}, J} = \|u^{1/p}\|_{L^p(\log L)^{2p-1/2}, J} \leq a_n^{1/p}.$$

Also, since v is constant on I^3 ,

$$\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1/2}, J} = \|v^{-1/p}\|_{L^{p'}(\log L)^{p'-1/2}, J} = (a_n \log^{3p}(1/a_n))^{-1/p}.$$

In both cases (7.15) and (7.16) hold.

- Suppose that $J \subset I^2$. If $p' \geq p$ then $u = 0$ on J . Otherwise $u \leq a_n$ on J , and hence, in both cases (7.18) holds. Also,

$$\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1/2}, J} = \|v^{-1/p}\|_{L^{p'}(\log L)^{p'-1/2}, J} = a_n^{1/p},$$

and hence we again obtain (7.15) and (7.16).

(ii) $J \cap I^i \neq \emptyset$ for every i . Then $|J| \sim |I_n|$. In this case (7.15) and (7.16) follow from a combination of Propositions 7.2 and 7.4.

(iii) $J \cap I^1 \neq \emptyset$, $J \cap I^2 \neq \emptyset$ and $J \cap I^3 = \emptyset$. Then (7.17) holds, and also, by Proposition 7.3,

$$\max(\|u^{1/p}\|_{L^p(\log L)^{p-1/2}, J}, \|u^{1/p}\|_{L^p(\log L)^{2p-1/2}, J}) \lesssim a_n^{-1/p},$$

which implies (7.15) and (7.16).

(iv) $J \cap I^1 = \emptyset$, $J \cap I^2 \neq \emptyset$ and $J \cap I^3 \neq \emptyset$. Then, arguing as above,

$$\max(\|u^{1/p}\|_{L^p(\log L)^{p-1/2}, J}, \|u^{1/p}\|_{L^p(\log L)^{2p-1/2}, J}) \lesssim a_n^{1/p}$$

and

$$\max(\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1/2}, J}, \|v^{-1/p}\|_{L^{p'}(\log L)^{p'-1/2}, J}) \lesssim (a_n \log^{3p}(1/a_n))^{-1/p},$$

and therefore, (7.15) and (7.16) hold.

Case 2: $J \cap I_n \neq \emptyset$ for just one I_n but $J \not\subset I_n$. In this case, by Proposition 7.3,

$$\|u^{1/p}\|_{L^p(\log L)^{p-1/2}, J} \lesssim \|u^{1/p}\|_{L^p(\log L)^{p-1/2}, J \cap I_n}$$

and

$$\|u^{1/p}\|_{L^p(\log L)^{2p-1/2}, J} \lesssim \|u^{1/p}\|_{L^p(\log L)^{2p-1/2}, J \cap I_n}.$$

On the other hand we note that for any $x, y \in J$ with $x \in J \setminus I_n$ and $y \in I_n$ $v^{-1/p}(x) \lesssim v^{-1/p}(y)$. Hence,

$$\|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1/2}, J} \lesssim \|v^{-1/p}\|_{L^{p'}(\log L)^{2p'-1/2}, J \cap I_n}$$

and

$$\|v^{-1/p}\|_{L^{p'}(\log L)^{p'-1/2}, J} \lesssim \|v^{-1/p}\|_{L^{p'}(\log L)^{p'-1/2}, J \cap I_n}.$$

This reduces us to the previous case and hence we are done.

Case 3: $J \cap I_n \neq \emptyset$ for more than one I_n . Using that

$$\|f\|_{L(\log L)^\alpha, Q} \lesssim \|f\|_{L^r, Q}$$

for $r > 1$, it suffices to show that for $r > 1$ small enough

$$\|u\|_{L^r, J} \lesssim 1 \quad \text{and} \quad \|v^{1-p'}\|_{L^r, J} \lesssim 1.$$

Let n_0 and n_1 be the smallest and the largest integers such that $J \cap I_{n_0} \neq \emptyset$ and $J \cap I_{n_1} \neq \emptyset$. If $n_0 > N$, then $J \subset [e^N, \infty)$ and $|J| \sim e^{n_1}$. If $n_0 = N$, then one can write $J = L \cup R$, where $L \subset (-\infty, e^N)$ (possibly $L = \emptyset$) and $R \subset [e^N, \infty)$ with $|R| \sim e^{n_1}$.

Since u is supported in $\cup_{n \geq N} I_n$, we obtain that

$$\int_J u^2 \leq \sum_{n=n_0}^{n_1} (e^n + e^{-3n}) \lesssim e^{n_1} \lesssim |J|.$$

Now, if $n_0 > N$, then for $r \leq p$,

$$\int_J v^{(1-p')r} \lesssim \sum_{n=n_0-1}^{n_1+1} \left(e^{nr(1-p')} e^n + \left(\frac{n^{3p}}{e^n} \right)^{r(1-p')} e^{-n(p'-1)} \right) \lesssim e^{n_1} \lesssim |J|.$$

In the case if $n_0 = N$ and $J = L \cup R$ represented as above, then

$$\int_J v^{(1-p')r} = |L| + \int_R v^{(1-p')r} \lesssim |L| + |R| = |J|.$$

This completes the proof.

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