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On the φ-index of inclusion: studying the structure generated by a subset of indexes

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Abstract

The φ-index of inclusion has proven to be a suitable generalization of the inclusion in the fuzzy setting. In this paper, the properties of the φ-index of inclusion, when its definition is restricted to a subset of indexes, are analyzed. The theoretical results obtained in this work are necessary in order to develop fuzzy inference systems based on the φ-index of inclusion.

KEYWORDS:
Inclusion measure, Fuzzy inclusion, Fuzzy sets, φ-index of inclusion.

1 | INTRODUCTION

Fuzzy set theory is a fruitful area of theoretical and practical research. At the theoretical level it has been used in linear programming [1], dynamical systems [2], measures of consistency [3], systems of relational equations [4], boundary value problems [5]; at the practical level one can find fuzzy-based notions in a variety of different applications to real-world problems.

Research on fuzzy inclusions is of significant interest due to its broad applications in fields such as decision-making, pattern recognition, and knowledge representation. Different approaches to the notion of fuzzy inclusions can be found in the literature, but all of them have a common feature: they provide a numerical value to represent the degree of inclusion of one fuzzy set into another. The φ-index of inclusion was presented originally [6] as a novel approach to model the inclusion between fuzzy sets and it was extended to general L-fuzzy sets [7]. The main difference with respect to the existing approaches in the literature is that the inclusion between fuzzy sets are represented by mappings, instead of by values in the lattice of truth degrees.

In this paper, we focus on the properties of the φ-index of inclusion, when its definition is restricted to a subset of indexes. The theoretical results obtained in this work are necessary in order to develop fuzzy inference systems based on the φ-index of inclusion. The potential uses of these results are innumerable, just take into account the different uses of fuzzy reasoning in real-world applications: control systems [8,9], expert systems [10], fuzzy modelling [11,12], image processing [13,14], decision support [15,16], natural language processing and opinion-mining [17,18], robotics [19], etc.

Since the introduction of the φ index of inclusion [6], many different results have been obtained to support its use as a good generalization in the fuzzy setting of the classical inclusion of sets. For example, in [20] it was proved that the φ-index of inclusion satisfies almost all the axioms proposed by Fan-Xie-Pei [21], Sinha-Dougherty [22] and Kitainik [23], after a convenient rewrite according to the functional structure of Ω. In [24], it was shown that the φ-index of inclusion and the φ-index of contradiction [25] can be used to define a square of opposition, in the line of the Aristotelian one. Another interesting result, introduced in [26], shows that the mapping that defines the φ-index of inclusion can be used in a *modus ponens*-like inference which is optimal with respect to any other modus ponens inference defined from residuated pairs.

The results obtained so far for the φ-index of inclusion motivate its use for developing fuzzy inference systems. With this goal in mind, we have realized that it is necessary to analyze the properties of the φ-index of inclusion when its definition is restricted to a subset of indexes. This paper provides the first steps in this direction.

2 | PRELIMINARY RESULTS: THE NOTION OF F-INCLUSION

In this section, we recall some preliminary notions needed in this work. To begin with, it is worth to note that fuzzy sets are defined as usual, that is a *fuzzy set* A is defined by means of its membership function $A : \mathcal{U} \to [0,1]$, where $\mathcal{U}$ is a set called universe. The standard operations between sets are extended as follows: given two fuzzy sets A and B, we define the fuzzy sets

- $(A \cup B)(u) = \max\{A(u), B(u)\}$;
- $(A \cap B)(u) = \min\{A(u), B(u)\}$;
- $A^c(u) = 1 - A(u)$.

In this paper, we will deal with some well-known notions of lattice theory. Below, we recall the chain conditions on lattices.

Definition 1 ([27])**.** Let $(L, \preceq)$ be a lattice. L is said to satisfy the *ascending chain condition*, denoted as ACC, if any sequence $x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq \dots$ of elements of L is eventually stationary (this is, there exists $k \in \mathbb{N}$ such that $x_k = x_{k+1} = \dots$). The dual of the ascending chain condition is the *descending chain condition*, denoted as DCC.

Let us recall now some notions needed to define the φ-index of inclusion. The first one is the set of indexes of inclusion, which plays an essential role in the definition of the φ-index of inclusion[6].

Definition 2. The set of *indexes of inclusion*, denoted by Ω, consists of all the deflationary and monotonically increasing mapping; that is, any mapping $f : [0,1] \to [0,1]$ such that

- $f(x) \le x$, for all $x \in [0,1]$ and
- $x \le y$ implies $f(x) \le f(y)$, for all $x, y \in [0,1]$.

Another necessary notion to define the φ-index of inclusion is the definition of f-inclusion which is recalled below.

Definition 3. Let A and B be two fuzzy sets and consider $f \in \Omega$. We say that *A is f-included in B*, denoted by $A \subseteq_f B$, if and only if the inequality $f(A(u)) \le B(u)$ holds for all $u \in \mathcal{U}$.

The notion of f-inclusion can be used to define a degree of inclusion between two sets, which is called φ-index of inclusion. In the following, the formal definition of this notion is given.

Definition 4. Let A and B be two fuzzy sets, the φ-index of inclusion, denoted by $\mathrm{Inc}(A, B)$, is defined as

$$\mathrm{Inc}(A, B) = \sup\{f \in \Omega \mid A \subseteq_f B\}.$$

It is convenient to note that each φ-index in Ω represents the inclusion according to a criterion based on the f-inclusion. In this way, fixed a fuzzy set A, the greater the $f \in \Omega$, the stronger the restriction imposed to the fuzzy set B by the f-inclusion. For a more detailed information related to the previous definitions, we refer the reader to[6,7].

The following theorem summarizes the main properties of Inc that are aligned to the axiomatic approaches of Fan-Xie-Pei[21], Sinha-Dougherty[22] and Kitainik[23].

Theorem 1 ([7])**.** Let A, B and C be fuzzy sets,

1. (Full inclusion) $Inc(A, B) = id$ if and only if $A(u) \le B(u)$ for all $u \in \mathcal{U}$.
2. (Null inclusion) $Inc(A, B) = \bot$ if and only if there is a set of elements in the universe $\{u_i\}_{i \in I} \subseteq \mathcal{U}$ such that $A(u_i) = 1$ for all $i \in I$ and $\bigwedge_{i \in I} B(u_i) = 0$.
3. (Pseudo transitivity) $Inc(B, C) \circ Inc(A, B) \le Inc(A, C)$.
4. (Monotonicity) If $B(u) \le C(u)$ for all $u \in \mathcal{U}$ then, $Inc(C, A) \le Inc(B, A)$.
5. (Monotonicity) If $B(u) \le C(u)$ for all $u \in \mathcal{U}$ then, $Inc(A, B) \le Inc(A, C)$.
6. (Transformation Invariance) Let $T : \mathcal{U} \to \mathcal{U}$ be a bijective mapping[1] on $\mathcal{U}$, then $Inc(A, B) = Inc(T(A), T(B))$.

[1] We use the term transformation to refer to bijective mappings in $\mathcal{U}$, following the usual approaches related to measures of inclusion, e.g.,[22].

7. (Relationship with intersection) $Inc(A, B \cap C) = Inc(A, B) \wedge Inc(A, C)$.
8. (Relationship with union) $Inc(A \cup B, C) = Inc(A, C) \wedge Inc(B, C)$.

The axiom related to the complement, that is, the contraposition rule, is the only one which is not directly satisfied by $\mathrm{Inc}(A, B)$. In general, we have that $\mathrm{Inc}(A, B) \neq \mathrm{Inc}(B^c, A^c)$, but this property is also captured by the φ-index of inclusion in terms of Galois connections as follows:

Proposition 1 ([20])**.** Let (f, g) be an isotone Galois connection, then $A \subseteq_f B$ if and only if $B^c \subseteq_{1-g(1-x)} A^c$.

2.1 | The f-index of inclusion restricted to a complete join-semilattice of Ω

The standard operators in fuzzy logic (such as t-norms) define subsets of indexes of inclusion satisfying that, under such restrictions, the φ-index of inclusion coincides with the standard measure of inclusion[26]. This fact motivates the consideration of subsets in Ω for the definition of the φ-index of inclusion.

Definition 5 ([26])**.** Let A and B be two fuzzy sets and Θ be a complete join-semilattice of Ω (i.e., Θ is closed under arbitrary suprema and contains $\bot$ and id), the *φ-index of inclusion restricted to* Θ, denoted by $\mathrm{Inc}_\Theta(A, B)$, is defined as

$$\mathrm{Inc}_\Theta(A, B) = \sup\{f \in \Theta \mid A \subseteq_f B\}.$$

The following example makes use of the product t-norm to illustrate this fact. Note that the same conclusions are obtained when other t-norms, e.g., Gödel or Łukasiewicz, are considered.

Example 1. We define the set of indexes of inclusions by means of the product t-norm, $x *_P y = x \cdot y$ with $x, y \in [0, 1]$, as the set $\Theta_P = \{\alpha *_P x \mid \alpha \in [0, 1]\}$. Then, it is proven in[26] Theorem 3 that, given two fuzzy sets A and B, we have that

$$\mathrm{Inc}_{\Theta_P}(A, B)(x) = \left(\bigwedge_{u \in \mathcal{U}} A(u) \rightarrow_P B(u) \right) *_P x$$

where $\rightarrow_P$ denotes the residuated implication of the product t-norm. As a result, the computation and meaning of $\mathrm{Inc}_{\Theta_P}(A, B)$ is equivalent to:

$$\bigwedge_{u \in \mathcal{U}} A(u) \rightarrow_P B(u)$$

which is the standard measure of inclusion with respect to the product residuated implication.

The properties of Inc_Θ related to those presented in Theorem 1 were analyzed in[28], are the following:

Theorem 2. Let A, B and C three fuzzy sets and let $\Theta \subseteq \Omega$ a join-semilattice of Ω, then

1. $\mathrm{Inc}_\Theta(A, B) \leq \mathrm{Inc}(A, B)$;
2. $Inc_\Theta(A, B) = id$ if and only if $A(u) \leq B(u)$ for all $u \in \mathcal{U}$.;
3. $Inc_\Theta(A, B) = \bot$ if and only if, for each $f \in \Theta$ with $f \neq \bot$, we have that there exists an element $u \in \mathcal{U}$ such that $f(A(u)) > B(u)$;
4. If Θ is closed under composition (i.e., for all $f, g \in \Theta$ we have $f \circ g \in \Theta$) then, $Inc_\Theta(B, C) \circ Inc_\Theta(A, B) \leq Inc_\Theta(A, C)$.
5. If $B(u) \leq C(u)$, for every $u \in \mathcal{U}$, then $Inc_\Theta(C, A) \leq Inc_\Theta(B, A)$;
6. If $B(u) \leq C(u)$, for every $u \in \mathcal{U}$, then $Inc_\Theta(A, B) \leq Inc_\Theta(A, C)$;
7. Let $T : \mathcal{U} \rightarrow \mathcal{U}$ be a bijective mapping on $\mathcal{U}$, then $Inc_\Theta(A, B) = Inc_\Theta(T(A), T(B))$;
8. $Inc_\Theta(A, B \cap C) \geq Inc_\Theta(A, B) \wedge Inc_\Theta(A, C)$;
9. $Inc_\Theta(A \cup B, C) \geq Inc_\Theta(A, C) \wedge Inc_\Theta(B, C)$;

The main difference between Theorem 1 and Theorem 2, concerns items related to intersections and unions; in Theorem 2 such properties are written with inequalities instead of equalities. In this line, it is worth recalling the following result related to complete lattices, since it guarantees that those Θ considered in Definition 5 have a complete lattice structure.

Lemma 1 ([27])**.** Let $(L, \leq)$ be a join complete semilattice with a minimum element, then $(L, \leq)$ is a complete lattice.

Therefore, those items related to intersection and unions turn into equalities when the infimum considered is the one of Θ.

Proposition 2. Let A, B and C be fuzzy sets, and let $(\Theta, \leq, \wedge_\Theta, \vee_\Theta)$ be the complete lattice structure with respect to the ordering $\leq$ in Ω. Then

- $Inc_\Theta(A, B \cap C) = Inc_\Theta(A, B) \wedge_\Theta Inc_\Theta(A, C)$.
- $Inc_\Theta(A \cup B, C) = Inc_\Theta(A, C) \wedge_\Theta Inc_\Theta(B, C)$.

The following proposition motivates the use of restricted subsets of Ω, since it allows to rewrite Proposition 1 directly in terms of the φ-index of inclusion.

Proposition 3. Let A and B be two fuzzy sets, let $n(x) = 1 - x$ and let us assume that, for every $f \in \Theta$:

- there exists $\overline{f}\colon [0, 1] \to [0, 1]$ such that $(f, \overline{f})$ is an isotone Galois connection;
- $n \circ \overline{f} \circ n \in \Theta$.

if $\overline{\mathrm{Inc}_\Theta(A, B)}$ is the only mapping such that $(\mathrm{Inc}_\Theta(A, B), \overline{\mathrm{Inc}_\Theta(A, B)})$ forms an isotone Galois connection, then

$$\mathrm{Inc}_\Theta(B^c, A^c) = n \circ \overline{\mathrm{Inc}_\Theta(A, B)} \circ n\,.$$

2.2 | An initial example

Our approach proposes the use of specific φ-indexes of inclusion restricted to certain subsets $\Theta \subset \Omega$. The motivation lies in the possible inferences we can make, since the results of these inferences are reduced to a subset of Ω. Then, the original φ-indexes of inclusion and the restricted ones are equivalent. On the other hand, considering only those possible φ-indexes of inclusion allows to obtain more algebraic properties, for example, we motivate our study to guarantee that an algorithm based on the φ-index of inclusion ends; which will be detailed in subsequent sections.

Let us start with our illustrative example. Let us consider four fuzzy sets A, B, C and D on an arbitrary (and unknown) universe $\mathcal{U}$. Let us assume that we know that:

- $\mathrm{Inc}(A, B) = f$;
- $\mathrm{Inc}(B, C) = g$;
- $\mathrm{Inc}(A, D) = h$;
- $\mathrm{Inc}(D, C) = m$;

Now, the following question arises:

What information about the φ-index of inclusion of A, B, C and D can we infer from the available information?

Obviously, we can make use of properties in Theorem 1. For example, from item 1 in the referred theorem, we can obtain trivial inferences as: $\mathrm{Inc}(A, A) = \mathrm{Inc}(B, B) = \mathrm{Inc}(C, C) = \mathrm{Inc}(D, D) = id$.

Certainly, we can also obtain some non-trivial information. From item 3 in Theorem 1, we can apply the pseudo-transitivity and find lower bounds of the f-index of inclusion $\mathrm{Inc}(A, C)$ as follows

- from $\mathrm{Inc}(A, B) = f$ and $\mathrm{Inc}(B, C) = g$ we get $\mathrm{Inc}(A, C) \geq g \circ f$;
- from $\mathrm{Inc}(A, D) = h$ and $\mathrm{Inc}(D, C) = m$ we get $\mathrm{Inc}(A, C) \geq h \circ m$;
- joining both previous items we get $\mathrm{Inc}(A, C) \geq (g \circ f) \vee (h \circ m)$;

In addition, from items 4 and 5 we can obtain direct lower bounds by decreasing the left and increasing the right part of the operator Inc. For instance, from $\mathrm{Inc}(A, B) = f$ we get directly that all $\mathrm{Inc}(A, B \cup C)$, $\mathrm{Inc}(A, B \cup D)$, $\mathrm{Inc}(A, B \cup C \cup D)$, $\mathrm{Inc}(A \cap C, B)$, $\mathrm{Inc}(A \cap D, B)$ and $\mathrm{Inc}(A \cap C \cap D, B)$ can be lower bounded by f; a similar inference can be done for the rest of combinations. Mixing all these inferences obtained from items 4 and 5, we can get other more interesting inequalities, for example:

- from item 4, $\text{Inc}(B, C) = g$ and $\text{Inc}(D, C) = m$ we get $\text{Inc}(B \cap D, C) \geq g \vee m$;
- from item 5, $\text{Inc}(A, B) = f$ and $\text{Inc}(A, D) = h$ we get $\text{Inc}(A, B \cup D) \geq f \vee h$;

On the other hand, items 7 and 8 provide equalities, that is, a more accurate inference, for union and intersection of the fuzzy sets A, B, C and D. Specifically, we can obtain the following inferences:

- from item 7, $\text{Inc}(A, B) = f$ and $\text{Inc}(A, D) = h$ we get $\text{Inc}(A, B \cap D) = f \wedge h$;
- from item 8, $\text{Inc}(B, C) = g$ and $\text{Inc}(D, C) = m$ we get $\text{Inc}(B \cup D, C) = g \wedge m$;

Note that mixing items 4,5,7 and 8 we can obtain, for instance, $\text{Inc}(A \cap C, B \cap D) \geq f \wedge h$.

Finally, if we impose also the existence of four mappings $\overline{f}, \overline{g}, \overline{h}$ and $\overline{m}$ such that $(f, \overline{f})$, $(g, \overline{g})$, $(h, \overline{h})$ and $(m, \overline{m})$ are isotone Galois connection, then we can also do inferences where the complement of fuzzy sets is involved. For example, from Proposition 3 and $\text{Inc}(A, B) = f$ we can obtain that $\text{Inc}(B^c, A^c) = n \circ \overline{f} \circ n$ (where n denotes the standard negation $n(x) = 1 - x$). Note that we can combine the inferences from Theorem 1 for union and intersection with those of Proposition 3 for the complements and obtain more complex inferences, for example:

- $\text{Inc}(B^c, A^c \cap D) = n \circ \overline{f} \circ n$
- $\text{Inc}(B^c \cup D^c, A^c) = \left(n \circ \overline{f} \circ n\right) \wedge \left(n \circ \overline{h} \circ n\right)$.

Conclusions from the example

Note that, taking into account the properties of Theorem 1, a subset of Ω can be obtained during the inference process. Specifically, from f, g, h and m we can obtain the identity and chains of mappings in $\{f, g, h, m\}$ connected by infimum, supremum and composition. Consequently, the mathematical analysis of this set of inferred indexes is of interest in order to know the expressive power of the proposed inference based on the φ-index of inclusion.

Another motivational aspect for the theoretical development presented in subsequent sections is the restriction of the definition of the φ-index of inclusion to certain subsets of indexes. This kind of restriction is interesting in the application of the φ-index of inclusion for approximate reasoning, in tasks where obtaining $\text{Inc}(A, B)$ could be computationally demanding. In this line, it is important not to lose the strengths that properties in Theorem 1 provides for reasoning. In other words, it is convenient to choose subsets $\Theta \subseteq \Omega$ aimed at keeping the same properties for Inc_Θ as for Θ. It has been mentioned before that when subsets $\Theta \subseteq \Omega$ with the structure of join-semilattice are considered, that is the case. Therefore, we are interested in subsets $\Theta \subseteq \Omega$ with the following properties:

- Closed under arbitrary supremum in Ω, in order to have a correct and practical definition of Inc_Θ;
- Closed under composition, in order to keep inferences associated to the $\circ$-transitivity (item 4, Theorem 2);
- Closed under infimum, in order to keep the inferences associated to union and intersection of fuzzy sets. Note that, in this case, the infimum might not coincide with that in Ω. Hence, we have two alternatives to proceed. On the one hand, we can consider Θ to be closed under the infimum of Ω, that is, Θ is a sublattice of Ω. On the other hand, we can consider the infimum in Θ which existence is guaranteed by Lemma 1, that is, the infimum in Θ may differ from the infimum in Ω. In the latter case, note that the inference with respect union and intersections is given by means of Proposition 2.
- Last but not least, if we wish to carry out inferences related to complement of fuzzy sets, we require each mapping in Θ to be a part of a Galois connection and be closed under the following construction: if $f \in \Theta$, then $n \circ \overline{f} \circ n$ is in Θ as well, where $\overline{f}$ is the only mapping such that $(f, \overline{f})$ forms a Galois connection.

In the subsequent section we will provide several constructions of subsets of Ω, satisfying the previous constraints and we will analyze more properties of such constructions.

3 | SETS OF INDEXES GENERATED BY FINITELY MANY INDEXES

This section is divided into two parts. The first one is devoted to the construction of several subsets of indexes from a finite set of indexes; the motivation for this construction is to consider the smallest set of indexes that will be used in an inference according to the example introduced in the previous section. The second part focuses on the representation of elements in those subsets of indexes. Specifically, we will introduce a representation theorem which resembles the disjunctive normal form.

3.1 | Definitions

From the conclusions obtained in Section 2.2, it is interesting to study the properties of the smallest join-subsemilattice generated by a finite set of mappings in Ω. Formally, given a finite subset $S \subseteq \Omega$, we define recursively the following subset of Ω from S, denoted by $\langle S \rangle$.

Definition 6. Let S be a finite subset of Ω, then $\langle S \rangle$ is defined recursively as follows:

- $id \in \langle S \rangle$;
- $\bot \in \langle S \rangle$
- if $f \in S$ then $f \in \langle S \rangle$;
- if $\phi, \psi \in \langle S \rangle$ then $\phi \circ \psi \in \langle S \rangle$;
- if $\phi, \psi \in \langle S \rangle$ then $\phi \vee \psi \in \langle S \rangle$;
- if $\phi, \psi \in \langle S \rangle$ then $\phi \wedge \psi \in \langle S \rangle$;

For notational purposes, as done above, we will denote by roman letters (possibly subscripted) f, g the mappings in S and by greek letters (possibly subscripted) ϕ, ψ the mappings in $\langle S \rangle$.

Note that, although $\langle S \rangle$ may contain infinitely many elements, each element is the combination of mappings in S by using a finitely many suprema, infima and composition operators. This is because the core of this approach is computationally oriented and then, the consideration of a transfinite number of operations is not an admissible scenario. The following example illustrates the structure of $\langle S \rangle$ for finite subsets $S \subset \Omega$.

Example 2. Let us consider firstly a simple case of a singleton $S = \{f\}$, whereas f is the mapping $f(x) = x^2$. In this case, the composition creates mappings of the form $f^n(x) = x^{2n}$, for $n \in \mathbb{N}$, and the supremum and infimum does not create more mappings in $\langle S \rangle$, since for $n \leq m$ we have $f^n \wedge f^m(x) = f^n(x) = x^{2n}$ and $f^n \vee f^m(x) = f^m(x) = x^{2m}$. Therefore, $\langle S \rangle = \{x^{2n} \mid n \in \mathbb{N}\} \cup \{\bot\}$.

Example 3. Let us consider now $S = \{f_1, f_2\}$ where $f_1(x) = 0.8 \cdot x$ and $f_2(x) = \min(x, 0.4)$. Note that f_2 is idempotent, so the composition of f_2 with itself does not generate new mappings in $\langle S \rangle$. The reiterative composition of f_1 generates mappings of the form $f_1^n(x) = (0.8)^n \cdot x$. The iterative composition of f_1 and f_2 is more intricate. For example, the first compositions return the mappings $f_2 \circ f_1 = \min(0.4, 0.8 \cdot x)$ and $f_1 \circ f_2(x) = \min(0.32, 0.8x)$. A second round of compositions generates the following mappings: $f_1^2 \circ f_2(x) = \min(0.4 \cdot (0.8)^2, (0.8)^2 \cdot x)$; $f_1 \circ f_2 \circ f_1(x) = f_2 \circ f_1 \circ f_1(x) = \min(0.32, (0.8)^2 \cdot x)$; $f_2 \circ f_1 \circ f_2 = f_1 \circ f_2^2 = f_1 \circ f_2$; and so on. Hence, the iterative composition may create new mappings or not, as the use of infima and suprema, since:

$$f_1 \vee f_2(x) = \begin{cases} x & \text{if } 0 \leq x \leq 0.4 \\ 0.4 & \text{if } 0.4 < x \leq 0.5 \\ 0.8 \cdot x & \text{if } 0.5 < x \leq 1 \end{cases}$$

but $f_1 \wedge f_2(x) = f_2 \circ f_1(x)$. Another interesting issue of this example is that the construction of mappings by composition, infimum and supremum collapses at one point and only iterative composition of f_1 generates new mappings. This is due to several facts: 1) the particular expression of f_2 where $f_2(x)$ is either a constant or the identity, 2) the inequality $f_1^5 < f_2$, and 3) the continuity of f_1. In this way, if $\phi < f_2$, then $f_2 \circ \phi = \phi$ and, as a consequence, $f_2 \vee \phi = f_2$, $f_2 \wedge \phi = \phi$, $f_2 \circ f_1^n \circ f_2 = f_1^n \circ f_2$ for all $n \in \mathbb{N}$ and $f_2 \circ f^m = f^m$ for all $m \geq 5$.

This example shows that determining the structure of $\langle S \rangle$ may be very complex even in the case where S is formed by two simple mappings.

Among the operators $\vee$, $\wedge$, and $\circ$ in Ω, the composition operator $\circ$ plays a principal role. For such a reason, we define the length of a mapping in $\langle S \rangle$ as the number of composition required to define it from S. This consideration defines intrinsically a stratification in $\langle S \rangle$ that might be useful to prove results by induction.

Definition 7. Let S be a finite subset of Ω and $\phi \in \langle S \rangle$. We say that ϕ *has length* n if the minimal number of compositions required to construct ϕ from S is n.

The following example illustrates the previous definition.

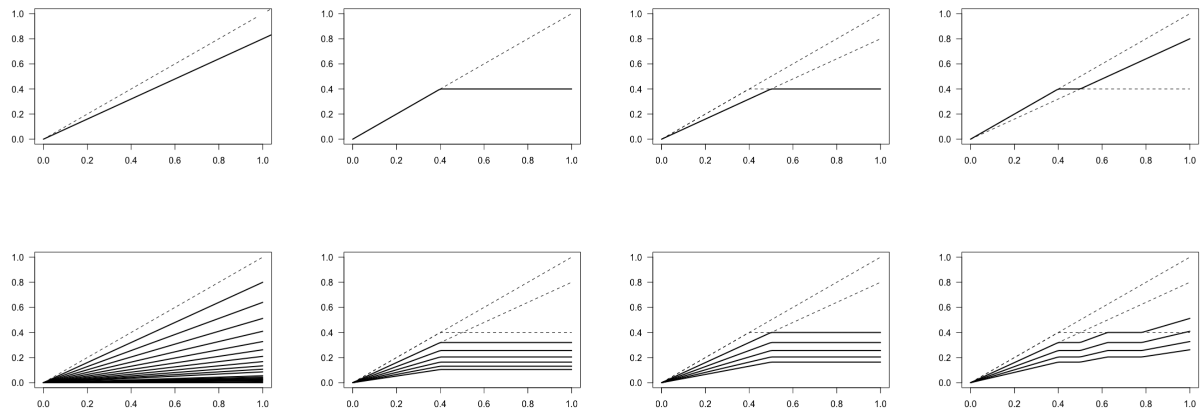

FIGURE 1 Graphs of several mappings in $\langle S\rangle$ of Example 1 . Upper row, from left to right: f_1; f_2; $f_1 \wedge f_2$; and $f_1 \vee f_2$. The lower row displays subsets of mappings in $\langle S\rangle$ according to an iterative composition of f_1. From left to right: mappings of the form $f_1 \circ \overset{n}{\cdots} \circ f_1$; mappings of the form $f_1 \circ \overset{n}{\cdots} \circ f_1 \circ f_2$; mappings of the form $f_1 \circ \overset{n}{\cdots} \circ f_1 \circ (f_1 \wedge f_2)$; and mappings of the form $f_1 \circ \overset{n}{\cdots} \circ f_1 \circ ((f_1 \circ f_2) \vee (f_2 \wedge (f_1 \circ f_1) \vee (f_1 \circ f_1 \circ f_1))$. Note that id, f_1 and f_2 are drawn by dashes in some graphs in order to serve as references.

Example 4. Note that if we consider $S = \{f_1, f_2, f_3\}$ where $f_1(x) = 0.5 \cdot x$, $f_2(x) = 0.125 \cdot x$ and $f_3(x) = x^2$. Then, obviously f_1, f_2 and f_3 have length 0. Note that $f_1 \vee f_3$ has length 0 as well. The mappings $f_1 \circ f_1(x) = 0.25 \cdot x$ and $f_1 \circ f_2 \vee f_3$ have length 1. Note that $f_2 = f_1 \circ f_1 \circ f_1$, but since the length is defined via the minimal number of iterations, f_2 has length 0.

3.2 | Normal forms in $\langle S\rangle$

In this section we show that elements in $\langle S\rangle$ can be written in a normal form that resembles the so-called disjunctive normal form (DNF) of propositional logic. The mentioned normal forms are written with respect to functions generated only by composition of mappings in S, so let us introduce a notation to refer to this set of compositions.

Definition 8. Let $S \subseteq \Omega$, then the set of compositions generated by S is denoted as

$$C(S) = \{f_1 \circ f_2 \circ \ldots \circ f_n \mid f_1, \ldots, f_n \in S\}$$

where f_i and f_j need not be necessarily different.

The possibility of writing elements in $\langle S\rangle$ by a normal form is partially due to the distributivity of Ω, which is inherited from the distributivity of $([0, 1], \vee, \wedge)$, as the following proposition shows.

Proposition 4. Ω is a distributive lattice.

Proof. It is straightforward. Let $f, g, h \in \Omega$, then for all $x \in [0, 1]$ we have

$$f \vee (g \wedge h)(x) = f(x) \vee (g(x) \wedge h(x)) = (f(x) \vee g(x)) \wedge (f(x) \vee h(x)) = (f \vee g)(x) \wedge (f \vee h)(x)$$

and

$$f \wedge (g \vee h)(x) = f(x) \wedge (g(x) \vee h(x)) = (f(x) \vee g(x)) \wedge (f(x) \vee h(x)) = (f \wedge g)(x) \vee (f \wedge h)(x)$$

Therefore, Ω is distributive. □

The following result shows that $\langle S\rangle$ preserves the distributivity of Ω.

Corollary 1. Let S be a finite subset of Ω, then $\langle S\rangle$ is a distributive lattice.

Proof. It is a direct consequence of $\langle S\rangle$ being a sublattice of a Ω and Proposition 4. □

The following result shows that we can write elements in $\langle S\rangle$ in disjuntive normal form (DNF) with respect to elements in $C(S)$.

Theorem 3. Let S be a finite subset of Ω, then for all $\phi \in \langle S\rangle$ there exists $\{p_{ij}\}_{i\in\{1,\dots,k\},j\in\{1,\dots,k_i\}} \subseteq C(S)$ such that

$$\phi = \bigvee_{i=1}^{k} p_{i1} \wedge \cdots \wedge p_{ik_i}$$

Proof. Let us prove the result by induction over the length of $\phi \in \langle S\rangle$. If the length of ϕ is 0, then the results comes directly by the distributivity of $\langle S\rangle$. Let us assume that the result is true for all $\psi \in \langle S\rangle$ with length less than or equal to n, and let us prove the result for a given $\phi \in \langle S\rangle$ of length $n+1$. By the distributivity of $\langle S\rangle$ we can assume that

$$\phi = \bigvee_{i=1}^{t} \psi_{i1} \wedge \cdots \wedge \psi_{it_i}$$

for certain subset $\{\psi_{ij}\}_{i\in\{1,\dots,t\},j\in\{1,\dots,t_i\}} \subseteq \langle S\rangle$ such that for each ψ_{ij} there exists $f_{ij} \in S$ and $\overline{\psi}_{ij} \in \langle S\rangle$ such that

$$\psi_{ij} = f_{ij}\circ\overline{\psi}_{ij}$$

and such that the length of $\overline{\psi}_{ij}$ is strictly less than $n+1$. By hypothesis of induction, we can ensure that

$$\psi_{ij} = f_{ij}\circ\overline{\psi}_{ij} = f_{ij}\circ\left(\bigvee_{h=1}^{\bar{t}} \overline{p}_{h1} \wedge \cdots \wedge \overline{p}_{ht_h}\right) \tag{1}$$

for certain subset $\{\overline{p}_{hj}\} \subseteq C(S)$. Let us prove now the equality

$$f_{ij}\circ\left(\bigvee_{h=1}^{\bar{t}} \overline{p}_{h1} \wedge \cdots \wedge \overline{p}_{ht_h}\right) = \bigvee_{h=1}^{\bar{t}} f_{ij}\circ\overline{p}_{h1} \wedge \cdots \wedge f_{ij}\circ\overline{p}_{ht_h}. \tag{2}$$

Consider $x \in [0,1]$, by the monotonicity of $f_{ij} \in S$ and the finiteness of the suprema and infima involved in the expressions, then we have

$$\begin{aligned}
f_{ij}\circ\left(\bigvee_{h=1}^{\bar{t}} \overline{p}_{h1} \wedge \cdots \wedge \overline{p}_{ht_h}\right)(x) &= f_{ij}\left(\max_{h\in\{1,\dots,\bar{t}\}} \left\{\overline{p}_{h1} \wedge \cdots \wedge \overline{p}_{ht_h}(x)\right\}\right)\\
&= \max_{h\in\{1,\dots,\bar{t}\}} \left\{ f_{ij}\left(\overline{p}_{h1} \wedge \cdots \wedge \overline{p}_{ht_h}(x)\right)\right\} = \max_{h\in\{1,\dots,\bar{t}\}} \left\{ f_{ij}\left(\min\{\overline{p}_{h1}(x),\dots,\overline{p}_{ht_h}(x)\}\right)\right\}\\
&= \max_{h\in\{1,\dots,\bar{t}\}} \left\{ \min_{h_t}\{f_{ij}\circ\overline{p}_{h1}(x),\dots,f_{ij}\circ\overline{p}_{ht_h}(x)\}\right\} = \left(\bigvee_{h=1}^{\bar{t}} f_{ij}\circ\overline{p}_{h1} \wedge \cdots \wedge f_{ij}\circ\overline{p}_{ht_h}\right)(x).
\end{aligned}$$

Since $f_{ij} \in S$ and $\{\overline{p}_{hj}\} \subseteq C(S)$, necessarily each $f_{ij}\circ\overline{p}_{hj} \in C(S)$. Finally, the result is trivially reached by distributivity of $\langle S\rangle$ and Equations 1 and 2. □

Example 5. Let us reconsider Example 1 , where S was formed by the mappings $f_1(x) = 0.8\cdot x$ and $f_2(x) = \min(x, 0.4)$ and the mapping g In $\langle S\rangle$ given by

$$g(x) = f_1(f_1 \vee f_2) \wedge f_2(x) = \begin{cases} 0.8\cdot x & \text{if } 0 \le x \le 0.4\\ 0.32 & \text{if } 0.4 < x \le 0.5\\ 0.64\cdot x & \text{if } 0.5 < x \le 0.625\\ 0.4 & \text{if } 0.625 < x \le 1 \end{cases}$$

It can be easily chekced that we can rewrite g in normal form as follows $g = ((f_1\circ f_1) \wedge f_2)) \vee (f_1\circ f_2)$.

Note that the proof of the previous theorem establishes an algorithm to construct a normal form a mapping $\phi \in \langle S\rangle$:

(I) if ϕ is in normal form we have finished, otherwise go to step (II);

(II) if we (I) does not hold, apply distributivity of Ω as much as possible to obtain a supremum of infima of mappings in $\langle S \rangle$ and get a formula with the following form

$$\phi = \bigvee_{i=1}^{k} \psi_{i1} \wedge \cdots \wedge \psi_{ik_i}$$

for $\psi_{ij} \in \langle S \rangle$. Go to step (III);

(III) For all $\psi_{ij} \notin C(S)$ in the formula of step (II) we have that necessarily $\psi_{ij} = f \circ \psi$ for certain $f \in S$ (otherwise we can apply distributivity in step (II)). Now, move f "inside" ψ and rewrite each ψ_{ij} with the form

$$\psi_{ij} = \bigvee_{h=1}^{t} f \circ \psi_{h1} \wedge \cdots \wedge f \circ \psi_{ht_h}.$$

Once this step is applied for all $\psi_{ij} \notin C(S)$ in the formula of step (II), go to step (I).

One fourth step (IV) can be added by applying a simplification of mappings according to the ordering in Ω. In other words, if after applying the algorithm, we get $(f \circ g) \wedge f$, we can use that $(f \circ g) \leq f$ and write directly $f \circ g$.

The following example illustrates the application of the previous algorithm.

Example 6. Let us consider an abstract example where S is formed by three arbitrary mappings; i.e., $S = \{f, g, h\}$. Then, the mapping given by $\phi = (f \circ g \circ (f \vee g)) \wedge ((g \circ f) \vee (h \circ g))$ can be written in normal form as follows:

- we apply (I) and since $\phi = (f \circ g \circ (f \vee g)) \wedge ((g \circ f) \vee (h \circ g))$ is not in normal form, we go to step (II);
- we apply (II) and by distributivity in Ω we get
$$\phi = (f \circ g \circ (f \vee g)) \wedge ((g \circ f) \vee (h \circ g)) = \big((f \circ g \circ (f \vee g)) \wedge (g \circ f)\big) \vee \big((f \circ g \circ (f \vee g)) \wedge (h \circ g)\big)$$
- we apply (III), and since $(f \circ g \circ (f \vee g)) = (f \circ g \circ f) \vee (f \circ g \circ g)$ we can write ϕ as follows:
$$\phi = \Big(\big((f \circ g \circ f) \vee (f \circ g \circ g)\big) \wedge (g \circ f)\Big) \vee \Big(\big((f \circ g \circ f) \vee (f \circ g \circ g)\big) \wedge (h \circ g)\Big)$$
- we apply again (I) and since ϕ is not in normal form yet, we go to step (II)
- we apply (II) and get
$$\phi = \Big(\big((f \circ g \circ f) \wedge (g \circ f)\big) \vee \big((f \circ g \circ g)\big) \wedge (g \circ f)\big)\Big) \vee \Big(\big((f \circ g \circ f) \wedge (h \circ g)\big) \vee \big((f \circ g \circ g) \wedge (h \circ g)\big)\Big)$$
- step (III) does not produce any modification and since ϕ is in normal form, step (I) informs us that we have finished.

We can apply an extra step of simplification in the algorithm taking into account to considerations. Firstly, that mappings in Ω are inflationary (i.e., lesser than id) and that as a result we have that $\psi_1 \circ \psi_2 \leq \psi_1$ and $\psi_1 \circ \psi_2 \leq \psi_2$. In our particular example, we have that $(f \circ g \circ f) \leq (g \circ f)$ and then $(f \circ g \circ f) \wedge (g \circ f) = f \circ g \circ f$. After the application of this first simplification step, we may get expressions of the kind $\psi_1 \vee (\psi_1 \wedge \psi_2)$ for certain $\psi_1, \psi_2 \in \langle S \rangle$, which in turns can be transformed in ψ_1 by removing $\psi_1 \wedge \psi_2$ from the expression. In our particular example, we would get $(f \circ g \circ f) \vee ((f \circ g \circ f) \wedge (h \circ g)) = f \circ g \circ f$. In summary, ϕ can be rewritten with the following normal form:

$$\phi = \big(f \circ g \circ f\big) \vee \big((f \circ g \circ g)\big) \wedge (g \circ f)\big) \vee \big((f \circ g \circ g) \wedge (h \circ g)\big)$$

4 | ON ASCENDING CHAINS IN $\langle S \rangle$.

In this section we will focus several situations which guarantee that the lattice $\langle S \rangle$ satisfies the Ascending Chain Condition (ACC). As mentioned above, this condition is important since it can ensure the termination of algorithms based on the inference rules obtained from the properties of Inc and described in Section 2 (since the inference constructs an increasing chain in $\langle S \rangle$). We provide proofs for some interesting cases: S is a singleton; S is formed by idempotent mappings; the iterative composition of mappings in S reach $\bot$. So far no examples have been found of finite sets S such that does not hold in $\langle S \rangle$, but the proof for the general case is still an open problem.

4.1 | ACC of $\langle S\rangle$ with S a singleton

Let us begin by studying the structure of $\langle S\rangle$ for the case of a singleton $S = \{f\}$, for $f \in \Omega$. For the sake of simplicity, since there is no confusion possible, we denote $\langle\{f\}\rangle$ directly by $\langle f\rangle$.

The first result of this section shows that suprema and infima of elements in $\mathcal{C}(\{f\})$ are elements of $\mathcal{C}(\{f\})$ as well.

Lemma 2. Let $f \in \Omega$, then the following equalities hold:

- $(f\circ\overset{n}{\cdots}\circ f) \vee (f\circ\overset{m}{\cdots}\circ f) = (f\circ\overset{\min(n,m)}{\cdots}\circ f)$.
- $(f\circ\overset{n}{\cdots}\circ f) \wedge (f\circ\overset{m}{\cdots}\circ f) = (f\circ\overset{\max(n,m)}{\cdots}\circ f)$

Proof. Since f is inflationary (i.e., $f \leq id$), we have that the result directly holds from the equalities below:

$$(f\circ\overset{n}{\cdots}\circ f) \vee (f\circ\overset{m}{\cdots}\circ f)(x) = \max(f\circ\overset{n}{\cdots}\circ f(x), f\circ\overset{m}{\cdots}\circ f(x)) = (f\circ\overset{\min(n,m)}{\cdots}\circ f)(x)$$
$$(f\circ\overset{n}{\cdots}\circ f) \wedge (f\circ\overset{m}{\cdots}\circ f)(x) = \min(f\circ\overset{n}{\cdots}\circ f(x), f\circ\overset{m}{\cdots}\circ f(x)) = (f\circ\overset{\max(n,m)}{\cdots}\circ f)(x)$$

□

The following proposition shows that all the mappings in $\langle f\rangle$ correspond to a concatenation of compositions of f; that is, $\langle f\rangle = \mathcal{C}(\{f\}) \cup \{id, \bot\}$

Proposition 5. Let $f \in \Omega$ and let $\phi \in \langle f\rangle$, then either $\phi = id$, or $\phi = \bot$ or there exists $n \in \mathbb{N}$ such that $\phi = f\circ f\circ\overset{n}{\cdots}\circ f$.

Proof. Lemma 2 implies, since the use of suprema and infima does not generate new mappings, that mappings other than top and bottom elements in $\langle f\rangle$ can be represented by using just compositions. □

Example 7. Let us consider the mapping $f(x) = 0.9 \cdot x$. Then, by Lemma 2 we can assert that $\langle f\rangle = \mathcal{C}(\{f\})$ and, in this particular case $\langle f\rangle$ coincides with the following infinite decreasing chain:

$$id > f > f\circ f > f\circ f\circ f > \cdots > f\circ f\circ\overset{n}{\cdots}\circ f > \cdots > \bot$$

In general, it is not true that $\langle f\rangle$ is an infinite chain isomorphic to $\mathbb{N} \cup \{\infty\}$, as the following examples show.

Example 8. Let us consider the mapping $f(x) = \min(x, 0.5)$. Note that f is idempotent, so $f\circ f = f$. As a result, $f\circ f\circ\overset{n}{\cdots}\circ f = f$ and therefore $\langle f\rangle$ coincides with the chain $\bot < f < id$.

Example 9. Let us consider the mapping in Ω:

$$f(x) = \begin{cases} x & \text{if } 0 \leq x \leq 0.2 \\ 0.2 & \text{if } 0.2 < x \leq 0.6 \\ 0.4 & \text{if } 0.6 < x \leq 0.8 \\ 0.7 & \text{if } 0.8 < x \leq 1 \end{cases}$$

In this case, f is not idempotent. However, it can be proven that for $n \geq 3$ we have $f\circ\overset{n}{\cdots}\circ f(x) = f\circ f\circ f(x) = \min(x, 0.2)$. As a result, we can state that $\langle f\rangle$ coincides with the chain

$$\bot < f\circ f\circ f < f\circ f < f < id$$

What we can establish in general is a lattice-isomorphism between $\langle f\rangle$ and sublattices of $(\mathbb{N} \cup \{\infty\}, \geq)$ (i.e., the dual of $(\mathbb{N} \cup \{\infty\}, \leq)$).

Theorem 4. Let $f \in \Omega$, then there exists $N \subseteq \mathbb{N} \cup \{\infty\}$ such that $\langle f\rangle$ is isomorphic to $(N, \geq)$.

Proof. Let us define the following mapping between $\langle f\rangle$ and $\mathbb{N} \cup \{\infty\}$:

$$\begin{aligned} \Psi :\ & \langle f\rangle \longrightarrow \mathbb{N} \\ & id \mapsto 0 \\ & \phi \mapsto \min\{n \in \mathbb{N} \mid \phi = f\circ f\circ\overset{n}{\cdots}\circ f\} \end{aligned}$$

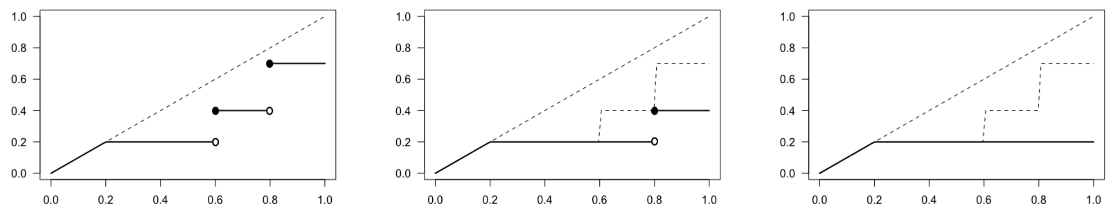

FIGURE 2 The tree iterative compositions of f in Example 9 that determines the structure of $\langle f\rangle$, which is isomorphic to the chain $0 < 1 < 2 < 3 < 4$. id and f are drawn with dashes in all the graphs in order to serve as references.

where $\min \varnothing = \infty$. Consider N, to be the image of the mapping Ψ. Then, Ψ clearly defines a lattice isomorphism between $\langle f\rangle$ and $(N, \geq)$. □

We can prove also the converse of Theorem 4, that is, for any chain $(N, \geq)$ in $\mathbb{N} \cup \{\infty\}$, there exists $f \in \Omega$ such that $\langle f\rangle$ is isomorphic to $(N, \geq)$.

Theorem 5. Let $(N, \geq) \subseteq (\mathbb{N} \cup \{\infty\}, \geq)$ with $|N| \geq 2$, then there exists $f \in \Omega$ such that $\langle f\rangle$ is isomorphic to $(N, \geq)$.

Proof. If N is an infinite subchain of $\mathbb{N}$, then we have $(N, \geq) \cong (\mathbb{N} \cup \{\infty\}, \geq)$. As a result, without loss of generality, we can assume $(N, \geq) = (\mathbb{N} \cup \{\infty\}, \geq)$. Now, considering $f(x) = x^2$ we have that $\langle f\rangle$ and $(\mathbb{N} \cup \{\infty\}, \geq)$ are isomorphic.

Now, consider that $(N, \geq)$ is a finite chain of elements in $\mathbb{N} \cup \{\infty\}$; that is $N = a_1 < a_2 < \cdots < a_n$. We can consider the following chain $0 = \alpha_1 < \alpha_2 < \cdots < \alpha_n = 1$ in the unit interval, which is certainly isomorphic to N. Let us define the mapping in Ω given by

$$f(x) = \begin{cases} 0 & \text{if } x \in [\alpha_1, \alpha_2] \\ \alpha_i & \text{if } x \in (\alpha_i, \alpha_{i+1}] \quad \text{for } i \in \{2, 3, \ldots, n-1\} \end{cases}.$$

Note that f is well defined and that $f \in \Omega$ because $\alpha_i < \alpha_{i+1}$ for all $i \in \{1, \ldots, n-1\}$. Note also that $f(\alpha_i) = \alpha_{i-1}$ for all $i \in \{2, \ldots, n\}$. We will prove that $\langle f\rangle$ has exactly n elements and then, by Theorem 4, we will be done.

Proving that $\langle f\rangle$ has exactly n elements is equivalent to prove that $f^k \neq f^{k-1}$ for all $k \in \{1, 2, \ldots, n-1\}$ and that $f^n = \bot$. By a simple calculation it can be proved that, for $k \in \{1, 2, \ldots, n\}$, we have:

$$f^k(x) = f \circ \overset{k}{\cdots} \circ f(x) = \begin{cases} 0 & \text{if } x \in [\alpha_1, \alpha_{1+k}] \\ \alpha_i & \text{if } x \in (\alpha_{i+k-1}, \alpha_{i+k}] \quad \text{for } i \in \{2, 3, \ldots, n-k\} \end{cases}.$$

As a result, $f^k \neq f^{k-1}$ for all $k \in \{1, 2, \ldots, n-1\}$ and $f^n = \bot$. □

The following example illustrates the constructive proof of the previous result.

Example 10. Let us consider a finite chain of 5 elements in $\mathbb{N}$. Such a chain is isomorphic to $0 < 0.25 < 0.5 < 0.75 < 1$ and according to it, we can construct the following mapping:

$$f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 0.25 \\ 0.25 & \text{if } 0.25 < x \leq 0.5 \\ 0.5 & \text{if } 0.5 < x \leq 0.75 \\ 0.75 & \text{if } 0.75 < x \leq 1 \end{cases}$$

Then, the sequence of mappings obtained by composition is

$$f \circ f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 0.5 \\ 0.25 & \text{if } 0.5 < x \leq 0.75 \\ 0.5 & \text{if } 0.75 < x \leq 1 \end{cases} \qquad f \circ f \circ f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 0.75 \\ 0.5 & \text{if } 0.75 < x \leq 1 \end{cases} \qquad f \circ f \circ f \circ f = \bot$$

As a result, $\langle f\rangle$ is the chain $\bot < f \circ f \circ f < f \circ f < f < id$, which is isomorphic to any chain of 5 elements in $\mathbb{N}$.

Finally, since $\mathbb{N} \cup \{\infty\}$ satisfies DCC (i.e., Descending Chain Condition with the usual ordering)), $\mathbb{N} \cup \{\infty\}$ satisfies ACC with the dual orderings and necessarily, $\langle f \rangle$ satisfies ACC,

Corollary 2. Let $f \in \Omega$, then $\langle f \rangle$ satisfies ACC.

4.2 | ACC of $\langle S \rangle$ with $C(S)$ finite

Taking as a reference the normal form provided in Section 3.2, we can obtain sufficient conditions for the ACC of $\langle S \rangle$ in terms of restrictions on $C(S)$. For instance, if a given set S generates a finite set $C(S)$, then we can ensure that $\langle S \rangle$ is finite and, then, it satisfies ACC.

Theorem 6. Consider $S \subseteq \Omega$ such that $C(S)$ is finite. Then, $\langle S \rangle$ satisfies ACC.

Proof. By Theorem 3 we have that for every element $\phi \in \langle S \rangle$ there exist $\{p_{ij}\}_{i \in I, j \in \mathbb{N}} \subseteq C(S)$ such that

$$\phi = \bigvee_{i \in I} p_{i1} \wedge \cdots \wedge p_{ik_i}$$

Now, since $C(S)$ is finite, there are only a finite set of possibilities to write mappings of $\langle S \rangle$ in their normal form. Consequently, $\langle S \rangle$ is finite. The ACC is directly obtained from the finiteness of $\langle S \rangle$. ☐

In the previous section we have already shown that $C(S)$ may be finite for many different singleton sets S. Below, we show that the finitude of $C(S)$ can be also reached for greater sets of mappings S.

Example 11. Let us consider the mappings $f(x) = \min(x, 0.9)$ and $g(x) = \max(0, x - 0.2)$. Note firstly that f is idempotent and that $f \circ g(x) = \min(\max(0, x - 0.2), 0.9) = \max(\min(0, 0.9), \min(x - 0.2, 0.9)) = \max(0, x - 0.2) = g(x)$. Secondly, note that the concatenation of compositions of g reaches $\bot$ in 5 steps, since: $g^2(x) = g \circ g(x) = \max(0, x - 0.4)$; $g^3(x) = \max(0, x - 0.6)$; $g^4(x) = \max(0, x - 0.8)$; $g^5 = \bot$. As a result, $C(S)$ is the following finite set of mappings:

$$C(S) = \left\{f, g, g^2, g^3, g^4, g \circ f, g^2 \circ f, g^3 \circ f, g^4 \circ f, \bot\right\}$$

In Figure 3 , the reader can see the graphs of the 9 mappings in $C(S)$. Note that, since $C(S)$ is closed by the infimum and supremum, we actually have the equality $\langle S \rangle = C(S) \cup \{id\}$.

The previous requirement (i.e., $C(S)$ to be finite) may look strong, but it holds when mappings of S are considered to be idempotent and/or mappings that reaches $\bot$ after a finite number of iterative self-compositions. The following two subsections go deply into these two cases.

4.2.1 | The case of idempotent mappings

It is clear, from the previous subsection 4.1, that if S is composed only of one idempotent mapping $f \in \Omega$, then $\langle S \rangle = \{id, f, \bot\}$. The problem comes when we consider S composed of several idempotent mappings, since the composition of idempotent mappings may not provide an idempotent mapping, as the following example shows.

Example 12. Let us consider the mappings

$$f(x) = \begin{cases} x & \text{if } 0 \le x \le 0.1 \\ 0.1 & \text{if } 0.1 < x \le 0.6 \\ x & \text{if } 0.6 < x \le 1 \end{cases} \qquad g(x) = \begin{cases} x & \text{if } 0 \le x \le 0.4 \\ 0.4 & \text{if } 0.4 < x \le 0.8 \\ x & \text{if } 0.8 < x \le 1 \end{cases}$$

The reader can check that both mappings are idempotent, that is, $f \circ f = f$ and $g \circ g = g$, however

$$f \circ g(x) = \begin{cases} x & \text{if } 0 \le x \le 0.1 \\ 0.1 & \text{if } 0.1 < x \le 0.6 \\ 0.4 & \text{if } 0.6 < x \le 0.8 \\ x & \text{if } 0.8 < x \le 1 \end{cases}$$

is not idempotent, since $(f \circ g)(0.7) = 0.4$ differs from $(f \circ g) \circ (f \circ g)(0.7) = 0.1$.

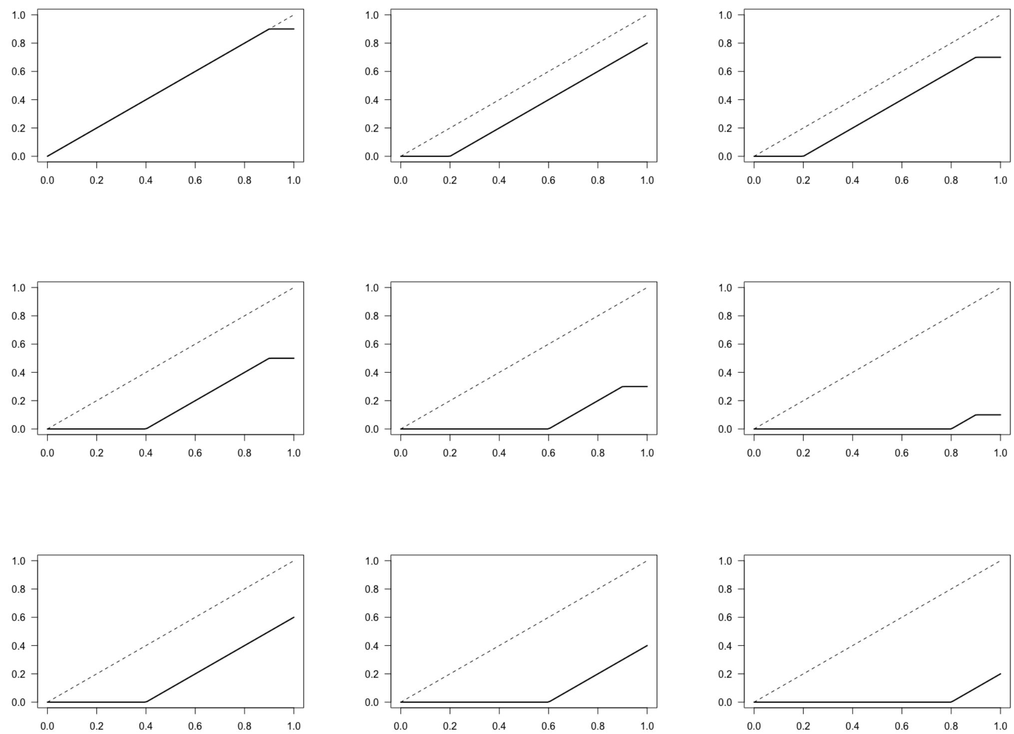

FIGURE 3 Graphs of the nine mappings in $C(S)$ of Example 11. The structure of $\langle S\rangle$ in this example is formed by these nine mappings plus $\bot$ and id. Note that id is drawn with dashes in each graph in order to serve as a reference.

The problem with the use of idempotent mappings in S is that $\langle S\rangle$ can contain non-idempotent mappings. In the rest of this section we identify some specific situations in which the generation of mappings by means of composition of idempotent mappings leads to a finite set of mappings; i.e., $C(S)$ finite. In order to prove this fact, let us provide firstly two technical lemmas.

Lemma 3. Let $f \in \Omega$ be an idempotent mapping. Then, f is defined piecewise, namely, for all $x \in [0, 1]$ either $f(x) = x$ or $f(x) = \alpha$ for certain $\alpha \in [0, 1]$ satisfying that $f(y) = \alpha$ of all $y \in [\alpha, x]$.

Proof. Consider $x \in [0, 1]$ such that $f(x) = \alpha \neq x$. Since f is in Ω, then necessarily $\alpha < x$. Moreover, by the idempotence of f we have $\alpha = f(x) = f \circ f(x) = f(\alpha)$. Finally, by monotony of f, for all $y \in [\alpha, x]$ we have that $\alpha = f(\alpha) \leq f(y) \leq f(x) = \alpha$. In other words, $f(y) = \alpha$. □

Note that all the conditions of mappings in Ω have been used in the proof, apart from the idempotency. The second technical lemma focuses on the "number of pieces" of each $f \in \Omega$. The following lemma shows that if S is formed by idempotent mappings with only a finite number of discontinuities, then $C(S)$ is finite.

Lemma 4. Let $S \subseteq \Omega$ such that each $f \in S$ is idempotent with a finite number of discontinuities, then $C(S)$ is finite.

Proof. For each $f_j \in S$, let us denote by $\{[\alpha_{ij}, \beta_{ij}) \subseteq [0, 1]\}_{i\in\{1,\dots,n\}}$ the respective intervals which give the different pieces. Then, all the compositions g in $C(S)$ are piecewise functions divided in intervals I constructed by intersections and union of intervals in $\{[\alpha_{ij}, \beta_{ij}) \subseteq [0, 1]\}_{i\in\{1,\dots,n\}}$, where g is either the identity for all x in I (i.e., $g(x) = x$ for all $x \in I$) or constant for one α_{ij} (i.e., $g(x) = \alpha_{ij}$ for all $x \in I$). Since the possibilities are finite, then $C(S)$ only contains a finite number of mappings. □

As a consequence of the previous results, we can ensure the ACC when S is formed by idempotent mappings with only a finite number of discontinuities.

Corollary 3. Let $S \subseteq \Omega$ such that each $f \in S$ is idempotent and with a finite number of discontinuities, then $\langle S\rangle$ satisfies ACC

Proof. The result is a direct consequence of Theorem 6 and Lemmas 3 and 4. □

Remark: It is worth noting that the mapping Inc(A, B), for a given pair of fuzzy sets A and B on a finite universe, can be seen as the infimum of idempotent mappings. Therefore, if S is built from φ-indices of inclusion of fuzzy sets on finite universes, then $\langle S \rangle$ satisfies ACC.

4.2.2 | The case of S formed by mappings reaching $\bot$

Assumption 1. Let us assume that every mapping $f \in S$ satisfies one of the following properties:

- There exists n such that $(f \circ \overset{(n)}{\cdots} \circ f)(x) = 0$ for all $x \in [0, 1)$.
- f is idempotent, i.e., $f \circ f = f$.

Proposition 6. Consider $S \subset \Omega$ satisfying Assumption 1, then for all $f \in S$ there exists n such that for all $m \geq n$ we have either

1. $f \circ \overset{(m)}{\cdots} \circ f = f$
2. $f \circ \overset{(m)}{\cdots} \circ f = \bot$
3. $f \circ \overset{(m)}{\cdots} \circ f(1) = 1$ and $f \circ \overset{(n)}{\cdots} \circ f(x) = 0$ for $x \in [0, 1)$.

Proof.

1. If f is idempotent, then $f \circ \overset{(m)}{\cdots} \circ f = f$ for all $m \in \mathbb{N}$.
   If f is not idempotent, we distinguish two further cases.
2. Let us assume firstly that $f(1) \neq 1$, recall that by Assumption 1 we have that there exists n_f such that $(f \circ \overset{(n_f)}{\cdots} \circ f)(x) = 0$ for all $x \in [0, 1)$. In particular, since $f(1) < 1$, we have:
   $$(f \circ \overset{(n_f+1)}{\cdots} \circ f)(1) = (f \circ \overset{(n_f)}{\cdots} \circ f)(f(1)) = 0$$
   which implies that $f \circ \overset{(n_f+1)}{\cdots} \circ f$. Hence, we have that for all $m \geq n_f + 1$ we have $f \circ \overset{(m)}{\cdots} \circ f = \bot$.
3. Let us assume now that f is not idempotent and that $f(1) = 1$. then by Assumption 1 we have that there exists n_f such that $(f \circ \overset{(n_f)}{\cdots} \circ f)(x) = 0$ for all $x \in [0, 1)$. Moreover, since $f(1) = 1$, we have $f \circ \overset{(n)}{\cdots} \circ f(1) = 1$ for all $n \in \mathbb{N}$. As a result, for all $m \geq n_f$ we have $f \circ \overset{(m)}{\cdots} \circ f(1) = 1$ and $f \circ \overset{(n)}{\cdots} \circ f(x) = 0$ for $x \in [0, 1)$.

□

Corollary 4. Consider $S \subset \Omega$ satisfying Assumption 1, then for all $f \in S$ there exists n such that $f \circ \overset{(n)}{\cdots} \circ f$ is idempotent.

Proof. It is a direct consequence of Proposition 6, since the three possibilities given in the proposition are indeed idempotent mappings. □

Theorem 7. Consider $S \subset \Omega$ satisfying Assumption 1, then $\mathcal{C}(S)$ is finite.

Proof. Let us assume that $\mathcal{C}(S)$ is infinite. Then there exists $f \in S$ and an infinite subset $\{\phi_n\}_{n \in \mathbb{N}} \subseteq \mathcal{C}(S)$ such that f appears in ϕ_n at least n times and $\phi_n \neq \phi_{n+1}$. We assume also without loss of generality that each f in ϕ_n is *strictly necessary*, that is, that if we remove f from ϕ_n the mapping obtained is different. Certainly, this f cannot be idempotent, otherwise the number of possible constructions would be finite and such an infinite subset would not exist.

By Proposition 6, we have that there exists n_f such that for all $m \geq n_f$ we have either

i) $f \circ \overset{(m)}{\cdots} \circ f = \bot$

ii) $f \circ \overset{(m)}{\cdots} \circ f(1) = 1$ and $f \circ \overset{(n)}{\cdots} \circ f(x) = 0$ for $x \in [0, 1)$.

iii) $f \circ \overset{(m)}{\cdots} \circ f = f$

As f is not idempotent, we know that item iii is a not possible case [2].

Assuming $m \geq 2n_f+1$ we will show that f is redundant in the definition of ϕ_m, obtaining a contradiction with the assumptions on the mappings in $\{\phi_n\}_{n\in\mathbb{N}}$.

To begin with, firstly suppose that f satisfies item i), then necessarily $\phi_m \leq f \circ \overset{(m)}{\cdots} \circ f = f \circ \overset{(n_f)}{\cdots} \circ f = \bot$. Therefore, removing one f from ϕ_m will return again the mapping $\bot$ and it contradict that all f's in ϕ_m are necessary to define ϕ_m.

Now, suppose that f satisfies item ii). Without loss of generality, we can write

$$\phi_m(x) = \psi_1 \circ f \circ \psi_2 \circ f \circ \cdots \circ f \circ \psi_m \circ f \circ \psi_{m+1}(x)$$

with $\psi_i \in \mathcal{C}(S) \cup \{id\}$. Obviously, for $x \in [0,1)$ we have

$$\phi_m(x) \leq (f \circ \overset{(m)}{\cdots} \circ f)(x) = 0$$

The only potential problem is the value of $\phi_m(1)$. If $\psi_k(1) < 1$ for some $k \geq n_f + 1$, we would have

$$\phi_m(1) \leq (f \circ \overset{(n_f)}{\cdots} \circ f) \circ (\psi_k \circ f \circ \psi_{k+1} \circ f \circ \cdots \circ f \circ \psi_m \circ f \circ \psi_{m+1}(1)) \leq (f \circ \overset{(n_f)}{\cdots} \circ f)(\psi_k(1)) = 0$$

and we would get again $\phi_m = \bot$. In this case, $m - k + 1$ f's are redundant for the definition of ϕ_m.

If $\psi_k(1) = 1$ for all $k \geq n_f + 1$, then we have that

$$\overline{\phi}(x) = \psi_{n_f+1} \circ f \circ \psi_{n_f+2} \circ f \circ \cdots \circ f \circ \psi_m \circ f \circ \psi_{m+1}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

and then,

$$\phi_m(x) = (\psi_1 \circ f \circ \psi_2 \circ f \circ \cdots \circ f \circ \psi_{k-1} \circ f)(\overline{\phi}(x))$$

But in this case we get that all the mappings f in the definition of $\overline{\phi}$ are redundant, since by a similar reasoning to the previous one we have for all $x \in [0,1)$

$$\phi_m(x) = (\psi_1 \circ f \circ \psi_2 \circ f \circ \cdots \circ f \circ \psi_{k-1} \circ f)(\overline{\phi}(x)) = \psi_1 \circ f \circ \psi_2 \circ f \circ \cdots \circ f \circ \psi_{k-1} \circ f(x)$$

and for $x = 1$ that

$$\phi_m(1) = (\psi_1 \circ f \circ \psi_2 \circ f \circ \cdots \circ f \circ \psi_{k-1} \circ f)(\overline{\phi}(1)) = \psi_1 \circ f \circ \psi_2 \circ f \circ \cdots \circ f \circ \psi_{k-1} \circ f(1)$$

which implies a contradiction with respect to the necessity of using each f in the definition of ϕ_m. □

Corollary 5. Consider $S \subset \Omega$ satisfying Assumption 1, then $\langle S \rangle$ satisfies ACC.

The reader can find in Example 11 a set of mappings satisfying Assumption 1. The following result shows that there are plenty of mappings in Ω that satisfy Assumption 1.

Proposition 7. Let $f \in \Omega$ such that there exists $\alpha \neq 0$ satisfying $f(x) \leq \max\{0, x - \alpha\}$, then f satisfies Assumption 1.

Proof. Since $f(x) \leq \max\{0, x - \alpha\}$, we have that

$$f \circ \overset{(n)}{\cdots} \circ f(x) \leq \max\{0, x - n \cdot \alpha\}$$

Then, for all $x \leq n \cdot \alpha$ we have that $f \circ \overset{(n)}{\cdots} \circ f(x) = 0$. As a result, if we consider $n_0 \geq \frac{1}{\alpha}$ we have for all $x \in [0,1]$ that $x \leq 1 \leq n_0 \cdot \alpha$ and then $f \circ \overset{(n_0)}{\cdots} \circ f(x) = 0$. □

Note that $\max\{0, x - \alpha\}$ can be approximated to id as much as wished, so $f(x) \leq \max\{0, x - \alpha\}$ does not impose a strong restriction indeed. For example, $f(x) = x^2$ does not satisfies Assumption 1, but if we consider the mapping associated to the previous proposition with respect to $\alpha = 0.01$, we get that the mapping

$$x^2 \wedge (\max\{0, x - \alpha\}) = \begin{cases} 0 & \text{if } x \leq \frac{5-2\sqrt{6}}{10} \\ x^2 & \text{if } \frac{5-2\sqrt{6}}{10} < x \leq \frac{5+2\sqrt{6}}{10} \\ x - 0.01 & \text{if } x > \frac{5+2\sqrt{6}}{10} \end{cases}$$

[2] See proof of Proposition 6

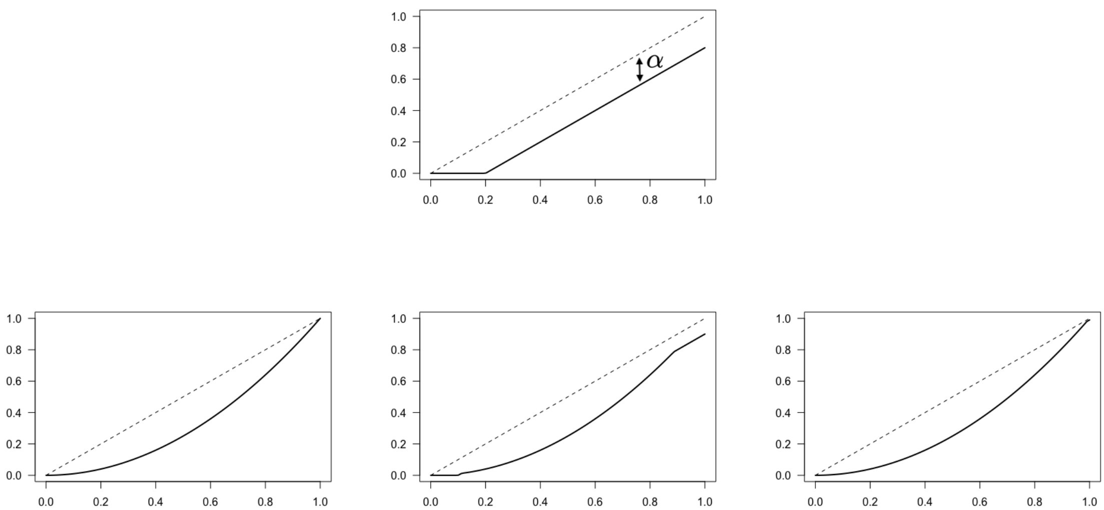

FIGURE 4 Upper row, graph of $\max\{0, x-\alpha\}$ with $\alpha = 0.2$ with the representation of the geometrical meaning of α (i.e., a kind of distance between id and $\max\{0, x-\alpha\}$). Lower row, approximation of the mapping $f(x) = x^2$ by means of the mapping given in Proposition 7. From left to right: mapping $f(x) = x^2$, $x^2 \wedge \max\{0, x-0.1\}$ and $x^2 \wedge \max\{0, x-0.01\}$. In the three graphs we have included the mapping id, drawn by dashes, to serve as reference.

satisfies Assumption 1 and coincides with $f(x) = x^2$ in the interval $[0.0102, 0.9898]$. Note firstly that the approximation can be improved by choosing a value α closer to 0 and secondly, that the same procedure can be used for all mapping in Ω. In Figure 4 we illustrate the previous approximation of $f(x) = x^2$ for $\alpha = 0.1$ and $\alpha = 0.01$.

Finally, note that thanks to the use of idempotent mappings, we can also improve the approximation of mappings that coincide with the identity in some intervals (i.e., such that $f(x) = x$ for all x in certain interval $[a, b]$). For example, the mapping $f(x) = \max(0, \min(x, 5x-2))$ (displayed in Figure 5) that coincide with id for all $x \in [0.5, 1]$ can be approximated by $f \wedge \max(0, x-0.1)$ and the supremum of the idempotent mapping

$$g(x) = \begin{cases} 0 & \text{if } x < 0.5 \\ x & \text{if } x \geq 0.5 \end{cases}$$

that is, $f \approx (f \wedge \max(0, x-0.1)) \vee g$. In other word, we can approximate nicely f in $\langle S \rangle$ for a set S satisfying Assumption 1. Figure 5 shows the proposed approximation.

5 | CONCLUSIONS AND FUTURE WORK

In this paper, we have recalled that we can define the φ-index of inclusion restricted to a certain set of f-indexes of inclusion Θ; named Inc_Θ, and that most of the properties required by the axiomatic approach of Sinha-Dougherty[22] hold for Inc_Θ. The only problematic cases can be solved easily by requiring some inner structure in Θ; e.g, transitivity can be solved by requiring that Θ is closed by composition of mappings. As a result, we focused on procedures to construct suitable subsets Θ of f-indexes, particularly, $\langle S \rangle$ and $C(S)$ for a finite set S of indexes.

Our main contributions here are related to the study of sufficient conditions for the Ascending Chain Condition of both $\langle S \rangle$ and $C(S)$. We intend to construct optimal subsets of f-indexes of inclusion for computational requirements (e.g., to simplify the computation of Inc_Θ). As future work, we will develop a fuzzy inference system based on the generalized modus ponens that defines the φ-index of inclusion; and we intend to provide a semantics based on the φ-index of inclusion in the context of fuzzy description logic.

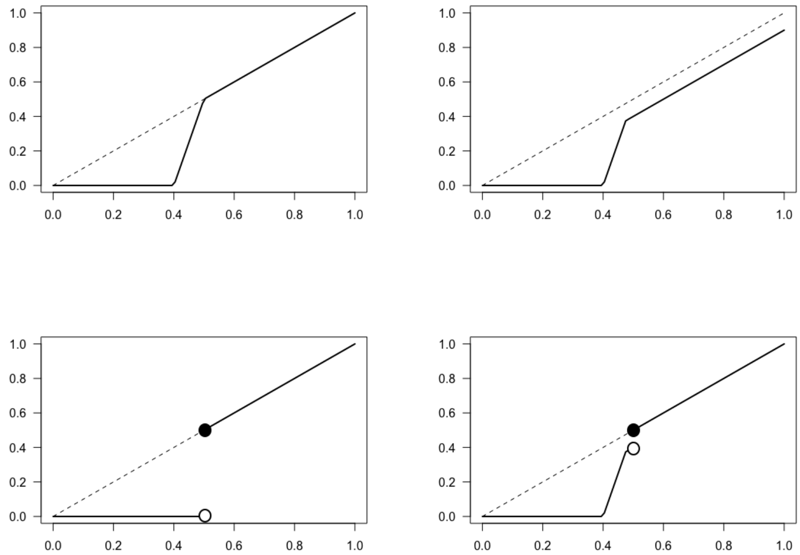

FIGURE 5 From top to bottom and from left to right. Mapping $f(x) = \max(0, \min(x, 5x-2))$ that do not satisfies Assumption 1; approximation of f by mean of Proposition 7; Idempotent mapping g; final approxximation.

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