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ORIGINAL PAPER

Mackey functors for posets

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Abstract

We define Mackey functors over posets mimicking the classical notion and introduce a weak version of them. Then we show that they are acyclic by analyzing cofibrant and pseudo-projective objects in the category of functors indexed in a filtered poset. As application, we study homotopy colimits over posets and we give a homology decomposition for the classifying space of the Bianchi group Γ_1.

Keywords Higher limit · Model category · Pseudo-projective · Mackey functor · Acyclic functor · Homology decomposition · Bianchi group

Mathematics Subject Classification 55P99 · 55R35 · 55N30 · 18G10

1 Introduction

Functors with values in abelian groups, as well as their higher direct and inverse limits, often occur in homotopy theory and group theory. For instance, in (co)homological decompositions [9], obstruction theory for maps [14], existence and uniqueness of the classifying space for fusion systems [4] or Alperin weight conjecture [21]. Fundamental vanishing results for higher limits, i.e., for colim-acyclicity or lim-acyclicity of a functor, are related to Mackey functors [6, 15] and Λ-functors [17, 18]. Under mild assumptions, higher limits over a category may be reduced to higher limits over a poset [26] and, for posets, well-known conditions for acyclicity are that either the functor satisfies the Mittag-Leffler condition or that the poset is directed.

Within the context of homological algebra, higher limits are the derived functors of direct and inverse limits. In this work, we follow the point of view of homotopy to study these derived functors over posets. More precisely, let $\mathcal{P}$ be a filtered poset, i.e., a poset equipped with an order-preserving map $\mathcal{P} \to \mathbb{N}$, and let $\mathrm{Ch}(R)$ be the category of unbounded chain complexes over a commutative ring with unit R. Then we endow the category of functors $\mathrm{Fun}(\mathcal{P}, \mathrm{Ch}(R))$ with a structure of model category such that a cofibrant functor $F : \mathcal{P} \to R\text{-mod}$ is colim-acyclic, where we see F concentrated in degree 0. We show that a functor is cofibrant if and only if it satisfies the following notion.

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Definition 1.1 A functor $F : \mathcal{P} \to R$-mod over a filtered poset is *pseudo-projective* at $i \in \mathcal{P}$ if, for every finite subset $J \subseteq \mathcal{P}_{\leq i}$ and every element $\oplus_{j\in J} x_j \in \bigoplus_{j\in J} F(j)$, the condition

$$\sum_{j\in J} F(j \leq i)(x_j) = 0 \tag{1}$$

implies that $x_j \in \mathrm{Im}_F(j) = \sum_{k<j} \mathrm{Im}\, F(k \leq j)$ for every $j \in \max J$. We say that F is *pseudo-projective* if F is pseudo-projective at $i \in \mathcal{P}$ for every $i \in \mathcal{P}$.

Note that, if $i \in J$, then $\max J = \{i\}$ and (1) implies directly that $x_i \in \mathrm{Im}_F(i)$. Pseudo-projectivity first appeared in [5], where it is shown that, for a poset $\mathcal{P}$ satisfying the descending chain condition (DCC for short), a functor $F : \mathcal{P} \to R$-mod is projective if and only if it is pseudo-projective and $F(i)/\,\mathrm{Im}_F(i)$ is free for any $i \in P$.

Theorem A *Let $\mathcal{P}$ be a filtered poset and $F : \mathcal{P} \to R$*-mod *be a functor. Then F is cofibrant if and only if F is pseudo-projective. Moreover, in this case, F is* colim*-acyclic.*

The part of colim-acyclicity easily follows from model category theory, hence providing an alternative proof of [5, Theorem B] when $\mathcal{P}$ is filtered. In addition, this theorem is useful in applications as it is often easier to study pseudo-projectivity than cofibrancy. As a first application of pseudo-projectivity, and inspired by the following example, we provide a definition of Mackey functors for posets below.

Example 1.2 Given a commutative ring with unit R, a *Mackey functor* for a finite group G, see [7], consists of a pair of functors (M_*, M^*) from the category of finite G-sets to the category of R-modules, such that M_* is covariant, M^* is contravariant, both coincide on objects and take disjoint union to direct sum, and they satisfy the Mackey formula:

$$M^*([\iota^U_V]) \circ M_*[\iota^U_W]) = \sum_{x\in[V\backslash U/W]} M_*([\iota^V_{V\cap {}^xW}]) \circ M_*([c_x]) \circ M^*([\iota^W_{V^x\cap W}]) \tag{2}$$

where $V, W \leq U$, $\iota^{\cdot}_{\cdot}$ denotes inclusion, c_x denotes conjugation by x, and $[V\backslash U/W]$ is a set of representatives in G for the double cosets $V\backslash U/W$. If we restrict these functors to the meet-semilattice of normal subgroups of G, formula (2) becomes

$$M^*([\iota^U_V]) \circ M_*([\iota^U_W]) = M_*([\iota^V_{V\cap W}]) \circ \left(\sum_{x\in[V\backslash U/W]} M_*([c_x]) \right) \circ M^*([\iota^W_{V\cap W}]).$$

Definition 1.3 For $\mathcal{P}$ a filtered meet-semilattice and a category $\mathcal{C}$, a pair of functors (F, G) is a *Mackey functor* if $F : \mathcal{P} \to \mathcal{C}$ is covariant, $G : \mathcal{P}^{op} \to \mathcal{C}$ is contravariant, $F(i) = G(i)$ for all $i \in \mathcal{P}$, and for all $j \leq i, k \leq i$ there exist

$$\alpha(i, j, k) \in \mathrm{End}_{\mathcal{C}}(F(j)),\ \beta(i, j, k) \in \mathrm{End}_{\mathcal{C}}(F(k \wedge j)), \text{and } \gamma(i, j, k) \in \mathrm{End}_{\mathcal{C}}(F(k))$$

such that

$$\begin{aligned} G(j \leq i) \circ F(k \leq i) =& \alpha(i, j, k) \circ F(k \wedge j \leq j) \circ G(k \wedge j \leq k) \\ =& F(k \wedge j \leq j) \circ \beta(i, j, k) \circ G(k \wedge j \leq k) \\ =& F(k \wedge j \leq j) \circ G(k \wedge j \leq k) \circ \gamma(i, j, k). \end{aligned}$$

We say that (F, G) has a *quasi-unit* if $\alpha(i, j, j) = \gamma(i, j, j) = \beta(i, j, j) \in \mathrm{Aut}_{\mathcal{C}}(F(j))$ for $j \leq i$.

The term *quasi-unit* is borrowed from [16, 5.7 Definition] and implies acyclicity. Example 1.2 restricted to the meet-semilattice of central subgroups is a Mackey functor with $\alpha(U, V, V)$ equal to multiplication by $|U/V|$, so that this Mackey functor has a quasi-unit if and only if these numbers are invertible in the ring R.

In addition, for $\mathcal{C} = R$-mod, we also define *weak Mackey functor* by dropping the contravariant functoriality and the meet-semilattice constraint; see Definition 3.2 for details. The covariant part of a Mackey functor is then a weak Mackey functor, and we have the following result.

Theorem B *Let* $F : \mathcal{P} \to R\text{-mod}$ *be a weak Mackey functor with a quasi-unit (or the covariant part of a Mackey functor with a quasi-unit). Then* F *is* colim*-acyclic.*

Notice that any functor $H : \mathcal{C} \to \mathcal{D}$ takes Mackey functors over $\mathcal{C}$ to Mackey functors over $\mathcal{D}$. A natural example of such a functor H is given by the homology functor $H_* : \mathrm{SSets} \to R\text{-mod}$. In the category of spaces SSets, as before, we can drop the contravariant functoriality and define weak Mackey functors over SSets, see Definition 3.4. We deduce the following topological application involving homotopy colimits and their homology.

Theorem C *Let* $F : \mathcal{P} \to \mathrm{SSets}$ *be a weak Mackey functor with a quasi-unit (or the covariant part of a Mackey functor with a quasi-unit). Then* $\mathrm{hocolim}_{\mathcal{P}} F \simeq \mathrm{colim}_{\mathcal{P}} F$ *and* $H_*(\mathrm{hocolim}_{\mathcal{P}} F) = \mathrm{colim}\, H_*(F)$.

In the language of [10], Theorem C asserts that the corresponding integral homological decomposition is *sharp*. This theorem may be applied to the construction of Davis-Januszkiewicz spaces and its generalization to polyhedral products over finite posets, see Examples 3.8 and 5.6. As a second application of pseudo-projectivity, we provide an integral homology decomposition of the classifying space of the Bianchi group $\Gamma_1 = \mathrm{PSL}_2(\mathcal{O}_1)$, where $\mathcal{O}_1$ is the ring of integers of the quadratic field $\mathbb{Q}(\sqrt{-1})$.

Theorem D *There exists a poset* $\mathcal{P}_1$ *of proper subgroups of* Γ_1 *and a homotopy equivalence*

$$\mathrm{hocolim}_{U \in \mathcal{P}_1} B(U) \xrightarrow{\simeq} B(\Gamma_1).$$

Here, the morphisms in $\mathcal{P}_1$ consists of the inclusions between the given subgroups. The notion of pseudo-projectivity is employed here to study the fiber of the natural map from the homotopy colimit to $B(\Gamma_1)$. It turns out that the homology of this fiber can be computed as the higher direct limits of an acyclic functor that is pseudo-projective at several but not all objects of $\mathcal{P}_1$. As a direct consequence of this theorem, we can determine the homology and cohomology of Γ_1 using the classical Bousfield–Kan spectral sequence, see Example 4.2. This contrasts with earlier computations of these (co)homology groups, in which the equivariant cohomology spectral sequence for an action of Γ_1 on a 2-complex is employed, see [1, 2, 25].

Notation: We denote by R a commutative ring with unit and by $\mathcal{P}$ a filtered poset unless stated otherwise. If I and J are subsets of $\mathcal{P}$, we write $J \leq I$ if every element of J is smaller than or equal to some element of I, and we define $J < I$ analogously. If $\mathcal{C}$ is a co-complete category, $F : \mathcal{P} \to \mathcal{C}$ is a functor, and $i \in \mathcal{P}$, we denote the natural map from the colimit as $\epsilon : \mathrm{colim}_{\mathcal{P}_{<i}} F \to F(i)$, and we write $[x]$ to denote the class in this colimit with representative x. If $\mathcal{C} = R$-mod and $x = \oplus_{i \in \mathcal{P}} x_i \in \bigoplus_{i \in \mathcal{P}} F(i)$, we set $\mathrm{supp}(x) = \{i \in \mathcal{P} \mid x_i \neq 0\}$.

Outline of the paper: We prove Theorem A in Sects. 2 and 6. In Sect. 3, we prove Theorems B and C. Posets of subgroups and the Bianchi group Γ_1 are studied in Sect. 4. The dual results are presented without proof in Sect. 5.

2 Model categories and homotopy theory

In this section, we equip the category of functors from the filtered poset $\mathcal{P}$ to unbounded chain complexes or to spaces with a model category structure.

Theorem 2.1 *There exists a model category on* $\operatorname{Fun}(\mathcal{P}, \operatorname{Ch}(R))$ *such that:*

(1) *a natural transformation* $\eta : F \Rightarrow G$ *is a weak equivalence if and only if* η_p *is a weak equivalence for every* $p \in \mathcal{P}$*;*
(2) *a functor* F *is cofibrant if only if the natural map* $\operatorname{colim}_{\mathcal{P}_{<i}} F \to F(i)$ *is injective for every* $i \in \mathcal{P}$*; and*
(3) *if a functor* $F : \mathcal{P} \to R\text{-mod}$ *is cofibrant, then it is* colim*-acylic, i.e.,* $\operatorname{colim}_n F = 0$ *for all* $n > 0$.

Proof We see $\mathcal{P}$ as Reedy category as in [24, 2.3 Example], i.e., we set $\overrightarrow{\mathcal{P}} = \mathcal{P}$ and $\overleftarrow{\mathcal{P}}$ equal to the underlying discrete category, and we endow $\operatorname{Ch}(R)$ with the injective model category structure, see [22, Theorem 18.5.4]. Hence, the weak equivalences and the cofibrations in $\operatorname{Ch}(R)$ are, respectively, the quasi-isomorphisms and the monomorphisms. Therefore, by the result of D. M. Kan [13, Theorem 15.3.4], we can equip $\operatorname{Fun}(\mathcal{P}, \operatorname{Ch}(R))$ with a model category structure satisfying points (1) and (2) of the statement and the following,

$$\eta : F \Rightarrow G \text{ is a fibration if and only if } \eta_p \text{ is a fibration in } \operatorname{Ch}(R) \text{ for every } p \in \mathcal{P}.$$

Now, the adjoint pair, $\operatorname{colim}_{\mathcal{P}} : \operatorname{Fun}(\mathcal{P}, \operatorname{Ch}(R)) \Leftrightarrow \operatorname{Ch}(R) : \Delta$, where Δ is the diagonal functor, is a Quillen pair as, by the comments above, Δ preserves acyclic fibrations and fibrations, see Theorem 9.7 and Remark 9.8 in [8]. Thus, by [13, Lemma 8.5.9], the total left derived functor $\operatorname{Ho}(\operatorname{Fun}(\mathcal{P}, \operatorname{Ch}(R))) \to \operatorname{Ho}(\operatorname{Ch}(R))$ exists and takes $[F]$ to $[\operatorname{colim}(QF)]$, where QF is a cofibrant replacement of F. As there exist enough projectives in $\operatorname{Fun}(\mathcal{P}, \operatorname{Ch}(R))$, we recover its usual i-th left derived functors as $\operatorname{colim}_i F = H_i(\operatorname{colim} QF)$. If F is cofibrant, we may choose $QF = F$ so that QF is concentrated in degree 0 and hence $H_i(\operatorname{colim} QF) = 0$ for $i > 0$. This proves (3) of the statement. □

Theorem A *Let* $\mathcal{P}$ *be a filtered poset, and* $F : \mathcal{P} \to R\text{-mod}$ *be a functor. Then,* F *is cofibrant if and only if* F *is pseudo-projective. Moreover, in this case,* F *is* colim*-acyclic.*

Proof The equivalence between cofibrant and pseudo-projective functors is postponed to Sect. 6. The latter conclusion of the theorem is exactly (3) in Theorem 2.1. □

Next, we prove a version of Theorem 2.1 for the category SSets of simplicial sets.

Theorem 2.2 *If* $\mathcal{P}$ *is a filtered poset, there exists a model category on* $\operatorname{Fun}(\mathcal{P}, \mathrm{SSets})$ *such that*

1. *a natural transformation* $\eta : F \Rightarrow G$ *is a weak equivalence if and only if* η_p *is a weak equivalence for every* $p \in \mathcal{P}$*;*
2. *a functor* F *is cofibrant if only if the natural map* $\operatorname{colim}_{\mathcal{P}_{<i}} F \to F(i)$ *is injective for every* $i \in \mathcal{P}$*; and*
3. *a cofibrant functor* $F : \mathcal{P} \to \mathrm{SSets}$ *satisfies that* $\operatorname{hocolim}_{\mathcal{P}} F \simeq \operatorname{colim}_{\mathcal{P}} F$.

Proof We consider $\mathcal{P}$ as a Reedy category as in the proof of Theorem 2.1. In SSets, we consider the usual model category structure [8, 11.1]: the weak equivalences are those simplicial maps whose topological realization is a weak homotopy equivalence, cofibrations are the (degreewise) injections, and the fibrations are the Kan fibrations. Again by [13, Theorem

15.3.4], we equip $\mathrm{Fun}(\mathcal{P}, \mathrm{SSets})$ with a model category structure satisfying points (1) and (2) of the statement and the following,

$$\eta : F \Rightarrow G \text{ is a fibration if and only if } \eta_p \text{ is a fibration in SSets for every } p \in \mathcal{P}.$$

Now, the adjoint pair, $\mathrm{colim}_{\mathcal{P}} : \mathrm{Fun}(\mathcal{P}, \mathrm{SSets}) \Leftrightarrow \mathrm{SSets} : \Delta$, where Δ is the diagonal functor, is a Quillen pair as Δ preserves acyclic fibrations and fibrations. By [13, Lemma 8.5.9], the total left derived functor $\mathrm{hocolim}_{\mathcal{P}} : \mathrm{Ho}(\mathrm{Fun}(\mathcal{P}, \mathrm{SSets})) \to \mathrm{Ho}(\mathrm{SSets}))$ exists and maps $[F]$ to $[\mathrm{colim}(QF)]$, where $F : \mathcal{P} \to \mathrm{SSets}$ and QF is its cofibrant replacement. If F is cofibrant we may choose $QF = F$ so that $\mathrm{hocolim}_{\mathcal{P}} F \simeq \mathrm{colim}_{\mathcal{P}} F$, obtaining (3) of the statement.

3 Weak Mackey functors

In this section, we weaken Definition 1.3 for $\mathcal{C} = R$-mod and $\mathcal{C} = \mathrm{SSets}$. We start by defining a certain class of morphisms that commute with a given functor.

Definition 3.1 Let $F : \mathcal{P} \to \mathcal{C}$ be a functor between a poset $\mathcal{P}$ and a category $\mathcal{C}$, and $i \in \mathcal{P}$. We say that $\alpha \in \mathrm{End}_{\mathcal{C}}(F(i))$ ($\alpha \in \mathrm{Aut}_{\mathcal{C}}(F(i))$) is *$F$-linear* if for all $j \leq i$

$$\alpha \circ F(j \leq i) = F(j \leq i) \circ \beta$$

for some $\beta \in \mathrm{End}_{\mathcal{C}}(F(j))$ ($\beta \in \mathrm{Aut}_{\mathcal{C}}(F(j))$). We denote by $\mathrm{End}^F_{\mathcal{C}}(i)$ ($\mathrm{Aut}^F_{\mathcal{C}}(i)$) the submonoid (subgroup) of F-linear endomorphisms (automorphisms) of $F(i)$.

Regarding Definition 1.3, it is not hard to see that the morphisms $\alpha(i, j, j)$ are F-linear, see Remark 3.3 for details. Moreover, for $\mathcal{C} = R$-mod, and $j \nleq k$, we have $k \wedge j \neq j$ and $G(j \leq i) \circ F(k \leq i)$ lands in $\mathrm{Im}_F(j)$. This fact and earlier comments motivate the following definition, where we recall that the meet-semilattice condition is dropped.

Definition 3.2 Let $\mathcal{P}$ be a filtered poset and let $F : \mathcal{P} \to R$-mod a covariant functor. We say that F is a *weak Mackey functor* if for all $j \leq i$ there exists a morphism in R-mod, $G(j \leq i) : F(i) \to F(j)$, such that $G(j \leq i) \circ F(j \leq i) = \alpha(i, j)$ with $\alpha(i, j) \in \mathrm{End}^F_{R\text{-mod}}(j)$, and, for $k < i$, $j \nleq k$,

$$\mathrm{Im}(G(j \leq i) \circ F(k \leq i)) \subseteq \mathrm{Im}_F(j).$$

We say that F has a *quasi-unit* if $\alpha(i, j) \in \mathrm{Aut}^F_{R\text{-mod}}(j)$ for every $j \leq i$.

Remark 3.3 If (F, G) is a Mackey functor in the sense of Definition 1.3, then $\alpha(i, j, j)$ is F-linear: If we let $j = k$ in Definition 1.3, we obtain

$$G(j \leq i) \circ F(j \leq i) = \alpha(i, j, j) = \beta(i, j, j) = \gamma(i, j, j),$$

if we let $k \leq j \leq i$ in that definition, we get to

$$G(j \leq i) \circ F(k \leq i) = F(k \leq j) \circ \beta(i, j, k),$$

and a short computation shows that

$$G(j \leq i) \circ F(k \leq i) = G(j \leq i) \circ F(j \leq i) \circ F(k \leq j) = \alpha(i, j, j) \circ F(k \leq j).$$

Thus, $\alpha(i, j, j)$ is F-linear. In the particular case $\mathcal{C} = R$-mod, we obtain that F is a weak Mackey functor by setting $\alpha(i, j) = \alpha(i, j, j)$. Moreover, if (F, G) has quasi-unit, then F has quasi-unit as a weak Mackey functor.

Theorem B *Let* $F : \mathcal{P} \to R$-mod *be a weak Mackey functor with a quasi-unit. Then* F *is* colim-*acyclic.*

Proof By Theorem A, it is enough to show that F is pseudo-projective. So let $i \in \mathcal{P}$, $J \subseteq \mathcal{P}_{\leq i}$, J finite, and $\oplus_{j \in J} x_j \in \bigoplus_{j \in J} F(j)$ such that

$$\sum_{j \in J} F(j \leq i)(x_j) = 0.$$

For $k \in \max J$, we want to prove that $x_k \in \operatorname{Im}_F(k)$. Applying the morphism $G(k \leq i)$ to the equation above we get

$$\begin{aligned} 0 = G(k \leq i)(\sum_{j \in J} F(j \leq i)(x_j)) &= \sum_{j \in J} (G(k \leq i) \circ F(j \leq i))(x_j) \\ &= (G(k \leq i) \circ F(k \leq i)(x_k) + \sum_{j \in J,\, k \not\leq j} (G(k \leq i) \circ F(j \leq i))(x_j), \end{aligned}$$

where the first summand corresponds to $j = k$. Now, as F is weak Mackey, this summand is equal to $\alpha(i, k)(x_k)$, and every other other summand belongs to $\operatorname{Im}_F(k)$ by Definition 3.2, so that

$$\alpha(i, k)(x_k) = \sum_{l < k} F(l \leq k)(y_l)$$

for some elements $y_l \in F(l)$. As $\alpha(i, k)$ is invertible and F-linear, we can solve for x_k as follows,

$$x_k = \sum_{l < k} (\alpha(i, k)^{-1} \circ F(l \leq k))(y_l) = \sum_{l < k} (F(l \leq k) \circ \beta_l)(y_l)$$

for some automorphisms $\beta_l \in \operatorname{Aut}_{R\text{-mod}}(F(l))$. Hence $x_k \in \operatorname{Im}_F(k)$ and we are done. □

By dropping the contravariant functoriality in Definition 1.3, we obtain the following weaker notion for $\mathcal{C} = \text{SSets}$.

Definition 3.4 For $\mathcal{P}$ a filtered meet-semilattice and $F : \mathcal{P} \to \text{SSets}$ a covariant functor. We say that F is a *weak Mackey functor* if there exists a collection of morphisms,

$$\{G(j \leq i) : F(i) \to F(j)\}_{j \leq i},$$

such that for all $j \leq i, k \leq i$ there exist

$$\alpha(i, j, k) \in \operatorname{End}_{\text{SSets}}(F(j)) \text{ and } \beta(i, j, k) \in \operatorname{End}_{\text{SSets}}(F(k \wedge j))$$

with

$$\begin{aligned} G(j \leq i) \circ F(k \leq i) =& \alpha(i, j, k) \circ F(k \wedge j \leq j) \circ G(k \wedge j \leq k) \\ =& F(k \wedge j \leq j) \circ \beta(i, j, k) \circ G(k \wedge j \leq k). \end{aligned}$$

We say that F has a *quasi-unit* if $\alpha(i, j, j) \in \operatorname{Aut}_{\text{SSets}}(F(j))$ for all $j \leq i$.

Remark 3.5 Notice that the covariant part of a Mackey functor over SSets is a weak Mackey functor and, by Remark 3.3, $\alpha(i, j, j)$ in Definition 3.4 is F-linear. Moreover, for any functor $H : \text{SSets} \to R$-mod and any weak Mackey functor $F : \mathcal{P} \to \text{SSets}$, the composition $H \circ F : \mathcal{P} \to R$-mod is a weak Mackey functor.

Remark 3.6 If $F : \mathcal{P} \to \mathcal{C}$ is a weak Mackey functor for $\mathcal{C} = R$-mod or $\mathcal{C} =$ SSets and it has a quasi-unit, then F is *monic*, i.e., $F(i \leq j)$ is injective for all $i \leq j$: In Definition 3.2, we get that $G(j \leq i) \circ F(i \leq j)$ is the automorphism $\alpha(i, j)$ and, in Definition 3.4, the case $j = k$ gives that $G(j \leq i) \circ F(j \leq i)$ is the automorphism $\alpha(i, j, j)$.

The next lemma will be employed to prove Theorem C below.

Lemma 3.7 *If* $F : \mathcal{P} \to$ SSets *is a weak Mackey functor with a quasi-unit, then F is cofibrant.*

Proof Let $i \in \mathcal{P}$. We want to check that the map of simplicial sets $\epsilon : \operatorname{colim}_{\mathcal{P}_{<i}} F \to F(i)$ is (degreewise) injective. We choose a degree n and we omit the sub-index n to denote n-simplices in order to avoid cluttering. Recall that

$$\operatorname{colim}_{\mathcal{P}_{<i}} F = \bigcup_{j<i} F(j)/\sim,$$

where $x \sim F(k \leq j)(x)$ for $k < j < i$. We denote by $[x]$ the image by $F(j) \to \operatorname{colim}_{\mathcal{P}_{<i}} F$ of $x \in F(j)$. The natural map ϵ is defined by $\epsilon([x]) = F(j \leq i)(x)$ if $x \in F(j)$. Assume that $y = \epsilon([x]) = \epsilon([x'])$ for $x \in F(j)$, $x' \in F(j')$. Applying the definition of Mackey functor we have then

$$\begin{aligned}
&G(j \leq i)(y) = G(j \leq i)(F(j \leq i))(x)) = \alpha(i, j, j)(x), \text{ and}\\
&G(j \leq i)(y) = G(j \leq i)(F(j' \leq i))(x')) = \alpha(i, j, j')(F(j \wedge j' \leq j)(G(j \wedge j' \leq j')(x'))).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\alpha(i, j, j)(x) &= \alpha(i, j, j')(F(j \wedge j' \leq j)(G(j \wedge j' \leq j')(x')))\\
&= F(j \wedge j' \leq j)(\beta(i, j, j')(G(j \wedge j' \leq j')(x')))
\end{aligned}$$

and, as $\alpha(i, j, j)$ is an F-linear automorphism, we get to

$$x = F(j \wedge j' \leq j)(x'')$$

for some $x'' \in F(j \wedge j')$. In particular,

$$F(j \wedge j' \leq i)(x'') = F(j \leq i)(F(j \wedge j' \leq j)(x'')) = F(j \leq i)(x) = y.$$

Switching the roles of x and x' we obtain $x''' \in F(j \wedge j')$ with $F(j \wedge j' \leq i)(x''') = y$. As $F(j \wedge j' \leq i)$ is injective, see Remark 3.6, we get that $x'' = x'''$. So we have $x \sim x'' = x''' \sim x'$, i.e., $[x] = [x']$. □

Theorem C *Let* $F : \mathcal{P} \to$ SSets *be a weak Mackey functor with a quasi-unit. Then* $\operatorname{hocolim}_{\mathcal{P}} F \simeq \operatorname{colim}_{\mathcal{P}} F$ *and* $H_*(\operatorname{hocolim}_{\mathcal{P}} F) = \operatorname{colim} H_*(F)$.

Proof The fact that $\operatorname{hocolim}_{\mathcal{P}} F \simeq \operatorname{colim}_{\mathcal{P}} F$ is a consequence of Theorem 2.2(3) and Lemma 3.7. In addition, by Remark 3.5, for all $n \geq 0$, $H_n \circ F$ is a weak Mackey functor with codomain R-mod, where H_n is homology in degree n. Applying then Theorem B we get that $\operatorname{colim}_i(H_n \circ F) = 0$ for $i > 0$. Then the Bousfield–Kan homology spectral sequence for the homotopy colimit $\operatorname{hocolim}_{\mathcal{P}} F$ collapses and gives the statement that $H_*(\operatorname{hocolim}_{\mathcal{P}} F) = \operatorname{colim} H_*(F)$. □

Example 3.8 One example of Mackey functors for posets are twin functors for the face poset of a simplicial complex. For any simplicial complex K and its face poset $\mathrm{CAT}(K)$, the twin functors $X^K : \mathrm{CAT}(K) \to \mathrm{Top}$ and $X_K : \mathrm{CAT}(K)^{\mathrm{op}} \to \mathrm{Top}$ in [23, Definition 2.6] form a

Mackey functor in the sense of Definition 1.3 with a quasi-unit and with $\alpha(i, j, k) = 1_{F(j)}$ for all $j \leq i, k \leq i$ of $\mathrm{CAT}(K)$. Hence, [23, Equation 2.8],

$$\operatorname{hocolim} X^K \to \operatorname{colim} X^K \text{ is a homotopy equivalence,}$$

is a direct consequence of Theorem C.

4 Poset of subgroups

In this section, we fix a group G and denote by $\mathcal{P}$ a subposet of the poset of the subgroups of G. We require that $1 \in \mathcal{P}$, that $G \cong \operatorname{colim}_{U\in\mathcal{P}} U$, and that the inclusions

$$U \hookrightarrow G$$

for each $U \in \mathcal{P}$ form the natural transformation associated to the colimit. Under these conditions, the arguments in the proof [5, Theorem 5.1] give a fibration

$$F \to \operatorname{hocolim}_{U\in\mathcal{P}} B(U) \to B(G), \tag{3}$$

where F is simply connected. Moreover, if we define $H : \mathcal{P} \to \mathrm{Ab}$ on $U \in \mathcal{P}$ by

$$H(U) = \left\{ \sum_{g\in G} n_g \cdot g \in \mathbb{Z}[G] \mid \sum_{u\in U} n_{ug} = 0 \text{ for all } g \in G \right\}$$

and by sending inclusions in $\mathcal{P}$ to inclusions in Ab, then we have

$$H_n(F; \mathbb{Z}) = \operatorname{colim}_{n-1} H \text{ for } n \geq 2.$$

Note that if H is acyclic then $F \simeq *$ and we have a homotopy equivalence between the homotopy colimit and $B(G)$ in (3). The following result gives conditions on the subgroups in the ray $\mathcal{P}_{\leq U}$ that are sufficient for H to be pseudo-projective at U. Recall that, since every element V of $\mathcal{P}$ is a subgroup of G, it makes sense to consider the subgroup of G generated by a collection M of subgroups of G,

$$\langle V \mid V \in M \rangle \leq G.$$

Proposition 4.1 *If for every finite subset $J \subseteq \mathcal{P}_{\leq U}$, the subgroup $\langle V \mid V \in \max J\rangle$ is the amalgamated product of the groups $\{V\}_{V\in\max J}$ over the subgroup $\bigcap_{V\in\max J} V \in \mathcal{P}$, then H is pseudo-projective at U.*

Proof We write $M = \max J$ for simplicity. Given

$$\oplus_{V\in J} x_V \in \bigoplus_{V\in J} H(V) \text{ with } \sum_{V\in J} x_V = 0,$$

we need to show that

$$x_V \in \operatorname{Im}_H(V) \text{ for every } V \in M.$$

As $J \leq M$, we may rewrite the sum $\sum_{V\in J} x_j$ as $\sum_{V\in M} x'_V$, where $x'_V \in H(V)$ for each $V \in M$. Note that $x_V - x'_V \in \operatorname{Im}_H(V)$ for all $V \in M$, so that it is enough to show that

$$x'_V \in \operatorname{Im}_H(V) \text{ for every } V \in M$$

subject to the condition

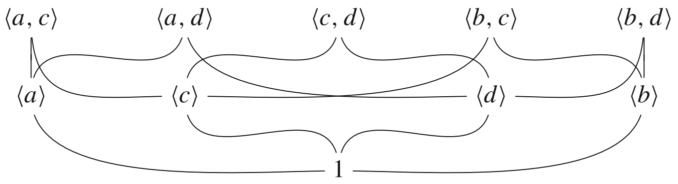
$$\sum_{V\in M} x'_V = 0.$$

This follows by the proof of [20, Theorem 1] adapted to an arbitrary number $|M|$ of subgroups.

In the rest of this section, we show that for the Bianchi group Γ_1 and a suitable chosen poset, the functor H is acyclic but not pseudo-projective. In fact, consider the following presentation of Γ_1, see [11],

$$\Gamma_1 = \langle a, b, c, d \mid a^3 = b^2 = c^3 = d^2 = (ac)^2 = (ad)^2 = (bc)^2 = (bd)^2 = 1\rangle,$$

as well as the following poset $\mathcal{P}_1$ of subgroups of Γ_1,



Theorem D *There is a homotopy equivalence,*

$$\operatorname{hocolim}_{U\in\mathcal{P}_1} B(U) \xrightarrow{\simeq} B(\Gamma_1).$$

Proof The usual chain complex [12] gives that $\operatorname{colim}_n H = 0$ for $n \geq 2$ and that $\operatorname{colim}_1 H$ consists of the tuples

$$(x_a, x'_a, x_b, x'_b, x_c, x'_c, x''_c, x_d, x'_d, x''_d),$$

where $x^{\cdot}_y \in H(\langle y\rangle)$, such that

$$\begin{aligned} &x_a + x'_a = 0, x_b + x'_b = 0, x_c + x'_c + x''_c = 0, x_d + x'_d + x''_d = 0, \text{ and}\\ &x_a + x_c = 0, x'_a + x_d = 0, x'_c + x'_d = 0, x_b + x''_c = 0, x'_b + x''_d = 0. \end{aligned} \tag{4}$$

As $\langle c, d\rangle = \langle c\rangle * \langle d\rangle \cong \mathrm{PSL}_2(\mathbb{Z})$, by Proposition 4.1 we have that H is pseudo-projective at this subgroup. As $H(1) = 0$, we have $\operatorname{Im}_H(\langle c\rangle) = \operatorname{Im}_H(\langle d\rangle) = 0$, and hence

$$C + D = 0 \Rightarrow C = D = 0 \text{ for any } C \in H(\langle c\rangle) \text{ and any } D \in H(\langle d\rangle). \tag{5}$$

From (4), it is easy to deduce that

$$x_c + x_d = x'_c + x'_d = x''_c + x''_d = 0,$$

and hence, from (5), we get to $x_c = x_d = x'_c = x'_d = x''_c = x''_d = 0$. Using (4) again, we have that $x_a = x'_a = x_b = x'_b = 0$ too, i.e., $\operatorname{colim}_1 H = 0$. □

Example 4.2 The Bousfield–Kan spectral sequence [3, XII.5.7] of the homotopy colimit in Theorem D converges to $H_*(B(\Gamma_1); \mathbb{Z})$ and has second page

$$E^2_{p,q} = \operatorname{colim}_p H_q(B(\cdot); \mathbb{Z}).$$

In addition, we have $\langle a, c\rangle \cong A_4$, $\langle a, d\rangle \cong \langle b, c\rangle \cong S_3$, and $\langle b, d\rangle \cong K$, where A_4 is alternating on 4 letters, S_3 is symmetric on 3 letters and K is the Klein group. Note that, by the descriptions of these subgroups and [20, Theorem 1], H is not pseudo-projective. Next, we refer the reader to [25, Section 4] for the homology of A_4, S_3, and K, and determine $E^2_{p,q}$.

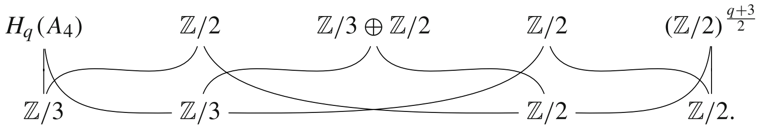
For $q = 0$ we get a constant diagram over a contractible poset, so $E^2_{p,0} = \mathbb{Z}$ for $p = 0$ and $E^2_{p,0} = 0$ for $p > 0$. For q even, $q > 0$, $H_q(B(\cdot); \mathbb{Z})$ takes values different from 0 only on $\langle a, c\rangle$ and $\langle b, d\rangle$. Thus,

$$\operatorname{colim}_0 H_q(B(\cdot); \mathbb{Z}) = H_q(A_4) \oplus H_q(K) = (\mathbb{Z}/2)^l \text{ for some } l, \tag{6}$$

and $E^2_{p,q} = 0$ for $p > 0$. For q odd, we have that

$$\operatorname{colim}_1 H_q(B(\cdot); \mathbb{Z}) \text{ is 3-torsion.} \tag{7}$$

In fact, if $q \equiv 1 \pmod 4$, the functor $H_q(B(\cdot); \mathbb{Z})$ can be depicted as follows,



It is easily checked that this functor is pseudo-projective at $\langle c, d\rangle$ and $\langle b, d\rangle$. Then, a similar computation to that in the proof of Theorem D, yields that there is no 2-torsion in $\operatorname{colim}_1 H_q(B(\cdot); \mathbb{Z})$. For $q \equiv 3 \pmod 4$, the computation is easier as the functor turns out to be pseudo-projective at $\langle a, d\rangle$, $\langle c, d\rangle$, $\langle b, c\rangle$, and $\langle b, d\rangle$.

Summing up, $E^2_{p,q}$ is non-zero only at $(p, q) \in \{(0, k), (1, 2k + 1)\}_{k\geq 0}$. In particular, the spectral sequence collapses at E_2. In addition, Eqs. (6) and (7) ensure that we can solve the extension problems and then get

$$H_n(B(\Gamma_1); \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0, \\ E^2_{0,n}, & \text{if } nisodd, \\ E^2_{0,n} \oplus E^2_{1,n-1}, & \text{if } niseven, n > 0. \end{cases}$$

Explicit formulae for this result can be found in [25, Theorem 5.5].

5 Dual

We devote this section to dualize the previous results and we do not include the corresponding proofs. For a filtered poset $\mathcal{P}$, it is immediate to dualize Theorem 2.1 to obtain a model category on $\operatorname{Fun}(\mathcal{P}^{\mathrm{op}}, \operatorname{Ch}(R))$ such that $G : \mathcal{P}^{\mathrm{op}} \to R$-mod is fibrant if and only if

$$G(i) \to \lim_{\mathcal{P}_{<i}} G \text{ is an epimorphism for every } i \in \mathcal{P}.$$

This condition is equivalent to the following notion.

Definition 5.1 A functor $G : \mathcal{P}^{\mathrm{op}} \to R$-mod over a filtered poset is *pseudo-injective* at $i \in \mathcal{P}$ if, for every subset $J \subset \mathcal{P}_{\leq i}$, and elements $x_j \in \ker_G(j) = \bigcap_{k<j} \ker(G(k \leq j))$ with $j \in J$, there exists $x \in G(i)$ with $G(j \leq i)(x) = x_j$ for every $j \in \max J$. We say that G is *pseudo-injective* if G is pseudo-injective at $i \in \mathcal{P}$ for every $i \in \mathcal{P}$.

Theorem A* *Let $\mathcal{P}$ be a filtered poset, and $G : \mathcal{P}^{\mathrm{op}} \to R$-mod be a functor. Then G is fibrant if and only if G is pseudo-injective. Moreover, in this case, G is* lim*-acyclic.*

Definition 5.2 Let $G : \mathcal{P}^{\mathrm{op}} \to \mathcal{C}$ be a functor between a poset $\mathcal{P}$ and a category $\mathcal{C}$, and $i \in \mathcal{P}$. We say that $\gamma \in \operatorname{End}_{\mathcal{C}}(G(i))$ ($\gamma \in \operatorname{Aut}_{\mathcal{C}}(G(i))$) is *$G$-linear* if for all $j < i$

$$G(j \leq i) \circ \gamma = \beta \circ G(j \leq i)$$

for some $\beta \in \mathrm{End}_{\mathcal{C}}(G(j))$ ($\beta \in \mathrm{Aut}_{\mathcal{C}}(G(j))$). We denote by $\mathrm{End}^G_{\mathcal{C}}(i)$ ($\mathrm{Aut}^G_{\mathcal{C}}(i)$) the submonoid (subgroup) of G-linear endomorphisms (automorphisms) of $G(i)$.

Definition 5.3 Let $\mathcal{P}$ be a filtered poset and let $G : \mathcal{P}^{\mathrm{op}} \to R$-mod a contravariant functor. We say that G is a *weak Mackey functor* if for all $j < i$ there exists a morphism in R-mod, $F(j \leq i) : G(j) \to G(i)$, such that $G(j \leq i) \circ F(j \leq i) = \alpha(i, j)$ with $\alpha(i, j) \in \mathrm{End}^G_{R\text{-mod}}(j)$, and, for $k < i$, $j \not< k$,

$$\ker_G(k) \subseteq \ker(G(j \leq i) \circ F(k \leq i))$$

We say that G has a *quasi-unit* if $\alpha(i, j) \in \mathrm{Aut}^G_{R\text{-mod}}(j)$.

Remark 5.4 The contravariant part of a Mackey functor over R-mod is a weak Mackey functor.

Theorem B* *Let $G : \mathcal{P}^{op} \to R$-mod be a weak Mackey functor with a quasi-unit (or the contravariant part of a Mackey functor with a quasi-unit). Then G is* lim-*acyclic.*

Remark 5.5 If $F : \mathcal{P} \to$ SSets is the covariant part of a Mackey functor (or a weak Mackey functor) with quasi-unit, and $H : \mathrm{SSets}^{\mathrm{op}} \to R$-mod is a contravariant functor, then $H \circ F : \mathcal{P}^{\mathrm{op}} \to R$-mod is a contravariant weak Mackey functor.

Example 5.6 For a finite poset $\mathcal{P}$ with an initial object, a notion of functor $G : \mathcal{P}^{op} \to \mathrm{Ab}$ with a *lower factoring section* is given in [19, Definition 2.5]. If $\mathcal{P}$ is filtered, it is immediate that such a functor is a weak Mackey functor according to Definition 5.3, taking as quasi-unit $\alpha(i, j) = 1_{F(j)}$. Then [19, Theorem A] is a direct consequence of Theorem B*.

6 The proof of Theorem A

We start noticing that Definition 1.1 makes sense for functors $F : \mathcal{P} \to R$-mod, where $\mathcal{P}$ is an arbitrary poset (not necessarily filtered). In addition, over such a poset, we say that F is *cofibrant* at $i \in \mathcal{P}$ if the natural map

$$\operatorname*{colim}_{\mathcal{P}_{<i}} F \to F(i)$$

is injective. We say that F is cofibrant if it is cofibrant at i for every $i \in \mathcal{P}$. Note that we have abused notation and termed some functors *cofibrant* without any reference to a model category structure. If $\mathcal{P}$ is filtered, then cofibrant has the usual meaning because of Theorem 2.1(2).

In the next result, we prove the equivalence between cofibrant functors and pseudo-projective functors over any DCC poset. This includes the particular case of Theorem A for filtered posets. The equivalence in Theorem A* between fibrant and pseudo-injective functors may be extended to DCC posets with the additional assumption that the functor takes Artinian R-modules as values. We do not know whether it is true for arbitrary functors over DCC posets.

Theorem 6.1 *Let $\mathcal{P}$ be a* DCC *poset and $F : \mathcal{P} \to R$*-mod *be a functor. Then F is cofibrant if and only if F is pseudo-projective.*

We divide the proof into several lemmas.

Lemma 6.2 *Let $\mathcal{P}$ be a* DCC *poset, $i \in \mathcal{P}$ and $F : \mathcal{P} \to R$-*mod *be a functor such that F is pseudo-projective at i. Let $x = \oplus_{j<i} x_j \in \bigoplus_{j<i} F(j)$ satisfy*

$$\sum_{j<i} F(j \leq i)(x_j) = 0.$$

Then, there is a sequence $\{x^n\}_{n\geq 0}$, $x^n = \bigoplus_{j<i} x^n_j \in \bigoplus_{j<i} F(j)$, with $x^0 = x$,

$$\sum_{j<i} F(j \leq i)(x^n_j) = 0, \qquad x^{n+1} - x^n = \sum_{k<j\in \max \operatorname{supp}(x^n)} y_{k,j} \oplus -F(k \leq j)(y_{k,j}),$$

$[x^{n+1}] = [x^n]$ in $\operatorname{colim}_{j<i} F(j)$, and $\operatorname{supp}(x^{n+1}) < \operatorname{supp}(x^n)$, for any $n \geq 0$, where $y_{k,j} \in F(k)$. In addition, there exists $N > 0$ such that $x^n_j = 0$ for all $j < i$ if $n \geq N$.

Proof This is a finer reformulation of [5, Lemma 2.3] and we provide details. We define $x^{-1} = 0$ and work by induction on $n \geq 0$ assuming that x^n satisfying the properties in the statement has been already constructed. Then, as $\sum_{j<i} F(j \leq i)(x^n_j) = 0$ and F is pseudo-projective at i, for every $j \in \max \operatorname{supp}(x^n)$ we have that $x^n_j \in \operatorname{Im}_F(j)$, i.e., there exists $\oplus_{k<j} y_{k,j} \in \bigoplus_{k<j} F(k)$ such that

$$x^n_j = \sum_{k<j} F(k \leq j)(y_{k,j}).$$

For every pair (k, j) with $k < j < i$, we set

$$x_{k,j} = \begin{cases} y_{k,j} & \text{if } k < j \in \max \operatorname{supp}(x^n), \\ x^n_j & \text{if } k = j \notin \max \operatorname{supp}(x^n), \\ 0 & \text{otherwise,} \end{cases}$$

and we define $x^{n+1} = \bigoplus_{j<i} x^{n+1}_j$ by

$$x^{n+1}_j = \sum_{k\geq j} x_{j,k}.$$

Then

$$\sum_{j<i} F(j \leq i)(x^{n+1}_j) = \sum_{j<i} F(j \leq i) \left(\sum_{k\geq j} x_{j,k} \right) = \sum_{j\leq k<i} F(j \leq i)(x_{j,k}). \tag{8}$$

In this last sum, if $j = k \notin \max \operatorname{supp}(x^n)$, the corresponding summand is $F(j \leq i)(x^n_j)$. The other summands can be reordered as follows,

$$\sum_{\substack{j<k<i \\ k\in \max \operatorname{supp}(x^n)}} F(j \leq i)(y_{j,k}) = \sum_{\substack{k<i \\ k\in \max \operatorname{supp}(x^n)}} F(k \leq i) \left(\sum_{j<k} F(j \leq k)(y_{j,k}) \right) = \sum_{\substack{k<i \\ k\in \max \operatorname{supp}(x^n)}} F(k \leq i)(x^n_k).$$

Hence the sum (8) equals $\sum_{j<i} F(j \leq i)(x^n_j)$, and this is 0 by hypothesis. From the construction above it easily follows that

$$x^{n+1} - x^n = \sum_{k<j\in \max \operatorname{supp}(x^n)} y_{k,j} \oplus -F(k \leq j)(y_{k,j}),$$

and, from here, it is clear that $[x^{n+1}] = [x^n]$ in $\operatorname{colim}_{j<i} F(j)$ and $\operatorname{supp}(x^{n+1}) < \operatorname{supp}(x^n)$. From this latter condition and [5, Lemma 2.6] we obtain $N > 0$ with the stated property.

Lemma 6.3 *Let $\mathcal{P}$ be a DCC poset, $F : \mathcal{P} \to R\text{-mod}$, and $i \in \mathcal{P}$. If F is pseudo-projective at i, then F is cofibrant at i.*

Proof Let $\epsilon : \operatorname{colim}_{\mathcal{P}_{<i}} F \to F(i)$ be the natural map and consider $[x] \in \ker(\epsilon)$ with $x \in \bigoplus_{j<i} F(j)$. By Lemma 6.2, there exists a sequence $\{x^n\}_{n \geq 0}$ with $x^n \in \bigoplus_{j<i} F(j)$ such that $x^0 = x$, $[x^{n+1}] = [x^n]$ and $x^N = 0$ for N big enough. Hence $[x^0] = [x^N] = [0] = 0$ and the Lemma is proven. □

Lemma 6.4 *Let $\mathcal{P}$ be a* DCC *poset, $F : \mathcal{P} \to R\text{-mod}$, and $i \in \mathcal{P}$. If F is cofibrant at j for every $j \leq i$, then F is pseudo-projective at j for every $j \leq i$.*

Proof Since $\mathcal{P}_{\leq i}$ is a DCC poset, we proceed by induction. If j is minimal in $\mathcal{P}_{\leq i}$, then F is pseudo-projective at j by definition. Thus, consider now $j \leq i$ such that F is pseudo-projective at k for all $k < j$. We show that F is pseudo-projective at j too. So let $x = \oplus_{k<j} x_k \in \bigoplus_{k<j} F(k)$ be such that

$$\sum_{k<j} F(k \leq j)(x_k) = 0.$$

This is equivalent to that $\epsilon([x]) = 0$ for the natural map $\epsilon : \operatorname{colim}_{k<j} F \to F(j)$. By hypothesis, F is cofibrant at j, and hence $[x] = 0$. In turn, this equality is equivalent to the existence of elements $y_{l,k} \in F(l)$ for $l < k < j$ such that finitely many of them are different from zero and with

$$x = \oplus_{k<j} x_k = \sum_{l<k<j} y_{l,k} \oplus -F(l \leq k)(y_{l,k}), \tag{9}$$

which implies that, for any $k < j$,

$$x_k = \sum_{k<l} y_{k,l} - \sum_{l<k} F(l \leq k)(y_{l,k}). \tag{10}$$

Let $K = \{k \in \mathcal{P}_{<j} \mid \exists\, l < k \text{ with } y_{l,k} \neq 0\}$. We wish to show that we can choose the elements $y_{l,k}$'s appearing in (9) subject to the constraint that $\max K \subseteq \operatorname{supp}(x)$. Thus let $m \in \max K \setminus \operatorname{supp}(x)$. Then

$$x_m = 0 = -\sum_{l<m} F(l \leq m)(y_{l,m}). \tag{11}$$

We can rewrite Eq. (9) as follows,

$$\begin{aligned} x = &\sum_{l<m<j} y_{l,m} \oplus -F(l \leq m)(y_{l,m}) + \sum_{\substack{l<k<j \\ k \neq m}} y_{l,k} \oplus -F(l \leq k)(y_{l,k}) \\ =&\Big(\oplus_{l<m}\, y_{l,m} \Big) - \Big(\oplus_m \sum_{l<m} F(l \leq m)(y_{l,m}) \Big) + \sum_{\substack{l<k<j \\ k \neq m}} y_{l,k} \oplus -F(l \leq k)(x_{l,k}). \end{aligned}$$

which, by Eq. (11), we can simplify to

$$x = y + \sum_{\substack{l<k<j \\ k \neq m}} y_{l,k} \oplus -F(l \leq k)(y_{l,k}), \tag{12}$$

where $y = \oplus_{l<m} y_{l,m} \in \oplus_{l<m} F(l)$. As F is pseudo-projective at $m < j$ by induction hypothesis, we may apply Lemma 6.2 to the element y to obtain a sequence of elements

$\{y^n\}_{n\geq 0}$ such that $y^0 = y$, $[y^{n+1}] = [y^n]$ for all $n \geq 0$, and $y^N = 0$ for N big enough. In addition, as $\operatorname{supp}(y^{n+1}) < \operatorname{supp}(y^n)$ and $\operatorname{supp}(y) < \{m\}$, we obtain that

$$y^n - y^{n+1} = \sum_{l<k\in\max\operatorname{supp}(y^n)} z_{l,k} \oplus -F(l\leq k)(z_{l,k}) = \sum_{l<k<m} z_{l,k} \oplus -F(l\leq k)(z_{l,k}) \tag{13}$$

for elements $z_{l,k} \in F(l)$. Define $y^0_{l,k} = y_{l,k}$, assume by induction that

$$x = y^n + \sum_{\substack{l<k<j\\ k\neq m}} y^n_{l,k} \oplus -F(l\leq k)(y^n_{l,k}), \tag{14}$$

for elements $y^n_{l,k} \in F(l)$, and note that this holds for $n = 0$ by Eq. (12). For the induction step, we may write

$$\begin{aligned} x &= y^{n+1} - y^{n+1} + y^n + \sum_{\substack{l<k<j\\ k\neq m}} y^n_{l,k} \oplus -F(l\leq k)(y^n_{l,k}) = y^{n+1} \\ &\quad + \sum_{\substack{l<k<j\\ k\neq m}} y^{n+1}_{l,k} \oplus -F(l\leq k)(y^{n+1}_{l,k}), \end{aligned}$$

for elements $y^{n+1}_{l,k} \in F(l)$, where in the last equality we have employed Eq. (13). Hence, for $n = N$, Eq. (14) simplifies to

$$x = \sum_{\substack{l<k<j\\ k\neq m}} y^N_{l,k} \oplus -F(l\leq k)(y^N_{l,k}).$$

Repeating this process for every element $m \in \max K \setminus \operatorname{supp}(x)$ we find a decomposition similar to Eq. (9),

$$x = \sum_{l<k<j} y'_{l,k} \oplus -F(l\leq k)(y'_{l,k}),$$

and satisfying that, for $K' = \{k \in \mathcal{P}_{<j} \mid \exists\, l < k \text{ with } y'_{l,k} \neq 0\}$, we have

$$\max K' \setminus \operatorname{supp}(x) < \max K \setminus \operatorname{supp}(x).$$

Iterating this procedure we obtain a sequence of sets $\{K^n\}_{n\geq 0}$ and decompositions similar to Eq. (9) with $K^0 = K$, $K^1 = K'$, and such that

$$\max K^{n+1} \setminus \operatorname{supp}(x) < \max K^n \setminus \operatorname{supp}(x).$$

Setting $J^n = \max K^n \setminus \operatorname{supp}(x)$ and applying [5, Lemma 2.6] we find N such that $J^N = \emptyset$, i.e., $\max K^N \subseteq \operatorname{supp}(x)$. For the corresponding decomposition,

$$x = \sum_{l<k<j} \hat{y}_{l,k} \oplus -F(l\leq k)(\hat{y}_{l,k}),$$

let k belong to $\max\operatorname{supp}(x)$ so that we have, similarly to Eq. (10),

$$x_k = \sum_{k<l} \hat{y}_{k,l} - \sum_{l<k} F(l\leq k)(\hat{y}_{l,k}).$$

If $\sum_{k<l} \hat{y}_{k,l} \neq 0$, there exists some $l > k$ such that $\hat{y}_{l,k} \neq 0$, which is a contradiction with $\max K^N \subseteq \operatorname{supp}(x)$. Hence, $\sum_{k<l} \hat{y}_{k,l} = 0$ and $x_k \in \operatorname{Im}_F(k)$. ☐

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