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Solution of one mixed problem of plate bending for a domain with partially unknown boundary

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Abstract The paper addresses the problem of finding a full-strength contour in the problem of plate bending for a cycle–symmetric doubly connected domain. An isotropic elastic plate, bounded by a regular polygon, is weakened by a required full-strength hole whose symmetric axes are the regular polygon diagonals. Rigid bars are attached to each component of the broken line of the outer boundary of the plate. The plate bends under the action of concentrated moments applied to the middle points of the bars. An unknown part of the boundary is free from external forces. Using the methods of complex analysis, the analytical image of Kolosov–Muskhelishvili's complex potentials (characterising an elastic equilibrium of the body) and of an unknown full-strength contour are determined. A numerical analysis is performed and the corresponding plots are obtained by means of the Mathcad system.

Keywords: plate bending theory, complex variable theory

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1 Introduction

Partially unknown boundary problems of elasticity theory and plate bending tasks (equi-strong contours finding problem) are considered to be most important problems and their solvability allows for controlling stress optimal distribution by selecting the appropriate hole boundary. In general, tangential normal stresses and moments, whose values depend on external loads and hole shapes, play an important role in the plasticity zone formation in plates with holes and also in the neighbourhood of the plates' hole boundary.

Proceeding from the above, the following procedure is applied: for given external loads the shapes of the holes in plates are chosen so that on the boundaries the maximum value of the tangential normal stresses (tangential normal moments) modulus is the same and minimum; for the same body in all other possible holes tangential normal stresses (tangential normal moments) has maximum modulus. It is proven that for infinite domains the minimum of maximum values of tangential normal stresses (tangential normal moments) is obtained on such contours, where these values are constant; this contours are named equi-strong contours. Bodies with such contours redistribute an optimal stress in the entire body, so that the plasticity zones primarily appear simultaneously along the whole contour.

Applications of elastic plates weakened with full-strength holes are of great interest in several mechanical constructions (building, mechanical engineering, shipbuilding, aircraft construction, etc).

It has been proved that the weight of a plate weakened by a hole with equal full-strength contours is less than 40% of the weight of one weakened by a circular hole with the same strength. Hence, finding the optimal shape is of great practical importance.

Boundary-value problems of the plane theory of elasticity and plate bending for infinite plates weakened by unknown full-strength holes, with normal stresses acting on their boundaries and forces applied at infinity, were analysed in [1, 2, 8, 9, 13, 19].

Boundary-value problems for a finite doubly-connected domain with a part of its boundary being unknown full-strength and the other part being a polygonal line are solved in [6, 7].

The axis-symmetric and cycle symmetric problems of the plane theory of elasticity and plate bending with partially unknown boundaries are studied in [3–5, 16, 14, 15].

The novelty and complication of the problem posed here is that it has both a known and unknown boundary. So, generally this problem is reduced to a non linear problem.

In the plane theory of elasticity and plate bending for solving problems of finding full-strength contours, the most effective method turned out to be that of boundary value problems of the theory of analytic functions (the problems of linear conjugation, Riemann-Hilbert, Diriclet, etc.).

2 Statement of the Problem

Consider a middle surface of a homogeneous isotropic elastic plate occupying a doubly connected domain on complex plane $z = x + iy$. Its external boundary L_1

is a regular polygon with centre $z = 0$ and m vertices (m is an even number). The internal boundary L_2 is a required full-strength contour, whose symmetric axes are the regular polygon diagonals (Fig. 1).

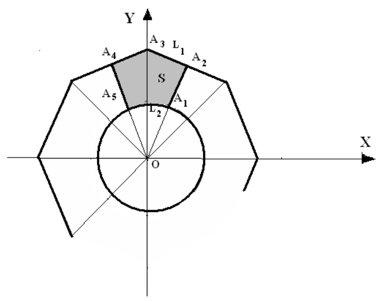


Fig. 1 Graph of the posed problem

The side length of the regular polygon is assumed to be equal to a. Rigid bars are attached to each component of the broken line of outer boundary L_1 of the plate. This plate bends under the action of concentrated moments M applied to the middle points of the bars. It is evident that the normals of the middle surface drawn at the points of rectilinear segments of the boundary turn through one and the same constant angle. The unknown part of boundary L_2 is free from outer actions.

Consider the following problem:

Problem 1 Find plate deflection $w(x, y)$ and the unknown part of its boundary such that the tangential normal moment acting on it takes a constant value $M_s = M_0 = \text{const}$.

According to the approximation theory of plate bending, the deflection of the middle surface $w(x, y)$ in the case under consideration is a biharmonic function. Since the angles of rotation of the midsurface normals $\frac{\partial w}{\partial n}$ are constants on each rectilinear section of boundary L_1, it is not difficult to see that the torque $H_{ns} = 0$ and shearing force $N_n = 0$. Thus on the entire boundary $H_{ns} = N_n = 0$. On the known part of plate boundary L_1: $\frac{\partial w}{\partial n} = \text{const}$ and on the unknown part of plate boundary L_2, the normal bending moment $M_n = 0$ and tangential normal moment $M_s = \text{const}$.

Since the problem is cyclically symmetric, it is easy to show that on segments $[A_1, A_2]$ and $[A_4, A_5]$ (A_2, A_4 are middle points of the sides of the polygon):

$$\frac{\partial w}{\partial n} = 0, \qquad H_{ns} = N_n = 0$$

To study the above-formulated problem, it is sufficient to consider a curvilinear pentagon $A_1A_2A_3A_4A_5$ which is denoted by S. Introduce the notation

$$\Gamma_1 = A_1A_2, \quad \Gamma_2 = A_2A_3, \qquad \Gamma_3 = A_3A_4,$$
$$\Gamma_4 = A_4A_5, \quad \Gamma = \cup_{j=1}^{4}\Gamma_j.$$

Denote the arc abscissa of point $t \in \Gamma \cup A_1A_5$ counted from point A_1 by s. The arc abscissas of points A_k are denoted by s_k, $k = 1, 2, 3, 4, 5$. By M_1, M_2 we denote the principal bending moments applied to Γ_1, Γ_4, and by $\frac{M}{2}$ those applied to Γ_2, Γ_3. Due to the equilibrium of the cut out part of the plate, we have:

$$\int_{\Gamma_2} M_n\, ds = \int_{\Gamma_3} M_n\, ds = \frac{M}{2},$$
$$M_1 = \int_{\Gamma_1} M_n\, ds = M_2 = \int_{\Gamma_4} M_n\, ds = \frac{M}{2}\cot\frac{\pi}{m}. \tag{1}$$

The boundary conditions of the problem can be written in the following way:

$$\frac{\partial w}{\partial n} = \begin{cases} d_1 = \text{const}, & t \in \Gamma_2 \cup \Gamma_3 \\ 0 & t \in \Gamma_1 \cup \Gamma_4 \end{cases} \tag{2}$$

$$M_s = \text{const}, \quad H_{ns} = M_n = N_n = 0, \quad t \in A_5A_1 \tag{3}$$

$$H_{ns} = N_n = 0, \quad t \in \Gamma. \tag{4}$$

3 Problem Solution

As is known, the biharmonic in S function $w(x, y)$ can be presented by Goursat's formula [17]:

$$w(x, y) = \text{Re}\left[\bar{z}\varphi(z) + \chi(z)\right] \tag{5}$$

where $z = x + iy$, $\bar{z} = x - iy$, $\varphi(z)$, $\chi(z)$ are holomorphic functions in S. We can replace formula (5) by a formula more convenient for our purpose [11]:

$$\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \qquad z \in S \tag{6}$$

where $\psi(z) = \chi'(z)$.

It is known that M_n, N_n, H_{ns} and M_s are expressed in terms of the functions $\varphi(z)$, $\chi(z)$ by formulas analogous to those of Kolosov-Muskhelishvili (see [11, 18]):

$$\frac{1}{D(1-\nu)}\int_{z_0}^{z}\left[M_n + i\int_{s_0}^{s}\left(N_n + \frac{\partial H_{ns}}{\partial s}\, ds\right) dz\right] = l\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \quad z \in S \tag{7}$$

$$(M_n + M_s) = M_x + M_y = -4D(l + \nu)\text{Re}\varphi'(z), \qquad z \in S \tag{8}$$

$$N_x + iN_y = -4D\varphi''(z), \qquad z \in S \tag{9}$$

where $l = -\frac{3+\nu}{1-\nu}$, ν is the Poisson coefficient and D the cylindrical rigidity.

Taking into account formulas (6), (7), (8), the boundary conditions to problem (2)-(4) can be presented in the following way:

$$\operatorname{Re}\left[e^{-i\alpha(t)}\left(\varphi(t)+t\overline{\varphi(t)}+\overline{\psi(t)}\right)\right]=\frac{\partial w}{\partial n}, \qquad t\in\Gamma,, \qquad (10)$$

$$\operatorname{Re}\left[e^{-\alpha(t)}\left(l\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}\right)\right]=C(t), \qquad t\in\Gamma, \qquad (11)$$

$$l\varphi(t)+t\overline{\varphi'(t)}+\overline{\psi(t)}=B, \qquad t\in A_5A_1, \qquad (12)$$

$$\operatorname{Re}\varphi'(t)=-\frac{M_0}{4D(1+\nu)}, \qquad t\in A_5A_1 \qquad (13)$$

where $B=\text{const}$,

$$C(t)=\operatorname{Re}\left(e^{-i\alpha(t)}\frac{i}{D(1-\nu)}\int_{A_1}^{t}M_n(s_0)e^{i\alpha(s_0)}\,ds_0\right)= \\ =\frac{1}{D(1-\nu)}\int_{A_1}^{t}M_n(s_0)\sin\left(\alpha(t)-\alpha(s_0)\right)\,ds_0$$

$\alpha(t)$ is the angle between the OX axis and the outward normal n of boundary L_1

$$\begin{aligned} &\alpha(t)=\alpha_k \qquad t\in A_kA_{k+1}, \qquad k=1,2,3,4.\\ &\alpha_1=-\frac{\pi}{m}, \qquad \alpha_2=\frac{\pi(m-2)}{2m}.\\ &\alpha_3=\frac{\pi(m+2)}{2m} \qquad \alpha_4=\frac{\pi(m+1)}{m}.\end{aligned} \qquad (14)$$

Taking into account (1), (3) and (14), we obtain

$$C(t)=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_4,\\ \frac{M}{2D(1-\nu)}\cot\frac{\pi}{m}, & t\in\Gamma_2\cup\Gamma_3\end{cases} \qquad (15)$$

Subtracting (10) from (11) and differentiating with respect to arc abscissa s and taking into account that $C(t)$, $\alpha(t)$ and $\frac{\partial w}{\partial n}$ are the piecewise-constant on Γ, we obtain

$$\operatorname{Im}\varphi'(t)=0, \qquad t\in\Gamma \qquad (16)$$

In what follows, functions $\varphi(z)$, $\bar{z}\varphi'(z)+\psi(z)$ will be assumed to be continuous in closed domain S, while functions $\varphi'(z)$, $\psi(z)$ are continuously extensible on the boundary of S, with the exclusion of point A_3 in the neighbourhood of which the following estimation can be made:

$$|\varphi'(z)|<M|z-A_3|^{-\delta}, \qquad |\psi(z)|<M|z-A_3|^{-\delta}, \qquad 0\le\delta<1 \qquad (17)$$

Equalities (13) and (16) are, in fact, the Keldysh-Sedov problem for domain S:

$$\begin{aligned}&\operatorname{Re}\varphi'(t)=-\frac{M_0}{4D(1+\nu)}, \quad t\in A_5A_1,\\ &\operatorname{Im}\varphi'(t)=0, \qquad t\in\Gamma.\end{aligned} \qquad (18)$$

The problem (18) has a unique solution [10]:

$$\varphi'(z) = \frac{-M_0}{4D(1+\nu)}. \tag{19}$$

Hence, we obtain

$$\varphi(z) = -\frac{M_0}{4D(1+\nu)}z. \tag{20}$$

where the constant was neglected.

Substituting the values $\varphi(t)$ into the boundary conditions (10)-(12), we obtain the following problem:

$$\begin{aligned} \text{Re}\left[e^{-i\alpha(t)}\left(\frac{K}{2}t + \overline{\psi(t)}\right)\right] &= C(t), \quad t \in \varGamma_2 \cup \varGamma_3, \\ \text{Re}\left[e^{-i\alpha(t)}\left(qt + \overline{\psi(t)}\right)\right] &= 0, \qquad t \in \varGamma_1 \cup \varGamma_4, \end{aligned} \tag{21}$$

$$\frac{K}{2}t + \overline{\psi(t)} = B, \qquad t \in A_5A_1, \tag{22}$$

where

$$K = \frac{M_0}{D(1-\nu)}, \qquad q = -\frac{M_0}{2D(1+\nu)}. \tag{23}$$

Let $t \in A_kA_{k+1}$, $k = 1, 2, 3, 4$, then $t - A_k = i|t - A_k|e^{i\alpha_k}$. Hence, we obtain

$$\text{Re}te^{-i\alpha(t)} = \text{Re}e^{-i\alpha(t)}A(t),$$

where $A(t)$ is a piecewise-constant function, $A(t) = A_k$, $t \in A_kA_{k+1}$, $k = 1, 2, 3, 4$.

Thus, if $t \in \varGamma_k$, $k = 1, 2, 3, 4$, then

$$\begin{aligned} \text{Re}te^{-i\alpha(t)} &= \frac{a}{2}\cot\frac{\pi}{m}, \quad t \in \varGamma_2 \cup \varGamma_3, \\ \text{Re}te^{-i\alpha(t)} &= 0, \qquad\qquad t \in \varGamma_1 \cup \varGamma_4 \end{aligned} \tag{24}$$

Taking into account (24) and (14), condition (21) can be written as follows:

$$\begin{aligned} \text{Re}e^{-i\alpha_1}\overline{\psi(t))} &= 0, & t &\in \varGamma_1 \\ \text{Re}e^{-i\alpha_2}\overline{\psi(t))} &= \frac{M}{2D(1-\nu)}\cot\frac{\pi}{m} - \frac{K}{2}a\cot\frac{\pi}{m}, & t &\in \varGamma_2 \\ \text{Re}e^{-i\alpha_3}\overline{\psi(t))} &= \frac{M}{2D(1-\nu)}\cot\frac{\pi}{m} - \frac{K}{2}a\cot\frac{\pi}{m}, & t &\in \varGamma_3 \\ \text{Re}e^{-i\alpha_4}\overline{\psi(t))} &= 0, & t &\in \varGamma_4 \end{aligned} \tag{25}$$

Taking into account (14), condition (24) can be written in the following way:

$$\begin{aligned} \text{Re}e^{-i\alpha_1}t &= 0, & t &\in \varGamma_1 \\ \text{Re}e^{-i\alpha_2}t &= \frac{a}{2}\cot\frac{\pi}{m}, & t &\in \varGamma_2 \\ \text{Re}e^{-i\alpha_3}t &= \frac{a}{2}\cot\frac{\pi}{m}, & t &\in \varGamma_3 \\ \text{Re}e^{-i\alpha_4}t &= 0, & t &\in \varGamma_4 \end{aligned} \tag{26}$$

Let $z = \omega(\zeta)$, $\zeta = \xi + i\eta$ map the half-disk $|\zeta| < 1$, $\text{Im}\zeta > 0$ onto domain S so that vertex A_k of the polygonal line corresponds to points $a_k = \omega^{-1}(A_k)$, $k = 1, 2, 3, 4, 5$ of the semicircle γ_0: $|\zeta| = 1$, $\text{Im}\zeta > 0$.

Let $a_1 = 1$ and $a_5 = -1$. Then the arc A_5A_1 is mapped to the diameter $-1 \le \xi \le 1$ of the circle, and the polygonal line Γ to the semicircle γ_0 , $a_3 = i$. The points $a_2 = e^{\theta_2 i}$, $a_4 = e^{(\pi-\theta_2)i}$, $0 < \theta_2 < \frac{\pi}{2}$ can be determined during the solution of the problem. As the problem is axially and cycle symmetric, $a_4 = -\bar{a}_2$.

By virtue of (25) and (26) for functions $\psi_0(\zeta) = \psi(\omega(\zeta))$ and $\omega(\zeta)$, we obtain:

$$\begin{aligned}
&\text{Re} e^{-i\alpha_1}\overline{\psi_0(\sigma)} = 0, && \sigma \in (a_1, a_2)\\
&\text{Re} e^{-i\alpha_2}\overline{\psi_0(\sigma)} = \frac{M}{2D(1-\nu)}\cot\frac{\pi}{m} - \frac{K}{4}a\cot\frac{\pi}{m}, && \sigma \in (a_2, a_3)\\
&\text{Re} e^{-i\alpha_3}\overline{\psi_0(\sigma)} = \frac{M}{2D(1-\nu)}\cot\frac{\pi}{m} - \frac{K}{4}a\cot\frac{\pi}{m}, && \sigma \in (a_3, a_4)\\
&\text{Re} e^{-i\alpha_4}\overline{\psi_0(\sigma)} = 0, && \sigma \in (a_4, a_5)
\end{aligned} \tag{27}$$

and

$$\begin{aligned}
&\text{Re} e^{-i\alpha_1}\omega(\sigma) = 0, && \sigma \in (a_1, a_2)\\
&\text{Re} e^{-i\alpha_2}\omega(\sigma) = \frac{a}{2}\cot\frac{\pi}{m}, && \sigma \in (a_2, a_3)\\
&\text{Re} e^{-i\alpha_3}\omega(\sigma) = \frac{a}{2}\cot\frac{\pi}{m}, && \sigma \in (a_3, a_4)\\
&\text{Re} e^{-i\alpha_4}\omega(\sigma) = 0, && \sigma \in (a_4, a_5)
\end{aligned} \tag{28}$$

Condition (22) becomes

$$\frac{K}{2}\omega(\xi) + \overline{\psi_0(\xi)} = B, \qquad -1 < \xi < 1 \tag{29}$$

Let $B = B_1 + B_2 = i\beta_1 + i\beta_2$, $\beta_1 = \text{const}$, $\beta_2 = \text{const}$.

Let us introduce a piecewise-holomorphic function $W(\zeta)$ in the circle $|\zeta| < 1$:

$$W(\zeta) = \begin{cases} \frac{K}{2}\omega(\zeta) - i\beta_1, & |\zeta| < 1, \quad \text{Im}\zeta > 0,\\ -\overline{\psi_0(\bar{\zeta})} + i\beta_2, & |\zeta| < 1, \quad \text{Im}\zeta < 0 \end{cases} \tag{30}$$

From (29), we obtain

$$\begin{aligned}
&W^+(\xi) = \frac{K}{2}\omega(\xi) - i\beta_1\\
&W^-(\xi) = -\overline{\psi_0(\xi)} + i\beta_2, \qquad -1 < \xi < 1.
\end{aligned} \tag{31}$$

Then

$$\begin{aligned}
&W^+(\sigma) = \frac{K}{2}\omega(\sigma) - i\beta_1, \quad \sigma \in \gamma_0,\\
&W^-(\sigma) = -\overline{\psi_0(\bar{\sigma})} + i\beta_2, \quad \sigma \in \gamma_0^*
\end{aligned} \tag{32}$$

where the signs (+) and (−) mark, as usual, the limit values the function takes at the upper and lower edges and γ_0^* is the reflection image of γ_0 about the OX-axis.

Because of (29) and (31), we have $W^+(\xi) - W^-(\xi) = 0$, $-1 < \xi < 1$. Thus, $W(\zeta)$ is a holomorphic function into the circle $|\zeta| < 1$.

By virtue of (27) and (28), function $W(\zeta)$ defined by (30) satisfies boundary conditions

$$\begin{aligned}
\mathrm{Re}e^{-i\alpha(\sigma)}W(\sigma) &= \frac{K}{2}\mathrm{Re}e^{-i\alpha(\sigma)}\omega(\sigma) - \mathrm{Re}e^{i\alpha(\sigma)}B_1, && \sigma\in\gamma_0,\\
\mathrm{Re}e^{-i\alpha(\bar\sigma)}W(\sigma) &= -\mathrm{Re}e^{-i\alpha(\bar\sigma)}\overline{\psi_0(\bar\sigma)} + \mathrm{Re}e^{-i\alpha(\sigma)}B_2, && \sigma\in\gamma_0^*,\\
\alpha(\sigma) &= \alpha(\bar\sigma), && \sigma\in\gamma_0^*.
\end{aligned}$$

Taking into account (27) and (28) for $\sigma\in\gamma_0$

$$\mathrm{Re}e^{-i\alpha_1}W(\sigma) = \frac{K}{2}\mathrm{Re}e^{-i\alpha_1}\omega(\sigma) - \mathrm{Re}e^{-i\alpha_1}B_1 = -\beta_1\sin\alpha_1, \quad \alpha\in(a_1,a_2), \tag{33}$$

$$\begin{aligned}
\mathrm{Re}e^{-i\alpha_2}W(\sigma) &= \frac{K}{2}\mathrm{Re}e^{-i\alpha_2}\omega(\sigma) - \mathrm{Re}e^{-i\alpha_2}B_1 =\\
&= \frac{Ka}{4}\cot\frac{\pi}{m} - \beta_1\sin\alpha_2, \qquad \alpha\in(a_2,a_3),
\end{aligned} \tag{34}$$

$$\begin{aligned}
\mathrm{Re}e^{-i\alpha_3}W(\sigma) &= \frac{K}{2}\mathrm{Re}e^{-i\alpha_3}\omega(\sigma) - \mathrm{Re}e^{-i\alpha_3}B_1 =\\
&= \frac{K}{4}a\cot\frac{\pi}{m} - \beta_1\sin\alpha_3, \qquad \alpha\in(a_3,a_4),
\end{aligned} \tag{35}$$

$$\mathrm{Re}e^{-i\alpha_4}W(\sigma) = \frac{K}{2}\mathrm{Re}e^{-i\alpha_4}\omega(\sigma) - \mathrm{Re}e^{-i\alpha_4}B_1 = -\beta_1\sin\alpha_4, \quad \alpha\in(a_1,a_2), \tag{36}$$

and for $\sigma\in\gamma_0^*$

$$\mathrm{Re}e^{-i\alpha_4}W(\sigma) = -\mathrm{Re}e^{-i\alpha_4}\overline{\psi_0(\bar\sigma)} + \mathrm{Re}e^{-i\alpha_4}B_2 = \beta_2\sin\alpha_4, \qquad \sigma\in(a_5,\bar a_4), \tag{37}$$

$$\begin{aligned}
\mathrm{Re}e^{-i\alpha_3}W(\sigma) &= -\mathrm{Re}e^{-i\alpha_3}\overline{\psi_0(\bar\sigma)} + \mathrm{Re}e^{-i\alpha_3}B_2 =\\
&= -\frac{M}{2D(1-\nu)}\cot\frac{\pi}{m} + \frac{Ka}{4}\cot\frac{\pi}{m} + \beta_2\sin\alpha_3, \quad \sigma\in(\bar a_4,\bar a_3),
\end{aligned} \tag{38}$$

$$\begin{aligned}
\mathrm{Re}e^{-i\alpha_2}W(\sigma) &= -\mathrm{Re}e^{-i\alpha_2}\overline{\psi_0(\bar\sigma)} + \mathrm{Re}e^{-i\alpha_2}B_2 =\\
&= -\frac{M}{2D(1-\nu)}\cot\frac{\pi}{m} + \frac{K}{4}a\cot\frac{\pi}{m} + \beta_2\sin\alpha_2, \quad \sigma\in(\bar a_3,\bar a_2),
\end{aligned} \tag{39}$$

$$\mathrm{Re}e^{-i\alpha_1}W(\sigma) = -\mathrm{Re}e^{-i\alpha_1}\overline{\psi_0(\bar\sigma)} + \mathrm{Re}e^{-i\alpha_1}B_2 = \beta_2\sin\alpha_1, \qquad \sigma\in(\bar a_2,a_1), \tag{40}$$

On account of continuity of function $W(\zeta)$, we have

$$W^+(\pm1) = W^-(\pm1) \qquad \text{for} \qquad \sigma\to\pm1$$

that is, taking into account conditions (33),(35) and (37), (39) we have

$$\beta_2 = -\beta_1$$

at points a_1, a_5. On account of (14), we obtain

$$\sin\alpha_1 = \sin\alpha_4, \qquad \sin\alpha_2 = \sin\alpha_3$$

Conditions (33)-(40) can be rearranged as follows:

$$\mathrm{Re} e^{-i\alpha(\sigma)} W(\sigma) = f(\sigma)), \qquad \sigma \in \gamma, \tag{41}$$

where $\gamma = \gamma_0 \cup \gamma_0^*$, $\alpha(\sigma) = \alpha(\bar{\sigma})$, $\sigma \in \gamma_0^*$.

Formula (41) presents the Riemann-Hilbert problem with piecewise-constant coefficients.

For $\sigma \in \gamma_0$

$$f(\sigma) = \begin{cases} -\beta_1 \sin\alpha_1, & \sigma \in (a_1, a_2) \cup (a_4, a_5) \\ \frac{aK}{4} \cot\frac{\pi}{m} - \beta_1 \sin\alpha_2, & \sigma \in (a_2, a_3) \cup (a_3, a_4) \end{cases}$$

For $\sigma \in \gamma_0^*$

$$f(\sigma) = \begin{cases} -\beta_1 \sin\alpha_1, & \sigma \in (\bar{a}_2, a_1) \cup (a_5, \bar{a}_4) \\ -\frac{M}{2D(1-\nu)} \cot\frac{\pi}{m} + \frac{K}{4} a \cot\frac{\pi}{m} - \beta_1 \sin\alpha_2, & \sigma \in (\bar{a}_4, \bar{a}_3) \cup (\bar{a}_3, \bar{a}_2). \end{cases}$$

Thus, the posed problem has been reduced to the Riemann-Hilbert problem with piecewise-constant coefficients. The solution of this problem was obtained in [12]. In this paper, the problem is reduced to the Dirichlet's problem for a circle, and its solution is represented in a simple form by the Schwarz integral.

Function $e^{2i\alpha(\sigma)}$ is given by

$$e^{2i\alpha(\sigma)} = \frac{X(\sigma)}{\overline{X(\sigma)}}, \qquad |\sigma| = 1, \tag{42}$$

where $X(\zeta)$ is a holomorphic function on the circle $|\zeta| < 1$ and has the following form:

$$\begin{aligned} X(\zeta) =& X_1(\zeta) X_1(0)^{\frac{1}{2}}, \\ X_1(\zeta) =& \left(\frac{\zeta - a_3}{\zeta - a_2}\right)^{\frac{\alpha_2}{\pi}} \cdot \left(\frac{\zeta - a_4}{\zeta - a_3}\right)^{\frac{\alpha_3}{\pi}} \cdot \left(\frac{\zeta - \bar{a}_4}{\zeta - a_4}\right)^{\frac{\alpha_4}{\pi}} \cdot \\ & \cdot \left(\frac{\zeta - \bar{a}_3}{\zeta - \bar{a}_4}\right)^{\frac{\alpha_3}{\pi}} \cdot \left(\frac{\zeta - \bar{a}_2}{\zeta - \bar{a}_3}\right)^{\frac{\alpha_2}{\pi}} \cdot \left(\frac{\zeta - a_2}{\zeta - \bar{a}_2}\right)^{\frac{\alpha_1}{\pi}} \end{aligned} \tag{43}$$

It is easy to prove

$$\begin{aligned} & X(\sigma) = |X(\sigma)| e^{i\alpha(\sigma)}, \\ & X(\sigma) = |X(-\bar{\sigma})|, \qquad \sigma = e^{i\theta}, \quad 0 < \theta < 2\pi. \end{aligned} \tag{44}$$

By virtue of (44), we obtain

$$|X_1(\sigma)| = \left|\frac{\sigma - \bar{a}_2}{\sigma - a_2}\right|^{\frac{a_2}{\pi} - \frac{a_1}{\pi}} \left|\frac{\sigma - \bar{a}_3}{\sigma - a_3}\right|^{\frac{a_3}{\pi} - \frac{a_2}{\pi}} \left|\frac{\sigma - \bar{a}_4}{\sigma - a_4}\right|^{\frac{a_4}{\pi} - \frac{a_3}{\pi}}$$

By virtue of (42), condition (41) becomes

$$\frac{W(\sigma)}{X(\sigma)} + \frac{\overline{W(\sigma)}}{\overline{X(\sigma)}} = \frac{2fe^{i\alpha(\sigma)}}{X(\sigma)} \tag{45}$$

Condition (45) is the boundary condition of the Dirichlet problem with the solution given as follows

$$\frac{W(\zeta)}{X(\zeta)} = \frac{1}{2\pi i}\int_\gamma \frac{f(\sigma)e^{i\alpha(\sigma)}(\sigma+\zeta)\,d\sigma}{X(\sigma)\sigma(\sigma-\zeta)} \tag{46}$$

Taking into account (44), formula (46) has the following form:

$$\frac{W(\zeta)}{X(\zeta)} = \frac{1}{2\pi i}\int_\gamma \frac{(\sigma+\zeta)}{(\sigma-\zeta)}\frac{f(\sigma)}{|X(\sigma)|}\frac{d\sigma}{\sigma}$$

Since the function $X(\zeta)$ has singularities of orders $\frac{\alpha_2}{\pi} - \frac{\alpha_1}{\pi}$ and $\frac{\alpha_4}{\pi} - \frac{\alpha_3}{\pi}$ at points $a_2, \bar{a}_2$ ($a_4 = -\bar{a}_2$) and of the order $\frac{\alpha_3}{\pi} - \frac{\alpha_2}{\pi}$ at point a_3, in this case the bounded solution $W(\zeta)$

$$W(\zeta) = \frac{X(\zeta)}{2\pi i}\int_\gamma \frac{(\sigma+\zeta)}{(\sigma-\zeta)}\frac{f(\sigma)}{|X(\sigma)|}\frac{d\sigma}{\sigma} \tag{47}$$

exists only if

$$\int_\gamma \frac{(\sigma+\zeta)}{(\sigma-\zeta)}\frac{f(\sigma)}{|X(\sigma)|}\frac{d\sigma}{\sigma} = 0, \qquad \text{for} \qquad \zeta \to a_2, \quad \zeta \to -\bar{a}_2, \quad \zeta \to a_3 \tag{48}$$

Since the density of the integral vanishes at points $\zeta = a_2$, $\zeta = -\bar{a}_2$, $\zeta = a_3$, based on the Sokhotsky-Plemelj formulas [15] we get

$$\begin{aligned} &\int_\gamma \frac{(\sigma+a_2)}{(\sigma-a_2)}\frac{f(\sigma)}{|X(\sigma)|}\frac{d\sigma}{\sigma} = 0,\\ &\int_\gamma \frac{(\sigma-\bar{a}_2)}{(\sigma+\bar{a}_2)}\frac{f(\sigma)}{|X(\sigma)|}\frac{d\sigma}{\sigma} = 0,\\ &\int_\gamma \frac{(\sigma+a_3)}{(\sigma-a_3)}\frac{f(\sigma)}{|X(\sigma)|}\frac{d\sigma}{\sigma} = 0, \end{aligned} \tag{49}$$

Taking into account $|X(\sigma)| = |X(-\bar{\sigma})|$ of (44), the first and second equations of (49) are equivalent, so it is sufficient to consider the system of two equations with respect to two unknown parameters K, β_1 for given $a_2 = e^{i\theta_2}$, $\theta_2 \in (0, \frac{\pi}{2})$

$$\begin{aligned} &\int_\gamma \frac{(\sigma+a_2)}{(\sigma-a_2)}\frac{f(\sigma)}{|X(\sigma)|}\frac{d\sigma}{\sigma} = 0,\\ &\int_\gamma \frac{(\sigma+a_3)}{(\sigma-a_3)}\frac{f(\sigma)}{|X(\sigma)|}\frac{d\sigma}{\sigma} = 0, \end{aligned} \tag{50}$$

From (50), we obtain

$$K = \frac{4}{\operatorname{arctg}\frac{\pi}{m}}\left(\frac{RT^{*}-R^{*}T}{QT^{*}-TQ^{*}}\right), \tag{51}$$

$$\beta_1 = \frac{R^{*}Q-RQ^{*}}{QT^{*}-TQ^{*}}, \tag{52}$$

where

$$T = -\sin\alpha_1(A+B) - Q\sin\alpha_2, \qquad Q = C+H, \qquad R = \frac{M\cot\frac{\pi}{m}}{2D(1-\nu)}H,$$

$$T^{*} = -\sin\alpha_1(A^{*}+B^{*}) - Q^{*}\sin\alpha_2, \qquad Q^{*} = C^{*}+H^{*}, \qquad R^{*} = \frac{M\cot\frac{\pi}{m}}{2D(1-\nu)}H^{*}$$

$$A = \int_{2\pi-\theta_2}^{\theta_2}\cot\frac{\theta-\theta_2}{2}\frac{d\theta}{|X(e^{i\theta})|}, \qquad B = \int_{\pi-\theta_2}^{\pi+\theta_2}\cot\frac{\theta-\theta_2}{2}\frac{d\theta}{|X(e^{i\theta})|},$$

$$C = \int_{\theta_2}^{\pi-\theta_2}\cot\frac{\theta-\theta_2}{2}\frac{d\theta}{|X(e^{i\theta})|}, \qquad H = \int_{\pi+\theta_2}^{2\pi-\theta_2}\cot\frac{\theta-\theta_2}{2}\frac{d\theta}{|X(e^{i\theta})|},$$

$$A^{*} = \int_{2\pi-\theta_2}^{\theta_2}\cot\frac{\theta-\frac{\pi}{2}}{2}\frac{d\theta}{|X(e^{i\theta})|}, \qquad B^{*} = \int_{\pi-\theta_2}^{\pi+\theta_2}\cot\frac{\theta-\frac{\pi}{2}}{2}\frac{d\theta}{|X(e^{i\theta})|},$$

$$C^{*} = \int_{\theta_2}^{\pi-\theta_2}\cot\frac{\theta-\frac{\pi}{2}}{2}\frac{d\theta}{|X(e^{i\theta})|}, \qquad H^{*} = \int_{\pi+\theta_2}^{2\pi-\theta_2}\cot\frac{\theta-\frac{\pi}{2}}{2}\frac{d\theta}{|X(e^{i\theta})|},$$

For the purpose of computation, it is expedient to specify K, β_1 for a given M after selecting the parameter θ_2 for $0<\theta_2<\frac{\pi}{2}$, $a_2 = e^{\theta_2 i}$, $a_4 = e^{(\pi-\theta_2)i}$.

By formula (30) we determine the equation of part of the unknown full-strength boundary

$$\omega(\xi) = \frac{2(W(\xi)+i\beta_1)}{K}, \qquad -1<\xi<1. \tag{53}$$

where function $W(\xi)$ is defined by (47).

Remark From equality

$$K = \frac{4}{\operatorname{arctg}\frac{\pi}{m}}\left(\frac{RT^{*}-R^{*}T}{QT^{*}-TQ^{*}}\right),$$

it follows that the solution of problem exists if

$$QT^{*}-TQ^{*} \neq 0$$

Concrete case ($m = 6$) .

Let us construct a full-strength hole for a specific case, for example, for hexagon ($m = 6$).The equation of part of the unknown full-strength boundary is defined by formula (53), where function $W(\xi)$ has the form

$$W(\xi) = \frac{X(\xi)}{2\pi}\left(\frac{\beta_1}{2}\int_{2\pi-\theta_2}^{\theta_2}\frac{e^{i\theta}+\xi}{e^{i\theta}-\xi}\frac{d\theta}{|X(e^{i\theta})|}+\right.$$
$$+\left(\frac{\sqrt{3}}{4}Ka-\frac{\sqrt{3}}{2}\beta_1\right)\int_{\theta_2}^{\pi-\theta_2}\frac{e^{i\theta}+\xi}{e^{i\theta}-\xi}\frac{d\theta}{|X(e^{i\theta})|}+\frac{\beta_1}{2}\int_{\pi-\theta_2}^{\pi+\theta_2}\frac{e^{i\theta}+\xi}{e^{i\theta}-\xi}\frac{d\theta}{|X(e^{i\theta})|}+$$
$$\left.+\left(\frac{\sqrt{3}}{4}Ka-\frac{\sqrt{3}}{2}\frac{M}{D(1-\nu)}-\frac{\sqrt{3}}{2}\beta_1\right)\int_{\pi+\theta_2}^{2\pi-\theta_2}\frac{e^{i\theta}+\xi}{e^{i\theta}-\xi}\frac{d\theta}{|X(e^{i\theta})|}\right), \qquad -1<\xi<1.$$

where

$$|X(\sigma)| = \left|\frac{\sigma-\bar{a}_2}{\sigma-a_2}\right|^{\frac{1}{2}}\left|\frac{\sigma-\bar{a}_3}{\sigma-a_3}\right|^{\frac{1}{3}}\left|\frac{\sigma+a_2}{\sigma+\bar{a}_2}\right|^{\frac{1}{2}}, \qquad \sigma = e^{i\theta}, \qquad 0<\theta<2\pi$$

$$K = \frac{4}{\sqrt{3}}\frac{RT^*-R^*T}{a(QT^*-TQ^*)}, \qquad \beta_1 = \frac{R^*Q-RQ^*}{QT^*-TQ^*}$$

$$T = \frac{A+B}{2}-\frac{\sqrt{3}}{2}Q, \qquad Q = C+H, \qquad R = \frac{\sqrt{3}M}{2D(1-\nu)}H,$$
$$T^* = \frac{A^*+B^*}{2}-\frac{\sqrt{3}}{2}Q^*, \qquad Q^* = C^*+H^*, \qquad R^* = \frac{\sqrt{3}M}{2D(1-\nu)}H^*,$$

A, A^*, B, B^*, C, C^*, H, H^* are defined as above.

Since the problem is cyclically symmetric, the other parts of the boundary can be obtained by rotating the graph $\omega(\xi)$ through an angle of $\frac{\pi}{3}$:

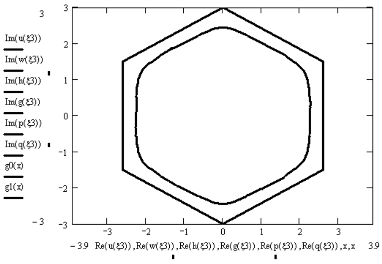
$$u(\xi) = \omega(\xi)e^{\frac{\pi}{3}i}, \qquad h(\xi) = u(\xi)e^{\frac{\pi}{3}i}, \qquad g(\xi) = h(\xi)e^{\frac{\pi}{3}i},$$
$$p(\xi) = g(\xi)e^{\frac{\pi}{3}i}, \qquad q(\xi) = p(\xi)e^{\frac{\pi}{3}i},$$

The hexagon boundary is determined by functions $g_0(x)$, $g_1(x)$. The side length of the hexagon is assumed to be equal to 3. The computations have been carried out and the plots have been obtained (see Fig 2).

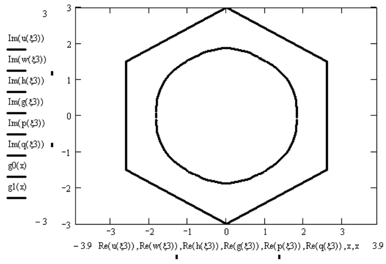
4 Conclusion

The most effective methods for studying this problem are those of the theory of analytical functions of a complex variable. This problem involves both mechanics and geometry, since the shape of a hole of a plate is required and the conformal mapping function is used to define it. Thus, formulas of Kolosov-Muskhelishvili are used for investigating this problem. Using the conformal mapping, the investigation of the problem is reduced to a Riemann-Hilbert problem. The solution is written in quadratures. The unknown full-strength part of the plate boundary is constructed. The conformal mapping function is a generalisation of Christoffel–Schwarz formulas.

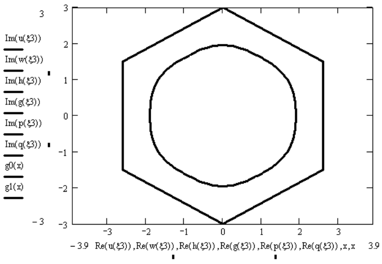
$$M = 5, \quad \nu = \frac{1}{3}, \quad a = 3, \quad \theta_2 = \frac{\pi}{200}, \quad K = 18.513, \quad \beta_1 = -1.348 \times 10^{-8}$$



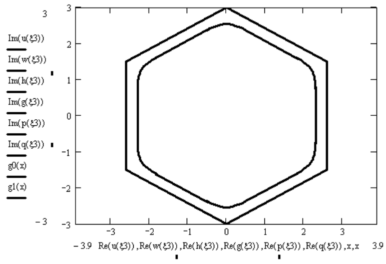
$$M = 5, \quad \nu = \frac{1}{3}, \quad a = 3, \quad \theta_2 = \frac{\pi}{7}, \quad K = 9.225, \quad \beta_1 = 1.007 \times 10^{-10}$$



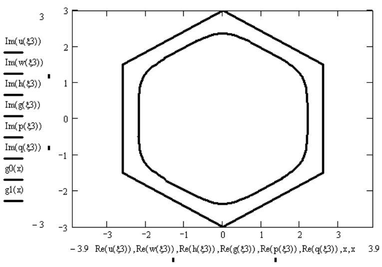
$$M = 500, \quad \nu = \frac{1}{3}, \quad a = 3, \quad \theta_2 = \frac{\pi}{9}, \quad K = 993.151, \quad \beta_1 = -2.273 \times 10^{-8}$$



$$M = 500, \quad \nu = \frac{1}{3}, \quad a = 3, \quad \theta_2 = \frac{\pi}{900}, \quad K = 2.267 \times 10^3, \quad \beta_1 = -7.867 \times 10^{-5}$$



$$M = 500, \quad \nu = \frac{1}{3}, \quad a = 3, \quad \theta_2 = \frac{\pi}{90}, \quad K = 1.631 \times 10^3, \quad \beta_1 = 2.75 \times 10^{-6}$$



$$M = 500, \quad \nu = \frac{1}{3}, \quad a = 3, \quad \theta_2 = \frac{\pi}{3}, \quad K = 671.971, \quad \beta_1 = -1.18 \times 10^{-7}$$

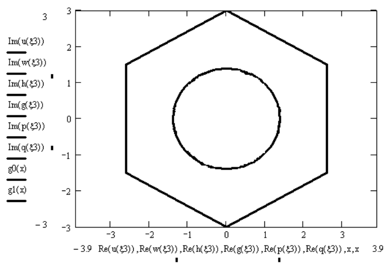


Fig. 2 Graph of a full-strength boundary for different parameters

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