WEIGHTED MODULAR INEQUALITIES FOR HARDY-STEKLOV OPERATORS

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Abstract. We characterize weighted modular inequalities of weak and strong type for the Hardy-Steklov operators T defined by $Tf(x) = g(x)\int_{s(x)}^{h(x)} f(t)dt$, where g is a positive function and s, h are increasing and continuous functions such that $s(x) \le h(x)$ for all x.

1. Introduction and results.

Let $-\infty \le a < b \le \infty$ and let $s, h : (a,b) \to \mathbf{R}$ be increasing and continuous functions such that $s(x) \le h(x)$ for all $x \in (a,b)$. Let g be a positive function defined on (a,b). Let T be the Hardy-Steklov operator defined by

$$Tf(x) = g(x)\int_{s(x)}^{h(x)} f(t)dt.$$

Particular important cases are the Hardy operator $Tf(x) = \int_0^x f$, the Hardy averaging operators $Tf(x) = x^{\alpha}\int_0^x f$, the moving averaging operators $Tf(x) = \frac{1}{h(x)-s(x)}\int_{s(x)}^{h(x)} f$ and the Steklov operator $Tf(x) = \int_{x-1}^{x+1} f$.

Hardy-Steklov operators arise naturally in the theory of delay differential equations and the knowledge of their behaviour may be useful in the study of some Cauchy problems (see [7]).

The weighted strong and weak type (p,q) inequalities for T have been characterized in [1], [3], [4] and [5].

In this paper we will characterize weighted modular inequalities of strong and weak type for T, i. e., inequalities of the forms

$$\Phi_2^{-1}\left(\int_a^b \Phi_2\left(Tf(x)\right)u(x)dx\right) \le \Phi_1^{-1}\left(\int_{s(a)}^{h(b)} \Phi_1(Cf(x))v(x)dx\right) \tag{1.1}$$

and

$$\Phi_2^{-1}\left(\Phi_2(\lambda)\int_{\{x\in(a,b):Tf(x)>\lambda\}} u\right) \le \Phi_1^{-1}\left(\int_{s(a)}^{h(b)} \Phi_1(Cf)v\right), \tag{1.2}$$

where Φ_1 and Φ_2 are positive, strictly increasing functions defined on $[0,\infty)$ and u, v are non-negative functions defined on (a,b) and $(s(a),h(b))$ respectively.

2000 *Mathematics Subject Classification.* 26D15.

Key words and phrases. weights, modular inequalities, Hardy operators, Hardy-Steklov operators.

This research has been supported in part by MICYT, BFM 2001-1638, and Junta de Andalucía.

The weighted modular inequalities for operators like maximal functions, singular integral, etc, have been extensively studied (see [6] and the references given there). In particular the weighted modular inequalities for the Hardy operator and the generalized Hardy operators were characterized in [2], [8], [9] and [10].

In the statements and proofs of the results we will need some concepts and properties related to N-functions. By a N-function we mean a continuous and convex function Φ defined on $[0,\infty)$ such that $\Phi(s)>0$ if $s>0$, $\frac{\Phi(s)}{s}\to 0$ when $s\to 0$ and $\frac{\Phi(s)}{s}\to\infty$ when $s\to\infty$. Every N-function Φ admits a representation of the form $\Phi(x)=\int_0^x\varphi(t)dt$, where φ is increasing, continuous by the right at every point and verifies $\varphi(0)=0$, $\varphi(s)>0$ if $s>0$ and $\varphi(s)\to\infty$ when $s\to\infty$. The function φ is called the density function of Φ. Given a N-function Φ, the function $\Psi:[0,\infty)\to\mathbf{R}$ defined by $\Psi(t)=\sup_{s\geq 0}(st-\Phi(s))$ is also a N-function called the complementary function of Φ. Two complementary N-functions Φ and Ψ verify Young's inequality: if $s,t\geq 0$, then $st\leq\Phi(s)+\Psi(t)$.

Our results are the following ones:

Theorem 1. *Let Φ_1 be a N-function and let $\Phi_2:[0,\infty)\to\mathbf{R}$ be a positive strictly increasing continuous function such that $\Phi_2(0)=0$ and $\lim_{t\to\infty}\Phi_2(t)=\infty$. Let us suppose that $\Phi_1\circ\Phi_2^{-1}$ is subadditive. Let Ψ_1 be the complementary N-function of Φ_1. Let u and v be non-negative functions defined on (a,b) and $(s(a),h(b))$ respectively. The following statements are equivalent:*

(i) *There exists $C>0$ such that inequality (1.1) holds for all positive functions f.*

(ii) *There exists $C>0$ such that the inequality*

$$\Phi_2^{-1}\left(\int_{\left\{x\in(a,b):\int_{s(x)}^{h(x)}f>\lambda\right\}}\Phi_2(\lambda g)u\right)\leq\Phi_1^{-1}\left(\int_{s(a)}^{h(b)}\Phi_1(Cf(x))v(x)dx\right)\tag{1.3}$$

holds for all $\lambda>0$ and all positive functions f.

(iii) *There exists $C>0$ such that*

$$\int_{s(y)}^{h(x)}\Psi_1\left(\frac{\alpha(\lambda,x,y)}{C\lambda v}\right)v\leq\alpha(\lambda,x,y)<\infty$$

holds for all $\lambda>0$ and all $x,y\in(a,b)$ with $x<y$ and $s(y)\leq h(x)$, where

$$\alpha(\lambda,x,y)=\Phi_1\circ\Phi_2^{-1}\left(\int_x^y\Phi_2(\lambda g)u\right).$$

Theorem 2. *Let Φ_1, Φ_2, Ψ_1, u and v be as in Theorem 1. Let us suppose that g is monotone. The following statements are equivalent:*

(i) *There exists $C>0$ such that inequality (1.2) holds for all $\lambda>0$ and all positive functions f.*

(ii) *There exists $C>0$ such that*

$$\int_{s(y)}^{h(x)}\Psi_1\left(\frac{(\inf_{(x,y)}g)\beta(\lambda,x,y)}{C\lambda v}\right)v\leq\beta(\lambda,x,y)<\infty$$

holds for all $\lambda > 0$ *and all* $x, y \in (a,b)$ *with* $x < y$ *and* $s(y) \le h(x)$, *where*

$$\beta(\lambda, x, y) = (\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_x^y u \right).$$

Observe that Theorems 1 and 2 include as particular cases the weighted strong and weak type (p,q) inequalities for $1 < p \le q < \infty$. Observe also that if $g \equiv 1$, then the strong type inequality (1.1) and the weak type inequality (1.2) are equivalent. However, for general monotone g, (1.1) and (1.2) are not equivalent, even if $\Phi_1(t) = t^p$ and $\Phi_2(t) = t^q$, $1 < p \le q < \infty$.

In order to prove the theorems, we will need the following lemma, whose proof can be found in [1]:

Lemma 1. *Let* $\{(a_j, b_j)\}_j$ *be the connected components of the open set* $\Omega = \{x \in (a,b) : s(x) < h(x)\}$. *Then*

- (a) $(s(a_j), h(b_j)) \cap (s(a_i), h(b_i)) = \emptyset$ *for all* $j \ne i$.
- (b) *For every* j *there exists a (finite or infinite) sequence* $\{m_k^j\}_k$ *of real numbers such that:*
  - (i) $a_j \le m_k^j < m_{k+1}^j \le b_j$ *for all* k *and* j;
  - (ii) $(a_j, b_j) = \cup_k (m_k^j, m_{k+1}^j)$ *a.e. for all* j;
  - (iii) $s(m_{k+1}^j) \le h(m_k^j)$ *for all* j *and* k *and* $s(m_{k+1}^j) = h(m_k^j)$ *if* $a_j < m_k^j < m_{k+1}^j < b_j$.

The proof of theorem 1 is included in section 2 and the proof of theorem 2 can be found in section 3.

2. PROOF OF THEOREM 1.

The proof of *(i)* ⇒ *(ii)* is trivial.

(ii) ⇒ *(iii)*. Let $\lambda > 0$, $n \in \mathbf{N}$ and $x, y \in (a,b)$ with $x < y$ and $s(y) \le h(x)$.If $s(y) = h(x)$, there is nothing to prove. Suppose that $s(y) < h(x)$. Since the function $\frac{\Psi_1(t)}{t}$ increases taking all values from 0 to ∞, there exists $\varepsilon > 0$ such that

$$\int_{s(y)}^{h(x)} \Psi_1 \left(\frac{\varepsilon}{v + \frac{1}{n}} \right) \frac{v + \frac{1}{n}}{\varepsilon} = 2C\lambda, \tag{2.1}$$

where C is the constant of inequality (1.1).

Let f be the function defined by

$$f = \frac{1}{C} \Psi_1 \left(\frac{\varepsilon}{v + \frac{1}{n}} \right) \frac{v + \frac{1}{n}}{\varepsilon} \chi_{(s(y), h(x))}.$$

If $z \in (x,y)$, we have

$$\int_{s(z)}^{h(z)} f \ge \int_{s(y)}^{h(x)} f = \int_{s(y)}^{h(x)} \frac{1}{C} \Psi_1 \left(\frac{\varepsilon}{v + \frac{1}{n}} \right) \frac{v + \frac{1}{n}}{\varepsilon} = 2\lambda > \lambda.$$

This shows that $(x,y) \subset \left\{z \in (a,b) : \int_{s(z)}^{h(z)} f > \lambda\right\}$. Then *(ii)*, the inequality $\Phi_1\left(\frac{\Psi_1(t)}{t}\right) \leq \Psi_1(t)$ and (2.1) give

$$\begin{aligned} \alpha(\lambda,x,y) &= \Phi_1 \circ \Phi_2^{-1}\left(\int_x^y \Phi_2(\lambda g)u\right) \\ &\leq \Phi_1 \circ \Phi_2^{-1}\left(\int_{\left\{z\in(a,b):\int_{s(z)}^{h(z)} f>\lambda\right\}} \Phi_2(\lambda g)u\right) \\ &\leq \int_{s(a)}^{h(b)} \Phi_1(Cf(t))\left(v(t)+\frac{1}{n}\right)dt \qquad (2.2) \\ &= \int_{s(y)}^{h(x)} \Phi_1\left(\Psi_1\left(\frac{\varepsilon}{v+\frac{1}{n}}\right)\frac{v+\frac{1}{n}}{\varepsilon}\right)\left(v+\frac{1}{n}\right) \\ &\leq \int_{s(y)}^{h(x)} \Psi_1\left(\frac{\varepsilon}{v+\frac{1}{n}}\right)\left(v+\frac{1}{n}\right) = 2C\lambda\varepsilon. \end{aligned}$$

This inequality ensures $\alpha(\lambda,x,y) < \infty$.

If ψ_1 is the density function of Ψ_1, it is known that

$$\Psi_1(x) \leq x\psi_1(x) \leq \Psi_1(2x). \tag{2.3}$$

On one hand, by (2.2), the right-hand side inequality in (2.3) and (2.1), we have

$$\begin{aligned} J &= \int_{s(y)}^{h(x)} \psi_1\left(\frac{\alpha(\lambda,x,y)}{4C\lambda\left(v+\frac{1}{n}\right)}\right) \leq \int_{s(y)}^{h(x)} \psi_1\left(\frac{\varepsilon}{2\left(v+\frac{1}{n}\right)}\right) \\ &\leq 2\int_{s(y)}^{h(x)} \Psi_1\left(\frac{\varepsilon}{v+\frac{1}{n}}\right)\frac{v+\frac{1}{n}}{\varepsilon} = 4C\lambda. \end{aligned} \tag{2.4}$$

On the other hand, the left-hand side inequality in (2.3) yields

$$\begin{aligned} J &\geq \int_{s(y)}^{h(x)} \Psi_1\left(\frac{\alpha(\lambda,x,y)}{4C\lambda\left(v+\frac{1}{n}\right)}\right)\frac{4C\lambda\left(v+\frac{1}{n}\right)}{\alpha(\lambda,x,y)} \\ &= \frac{4C\lambda}{\alpha(\lambda,x,y)}\int_{s(y)}^{h(x)} \Psi_1\left(\frac{\alpha(\lambda,x,y)}{4C\lambda\left(v+\frac{1}{n}\right)}\right)\left(v+\frac{1}{n}\right). \end{aligned} \tag{2.5}$$

Putting together (2.4) and (2.5) we obtain

$$\frac{4C\lambda}{\alpha(\lambda,x,y)}\int_{s(y)}^{h(x)} \Psi_1\left(\frac{\alpha(\lambda,x,y)}{4C\lambda\left(v+\frac{1}{n}\right)}\right)\left(v+\frac{1}{n}\right) \leq J \leq 4C\lambda.$$

Letting $n \to \infty$ and applying the monotone convergence theorem, we get

$$\int_{s(y)}^{h(x)} \Psi_1\left(\frac{\alpha(\lambda,x,y)}{4C\lambda v}\right)v \leq \alpha(\lambda,x,y).$$

(iii) ⇒ (i). If $s(x) = h(x)$ for all $x \in (a,b)$, there is nothing to prove.

Let us suppose that there exists $z \in (a,b)$ such that $s(z) < h(z)$. Then $\Omega = \{x \in (a,b) : s(x) < h(x)\}$ is a nonempty open set. Let $\{(a_j,b_j)\}_j$ be the collection of the connected components of Ω and, for every j, let $\{m_k^j\}$ be the sequence given by lemma 1.

For fixed j, k and $x \in (m_k^j, m_{k+1}^j)$ we have

$$Tf(x) = g(x)\int_{s(x)}^{h(x)} f = g(x)\int_{s(x)}^{s(m_{k+1}^j)} f + g(x)\int_{s(m_{k+1}^j)}^{h(m_k^j)} f + g(x)\int_{h(m_k^j)}^{h(x)} f.$$

Then

$$\begin{aligned}
&\int_a^b \Phi_2(Tf)u \\
&= \sum_{j,k}\int_{m_k^j}^{m_{k+1}^j} \Phi_2\left(g(x)\int_{s(x)}^{s(m_{k+1}^j)} f + g(x)\int_{s(m_{k+1}^j)}^{h(m_k^j)} f + g(x)\int_{h(m_k^j)}^{h(x)} f\right) u(x)dx \\
&\leq \sum_{j,k}\int_{m_k^j}^{m_{k+1}^j} \Phi_2\left(g(x)\int_{s(x)}^{s(m_{k+1}^j)} 3f\right) u(x)dx \\
&+ \sum_{j,k}\int_{m_k^j}^{m_{k+1}^j} \Phi_2\left(g(x)\int_{s(m_{k+1}^j)}^{h(m_k^j)} 3f\right) u(x)dx \\
&+ \sum_{j,k}\int_{m_k^j}^{m_{k+1}^j} \Phi_2\left(g(x)\int_{h(m_k^j)}^{h(x)} 3f\right) u(x)dx = (I) + (II) + (III).
\end{aligned}$$

Let us estimate (III). Let us fix j, k and consider the sequence $\{x_n\}$ defined by $x_0 = m_{k+1}^j$ and $\int_{h(m_k^j)}^{h(x_n)} f = \int_{h(x_n)}^{h(x_{n-1})} f$. This sequence verifies

$$\int_{h(x_{n+2})}^{h(x_{n+1})} f = \frac{1}{4}\int_{h(m_k^j)}^{h(x_n)} f.$$

Let, for every $n \in \mathbf{N}$, $f_n = f\chi_{(h(x_{n+2}),h(x_{n+1}))}$. If $x \in (x_{n+1}, x_n)$ then, by the definition of the sequence $\{x_n\}$, we have

$$\int_{h(m_k^j)}^{h(x)} 4f_n \geq \int_{h(m_k^j)}^{h(x_{n+1})} 4f_n = 4\int_{h(x_{n+2})}^{h(x_{n+1})} f = \int_{h(m_k^j)}^{h(x_n)} f.$$

This shows that

$$(x_{n+1}, x_n) \subset E_n = \left\{x \in (m_k^j, m_{k+1}^j) : \int_{h(m_k^j)}^{h(x)} 12f_n > \lambda_n\right\}, \tag{2.6}$$

where $\lambda_n = \int_{h(m_k^j)}^{h(x_n)} 3f$.

By the monotonicity of $\int_{h(m_k^j)}^{h(x)} 12f_n$, it is clear that E_n is an interval of the form (γ, m_{k+1}^j). Let $x \in E_n$. Then,

$$\lambda_n < \int_{h(m_k^j)}^{h(x)} 12f_n = \int_{h(m_k^j)}^{h(x)} 12Cf_n\frac{\alpha(\lambda_n, x, m_{k+1}^j)}{Cv\alpha(\lambda_n, x, m_{k+1}^j)}v$$

or, equivalently,

$$2\alpha(\lambda_n, x, m^j_{k+1}) \leq \int_{h(m^j_k)}^{h(x)} 24Cf_n \frac{\alpha(\lambda_n, x, m^j_{k+1})}{\lambda_n Cv} v.$$

Applying Young's inequality and *(iii)* we obtain

$$\begin{aligned} 2\alpha(\lambda_n, x, m^j_{k+1}) &\leq \int_{h(m^j_k)}^{h(x)} \Phi_1(24Cf_n)v + \int_{h(m^j_k)}^{h(x)} \Psi_1\left(\frac{\alpha(\lambda_n, x, m^j_{k+1})}{\lambda_n Cv}\right) v \\ &\leq \int_{h(m^j_k)}^{h(x)} \Phi_1(24Cf_n)v + \alpha(\lambda_n, x, m^j_{k+1}), \end{aligned}$$

which gives

$$\alpha(\lambda_n, x, m^j_{k+1}) \leq \int_{h(m^j_k)}^{h(x)} \Phi_1(24Cf_n)v.$$

Since the above inequality holds for all $x \in E_n$, taking infimum we get

$$\int_{E_n} \Phi_2(\lambda_n g)u \leq \Phi_2 \circ \Phi_1^{-1} \left(\int_{h(m^j_k)}^{h(m^j_{k+1})} \Phi_1(24Cf_n)v \right). \tag{2.7}$$

By (2.6), (2.7), the definition of f_n and the subadditivity of $\Phi_1 \circ \Phi_2^{-1}$, we conclude

$$\begin{aligned} (III) &= \sum_{j,k} \int_{m^j_k}^{m^j_{k+1}} \Phi_2\left(g(x)\int_{h(m^j_k)}^{h(x)} 3f\right) u(x)dx \\ &= \sum_{j,k}\sum_n \int_{x_{n+1}}^{x_n} \Phi_2\left(g(x)\int_{h(m^j_k)}^{h(x)} 3f\right) u(x)dx \\ &\leq \sum_{j,k}\sum_n \int_{x_{n+1}}^{x_n} \Phi_2\left(g(x)\int_{h(m^j_k)}^{h(x_n)} 3f\right) u(x)dx \\ &\leq \sum_{j,k}\sum_n \int_{E_n} \Phi_2(g(x)\lambda_n)u(x)dx \\ &\leq \sum_{j,k}\sum_n (\Phi_2 \circ \Phi_1^{-1})\left(\int_{h(m^j_k)}^{h(m^j_{k+1})} \Phi_1(24Cf_n)v\right) \\ &= \sum_{j,k}\sum_n (\Phi_2 \circ \Phi_1^{-1})\left(\int_{h(x_{n+2})}^{h(x_{n+1})} \Phi_1(24Cf)v\right) \\ &= \sum_{j,k} (\Phi_2 \circ \Phi_1^{-1})\left(\int_{h(m^j_k)}^{h(m^j_{k+1})} \Phi_1(24Cf)v\right). \end{aligned}$$

The estimation of (I) can be done in a similar way obtaining

$$(I) \leq \sum_{j,k} (\Phi_2 \circ \Phi_1^{-1})\left(\int_{s(m^j_k)}^{s(m^j_{k+1})} \Phi_1(24Cf)v\right).$$

In order to estimate (II), let $\lambda_{j,k} = \int_{s(m^j_{k+1})}^{h(m^j_k)} 3f$. By Young's inequality and *(iii)* we have

$$\begin{aligned}2\alpha(\lambda_{j,k}, m^j_k, m^j_{k+1}) &= \int_{s(m^j_{k+1})}^{h(m^j_k)} 6Cf\frac{\alpha(\lambda_{j,k}, m^j_k, m^j_{k+1})}{C\lambda_{j,k}v}v \\ &\le \int_{s(m^j_{k+1})}^{h(m^j_k)} \Phi_1(6Cf)v + \int_{s(m^j_{k+1})}^{h(m^j_k)} \Psi_1\left(\frac{\alpha(\lambda_{j,k}, m^j_k, m^j_{k+1})}{C\lambda_{j,k}v}\right)v \\ &\le \int_{s(m^j_{k+1})}^{h(m^j_k)} \Phi_1(6Cf)v + \alpha(\lambda_{j,k}, m^j_k, m^j_{k+1}).\end{aligned}$$

Therefore

$$\alpha(\lambda_{j,k}, m^j_k, m^j_{k+1}) \le \int_{s(m^j_{k+1})}^{h(m^j_k)} \Phi_1(6Cf)v,$$

and this implies

$$(II) = \sum_{j,k}\int_{m^j_k}^{m^j_{k+1}} \Phi_2\left(g\int_{s(m^j_{k+1})}^{h(m^j_k)} 3f\right)u \le \sum_{j,k}(\Phi_2\circ\Phi_1^{-1})\left(\int_{s(m^j_{k+1})}^{h(m^j_k)} \Phi_1(6Cf)v\right).$$

Putting together the estimations of (I), (II) and (III), summing up in j and k and applying the subadditivity of $\Phi_1\circ\Phi_2^{-1}$, we get *(i)*.

□

3. Proof of theorem 2.

(i) ⇒ (ii). Let $\lambda > 0$ and let $x, y \in (a,b)$ with $x < y$ and $s(y) \le h(x)$. If $s(y) = h(x)$, there is nothing to prove. Let us suppose $s(y) < h(x)$. Let ρ be a positive number and $n \in \mathbf{N}$. There exists $\varepsilon > 0$ such that

$$\int_{s(y)}^{h(x)} \Psi_1\left(\frac{\varepsilon(\inf_{(x,y)}g)}{v+\frac{1}{n}}\right)\frac{v+\frac{1}{n}}{\varepsilon} = (1+\rho)C\lambda, \tag{3.1}$$

where C is the constant of inequality (1.2).

Let f be the function defined by

$$f = \frac{1}{C}\Psi_1\left(\frac{\varepsilon(\inf_{(x,y)}g)}{v+\frac{1}{n}}\right)\frac{v+\frac{1}{n}}{\varepsilon(\inf_{(x,y)}g)}\chi_{(s(y),h(x))}.$$

If $z \in (x,y)$ we have, by (3.1),

$$\begin{aligned}Tf(z) &= g(z)\int_{s(z)}^{h(z)} f \ge g(z)\int_{s(y)}^{h(x)} f = g(z)\int_{s(y)}^{h(x)} \frac{1}{C}\Psi_1\left(\frac{\varepsilon(\inf_{(x,y)}g)}{v+\frac{1}{n}}\right)\frac{v+\frac{1}{n}}{\varepsilon(\inf_{(x,y)}g)} \\ &\ge \int_{s(y)}^{h(x)} \frac{1}{C}\Psi_1\left(\frac{\varepsilon(\inf_{(x,y)}g)}{v+\frac{1}{n}}\right)\frac{v+\frac{1}{n}}{\varepsilon} = (1+\rho)\lambda > \lambda.\end{aligned}$$

We have seen that $(x,y) \subset \{z \in (a,b) : Tf(z) > \lambda\}$.

Applying *(i)*, the inequality $\Phi_1\left(\frac{\Psi_1(t)}{t}\right) \leq \Psi_1(t)$ and (3.1), we obtain

$$
\begin{aligned}
\beta(\lambda, x, y) &= (\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_x^y u \right) \leq (\Phi_1 \circ \Phi_2^{-1}) \left(\Phi_2(\lambda) \int_{\{z \in (a,b): Tf(z) > \lambda\}} u \right) \\
&\leq \int_{s(a)}^{h(b)} \Phi_1(Cf)v = \int_{s(y)}^{h(x)} \Phi_1 \left(\Psi_1 \left(\frac{\varepsilon(\inf_{(x,y)} g)}{v + \frac{1}{n}} \right) \frac{v + \frac{1}{n}}{\varepsilon(\inf_{(x,y)} g)} \right) v \\
&\leq \int_{s(y)}^{h(x)} \Psi_1 \left(\frac{\varepsilon(\inf_{(x,y)} g)}{v + \frac{1}{n}} \right) v \leq \int_{s(y)}^{h(x)} \Psi_1 \left(\frac{\varepsilon(\inf_{(x,y)} g)}{v + \frac{1}{n}} \right) (v + \frac{1}{n}) \\
&= (1 + \rho) C \lambda \varepsilon.
\end{aligned} \tag{3.2}
$$

The fact that the function $\frac{\Psi_1(t)}{t}$ increases, together with (3.1) and (3.2) give

$$
\begin{aligned}
&\int_{s(y)}^{h(x)} \Psi_1 \left(\frac{(\inf_{(x,y)} g) \beta(\lambda, x, y)}{(1 + \rho) C \lambda \left(v + \frac{1}{n}\right)} \right) \frac{v + \frac{1}{n}}{\beta(\lambda, x, y)} \\
&= \int_{s(y)}^{h(x)} \Psi_1 \left(\frac{(\inf_{(x,y)} g) \beta(\lambda, x, y)}{(1 + \rho) C \lambda \left(v + \frac{1}{n}\right)} \right) \frac{\left(v + \frac{1}{n}\right) (1 + \rho) C \lambda (\inf_{(x,y)} g)}{\beta(\lambda, x, y) (\inf_{(x,y)} g) (1 + \rho) C \lambda} \\
&\leq \int_{s(y)}^{h(x)} \Psi_1 \left(\frac{(\inf_{(x,y)} g) \varepsilon}{v + \frac{1}{n}} \right) \frac{\left(v + \frac{1}{n}\right) (\inf_{(x,y)} g)}{(\inf_{(x,y)} g) \varepsilon (1 + \rho) C \lambda} \\
&= \int_{s(y)}^{h(x)} \Psi_1 \left(\frac{(\inf_{(x,y)} g) \varepsilon}{v + \frac{1}{n}} \right) \frac{v + \frac{1}{n}}{\varepsilon (1 + \rho) C \lambda} = 1.
\end{aligned}
$$

Letting $n \to \infty$ and then $\rho \to 0$ we obtain

$$
\int_{s(y)}^{h(x)} \Psi_1 \left(\frac{(\inf_{(x,y)} g) \beta(\lambda, x, y)}{C \lambda v} \right) \frac{v}{\beta(\lambda, x, y)} \leq 1.
$$

(ii) $\Rightarrow$ *(i)*. If $\{m_k^j\}$ is the sequence given by lemma 1,

$$
u(\{x \in (a, b) : Tf(x) > \lambda\}) = \sum_{j,k} u(\{x \in (m_k^j, m_{k+1}^j) : Tf(x) > \lambda\}).
$$

For fixed k and j we have that if $x \in (m_k^j, m_{k+1}^j)$, then

$$
Tf(x) = g(x) \int_{s(x)}^{h(x)} f = g(x) \int_{s(x)}^{s(m_{k+1}^j)} f + g(x) \int_{s(m_{k+1}^j)}^{h(m_k^j)} f + g(x) \int_{h(m_k^j)}^{h(x)} f.
$$

By the subadditivity of $\Phi_1 \circ \Phi_2^{-1}$, it is clear that

$$\begin{aligned}
&\Phi_1 \circ \Phi_2^{-1}\left(\Phi_2(\lambda)\int_{\{x\in(m_k^j,m_{k+1}^j):Tf(x)>\lambda\}} u\right)\\
&\leq \Phi_1 \circ \Phi_2^{-1}\left(\Phi_2(\lambda)\int_{\left\{x\in(m_k^j,m_{k+1}^j):g(x)\int_{s(x)}^{s(m_{k+1}^j)} f>\frac{\lambda}{3}\right\}} u\right)\\
&+\Phi_1 \circ \Phi_2^{-1}\left(\Phi_2(\lambda)\int_{\left\{x\in(m_k^j,m_{k+1}^j):g(x)\int_{s(m_{k+1}^j)}^{h(m_k^j)} f>\frac{\lambda}{3}\right\}} u\right)\\
&+\Phi_1 \circ \Phi_2^{-1}\left(\Phi_2(\lambda)\int_{\left\{x\in(m_k^j,m_{k+1}^j):g(x)\int_{h(m_k^j)}^{h(x)} f>\frac{\lambda}{3}\right\}} u\right)\\
&= (I) + (II) + (III).
\end{aligned}$$

Let us estimate (III). Let $\{x_n\}$ be the sequence defined as in the proof of theorem 1. Let

$$E_n = (x_{n+1}, x_n) \cap \left\{x \in (m_k^j, m_{k+1}^j) : g(x)\int_{h(m_k^j)}^{h(x)} f > \frac{\lambda}{3}\right\}.$$

If $x \in E_n$, then

$$\frac{\lambda}{3} < g(x)\int_{h(m_k^j)}^{h(x)} f \leq g(x)\int_{h(m_k^j)}^{h(x_n)} f = 4g(x)\int_{h(x_{n+2})}^{h(x_{n+1})} f.$$

This implies

$$\lambda \leq 12(\inf_{E_n} g)\int_{h(x_{n+2})}^{h(x_{n+1})} f.$$

Let $\delta_n = \inf E_n$ and $\gamma_n = \sup E_n$. Since g is monotone we can ensure

$$\lambda \leq 12(\inf_{(\delta_n,\gamma_n)} g)\int_{h(x_{n+2})}^{h(x_{n+1})} f.$$

Applying this property and Young's inequality we obtain

$$\begin{aligned}
2\beta(\lambda,\delta_n,\gamma_n) &\leq \beta(\lambda,\delta_n,\gamma_n)\frac{24}{\lambda}\left(\inf_{(\delta_n,\gamma_n)} g\right)\int_{h(x_{n+2})}^{h(x_{n+1})} f\\
&= \int_{h(x_{n+2})}^{h(x_{n+1})} 24Cf\,\frac{\beta(\lambda,\delta_n,\gamma_n)\inf_{(\delta_n,\gamma_n)} g}{C\lambda v} v.\\
&\leq \int_{h(x_{n+2})}^{h(x_{n+1})} \Phi_1(24Cf)v + \int_{h(x_{n+2})}^{h(x_{n+1})} \Psi_1\left(\frac{\beta(\lambda,\delta_n,\gamma_n)\inf_{(\delta_n,\gamma_n)} g}{C\lambda v}\right) v.
\end{aligned} \tag{3.3}$$

Since $s(\gamma_n) \le s(m^j_{k+1}) \le h(m^j_k) \le h(x_{n+2}) \le h(x_{n+1}) \le h(\delta_n)$, condition *(ii)* gives

$$\begin{aligned}&\int_{h(x_{n+2})}^{h(x_{n+1})} \Psi_1\left(\frac{\beta(\lambda,\delta_n,\gamma_n)\left(\inf_{(\delta_n,\gamma_n)} g\right)}{C\lambda v}\right) v\\ &\le \int_{s(\gamma_n)}^{h(\delta_n)} \Psi_1\left(\frac{\beta(\lambda,\delta_n,\gamma_n)\left(\inf_{(\delta_n,\gamma_n)} g\right)}{C\lambda v}\right) v\\ &\le \beta(\lambda,\delta_n,\gamma_n).\end{aligned}$$

Taking away this inequality to (3.3) we obtain

$$(\Phi_1 \circ \Phi_2^{-1})\left(\Phi_2(\lambda)\int_{\delta_n}^{\gamma_n} u\right) \le \int_{h(x_{n+2})}^{h(x_{n+1})} \Phi_1(24Cf)v,$$

which implies

$$(\Phi_1 \circ \Phi_2^{-1})\left(\Phi_2(\lambda)\int_{E_n} u\right) \le \int_{h(x_{n+2})}^{h(x_{n+1})} \Phi_1(24Cf)v.$$

Summing up in n and applying the subadditivity of $\Phi_1 \circ \Phi_2^{-1}$ we get

$$(III) = (\Phi_1 \circ \Phi_2^{-1})\left(\Phi_2(\lambda)\int_{\left\{x\in(m^j_k,m^j_{k+1}):g(x)\int_{h(m^j_k)}^{h(x)} f>\frac{\lambda}{3}\right\}} u\right) \le \int_{h(m^j_k)}^{h(m^j_{k+1})} \Phi_1(24Cf)v.$$

In a similar way, we have

$$(I) \le \int_{s(m^j_k)}^{s(m^j_{k+1})} \Phi_1(24Cf)v.$$

In order to estimate (II), let

$$E_{j,k} = \left\{x \in (m^j_k, m^j_{k+1}) : g(x)\int_{s(m^j_{k+1})}^{h(m^j_k)} f > \frac{\lambda}{3}\right\}.$$

Since g is monotone, the set $E_{j,k}$ is an interval. Working as in the estimation of (III), we prove

$$(II) = (\Phi_1 \circ \Phi_2^{-1})\left(\Phi_2(\lambda)\int_{E_{j,k}} u\right) \le \int_{s(m^j_{k+1})}^{h(m^j_k)} \Phi_1(24Cf)v.$$

From the estimations of (I), (II) and (III), we deduce

$$(\Phi_1 \circ \Phi_2^{-1})\left(\Phi_2(\lambda)\int_{\{x\in(m^j_k,m^j_{k+1}):Tf(x)>\lambda\}} u\right) \le \int_{s(m^j_k)}^{h(m^j_{k+1})} \Phi_1(24Cf)v.$$

Summing up in k and j and taking into account the subadditivity of $\Phi_1 \circ \Phi_2^{-1}$, we get

$$(\Phi_1 \circ \Phi_2^{-1})\left(\Phi_2(\lambda)\int_{\{x\in(a,b):Tf(x)>\lambda\}} u\right) \le \int_{s(a)}^{h(b)} \Phi_1(24Cf)v.$$

$\square$

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