



# Endpoint Entropy Fefferman–Stein Bounds for Commutators

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Received: 5 November 2022 / Revised: 2 June 2023 / Accepted: 9 August 2023 /

Published online: 15 September 2023

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## Abstract

In this paper endpoint entropy Fefferman–Stein bounds for Calderón–Zygmund operators introduced by Rahm (J Math Anal Appl 504(1):Paper No. 125372, 2021) are extended to iterated Coifman–Rochberg–Weiss commutators.

**Keywords** Commutators · Endpoint estimates · Entropy bumps · Quantitative weighted estimates

**Mathematics Subject Classification** 42B20 · 42B25

## 1 Introduction and Main Result

In the last decade, quantitative weighted estimates have been an important topic of study in harmonic analysis. The motivation of the results that we present here can be traced back to the so called Muckenhoupt–Wheeden conjecture. It is a classical result due to Fefferman and Stein that if  $w$  is any weight, namely a non negative locally integrable function, then

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Communicated by Rodolfo H. Torres.

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P. A. Muller and I. P. Rivera-Ríos were partially supported by FONCyT PICT 2018-02501 and PICT 2019-00018. The author I. P. Rivera-Ríos was partially supported by Junta de Andalucía UMA18FEDERJA002, FQM 354.

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$$w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq c_n \frac{1}{t} \int_{\mathbb{R}^n} |f| Mw, \tag{1.1}$$

where  $c_n$  is a constant depending just on  $n$  and  $M$  stands for the classical Hardy–Littlewood maximal function,

$$Mf(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f|,$$

where each  $Q$  is a cube with its sides parallel to the axis. The Muckenhoupt–Wheeden conjecture considered the possibility of replacing  $M$  by the Hilbert transform on the left-hand side of (1.1). In the case of dyadic models that conjecture was disproved by Reguera in [15] and for the Hilbert transform by Reguera and Thiele in [16].

Being that conjecture disproved a natural question would be whether (1.1) could hold for Calderón–Zygmund operators or at least for the Hilbert transform with the maximal operator in the right hand side replaced by a slightly larger one. That direction of research had been already followed in the 90s by Pérez [11], who showed that the following inequality holds

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq c_{n,\rho} \int_{\mathbb{R}^n} \frac{|f|}{t} M_{L(\log L)^\rho} w \quad \rho > 0 \tag{1.2}$$

where  $T$  stands for any Calderón–Zygmund,  $c_{n,\rho}$  is a constant that blows up when  $\rho \rightarrow 0$ , and

$$M_{L(\log L)^\rho} w = \sup_{x \in Q} \|w\|_{L(\log L)^\rho, Q}.$$

In order to make sense of  $M_{L(\log L)^\rho} w$ , we recall that, given a Young function  $A : [0, \infty) \rightarrow [0, \infty)$ , namely a convex increasing function such that  $\lim_{t \rightarrow \infty} \frac{A(t)}{t} = \infty$  and  $A(0) = 0$  we define

$$\|f\|_{A(L), Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_Q A\left(\frac{|f|}{\lambda}\right) \leq 1 \right\}.$$

Abusing of notation we shall denote  $\|f\|_{L(\log L)^\gamma, Q}$  in the case in which  $A(t) = t \log^\gamma(e + t)$  and analogously, for instance  $\|f\|_{L(\log^\gamma \log L), Q}$ , for the case  $A(t) = t \log^\gamma(e^e + \log(e + t))$ . A fundamental property of these averages is that if  $A(t) \leq B(t)$  for every  $t \geq t_0$  for a certain  $t_0 \geq 0$ , then

$$\|f\|_{A(L), Q} \lesssim \|f\|_{B(L), Q}.$$

Furthermore, they satisfy a generalized Hölder inequality. If  $A, B, C$  are Young functions such that  $A^{-1}(t)B^{-1}(t) \lesssim C^{-1}(t)$ , then

$$\|fg\|_{C, Q} \lesssim \|f\|_{A, Q} \|g\|_{B, Q}. \tag{1.3}$$

Coming back to our discussion, it is worth noting that the development of sparse domination theory led, directly or indirectly, to several improvements for (1.2).

- In [6] it was established that  $c_{n,\rho} \simeq c_n \frac{1}{\rho}$  in (1.2). That blow up in  $\rho$  is sharp, for instance, due to the sharp dependence on the  $A_1$  constant for the Hilbert transform settled in [10].
- In [3] it was settled that  $M_{L(\log L)^\rho}$  in (1.2) could be replaced for even smaller operators such as  $M_{L(\log \log L)^{1+\rho}}$  keeping  $c_{n,\rho} \simeq c_n \frac{1}{\rho}$  as well.
- In [1] it was shown that if

$$\lim_{t \rightarrow \infty} \frac{\phi(t)}{t \log \log(t)} = 0$$

then (1.2) with  $M_\phi$  in place of  $M_{L(\log L)^\rho}$  cannot hold. Up until now the whether (1.2) holds with  $M_{L \log \log L}$  in the right hand side remains an open question.

Quite recently another line of research related to Fefferman Stein estimates was initiated by Rahm in [14]. The new approach consisted in replacing in (1.2)  $M_{L(\log L)^\rho}$  by a suitable entropy bump type maximal operator encoding  $A_\infty$  type information of the weight. Entropy bump conditions were introduced by Treil and Volberg [17] to obtain sufficient conditions for the two weight boundedness of Calderón–Zygmund operators. Also in [17] it was shown for the case  $p = 2$  that entropy bump conditions are slightly more general than the bump conditions introduced by Pérez in [13]. An easy approach to entropy bump estimates relying upon sparse domination results was provided by Lacey and Spencer in [8].

Let us recall now Rahm’s result. Given a weight  $w$ , let  $\rho_w(Q) = \frac{1}{w(Q)} \int_Q M(\chi_Q w)$  and assume that  $\varepsilon : [1, \infty) \rightarrow [1, \infty)$  an increasing function. Then we define

$$M_\varepsilon w(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q w(y) dy \log_2(2 + \rho_w(Q)) \varepsilon(\rho_w(Q)).$$

As we mentioned above, the operator  $M_\varepsilon$  encodes  $A_\infty$  type information since the  $A_\infty$  constant is defined precisely in terms of  $\rho_w(Q)$ . To be more precise  $w \in A_\infty$  if  $[w]_{A_\infty} = \sup_Q \rho_w(Q) < \infty$ . Rahm shows that for this operator  $M_\varepsilon$

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > t\}) \leq c_n \sum_{k=1}^\infty \frac{1}{\varepsilon(2^{2^k})} \int_{\mathbb{R}^n} \frac{|f|}{t} M_\varepsilon w.$$

Observe that  $M_\varepsilon$  introduces a whole new scale of maximal operators suitable for endpoint estimates. It is not known if  $M_\varepsilon$  is comparable to any Orlicz maximal operator as the ones mentioned above.

Now we turn our attention to our contribution. We recall that given a Calderón Zygmund operator  $T$  and  $b \in BMO$ , the iterated commutator  $T_b^m$  is defined as

$$T_b^m f(x) = [b, T_b^{m-1}]f(x)$$

where

$$T_b^1 f(x) = [b, T]f(x) = b(x)Tf(x) - T(bf)(x)$$

is the Coifman–Rochberg–Weiss commutator.

Endpoint Fefferman–Stein type estimates for commutators have been explored as well. The best known result up until now is the following [7, 9]. If  $w$  is an arbitrary weight and  $b \in BMO$  then

$$\begin{aligned} & w(\{x \in \mathbb{R}^n : |T_b^m f(x)| > t\}) \\ & \leq c_{n,T} \frac{1}{\rho} \int_{\mathbb{R}^n} \Phi_m \left( \frac{\|b\|_{BMO}^m |f|}{t} \right) M_{L(\log L)^m(\log L)^{1+\rho}} w \quad \rho > 0. \end{aligned}$$

where  $\Phi_m(t) = t \log^m(e + t)$ . Our purpose in this note is to explore endpoint entropy bump weighted estimates for  $T_b^m$ . Before presenting our results we need a few more definitions. As we noted above, given a weight  $w$  Rahm defines  $\rho_w(Q)$

$$\rho_w(Q) = \frac{1}{w(Q)} \int_Q M(\chi_Q w).$$

Note that, since  $\frac{1}{|Q|} \int_Q M(\chi_Q w) \simeq \|w\|_{L \log L, Q}$ , we can rephrase this condition as

$$\rho_{1,w}(Q) = \frac{\|w\|_{L \log L, Q}}{\langle w \rangle_Q}$$

in the sense that  $\rho_w(Q) \simeq \rho_{1,w}(Q)$ . Hence it is natural to generalize such a condition as follows. Given a positive integer  $k$  we define

$$\rho_{k,w}(Q) = \frac{\|w\|_{L(\log L)^k, Q}}{\langle w \rangle_Q}.$$

Having that notation at our disposal we can also generalize the entropy maximal function due to Rahm as follows. Given a Young  $A$ , a non-negative integer  $k$  and an increasing function  $\varepsilon : [1, \infty) \rightarrow [1, \infty)$ , we define

$$M_{\varepsilon,A,k} w(x) = \sup_{x \in Q} \langle w \rangle_{A,Q} \log_2 (2 + \rho_{k,w}(Q)) \varepsilon (\rho_{k,w}(Q)).$$

If  $A(t) = t$ , we shall drop the subscript  $A$ . On the other hand, if besides  $A(t) = t$  we have that  $k = 1$  as well this operator reduces to Rahm’s  $M_\varepsilon$ .

Armed with the preceding definitions we can finally state the Theorem of this paper.

**Theorem 1** *Let  $m$  be a positive integer. Let  $b \in BMO$  and assume that  $T$  is a Calderón–Zygmund operator and that  $w$  is a weight. Then*

$$w(\{x \in \mathbb{R}^n : |T_b^m f(x)| > t\}) \leq \kappa_\varepsilon \int_{\mathbb{R}^n} \Phi_m \left( \frac{\|b\|_{BMO}^m |f|}{t} \right) M_{\varepsilon,L(\log L)^m,m+1} w$$

where  $\kappa_\varepsilon = c_{n,T,m} \max \left\{ \sum_{r=0}^\infty \frac{1}{\varepsilon(2^{2^r})}, 1 \right\}$ .

The remainder of the paper is devoted to the proof of this result.

## 2 Proof of the Main Result

Our proof relies upon the sparse domination result that was settled in [7, 9].

**Theorem 2** *Let  $b \in L^1_{loc}$  and let  $T$  be a Calderón–Zygmund operator. Then there exist  $N_\alpha$   $\alpha$ -Carleson families  $S_j$  contained in  $3^n$  dyadic lattices such that*

$$|T_b^m f(x)| \leq c_{n,m,T} \sum_{j=1}^{N_\alpha} \sum_{h=0}^m \mathcal{T}_{b,S_j}^{h,m} |f|(x)$$

where

$$\mathcal{T}_{b,S}^{h,m} f(x) = \sum_{Q \in S} |b(x) - b_Q|^h \frac{1}{|Q|} \int_Q |b - b_Q|^{m-h} f.$$

Observe that, in fact, it suffices to study  $\mathcal{T}_{b,S}^{m,m}$  and  $\mathcal{T}_{b,S}^{0,m}$ , since, as it was shown in [2, Lemma 2.2], for  $f \geq 0$

$$\mathcal{T}_{b,S}^h f(x) \leq \mathcal{T}_{b,S}^{m,m} f(x) + \mathcal{T}_{b,S}^{0,m} f(x)$$

for every  $h \in \{0, \dots, m\}$ .

Hence the proof of Theorem 1 boils down to obtaining estimates just for  $\mathcal{T}_{b,S_j}^{m,m}$  and  $\mathcal{T}_{b,S_j}^{0,m}$ .

For  $\mathcal{T}_{b,S}^{m,m}$  we provide the following result.

**Theorem 3** *Let  $S$  be a  $\alpha$ -Carleson family with  $0 < 56^m(\alpha - 1) < 1$  and  $b \in BMO$ . Then,*

$$\|\mathcal{T}_{b,S}^{m,m} f\|_{L^{1,\infty}(w)} \leq c_{n,\alpha} \|b\|_{BMO}^m \sum_{r=0}^\infty \frac{1}{\varepsilon(2^{2^r})} \|f\|_{L^1(M_{\varepsilon,L}(\log L)^{m,m+1} w)}$$

where  $\varepsilon : [1, \infty) \rightarrow [1, \infty)$  is an increasing function.

On the other hand we note that for  $\mathcal{T}_{b,S}^{0,m}$  the following estimate can be recovered from arguments in [7, 9].

**Theorem 4** *Let  $S$  be a Carleson family and let  $b \in BMO$ . Then*

$$w \left( \left\{ x \in \mathbb{R}^n : \mathcal{T}_{b,S}^{0,m} |f| > t \right\} \right) \leq c_{n,m,\alpha} \int_{\mathbb{R}^n} \Phi_m \left( \frac{\|b\|_{BMO}^m |f|}{t} \right) M_{L(\log L)^m} w.$$

Since  $M_{L(\log L)^m} w \leq M_{\varepsilon, L(\log L)^m, m+1} w$  the estimate in the preceding result is good enough for our purposes due to the fact that the main theorem readily follows from the combination of the results above. Hence all we are left to settle our main result is to establish Theorem 3. We devote the remainder of the section and of the paper to that purpose.

### 2.1 Lemmata

Arguing as in [5, 6.6 Lemma] we can get the following lemma.

**Lemma 1** *For a cube  $Q$  and a subset  $E \subsetneq Q$  we have that*

$$w(E) \leq \frac{2^{n+3} \rho_w(Q)}{\log\left(\frac{|Q|}{|E|}\right)} w(Q).$$

Lemma 1 is an important tool in [14]. In the following lines we present a result generalizes the lemma above. Before that we recall that it is a well known fact that

$$\|w\|_{L \log L^k, Q} \simeq \frac{1}{|Q|} \int_Q w \log\left(e + \frac{w}{w_Q}\right)^k$$

(see for instance [18, p. 179, Theorem 10.8]) and it is not hard to check that there exists  $\kappa_k \geq 1$  such that for every  $a, b \geq 0$

$$\Phi_k(ab) \leq \kappa_k \Phi_k(a) \Phi_k(b)$$

where  $\Phi_k(t) = t \log^k(e + t)$ . Bearing those facts in mind we can settle the following Lemma.

**Lemma 2** *Let  $Q$  be a cube and  $E \subsetneq Q$ . Then there exists  $c > 0$  depending just on  $k$  and  $n$ , such that*

$$\|w \chi_E\|_{L \log^k L, Q} \leq c \frac{\log\left(e + \log\left(\frac{|Q|}{|E|}\right)\right)^k}{\log\left(\frac{|Q|}{|E|}\right)} \|w\|_{L \log^{k+1} L, Q}$$

and consequently

$$\|w \chi_E\|_{L \log^k L, Q} \leq c \frac{\log\left(e + \log\left(\frac{|Q|}{|E|}\right)\right)^k}{\log\left(\frac{|Q|}{|E|}\right)} \rho_{k+1, w}(Q) \langle w \rangle_Q.$$

**Proof** Let  $J_\gamma = \{x \in Q : w(x) > e^\gamma \langle w \rangle_Q\}$ . First we observe that

$$\begin{aligned} & \frac{1}{|Q|} \int_{J_\gamma} \frac{w}{\lambda_0 \langle w \rangle_Q} \log^k \left( e + \frac{w}{\lambda_0 \langle w \rangle_Q} \right) \\ &= \frac{1}{|Q|} \int_{J_\gamma} \frac{w}{\lambda_0 \langle w \rangle_Q} \log^k \left( e + \frac{w}{\lambda_0 \langle w \rangle_Q} \right) \frac{\log \left( e + \frac{w}{\langle w \rangle_Q} \right)}{\log \left( e + \frac{w}{\langle w \rangle_Q} \right)} \\ &\leq \frac{1}{\gamma} \frac{1}{|Q|} \int_Q \frac{w}{\lambda_0 \langle w \rangle_Q} \log^k \left( e + \frac{w}{\lambda_0 \langle w \rangle_Q} \right) \log \left( e + \frac{w}{\langle w \rangle_Q} \right) \\ &\leq \frac{c_n \kappa}{\gamma} \Phi_k \left( \frac{1}{\lambda_0} \right) \frac{1}{\langle w \rangle_Q} \|w\|_{L \log^{k+1} L, Q}. \end{aligned}$$

Now note that we have that

$$\begin{aligned} \frac{c_n \kappa}{\gamma} \Phi_k \left( \frac{1}{\lambda_0} \right) \frac{1}{\langle w \rangle_Q} \|w\|_{L \log^{k+1} L, Q} \leq 1 &\iff \Phi_k \left( \frac{1}{\lambda_0} \right) \leq \frac{\langle w \rangle_Q \gamma}{c_n \kappa \|w\|_{L \log^{k+1} L, Q}} \\ \iff \frac{1}{\lambda_0} \leq \Phi_k^{-1} \left( \frac{\langle w \rangle_Q \gamma}{c_n \kappa \|w\|_{L \log^{k+1} L, Q}} \right) &\iff \frac{1}{\Phi_k^{-1} \left( \frac{\langle w \rangle_Q \gamma}{c_n \kappa \|w\|_{L \log^{k+1} L, Q}} \right)} \leq \lambda_0 \end{aligned}$$

and this yields

$$\begin{aligned} \|w \chi_{J_\gamma}\|_{L \log^k L, Q} &\leq \frac{\langle w \rangle_Q}{\Phi_k^{-1} \left( \frac{\langle w \rangle_Q \gamma}{c_n \kappa \|w\|_{L \log^{k+1} L, Q}} \right)} \simeq \frac{\langle w \rangle_Q \log \left( e + \frac{\langle w \rangle_Q \gamma}{c_n \kappa \|w\|_{L \log^{k+1} L, Q}} \right)}{\frac{\langle w \rangle_Q \gamma}{c_n \kappa \|w\|_{L \log^{k+1} L, Q}}} \\ &\leq c_n \kappa \frac{\log^k (e + \gamma)}{\gamma} \|w\|_{L \log^{k+1} L, Q}. \end{aligned}$$

Having that estimate at our disposal now we can proceed as follows. Let

$$\frac{|E|}{|Q|} = e^{-\lambda}.$$

Then

$$\begin{aligned} \|w \chi_E\|_{L \log^k L, Q} &\leq \|w \chi_{E \cap J_{\lambda/2}}\|_{L \log^k L, Q} + \|w \chi_{E \setminus J_{\lambda/2}}\|_{L \log^k L, Q} \\ &\leq c_n \kappa \frac{\log^k (e + \lambda/2)}{\lambda/2} \|w\|_{L \log^{k+1} L, Q} + e^{\frac{\lambda}{2}} \frac{w(Q)}{|Q|} \frac{1}{\Phi_k^{-1} \left( \frac{|Q|}{|E|} \right)} \\ &\leq 2c_n \kappa \frac{\log^k (e + \lambda)}{\lambda} \|w\|_{L \log^{k+1} L, Q} + c_n e^{\frac{\lambda}{2}} \frac{w(Q)}{|Q|} \frac{\log^k \left( e + \frac{|Q|}{|E|} \right)}{\frac{|Q|}{|E|}} \end{aligned}$$

$$\begin{aligned}
 &= 2c_n \kappa \frac{\log^k(e + \lambda)}{\lambda} \|w\|_{L^{\log^{k+1} L, Q}} + c_n e^{\frac{\lambda}{2}} \frac{w(Q) \log^k(e + e^\lambda)}{|Q| e^\lambda} \\
 &\leq 2c_n \kappa \frac{\log^k(e + \lambda)}{\lambda} \|w\|_{L^{\log^{k+1} L, Q}} + c_n e^{-\frac{\lambda}{2}} \frac{w(Q)}{|Q|} \log^k(e + e^\lambda).
 \end{aligned}$$

Since we have that the first term is larger we are done.

We end this section recalling two equivalent ways to state the John–Nirenberg inequality. The first one (see for instance [4, p. 124]) tells us that if  $b \in BMO$  then

$$\{x \in Q : |b(x) - b_Q| > \lambda\} \leq e|Q|e^{-\frac{\lambda}{e^{2n}\|b\|_{BMO}}}. \tag{2.1}$$

An alternative way to formulate John–Nirenberg that follows from the one above is the following. There exists a constant  $\beta_n$  such that such that for every  $b \in BMO$

$$\left\| \frac{|b - b_Q|}{\|b\|_{BMO}} \right\|_{\exp(L), Q} \leq \beta_n \tag{2.2}$$

where  $\exp(L)$  stands for  $\varphi(t) = \exp(t) - 1$ .

### 2.2 Proof of Theorem 3

We shall assume that  $\|b\|_{BMO} = 1$  by homogeneity and also that  $f \geq 0$  since  $T_{b, S}^{m, m}$  is a positive operator. Observe that we can split the sparse family as  $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$  where  $\mathcal{S}_1$  contains the cubes for which  $1 \leq \rho_{m+1, w}(Q) < 2$  and  $\mathcal{S}_2$  the remaining ones. Then

$$w(\{T_{b, \mathcal{S}}^{m, m} f > t\}) \leq w\left(\left\{T_{b, \mathcal{S}_1}^{m, m} f > \frac{t}{2}\right\}\right) + w\left(\left\{T_{b, \mathcal{S}_2}^{m, m} f > \frac{t}{2}\right\}\right).$$

Observe that for the first term we have that  $w \in A_\infty$  with respect to the family  $\mathcal{S}_1$  and hence, arguments in [7, 9] show that

$$w\left(\left\{T_{b, \mathcal{S}_1}^{m, m} f > t\right\}\right) \lesssim \frac{1}{t} \int_{\mathbb{R}^n} f M w.$$

However we will provide an argument for that term as well the sake of completeness. We shall deal with those terms separately. We will be done provided we can show that

$$w\left(\left\{T_{b, \mathcal{S}_i}^{m, m} f > t\right\}\right) \lesssim \frac{1}{t} \int_{\mathbb{R}^n} f M_{\varepsilon, L(\log L)^m, m+1} w.$$

We shall proceed as follows. First recall that by homogeneity it suffices to show that for some  $t_0 > 0$

$$w(\{T_{b, \mathcal{S}_i}^{m, m} f > t_0\}) \lesssim \int_{\mathbb{R}^n} f M_{\varepsilon, L(\log L)^m, m+1} w.$$

We will argue as follows for both terms. Let  $\tau > 0$  such that  $\varphi(t) = \frac{\log^m(e+\log(t))}{\log^m(t)}$  is decreasing for  $t \geq e^{4^{1+\tau}-1}$ . Observe that

$$\begin{aligned} & w(\{T_{b,S_i}^{m,m} f > 4^{\tau m} 2^{nm} e^m \cdot 100\}) \\ &= w\left(\left\{T_{b,S_i}^{m,m} f > 4^{\tau m} 2^{nm} e^m \cdot 100, Mf \leq \frac{1}{56^m}\right\}\right. \\ &\quad \left.\cup \left\{T_{b,S_i}^{m,m} f > 4^{\alpha m} 2^{nm} e^m \cdot 100, Mf > \frac{1}{56^m}\right\}\right) \\ &= w\left(\left\{T_{b,S_i}^{m,m} f > 4^{\tau m} 2^{nm} e^m \cdot 100, Mf \leq \frac{1}{56^m}\right\}\right) + w\left(\left\{Mf > \frac{1}{56^m}\right\}\right) \\ &\leq w\left(\left\{T_{b,S_i}^{m,m} f > 4^{\tau m} 2^{nm} e^m \cdot 100, Mf \leq \frac{1}{56^m}\right\}\right) + 56^m \int_{\mathbb{R}^n} f M w. \end{aligned}$$

This reduces us to provide a suitable estimate for the first term. Let us call

$$G_i = \left\{T_{b,S_i}^{m,m} f > 4^{\tau m} 2^{nm} e^m \cdot 100, Mf \leq \frac{1}{56^m}\right\}.$$

We shall assume that  $w(G_i) < \infty$  since otherwise we already had that  $w(\{T_{b,S_i}^{m,m} f > 4^{\tau m} 2^{nm} e^m \cdot 100\}) = \infty$  and hence the estimate was trivial. Then it suffices to show that

$$w(G_i) \leq c \int_{\mathbb{R}^n} f M_{\varepsilon,L(\log L)^m,m+1} w + \nu_i w(G) \tag{2.3}$$

for some  $\nu_i \in (0, 1)$  in both cases. We devote the remainder of the subsection to that purpose.

### 2.2.1 Bound for $T_{b,S_1}^{m,m}$

We shall drop the subscripts of  $S_1$  and  $G_1$  for the sake of clarity. We split the family  $S$  as follows  $Q \in S_k$  if and only if

$$\frac{1}{56^{m(k+1)}} < \frac{1}{|Q|} \int_Q f \leq \frac{1}{56^{km}}.$$

Then,  $S = \bigcup_{k=1}^{\infty} S_k$ . We recall, as well, that  $1 \leq \rho_{m+1,w}(Q) < 2$  for every cube  $Q \in S$ .

Observe that then

$$w(G) \leq \frac{1}{4^{\tau m} 2^{nm} e^m \cdot 100} \sum_{k=1}^{\infty} \sum_{Q \in S_k} \frac{1}{|Q|} \int_Q f \int_{G \cap Q} |b - b_Q|^m w \tag{2.4}$$

and let us consider, as above, for  $Q \in \mathcal{S}_k$ ,

$$F_k(Q) = \left\{ x \in Q : |b - b_Q|^m > 2^{nm} e^m 4^{m(k+\tau)} \right\}.$$

By John–Nirenberg theorem (2.1), since  $b \in BMO$ ,

$$\frac{|F_k(Q)|}{|Q|} \leq e e^{-4^{k+\tau}}.$$

Now we argue as follows. Observe that

$$\begin{aligned} & \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f \int_{G \cap Q} |b - b_Q|^m w \\ & \leq \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w \\ & \quad + \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f \int_{G \cap (Q \setminus F_k(Q))} |b - b_Q|^m w \\ & \leq \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w \\ & \quad + 2^{nm} e^m \cdot 4^{\tau m} \sum_{k=1}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f \int_{G \cap Q} w \\ & = (L_1 + 2^{nm} e^m \cdot 4^{\tau m} L_2) \end{aligned}$$

Observe that if  $Q \in \mathcal{S}_k$  and we denote  $E_Q = Q \setminus \bigcup_{Q' \subsetneq Q, Q' \in \mathcal{S}_k} Q'$  then

$$\int_Q f \lesssim \int_{E_Q} f. \tag{2.5}$$

Indeed

$$\begin{aligned} \int_Q f &= \int_{E_Q} f + \sum_{\substack{Q' \subsetneq Q \\ Q' \in \mathcal{S}_k}} \int_{Q'} f \leq \int_{E_Q} f + \sum_{\substack{Q' \subsetneq Q \\ Q' \in \mathcal{S}_k}} \frac{1}{56^{km}} |Q'| \\ &\leq \int_{E_Q} f + 56^m (\alpha - 1) \frac{|Q|}{56^{(k+1)m}} \leq \int_{E_Q} f + 56^m (\alpha - 1) \int_Q f \end{aligned}$$

and since  $56^m (\alpha - 1) < 1$  we arrive to the desired conclusion.

First we deal with  $L_1$ . Since  $\varphi(t) = \frac{\log^m(e + \log(t))}{\log^m(t)}$  is decreasing for  $t \geq e^{4^{1+\tau}-1}$  taking into account, that  $\frac{|F_k(Q)|}{|Q|} \leq e e^{-4^{k+\tau}} \iff \frac{|Q|}{|F_k(Q)|} \geq e^{4^{k+\tau}-1}$ , we have that by

Lemma 2,

$$\begin{aligned} \|w\chi_{F_k(Q)}\|_{L \log L^m, Q} &\leq c \frac{\log\left(e + \log\left(\frac{|Q|}{|F_k(Q)|}\right)\right)^m}{\log\left(\frac{|Q|}{|F_k(Q)|}\right)} \rho_{m+1, w}(Q) \langle w \rangle_Q \\ &\leq c \frac{\log\left(e + \log\left(e^{4^{k+\tau}} - 1\right)\right)^m}{\log\left(e^{4^{k+\tau}} - 1\right)} \rho_{m+1, w}(Q) \langle w \rangle_Q \\ &\lesssim \frac{k^m}{4^k} \rho_{m+1, w}(Q) \langle w \rangle_Q, \end{aligned}$$

namely

$$\|w\chi_{F_k(Q)}\|_{L \log L^m, Q} \lesssim \frac{k^m}{4^k} \rho_{m+1, w}(Q) \langle w \rangle_Q. \tag{2.6}$$

Before continuing we note that if  $A(t) = \exp(L\frac{1}{m}) - 1$  we have that

$$A^{-1}(t) \frac{t}{\log^m(1+t)} = t$$

and if  $B^{-1}(t) = \frac{t}{\log^m(1+t)}$  we have that  $B(t) \simeq t \log^m(1+t) \leq t \log^m(e+t)$ . Hence, by generalized Hölder inequality (1.3) combined with (2.2), since  $\|b\|_{BMO} = 1$ ,

$$\begin{aligned} \frac{1}{|Q|} \int_{G \cap F_k(Q)} |b - b_Q|^m w &= \frac{1}{|Q|} \int_{G \cap F_k(Q)} |b - b_Q|^m w \\ &\leq \|(b - b_Q)^m\|_{\exp(L\frac{1}{m}), Q} \|w\chi_{F_k(Q)}\|_{L(\log L)^m, Q} \\ &= \left\| \frac{|b - b_Q|}{\|b\|_{BMO}} \right\|_{\exp(L), Q} \|f\|_{L(\log L)^m, Q} \leq \beta_m \|f\|_{L(\log L)^m, Q} \end{aligned}$$

Armed with the estimates above, since  $\rho_{m+1, w}(Q) \leq 2$  for every  $Q \in \mathcal{S}$ , we have that

$$\begin{aligned} \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w &\lesssim \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \int_{E_Q} f \|w\chi_{F_k(Q)}\|_{L \log L^m, Q} \\ &\lesssim \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \int_{E_Q} f \frac{k^m}{4^k} \rho_{m+1, w}(Q) \langle w \rangle_Q \lesssim \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \int_{E_Q} f \frac{2k^m}{4^k} \langle w \rangle_Q \\ &\lesssim \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_k} \int_{E_Q} f \frac{2 \cdot 2^k}{4^k} \langle w \rangle_Q \simeq \sum_{k=1}^{\infty} \frac{1}{2^k} \sum_{Q \in \mathcal{S}_k} \int_{E_Q} f M w \lesssim \int_{\mathbb{R}^n} f M w. \end{aligned}$$

Now we turn our attention to  $L_2$ . We begin discussing a suitable way to break into pieces a cube  $Q \in \mathcal{S}_k$ . We shall split  $\mathcal{S}_k^v$  where  $\mathcal{S}_k^0$  is the family of maximal cubes in  $\mathcal{S}_k$ ,  $\mathcal{S}_k^{j+1}$  is the family of maximal cubes contained in cubes of  $\mathcal{S}_k^j$  and so on. Let

$Q \in \mathcal{S}_k^j$ . Note that by the  $\alpha$ -Carleson condition

$$\sum_{Q' \in \mathcal{S}_k^{j+1}(Q)} |Q'| \leq (\alpha - 1)|Q|.$$

where  $\mathcal{S}_k^{j+1}(Q)$  stands for the family of cubes of  $\mathcal{S}_k^{j+1}$  contained in  $Q$ . Furthermore, iterating the left hand side,

$$\sum_{Q' \in \mathcal{S}_k^{j+t}(Q)} |Q'| \leq (\alpha - 1)^t |Q|.$$

Let us call  $Q^t = \cup_{Q' \in \mathcal{S}_{r,k}^{j+t}(Q)} Q'$ . Then we have that

$$Q = Q^t \cup \tilde{E}_Q \quad \text{where} \quad \tilde{E}_Q = \bigcup_{s=1}^t Q \setminus \cup_{P \in \mathcal{S}_k^{j+s}(Q)} P.$$

Note that for this choice of  $\tilde{E}_Q$ ,

$$\sum_{Q \in \mathcal{S}_k} \chi_{\tilde{E}_Q}(x) \leq t.$$

Let us choose  $t = 7^{km}$ . Observe that, then

$$\frac{|Q|}{|Q^t|} \geq \frac{1}{(\alpha - 1)^{7^{km}}} = \left(\frac{1}{\alpha - 1}\right)^{7^{km}}$$

and since  $1 < \alpha < 2$ ,

$$\log \left( 2 \frac{|Q|}{|Q^t|} \right) \geq 7^{km} \log \left( \frac{1}{\alpha - 1} \right).$$

Having the discussion above at our disposal we now provide our estimate for  $L_2$ . First we split the sum in two terms

$$L_2 = \sum_{k=1}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f w(G \cap Q^t) + \sum_{k=1}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f w(G \cap \tilde{E}_Q).$$

For the first term since we have that for every cube  $Q$ ,  $\rho_{m+1,w}(Q) \leq 2$

$$\frac{1}{|Q|} \int_Q f w(G \cap Q^t) \leq \frac{1}{|Q|} \int_Q f w(Q^t) \leq 2 \|w\|_{L \log L, Q} \| \chi_{Q^t} \|_{\exp(L^m), Q} \int_Q f$$

$$\begin{aligned}
 &= \frac{1}{\log\left(2\frac{|Q|}{|Q'|}\right)} \int_Q f \|w\|_{L \log L, Q} \lesssim \frac{1}{\log\left(2\frac{|Q|}{|Q'|}\right)} \int_Q f \|w\|_{L \log L, Q} \\
 &\lesssim \frac{1}{7^{km}} \int_{E_Q} f M w.
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \sum_{k=1}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f w(G \cap Q') &\lesssim \sum_{k=1}^{\infty} \frac{4^{km}}{7^{km}} \sum_{Q \in \mathcal{S}_k} \int_{E_Q} f M w \\
 &\lesssim \int_{\mathbb{R}^d} f M w \sum_{k=1}^{\infty} \left(\frac{4}{7}\right)^{km} \lesssim \int_{\mathbb{R}^d} f M w.
 \end{aligned}$$

For the remaining term

$$\begin{aligned}
 \sum_{k=1}^{\infty} 4^{mk} \sum_{Q \in \mathcal{S}_k} \frac{1}{|Q|} \int_Q f w(G \cap \tilde{E}_Q) &= \sum_{k=1}^{\infty} 4^{mk} \sum_{v=0}^{7^{km}} \sum_{Q \in \mathcal{S}_k} \sum_{Q' \in \mathcal{S}_k^v(Q)} \frac{1}{|Q|} \int_Q f w(G \cap Q') \\
 &\leq \sum_{k=1}^{\infty} \frac{4^{mk}}{56^{mk}} \sum_{v=0}^{7^{km}} \sum_{Q \in \mathcal{S}_k} \sum_{Q' \in \mathcal{S}_k^v(Q)} w(G \cap Q') \leq \sum_{k=1}^{\infty} \frac{4^{km}}{56^{km}} 7^{km} w(G) \\
 &\leq \sum_{k=1}^{\infty} \frac{1}{2^{km}} w(G) \leq w(G)
 \end{aligned}$$

and hence we are done.

### 2.2.2 Bound for $T_{b, \mathcal{S}_2}^{m,m}$

Again, we shall drop the subscripts of  $\mathcal{S}_2$  and  $G_2$  for the sake of clarity. First we split the sparse family  $\mathcal{S}$  as follows.  $Q \in \mathcal{S}_{r,k}$  if

$$2^{2^r} \leq \rho_{m+1, w}(Q) < 2^{2^{r+1}} \quad \text{and} \quad \frac{1}{56^{m(k+1)}} < \frac{1}{|Q|} \int_Q f \leq \frac{1}{56^{km}}.$$

Then,  $\mathcal{S} = \bigcup_{r=0}^{\infty} \bigcup_{k=1}^{\infty} \mathcal{S}_{r,k}$ . Observe that

$$w(G) \leq \frac{1}{4^{\tau m} 2^{nm} e^m \cdot 100} \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap Q} |b - b_Q|^m w. \tag{2.7}$$

Now we further consider for  $Q \in \mathcal{S}_{r,k}$

$$F_k(Q) = \left\{ x \in Q : |b - b_Q|^m > 2^{nm} e^m 4^{m(k+\tau)} \right\}.$$

Note that due to the John–Nirenberg inequality this yields

$$\frac{|F_k(Q)|}{|Q|} \leq ee^{-4^{k+\tau}}.$$

Then

$$\begin{aligned} & \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap Q} |b - b_Q|^m w \\ & \leq \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w \\ & \quad + \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap (Q \setminus F_k(Q))} |b - b_Q|^m w \\ & \leq \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w \\ & \quad + 2^{nm} e^m \cdot 4^{\tau m} \sum_{r=0}^{\infty} \sum_{k=1}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap Q} w \\ & = (L_1 + 2^{nm} e^m \cdot 4^{\tau m} \cdot L_2). \end{aligned}$$

We observe that if  $Q \in \mathcal{S}_{r,k}$  then

$$\int_Q f \lesssim \int_{E_Q} f \tag{2.8}$$

where  $E_Q = Q \setminus \bigcup_{Q' \subsetneq Q, Q' \in \mathcal{S}_{r,k}} Q'$ . Note that it suffices to argue as we did to derive (2.5) since we only used information relative to the splitting in  $k$  of the sparse family there.

Let us deal now with  $L_1$ . We split the sum in  $k$  as follows

$$\begin{aligned} L_1 &= \sum_{r=0}^{\infty} \sum_{k=1}^{\log_2(2^{2^{r+1}})} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w \\ & \quad + \sum_{r=0}^{\infty} \sum_{k=\log_2(2^{2^{r+1}})}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w \\ & = L_{11} + L_{12}. \end{aligned}$$

Let us focus first on  $L_{11}$ . Observe that

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \sum_{k=1}^{\log_2(2^{2r+1})} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w \\
 & \lesssim \sum_{r=0}^{\infty} \sum_{k=1}^{\log_2(2^{2r+1})} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f \|w \chi_{F_k(Q)}\|_{L \log L^m, Q} \|b - b_Q\|_{\exp L^{\frac{1}{m}}, Q}^m \\
 & \lesssim \sum_{r=0}^{\infty} \sum_{k=1}^{\log_2(2^{2r+1})} \sum_{Q \in \mathcal{S}_{r,k}} \frac{\log_2(2 + \rho_{m+1,w}(Q)) \varepsilon(\rho_{m+1,w}(Q))}{\log_2(2 + \rho_{m+1,w}(Q)) \varepsilon(\rho_{m+1,w}(Q))} \\
 & \quad \times \int_{E_Q} f \|w \chi_{F_k(Q)}\|_{L \log L^m, Q} \\
 & \lesssim \sum_{r=0}^{\infty} \sum_{k=1}^{\log_2(2^{2r+1})} \frac{1}{\log_2(2 + 2^{2r}) \varepsilon(2^{2r})} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f M_{\varepsilon, L \log L^m} w \\
 & \lesssim \sum_{r=0}^{\infty} \sum_{k=1}^{\log_2(2^{2r+1})} \frac{1}{\log_2(2 + 2^{2r}) \varepsilon(2^{2r})} \int_{\mathbb{R}^d} f M_{\varepsilon, L \log L^m} w \\
 & \lesssim \sum_{r=0}^{\infty} \frac{1}{\varepsilon(2^{2r})} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f M_{\varepsilon, L \log L^m} w.
 \end{aligned}$$

Now we turn our attention to  $L_{12}$ . Arguing as we did to settle (2.6), we have that by Lemma 2, for  $Q \in \mathcal{S}_{r,k}$

$$\|w \chi_{F_k(Q)}\|_{L \log L^m, Q} \lesssim \frac{k^m}{4^k} \rho_{m+1,w}(Q) \langle w \rangle_Q.$$

Hence

$$\begin{aligned}
 & \sum_{r=0}^{\infty} \sum_{k=\log_2(2^{2r+1})}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f \int_{G \cap F_k(Q)} |b - b_Q|^m w \\
 & \lesssim \sum_{r=0}^{\infty} f \sum_{k=\log_2(2^{2r+1})}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f \|w \chi_{F_k(Q)}\|_{L \log L^m, Q} \\
 & \lesssim \sum_{r=0}^{\infty} \sum_{k=\log_2(2^{2r+1})}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f \frac{k^m}{4^k} \rho_{m+1,w}(Q) \langle w \rangle_Q \\
 & \lesssim \sum_{r=0}^{\infty} \sum_{k=\log_2(2^{2r+1})}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f \frac{2^{2r+1} k^m \varepsilon(\rho_{m+1,w}(Q))}{4^k \varepsilon(\rho_{m+1,w}(Q))} \langle w \rangle_Q
 \end{aligned}$$

$$\begin{aligned}
 &\lesssim \sum_{r=0}^{\infty} \sum_{k=\log_2(2^{2^{r+1}})}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f \frac{2^{2^{r+1}} 2^k}{4^k} \frac{\log_2(2 + \rho_{m+1,w}(Q)) \varepsilon(\rho_{m+1,w}(Q))}{\log_2(2 + \rho_{m+1,w}(Q)) \varepsilon(\rho_{m+1,w}(Q))} \langle w \rangle_Q \\
 &\lesssim \sum_{r=0}^{\infty} \frac{1}{\log_2(2 + 2^{2^r}) \varepsilon(2^{2^r})} \sum_{k=\log_2(2^{2^{r+1}})}^{\infty} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f M_{\varepsilon,L \log L,m+1} w \frac{2^{2^{r+1}}}{2^k} \\
 &\lesssim \sum_{r=0}^{\infty} \frac{2^{2^{r+1}}}{2^r \varepsilon(2^{2^r})} \sum_{k=\log_2(2^{2^{r+1}})}^{\infty} \frac{1}{2^k} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f M_{\varepsilon,L \log L,m+1} w \\
 &\lesssim \sum_{r=0}^{\infty} \frac{2^{2^{r+1}}}{2^r \varepsilon(2^{2^r})} \sum_{k=\log_2(2^{2^{r+1}})}^{\infty} \frac{1}{2^k} \int_{\mathbb{R}^n} f M_{\varepsilon,L \log L,m+1} w \\
 &\lesssim \sum_{r=0}^{\infty} \frac{2^{2^{r+1}}}{2^r \varepsilon(2^{2^r}) 2^{2^{r+1}}} \int_{\mathbb{R}^n} f M_{\varepsilon,L \log L,m+1} w \simeq \sum_{r=0}^{\infty} \frac{1}{2^r \varepsilon(2^{2^r})} \int_{\mathbb{R}^n} f M_{\varepsilon,L \log L,m+1} w.
 \end{aligned}$$

To provide our estimate for  $L_2$ , we split again in two sums.

$$\begin{aligned}
 L_2 &\leq \sum_{r=0}^{\infty} \sum_{k=1}^{\lfloor \frac{r}{2m} \rfloor} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f w(G \cap Q) \\
 &\quad + \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2m} \rfloor}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f w(G \cap Q) = L_{21} + L_{22}.
 \end{aligned}$$

To bound  $L_{21}$  we observe that

$$\begin{aligned}
 L_{21} &= \sum_{r=0}^{\infty} \sum_{k=1}^{\lfloor \frac{r}{2m} \rfloor} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f w(G \cap Q) \\
 &\lesssim \sum_{r=0}^{\infty} \sum_{k=1}^{\lfloor \frac{r}{2m} \rfloor} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f \frac{w(G \cap Q)}{|Q|} \\
 &\lesssim \sum_{r=0}^{\infty} \sum_{k=1}^{\lfloor \frac{r}{2m} \rfloor} \sum_{Q \in \mathcal{S}_{r,k}} \frac{\log_2(2 + \rho_{m+1,w}(Q)) \varepsilon(\rho_{m+1,w}(Q))}{\log_2(2 + \rho_{m+1,w}(Q)) \varepsilon(\rho_{m+1,w}(Q))} \int_{E_Q} f \frac{w(G \cap Q)}{|Q|} \\
 &\lesssim \sum_{r=0}^{\infty} \frac{1}{\log_2(2 + 2^{2^r}) \varepsilon(2^{2^r})} \sum_{k=1}^{\lfloor \frac{r}{2m} \rfloor} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f M_{\varepsilon,L,m+1} w \\
 &\lesssim \sum_{r=0}^{\infty} \frac{2^r}{\log_2(2 + 2^{2^r}) \varepsilon(2^{2^r})} \int_{\mathbb{R}^d} f M_{\varepsilon,L,m+1} w \\
 &\lesssim \sum_{r=0}^{\infty} \frac{1}{\varepsilon(2^{2^r})} \int_{\mathbb{R}^d} f M_{\varepsilon,L,m+1} w \lesssim \sum_{r=0}^{\infty} \frac{1}{\varepsilon(2^{2^r})} \int_{\mathbb{R}^d} f M_{\varepsilon,L(\log L)^m,m+1} w
 \end{aligned}$$

and hence we are done for this term and it remains to deal with  $L_{22}$ . Note that arguing as we did in the previous subsection, for every cube  $Q \in \mathcal{S}_{r,k}$  we have that

$$Q = Q^t \cup \tilde{E}_Q$$

where

$$\sum_{Q \in \mathcal{S}_{r,k}} \chi_{\tilde{E}_Q}(x) \leq t \quad \text{and} \quad \log \left( 2 \frac{|Q|}{|Q^t|} \right) \geq 7^{km} \log \left( \frac{1}{\alpha - 1} \right).$$

Bearing those properties in mind we provide our estimate for  $L_{22}$ . We consider the following terms

$$\begin{aligned} L_{22} &= \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2^m} \rfloor}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f w(G \cap Q^t) \\ &\quad + \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2^m} \rfloor}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f w(G \cap \tilde{E}_Q) \\ &= L_{221} + L_{222}. \end{aligned}$$

For  $L_{221}$  we observe that

$$\begin{aligned} \frac{1}{|Q|} \int_Q f w(G \cap Q^t) &\leq \frac{1}{|Q|} \int_Q f w(Q^t) \leq 2 \|w\|_{L \log L, Q} \|\chi_{Q^t}\|_{\exp L, Q} \int_Q f \\ &= \frac{1}{\log \left( 2 \frac{|Q|}{|Q^t|} \right)} \int_Q f \|w\|_{L \log L} \lesssim \frac{1}{7^{km}} \int_{E_Q} f M_{L \log L} w. \end{aligned}$$

Hence

$$\begin{aligned} \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2^m} \rfloor}^{\infty} 4^{km} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f w(G \cap Q^t) &\lesssim \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2^m} \rfloor}^{\infty} \frac{4^{km}}{7^{km}} \sum_{Q \in \mathcal{S}_{r,k}} \int_{E_Q} f M_{L \log L} w \\ &\lesssim \int_{\mathbb{R}^d} f M_{L \log L} w \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2^m} \rfloor}^{\infty} \left( \frac{4}{7} \right)^{km} \lesssim \int_{\mathbb{R}^d} f M_{L \log L} w \sum_{r=0}^{\infty} \left( \frac{4}{7} \right)^{\frac{r}{2}} \lesssim \int_{\mathbb{R}^d} f M_{L \log L} w. \end{aligned}$$

Finally, for  $L_{222}$ ,

$$\begin{aligned} &\sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2^m} \rfloor}^{\infty} 4^{mk} \sum_{Q \in \mathcal{S}_{r,k}} \frac{1}{|Q|} \int_Q f w(G \cap \tilde{E}_Q) \\ &= \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2^m} \rfloor}^{\infty} 4^{mk} \sum_{v=0}^{7^{km}} \sum_{Q \in \mathcal{S}_{r,k}} \sum_{Q' \in \mathcal{S}_{r,k}^v(Q)} \frac{1}{|Q|} \int_Q f w(G \cap Q') \end{aligned}$$

$$\begin{aligned}
&\leq \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2m} \rfloor}^{\infty} \frac{4^{mk}}{56^{mk}} \sum_{v=0}^{7^{km}} \sum_{Q \in \mathcal{S}_{r,k}} \sum_{Q' \in \mathcal{S}_{r,k}^v(Q)} w(G \cap Q') \\
&\leq \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2m} \rfloor}^{\infty} \frac{4^{km}}{56^{km}} 7^{km} w(G) \leq \sum_{r=0}^{\infty} \sum_{k=\lfloor \frac{r}{2m} \rfloor}^{\infty} \frac{1}{2^{km}} w(G) \leq 8w(G).
\end{aligned}$$

and hence, combining the estimates above we are done.

This ends the proof of Theorem 3.

**Acknowledgements** This work will be part of the first author's Ph.D. thesis at Universidad Nacional del Sur. The authors would like to thank the anonymous reviewers for remarks that improved the readability of the paper.

**Funding** Funding for open access publishing: Universidad Málaga/CBUA

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