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Three-Sasakian manifolds & the conformal group

AFFINE CONNECTIONS

(M,g) is a Riemannian manifold ($g \in \mathcal{T}_{0,2}(M)$):

Definition: An affine connection $\nabla\colon \mathfrak{X}(M)\times\mathfrak{X}(M)\to\mathfrak{X}(M)$ is a map satisfying:

$$\nabla_{fX}Y = f\nabla_X Y,\qquad \nabla_X fY = f\nabla_X Y + X(f)Y$$

(object which connects nearby tangent spaces)

Definition: ∇ is compatible with the metric if $\nabla g = 0$, that is,

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

(every parallel transport is an isometry)

Definition: The torsion tensor field is defined by

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

(how tangent spaces twist about a curve when they are parallel transported)

The metric connections are determined by the torsion:

- ☞ There is a unique torsion-free metric connection, called the Levi-Civita connection, denoted here by ∇^g.
- ☞ If ∇ is any metric connection, then $\nabla = \nabla^g + \frac{1}{2}T^\nabla$.

Definition: There is a related (also called torsion) tensor $\omega^\nabla \in \mathcal{T}_{0,3}(M)$ (which determines T^∇) given by

$$\omega^\nabla(X,Y,Z) = g(T^\nabla(X,Y),Z).$$

Thus, the torsion T^∇ is said totally skew-symmetric if

$$\omega^\nabla \in \Lambda^3(M)\quad \text{(a 3-form)}.$$

(This happens iff ∇ and ∇^g have the same geodesics)

Definition: If ∇ is a metric connection with totally skew-symmetric torsion, then it is called ∇-Einstein if

$$\mathrm{Sym}(\mathrm{Ric}^\nabla) = \frac{s^\nabla}{\dim M}\, g.$$

(This solves a variational problem related with the scalar curvature)

It generalizes Einstein manifolds (now with torsion!)

HOMOGENEOUS SPACES

Definition. A manifold M is a homogeneous space if there is a Lie group G and a transitive action (only one orbit) $G\times M\to M$.

In this case, for each fixed point $o\in M$, the isotropy subgroup $H_o = \{\sigma\in G \mid \sigma\cdot o = o\}$ is closed and $M\cong G/H_o$.

In particular, $\tau_\sigma\colon M\to M, p\mapsto \sigma\cdot p$ is a diffeomorphism of M.

Definition. An affine connection ∇ in a homogeneous space G/H is said G-invariant if $\nabla_{\tau_\sigma X}\tau_\sigma Y = \tau_\sigma(\nabla_X Y)$ for all $X,Y\in\mathfrak{X}(M)$ (for $(\tau_\sigma X)_p := (\tau_\sigma)_* X_{\sigma^{-1}.p}$)

Definition. If H is connected, the homogeneous space G/H is reductive if, for $\mathfrak{g} = \mathrm{Lie}(G)$ and $\mathfrak{h} = \mathrm{Lie}(G)$, there is a subspace $\mathfrak{m}$ with $\mathfrak{g} = \mathfrak{h}\oplus\mathfrak{m}$ and $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$.

Nomizu's Theorem. If G/H is a reductive homogeneous space with $\mathfrak{g}=\mathfrak{h}\oplus\mathfrak{m}$ a fixed reductive decomposition and H connected, then there is a bijective correspondence:

Invariant affine connections on M $\xleftrightarrow{\text{1-1}}$ $\mathrm{hom}_{\mathfrak{h}}(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m})$

Moreover, if the metric g is G-invariant (τ_σ isometries),

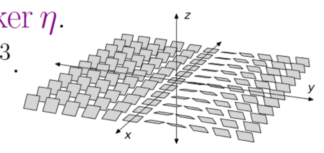
Invariant affine metric connections on M $\xleftrightarrow{\text{1-1}}$ $\mathrm{hom}_{\mathfrak{h}}(\mathfrak{m},\mathfrak{m}\wedge\mathfrak{m})$

Invariant affine metric connections on M with skew-torsion $\xleftrightarrow{\text{1-1}}$ $\mathrm{hom}_{\mathfrak{h}}(\Lambda^3\mathfrak{m},\mathbb{R})$

CONTACT MANIFOLDS

Definition. A contact form on a manifold $M = M^{2n+1}$ is $\eta\in\Lambda^1(M)$ such that $\eta\wedge d\eta^{2n}\neq 0$. It defines a (non-integrable) contact distribution $\mathcal{D} = \ker\eta$.

Example. $\eta = dz - ydx$ on $M=\mathbb{R}^3$.



Definition. $(M^{2n+1}, g,\xi,\eta,\varphi)$ is an almost contact metric structure if $\varphi\in\mathcal{T}_{1,1}(M)$, ξ is a (called *Reeb*) vector field and η is the dual 1-form, satisfying

- $\varphi^2 = -\mathrm{id} + \eta\otimes\xi$,
- $g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y)$

M has a preferred direction and φ behaves like an almost complex structure on the orthogonal distribution

Example. The sphere $\mathbb{S}^{2n+1}\subset\mathbb{C}^{n+1}$, with $\xi(z) = -\mathbf{i}z$, $\eta = g(\xi,-)$ and φ given by $\mathbf{i}X = \varphi(X) + \eta(X)N$ for N the unit outward normal vector field to $\mathbb{S}^{2n+1}$.

Definition. A 3-Sasakian manifold M (necessarily dimension $4n+3$) is that one with (φ_i,ξ_i,η_i), $i=1,2,3$, three almost contact metric structures satisfying the compatibility condition:

$$\varphi_{i+2} = \varphi_i\varphi_{i+1} - \eta_{i+1}\otimes\xi_i = -\varphi_{i+1}\varphi_i + \eta_i\otimes\xi_{i+1},$$
$$\xi_{i+2} = \varphi_i\xi_{i+1} = -\varphi_{i+1}\xi_i.$$

This implies that η_i's are contact forms

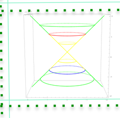
Example. The sphere $\mathbb{S}^{4n+3}\subset\mathbb{H}^{n+1}$: $\xi_1(z) = -\mathbf{i}z$, $\xi_2(z) = -\mathbf{j}z$, $\xi_3(z) = -\mathbf{k}z$.

Theorem. A homogeneous 3-Sasakian manifold is isometric to one of the following list:

- ✡ $\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}\cong\mathbb{S}^{4n+3}$, $\mathbb{S}^{4n+3}/\mathbb{Z}_2\cong\mathbb{RP}^{4n+3}$,
- ✡ $\frac{\mathrm{SU}(m)}{S(\mathrm{U}(m-2)\times\mathrm{U}(1))}$,
- ✡ $\frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4)\times\mathrm{Sp}(1)}$,
- ✡ $G_2/\mathrm{Sp}(1)$, F_4/C_3, $E_6/\mathrm{SU}(10)$, $E_7/\mathrm{Spin}(12)$, E_8/E_7.

MOTIVATION: *The sphere of dimension 7 has been studied as homogeneous space from different viewpoints. Each one increases the size and structure of the set of ∇-Einstein manifolds. Will be this fact more general?*

OUR AIM: *To find out ∇-Einstein manifolds on homogeneous 3-Sasakian manifolds*

$\mathbb{S}^7$	$\frac{\mathrm{Spin}(7)}{G_2}$	$\frac{\mathrm{SU}(4)}{\mathrm{SU}(3)}$	$\frac{\mathrm{Sp}(2)}{\mathrm{Sp}(1)}$
$\mathfrak{g}$	$\mathfrak{so}(\mathbb{O}_0)\cong\mathfrak{b}_3$	$\mathfrak{su}(4) = \{A\in M_4(\mathbb{C}) \mid A+\bar{A}^t = 0, \mathrm{tr}(A)=0\}$	$\mathfrak{sp}(2) = \{A\in\mathrm{Mat}_{2\times 2}(\mathbb{H}) \mid A+\bar{A}^t = 0\}$
$\mathfrak{h}$	$\mathfrak{der}(\mathbb{O})\cong\mathfrak{g}_2$	$\mathfrak{su}(3)$	$\mathfrak{sp}(1)\cong\mathbb{H}_0$
$\mathfrak{m}$	$\mathbb{O}_0$	$\mathbb{C}^3\oplus\mathbb{R}\mathbf{i}$	$\mathbb{H}\oplus\mathbb{H}_0$
Invariant	1	9	63
Metric	1	5	30
Skew-Torsion	1 $\omega^\nabla = s(\eta_1\wedge d\eta_1 - \eta_2\wedge d\eta_2 + \eta_3\wedge d\eta_3)$	3 $\omega^\nabla = s_1\eta_1\wedge d\eta_1 + s_2(\eta_2\wedge d\eta_2 - \eta_3\wedge d\eta_3) + s_3(\eta_2\wedge d\eta_3 + \eta_3\wedge d\eta_2)$	10 $\omega^\nabla = b\,\eta_1\wedge\eta_2\wedge\eta_3 + \sum_{i,j=1}^3 b_{i,j}\,\eta_i\wedge d\eta_j$
∇-Einstein	$s\in\mathbb{R}$ Line	 $s_1^2 = s_2^2+s_3^2$ Cone	$B=(b_{ij})\in\mathrm{CO}(3)$, i.e. $BB^t = \lambda I_3$ and $b = \mathrm{tr}(B)\pm\sqrt{\lambda}$ $\boxed{\mathbb{Z}_2\times\mathrm{CO}(3)}$
Ricci-Flat	$s=\pm 1$	$s_1^2 = 1 = s_2^2+s_3^2$	$BB^t = I_3$, $b = \mathrm{tr}(B)\pm 1$ $\boxed{\mathbb{Z}_2\times\mathrm{O}(3)}$
Flat	$s=1$	$s_1 = 1 = s_2^2+s_3^2$	$BB^t = I_3$, $b=\mathrm{tr}(B)+1$ $\boxed{\mathrm{O}(3)}$
$+\nabla T^\nabla=0$	$s=1/3$	$s_1 = \frac{1}{3}$, $s_2^2+s_3^2 = \frac{1}{9}$	$BB^t = \frac{1}{9}I_3$, $b = \mathrm{tr}(B)+\frac{1}{3}$ $\boxed{\mathrm{O}(3)}$

∇-Einstein ($\neq\nabla^g$) and pseudoeinstein

∇	INVAR	MET	SKEW-T
$\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n)}$	63	30	10
$\frac{\mathrm{SU}(m)}{S(U(m-2)\times U(1))}$	99	55	13
$\frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4)\times\mathrm{Sp}(1)}$	63	30	10
$G_2/\mathrm{Sp}(1)$	63	30	10
F_4/C_3	63	30	10
$E_6/\mathrm{SU}(10)$	63	30	10
$E_7/\mathrm{Spin}(12)$	63	30	10
E_8/E_7	63	30	10

For ∇ a G-invariant connection on a 3-Sasakian homogeneous manifold $M=G/H$,

$$\mathrm{Sym}(\mathrm{Ric}^\nabla)\in\mathrm{hom}_{\mathfrak{h}}(S^2(\mathfrak{m}),\mathbb{R}) = \langle\{g,\eta_i\otimes\eta_{i+1}+\eta_{i+1}\otimes\eta_i,\eta_i\otimes\eta_i \mid i=1,2,3\}\rangle \quad \text{(dim 7)}$$

If ∇ is metric and the torsion is totally-skew: 10 free parameters →

$$\omega^\nabla = a\,\eta_1\wedge\eta_2\wedge\eta_3 + \sum_{i,j=1}^3 b_{i,j}\,\eta_i\wedge d\eta_j$$

Denote $B=(b_{ij})\in M_3(\mathbb{R})$, $\|B\| = \sum_{i,j=1}^3 b_{ij}^2$, $\mathrm{tr}\,B = \sum_{i=1}^3 b_{ii}$

We compute:

$$\mathrm{Sym}(\mathrm{Ric}^\nabla) = (4n+2-\tfrac{1}{2}\|B\|)g - n\textstyle\sum_{i,s=1}^3 b_{is}b_{i+1,s}(\eta_i\otimes\eta_{i+1}+\eta_{i+1}\otimes\eta_i) - \sum_{i,s=1}^3\left(\tfrac{1}{2}(a-\mathrm{tr}(B))^2 - \tfrac{1}{2}\|B\| + nb_{is}^2\right)\eta_i\otimes\eta_i.$$

Hence $\mathrm{Sym}(\mathrm{Ric}^\nabla)|_{\mathfrak{m}_1\times\mathfrak{m}_1} = \dot{g}|_{\mathfrak{m}_1\times\mathfrak{m}_1} \Leftrightarrow B\in\mathrm{CO}(3)$

and in this case

$$\mathrm{Sym}(\mathrm{Ric}^\nabla) = (4n+2-\tfrac{1}{2}\|B\|)g - \left(\tfrac{1}{2}(a-\mathrm{tr}(B))^2 - \tfrac{1}{2}\|B\| + n\tfrac{\|B\|}{3}\right)\sum_{i=1}^3\eta_i\otimes\eta_i.$$

This means that

$$\mathrm{Sym}(\mathrm{Ric}^\nabla) = \dot{g} \Longleftrightarrow \begin{cases} B\in\mathrm{CO}(3),\\ \frac{1}{2}(a-\mathrm{tr}(B))^2 + \frac{1}{6}(2n-3)\|B\| = 0\end{cases} \Leftrightarrow \begin{cases}\dim M = 7\\ B\in\mathrm{CO}(3)\\ a = \mathrm{tr}\,B\pm\sqrt{\frac{\|B\|}{3}}\end{cases} \xleftrightarrow{\text{1-1}} \mathbb{Z}_2\times\mathrm{CO}(3)$$

? We are still checking that Ric is symmetric in ALL the cases (already spheres)

Conclusion:

- ⭐ Pseudo-Einstein $\begin{pmatrix}\mathrm{Ric}^\nabla|_{\mathfrak{m}_1} = \dot{g}|_{\mathfrak{m}_1}\\ \mathrm{Ric}^\nabla|_{\mathfrak{m}_1^\perp} = \dot{g}|_{\mathfrak{m}_1^\perp}\end{pmatrix}$ parametrized by $\mathrm{CO}(3)\times\mathbb{R}$ for any M
- ⭐ ∇-Einstein ($\mathrm{Ric}^\nabla = \dot{g}$) only in dimension 7: $\begin{cases}\text{parametrized by } \mathrm{CO}(3)\times\mathbb{Z}_2 \text{ for } M=\mathbb{S}^7\\ \text{at least } \mathrm{CO}(3)\times\mathbb{Z}_2 \text{ for } M = \text{Aloff-Wallach}\end{cases}$

Exceptional

TITS CONSTRUCTION OF THE EXCEPTIONAL LIE ALGEBRAS

$\mathbb{K},\mathbb{K}'\in\{\mathbb{R},\mathbb{C},\mathbb{H},\mathbb{O}\}$ (composition algebras),
$\mathcal{J} = \mathcal{H}_3(\mathbb{K}')$ (hermitian matrices, a Jordan algebra)

$$\mathcal{T}(\mathbb{K},\mathcal{J}) = \mathrm{Der}\,\mathbb{K}\ \oplus\ (\mathbb{K}_0\otimes\mathcal{J}_0)\ \oplus\ \mathrm{Der}\,\mathcal{J}$$

with product

- ☞ $[\mathrm{Der}\,\mathbb{K},\mathrm{Der}\,\mathcal{J}] = 0$,
- ☞ $[d, a\otimes x] = d(a)\otimes x$,
- ☞ $[D, a\otimes x] = a\otimes D(x)$
- ☞ $[a\otimes x, b\otimes y] = \frac{1}{3}\mathrm{tr}(\underbrace{x\cdot y}_{\frac{xy+yx}{2}})D_{a,b} + \underbrace{[a,b]}_{ab-ba}\otimes\underbrace{\mathrm{pr}_{\mathcal{J}_0}(x\cdot y)}_{x\cdot y - \frac{\mathrm{tr}(x\cdot y)}{3}1} + 2\,\underbrace{\mathrm{tr}(ab)}_{ab+\overline{ab}}[R_x,R_y]$

for any $d\in\mathrm{Der}\,\mathbb{K}$, $D\in\mathrm{Der}\,\mathcal{J}$, $a,b\in\mathbb{K}$, $x,y\in\mathcal{J}$, where

$$\left.\begin{array}{l} R_x\colon\mathcal{J}\to\mathcal{J},\ R_x(y) = y\cdot x\\ r_x\colon\mathbb{K}\to\mathbb{K},\ r_x(y) = yx\\ l_x\colon\mathbb{K}\to\mathbb{K},\ l_x(y) = xy\end{array}\right\}\to\left\{\begin{array}{l} D_{a,b} = [l_a,l_b]+[l_a,r_b]+[r_a,r_b]\in\mathrm{Der}\,\mathbb{K}\\ [R_x,R_y]\in\mathrm{Der}\,\mathcal{J}\end{array}\right.$$

Case G_2

$$\mathbb{O} = \underbrace{\mathbb{H}}_{\mathbb{O}_{\bar 0}}\oplus\underbrace{\mathbb{H}\ell}_{\mathbb{O}_{\bar 1}},\quad \ell^2 = -1,\ \mathbb{Z}_2\text{-grading}$$

$$\Rightarrow \mathrm{Der}\,\mathbb{O} = \{d\in\mathrm{Der}\,\mathbb{O}\mid d(\mathbb{O}_{\bar i})\subset\mathbb{O}_{\bar i}\}\ \oplus\ \{d\in\mathrm{Der}\,\mathbb{O}\mid d(\mathbb{O}_{\bar i})\subset\mathbb{O}_{\overline{i+1}}\}$$

$$\{d\in\mathrm{Der}\,\mathbb{O}\mid d(\mathbb{O}_{\bar i})\subset\mathbb{O}_{\bar i}\} = \underbrace{D_{\mathbb{H},\mathbb{H}}}_{\mathfrak{h}}\oplus\underbrace{\{d\in\mathrm{Der}\,\mathbb{O}\mid d(\mathbb{H})=0\}}_{\mathfrak{m}_1},\qquad \{d\in\mathrm{Der}\,\mathbb{O}\mid d(\mathbb{O}_{\bar i})\subset\mathbb{O}_{\overline{i+1}}\} = \underbrace{D_{\mathbb{H},\mathbb{H}\ell}}_{\mathfrak{m}_2}$$

Thus $D_{\mathbb{H},\mathbb{H}}\cong\mathrm{Der}\,\mathbb{H}$
and $\{d\in\mathrm{Der}\,\mathbb{O}\mid d(\mathbb{H})=0\} = \{q^l\mid q\in\mathbb{H}_0 = \langle\mathbf{i},\mathbf{j},\mathbf{k}\rangle\} =: \mathbb{H}_0^l$
where $q^l\colon\mathbb{O}\to\mathbb{O}$ given by $q^l(\mathbb{H}) = 0$ and $q^l(p\ell) = (qp)\ell$.

Freudenthal magic Square

$\mathbb{K}/J$	$\mathbb{R}$	$\mathcal{H}_3(\mathbb{R})$	$\mathcal{H}_3(\mathbb{C})$	$\mathcal{H}_3(\mathbb{H})$	$\mathcal{H}_3(\mathbb{O})$
$\mathbb{R}$	0	$\mathfrak{so}_3$	$\mathfrak{su}_3$	$\mathfrak{sp}_3$	$\mathfrak{f}_4$
$\mathbb{C}$	0	$\mathfrak{su}_3$	$\mathfrak{su}_3\oplus\mathfrak{su}_3$	$\mathfrak{su}_6$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}_1$	$\mathfrak{sp}_3$	$\mathfrak{su}_6$	$\mathfrak{so}_{12}$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{g}_2$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Algebraic model of the Tangent space Of any 3-Sasakian exceptional manifold

$\mathcal{J} = \mathbb{R},\mathcal{H}_3(\mathbb{R}),\mathcal{H}_3(\mathbb{C}),\mathcal{H}_3(\mathbb{H}),\mathcal{H}_3(\mathbb{O})$

$$\mathfrak{g} = \mathcal{T}(\mathbb{O},\mathcal{J}) = \mathrm{Der}\,\mathbb{O}\ \oplus\ \mathbb{O}_0\otimes\mathcal{J}_0\ \oplus\ \mathrm{Der}\,\mathcal{J}$$
$$\mathfrak{h} = \mathcal{T}(\mathbb{H},\mathcal{J}) = D_{\mathbb{H},\mathbb{H}}\ \oplus\ \mathbb{H}_0\otimes\mathcal{J}_0\ \oplus\ \mathrm{Der}\,\mathcal{J}\subset\mathfrak{g}$$
$$\mathfrak{m} = \underbrace{\mathbb{H}_0^l}_{\mathfrak{m}_1}\oplus\underbrace{D_{\mathbb{H},\mathbb{H}\ell}\oplus\mathbb{H}\ell\otimes\mathcal{J}_0}_{\mathfrak{m}_2}$$

$\xi_i\in\mathfrak{m}$: $\xi_1 = \mathbf{i}^l$, $\xi_2 = \mathbf{j}^l$, $\xi_3 = \mathbf{k}^l$
$\eta_i\colon\mathfrak{m}\to\mathbb{R}$,
$\eta_i(x_1\mathbf{i}^l + x_2\mathbf{j}^l + x_3\mathbf{k}^l + D_{a,q\ell} + p\ell\otimes y) = x_i$
$\varphi_i\colon\mathfrak{m}\to\mathfrak{m}\ \begin{cases}\varphi_i|_{\mathfrak{m}_1} = \frac{1}{2}\mathrm{ad}(\xi_i)\\ \varphi_i|_{\mathfrak{m}_2} = \mathrm{ad}(\xi_i)\end{cases}$

Invariant connections

$\mathfrak{m}_1$ trivial $\mathfrak{h}$-module and $\mathfrak{m}_2^{\mathbb{C}}\cong 2V(\lambda_i)$ for some fundamental weight

Invariant: $\mathfrak{m}^{\mathbb{C}}\otimes\mathfrak{m}^{\mathbb{C}}\cong 4\underbrace{V\otimes V}_{0V,1\mathbb{C}} + 12V + 9\mathbb{C} \Rightarrow \dim\mathrm{hom}_{\mathfrak{h}}(\mathfrak{m}\otimes\mathfrak{m},\mathfrak{m}) = 2\cdot 12 + 3\cdot 13 = 63$

+Compatible with metric: $\mathfrak{m}^{\mathbb{C}}\wedge\mathfrak{m}^{\mathbb{C}}\cong 3\underbrace{\Lambda^2 V}_{1\mathbb{C}} + S^2V + 6V + 3\mathbb{C} \Rightarrow \dim\mathrm{hom}_{\mathfrak{h}}(\mathfrak{m},\mathfrak{m}\wedge\mathfrak{m}) = 2\cdot 6 + 3\cdot 6 = 30$

$\Lambda^3\mathfrak{m}^{\mathbb{C}}\cong 2\Lambda^3V + 2\Lambda^2V\otimes V + 9\underbrace{\Lambda^2V}_{1\mathbb{C}} + 3S^2V + 6V + \mathbb{C} \Rightarrow \dim\mathrm{hom}_{\mathfrak{h}}(\Lambda^3\mathfrak{m},\mathbb{R}) = 10$ +Skew-torsion

ALL: Inv+Metric+Skew-torsion: $\omega^\nabla = b\,\eta_1\wedge\eta_2\wedge\eta_3 + \sum_{i,j=1}^3 b_{i,j}\,\eta_i\wedge d\eta_j$