

A spatial functional count model for heterogeneity analysis in time

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Abstract A spatial curve dynamical model framework is adopted for functional prediction of counts in a spatiotemporal log-Gaussian Cox process model. Our spatial functional estimation approach handles both wavelet-based heterogeneity analysis in time, and spectral analysis in space. Specifically, model fitting is achieved by minimising the information divergence or relative entropy between the multiscale model underlying the data, and the corresponding candidates in the spatial spectral domain. A simulation study is carried out within the family of log-Gaussian Spatial Autoregressive ℓ^2 -valued processes (SAR ℓ^2 processes) to illustrate the asymptotic properties of the proposed spatial functional estimators. We apply our modelling strategy to spatiotemporal prediction of respiratory disease mortality.

Keywords Cox processes in Hilbert spaces · Spatial functional estimation · Spectral wavelet-based analysis

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1 Introduction

Count and aggregated data can be generally found in problems of disease incidence, mortality, population dynamics, or wildfire occurrences that span the scientific fields of Environmental Health, Ecology, Epidemiology, and Atmospheric Environment, to mention just a few. In such cases, stochastic modelling of counts allows for a deeper understanding and accurate predictions for risk assessment and management (see [6]; [7]; [11]; [8]; [12]; [9]; [16]; [18]; [29]; [31], and the references therein).

In most of these cases, the term aggregated point process data (or aggregated data, for short) is used to refer to discretely observed data which in reality most likely arose from an underlying spatially- or spatiotemporally-continuous point process (see [19] and [54]). These later authors argue that it is possible to fit a discrete model and obtain spatially- or spatiotemporally-continuous inference via spatial prediction. We refer the reader to [41] for background material on spatial point processes and the corresponding theoretical details.

In particular, the family of spatial Cox processes (see [14], [27]) has been extensively considered in point pattern analysis. The log-normal intensity model adopted here provides a flexible modelling framework (see [21]; [26], and [39], among others). Its complete characterisation by the intensity and pair correlation functions makes possible its application to different environmental fields (see, e.g., [46] in pine forest; [52] in wildfire occurrences; [58]; [59] in epidemic dynamics modelling, or [34] in disease mapping). Extended models can be found, for instance, in [40]; [53]; and [57]. It is well-known that log-Gaussian Cox processes allow the application of parametric (likelihood, pseudo-likelihood, composite likelihood), semi-parametric, and classical and Bayesian estimation methodologies, avoiding biased estimations, as observed in kernel estimators (see [3]; [20]; [25]; [28]; [32], to mention a few).

The distribution of the hidden environmental fields driving the counts usually displays significant variability and uncertainties across space and time. The characterisation of these fields depends on the spatial scale at which the phenomenon is considered, that could be different from the measurement scale. The effect of heterogeneities at different geographical scales on the spatial distribution of counts has been already examined in [10]; [13]; and [35]. Another issue to be addressed, when inference comes to play, is the size and resolution of the temporal window, quantifying temporal rate fluctuations at the spatial regions (see, e.g., [4]; [33]; [51]). The approach presented in this paper addresses this problem in a Functional Data Analysis (FDA) framework, incorporating spatial correlations between curve rate parameters, at the considered regions. The resulting functional predictions reflect spatial point pattern evolution at any time. Note that FDA techniques are well suited to estimate summary statistics, which are functional in nature. In particular, point process data classification, based on second-order statistics, can be performed applying FDA methodologies (see, e.g., pp. 135–150 in [3], and [31]). However, FDA is a relatively new branch in point pattern analysis. We note the contributions of [59], where a functional statistical approach is adopted in the approximation of the distribution of the random event times observed over a fixed time interval, and the recent one by [15], where a new framework to handle functional marked point processes is derived.

One of the most important challenges in point pattern analysis from a FDA framework is the suitable definition of the process that generates the points. An ℓ^2 -valued homogeneous Poisson process is introduced in [5], where its functional parameter estimation and prediction are addressed from both, a componentwise Bayesian and classical frameworks. The asymptotic efficiency and equivalence of both estimation approaches are also shown. In [55], sufficient conditions are derived for the existence and proper definition of an ℓ^2 -valued temporal log-Gaussian Cox process, with infinite-dimensional log-intensity given by a Hilbert-valued Ornstein-Uhlenbeck process. Its estimation is achieved using a discrete ARH(1) approximation of such process in time.

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The present paper establishes sufficient conditions to introduce a new class of spatial ℓ^2 -valued log-Gaussian Cox processes. These conditions entail the corresponding random intensity process to live in a real separable Hilbert space. Note that, recently, in [22], under mild conditions, a new class of spatial Cox processes has been introduced, driven by a log-intensity process lying in a real separable Hilbert space. However, its intensity process does not necessarily belong to such a space. This paper attempts to cover this gap. The derived conditions allow to perform a multiscale analysis of the functional variance of the random intensity process. The range of temporal fluctuations is then analysed through different scales. In our case, we choose a compactly supported wavelet basis. A more accurate fitting of the local variability displayed by curve data is obtained with this multiscale analysis. Note that, B-splines bases have been widely used in Functional Data Analysis (FDA) preprocessing leading, in some cases, to an over-smoothing of the analysed curve data.

The present paper also proposes an alternative spectral-based multiscale spatial functional estimation methodology, in contrast with the Whittle-based parametric one adopted in [22]. Indeed, this methodology involves the relative entropy minimization criterion, to obtain the optimal multiscale model, underlying the data, in the spatial spectral domain, from the computation of the periodogram operator at different temporal resolution levels. The properties of the derived multiscale estimators are analysed in the simulation study. The validation results obtained in the real-data application illustrate the good properties of the estimation approach presented in the reconstruction of the log-intensity field at different temporal scales.

Summarising, the main ingredients used in the introduction of a new class of multiscale spatial log-Gaussian Cox processes in ℓ^2 spaces can be found in Section 2. The theoretical results for a multiscale analysis of the functional variance are provided in Section 3. In Section 4, a temporal multiresolution estimation approach is adopted in the spatial spectral domain. The class of log-Gaussian SAR ℓ^2 (1) intensity processes is considered in the implementation of this estimation framework. The multiscale analysis, and the asymptotic properties of the proposed componentwise estimators, in the spectral domain, are illustrated in the simulation study carried out in Section 5. The introduced spatial functional estimation methodology is then implemented for prediction of respiratory disease mortality, in a real-data application in Section 6.

2 Spatial log-Gaussian Cox processes in infinite dimensions

Let (Ω, \mathcal{A}, P) be the basic probability space, where all the random variables are subsequently defined on. Denote by \mathcal{H} an arbitrary real separable Hilbert space of functions, with the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$, and the associated norm $\|\cdot\|_{\mathcal{H}}$. Let $\mathbf{X} = \{X_{\mathbf{z}}, \mathbf{z} \in \mathbb{R}^d\}$ be a spatial stationary zero-mean Gaussian random field, with values in \mathcal{H} . Hence, $\sigma^2 = E\|X_{\mathbf{z}}\|_{\mathcal{H}}^2 < \infty$, and $P[X_{\mathbf{z}} \in \mathcal{H}] = 1$, for each $\mathbf{z} \in \mathbb{R}^d$. That is, $X_{\mathbf{z}}$ defines a random element in \mathcal{H} , for every $\mathbf{z} \in \mathbb{R}^d$.

The nuclear cross-covariance operator

$$\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}(f)(g) = E(X_{\mathbf{z}} \otimes X_{\mathbf{y}})(f)(g) = \langle E(X_{\mathbf{z}} \otimes X_{\mathbf{y}})(f), g \rangle_{\mathcal{H}}, \quad f, g \in \mathcal{H}. \quad (1)$$

defines the spatial functional dependence structure of the infinite-dimensional Gaussian random field \mathbf{X} . We have applied Riesz representation theorem to define $\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}(f)(g)$ as the dual element of $\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}(f)$ acting on $g \in \mathcal{H}$, for every $f, g \in \mathcal{H}$. Here, we are restricting our attention to the class of nuclear or trace operators, i.e., in the space $\ell^1(\mathcal{H})$, satisfying

$$\|\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}\|_{\ell^1(\mathcal{H})} = \sum_{j=1}^{\infty} \left\langle \left([\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}]^* \mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}} \right)^{1/2} (\varphi_j), \varphi_j \right\rangle_{\mathcal{H}} < \infty,$$

for any orthonormal basis $\{\varphi_j\}_{j \geq 1}$ in \mathcal{H} .

Remark 1 Note that the approach presented is focused on modelling spatial functional (curve) dependence and variability in an \mathcal{H} -framework. This is the reason why, in our initial assumptions, the infinite-dimensional spatial field \mathbf{X} is considered to be previously detrended, under spatial homogeneity. We refer the reader to Section 4 in [5], for instance, where a componentwise approach is adopted in the estimation of the functional mean of an infinite-dimensional Gaussian population, under classical and Bayesian frameworks.

From (1), $\mathcal{R}_0^{\mathbf{X}}$, with kernel $r_0^{\mathbf{X}}$, is a self-adjoint (symmetric) trace operator, satisfying

$$\mathcal{R}_0^{\mathbf{X}}(\phi_j) = \lambda_j(\mathcal{R}_0^{\mathbf{X}})\phi_j, \quad j \geq 1, \quad (2)$$

where $\{\phi_j, j \geq 1\}$ denotes the orthonormal system of eigenvectors of $\mathcal{R}_0^{\mathbf{X}}$ in \mathcal{H} . For each $\mathbf{z} \in \mathbb{R}^d$, $X_{\mathbf{z}}$ admits the following orthogonal expansion in $\mathcal{L}_{\mathcal{H}}^2(\Omega, \mathcal{A}, P)$ (see [2])

$$\begin{aligned} X_{\mathbf{z}} &= \sum_{j=1}^{\infty} \langle X_{\mathbf{z}}, \phi_j \rangle_{\mathcal{H}} \phi_j = \sum_{j=1}^{\infty} X_{\mathbf{z}}(\phi_j) \phi_j \\ r_0^{\mathbf{X}} &= \sum_{\mathcal{H} \otimes \mathcal{H}} \sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_0^{\mathbf{X}}) \phi_j \otimes \phi_j. \end{aligned} \quad (3)$$

That is,

$$E \left\| X_{\mathbf{z}} - \sum_{j=1}^M \langle X_{\mathbf{z}}, \phi_j \rangle_{\mathcal{H}} \phi_j \right\|_{\mathcal{H}}^2 \rightarrow 0, \quad M \rightarrow \infty,$$

with $E[\langle X_{\mathbf{z}}, \phi_j \rangle_{\mathcal{H}} \langle X_{\mathbf{z}}, \phi_p \rangle_{\mathcal{H}}] = \delta_{j,p} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}})$, $j \geq 1$, for each $\mathbf{z} \in \mathbb{R}^d$. Here, $\delta_{j,p}$ denotes the Kronecker delta function.

Assume that \mathbf{X} is such that, for every $\mathbf{z} \in \mathbb{R}^d$, the functional values of $X_{\mathbf{z}}$ have almost surely (a.s.) their support in the bounded temporal interval $\mathcal{T} \subset \mathbb{R}_+$. Define, for each fixed $\mathbf{z} \in \mathbb{R}^d$,

$$\Lambda_{\mathbf{z}}(t) = \exp(X_{\mathbf{z}}(t)) = \sum_{p=0}^{\infty} \frac{C_p}{p!} H_p(X_{\mathbf{z}}(t)), \quad \forall t \in \mathcal{T}, \quad \text{a.s.} \quad (4)$$

where the last equality follows from Hermite polynomial expansion in the space $L_2(\mathbb{R}, \varphi(u)du)$, with $\varphi(u) = (1/\sqrt{2\pi}) \exp(-u^2/2)$. Here, for every $p \geq 0$, H_p denotes the p th Hermite polynomial, and C_p is the associated coefficient of function $G(u) = \exp(u)$, by projection in the space $L_2(\mathbb{R}, \varphi(u)du)$.

The next condition on $\Lambda_{\mathbf{z}}$, $\mathbf{z} \in \mathbb{R}^d$, allows the introduction from (4) of our functional model for the spatial counting random density in the $\mathcal{L}_{\mathcal{H}}^p(\Omega, \mathcal{A}, P)$ sense, $p > 1$.

Condition C1. Assume that for any bounded set $A \in \mathcal{B}^d$ of the Borel σ -algebra \mathcal{B}^d of \mathbb{R}^d , the following integral is almost surely (a.s.) finite:

$$\Lambda(A) = \int_A \int_{\mathcal{T}} \Lambda_{\mathbf{z}}(t) dt d\mathbf{z} < \infty, \quad (\text{a.s.}) \quad (5)$$

Given the observations $\Psi_{\mathbf{z}, \omega_0} = \int_{\mathcal{T}} \Lambda_{\mathbf{z}, \omega_0}(t) dt$, $\mathbf{z} \in A \subset \mathbb{R}^d$, for certain $\omega_0 \in \Omega$, the number of events $\mathcal{C}(A)$, that occur, during the period \mathcal{T} , at the region A , follows a Poisson probability distribution with mean $\Lambda(A)$. Note that, the least-squares predictor of $\mathcal{C}(A)$ is given by $\Lambda(A)$, introduced in (5), for any bounded Borel set $A \in \mathcal{B}^d$. From (4), equation (3) leads to the following expression of the second-order variation of $\Psi_{\mathbf{z}}$:

$$E[\Psi_{\mathbf{z}}^2] = \sum_{p=0}^{\infty} \int_{\mathcal{T} \times \mathcal{T}} \frac{C_p(t)C_p(s)}{p!} \left[\sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}) \phi_j \otimes \phi_j(t, s) \right]^p dt ds. \quad (6)$$

3 Spatial second-order analysis at different temporal scales

Consider the special case where $\mathcal{H} = L^2(\mathcal{T})$, the space of square integrable functions on \mathcal{T} .

Theorem 1 Under Condition C1, if $\{\phi_j, j \geq 1\}$, in equation (3), are uniformly bounded in \mathcal{T} , $\Psi_{\mathbf{z}}$ defines a spatial a.s. locally absolute integrable second-order random density.

Proof From equation (6), applying Proposition 4.9 in p. 92 in [38], after considering Cauchy-Schwarz inequality, in terms of the inner product introduced in Formula

(4.7) in p.89 of [38], Hermite expansion properties lead to

$$\begin{aligned}
E[\Psi_{\mathbf{z}}^2] &= \sum_{p=0}^{\infty} \int_{\mathcal{T} \times \mathcal{T}} \frac{C_p(t)C_p(s)}{p!} \left[\sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}) \phi_j \otimes \phi_j(t, s) \right]^p dt ds \\
&\leq \sum_{p=0}^{\infty} \frac{1}{p!} \int_{\mathcal{T} \times \mathcal{T}} \sqrt{E[\Lambda_{\mathbf{z}}(t)]^2 E[\Lambda_{\mathbf{z}}(s)]^2} \sqrt{E[H_p(X_{\mathbf{z}}(t))]^2 E[H_p(X_{\mathbf{z}}(s))]^2} \\
&\quad \times \left[\sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}) \phi_j \otimes \phi_j(t, t) \sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}) \phi_j \otimes \phi_j(s, s) \right]^{p/2} dt ds \\
&= \sum_{p=0}^{\infty} \frac{1}{p!} \int_{\mathcal{T} \times \mathcal{T}} \exp(r_0(t, t)/2 + r_0(s, s)/2) [r_0(t, t)r_0(s, s)]^{p/2} \\
&\quad \times \left[\sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}) \phi_j \otimes \phi_j(t, t) \sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}) \phi_j \otimes \phi_j(s, s) \right]^{p/2} dt ds \\
&= \sum_{p=0}^{\infty} \frac{1}{p!} \left\{ \int_{\mathcal{T}} \exp(r_0(t, t)/2) \left[\sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}) \phi_j \otimes \phi_j(t, t) \right]^p dt \right\}^2 \\
&\leq |\mathcal{T}|^2 \exp\left(2\mathcal{M}^2 \|\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}\|_{\ell^1(\mathcal{H})}\right) \left\{ \sum_{p=0}^{\infty} \frac{\mathcal{M}^{2p}}{p!} \left[\sum_{j=1}^{\infty} \lambda_j(\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}) \right]^p \right\}^2 \\
&= |\mathcal{T}|^2 \exp\left(4\|\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}\|_{\ell^1(\mathcal{H})}\mathcal{M}^2\right) < \infty,
\end{aligned}$$

where $\mathcal{M} > 0$, is such that $\sup_{t \in \mathcal{T}} |\phi_j(t)| \leq \mathcal{M}$, for any $j \geq 1$.

Let $\{\psi_{i:k}, k \in \Gamma_j, j \in \mathbb{Z}\}$ be an orthonormal basis of wavelets, providing a multiresolution analysis of $L^2(\mathcal{T})$ (see, e.g., [50]). For each $\mathbf{z} \in \mathbb{R}^d$, the zero-mean Gaussian random coefficient sequence $\{X_{\mathbf{z}}(\psi_{i:k}), k \in \Gamma_j, j \in \mathbb{Z}\}$, with $X_{\mathbf{z}}(\psi_{j:k}) = \langle X_{\mathbf{z}}, \psi_{j:k} \rangle_{L^2(\mathcal{T})}$, $k \in \Gamma_j, j \in \mathbb{Z}$, has covariance

$$E[X_{\mathbf{z}}(\psi_{i_1:k_1})X_{\mathbf{z}}(\psi_{i_2:k_2})] = \mathcal{R}_{\mathbf{0}}^{\mathbf{X}}(\psi_{i_1:k_1})(\psi_{i_2:k_2}), \quad k \in \Gamma_{j_i}, j_i \in \mathbb{Z}, i = 1, 2,$$

providing a multiscale analysis of the curve dependence structure at spatial location \mathbf{z} , through the autocovariance operator $\mathcal{R}_{\mathbf{0}}^{\mathbf{X}}$. In a similar way, for any $\mathbf{z}, \mathbf{v} \in \mathbb{R}^d$, a multiscale analysis is induced by

$$E[X_{\mathbf{z}}(\psi_{i_1:k_1})X_{\mathbf{v}}(\psi_{i_2:k_2})] = \mathcal{R}_{\mathbf{z}-\mathbf{v}}^{\mathbf{X}}(\psi_{i_1:k_1})(\psi_{i_2:k_2}), \quad k \in \Gamma_{j_i}, j_i \in \mathbb{Z}, i = 1, 2,$$

on the curve cross-dependence structure between the spatial locations \mathbf{z} and \mathbf{v} , through the cross-covariance operator $\mathcal{R}_{\mathbf{z}-\mathbf{v}}^{\mathbf{X}}$. The covariance structure of the log-Gaussian sequence $\{\exp(X_{\mathbf{z}}(\psi_{j:k})), k \in \Gamma_j, j \in \mathbb{Z}\}$ also displays a multiscale analysis in time, in the space $L^2(\mathcal{T})$, of the curve spatial dependence structure of the spatial infinite-dimensional intensity process $\{\Lambda_{\mathbf{z}}(t), t \in \mathcal{T}, \mathbf{z} \in \mathbb{R}^d\}$. Note that, from (4), applying Parseval identity in $[L^2(\mathcal{T})]^{\otimes p}$, $p \geq 1$, and Cauchy–Schwarz

inequality in $L^2(\mathcal{T})$,

$$\begin{aligned}
& \sum_{j=1}^{\infty} \sum_{k \in \Gamma_j} E [\exp(X_{\mathbf{z}}(\psi_{j:k})) \exp(X_{\mathbf{y}}(\psi_{j:k}))] \\
&= \sum_{j=1}^{\infty} \sum_{k \in \Gamma_j} \exp \left(\mathcal{R}_0^{\mathbf{X}}(\psi_{j:k})(\psi_{j:k}) + \frac{R_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}(\psi_{j:k})(\psi_{j:k}) + R_{\mathbf{y}-\mathbf{z}}^{\mathbf{X}}(\psi_{j:k})(\psi_{j:k})}{2} \right) \\
&= \sum_{j=1}^{\infty} \sum_{k \in \Gamma_j} \sum_{p_1, p_2, p_3} \frac{(1/2)^{p_2+p_3}}{p_1! p_2! p_3!} \sum_{h_1, \dots, h_{p_1}} \sum_{l_1, \dots, l_{p_2}} \sum_{q_1, \dots, q_{p_3}} \left| \lambda_{h_1}(\mathcal{R}_0^{\mathbf{X}}) \cdots \lambda_{h_{p_1}}(\mathcal{R}_0^{\mathbf{X}}) \right| \\
&\quad \times \left| \lambda_{l_1}(\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}) \cdots \lambda_{l_{p_2}}(\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}) \right| \left| \lambda_{q_1}(\mathcal{R}_{\mathbf{y}-\mathbf{z}}^{\mathbf{X}}) \cdots \lambda_{q_{p_3}}(\mathcal{R}_{\mathbf{y}-\mathbf{z}}^{\mathbf{X}}) \right| \\
&\quad \times \left| \phi_{h_1}(\psi_{j:k}) \cdots \phi_{h_{p_1}}(\psi_{j:k}) \right|^2 \\
&\quad \times \left| \psi_{l_1}^{\mathbf{z}-\mathbf{y}}(\psi_{j:k}) \cdots \psi_{l_{p_2}}^{\mathbf{z}-\mathbf{y}}(\psi_{j:k}) \varphi_{l_1}^{\mathbf{z}-\mathbf{y}}(\psi_{j:k}) \cdots \varphi_{l_{p_2}}^{\mathbf{z}-\mathbf{y}}(\psi_{j:k}) \right| \\
&\quad \times \left| \psi_{q_1}^{\mathbf{y}-\mathbf{z}}(\psi_{j:k}) \cdots \psi_{q_{p_3}}^{\mathbf{y}-\mathbf{z}}(\psi_{j:k}) \varphi_{q_1}^{\mathbf{y}-\mathbf{z}}(\psi_{j:k}) \cdots \varphi_{q_{p_3}}^{\mathbf{y}-\mathbf{z}}(\psi_{j:k}) \right| \\
&\leq \sum_{p_1, p_2, p_3} \frac{(1/2)^{p_2+p_3}}{p_1! p_2! p_3!} \left[\sum_{h=1}^{\infty} \left| \lambda_h(\mathcal{R}_0^{\mathbf{X}}) \right| \right]^{p_1} \left[\sum_{l=1}^{\infty} \left| \lambda_l(\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}) \right| \right]^{p_2} \left[\sum_{q=1}^{\infty} \left| \lambda_q(\mathcal{R}_{\mathbf{y}-\mathbf{z}}^{\mathbf{X}}) \right| \right]^{p_3} \\
&= \exp \left(\|\mathcal{R}_0^{\mathbf{X}}\|_{\ell^1(\mathcal{H})} + \frac{1}{2} \left[\|\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}\|_{\ell^1(\mathcal{H})} + \|\mathcal{R}_{\mathbf{y}-\mathbf{z}}^{\mathbf{X}}\|_{\ell^1(\mathcal{H})} \right] \right) < \infty, \quad \forall \mathbf{z}, \mathbf{y} \in \mathbb{R}^d, \quad (7)
\end{aligned}$$

which implies that the series

$$\sum_{j=1}^{\infty} \sum_{k \in \Gamma_j} E [\exp(X_{\mathbf{z}}(\psi_{j:k}))]^2 = \sum_{j=1}^{\infty} \sum_{k \in \Gamma_j} \exp \left(\mathcal{R}_0^{\mathbf{X}}(\psi_{j:k})(\psi_{j:k}) \right)$$

is convergent, for every $\mathbf{z} \in \mathbb{R}^d$. In (7), we have considered (3), i.e.,

$$\mathcal{R}_0^{\mathbf{X}} = \sum_{h=1}^{\infty} \lambda_h(\mathcal{R}_0^{\mathbf{X}}) \phi_h \otimes \phi_h. \quad (8)$$

Also, we have applied that, for any $\mathbf{z}, \mathbf{y} \in \mathbb{R}^d$, $\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}$ and $\mathcal{R}_{\mathbf{y}-\mathbf{z}}^{\mathbf{X}}$ are nuclear operators admitting a singular value decomposition, given by

$$\begin{aligned}
\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}} &= \sum_{l=1}^{\infty} \lambda_l(\mathcal{R}_{\mathbf{z}-\mathbf{y}}^{\mathbf{X}}) \psi_l^{\mathbf{z}-\mathbf{y}} \otimes \varphi_l^{\mathbf{z}-\mathbf{y}} \\
\mathcal{R}_{\mathbf{y}-\mathbf{z}}^{\mathbf{X}} &= \sum_{q=1}^{\infty} \lambda_q(\mathcal{R}_{\mathbf{y}-\mathbf{z}}^{\mathbf{X}}) \psi_q^{\mathbf{y}-\mathbf{z}} \otimes \varphi_q^{\mathbf{y}-\mathbf{z}}. \quad (9)
\end{aligned}$$

4 Multiresolution spatial functional estimation in the spectral domain

This section introduces the spatial functional estimation approach adopted in the spatial spectral domain following a multiscale componentwise parametric framework. In the next section, the Spatial Autoregressive Hilbertian model of order one (SARH(1) model) is first introduced, in a spatial curve and spectral model frameworks.

4.1 A spatial curve state space equation

Let $\mathbf{X} = \{X_{\mathbf{z}}, \mathbf{z} \in \mathbb{R}^d\}$ be the Gaussian spatial curve process introduced in Section 2. Without loss of generality, we restrict our attention here to the case $d = 2$, and $\mathcal{H} = L^2(\mathcal{T})$, $\mathcal{T} = [0, 1]$. Assume \mathbf{X} obeys a Spatial Autoregressive Hilbertian State Equation (SARH(1) equation), as given in [48]. Thus,

$$X_{p,q} = Y_{p,q} - R = L_1(X_{p-1,q}) + L_2(X_{p,q-1}) + L_3(X_{p-1,q-1}) + \epsilon_{p,q}, \quad (p, q) \in \mathbb{Z}^2, \quad (10)$$

where $R \in \mathcal{H}$ is the functional mean, that is estimated applying the methodology proposed in [5], from a compactly supported orthonormal wavelet basis $\{\psi_{j:k}, k \in I_j, j \in \mathbb{Z}\}$ in $L^2([0, 1])$. The autocorrelation operators L_i , $i = 1, 2, 3$, are assumed to be bounded on $L^2([0, 1])$. Random fluctuations, introduced by the external force, are represented in terms of the $L^2([0, 1])$ -valued zero-mean Gaussian innovation process $\epsilon = \{\epsilon_{p,q}, (p, q) \in \mathbb{Z}^2\}$. Under spatial homogeneity, this process displays constant functional variance $E\|\epsilon_{p,q}\|_{L^2([0,1])}^2 = \sigma^2$, through the spatial locations $(p, q) \in \mathbb{Z}^2$. The spatial functional dependence structure of \mathbf{X} is represented in terms of a nuclear covariance operator, given by $\mathcal{R}_{p,q}^\epsilon = E(\epsilon_{p+k,q+l} \otimes \epsilon_{k,l}) = E(\epsilon_{p,q} \otimes \epsilon_{0,0})$, for every $(p, q), (k, l) \in \mathbb{Z}^2$. In the following, we will work under the assumption of $\{\epsilon_{p,q}, (p, q) \in \mathbb{Z}^2\}$ being a strong Gaussian white noise in $L^2([0, 1])$. Hence, $\mathcal{R}_{p,q}^\epsilon = 0$, for $p \neq q$. In our framework, equation (10) is interpreted as the discrete approximation of a spatial functional log-intensity process over continuous space, by considering constant values within the quadrants of the regular grid defining the spatial observation network (see, e.g., [46], in the real-valued case). See also [43] on spline function approximation, to represent the first-order intensity of a marked inhomogeneous Poisson point process.

In the implementation of our wavelet based estimation, in the spectral domain, of the spatial functional dependence structure of $\{A_{\mathbf{z}}(\cdot), \mathbf{z} \in \mathbb{R}^d\}$, we work under the conditions assumed in Propositions 3 and 4 in [48], for the existence of a unique stationary solution to equation (10); additionally, we also consider the following assumption:

Condition C2. $\mathcal{R}_{p,q}^{\mathbf{X}}$ is such that $\sum_{(p,q) \in \mathbb{Z}^2} \|\mathcal{R}_{p,q}^{\mathbf{X}}\|_{L^1(\mathcal{H})} < \infty$.

Under **Condition C2**, the spectral density operator is given by

$$\mathcal{F}_{\omega_1, \omega_2} := \frac{1}{(2\pi)^2} \sum_{(p,q) \in \mathbb{Z}^2} \mathcal{R}_{p,q}^{\mathbf{X}} \exp(-i(p\omega_1 + q\omega_2)), \quad (\omega_1, \omega_2) \in [0, 2\pi) \times [0, 2\pi), \quad (11)$$

which is a trace non-negative self-adjoint operator.

For a given functional sample of size $N = S_1 \times S_2$, $\{X_{p,q}, p = 1, \dots, S_1, q = 1, \dots, S_2\}$, its functional Discrete Fourier Transform (fDFT) is defined as

$$\tilde{X}_{\omega_1, \omega_2}^N(\cdot) := \frac{1}{2\pi\sqrt{N}} \sum_{p=1}^{S_1} \sum_{q=1}^{S_2} X_{p,q}(\cdot) \exp(-i(p\omega_1 + q\omega_2)). \quad (12)$$

This transform is linear, periodic and Hermitian. Under suitable cumulant kernel conditions (see Theorem 2.2 in [44]), the fDFT (12) at frequencies $\omega_1 := \omega_{1,N} = 0$, $\omega_{2,N} := \omega_2 = \pi$, $\omega_{j,N} \in \left\{ \frac{2\pi}{N}, \dots, \frac{2\pi[(N-1)/2]}{N} \right\}$; $\omega_{j,N} \rightarrow \omega_j$, $N \rightarrow \infty$, $j =$

$3, \dots, J$, converges, as $N \rightarrow \infty$, to independent Gaussian elements in $L^2([0, 1], \mathbb{R})$, for $j = 1, 2$, and in $L^2([0, 1], \mathbb{C})$, for $j = 3, \dots, J$, with respective covariance operators \mathcal{F}_{ω_j} , $j = 1, \dots, J$ (see equation (11)).

From a functional sample of size N , the periodogram operator at frequency $(\omega_1, \omega_2) \in [0, 2\pi) \times [0, 2\pi)$ is given by

$$\begin{aligned} \mathcal{I}_{\omega_1, \omega_2}^N(\cdot, \cdot) \\ := \sum_{p=1}^{S_1} \sum_{q=1}^{S_2} \sum_{p'=1}^{S_1} \sum_{q'=1}^{S_2} \frac{X_{p,q} \otimes X_{p',q'}(\cdot, \cdot) \exp(-i[(p-p')\omega_1 + (q-q')\omega_2])}{(2\pi)^2 N}, \end{aligned} \quad (13)$$

or, equivalently by

$$\mathcal{I}_{\omega_1, \omega_2}^N := \tilde{X}_{\omega_1, \omega_2}^N \otimes \overline{\tilde{X}_{\omega_1, \omega_2}^N}. \quad (14)$$

For a given orthonormal basis of compactly supported wavelets $\{\psi_{j:k}, k \in \Gamma_j, j \in \mathbb{Z}\}$ in $L^2(\mathcal{T})$, from equations (12)–(14),

$$\tilde{X}_{\omega_1, \omega_2}^N(\psi_{j:k}) = \frac{1}{2\pi\sqrt{N}} \sum_{p=1}^{S_1} \sum_{q=1}^{S_2} X_{p,q}(\psi_{j:k}) \exp(-i(p\omega_1 + q\omega_2)), \quad k \in \Gamma_j, j \in \mathbb{Z} \quad (15)$$

$$\begin{aligned} \mathcal{I}_{\omega_1, \omega_2}^N(\psi_{j_1:k_1})(\psi_{j_2:k_2}) &= \tilde{X}_{\omega_1, \omega_2}^N(\psi_{j_1:k_1}) \overline{\tilde{X}_{\omega_1, \omega_2}^N(\psi_{j_2:k_2})} \\ &= \sum_{p=1}^{S_1} \sum_{q=1}^{S_2} \sum_{p'=1}^{S_1} \sum_{q'=1}^{S_2} \frac{X_{p,q}(\psi_{j_1:k_1}) X_{p',q'}(\psi_{j_2:k_2}) \exp(-i[(p-p')\omega_1 + (q-q')\omega_2])}{(2\pi)^2 N}, \end{aligned} \quad (16)$$

for any $k_i \in \Gamma_{j_i}, j_i \in \mathbb{Z}, i = 1, 2$. Thus, a multiscale analysis in time is considered in the spatial spectral domain.

we define

4.2 The estimation approach

From (10)–(16), define the diagonal wavelet coefficient parameter vector sequence

$$\begin{aligned} \boldsymbol{\theta}_{j:k} &= (\theta_{j:k,1}, \theta_{j:k,2}, \theta_{j:k,3}) \\ &= (L_1(\psi_{j:k})(\psi_{j:k}), L_2(\psi_{j:k})(\psi_{j:k}), L_3(\psi_{j:k})(\psi_{j:k})) \in \Theta_{j:k} \subset \Theta, \quad k \in \Gamma_j, j \in \mathbb{Z} \end{aligned} \quad (17)$$

and the multiresolution approximation

$$\{X_{p,q}(\psi_{j:k}), p = 1, \dots, S_1, q = 1, \dots, S_2, k \in \Gamma_j, j \in \mathbb{Z}\}$$

in time of the spatial sample information. Note that, here, for every $k \in \Gamma_j, j \in \mathbb{Z}$, $\Theta_{j:k}$ is finite, and $\Theta = \cup_{j \in \mathbb{Z}} \cup_{k \in \Gamma_j} \Theta_{j:k}$ is a compact set. For $k \in \Gamma_j, j \geq 1$, we also assume that the true parameter value $\boldsymbol{\theta}_{0,j:k}$ always lies in the interior of $\Theta_{j:k}$, and our spatial spectral model is identifiable in the wavelet domain.

For any node $k \in \Gamma_j$, at resolution level $j \in \mathbb{Z}$, one can consider **the parameter estimator** $\hat{\theta}_{N,j:k} = \left(\hat{\theta}_{N,j:k,1}, \hat{\theta}_{N,j:k,2}, \hat{\theta}_{N,j:k,3} \right)$ of $\theta_{j:k} = (\theta_{j:k,1}, \theta_{j:k,2}, \theta_{j:k,3})$, **computed from the loss function**

$$\begin{aligned} K_{j:k}(\theta_{0,j:k}, \theta_{j:k}) &:= \int_{[0,2\pi) \times [0,2\pi)} f_{j:k}(\varpi, \theta_{0,j:k}) \eta_{j:k}(\varpi) \log \frac{\Psi_{j:k}(\varpi, \theta_{0,j:k})}{\Psi_{j:k}(\varpi, \theta_{j:k})} d\varpi \\ &= U_{j:k}(\theta_{j:k}) - U_{j:k}(\theta_{0,j:k}), \end{aligned} \quad (18)$$

where $\theta_{0,j:k}$ denotes the true parameter value, associated with node k at scale $j \in \Gamma_j$. The **multiscale normalised spatial spectral density**

$$\{\Psi_{j:k}(\varpi, \theta_{j:k}), k \in \Gamma_j, j \in \mathbb{Z}, \varpi \in [0, 2\pi) \times [0, 2\pi)\}$$

is obtained from the identities

$$\begin{aligned} f_{j:k}(\varpi, \theta_{j:k}) &= \sigma^2(\theta_{j:k}) \Psi_{j:k}(\varpi, \theta_{j:k}) \\ &= \left[\int_{[0,2\pi) \times [0,2\pi)} f_{j:k}(\varpi, \theta_{j:k}) \eta_{j:k}(\varpi) d\varpi \right] \Psi_{j:k}(\varpi, \theta_{j:k}) \\ f_{j:k}(\varpi, \theta_{j:k}) &= \frac{\sigma_{\epsilon(\psi_{j:k})}^2}{2\pi^2} \left| 1 - \theta_{j:k,1} e^{i\varpi_1} - \theta_{j:k,2} e^{i\varpi_2} - \theta_{j:k,3} e^{i(\varpi_1 + \varpi_2)} \right|^{-2}, \end{aligned} \quad (19)$$

for every $\varpi = (\varpi_1, \varpi_2) \in [0, 2\pi) \times [0, 2\pi)$, with, for each $k \in \Gamma_j$, and $j \in \mathbb{Z}$, $\eta_{j:k}(\varpi)$ being a nonnegative symmetric spatial function, such that $\eta_{j:k}(\varpi) f_{j:k}(\varpi, \theta_{j:k}) \in L_1([0, 2\pi) \times [0, 2\pi))$, the space of absolute integrable functions on $[0, 2\pi) \times [0, 2\pi)$, for each $\theta_{j:k} \in \Theta_{j:k} \subset \Theta$.

The loss functions in (18) measure the discrepancy, at different temporal resolution levels, between the true spatial spectral parametric model $\Psi_{j:k}(\varpi, \theta_{0,j:k})$, underlying the data, and the parametric candidates $\Psi_{j:k}(\varpi, \theta_{j:k})$, $\theta_{j:k} \in \Theta_{j:k} \subset \Theta$, at node k , within the temporal variation scale $j \in \mathbb{Z}$. In the last identity in equation (18), for each scale $j \in \mathbb{Z}$,

$$U_{j:k}(\theta_{j:k}) := - \int_{[0,2\pi) \times [0,2\pi)} f_{j:k}(\varpi, \theta_{0,j:k}) \eta_{j:k}(\varpi) \log \Psi_{j:k}(\varpi, \theta_{j:k}) d\varpi, \quad k \in \Gamma_j. \quad (20)$$

In practice, we can then consider the **empirical multiscale functional**

$$\hat{U}_{N,j:k}(\theta_{j:k}) := - \int_{[0,2\pi) \times [0,2\pi)} I_{N,j:k}(\varpi) \eta_{j:k}(\varpi) \log \Psi_{j:k}(\varpi, \theta_{j:k}) d\varpi, \quad (21)$$

where $I_{N,j:k}(\varpi) = \mathcal{I}_{\varpi_1, \varpi_2}^N(\psi_{j:k})(\psi_{j:k})$ denotes, as before, **the multiscale periodogram introduced in (16)**, for $k \in \Gamma_j$, and $j \in \mathbb{Z}$.

For each $k \in \Gamma_j$, and $j \in \mathbb{Z}$, $\eta_{j:k}$ must satisfy suitable conditions (see Theorem 2.1 in [1]), such that the loss function (18) has a minimum at the true parameter value, for each node at any scale, and the following asymptotic behaviour holds (see, e.g., [1]):

$$\hat{U}_{N,j:k}(\theta_{j:k}) - \hat{U}_{N,j:k}(\theta_{0,j:k}) \rightarrow_{P_{0,j:k}} K_{j:k}(\theta_{0,j:k}, \theta_{j:k}), \quad N \rightarrow \infty, \quad (22)$$

for each $\theta_{j:k} \in \Theta_{j:k} \subset \Theta$, where $P_{0,j:k}$ denotes the measure associated with density function $f_{j:k}(\varpi, \theta_{0,j:k})$, for each $k \in \Gamma_j$, and $j \in \mathbb{Z}$. **To minimize the divergence in**

(18), in practice, we can compute the minimum of $\hat{U}_{N,j:k}(\boldsymbol{\theta}_{j:k})$ over $\boldsymbol{\theta}_{j:k} \in \Theta_{j:k}$, through the different nodes k at each scale $j \in \mathbb{Z}$. That is, we will consider the multiscale parameter estimators

$$\hat{\boldsymbol{\theta}}_{N,j:k} = \arg \min_{\boldsymbol{\theta}_{j:k} \in \Theta_{j:k}} \hat{U}_{N,j:k}(\boldsymbol{\theta}_{j:k}), \quad k \in \Gamma_j, j \in \mathbb{Z}. \quad (23)$$

The same estimation procedure, based on the multiscale periodogram in (16), is applied for the remaining coefficients in the two-dimensional wavelet transforms of operators L_i , $i = 1, 2, 3$, including the scaling function coefficients, with respect to the basis $\{\varphi_{j_0:\tilde{k}}, \tilde{k} \in \Upsilon_{j_0}\}$ of the space $V_0 \subset L^2([0, 1])$. That is, similar estimators are computed for the parameters

$$\begin{aligned} \boldsymbol{\theta}_{j_0:\tilde{k},\tilde{l}} &= \left(\theta_{j_0:\tilde{k},\tilde{l},1}, \theta_{j_0:\tilde{k},\tilde{l},2}, \theta_{j_0:\tilde{k},\tilde{l},3} \right) \\ &= \left(L_1(\varphi_{j_0:\tilde{l}})(\varphi_{j_0:\tilde{k}}), L_2(\varphi_{j_0:\tilde{l}})(\varphi_{j_0:\tilde{k}}), L_3(\varphi_{j_0:\tilde{l}})(\varphi_{j_0:\tilde{k}}) \right) \in \Theta_{j_0:\tilde{k}} \times \Theta_{j_0:\tilde{l}} \\ &\quad \Theta_{j_0:\tilde{k}} \times \Theta_{j_0:\tilde{l}} \subset \Theta \times \Theta, \quad \tilde{k}, \tilde{l} \in \Upsilon_{j_0} \\ \boldsymbol{\theta}_{j_0:\tilde{k};j:k} &= \left(\theta_{j_0:\tilde{k};j:k,1}, \theta_{j_0:\tilde{k};j:k,2}, \theta_{j_0:\tilde{k};j:k,3} \right) \\ &= \left(L_1(\psi_{j:k})(\varphi_{j_0:\tilde{k}}), L_2(\psi_{j:k})(\varphi_{j_0:\tilde{k}}), L_3(\psi_{j:k})(\varphi_{j_0:\tilde{k}}) \right) \in \Theta_{j_0:\tilde{k}} \times \Theta_{j:k} \\ &\quad \Theta_{j_0:\tilde{k}} \times \Theta_{j:k} \subset \Theta \times \Theta, \quad \tilde{k} \in \Upsilon_{j_0}; k \in \Gamma_j, j \geq j_0 \\ \boldsymbol{\theta}_{j:k;j_0:\tilde{k}} &= \left(\theta_{j:k;j_0:\tilde{k},1}, \theta_{j:k;j_0:\tilde{k},2}, \theta_{j:k;j_0:\tilde{k},3} \right) \\ &= \left(L_1(\varphi_{j_0:\tilde{k}})(\psi_{j:k}), L_2(\varphi_{j_0:\tilde{k}})(\psi_{j:k}), L_3(\varphi_{j_0:\tilde{k}})(\psi_{j:k}) \right) \in \Theta_{j:k} \times \Theta_{j_0:\tilde{k}} \\ &\quad \Theta_{j:k} \times \Theta_{j_0:\tilde{k}} \subset \Theta \times \Theta, \quad k \in \Gamma_j, j \geq j_0; \tilde{k} \in \Upsilon_{j_0} \\ \boldsymbol{\theta}_{j:k;l:h} &= \left(\theta_{j:k;l:h,1}, \theta_{j:k;l:h,2}, \theta_{j:k;l:h,3} \right) \\ &= \left(L_1(\psi_{j:k})(\psi_{l:h}), L_2(\psi_{j:k})(\psi_{l:h}), L_3(\psi_{j:k})(\psi_{l:h}) \right) \in \Theta_{j:k} \times \Theta_{l:h} \\ &\quad \Theta_{j:k} \times \Theta_{l:h} \subset \Theta \times \Theta, \quad k \in \Gamma_j, j \geq j_0; h \in \Gamma_l, l \geq j_0; (j,k) \neq (h,l). \end{aligned} \quad (24)$$

The resulting multiscale SAR $\ell^2(1)$ plug-in predictor is computed, for any spatial location (p, q) , as

$$\begin{aligned}
\widehat{X}_{N,p,q}(\cdot) &= \sum_{\tilde{k} \in \Upsilon_{j_0}} \sum_{\tilde{l} \in \Upsilon_{j_0}} \widehat{\theta}_{N,j_0:\tilde{k},\tilde{l},1} X_{p-1,q}(\varphi_{j_0:\tilde{l}}) \varphi_{j_0:\tilde{k}}(\cdot) \\
&+ \sum_{\tilde{k} \in \Upsilon_{j_0}} \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \widehat{\theta}_{N,j_0:\tilde{k};j:k,1} X_{p-1,q}(\psi_{j:k}) \varphi_{j_0:\tilde{k}}(\cdot) \\
&+ \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \sum_{\tilde{k} \in \Upsilon_{j_0}} \widehat{\theta}_{N,j:k;j_0:\tilde{k},1} X_{p-1,q}(\varphi_{j_0:\tilde{k}}) \psi_{j:k}(\cdot) \\
&+ \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \sum_{l \geq j_0} \sum_{h \in \Gamma_l} \widehat{\theta}_{N,j:k;l:h,1} X_{p-1,q}(\psi_{l:h}) \psi_{j:k}(\cdot) \\
&+ \sum_{\tilde{k} \in \Upsilon_{j_0}} \sum_{\tilde{l} \in \Upsilon_{j_0}} \widehat{\theta}_{N,j_0:\tilde{k},\tilde{l},2} X_{p,q-1}(\varphi_{j_0:\tilde{l}}) \varphi_{j_0:\tilde{k}}(\cdot) \\
&+ \sum_{\tilde{k} \in \Upsilon_{j_0}} \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \widehat{\theta}_{N,j_0:\tilde{k};j:k,2} X_{p,q-1}(\psi_{j:k}) \varphi_{j_0:\tilde{k}}(\cdot) \\
&+ \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \sum_{\tilde{k} \in \Upsilon_{j_0}} \widehat{\theta}_{N,j:k;j_0:\tilde{k},2} X_{p,q-1}(\varphi_{j_0:\tilde{k}}) \psi_{j:k}(\cdot) \\
&+ \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \sum_{l \geq j_0} \sum_{h \in \Gamma_l} \widehat{\theta}_{N,j:k;l:h,2} X_{p,q-1}(\psi_{l:h}) \psi_{j:k}(\cdot) \\
&+ \sum_{\tilde{k} \in \Upsilon_{j_0}} \sum_{\tilde{l} \in \Upsilon_{j_0}} \widehat{\theta}_{N,j_0:\tilde{k},\tilde{l},3} X_{p-1,q-1}(\varphi_{j_0:\tilde{l}}) \varphi_{j_0:\tilde{k}}(\cdot) \\
&+ \sum_{\tilde{k} \in \Upsilon_{j_0}} \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \widehat{\theta}_{N,j_0:\tilde{k};j:k,3} X_{p-1,q-1}(\psi_{j:k}) \varphi_{j_0:\tilde{k}}(\cdot) \\
&+ \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \sum_{\tilde{k} \in \Upsilon_{j_0}} \widehat{\theta}_{N,j:k;j_0:\tilde{k},3} X_{p-1,q-1}(\varphi_{j_0:\tilde{k}}) \psi_{j:k}(\cdot) \\
&+ \sum_{j \geq j_0} \sum_{k \in \Gamma_j} \sum_{l \geq j_0} \sum_{h \in \Gamma_l} \widehat{\theta}_{N,j:k;l:h,3} X_{p-1,q-1}(\psi_{l:h}) \psi_{j:k}(\cdot). \quad (25)
\end{aligned}$$

In practice, we select a finite number D of scales, according to the adopted discretisation step size in time, in the preprocessing procedure involved in the construction of our curve data set. Note that, as commented before, for a given scale $j \in \{1, \dots, D\}$, the corresponding number of nodes $k(j)$ is finite.

5 Simulation study

Falta un verbo?
Parece que no!!

To illustrate the asymptotic properties of the formulated multiscale estimators, an increasing spatial curve sample size sequence $N = 100, 900, 2500, 4900, 8100, 12100, 16900, 22500$, has been considered. The Haar wavelet system has been selected for our implementation (see, e.g., [17]). In particular, let $L_3 = -L_1 L_2$, and, as before, $T = [0, 1]$. Operators L_1 and L_2 are defined in terms of the common eigenvectors

$$\phi_p(t) = \sin(\pi p t), \quad t \in (0, 1), \quad p \geq 1, \quad (26)$$

Table 1: Eigenvalues $\lambda_{p1}, \lambda_{p2}, p = 1, \dots, 10$.

	λ_{p1}^0	λ_{p2}^0
$p = 1$	0.300	0.500
$p = 2$	0.270	0.470
$p = 3$	0.230	0.430
$p = 4$	0.200	0.400
$p = 5$	0.170	0.370
$p = 6$	0.130	0.330
$p = 7$	0.100	0.300
$p = 8$	0.030	0.230
$p = 9$	0.010	0.200
$p = 10$	0.005	0.150.

with $\phi_p(0) = \phi_p(1) = 0$. The corresponding systems of eigenvalues $\{\lambda_{pl}, p \geq 1, l = 1, 2\}$ satisfy conditions (i)–(iii) in Proposition 3 of [48], for the existence of a unique stationary solution to the SAR $\mathcal{H}(1)$ equation. Note that the conditions assumed in Theorem 1, and **Condition C2** also hold, under this scenario. In the orthogonal decomposition (3), we have considered the truncation parameter $k_N = k_{22500} = \lceil \ln(N) \rceil = \lceil \ln(22500) \rceil = 10$, where we have selected the most unfavorable case (i.e., the largest truncation order corresponding to the functional sample size $N = 22500$). Table 1 displays the $k_N = 10$ eigenvalues $\{\lambda_{pi}, p = 1, \dots, 10\}$ of operators $L_i, i = 1, 2$.

The large-scale sample properties (the draft) of \mathbf{X} are obtained by its projection onto the space $V_0 \subset L^2([0, 1])$, generated by the scaling functions

$$\{\varphi_{j_0:\tilde{k}}, \tilde{k} \in \mathcal{Y}_{j_0}\}$$

at the coarser scale j_0 . The sample local variability (details) of \mathbf{X} is reproduced at different resolution levels, by its projection onto the subspaces $W_j \subset L^2([0, 1])$, $j = j_0, \dots, D$, generated by the wavelet bases $\{\psi_{j:k}, k \in \Gamma_j\}$, $j = j_0, \dots, D$, respectively. Figures 1–4 show the displayed temporal variability at different scales of the generated curve data, over some of the nodes of a 30×30 spatial regular grid ($N = 900$).

In the estimation of the multiscale parameters (17) and (24), equations (16)–(23), and their non-diagonal counterparts are respectively computed. Function η is constant over the nodes of the D scales considered in the two-dimensional wavelet transform of operators $L_i, i = 1, 2$. In particular, the choice $\eta(\boldsymbol{\varpi}) = |\varpi_1|^2 |\varpi_2|^2$ has been made, for every $\boldsymbol{\varpi} = (\varpi_1, \varpi_2) \in [0, 2\pi) \times [0, 2\pi)$. The average by scale of the empirical mean quadratic errors, associated with the multiscale parameter estimators of (17) and (24), for $j_0 = 6, D = 9$, based on 100 generations of the functional samples of size

$$N = 100, 900, 2500, 4900, 8100, 12100, 16900, 22500,$$

are displayed in Tables 2–3.

Figure 5 shows the empirical mean quadratic errors, associated with the estimates $\{\hat{\lambda}_{N,p,1}, \hat{\lambda}_{N,p,2}, p = 1, \dots, k_N\}$ of the pure point spectra of L_1 and L_2 , computed from the empirical two-dimensional wavelet reconstructions of L_1 and

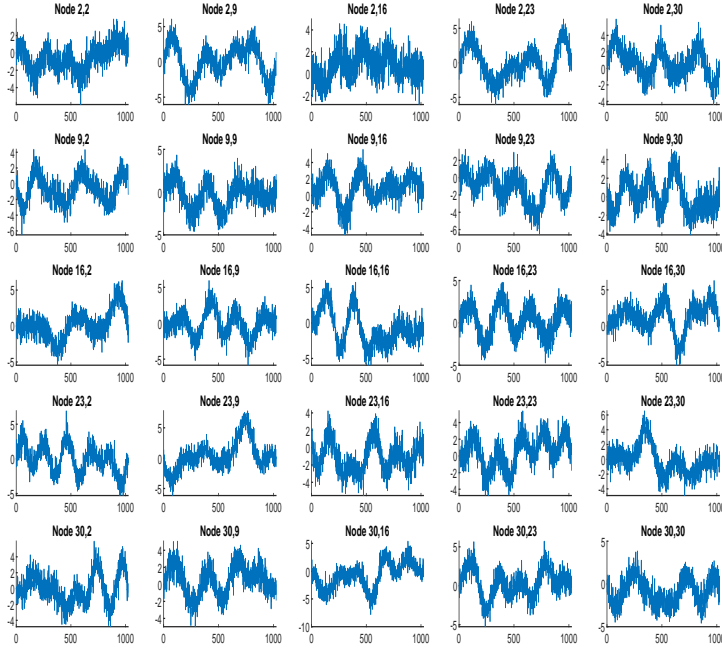


Fig. 1: Scale 10, $N = 900$. Curve data over some nodes of a 30×30 spatial regular grid

Table 2: Average by scale of empirical mean quadratic errors associated with the parameter estimators, involved in the approximation of L_1 , from 100 generations of the functional samples of size $N = 100, 900, 2500, 4900, 8100, 12100, 16900, 22500$. Scales $j = 6, 7, 8, 9$, and truncation parameter $k_N = 10$ are considered

N	Scale 6	Scale 7	Scale 8	Scale 9
100	6.828e-02	4.158e-03	2.606e-04	1.630e-05
900	9.171e-03	5.678e-04	3.558e-05	2.226e-06
2500	3.540e-03	2.174e-04	1.362e-05	8.520e-07
4900	1.930e-03	1.309e-04	8.205e-06	5.132e-07
8100	1.151e-03	7.640e-05	4.789e-06	2.995e-07
12100	9.036e-04	5.234e-05	3.280e-06	2.052e-07
16900	6.120e-04	3.712e-05	2.326e-06	1.455e-07
22500	4.663e-04	3.066e-05	1.921e-06	1.202e-07

L_2 at scale $D = 10$, based on 100 realisations of the multiscale parameter estimators. The boxplots of their sample values can be found in Figure 6. Finally, the true operators L_1 and L_2 , and their functional estimates, at scales $j = 7, 8, 9, 10$, are displayed in Figure 7. The contour plots in Figure 9 provide the multiscale (scales 7–10) description of the original and estimated spatial log-intensity field \mathbf{X} , at $t = 1/2$. At the same time, Figure 8 displays the smoothed original and estimated log-intensity values at different scales or resolution levels ($j = 7, 8, 9, 10$).

Aquí habría que diferenciar entre over a 30×30 spatial regular grid en 9 y 10 x 10 en Fig 8

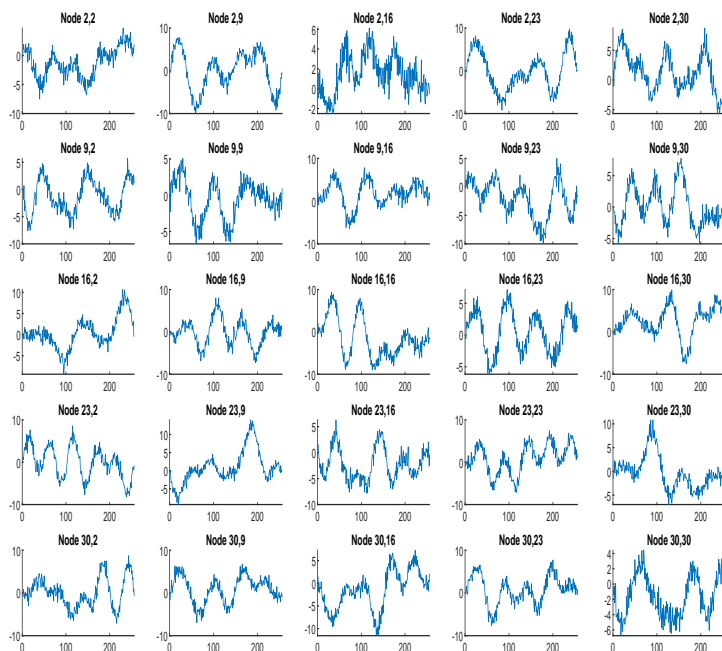


Fig. 2: Scale 9, $N = 900$. Curve data over some nodes of a 30×30 spatial regular grid

Table 3: Average by scale of empirical mean quadratic errors associated with the parameter estimators, involved in the approximation of L_2 , from 100 generations of the functional samples of size $N = 100, 900, 2500, 4900, 8100, 12100, 16900, 22500$. Scales $j = 6, 7, 8, 9$, and truncation parameter $k_N = 10$ are considered

N	Scale 6	Scale 7	Scale 8	Scale 9
100	7.080e-02	4.476e-03	2.806e-04	1.755e-05
900	9.360e-03	5.916e-04	3.708e-05	2.319e-06
2500	3.642e-03	2.302e-04	1.443e-05	9.025e-07
4900	1.787e-03	1.129e-04	7.078e-06	4.427e-07
8100	1.093e-03	6.909e-05	4.330e-06	2.708e-07
12100	9.795e-04	6.188e-05	3.878e-06	2.425e-07
16900	6.367e-04	4.022e-05	2.521e-06	1.577e-07
22500	4.472e-04	2.825e-05	1.771e-06	1.107e-07

One can observe the effect of the Functional Data Analysis (FDA) preprocessing procedure, and the effect of increasing the number of spatial nodes, when comparing the spatial patterns observed at different temporal scales in Figures 8 and 9.

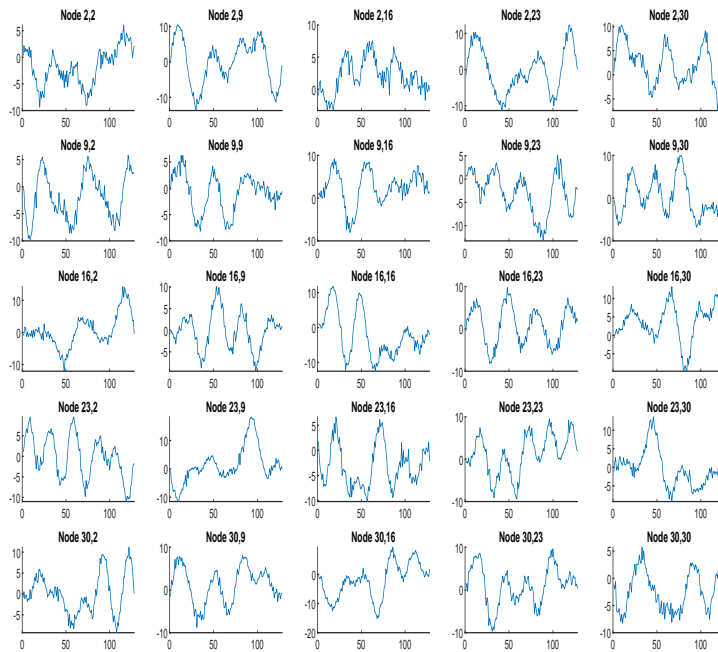


Fig. 3: Scale 8, $N = 900$. Curve data over some nodes of a 30×30 spatial regular grid

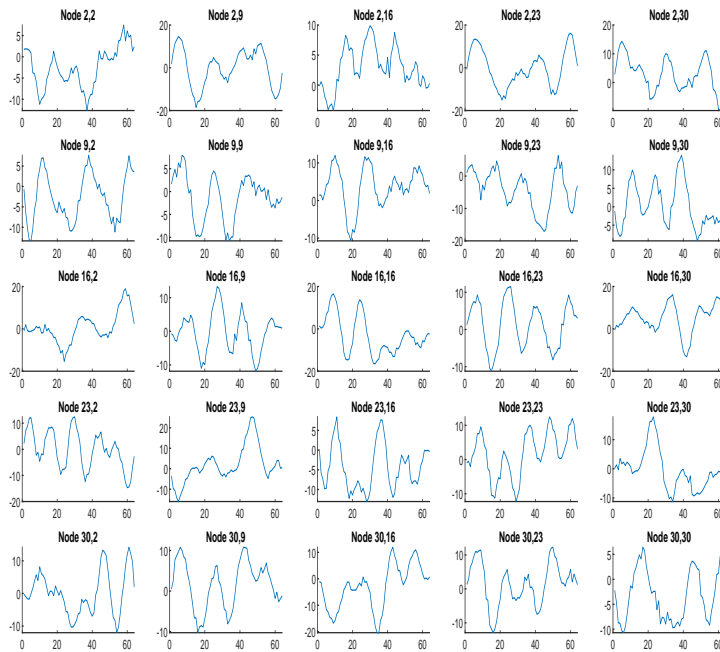


Fig. 4: Scale 7, $N = 900$. Curve data over some nodes of a 30×30 spatial regular grid

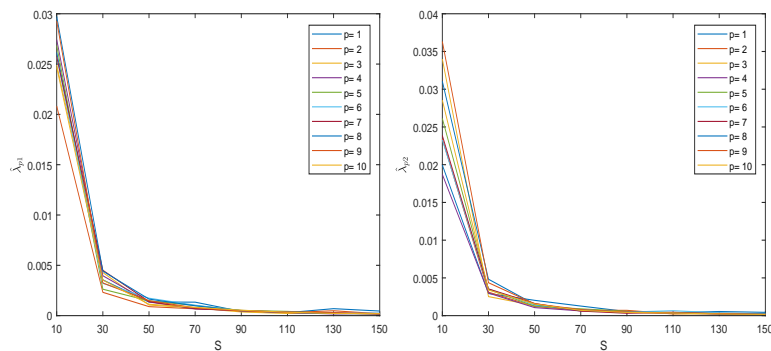


Fig. 5: Empirical mean quadratic errors (E.M.S.Es), associated with $\{\widehat{\lambda}_{N,p,1}, \widehat{\lambda}_{N,p,2}, p = 1, \dots, k_N\}$, $N = 100, 900, 2500, 4900, 8100, 12100, 16900, 22500$

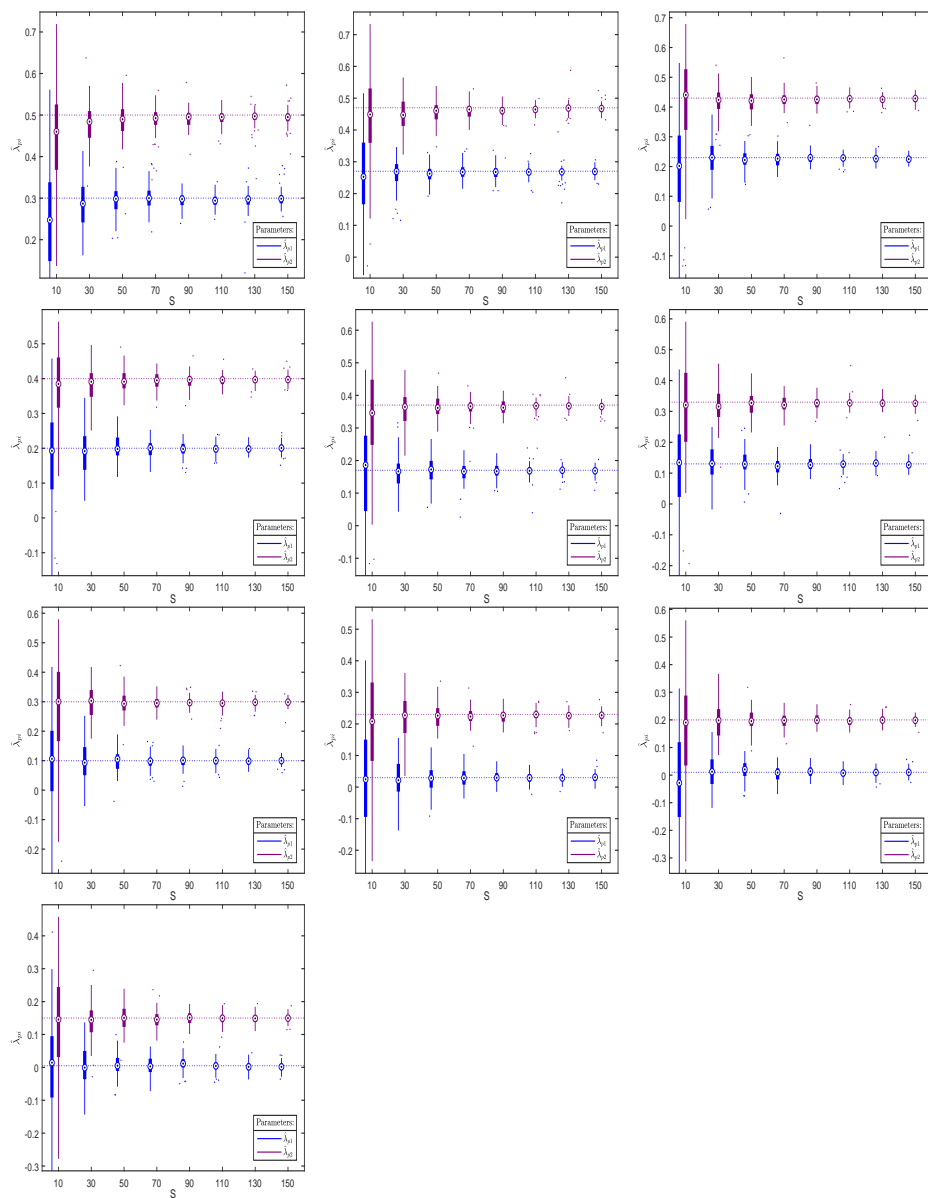


Fig. 6: Boxplots of the sample values of $\hat{\lambda}_{N,p,i}$, $p = 1, \dots, 10$, $i = 1, 2$ (from left to right, and from top to bottom), based on 100 generations of the functional samples of size $N = 100, 900, 2500, 4900, 8100, 12100, 16900, 22500$. The true parameter value is reflected in dotted line

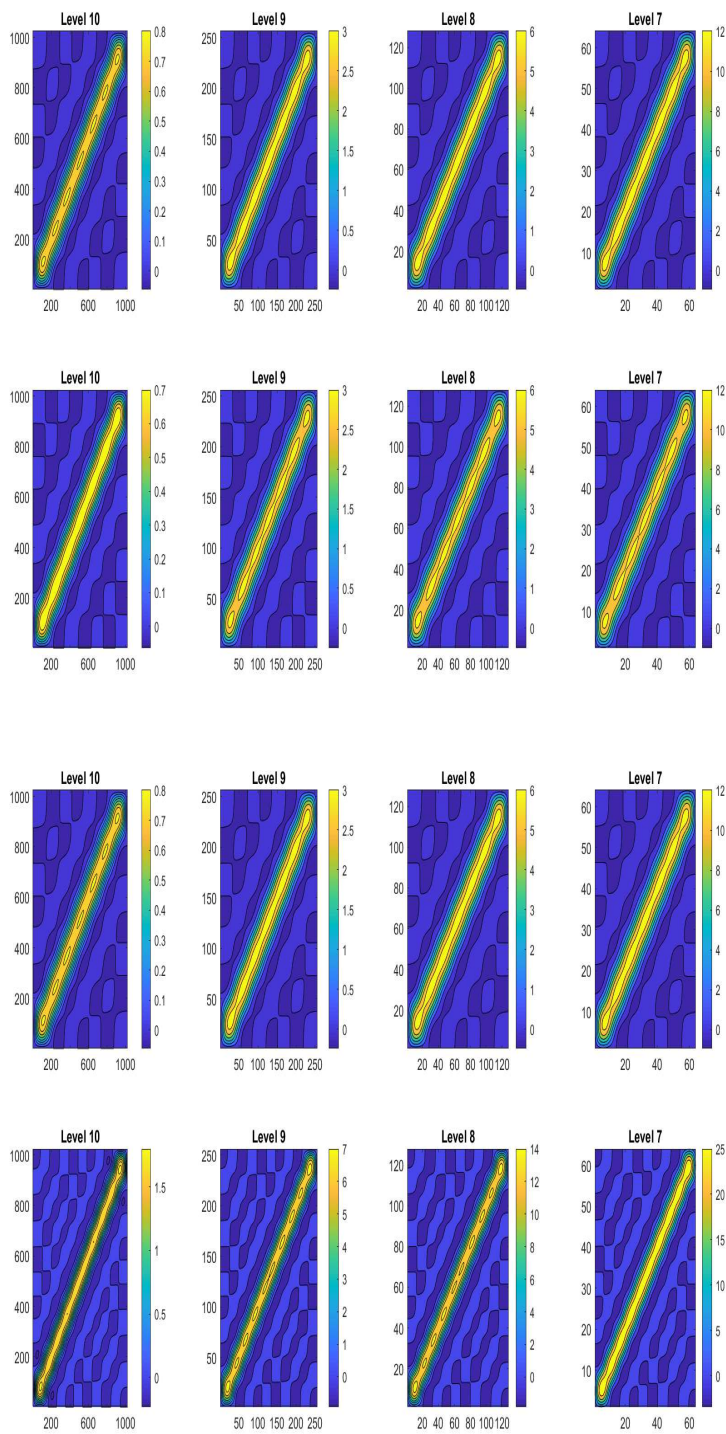


Fig. 7: True L_1 at the top row, and its multiscale estimate at the second row, for scales $j = 7, 8, 9, 10$ (from right to left). True L_2 at the third row, and its multiscale estimation at the bottom row, for scales $j = 7, 8, 9, 10$ (from right to left), over a 30×30 spatial regular grid

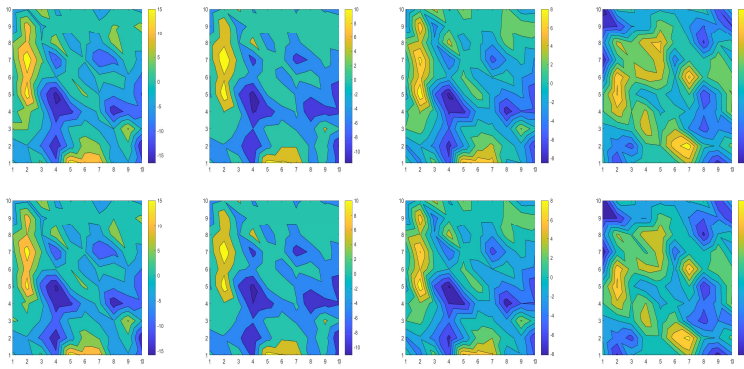


Fig. 8: Original (top-row) and estimated (bottom-row) spatial log-intensity field \mathbf{X} , at time $t = 1/2$, through the scales $j = 7, 8, 9, 10$ (from left to right), over a 10×10 spatial regular grid, from smoothed curve data

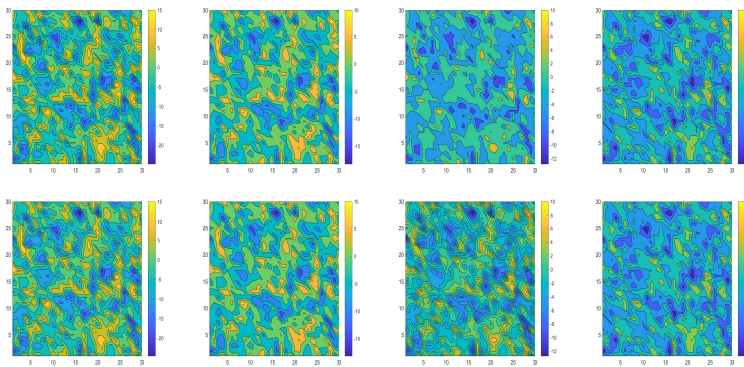


Fig. 9: Original, non-smoothed (top-row), and estimated (bottom-row) spatial log-intensity field \mathbf{X} , at time $t = 1/2$, through the scales $j = 7, 8, 9, 10$ (from left to right), over a 30×30 spatial regular grid

6 Real-data example

continental Europe

The Spanish Statistical National Institute provided the data on the observed cases of respiratory disease deaths, consisting of 432 monthly records, in the period 1980–2015, collected at the 48 Spanish provinces in the Iberian Peninsula. The data are temporal, and spatial interpolated over a 20×20 regular grid. Specifically, 1725 temporal nodes, and 400 spatial nodes are considered. A flexible fitting of the underlying local behaviour (or singularity) of the observed and interpolated data is obtained, from a suitable choice of the scale or resolution level (see Figures 10–11). Note that FDA preprocessing usually leads to an over-smoothing. That is the case of B-spline smoothing often applied to construct curve data sets (see Figure 12).

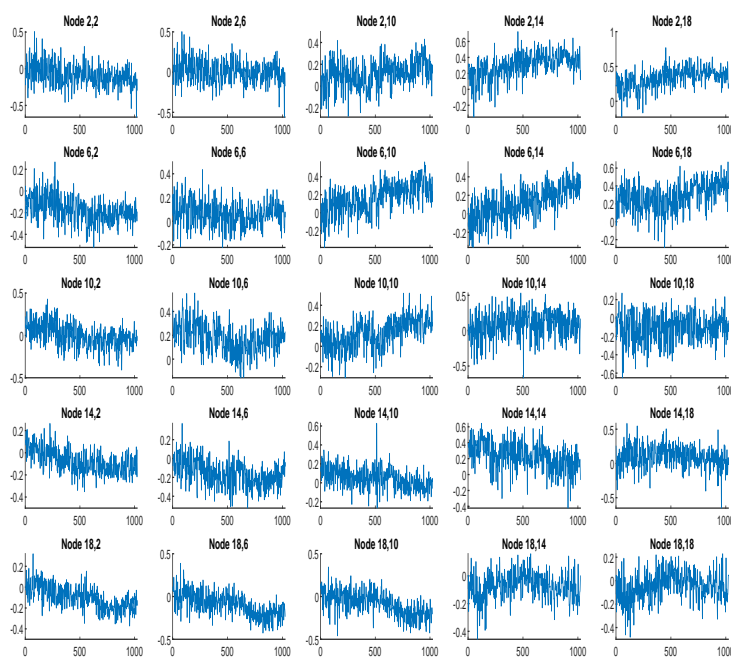


Fig. 10: Temporal and spatial interpolated data over a 20×20 spatial regular grid

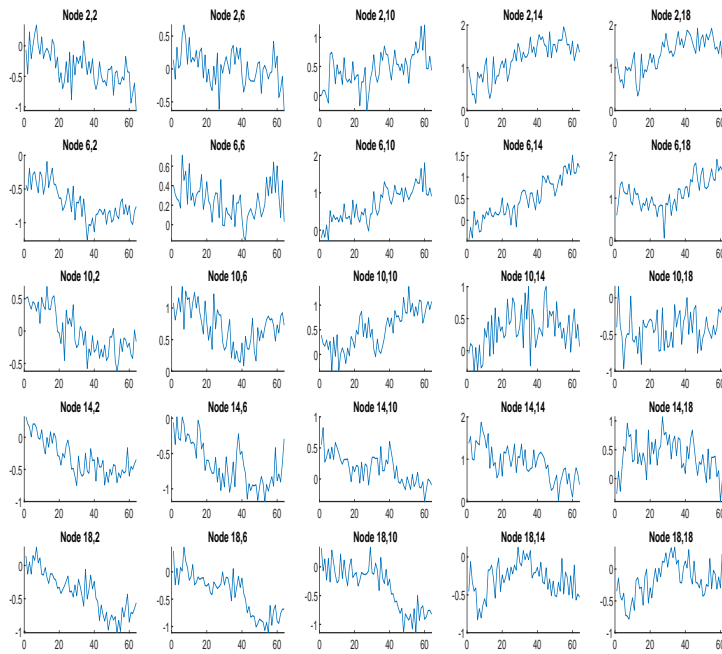
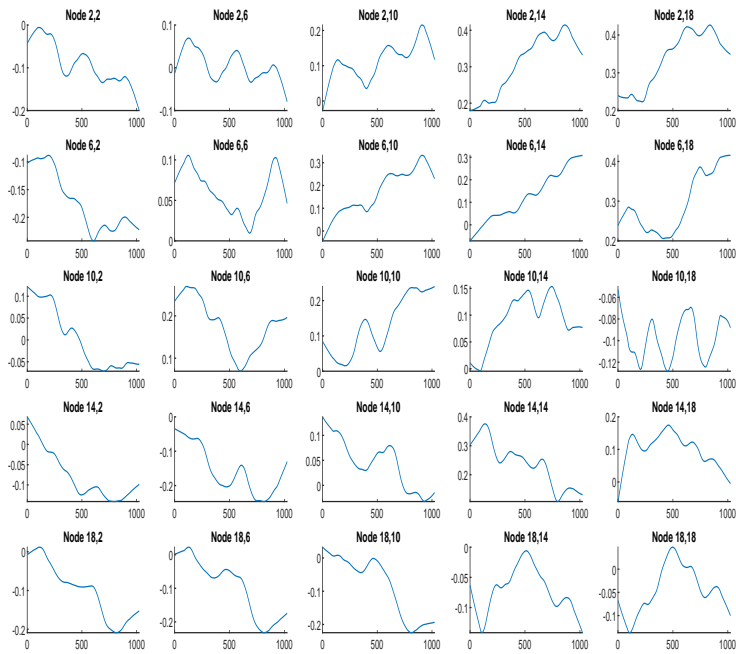


Fig. 11: Temporal and spatial interpolated data over a 20×20 spatial regular grid at scale (resolution level) 7

Fig. 12: B-spline smoothed curve data over a 20×20 spatial regular grid

6.1 Multiscale estimation

Equations (17)–(25) are implemented in terms of the empirical eigenvectors, and the Haar wavelet basis. The computed estimates at scales (resolution levels) $j = 7, 8, 9, 10$, of the autocorrelation operators L_1 and L_2 can be found in Figure 13, for $k_N = \lceil \ln(N) \rceil^- = \lceil \ln(400) \rceil^- = 5 = k_{400}$. Contour plots in Figure 14 display the spatial patterns of the observed and estimated log-intensity field over a 20×20 spatial regular grid, at monthly times $t = 108$ and $t = 216$, through scales $j = 7, 8, 9, 10$. Here, the multiscale analysis has been implemented from the interpolated non-smoothed data. Figure 15 shows the original and estimated values of the log-intensity field over the same temporal and spatial nodes, from the B-spline smoothed curve data. One can observe the loss of information in Figure 15, about spatial variability displayed by the log-intensity field at scales $j = 9, 10$, with respect to Figure 14. Thus, similar spatial patterns are observed, at scales $j = 7, 8, 9, 10$, when B-spline smoothed curve data are considered, hiding the heterogeneities that the log-intensity field presents through different scales.

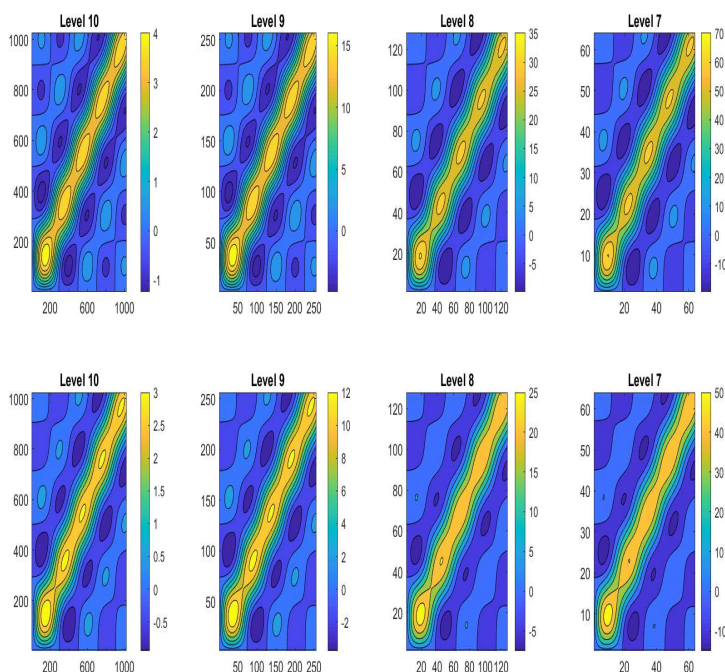


Fig. 13: Multiscale estimation of operator L_1 (top), and of L_2 (bottom) through scales (multiresolution levels) $j = 7, 8, 9, 10$ (from right to left)

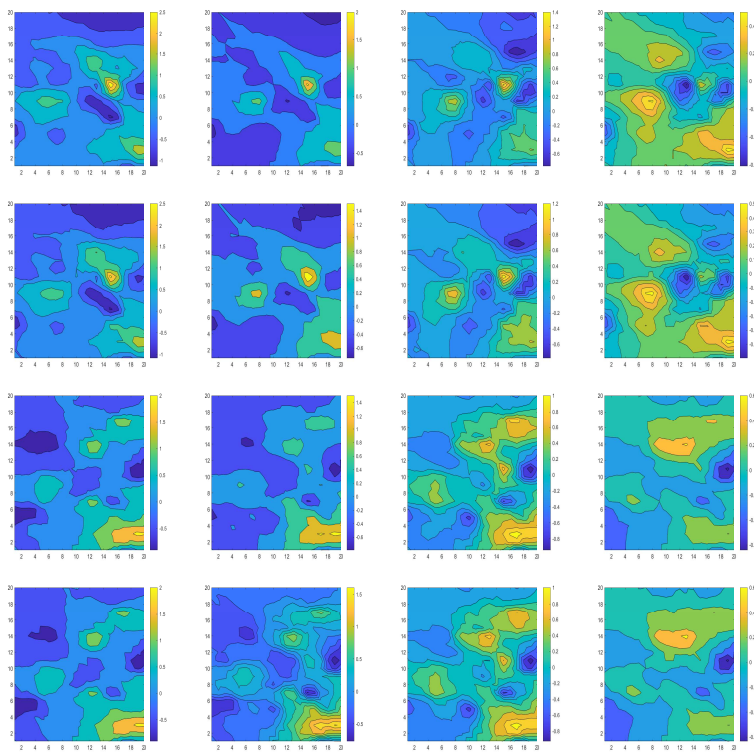


Fig. 14: Contour plots of the observed log-intensity field at monthly times $t = 108$ (top-row) and $t = 216$ (third-row), and the estimated log-intensity field at $t = 108$ (second-row) and $t = 216$ (bottom-row). Both observed and estimated values at times $t = 108$, and $t = 216$ are displayed through the scales $j = 7, 8, 9, 10$ (from left to right), in the Haar wavelet system, from the temporal and spatial interpolated data over a 20×20 regular grid

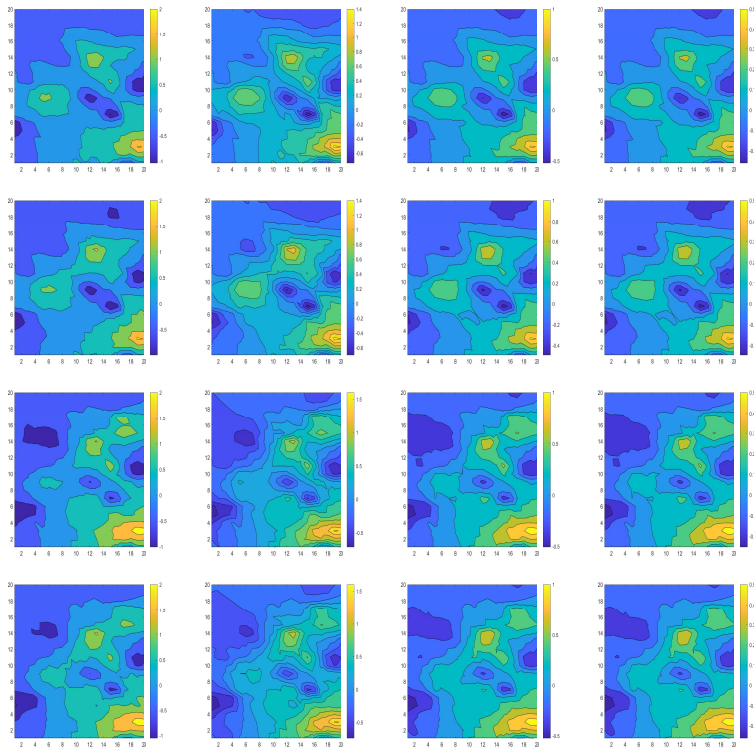


Fig. 15: Contour plots of the observed log-intensity field at monthly times $t = 108$ (top-row) and $t = 216$ (third-row), and the estimated log-intensity field at $t = 108$ (second-row) and $t = 216$ (bottom-row). Both observed and estimated values at times $t = 108$, and $t = 216$ are displayed through the scales $j = 7, 8, 9, 10$ (from left to right), in the Haar wavelet system, from the temporal interpolated and smoothed data over a 20×20 regular grid

Table 4: ALOOCVE. Pointwise annually averaged cross-validation errors.

Year	ALOOCVE	Year	ALOOCVE	Year	ALOOCVE
1980	0.0247	1992	0.0118	2004	0.0132
1981	0.0144	1993	0.0130	2005	0.0117
1982	0.0112	1994	0.0163	2006	0.0135
1983	0.0125	1995	0.0159	2007	0.0140
1984	0.0144	1996	0.0111	2008	0.0118
1985	0.0122	1997	0.0099	2009	0.0113
1986	0.0126	1998	0.0108	2010	0.0143
1987	0.0155	1999	0.0141	2011	0.0131
1988	0.0161	2000	0.0167	2012	0.0122
1989	0.0144	2001	0.0161	2013	0.0115
1990	0.0125	2002	0.0143	2014	0.0145
1991	0.0118	2003	0.0140	2015	0.0221

6.2 Validation results

Our multiscale spatial functional approach is now validated from the data. Specifically, by leaving aside the curves observed at the nodes in a neighbourhood of the province defining the region of interest (the validation functional data set), equations (17)–(25) are computed from the remaining functional observations, spatially distributed at the neighbourhoods of the rest of the Spanish provinces (the training functional data set). The corresponding multiscale $\text{SAR}\ell^2(1)$ componentwise parameter estimators and predictors are then obtained, from the empirical wavelet reconstruction formula at resolution level 10 (truncated version of equation (25)). This process is repeated 48 times. Thus, the cross-validation functional error is calculated as the mean of the absolute functional errors computed at each one of the 48 iterations. The annual pointwise mean of the computed cross-validation functional error can be found in Table 4 above. The original and estimated annually averaged number of deaths at each province, for each one of the years analysed, are also displayed in Figures 16 and 17.

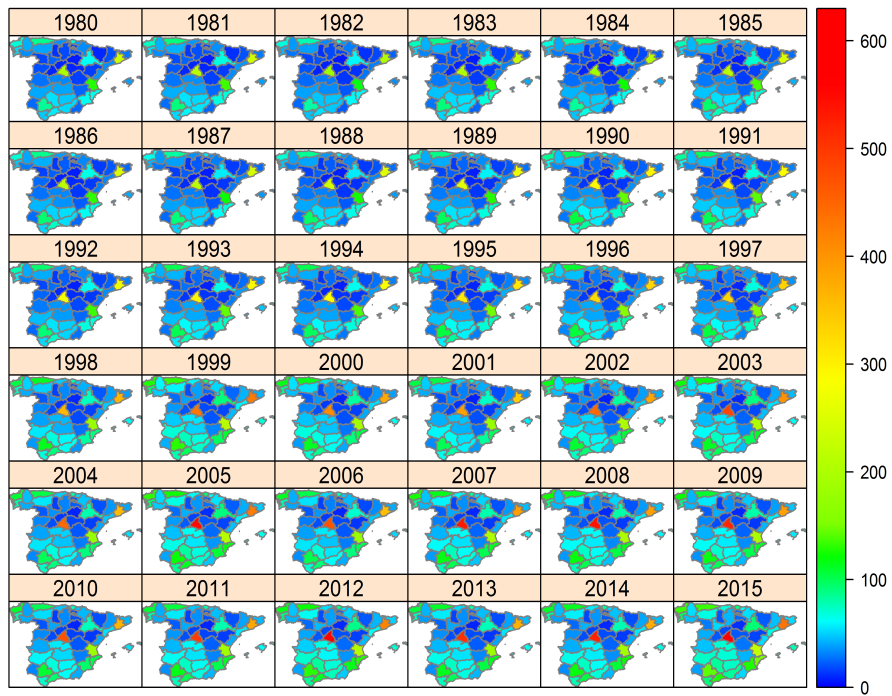


Fig. 16: Annually averaged observed number of respiratory disease deaths at each one of the 48 Spanish provinces from January 1980 to December 2015.

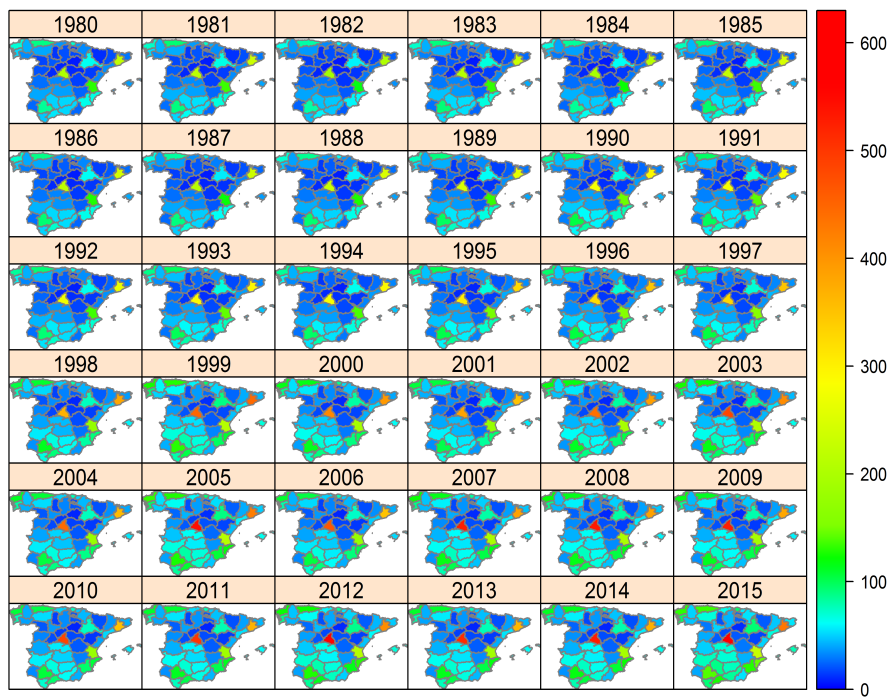


Fig. 17: Annually averaged estimates of the number of respiratory disease deaths, at each one of the 48 Spanish provinces from January 1980 to December 2015.

7 Concluding remarks

The multiscale spatial functional prediction methodology presented allows heterogeneity analysis over different temporal scales of the log-intensity field. It is well-known that FDA preprocessing techniques (e.g., B-spline smoothing) usually hide or eliminate local variation at high resolution levels (see, e.g., [42]; [30]; [23], among others). The estimation approach adopted in this paper overcomes this limitation, providing a more flexible framework. Thus, a suitable choice of the scale where the log-intensity field should be analysed can be performed, according to the aims of the study, and uncertainties in the counts associated with the lack of sample information.

The spatial functional frequency relative entropy based approach, we have presented in this paper, does not require additional information about the prior-probability distribution of the parameters, as in the Bayesian framework. Particularly, in [56], the authors present a novel approach on spatiotemporal Bayesian multivariate count modeling from areal data (see also the references therein for an overview on the state of the art). The FDA based reduction dimension technique considered here avoids heavy computational problems usually arising when high-dimensional covariance matrices, associated with latent Gaussian variables and hyperparameters, are involved in the Bayesian estimation methodology. Indeed, multivariate conditional autoregressive models (MCAR) have been widely developed under a fully Bayesian framework. Markov chain Monte Carlo (MCMC) based approaches, and integrated nested Laplace approximation (INLA) (see, e.g., [36], and [47]) are usually applied for model fitting in this context. However, several problems remain open regarding separability and spatiotemporal covariance modelling, when a FDA framework is adopted.

The presented estimation approach provides an alternative to INLA in the functional frequency domain, by considering the spatial truncated functional Cramér-wavelet expansion (see [45], where the temporal functional Cramér-Karhunen-Loève expansion is introduced). It is well-known that functional Cramér-Karhunen-Loève expansion provides an optimal representation of a stationary functional process in the second-order moment sense. This optimality criterion is not always satisfied by our functional Cramér-wavelet expansion, but this is the price we must pay for a more flexible and accurate description of local variation at high resolution levels. That, on the other hand, constitutes the aim of the present paper, regarding multiscale analysis in a FDA frequency framework.

Our approach can be extended to the case where a multiresolution analysis is also performed in space, for approximation of the hidden spatial continuous functional log-intensity process driving the counts, as an alternative to the usual spatial B-spline smoothing techniques. The resulting approach allows heterogeneity analysis through temporal and spatial scales, providing a multiresolution approximation of space-time interaction affecting the evolution of the log-intensity process. This topic constitutes the subject of a subsequent research paper.

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