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# The Borda and Condorcet winners coincide for lexicographic preferences

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ABSTRACT

We show that, for any lexicographic preference profile, the sets of Borda and Condorcet winners coincide. Moreover, the sets of Borda and Condorcet losers also coincide. However the binary relations induced by the Borda Count and the majority pairwise comparisons are not the same.

## 1. Introduction

When only two alternatives are at stake, majority voting is the best method, the one satisfying well-known properties (see May (1952)). The challenge is to extend majority voting among pairs to situations where there are more than two alternatives. Condorcet (1785) proposed choosing the alternative defeating every other alternative in pairwise comparisons. Borda (1781) proposed assigning points to each candidate according to preferences and choosing the alternative with the highest total score. Both proposals are used in many contexts, which is why they have been the subject of much attention, in particular to establish some type of relationship between them. In the case of unrestricted preference domains, for any number of voters and any number of alternatives greater than two, there exist Borda winners that are not Condorcet winners for some preference profiles and vice versa (see Fishburn (1973) and Moulin (1988)). Naturally, in the literature it has been explored in which restricted domains both proposals coincide. Recently, Berga et al. (2019) provide a partial answer for the single-peaked domain that depends on the number of alternatives and agents. As in Barberà et al. (1991) we consider situations in which there is a finite number of alternatives that can be described by a finite set of ordered attributes. We assume that agents agree that the first attribute is the most important, the second attribute the next most important, and so on (although they may disagree about whether each attribute is a good one or not).[1] Then, we define lexicographicity and show that, for any lexicographic preference profile (see Chipman (1960), Fishburn (1974) and Barberà et al. (2010)) the sets of Borda and Condorcet winners coincide for any number of alternatives and agents. This coincidence between Borda and Condorcet winners extends to the set of Borda and Condorcet losers. Finally, we also show that for any pair of alternatives it is not the case that either the alternative having a greater Borda score is the one defeating in pairwise majority comparisons the other or the other way around.

Section 2 presents the model, and Section 3 presents the main results and their proofs.

## 2. The model

Let $N = \{1, \dots, n\}$ be a finite set of agents. The set of alternatives is $A = \{0, 1\}^K$, where $K$ is the set of attributes, traits or alike and $\#K = k \in \mathbb{N}$. From now on we identify the set of attributes $K$ with the set $\{1, \dots, k\}$. Agent $i's$ preference relation is a linear ordering $\succ_i$ over the set of alternatives. A preference profile is denoted by $\succ = (\succ_1, \dots, \succ_n)$.

**Definition 1.** A preference relation $\succ_i$ is **lexicographic** relative to the ordering $(1, \dots, k)$ of $K$ if there exists a sequence of preference relations $(\succ_{i,j})_{j=1}^{k}$ over $\{0, 1\}$ such that for every two alternatives $x, y \in A$, $x \succ_i y$ if and only if $x_j \succ_{i,j} y_j$, where $j$ is the first component, according to the ordering $(1, \dots, k)$, in which $x$ and $y$ differ.

Note that lexicographicity implies a property called **separability** over the preference relations, which means that, for any $x, y, \bar{x}, \bar{y} \in A$ and $j \in K$ such that $x_j = \bar{x}_j$ and $y_j = \bar{y}_j$ and, for any $h \neq j$, $x_h = y_h$, $\bar{x}_h = \bar{y}_h$, then $x \succ_i y$ if and only if $\bar{x} \succ_i \bar{y}$.

Let $\succ$ be any lexicographic preference profile. Since $\succ$ is fixed in the remaining of the paper, we omit any reference to it. The alternative

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[1] Indeed lexicographicity in this context implies additive representability as defined in Barberà et al. (1991).

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$x$ **defeats** $y$ if $\#\{i \in N | y \succ_i x\} < \#\{i \in N | x \succ_i y\}$. A **Condorcet winner** is an alternative that is never defeated by any other alternative; a **Condorcet loser** is an alternative that does not defeat any other alternative.

The **Borda score** for an agent $i \in N$ of an alternative $x$ is $B^i_x \equiv \#\{y \in A | x \succ_i y\}$ and the **Borda score** of an alternative $x$ is $B_x \equiv \sum_{i=1}^n B^i_x$. A **Borda winner** is an alternative with the highest Borda score among all the alternatives in $A$, and a **Borda loser** is an alternative with the lowest Borda score among all the alternatives in $A$.

Finally, we introduce two binary relations, the majority binary relation and the Borda binary relation. Given two alternatives $x, y \in A$, we write $P(x, y)$ when $x$ is not defeated by $y$, and $B(x, y)$ when the Borda score of $x$ is greater or equal than the Borda score of $y$. The strict and indifferent parts of the two binary relations are defined as usual.

## 3. Results

In this section, we prove that the sets of Borda and Condorcet winners coincide. To do that we present two lemmas and their proofs.

Lemma 1 shows that the difference in absolute values of the Borda scores of any two alternatives that differ only in one component is always the same. We also provide the exact difference.

**Lemma 1.** *For any $i \in N$, any $x, y \in A$, $x \neq y$, and $j \in K$ such that $x_j \neq y_j$ and $x_h = y_h$ for any $h \neq j$, we have that $|B^i_x - B^i_y| = 2^{k-j}$.*

**Proof.** Without loss of generality, assume $x \succ_i y$, $x_j = 1$ and $y_j = 0$. Then

$$|B^i_x - B^i_y| = B^i_x - B^i_y = \#\{w \in A | x \succ_i w\} - \#\{w \in A | y \succ_i w\} \quad (1)$$

Let $D(x, y) = \{w \in A | x \succ_i w\} \setminus \{w \in A | y \succ_i w\} = \{w \in A | x \succ_i w \succsim_i y\}$. Note that $\{w \in A | x \succ_i w\} \supseteq \{w \in A | y \succ_i w\}$, so $\#D(x, y) = \#\{w \in A | x \succ_i w\} - \#\{w \in A | y \succ_i w\}$ and, by (1), $\#D(x, y) = |B^i_x - B^i_y|$.

Let $z \in A$ be an alternative such that $z_h \neq x_h = y_h$ for some $h < j$. By lexicographicity, either $x \succ_i y \succ_i z$ or $z \succ_i x \succ_i y$. In both cases $z \notin D(x, y)$.

Thus, only an alternative $z \in A$ such that $z_h = x_h = y_h$ for all $h < j$ can belong to $D(x, y)$. Let $W = \{z \in A | z_h = x_h = y_h \text{ for all } h < j\}$; note that $\#W = \frac{2^k}{2^{j-1}} = 2^{k-j+1}$.

We partition $W$ in pairs $(z^0, z^1)$ such that

(a) $z^1_h = z^0_h = x_h = y_h$ for all $h < j$,
(b) $z^0_j = 0$ and $z^1_j = 1$,
(c) $z^0_h = z^1_h$ for all $h > j$.

The number of such pairs is $\frac{\#W}{2} = 2^{k-j}$. Note that $(y, x)$ constitutes one of these pairs with $z^0 = y$ and $z^1 = x$ and that $x \notin D(x, y)$ while $y \in D(x, y)$. Let $(z^0, z^1) \neq (y, x)$ be any pair that respects (a), (b) and (c). By lexicographicity $z^1 \succ_i y$ and $x \succ_i z^0$, therefore $z^1$ belongs to $D(x, y)$ if and only if $x \succ_i z^1$. Let $\ell > j$ be the first component such that $z^1_\ell \neq x_\ell$. We have that $z^1_\ell = z^0_\ell$ and that $x_\ell = y_\ell$. Thus $x \succ_i z^1$ if and only if $y \succ_i z^0$ by lexicographicity. The latter is equivalent to $z^0 \notin D(x, y)$. To sum up $z^1 \in D(x, y) \Leftrightarrow z^0 \notin D(x, y)$. We can conclude that exactly one element in each pair belongs to $D(x, y)$ and that $\#D(x, y) = 2^{k-j}$. ■

Lemma 2 shows that for any two alternatives that differ only in one component, the Borda and the majority pairwise binary relation coincide.

**Lemma 2.** *Let $x, y \in A$ and $j \in K$ be such that $x_j \neq y_j$ and $x_h = y_h$ for any $h \neq j$. Then $P(x, y)$ if and only if $B(x, y)$.*

**Proof.** By definition $B(x, y)$ if and only if $B_x - B_y \geq 0$. The value $B_x - B_y$ is equal to $\sum_{i=1}^n B^i_x - \sum_{i=1}^n B^i_y = \sum_{i=1}^n (B^i_x - B^i_y)$. By Lemma 1, $B^i_x - B^i_y = 2^{k-j}$ if $x \succ_i y$, and $B^i_x - B^i_y = -2^{k-j}$ if $y \succ_i x$. Thus, $\sum_{i=1}^n (B^i_x - B^i_y) = 2^{k-j}\#\{i \in N | x \succ_i y\} - 2^{k-j}\#\{i \in N | y \succ_i x\}$. Note that, $2^{k-j}\#\{i \in N | x \succ_i y\} - 2^{k-j}\#\{i \in N | y \succ_i x\} \geq 0$ if and only if $\#\{i \in N | x \succ_i y\} - \#\{i \in N | y \succ_i x\} \geq 0$. Finally, by definition of $P(x, y)$, $\#\{i \in N | x \succ_i y\} \geq \#\{i \in N | y \succ_i x\}$ if and only if $P(x, y)$. ■

Equipped with Lemma 2, we prove our main result that states that for every lexicographic preference profile, the sets of Borda and Condorcet winners coincide.

**Theorem 1.** *For any alternative $x \in A$, $x$ is a Borda winner if and only if $x$ is a Condorcet winner.*

**Proof.** Let $x \in A$ be a Borda winner. We prove that $P(x, y)$ for every $y \in A$. If $x = y$ the statement is trivially true, so $y \neq x$ and let $j$ be the first component such that $x_j \neq y_j$. Let $\bar{x} \in A$ be the alternative such that $\bar{x}_h = x_h$ for every $h \neq j$ and $\bar{x}_j = y_j$. Since $x$ is a Borda winner, $B(x, \bar{x})$. By Lemma 2, $P(x, \bar{x})$. Now, by lexicographicity, $\{i \in N | x \succ_i \bar{x}\} = \{i \in N | x \succ_i y\}$ and $\{i \in N | x \prec_i \bar{x}\} = \{i \in N | x \prec_i y\}$, since $P(x, \bar{x})$, also $P(x, y)$. Since $y$ may be any alternative, this proves that $x$ is a Condorcet winner.

Let $x \in A$ be a Condorcet winner. We prove that $B(x, y)$ for every $y \in A$. Let $y \in A$ be any alternative and consider the sequence $(x^0, \ldots, x^k)$ of alternatives such that

- $x^j_h = x_h$ for any $h > j$
- $x^j_h = y_h$ for any $h \leq j$

Note that $x^0 = x$ and $x^k = y$ and that $x^j$ and $x^{j-1}$ may differ only in the $j$th component. we will prove by induction on $j$ that $B(x, x^j)$.

**Base case:** $B(x, x^0)$ holds because $x$ has an equal Borda score to itself.

**Inductive step** Induction hypothesis: let $j \geq 1$ such that $B(x, x^{j-1})$ holds. We need to show that $B(x, x^j)$ holds. To do so, consider $\bar{x}^j \in A$ be the alternative such that $\bar{x}^j_h = x_h$ for every $h \neq j$ and $\bar{x}^j_j = x^j_j$. Because $x$ is a Condorcet winner, $P(x, \bar{x}^j)$ holds. By separability, for every $i \in N$, $x \succ_i \bar{x}^j$ if and only if $x^{j-1} \succ_i x^j$. So $\#\{i \in N | x^{j-1} \succ_i x^j\} = \#\{i \in N | x \succ_i \bar{x}^j\}$ and $\#\{i \in N | x^{j-1} \prec_i x^j\} = \#\{i \in N | x \prec_i \bar{x}^j\}$. It follows that $P(x, \bar{x}^j)$ implies $P(x^{j-1}, x^j)$. By Lemma 2, $P(x^{j-1}, x^j)$ implies $B(x^{j-1}, x^j)$. So $B_{x^j} \leq B_{x^{j-1}} \leq B_x$, and this implies $B(x, x^j)$.

Since $x^k = y$, $B(x, x^k)$ means $B(x, y)$. Since $y$ may be any alternative, this proves that $x$ is a Borda winner. ■

Our last result is based on Theorem 1 and shows that the sets of Borda and Condorcet losers coincide.

**Corollary 1.** *For any alternative $x \in A$, $x$ is a Borda loser if and only if $x$ is a Condorcet loser.*

**Proof.** Let us consider the reverse preference profile $\succ^C$ such that for every pair of alternatives $x, y \in A$ and every agent $i \in N$, $x \succ^C_i y$ if and only if $y \succ_i x$. Then $\succ^C$ is a lexicographic preference profile and the sets of Borda winners and Condorcet winners according to $\succ^C$ are, respectively, the set of Borda losers and Condorcet losers according to $\succ$. By Theorem 1, these two sets coincide. ■

Even if the best and the worst alternatives according to the binary relations $B(\cdot, \cdot)$ and $P(\cdot, \cdot)$ are the same, as Theorem 1 and Corollary 1 state, these two binary relations do not coincide on every pair of alternatives, as the following example shows.

**Example 1.** Let $N = \{1, 2, 3\}$, $K = \{1, 2\}$ $A = \{0, 1\}^2$ and $\succ$ be the preference profile of Fig. 1.

Alternative 11 is both the only Condorcet winner and the only Borda winner with a Borda score of 7, while 00 is both the only Condorcet loser and the only Borda loser with a Borda score of 2. When comparing 01 and 10, $P(10, 01)$ because 01 is preferred to 10 by 2 out of 3 agents, but $B(10, \succ) = 4$ and $B(01, \succ) = 5$, so $B(01, 10)$. Hence, the two binary relations do not coincide when comparing these two alternatives.

| $\succ_1$ | $\succ_2$ | $\succ_3$ |
|---|---|---|
| 01 | 11 | 11 |
| 00 | 10 | 10 |
| 11 | 01 | 01 |
| 10 | 00 | 00 |

**Fig. 1.** Agents' preferences in Example 1.

## Data availability

No data was used for the research described in the article.


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