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ORIGINAL PAPER

Inequalities on tent spaces and closed range integration operators on spaces of average radial integrability

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Abstract

We deal with a reverse Carleson measure inequality for the tent spaces of analytic functions in the unit disc $\mathbb{D}$ of the complex plane. The tent spaces of measurable functions were introduced by Coifman, Meyer and Stein. Let $1 \leq p, q < \infty$ and consider the measurable set $G \subseteq \mathbb{D}$. We prove a necessary and sufficient condition on G in order to exist a constant $K > 0$ such that

$$\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi) \cap G} |f(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \geq K \int_{\mathbb{T}} \left(\int_{\Gamma_{1/2}(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi|,$$

for any analytic function f in $\mathbb{D}$ with the property, the right term of the inequality above is finite. Here $\mathbb{T}$ stands for the unit circle, $dm(z)$ is the area Lebesgue measure in $\mathbb{D}$ and $\Gamma_\beta(\xi)$ is the cone-like region

$$\Gamma_\beta(\xi) = \{z \in \mathbb{D} \ \ |z| < \beta\} \cup \bigcup_{|z|<\beta} [z, \xi), \quad \beta \in (0, 1),$$

with vertex at $\xi \in \mathbb{T}$. This work extends the study of D. Luecking on Bergman spaces to the analytic tent spaces. We apply this result in order to characterize the closed range property of the integration operator

$$T_g(f)(z) = \int_0^z f(w) g'(w) \, dw, \quad z \in \mathbb{D},$$

when acting on the average radial integrability spaces. The Hardy and the Bergman spaces form part of this family. The function g is a fixed analytic function in the unit disc. The operator T_g is known as Pommerenke operator. Moreover, for the first time, we provide examples of symbols g that introduce or not a closed range operator T_g in these spaces.

Keywords Radial integrability · Tent spaces · Reverse Carleson measures · Closed range integration operators

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Contents

1 Introduction

Let $\mathcal{H}(\mathbb{D})$ be the class of analytic functions in the unit disc $\mathbb{D}$ of the complex plane and $\mathbb{X} \subset \mathcal{H}(\mathbb{D})$ be a Banach space. Consider a bounded linear operator

$$Q : \mathbb{X} \to \mathbb{X}.$$

We call Q bounded below if there is a positive constant C such that

$$\|Q(f)\|_{\mathbb{X}} \ \geq \ C \ \ \|f\|_{\mathbb{X}} \,, \qquad f \in \mathbb{X}. \tag{1.1}$$

An operator that fulfills property (1.1) has closed range and it is not compact. Conversely, if Q is an one-to-one operator with closed range then it is bounded below. Typical examples of operators for which the closed range property is equivalent to (1.1) are the composition operators (see [13]). The same is true for the integration operator

$$T_g(f)(z) \ = \ \int_0^z \ f(\zeta)\, g'(\zeta)\, d\zeta \ , \qquad z \in \mathbb{D}$$

and its companion

$$S_g(f)(z) \ = \ \int_0^z \ f'(\zeta)\, g(\zeta)\, d\zeta \ , \qquad z \in \mathbb{D}.$$

The function $g \in \mathcal{H}(\mathbb{D})$ is called the symbol of the operators. The operator T_g is also known as Pommerenke operator. If T_g is acting boundedly on a Banach space $\mathbb{X}$ of analytic functions in $\mathbb{D}$ then condition (1.1) is equivalent to the closed range property. This is also true for S_g under the following modification. It is clear by its definition that S_g sends the constant functions to the zero function. Therefore, it makes sense to consider S_g on $\mathbb{X}/\mathbb{C}$. Otherwise, it is not one-to-one. Then, we can state that S_g is bounded below on $\mathbb{X}/\mathbb{C}$ if and only if it has closed range in $\mathbb{X}/\mathbb{C}$ (see [7]). Consequently, it is natural to look for those symbols g that introduce bounded below operators on Banach spaces of analytic functions.

The operators T_g, S_g are closely related to the pointwise multiplication operator

$$M_g(f) \ = \ f \cdot g\,, \qquad f \in \mathbb{X},$$

since

$$M_g(f) \ = \ f(0)g(0) + T_g(f) + S_g(f)\,, \qquad f \in \mathbb{X}. \tag{1.2}$$

Luecking, in [19], considered (1.1) for the M_g in the case of the Bergman spaces. We say that an $f \in \mathcal{H}(\mathbb{D})$ belongs to the Bergman space A^p, $p \geq 1$, if

$$\|f\|_{A^p}^p = \int_{\mathbb{D}} |f(z)|^p \, dm(z)$$
$$\asymp |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^p \, dm(z) < \infty, \tag{1.3}$$

where $dm(z)$ stands for the normalized area Lebesque measure in the unit disc. Through out this work the notation $A \asymp B$ states that there are positive constants k_1, k_2 such that $k_1 A \leq B \leq k_2 A$. If one of the inequalities is true then we use the symbol $\lesssim$. Let now a $g \in H^\infty$ that is a bounded analytic function in $\mathbb{D}$. This is exactly the class of symbols that introduces a bounded multiplication operator in Bergman spaces. Luecking proved that M_g is bounded below in A^p if and only if there is an $\eta \in (0,1)$, a $\delta > 0$ and a $c > 0$ such that

$$m(G_c \cap \Delta(\alpha, \eta)) \geq \delta \, m(\Delta(\alpha, \eta)), \qquad \forall \alpha \in \mathbb{D}, \tag{1.4}$$

where

$$G_c = \{z \in \mathbb{D} : |g(z)| > c\}. \tag{1.5}$$

The set

$$\Delta(\alpha, \eta) = \left\{ z \in \mathbb{D} \; : \; |\phi_\alpha(z)| = \left| \frac{\alpha - z}{1 - \bar{\alpha} z} \right| < \eta \right\}$$

is the pseudohyperbolic disc with "center" α and "radius" η. It is clear that $z \in \Delta(\alpha, \eta)$ if and only if $\alpha \in \Delta(z, \eta)$. Moreover,

$$(1-|\alpha|) \asymp (1-|z|), \quad \forall z \in \Delta(\alpha, \eta), \tag{1.6}$$

and

$$m(\Delta(\alpha, \eta)) \asymp (1-|\alpha|)^2. \tag{1.7}$$

The constants involved in (1.6) and (1.7) depend on η. See [15] for more information about the pseudohyperbolic metric in $\mathbb{D}$ and its properties.

However, the characterization of M_g as a bounded below operator on A^p is actually a consequence of the main result of [19] according to which, condition (1.4) is equivalent to a reverse Carleson measure type inequality for Bergman spaces. The following is presented as "Main Theorem" in [19, p. 2].

Theorem A *Let G be a measurable subset of $\mathbb{D}$ and $p > 0$. There is a constant $K > 0$ such that*

$$\int_G |f(z)|^p \, dm(z) \geq K \int_{\mathbb{D}} |f(z)|^p \, dm(z), \qquad f \in A^p \tag{1.8}$$

if and only if there is a $\eta \in (0,1)$ and a $\delta > 0$ such that (1.4) is true.

Employing the Bergman space norm, stated in terms of the derivative as in (1.3), and Theorem A we can also answer (1.1) for a bounded S_g in $\mathbb{X} = A^p/\mathbb{C}$ and for a bounded T_g in $\mathbb{X} = A^p$. We answer the problem in terms of condition (1.4) on the sets (1.5) and on the sets

$$G_c = \{z \in \mathbb{D} : |g'(z)| (1-|z|) > c\}$$

respectively. We recall that S_g is bounded on A^p if and only if $g \in H^\infty$. On the other hand, T_g is bounded if and only if the symbol g belongs to the Bloch space that is

$$\|f\|_{\mathcal{B}} = |g(0)| + \sup_{z\in\mathbb{D}} |g'(z)|\,(1-|z|^2) < \infty$$

(see [6]).

Anderson, in [7], proved that only the constant symbols introduce a bounded below operator T_g in the case of the Bloch, the BMOA and in that of the Hardy spaces H^p. The BMOA consists of those $f \in \mathcal{B}$ that have boundary values almost everywhere in the unit circle which define a square-integrable function on the unit circle with bounded mean oscillation. We say that an $f \in \mathcal{H}(\mathbb{D})$ belongs to H^p, $1 \le p < \infty$, if it has boundary values almost everywhere in the unit circle and these define a p-integrable function on the unit circle or equivalently if

$$\|f\|_{H^p}^p = \sup_{r\in(0,1)} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta < \infty.$$

In the theory of spaces of analytic functions, the Bloch and the BMOA space stand as the limit case of Bergman and Hardy spaces respectively when p grows to infinity. More information about the Bloch and the BMOA space can be found in [16, 27] and for the Hardy spaces in [14].

Based on the above, someone may come down to the conclusion that (1.1) for S_g, when acting on the Bloch, the BMOA and the Hardy spaces, gets trivial answer too. On the contrary, Anderson in [7] proves that this not the case when $\mathbb{X}$ is the Hilbert Hardy space H^2. Due to the fact that H^2 can be identified equivalently in terms of the Littlewood-Paley identity as

$$\|f\|_{H^2}^2 \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\,(1-|z|)\,dm(z) < \infty\,,$$

the author is able to apply the technique used in the case of Bergman spaces and completely answer the problem as follows. Let a $g \in H^\infty$, that is, consider S_g to be bounded in $H^2/\mathbb{C}$. The condition (1.4) on the sets (1.5) is necessary and sufficient in order for S_g to be bounded below in $H^2/\mathbb{C}$.

Looking carefully the result of the H^2 space and that of Bergman spaces, we can immediately state that S_g is bounded below in $H^2/\mathbb{C}$ if and only if S_g and M_g are bounded below in $A^p/\mathbb{C}$ and A^p, respectively. However, it is very interesting the following fact. Saying that S_g is bounded below in $H^2/\mathbb{C}$ is not equivalent to the boundedness from below of M_g in H^2. The latter is true if and only if the radial limit function of $g \in H^\infty$ is essentially bounded away from 0 on $\mathbb{T}$, which is a weaker condition (see [7, p. 93]).

Anderson goes further. He precisely characterizes the bounded functions under discussion. The functions $g \in H^\infty$, for which (1.1) is true for S_g in $H^2/\mathbb{C}$, are exactly those that can be factored as

$$g = B \cdot F,$$

where B is a finite product of interpolating Blaschke products and F, $1/F \in H^\infty$. In its turn this is equivalent to the existence of an $r > 0$ and a $\eta \in (0,1)$ such that

$$\sup_{z\in\Delta(\alpha,\eta)} |g(z)| > r\,, \qquad \forall \alpha \in \mathbb{D}.$$

Finally, he establishes that the boundedness from below of S_g in the $\mathcal{B}/\mathbb{C}$ is characterized by the same symbols g like those in the Hardy case (see [7, Theorem 3.9]).

Recently, Panteris extended the result of Anderson about S_g on $H^2/\mathbb{C}$ to any Hardy space $H^p/\mathbb{C}$, $p > 1$ [22]. His approach is based on the use of the following equivalent characterization of H^p due to Calderon (see [23, Theorem 1.3]):

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)} |f'(z)|^2 \, dm(z) \right)^{p/2} |d\xi| ,$$

where the symbol $\mathbb{T}$ stands for the unit circle and $\Gamma_\beta(\xi)$ is the cone-like region

$$\Gamma_\beta(\xi) = \{z \in \mathbb{D} : |z| < \beta\} \cup \bigcup_{|z|<\beta} [z, \xi), \quad \beta \in (0, 1),$$

with vertex at $\xi \in \mathbb{T}$. Panteris arrived to the same conclusion as Anderson using appropriately this tent space expression. He also deals with the case $\mathbb{X}$= BMOA and that of the Besov spaces.

Our aim is to extend the study of property (1.1) for T_g and S_g to the more general setting of the $RM(p, q)$ spaces, $1 \le p, q < \infty$. The first author, M. Contreras and L. Rodríguez-Piazza, motivated by the property of bounded radial integrability

$$\sup_{\theta\in[0,2\pi]} \int_0^1 |f(re^{i\theta})| \, dr < \infty, \tag{1.9}$$

introduced in [2] the spaces of average radial integrability $RM(p, q)$.

Let $1 \le p, q < \infty$, we say that an $f \in \mathcal{H}(\mathbb{D})$ belongs to $RM(p, q)$ if

$$\|f\|_{RM(p,q)}^q = \frac{1}{2\pi} \int_0^{2\pi} \left(\int_0^1 |f(re^{i\theta})|^p \, dr \right)^{q/p} d\theta < \infty.$$

Among them we can identify the Bergman spaces when $p = q$. This is justified only by a change in the order of integration. When $p \ne q$, an application of Holder's inequality leads to the conclusion that each $RM(p, q)$ space is located in between two Bergman spaces. Furthermore, when $p = \infty$, the $RM(\infty, q)$ spaces are the Hardy spaces H^q. Condition (1.9), in terms of the notation used for the spaces of average radial integrability, can be visualized as the limit case $RM(1, \infty)$ and, according to Fejer-Riesz inequality, (1.9) is satisfied by all $f \in H^1$ (see [14]). In [2] it is proved that $H^s \subseteq RM(p, q)$, $1 \le p, q < \infty$ if and only if $1/s \le 1/p + 1/q$. On the other hand, the $RM(p, q)$ spaces can be identified to the analytic tent spaces

$$AT_p^q = \mathcal{H}(\mathbb{D}) \cap T_p^q ,$$

for $1 \le p, q < \infty$ (see [4]). The tent spaces T_p^q, $1 \le p, q < \infty$, consist of those measurable functions f in $\mathbb{D}$ such that

$$\|f\|_{T_p^q}^q = \int_{\mathbb{T}} \left(\int_{\Gamma_{1/2}(\xi)} |f(z)|^p \, \frac{dm(z)}{1 - |z|} \right)^{q/p} |d\xi| < \infty.$$

Taking into account all the above, it is natural to consider problem (1.1) for the S_g, T_g and M_g operators when acting on the $RM(p, q)$ spaces. The key, in order to answer this question, is the following analogue of (1.8) for the AT_p^q spaces. This is our main result.

Theorem 1.1 *Let* $1 \le p, q < +\infty$ *and* $G \subset \mathbb{D}$ *be a measurable set. The following assertions are equivalent:*

(a) There is a $\beta \in (0,1)$ *and a constant* $K > 0$ *such that*

$$\left(\int_{\mathbb{T}}\left(\int_{\Gamma_\beta(\xi)\cap G} |f(z)|^p \frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\right)^{1/q} \geq K\left(\int_{\mathbb{T}}\left(\int_{\Gamma_{1/2}(\xi)} |f(z)|^p \frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\right)^{1/q},$$

for all $f \in AT^q_p$.

(b) There exist an $\eta \in (0,1)$ *and a* $\delta > 0$ *such that*

$$m(G \cap \Delta(\alpha,\eta)) \geq \delta\, m(\Delta(\alpha,\eta))$$

for all $\alpha \in \mathbb{D}$.

Now, we are in position to characterize the closed range operators T_g, S_g and M_g as follows.

Theorem 1.2 *Let* $1 \leq p, q < \infty$ *and* $g \in \mathcal{B}$. *The operator* $T_g : RM(p,q) \to RM(p,q)$ *has closed range if and only if there is an* $\eta \in (0,1)$, *a* $\delta > 0$ *and a* $c > 0$ *such that*

$$m(G_c \cap \Delta(\alpha,\eta)) \geq \delta m(\Delta(\alpha,\eta)), \quad \alpha \in \mathbb{D}, \tag{1.10}$$

where $G_c = \{z \in \mathbb{D} : |g'(z)|(1-|z|) > c\}$.

Theorem 1.3 *Let* $1 \leq p, q < \infty$ *and* $g \in H^\infty$. *The following are equivalent:*

(1) $M_g : RM(p,q) \to RM(p,q)$ *has closed range.*
(2) $S_g : RM(p,q)/\mathbb{C} \to RM(p,q)/\mathbb{C}$ *has closed range.*
(3) There is an $\eta \in (0,1)$, *a* $\delta > 0$ *and a* $c > 0$ *such that*

$$m(G_c \cap \Delta(\alpha,\eta)) \geq \delta\, m(\Delta(\alpha,\eta)), \quad \alpha \in \mathbb{D},$$

where $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$.

Notice that the symbols $g \in H^\infty$ of Theorem 1.3 are explicitly understood by the work of Anderson. In addition, Anderson proves that the symbols $g \in BMOA$ that introduce a closed range T_g on the Hardy spaces are exactly the constant functions. However, the constant functions are not among the symbols that introduce a closed range T_g on the $RM(p,q)$ spaces due to Theorem 1.2. As far as we know, there are no examples of Bloch functions that fulfill condition (1.10). That being the case, we provide for the first time non trivial examples of functions $g \in \mathcal{B}$ that serve as symbols of a closed range T_g in the $RM(p,q)$ spaces. Consequently, these symbols are valid for the case of Bergman spaces too.

Let a $g \in \mathcal{H}(\mathbb{D})$ defined in terms of lacunary series, that is

$$g(z) = \sum_{k=0}^{\infty} a_k z^{n_k}, \quad \frac{n_{k+1}}{n_k} \geq \lambda > 1.$$

Recall that g belongs to the Bloch space if and only if $\{a_k\} \in \ell^\infty$ (see [8, Lemma 2.1]). We are able to prove the following

Proposition 1.4 *Let* $1 \leq p, q < \infty$. *There are* $g \in \mathcal{B}$ *defined in terms of lacunary series that introduce a closed range operator* $T_g : RM(p,q) \to RM(p,q)$.

On the other hand, we proved that the univalent Bloch functions do not satisfy the condition (1.10).

This work has the following structure. The first part of Sect. 2 is about the definition of $RM(p,q)$, AT_p^q spaces and their connection. We also present equivalent expressions in terms of the derivative. These expressions are known in the literature as Littlewood-Paley type inequalities. In the rest of the section we discuss the background related to the action of T_g, S_g and M_g on the $RM(p,q)$. In Sect. 3, we introduce the auxiliary lemmas needed and we present the proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3. The last section is devoted to the proof of Proposition 1.4 and to the presentation of the argument that Bloch univalent functions do not introduce a closed range T_g.

2 Definitions and first properties

In this section we recall the definition of the spaces of analytic functions under discussion. Moreover, we compile some properties for the sake of being self-contained.

2.1 Average radial integrability spaces

These are the $RM(p,q)$ spaces introduced in [2]. Although we are interested for their Banach space version, below we present their definition for the full range of the values of p, q.

Definition 2.1 Let $0 < p, q \le +\infty$. We define the spaces of analytic functions

$$RM(p,q) = \{f \in \mathcal{H}(\mathbb{D}) : \rho_{p,q}(f) < +\infty\}$$

where

$$\rho_{p,q}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^1 |f(re^{i\theta})|^p\,dr\right)^{q/p} d\theta\right)^{1/q}, \quad \text{if } p, q < +\infty,$$

$$\rho_{p,\infty}(f) = \operatorname*{ess\,sup}_{t\in[0,2\pi)}\left(\int_0^1 |f(re^{i\theta})|^p\,dr\right)^{1/p}, \quad \text{if } p < +\infty,$$

$$\rho_{\infty,q}(f) = \left(\frac{1}{2\pi}\int_0^{2\pi}\left(\sup_{r\in[0,1)} |f(re^{i\theta})|\right)^{q} d\theta\right)^{1/q}, \quad \text{if } q < +\infty,$$

$$\rho_{\infty,\infty}(f) = \|f\|_{H^\infty} = \sup_{z\in\mathbb{D}}|f(z)|.$$

Our interest is restricted to the range $1 \le p, q < \infty$. These $RM(p,q)$ are Banach spaces under the norm

$$\|f\|_{RM(p,q)} = \rho_{p,q}(f).$$

Notice that, when $p = q$

$$RM(p,p) \equiv A^p.$$

A change in the order of integration is enough to prove it. In any other case, an appropriate application of Holder's inequality results to the containments

$$A^p \subset RM(p,q) \subset A^q, \quad q < p$$

or

$$A^q \subset RM(p,q) \subset A^p, \quad p < q.$$

The inclusions are strict.

The case $q = \infty$ stands for the bounded radial p-integrability of f. It makes no difference if instead of the essential supremum we use the supremum.

It is worth of mentioning that in the case $p = \infty$ we have

$$RM(\infty, q) \equiv H^q$$

due to the fact that the membership of an f in H^q can be equivalently described by the q-integrability over $[0, 2\pi)$ of the radial maximal function

$$Rf(e^{i\theta}) = \sup_{r\in[0,1)} |f(re^{i\theta})|$$

of the function f (see [14]).

Closing, we recall from [2] the pointwise growth estimates

$$|f(z)| \leq C_1 \frac{\|f\|_{RM(p,q)}}{(1-|z|)^{1/p+1/q}}, \quad z \in \mathbb{D}$$

and

$$|f'(z)| \leq C_2 \frac{\|f\|_{RM(p,q)}}{(1-|z|)^{1/p+1/q+1}}, \quad z \in \mathbb{D} \tag{2.1}$$

of the $RM(p,q)$, $1 \leq p, q < \infty$. The C_1, C_2 are positive constants independent of f and z.

2.2 Tent spaces

The work of Coifman, Meyer and Stein is considered as the starting point of the study of tent spaces [12]. Since then, they have been widely studied by many authors (see, e.g., [9, 11, 17, 18, 20, 21, 24]).

Let a $\xi \in \mathbb{T}$. We define the cone-like region

$$\Gamma_{1/2}(\xi) = \left\{z \in \mathbb{D} : |z| < 1/2\right\} \cup \bigcup_{|z|<1/2} [z, \xi).$$

Definition 2.2 Let $0 < p, q < +\infty$. The tent spaces T_p^q consist of measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{T_p^q} = \left\{\int_{\mathbb{T}} \left(\int_{\Gamma_{1/2}(\xi)} |f(z)|^p \frac{dA(z)}{(1-|z|^2)}\right)^{q/p} |d\xi|\right\}^{1/q} < +\infty. \tag{2.2}$$

It is of great importance that in (2.2) we can use any of the cone-like regions

$$\Gamma_\beta(\xi) = \{z \in \mathbb{D} : |z| < \beta\} \cup \bigcup_{|z|<\beta} [z, \zeta)\,, \quad \beta \in (0, 1).$$

In other words, we have

$$\|f\|_{T_p^q} \asymp \left\{ \int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)} |f(z)|^p \, \frac{dA(z)}{(1-|z|^2)} \right)^{q/p} |d\xi| \right\}^{1/q} .$$

Actually, we can use any of the non-tangential regions

$$\Gamma_M(\xi) = \left\{ z \in \mathbb{D} : |z - \xi| < M(1-|z|^2) \right\}, \quad M > \frac{1}{2}$$

as well. This is true due the following technical lemma, well known to the experts of the area. The symbol $\Gamma_C(\xi)$ stands for any of the $\Gamma_\beta(\xi)$ or $\Gamma_M(\xi)$.

Lemma 2.3 [9, Lemma 4, p. 66] *Let* $0 < p, q < +\infty$, $\lambda > \max\{1, p/q\}$ *and* μ *be a positive Borel measure on* $\mathbb{D}$. *There are constants* $K_i = K_i(p, q, \lambda, C)$, $i = 1, 2$ *such that*

$$K_1 \int_{\mathbb{T}} \mu(\Gamma_C(\xi))^{q/p} |d\xi| \le \int_{\mathbb{T}} \left(\int_{\mathbb{D}} \left(\frac{1-|z|}{|1-z\bar{\xi}|} \right)^{\lambda} d\mu(z) \right)^{q/p} |d\xi| \le K_2 \int_{\mathbb{T}} \mu(\Gamma_C(\xi))^{q/p} |d\xi| \,.$$

As in the case of the $RM(p, q)$, the tent spaces are defined for the limit values of the p, q as well. The T_∞^q consists of measurable functions f on $\mathbb{D}$ with

$$\|f\|_{T_\infty^q} = \left\{ \int_{\mathbb{T}} (\operatorname*{ess\,sup}_{z\in\Gamma_{1/2}(\xi)} |f(z)|)^q \, |d\xi| \right\}^{1/q} , \quad \text{if } q < +\infty \,.$$

It is known that the definition is independent of the type of the non-tangential region we use.

When $q = +\infty$ and $p < +\infty$, the tent space T_p^∞ consists of measurable functions f on $\mathbb{D}$ with

$$\|f\|_{T_p^\infty} = \sup_{\xi\in\mathbb{T}} \left(\sup_{w\in\Gamma(\xi)} \frac{1}{(1-|w|^2)} \int_{S(w)} |f(z)|^p \, dA(z) \right)^{1/p} < +\infty, \tag{2.3}$$

where

$$S(re^{i\theta}) = \left\{ \rho e^{it} : 1-\rho \le 1-r,\ |t-\theta| \le \frac{1-r}{2} \right\}$$

for $re^{i\theta} \in \mathbb{D} \setminus \{0\}$ and $S(0) = \mathbb{D}$. An equivalent way to state (2.3) is to say that the measure

$$d\mu(z) = |f(z)|^p \, dm(z)$$

is a Carleson measure. Recall that a positive Borel measure μ in $\mathbb{D}$ is called Carleson if and only if

$$\sup_{I\subset\mathbb{T}} \frac{\mu(S(I))}{|I|} < \infty \ .$$

The supremum is taken over all arcs I of the unit circle, $|I|$ is the arc length and

$$S(I) = \left\{ z \in \mathbb{D} :\ 1-|I| \le\ |z| < 1,\ \frac{z}{|z|} \in I \right\}.$$

Here, we are interested in the holomorphic version of the tent spaces. We denote that as

$$AT_p^q = T_p^q \cap \mathcal{H}(\mathbb{D}).$$

The limit case $p = \infty$ corresponds to the Hardy spaces, that is, $AT_\infty^q \equiv H^q$, since the membership of an $f \in H^q$ can be determined by the q-integrability over $[0, 2\pi)$ of the non-tangential maximal function

$$N_C f(e^{i\theta}) = \operatorname*{ess\,sup}_{z \in \Gamma_C(e^{i\theta})} |f(z)|$$

of the function f (see [14]).

2.3 Integration Operators on the RM(p,q) spaces

Let $g \in \mathcal{H}(\mathbb{D})$ and the integration operator

$$T_g(f)(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta\,, \qquad z \in \mathbb{D},$$

acting on $f \in \mathcal{H}(\mathbb{D})$. Pommerenke was the first who considered T_g on spaces of analytic functions by proving that it is bounded on H^2 if and only if $g \in BMOA$ [25]. Aleman and Siskakis proved in [5] that the same is true for any H^p, $1 \leq p < \infty$. Later on, in [6], they confronted the question of the boundedness on the Bergman spaces A^p, $1 \leq p < \infty$, by establishing the membership of the g in the Bloch space $\mathcal{B}$ as necessary and sufficient condition. Hardly speaking, the key for the proof of the sufficiency in any of these spaces is the use of an expression of the norm based on the derivative of f. Necessity comes out using an appropriate family of test functions.

In [3], among other things, the authors consider the problem of the boundedness of T_g in the $RM(p, q)$ spaces. Therefore, first they establish that the norm associated to the spaces can be described equivalently in terms of the derivative. This is what they call Littlewood-Paley type inequalities. Precisely,

$$\rho_{p,q}(f) \simeq |f(0)| + \rho_{p,q}(f'(z)(1 - |z|))\,, \quad 1 \leq p, q < \infty. \tag{2.4}$$

That being the case, they follow the procedure described above in order to prove that

$$T_g : RM(p, q) \to RM(p, q), \quad 1 \leq p, q < \infty$$

boundedly if and only if $g \in \mathcal{B}$.

It is well known among the experts of the area that the boundedness of

$$S_g(f)(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta\,, \qquad z \in \mathbb{D},$$

on Hardy and Bergman spaces is characterized by the condition $g \in H^\infty$. This is the case for the $RM(p, q)$, $1 \leq p, q < \infty$, too. However, this operator is not considered in [3]. Therefore we present the proof.

Proposition 2.4 *Let $1 \leq p, q < \infty$ and $g \in \mathcal{H}(\mathbb{D})$. The operator S_g is bounded in the $RM(p, q)$ spaces if and only if $g \in H^\infty$.*

Proof We begin with the assumption that $g \in H^\infty$ and we recall the (2.4). Then

$$\|S_g(f)\|_{RM(p,q)} = \rho_{p,q}(S_g(f)) \asymp |S_g(f)(0)| + \rho_{p,q}(S_g(f)'(z)(1 - |z|))$$

$$
\begin{aligned}
&= \rho_{p,q}(S_g(f)'(z)(1-|z|)) \\
&= \left(\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^1 |f'(re^{i\theta})|^p\,|g(re^{i\theta})|^p\,(1-r)^p\,dr\right)^{q/p} d\theta\right)^{1/q} \\
&\le \|g\|_{H^\infty}\left(\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^1 |f'(re^{i\theta})|^p\,(1-r)^p\,dr\right)^{q/p} d\theta\right)^{1/q} \\
&\lesssim \|g\|_{H^\infty}\,\|f\|_{RM(p,q)}\,,
\end{aligned}
$$

for every $f \in RM(p,q)$. So, the sufficiency is proved.

For the necessity we consider the operator bounded. So, there exists a constant $C > 0$ such that

$$\|S_g(f)\|_{RM(p,q)} \le C\|f\|_{RM(p,q)}\,, \quad f \in RM(p,q).$$

Combining that with (2.1) we get

$$|S_g(f)'(z)| \le C'\,\frac{\|f\|_{RM(p,q)}}{(1-|z|)^{1/p+1/q+1}}, \quad f \in RM(p,q),$$

where C' is a positive constant. Now, we employ the test functions

$$f_\alpha(z) = \frac{(1-|\alpha|)^{\gamma-1/p-1/q}}{(1-\bar{\alpha}z)^\gamma}, \quad \gamma > 1/p+1/q,\ \alpha \in \mathbb{D},\ z \in \mathbb{D}.$$

They have the property $\sup_\alpha \|f_\alpha\|_{RM(p,q)} < \infty$ (see [1]). As a consequence, for any $z \in \mathbb{D}$,

$$|f_\alpha'(z)|\,|g(z)| \lesssim \frac{1}{(1-|z|)^{1/p+1/q+1}}$$

for every $\alpha \in \mathbb{D}$. Choosing $z = \alpha$ implies that $g \in H^\infty$. □

Remember that M_g, T_g and S_g are connected as in (1.2) and that $H^\infty \subset \mathcal{B}$. Therefore, without to much effort, we can check that M_g is bounded in the $RM(p,q)$ if and only if $g \in H^\infty$.

As it is stated in the introduction, our concern is the characterization of T_g, S_g and M_g as bounded below operators when acting boundedly on the $RM(p,q)$. We are able to accomplish this task by studying the problem in the tent space version of the $RM(p,q)$. We claim that this is possible since, as it is established in [4], for $1 \le p, q < \infty$

$$
\begin{aligned}
\rho_{p,q}(f) &\asymp \|f\|_{T_p^q}\,, \\
\rho_{p,q}(f'(z)(1-|z|)) &\asymp \|f'(z)(1-|z|)\|_{T_p^q}
\end{aligned}
$$

and due to the Littlewood-Paley type inequalities

$$\|f\|_{T_p^q} \simeq \|f'(z)(1-|z|)\|_{T_p^q}\,, \quad 1 \le p, q < \infty$$

for every $f \in AT_p^q$ (see [24, Theorem 2, p. 9]).

3 Reverse Carleson type measures for the AT_p^q, $1 \le p, q < \infty$

In this section we prove our main result, Theorem 1.1, and its consequences. In the statement presented in the introduction we only make use of pseudohyperbolic discs. However, due to the technicalities we have to deal with, we need the euclidean discs

$$\Delta_\eta(\alpha) = \{ z \in \mathbb{D} : |z - \alpha| < \eta(1 - |\alpha|) \}, \quad \eta \in (0, 1), \alpha \in \mathbb{D},$$

as well. Notice the following remark.

Remark 3.1 It is known (see [19, p. 4]) that if $\eta \in (0, 1)$, $\alpha \in \mathbb{D}$ and $\frac{2\eta}{1+\eta^2} \le r < 1$ then

$$\Delta_\eta(\alpha) \subseteq \Delta(\alpha, r).$$

In addition to that, if $\eta \in (0, 1)$ then there is a $\beta \in (1/2, 1)$ such that

$$\cup_{\alpha \in \Gamma_{1/2}(\xi)} \Delta_\eta(\alpha) \subseteq \Gamma_\beta(\xi)$$

where $\xi \in \mathbb{T}$ is the same for both regions. This is the region $\Gamma_\beta(\xi)$ considered in the following two Lemmas.

Lemma 3.2 *Let* $1 \le p < \infty$, $\eta \in (0, 1)$, $\varepsilon > 0$, $f \in \mathcal{H}(\mathbb{D})$, *and*

$$\mathcal{A} = \left\{ \alpha \in \mathbb{D} : |f(\alpha)|^p < \frac{\varepsilon}{m(\Delta_\eta(\alpha))} \int_{\Delta_\eta(\alpha)} |f(z)|^p \, dm(z) \right\}.$$

Then, there is a constant $C_1 = C_1(\eta) > 0$ *such that*

$$\int_{\mathcal{A} \cap \Gamma_{1/2}(\xi)} |f(z)|^p \frac{dm(z)}{1 - |z|} \le \varepsilon C_1 \int_{\Gamma_\beta(\xi)} |f(z)|^p \frac{dm(z)}{1 - |z|},$$

for a non tangential region $\Gamma_\beta(\xi)$ *with vertex at the same point* $\xi \in \mathbb{T}$ *as* $\Gamma_{1/2}(\xi)$.

Proof If $\alpha \in \mathcal{A}$ then

$$|f(\alpha)|^p < \frac{\varepsilon}{m(\Delta_\eta(\alpha))} \int_{\Delta_\eta(\alpha)} |f(z)|^p \, dm(z) \, .$$

Hence, integrating both sides over $\mathcal{A} \cap \Gamma_{1/2}(\xi)$ and considering a $\Gamma_\beta(\xi)$ as in Remark 3.1

$$\begin{aligned} \int_{\mathcal{A} \cap \Gamma_{1/2}(\xi)} |f(\alpha)|^p \frac{dm(\alpha)}{1 - |\alpha|} &< \varepsilon \int_{\mathcal{A} \cap \Gamma_{1/2}(\xi)} \left(\frac{1}{m(\Delta_\eta(\alpha))} \int_{\Delta_\eta(\alpha)} |f(z)|^p \, dm(z) \right) \frac{dm(\alpha)}{1 - |\alpha|} \\ &\le \varepsilon \int_{\mathcal{A} \cap \Gamma_{1/2}(\xi)} \left(\frac{1}{m(\Delta_\eta(\alpha))} \int_{\Gamma_\beta(\xi)} |f(z)|^p \chi_{\Delta_\eta(\alpha)}(z) \, dm(z) \right) \frac{dm(\alpha)}{1 - |\alpha|} . \end{aligned}$$

Remark 3.1 also implies that $\chi_{\Delta_\eta(\alpha)}(z) \le \chi_{\Delta(\alpha,r)}(z)$ when $\frac{2\eta}{1+\eta^2} \le r < 1$. Therefore, applying Fubini's theorem

$$\int_{\mathcal{A} \cap \Gamma_{1/2}(\xi)} |f(\alpha)|^p \frac{dm(\alpha)}{1 - |\alpha|} < \varepsilon \int_{\Gamma_\beta(\xi)} |f(z)|^p \int_{\mathcal{A} \cap \Gamma_{1/2}(\xi)} \left(\frac{\chi_{\Delta(z,r)}(\alpha)}{m(\Delta_\eta(\alpha))} \right) \frac{dm(\alpha)}{1 - |\alpha|} \, dm(z) .$$

Remember that

$$1 - |\alpha| \asymp 1 - |z| , \quad \alpha \in \Delta(z, r)$$

and

$$m(\Delta_\eta(\alpha)) \asymp (1 - |\alpha|)^2 \asymp m(\Delta(z, r)) , \quad \alpha \in \Delta(z, r),$$

where the constants involved depend on η. Thus, we obtain that

$$\int_{\mathcal{A}\cap\Gamma_{1/2}(\xi)} |f(\alpha)|^p \, \frac{dm(\alpha)}{1-|\alpha|} \leq C_1 \, \varepsilon \int_{\Gamma_\beta(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} \, .$$

□

Let $1 \leq p < \infty$ and $\lambda \in (0,1)$. We define the set

$$E_\lambda(\alpha) = \left\{ z \in \Delta_\eta(\alpha) \ : \ |f(z)|^p \geq \lambda |f(\alpha)|^p \right\}$$

and the function

$$B_\lambda f(\alpha) = \frac{1}{m(E_\lambda(\alpha))} \int_{E_\lambda(\alpha)} |f(z)|^p \, dm(z) \, , \qquad \alpha \in \mathbb{D} \, .$$

The following is the analogue of Lemma 3.2 for $B_\lambda f$.

Lemma 3.3 *If* $p \geq 1$, $\varepsilon \in (0,1)$, $f \in \mathcal{H}(\mathbb{D})$, $\lambda \in (0, \frac{1}{2^p})$, *and*

$$B = \left\{ \alpha \in \mathbb{D} \ : |f(\alpha)|^p < \varepsilon^{2+\frac{2}{p}} B_\lambda(f)(\alpha) \right\} .$$

Then, there is a constant $C_2 = C_2(\eta) > 0$ *such that*

$$\int_{B\cap\Gamma_{1/2}(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} \leq \varepsilon \, C_2 \int_{\Gamma_\beta(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} \ ,$$

for a non tangential region $\Gamma_\beta(\xi)$ *with vertex at the same* $\xi \in \mathbb{T}$ *as* $\Gamma_{1/2}(\xi)$.

Proof Applying Lemma 3.2

$$\begin{aligned} \int_{B\cap\Gamma_{1/2}(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} &= \int_{B\cap\Gamma_{1/2}(\xi)\cap\mathcal{A}} |f(z)|^p \, \frac{dm(z)}{1-|z|} + \int_{(B\cap\Gamma_{1/2}(\xi))\setminus\mathcal{A}} |f(z)|^p \, \frac{dm(z)}{1-|z|} \\ &\leq C_1(\eta) \, \varepsilon \int_{\Gamma_\beta(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} + \int_{(B\cap\Gamma_{1/2}(\xi))\setminus\mathcal{A}} |f(z)|^p \, \frac{dm(z)}{1-|z|}. \end{aligned}$$

We are looking for a similar ε-estimate of the second additive. If $\alpha \in B$, then

$$|f(\alpha)|^p \ < \ \frac{\varepsilon^{2+\frac{2}{p}}}{m(E_\lambda(\alpha))} \int_{E_\lambda(\alpha)} |f(z)|^p \, dm(z) \, .$$

So, integrating both sides over $(B \cap \Gamma_{1/2}(\xi))\setminus\mathcal{A}$, we get

$$\int_{(B\cap\Gamma_{1/2}(\xi))\setminus\mathcal{A}} |f(\alpha)|^p \, \frac{dm(\alpha)}{1-|\alpha|} < \varepsilon^{2+\frac{2}{p}} \int_{(B\cap\Gamma_{1/2}(\xi))\setminus\mathcal{A}} \frac{1}{m(E_\lambda(\alpha))} \int_{E_\lambda(\alpha)} |f(z)|^p \, dm(z) \, \frac{dm(\alpha)}{1-|\alpha|} \ .$$

Assume that

$$\left\{ z \in \mathbb{D} : \ |z-\alpha| < \frac{\varepsilon^{1/p}\eta(1-|\alpha|)}{2C} \right\} \subset E_\lambda(\alpha) \, , \quad \forall \alpha \notin \mathcal{A}, \tag{3.1}$$

where C is a positive constant to be determined later, then

$$m(E_\lambda(\alpha)) \geq \frac{\varepsilon^{2/p}}{4C^2} \, m(\Delta_\eta(\alpha)) \ , \quad \forall \alpha \notin \mathcal{A} \, .$$

Thus,

$$\begin{aligned}
\int_{(B\cap\Gamma_{1/2}(\xi))\setminus\mathcal{A}} &|f(\alpha)|^p \,\frac{dm(\alpha)}{1-|\alpha|} \\
&\le 4C^2\,\varepsilon^2 \int_{(B\cap\Gamma_{1/2}(\xi))\setminus\mathcal{A}} \frac{1}{m(\Delta_\eta(\alpha))}\int_{E_\lambda(\alpha)} |f(z)|^p\,dm(z)\,\frac{dm(\alpha)}{1-|\alpha|} \\
&\le 4C^2\,\varepsilon^2 \int_{(B\cap\Gamma_{1/2}(\xi))\setminus\mathcal{A}} \frac{1}{m(\Delta_\eta(\alpha))}\int_{\Delta_\eta(\alpha)} |f(z)|^p\,dm(z)\,\frac{dm(\alpha)}{1-|\alpha|} \\
&\le 4C^2\,\varepsilon \int_{(B\cap\Gamma_{1/2}(\xi))\setminus\mathcal{A}} |f(\alpha)|^p\,\frac{dm(\alpha)}{1-|\alpha|} \\
&\le 4C^2\,\varepsilon \int_{\Gamma_\beta(\xi)} |f(\alpha)|^p\,\frac{dm(\alpha)}{1-|\alpha|}
\end{aligned}$$

and the lemma is proved.

To complete the picture, we proceed to the proof of inclusion (3.1). Let $\alpha\in\mathbb{D}$ and $z\in\Delta_{\frac{\eta}{4}}(\alpha)$. By Cauchy integral formula, we have

$$\begin{aligned}
|f(z)-f(\alpha)| &= \frac{1}{2\pi}\left|\int_{|w-\alpha|=\frac{\eta}{2}(1-|\alpha|)} f(w)\left(\frac{1}{w-z}-\frac{1}{w-\alpha}\right)dw\right| \\
&\le \frac{1}{2\pi}\int_{|w-\alpha|=\frac{\eta}{2}(1-|\alpha|)} |f(w)|\left|\frac{z-\alpha}{(w-z)(w-\alpha)}\right|\,|dw|\,. \qquad (3.2)
\end{aligned}$$

By subhamonicity, for $w\in\overline{\Delta_{\frac{\eta}{2}}(\alpha)}$, we have

$$|f(w)|\le \frac{C'}{m(\Delta_\eta(\alpha))}\int_{\Delta_\eta(\alpha)} |f(u)|\,dm(u)$$

where C' is an absolute positive constant. Applying that in (3.2)

$$|f(z)-f(\alpha)|\le \frac{4C'\,|z-\alpha|}{\eta(1-|\alpha|)}\,\frac{1}{m(\Delta_\eta(\alpha))}\int_{\Delta_\eta(\alpha)} |f(u)|\,dm(u)\,.$$

Now, if we consider $z\in\Delta_{\frac{\varepsilon^{\frac{1}{p}}\eta}{2C}}(\alpha)$ for a C large enough, then

$$|f(z)-f(\alpha)|\le \frac{\varepsilon^{\frac{1}{p}}}{2m(\Delta_\eta(\alpha))}\int_{\Delta_\eta(\alpha)} |f(u)|\,dm(u).$$

Therefore, if $\alpha\notin\mathcal{A}$ then it follows that

$$|f(z)|\ge |f(\alpha)|-|f(\alpha)-f(z)|\ge \frac{1}{2}|f(\alpha)|\,.$$

So that,

$$|f(z)|^p\ge \frac{|f(\alpha)|^p}{2^p} > \lambda|f(\alpha)|^p$$

and this implies the strength of inclusion (3.1). □

Remark 3.4 Notice that for $f \in \mathcal{H}(\mathbb{D})$, $\alpha \in \mathbb{D}$ and $\lambda \in (0,1)$, we have that

$$\frac{m(E_\lambda(\alpha))}{m(\Delta_\eta(\alpha))} \geq \frac{\log(1/\lambda)}{\log\left(\frac{B_\lambda(f)(\alpha)}{|f(\alpha)|^p}\right) + \log(1/\lambda)}.$$

Now, if $\alpha \in \mathbb{D} \setminus B$, $\varepsilon \in (0,1)$ and $\lambda \in (0, 1/2^p)$, it follows that

$$\frac{B_\lambda(f)(\alpha)}{|f(\alpha)|^p} \leq \frac{1}{\varepsilon^{2+\frac{2}{p}}}.$$

If $\delta \in (0,1)$, then choosing $\lambda < \varepsilon^{\frac{2}{\delta}(2+\frac{2}{p})}$, we obtain

$$\frac{m(E_\lambda(\alpha))}{m(\Delta_\eta(\alpha))} > \frac{(2/\delta)\log(1/\varepsilon^{2+\frac{2}{p}})}{\log\left(1/\varepsilon^{2+\frac{2}{p}}\right) + (2/\delta)\log\left(1/\varepsilon^{2+\frac{2}{p}}\right)} > 1 - \frac{\delta}{2},$$

that is,

$$m(E_\lambda(\alpha)) > \left(1 - \frac{\delta}{2}\right) m(\Delta_\eta(\alpha)).$$

Now, we state the last auxiliary lemma. In order to prove it we use an argument based on the family of functions $\{f_\alpha\}_{\alpha\in\mathbb{D}} \subset AT_p^q$, $1 \leq p, q < \infty$, of Proposition 2.4. Recall that

$$\|f_\alpha\|_{T_p^q} \asymp 1, \quad \forall \alpha \in \mathbb{D}.$$

Lemma 3.5 *Let* $0 < p, q < \infty$. *Given* $\varepsilon > 0$, *there exist* $\eta \in (0,1)$ *such that for all* $\alpha \in \mathbb{D}$ *there is a function* $f_\alpha \in AT_p^q$ *such that*

(1) $\|f_\alpha\|_{T_p^q} \asymp 1$, *and*
(2) $\|f_\alpha \chi_{\Delta^c(\alpha,\eta)}\|_{T_p^q} \leq \varepsilon$.

Proof Fix $\lambda > \max\{1, p/q\}$. First of all, we consider the family $\{f_\alpha\}$ of the functions

$$f_\alpha(z) = \frac{(1-|\alpha|^2)^{\gamma - \frac{1}{p} - \frac{1}{q}}}{(1-\overline{\alpha}z)^\gamma}, \; z \in \mathbb{D},$$

where $\gamma > \frac{\lambda+2}{p}$. As we have seen previously, these functions satisfies *(1)*. Let us see now that, for a given $\varepsilon > 0$, there is $\eta \in (0,1)$ such that

$$\mathbb{I} = \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)\setminus\Delta(\alpha,\eta)} |f_\alpha(z)|^p \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \leq \varepsilon^q.$$

Observe that

$$\begin{aligned}\mathbb{I} &= \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)\setminus\Delta(\alpha,\eta)} |f_\alpha(z)|^p \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \\ &= \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)} \chi_{\Delta^c(\alpha,\eta)}(z)|f_\alpha(z)|^p \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi|.\end{aligned}$$

Since $\lambda > \max\{1,\ p/q\}$ and $\chi_{\Gamma_{\frac{1}{2}}(\xi)}(z) \lesssim \frac{1-|z|^2}{|1-\bar{\xi}z|}$, we have that

$$\mathbb{I} \lesssim \int_{\mathbb{T}} \left(\int_{\mathbb{D}} \left(\frac{1-|z|^2}{|1-\bar{\xi}z|} \right)^{\lambda} \chi_{\Delta^c(\alpha,\eta)}(z) |f_\alpha(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} \, |d\xi|.$$

Applying in the inner integral the change of variable

$$z \mapsto \phi_\alpha(z),$$

where $\phi_\alpha(z) = \frac{\alpha - z}{1-\overline{\alpha}z}$, we get that

$$\begin{aligned}
\mathbb{I} &\lesssim \int_{\mathbb{T}} \left(\int_{\mathbb{D}\setminus\Delta(0,\eta)} \frac{(1-|\phi_\alpha(z)|^2)^{\lambda-1}}{|1-\bar{\xi}\phi_\alpha(z)|^{\lambda}} |f_\alpha(\phi_\alpha(z))|^p \, |\phi_\alpha'(z)|^2 \, dm(z) \right)^{q/p} \, |d\xi| \\
&= \int_{\mathbb{T}} \frac{1}{|1-\overline{\alpha}\xi|^{\lambda q/p}} \left(\int_{\mathbb{D}\setminus\Delta(0,\eta)} \frac{(1-|\phi_\alpha(z)|^2)^{\lambda-1} \, |1-\overline{\alpha}z|^{\lambda}}{|1-\overline{\phi_\alpha(\xi)}z|^{\lambda}} |f_\alpha(\phi_\alpha(z))|^p \, |\phi_\alpha'(z)|^2 \, dm(z) \right)^{q/p} \, |d\xi|.
\end{aligned}$$

Since

$$1-|\phi_\alpha(z)|^2 = |\phi_\alpha'(z)| \, (1-|z|^2), \quad \alpha, z \in \mathbb{D},$$

$$\mathbb{I} \lesssim \int_{\mathbb{T}} \frac{1}{|1-\overline{\alpha}\xi|^{\lambda q/p}} \left(\int_{\mathbb{D}\setminus\Delta(0,\eta)} \frac{(1-|z|^2)^{\lambda-1} \, |1-\overline{\alpha}z|^{\lambda}}{|1-\overline{\phi_\alpha(\xi)}z|^{\lambda}} |f_\alpha(\phi_\alpha(z))|^p \, |\phi_\alpha'(z)|^{\lambda+1} \, dm(z) \right)^{q/p} \, |d\xi|.$$

Verifying that

$$|1-\overline{\alpha}z|^{\lambda} \, |f_\alpha(\phi_\alpha(z))|^p \, |\phi_\alpha'(z)|^{\lambda+1} \; = \; (1-|\alpha|^2)^{\lambda-\frac{p}{q}} |1-\overline{\alpha}z|^{\gamma p-\lambda-2}$$

we get

$$\mathbb{I} \lesssim (1-|\alpha|^2)^{\lambda\frac{q}{p}-1} \int_{\mathbb{T}} \frac{1}{|1-\overline{\alpha}\xi|^{\lambda q/p}} \left(\int_{\mathbb{D}\setminus\Delta(0,\eta)} \frac{(1-|z|^2)^{\lambda-1}}{|1-\overline{\phi_\alpha(\xi)}z|^{\lambda}} |1-\overline{\alpha}z|^{\gamma p-\lambda-2} \, dm(z) \right)^{q/p} \, |d\xi|.$$

Since $\gamma > \frac{\lambda+2}{p}$,

$$|1-\overline{\alpha}z|^{\gamma p-\lambda-2} \leq 2^{\gamma p-\lambda-2}.$$

Thus,

$$\begin{aligned}
\mathbb{I} &\lesssim (1-|\alpha|^2)^{\lambda\frac{q}{p}-1} \int_{\mathbb{T}} \frac{1}{|1-\overline{\alpha}\xi|^{\lambda q/p}} \left(\int_{\mathbb{D}\setminus\Delta(0,\eta)} \frac{(1-|z|^2)^{\lambda-1}}{|1-\overline{\phi_\alpha(\xi)}z|^{\lambda}} \, dm(z) \right)^{q/p} \, |d\xi| \\
&= (1-|\alpha|^2)^{\lambda\frac{q}{p}-1} \int_{\mathbb{T}} \frac{1}{|1-\overline{\alpha}\xi|^{\lambda q/p}} \left(\int_{\eta}^{1} \frac{1}{\pi} \int_0^{2\pi} \frac{1}{|1-\overline{\phi_\alpha(\xi)}re^{i\theta}|^{\lambda}} \, d\theta \, (1-r^2)^{\lambda-1} \, r \, dr \right)^{q/p} \, |d\xi|.
\end{aligned}$$

Now, a standard argument based on Parseval's identity, Stirling's formula and the fact that $|\phi_\alpha(\xi)| = 1 = |\xi|$ implies

$$\mathbb{I} \lesssim (1-|\alpha|^2)^{\lambda\frac{q}{p}-1} \int_{\mathbb{T}} \frac{1}{|1-\overline{\alpha}\xi|^{\lambda q/p}} \; |d\xi| \, (1-\eta)^{q/p}.$$

Employing the same argument one more time, we obtain that

$$\mathbb{I} \lesssim (1-\eta)^{q/p}.$$

Therefore, we conclude that, for a given $\varepsilon > 0$, there is $\eta \in (0,1)$ such that *(2)* holds. □

Now we are in a position to present the complete version of our main result.

Theorem 3.6 *Let $1 \leq p, q < +\infty$ and G be a measurable subset of $\mathbb{D}$. The following assertions are equivalent:*

(a) There is a $\beta \in (0,1)$ and a constant $K > 0$ such that

$$\left(\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi) \cap G} |f(z)|^p \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{1/q} \geq K \left(\int_{\mathbb{T}} \left(\int_{\Gamma_{1/2}(\xi)} |f(z)|^p \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{1/q},$$

for every $f \in AT_p^q$.

(b) There exist an $\eta \in (0,1)$ and a $\delta > 0$ such that

$$m(G \cap \Delta(\alpha, \eta)) \geq \delta\, m(\Delta(\alpha, \eta))$$

for $\alpha \in \mathbb{D}$.

(c) There exist an $\eta \in (0,1)$ and a $\delta > 0$ such that

$$m(G \cap \Delta_\eta(\alpha)) \geq \delta\, m(\Delta_\eta(\alpha))$$

for $\alpha \in \mathbb{D}$.

Proof *(b) $\Leftrightarrow$ (c)* : The proof is the same as in [19].
(c) $\Rightarrow$ (a) : Assume *(c)* holds. We want to prove that there is a $\beta \in (0,1)$ and a constant $K > 0$ such that

$$\left(\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi) \cap G} |f(z)|^p \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{1/q} \geq K \left(\int_{\mathbb{T}} \left(\int_{\Gamma_{1/2}(\xi)} |f(z)|^p \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{1/q}$$

for all $f \in AT_p^q$.

Consider an $\varepsilon \in (0,1)$ and a $\lambda < \min\{1/2^p, \varepsilon^{\frac{2}{\delta}(2+2/p)}\}$. By Remark 3.4 we obtain that

$$\begin{aligned} m(G \cap E_\lambda(\alpha)) &= m(G \cap \Delta_\eta(\alpha)) - m(G \cap (\Delta_\eta(\alpha) \setminus E_\lambda(\alpha))) \\ &\geq \delta m(\Delta_\eta(\alpha)) - m(\Delta_\eta(\alpha) \setminus E_\lambda(\alpha)) \\ &= \delta m(\Delta_\eta(\alpha)) - m(\Delta_\eta(\alpha)) + m(E_\lambda(\alpha)) \\ &\geq \delta m(\Delta_\eta(\alpha)) - m(\Delta_\eta(\alpha)) + \left(1 - \frac{\delta}{2}\right) m(\Delta_\eta(\alpha)) \\ &= \frac{\delta}{2} m(\Delta_\eta(\alpha)) \end{aligned}$$

for any $\alpha \in \mathbb{D} \setminus B$, where B is the set in Lemma 3.3.

Let a pseudohyperbolic radius $r = r(\eta) \in (0,1)$ as in Remark 3.1. We can find a $\beta \in (1/2, 1)$ such that $\cup_{\alpha \in \Gamma_{1/2}(\xi)} \Delta(\alpha, r) \subset \Gamma_\beta(\xi)$.

If $\alpha \in \Gamma_{1/2}(\xi) \setminus B$ then using the fact that $E_\lambda(f)(\alpha) \subset \Delta_\eta(\alpha) \subset \Gamma_\beta(\xi)$

$$\frac{1}{m(\Delta_\eta(\alpha))} \int_{G \cap \Gamma_\beta(\xi)} \chi_{\Delta_\eta(\alpha)}(z) |f(z)|^p \frac{dm(z)}{1-|z|}$$

$$
\begin{aligned}
&\geq \frac{\delta}{2} \frac{1}{m(G \cap E_\lambda(\alpha))} \int_{G \cap \Gamma_\beta(\xi)} \chi_{\Delta_\eta(\alpha)}(z) |f(z)|^p \frac{dm(z)}{1-|z|} \\
&\geq \frac{\delta}{2} \frac{1}{m(G \cap E_\lambda(\alpha))} \int_{G \cap E_\lambda(\alpha)} \chi_{\Delta_\eta(\alpha)}(z) |f(z)|^p \frac{dm(z)}{1-|z|} \\
&\geq C_3 \frac{\delta\lambda}{2} \frac{|f(\alpha)|^p}{(1-|\alpha|)}\,.
\end{aligned}
$$

The constant C_3 depends on η. Integrating over the set $\Gamma_{1/2}(\xi) \setminus B$ and applying Fubini's theorem, we obtain

$$\int_{G \cap \Gamma_\beta(\xi)} |f(z)|^p \int_{\Gamma_{1/2}(\xi) \setminus B} \frac{\chi_{\Delta(z,r)}(\alpha)}{m(\Delta_\eta(\alpha))}\, dm(\alpha) \frac{dm(z)}{1-|z|} \geq\, C_3\, \frac{\delta\lambda}{2} \int_{\Gamma_{1/2}(\xi) \setminus B} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|}.$$

Using arguments we have seen before, we conclude that there is a positive constant $C_4 = C_4(\eta)$ such that

$$\int_{\Gamma_{1/2}(\xi) \setminus B} \frac{\chi_{\Delta(z,r)}(\alpha)}{m(\Delta(\alpha, \eta))}\, dm(\alpha) \leq C_4$$

for any $z \in \mathbb{D}$. Therefore,

$$\int_{G \cap \Gamma_\beta(\xi)} |f(z)|^p\, \frac{dm(z)}{1-|z|} \geq\, C_5\, \frac{\delta\lambda}{2} \int_{\Gamma_{1/2}(\xi) \setminus B} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|}.$$

Applying Lemma 3.3, it follows that

$$
\begin{aligned}
&\int_{G \cap \Gamma_\beta(\xi)} |f(z)|^p\, \frac{dm(z)}{1-|z|} \\
&\quad \geq\, C_5\, \frac{\delta\lambda}{2} \int_{\Gamma_{1/2}(\xi)} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|} -\, C_5\, \frac{\delta\lambda}{2} \int_{\Gamma_{1/2}(\xi) \cap B} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|} \\
&\quad \geq\, C_5\, \frac{\delta\lambda}{2} \int_{\Gamma_{1/2}(\xi)} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|} -\, C_5\, \frac{\delta\lambda\varepsilon C_2}{2} \int_{\Gamma_\beta(\xi)} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|}\,.
\end{aligned}
$$

So that,

$$
\begin{aligned}
&\left(\int_{G \cap \Gamma_\beta(\xi)} |f(z)|^p\, \frac{dm(z)}{1-|z|}\right)^{1/p} + \left(C_6\, \frac{\delta\lambda\varepsilon}{2}\right)^{1/p} \left(\int_{\Gamma_\beta(\xi)} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|}\right)^{1/p} \\
&\qquad\qquad \geq \left(C_5\, \frac{\delta\lambda}{2}\right)^{1/p} \left(\int_{\Gamma_{1/2}(\xi)} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|}\right)^{1/p}\,.
\end{aligned}
$$

By means of Minkowski's inequality we get that

$$
\begin{aligned}
&\left(\int_{\mathbb{T}} \left(\int_{G \cap \Gamma_\beta(\xi)} |f(z)|^p\, \frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\right)^{1/q} \\
&\quad + \left(C_6\, \frac{\delta\lambda\varepsilon}{2}\right)^{1/p} \left(\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)} |f(\alpha)|^p\, \frac{dm(\alpha)}{1-|\alpha|}\right)^{q/p} |d\xi|\right)^{1/q}
\end{aligned}
$$

$$\geq \left(C_5 \frac{\delta\lambda}{2} \right)^{1/p} \left(\int_{\mathbb{T}} \left(\int_{\Gamma_{1/2}(\xi)} |f(\alpha)|^p \, \frac{dm(\alpha)}{1-|\alpha|} \right)^{q/p} |d\xi| \right)^{1/q}.$$

According to Lemma 2.3, the AT_p^q-norm can be equivalently expressed by any cone-like region. So, if we choose ε small enough, then we end up that there is a constant $K > 0$ such that

$$\left(\int_{\mathbb{T}} \left(\int_{G\cap\Gamma_\beta(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{1/q} \geq K \|f\|_{T_p^q}.$$

(a) $\Rightarrow$ *(b)* : Before starting with the proof, we need to show the following estimate:

$$\left(\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)} \chi_{G\cap\Delta(\alpha,\eta)}(z) |f_\alpha(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{p/q} \lesssim \frac{m(G\cap\Delta(\alpha,\eta))}{m(\Delta(\alpha,\eta))},$$

where f_α is the function considered in Lemma 3.5. Applying Lemma 2.3 and properties of pseudo-hyperbolic disks (see, e.g., (1.6) and (1.7)), it follows that

$$\begin{aligned} &\left(\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)} \chi_{G\cap\Delta(\alpha,\eta)}(z) |f_\alpha(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{p/q} \\ &\quad \asymp \frac{(1-|\alpha|)^{1-p/q}}{m(\Delta(\alpha,\eta))} \left(\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)} \chi_{G\cap\Delta(\alpha,\eta)}(z) \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \right)^{p/q} \\ &\quad \asymp \frac{(1-|\alpha|)^{-p/q}}{m(\Delta(\alpha,\eta))} \left(\int_{\mathbb{T}} \left(\int_{G\cap\Delta(\alpha,\eta)} \frac{(1-|z|)^\lambda}{|1-z\overline{\xi}|^\lambda} \, dm(z) \right)^{q/p} |d\xi| \right)^{p/q}, \end{aligned} \tag{3.3}$$

where $\lambda > \max\{1, p/q\}$. Notice that, by means of rotations, we can assume the pseudo-hyperbolic disk centered at $\alpha \in (0,1)$ without loss of generality. Applying the change of variable $z \mapsto \phi_\alpha(z)$, where $\phi_\alpha(z) = \frac{\alpha - z}{1-\alpha z}$, it follows

$$\begin{aligned} &\left(\int_{\mathbb{T}} \left(\int_{G\cap\Delta(\alpha,\eta)} \frac{(1-|z|^2)^\lambda}{|1-z\overline{\xi}|^\lambda} \, dm(z) \right)^{q/p} |d\xi| \right)^{p/q} \\ &\quad = \left(\int_{\mathbb{T}} \left(\int_{\phi_\alpha(G\cap\Delta(\alpha,\eta))} \frac{(1-|\phi_\alpha(z)|^2)^\lambda}{|1-\overline{\xi}\phi_\alpha(z)|^\lambda} |\phi_\alpha'(z)|^2 \, dm(z) \right)^{q/p} |d\xi| \right)^{p/q} \\ &\quad = \left(\int_{\mathbb{T}} \frac{(1-|\alpha|^2)^{\lambda q/p}}{|1-\alpha\xi|^{\lambda q/p}} \left(\int_{\phi_\alpha(G\cap\Delta(\alpha,\eta))} \frac{(1-|z|^2)^\lambda |\phi_\alpha'(z)|^2}{|1-\alpha z|^\lambda |1-z\overline{\phi_\alpha(\xi)}|^\lambda} \, dm(z) \right)^{q/p} |d\xi| \right)^{p/q}. \end{aligned}$$

Applying the same change of variable again, we obtain

$$\begin{aligned} &\left(\int_{\mathbb{T}} \left(\int_{G\cap\Delta(\alpha,\eta)} \frac{(1-|z|^2)^\lambda}{|1-z\overline{\xi}|^\lambda} \, dm(z) \right)^{q/p} |d\xi| \right)^{p/q} \\ &\quad = \left(\int_{\mathbb{T}} \frac{(1-|\alpha|^2)^{\lambda q/p}}{|1-\alpha\xi|^{\lambda q/p}} \left(\int_{G\cap\Delta(\alpha,\eta)} \frac{(1-|\phi_\alpha(z)|^2)^\lambda}{|1-\alpha\phi_\alpha(z)|^\lambda |1-\phi_\alpha(z)\overline{\phi_\alpha(\xi)}|^\lambda} \, dm(z) \right)^{q/p} |d\xi| \right)^{p/q} \end{aligned}$$

$$\lesssim \left(\int_{\mathbb{T}} \frac{1}{|1-\alpha\xi|^{\lambda q/p}} \left(\int_{G\cap\Delta(\alpha,\eta)} |1-\alpha z|^{\lambda}\, dm(z)\right)^{q/p} |d\xi|\right)^{p/q}$$

$$\asymp m(G\cap\Delta(\alpha,\eta))(1-\alpha)^{\lambda} \left(\int_{\mathbb{T}} \frac{1}{|1-\alpha\xi|^{\lambda q/p}}\, |d\xi|\right)^{p/q}.$$

By a standard argument, the same as the one used at the end of the proof of Lemma 3.5, we get that

$$\left(\int_{\mathbb{T}} \left(\int_{G\cap\Delta(\alpha,\eta)} \frac{(1-|z|^2)^{\lambda}}{|1-z\bar{\xi}|^{\lambda}}\, dm(z)\right)^{q/p} |d\xi|\right)^{p/q} \lesssim m(G\cap\Delta(\alpha,\eta))(1-|\alpha|)^{p/q}. \quad (3.4)$$

Thus, putting together (3.3) and (3.4), we have obtained that

$$\|f\chi_{G\cap\Delta(\alpha,\eta)}\|_{T_p^q} \lesssim \left(\frac{m(G\cap\Delta(\alpha,\eta))}{m(\Delta(\alpha,\eta))}\right)^{1/p},$$

for $\alpha\in\mathbb{D}$ and $\eta\in(0,1)$.

Now, we continue with the proof of *(a)* $\Rightarrow$ *(b)*. Given $\varepsilon \le \frac{KC_1}{2}$. By Lemma 3.5, there is $\eta\in(0,1)$ such that for all pseudo-hyperbolic disks $\Delta(\alpha,\eta)$ there is a function f_α satisfying the following conditions:

(1) There are constants $C_1, C_2 > 0$ such that $C_1 \le \|f\|_{T_p^q} \le C_2$, and
(2) $\|f\chi_{\Delta(\alpha,\eta)^c}\|_{T_p^q} \le \varepsilon$.

So that, applying *(a)*, we have

$$\left(\frac{m(G\cap\Delta(\alpha,\eta))}{m(\Delta(\alpha,\eta))}\right)^{1/p} \gtrsim \|f\chi_{G\cap\Delta(\alpha,\eta)}\|_{T_p^q} \ge \|f\chi_G\|_{T_p^q} - \|f\chi_{\Delta(\alpha,\eta)^c}\|_{T_p^q}$$

$$\ge K\|f\|_{T_p^q} - \varepsilon \ge KC_1 - \varepsilon \ge \frac{KC_1}{2}.$$

Therefore, we have proved that there is an $\eta\in(0,1)$ and a $\delta>0$ such that

$$m(G\cap\Delta(\alpha,\eta)) \ge \delta m(\Delta(\alpha,\eta))$$

for every $\alpha\in\mathbb{D}$.

□

All the above lead to the following statement.

Theorem 3.7 *Let $1\le p,q<\infty$ and $g\in\mathcal{B}$. The operator $T_g : RM(p,q)\to RM(p,q)$ has closed range if and only if there are constants $\delta, c>0$ and $\eta\in(0,1)$ such that*

$$m(G_c\cap\Delta(\alpha,\eta)) \ge \delta\, m(\Delta(\alpha,\eta)), \quad (3.5)$$

for every $\alpha\in\mathbb{D}$, where $G_c = \{z\in\mathbb{D} : |g'(z)|(1-|z|) > c\}$.

Proof Assume first that there is a $c>0$, an $\eta\in(0,1)$ and a $\delta>0$ such that

$$m(G_c\cap\Delta(\alpha,\eta)) \ge \delta m(\Delta(\alpha,\eta)),$$

for every $\alpha\in\mathbb{D}$. We employ Theorem 3.6 and [4, Proposition 3.1] to get that there is a $\beta\in(0,1)$ and a constant $K'>0$ such that

$$\left\{\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)\cap G_c} |f(z)|^p \frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\right\}^{1/q} \ge K'\,\|f\|_{RM(p,q)}$$

for every $f \in RM(p,q)$.

Applying the Littlewood-Paley type inequalities, [4, Proposition 3.1] and that $T_g(f)(0) = 0$, we get that

$$\begin{aligned}\|T_g(f)\|_{RM(p,q)} &\asymp \left\{\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)} |f(z)|^p\,|g'(z)|^p\,(1-|z|)^p\,\frac{dm(z)}{1-|z|}\right)^{q/p}\right\}^{\frac{1}{q}}\\ &\geq \left\{\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)\cap G_c} |f(z)|^p\,|g'(z)|^p\,(1-|z|)^p\,\frac{dm(z)}{1-|z|}\right)^{q/p}\right\}^{\frac{1}{q}}\\ &\geq c\left\{\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)\cap G_c} |f(z)|^p\,\frac{dm(z)}{1-|z|}\right)^{q/p}\right\}^{\frac{1}{q}}.\end{aligned}$$

Combining the above, we conclude that there is a constant $C > 0$ such that

$$\|T_g(f)\|_{RM(p,q)} \geq C\,\|f\|_{RM(p,q)} \tag{3.6}$$

for every $f \in RM(p,q)$.

Now, assume there is a constant $C > 0$ such that (3.6) holds. Due to the Littlewood-Paley type inequalities, it is true that

$$\begin{aligned}&\int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)} |f(z)|^p\,|g'(z)|^p\,(1-|z|)^p\,\frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\\ &\quad\geq C'\int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)} |f(z)|^p\,\frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\end{aligned}$$

for every $f \in RM(p,q)$. Notice that

$$\begin{aligned}&\int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)} |f(z)|^p\,|g'(z)|^p\,(1-|z|)^p\,\frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\\ &\lesssim \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)\cap G_c} |f(z)|^p\,|g'(z)|^p\,(1-|z|)^p\,\frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\\ &\quad+\int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)\setminus G_c} |f(z)|^p\,|g'(z)|^p\,(1-|z|)^p\,\frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\\ &\leq \|g\|_{\mathcal{B}}^q \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)\cap G_c} |f(z)|^p\,\frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\\ &\quad+c^q \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)} |f(z)|^p\,\frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|.\end{aligned}$$

Taking into account all the above, we conclude that

$$\|g\|_{\mathcal{B}}^q \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)\cap G_c} |f(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| + c^q \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi|$$

$$\geq C'' \int_{\mathbb{T}} \left(\int_{\Gamma_{\frac{1}{2}}(\xi)} |f(z)|^p \, \frac{dm(z)}{1-|z|} \right)^{q/p} |d\xi| \, .$$

Therefore, choosing $c > 0$ small enough, we end up to the strength of condition (a) of Theorem 3.6. As a consequence, there exist an $\eta \in (0, 1)$ and a $\delta > 0$ such that

$$m(G_c \cap \Delta(\alpha, \eta)) \geq \delta m(\Delta(\alpha, \eta))$$

for every $\alpha \in \mathbb{D}$. □

The argument for M_g, where $g \in H^\infty$, is the same as that for T_g. We only have to use the level set $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$. The approach for S_g makes use of auxiliary lemmas based on the quantity $|f'(z)|(1-|z|)$ instead of $|f(z)|$. Nevertheless, these new versions of auxiliary lemmas are proved through minor modifications of those we have presented for T_g. Consequently, we only present the statements and we leave the details for the interested reader.

Lemma 3.8 *Let $1 \leq p < \infty$, $\eta \in (0, 1)$, $\varepsilon > 0$, $f \in \mathcal{H}(\mathbb{D})$ and*

$$\mathcal{A} = \left\{ \alpha \in \mathbb{D} \, : (1-|\alpha|)^p |f'(\alpha)|^p < \frac{\varepsilon}{m(\Delta_\eta(\alpha))} \int_{\Delta_\eta(\alpha)} (1-|z|)^p |f'(z)|^p \, dm(z) \right\}.$$

Then, there is a constant $C_1 = C_1(\eta) > 0$ such that

$$\int_{\mathcal{A}\cap\Gamma_{1/2}(\xi)} (1-|z|)^p |f'(z)|^p \, \frac{dm(z)}{1-|z|} \leq \varepsilon C_1 \int_{\Gamma_\beta(\xi)} (1-|z|)^p |f'(z)|^p \, \frac{dm(z)}{1-|z|}$$

for a non tangential region $\Gamma_\beta(\xi)$ with vertex at the same point $\xi \in \mathbb{T}$ as $\Gamma_{1/2}(\xi)$.

Let $1 \leq p < \infty$ and $\lambda \in (0, 1)$. We define the set

$$E_\lambda(\alpha) = \left\{ z \in \Delta_\eta(\alpha) \ : \ (1-|z|)^p |f'(z)|^p \geq \lambda (1-|\alpha|)^p |f'(\alpha)|^p \right\}$$

and

$$B_\lambda f(\alpha) = \frac{1}{m(E_\lambda(\alpha))} \int_{E_\lambda(\alpha)} |f'(z)|^p (1-|z|)^p \, dm(z), \quad \alpha \in \mathbb{D}.$$

Lemma 3.9 *If $1 \leq p < \infty$, $\varepsilon \in (0, 1)$, $f \in \mathcal{H}(\mathbb{D})$, $\lambda \in (0, \frac{1}{2^p})$ and*

$$B = \left\{ \alpha \in \mathbb{D} \, : |f'(\alpha)|(1-|\alpha|)^p < \varepsilon^{1+\frac{2}{p}} B_\lambda(f)(\alpha) \right\}.$$

Then, there is a constant $C_2 = C_2(\eta) > 0$ such that

$$\int_{B\cap\Gamma_{1/2}(\xi)} (1-|z|)^p |f'(z)|^p \, \frac{dm(z)}{1-|z|} \leq \varepsilon C_2 \int_{\Gamma_\beta(\xi)} (1-|z|)^p |f'(z)|^p \, \frac{dm(z)}{1-|z|}$$

for a non tangential region $\Gamma_\beta(\xi)$ with vertex at the same $\xi \in \mathbb{T}$ as $\Gamma_{1/2}(\xi)$.

So that, the analogue of Theorem 3.6 is the following result.

Theorem 3.10 *Let* $1 \leq p, q < +\infty$ *and* $G \subset \mathbb{D}$ *be a measurable subset. The following assertions are equivalent:*

(a) There is a $\beta \in (0, 1)$ *and a constant* $K > 0$ *such that*

$$\left(\int_{\mathbb{T}} \left(\int_{\Gamma_\beta(\xi)\cap G} |f'(z)|^p (1-|z|)^p \frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\right)^{1/q}$$
$$\geq K \left(\int_{\mathbb{T}} \left(\int_{\Gamma_{1/2}(\xi)} |f(z)|^p \frac{dm(z)}{1-|z|}\right)^{q/p} |d\xi|\right)^{1/q},$$

for all $f \in AT_p^q$.

(b) There exist an $\eta \in (0, 1)$ *and a* $\delta > 0$ *such that*

$$m(G \cap \Delta(\alpha, \eta)) \geq \delta m(\Delta(\alpha, \eta))$$

for $\alpha \in \mathbb{D}$.

(c) There exist an $\eta \in (0, 1)$ *and a* $\delta > 0$ *such that*

$$m(G \cap \Delta_\eta(\alpha)) \geq \delta m(\Delta_\eta(\alpha))$$

for $\alpha \in \mathbb{D}$.

Putting everything together, we get the desired result for the operator S_g.

Theorem 3.11 *Let* $1 \leq p, q < \infty$ *and* $g \in H^\infty$. *The operator* $S_g : RM(p,q)/\mathbb{C} \to RM(p,q)/\mathbb{C}$ *has closed range if and only if there is an* $\eta \in (0, 1)$, *a* $\delta > 0$ *and* $c > 0$ *such that*

$$m(G_c \cap \Delta(\alpha, \eta)) \geq \delta m(\Delta(\alpha, \eta)),$$

for every $\alpha \in \mathbb{D}$, *where* $G_c = \{z \in \mathbb{D} : |g(z)| > c\}$.

4 Examples

Example 4.1 The univalent functions do not satisfy the condition (3.5). Assume that there is a univalent function g satisfying this condition. Using [10, Theorem 3.4.9], we have that

$$|g'(z)|(1-|z|) \leq d(g(z), \partial g(\mathbb{D})) \leq \frac{1}{4}|g'(z)|(1-|z|),$$

for a univalent function g and $z \in \mathbb{D}$. So, this implies that $d(g(\mathbb{D}), \partial g(\mathbb{D})) \geq c$. Moreover, it is known that there is $\xi \in \mathbb{T}$ such that $\angle \lim_{z\to\xi} g(z) = L \in \mathbb{C}$. Then, by condition (3.5) we can find a sequence $\{z_k\}$ in the non-tangential region such that $z_k \to \xi$. However, we have

$$|g(z_k) - L| \geq d(g(z_k), \partial g(\mathbb{D})) \geq c$$

for all $k \in \mathbb{N}$. Therefore, we obtain a contradiction.

Proof of Proposition 1.4 Without loss of generality we can assume that $\alpha \in [0, 1)$. Following the proof of [26, Proposition 5.4], we can show that for a given large positive integer q we can find a function $g(z) = \sum_{j=0}^{\infty} z^{q^j} \in \mathcal{B}$ and a constant $c > 0$ such that

$$|g'(z)|(1-|z|) \geq c, \quad 1 - \frac{1}{q^k} \leq |z| \leq 1 - \frac{1}{q^{k+\frac{1}{2}}}, \quad k = 1, 2, \ldots.$$

First of all, we will show that if $\eta \geq 1 - \frac{1}{q}$, we have that

$$\frac{(1-\eta^2)\alpha}{1-\alpha^2\eta^2} + \frac{(1-\alpha^2)\eta}{1-\alpha^2\eta^2} = \frac{\alpha+\eta}{1+\alpha\eta} \geq 1 - \frac{1}{q^{k+\frac{1}{2}}}$$
$$\frac{(1-\eta^2)\alpha}{1-\alpha^2\eta^2} - \frac{(1-\alpha^2)\eta}{1-\alpha^2\eta^2} = \frac{\alpha-\eta}{1-\alpha\eta} \leq 1 - \frac{1}{q^{k}}$$

for $\alpha \in \left[1 - \frac{1}{q^{k-\frac{1}{2}}}, 1 - \frac{1}{q^{k+\frac{1}{2}}}\right]$, $k = 1, 2, \ldots$. Since the functions $h_\eta(\alpha) = \frac{\alpha+\eta}{1+\alpha\eta}$ and $g_\eta(\alpha) = \frac{\alpha-\eta}{1-\alpha\eta}$ are increasing functions, the previous inequalities have to satisfy the following

$$\frac{1 - \frac{1}{q^{k-\frac{1}{2}}} + \eta}{1 + \eta\left(1 - \frac{1}{q^{k-\frac{1}{2}}}\right)} \geq 1 - \frac{1}{q^{k+\frac{1}{2}}}, \quad \frac{1 - \frac{1}{q^{k+\frac{1}{2}}} - \eta}{1 - \eta\left(1 - \frac{1}{q^{k+\frac{1}{2}}}\right)} \leq 1 - \frac{1}{q^{k}}$$

for $k = 1, 2, \ldots$. Hence, we obtain that $\eta \geq \max\{1 - \frac{1}{q}, 1 - \frac{1}{q^{1/2}}\} = 1 - \frac{1}{q}$. Analogously, one can prove that if $\eta \geq 1 - \frac{1}{q^{3/2}}$, we obtain

$$\frac{(1-\eta^2)\alpha}{1-\alpha^2\eta^2} + \frac{(1-\alpha^2)\eta}{1-\alpha^2\eta^2} = \frac{\alpha+\eta}{1+\alpha\eta} \geq 1 - \frac{1}{q^{\frac{3}{2}}}$$
$$\frac{(1-\eta^2)\alpha}{1-\alpha^2\eta^2} - \frac{(1-\alpha^2)\eta}{1-\alpha^2\eta^2} = \frac{\alpha-\eta}{1-\alpha\eta} \leq 1 - \frac{1}{q}$$

for $\alpha \in [0, 1 - \frac{1}{q^{1/2}}]$.

Bearing this in mind, it follows that if $1 - \frac{1}{q^{k-\frac{1}{2}}} \leq \alpha \leq 1 - \frac{1}{q^{k+\frac{1}{2}}}$, $k = 1, 2, \ldots$, we have

$$\begin{aligned} m(G_c \cap \Delta(\alpha,\eta)) &\geq m\left(\left\{z \in \Delta(\alpha,\eta) : 1 - \frac{1}{q^k} \leq |z| \leq 1 - \frac{1}{q^{k+\frac{1}{2}}}, k = 1, 2, \ldots\right\}\right) \\ &\geq m\left(D\left(1 - \frac{q^{1/2}+1}{2q^{k+\frac{1}{2}}}, \frac{q^{1/2}-1}{2q^{k+\frac{1}{2}}}\right)\right) = \frac{(q^{1/2}-1)^2}{4q^{2k+1}} \\ &\geq \frac{(q^{1/2}-1)^2}{4q^{2k+1}} q^{2k-1}(1-\alpha)^2 = \frac{(q^{1/2}-1)^2}{4q^2}(1-\alpha)^2 \end{aligned}$$

for $\eta \geq 1 - \frac{1}{q}$. The remaining case $(0 \leq \alpha \leq 1 - \frac{1}{q^{1/2}})$ follows in the same manner, that is,

$$\begin{aligned} m(G_c \cap \Delta(\alpha,\eta)) &\geq m\left(\left\{z \in \Delta(\alpha,\eta) : 1 - \frac{1}{q^k} \leq |z| \leq 1 - \frac{1}{q^{k+\frac{1}{2}}}, k = 1, 2, \ldots\right\}\right) \\ &\geq m\left(D\left(1 - \frac{q^{1/2}+1}{2q^{\frac{3}{2}}}, \frac{q^{1/2}-1}{2q^{\frac{3}{2}}}\right)\right) = \frac{(q^{1/2}-1)^2}{4q^{3}} \\ &\geq \frac{(q^{1/2}-1)^2}{4q^3}(1-\alpha)^2 \end{aligned}$$

for $\eta \geq 1 - \frac{1}{q^{3/2}}$. □

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