HARDY OPERATORS ON WEIGHTED AMALGAMS

PEDRO ORTEGA SALVADOR AND CONSUELO RAMÍREZ TORREBLANCA

Abstract. We characterize the boundedness of the Hardy operator between weighted amalgams, a problem studied, but not completely solved, by C. Carton-Lebrun, H. P. Heinig and S. C. Hofmann. We also characterize the weighted weak type inequalities for modified Hardy operators on amalgams.

1. Introduction

If $1 \leq p, q < \infty$ and u is a positive measurable function of one real variable, the amalgam of $L^p(u)$ and ℓ^q is the space $(L^p(u), \ell^q)$ of functions f on the real line such that

$$\|f\|_{p,u,q} = \left\{ \sum_{n \in \mathbf{Z}} \left(\int_n^{n+1} |f(x)|^p u(x) dx \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} < \infty.$$

In this way, a function f belongs to $(L^p(u), \ell^q)$ if f is locally in $L^p(u)$ and globally in ℓ^q. Endowed with the norm $\| \cdot \|_{p,u,q}$, $(L^p(u), \ell^q)$ becomes a Banach space.

The amalgam spaces were introduced by N. Wiener in 1926 ([?]). A survey about the role played by amalgam spaces in Harmonic Analysis can be found in [?] (see also [?]).

C. Carton-Lebrun, H. Heinig and S. Hofmann studied in [?] weighted inequalities in amalgams for the Hardy operator P defined for nonnegative functions f by

$$Pf(x) = \int_{-\infty}^{x} f.$$

Specifically, they characterized the pairs of weights (u, v) such that the inequality

$$\|Pf\|_{p,u,q} \leq C \|f\|_{\overline{p},v,\overline{q}} \tag{1.1}$$

holds for all f with a constant C independent of f whenever $1 < \overline{q} \leq q < \infty$, while in the case $1 < q < \overline{q} < \infty$ they only got necessary conditions and different sufficient conditions. As far as we know, the full characterization in the case $1 < q < \overline{q} < \infty$ is an open problem.

The first purpose of this paper is to solve this problem, i.e., to characterize the pairs of weights (u, v) such that the inequality (??) holds in the case $1 < q < \overline{q} < \infty$, showing that the sufficient conditions given in [?] are also necessary. This will be done in section 3.

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We also study the boundedness of modified Hardy operators from an amalgam of $L^{\overline{p}}(v)$ and $\ell^{\overline{q}}$ to an amalgam of $L^{p,\infty}(u)$ and ℓ^q. By the amalgam of $L^{p,\infty}(u)$ and ℓ^q we mean the space $(L^{p,\infty}(u), \ell^q)$ of functions f such that

$$\|f\|_{p,\infty,u,q} = \left\{ \sum_{n\in\mathbf{Z}} \|\chi_{(n,n+1)} f\|_{p,\infty;u}^q \right\}^{\frac{1}{q}} < \infty,$$

where $\|g\|_{p,\infty;u} = \sup_{\lambda>0} \lambda \left(\int_{\{x:|g(x)|>\lambda\}} u \right)^{\frac{1}{p}}$ and $L^{p,\infty}(u) = \{g : \|g\|_{p,\infty;u} < \infty\}$.

If g is a positive function of one real variable, we consider the modified Hardy operator T_g defined for nonnegative functions f by

$$T_g f(x) = g(x) \int_{-\infty}^{x} f$$

and characterize, for all values of $p, \overline{p}, q, \overline{q}$, the pairs of weights (u, v) such that the inequality

$$\|T_g f\|_{p,\infty,u,q} \le C \|f\|_{\overline{p},v,\overline{q}} \tag{1.2}$$

holds for all nonnegative functions f with a constant C independent of f. The results will be stated and proved in section 4.

2. Notations and preliminaries

The notation we will use is standard. In particular, if $p > 1$, its conjugate exponent is $p' = \frac{p}{p-1}$. On the other hand, throughout the paper the letter C will design a positive constant, not necessarily the same at each occurrence.

In the statements of our results we will use the following notations, in which $n \in \mathbf{Z}$ and u, v are positive measurable functions on $\mathbf{R}$.

(i) If $1 < \overline{p} \le p < \infty$,

$$S_n = \sup_{\alpha\in(0,1)} \left(\int_{n+\alpha}^{n+1} u \right)^{\frac{1}{p}} \left(\int_{n}^{n+\alpha} v^{1-\overline{p}'} \right)^{\frac{1}{\overline{p}'}}.$$

(ii) If $1 < p < \overline{p} < \infty$ and $\frac{1}{r} = \frac{1}{p} - \frac{1}{\overline{p}}$,

$$S_n' = \left\{ \int_{n}^{n+1} \left(\int_{t}^{n+1} u \right)^{\frac{r}{p}} \left(\int_{n}^{t} v^{1-\overline{p}'} \right)^{\frac{r}{p'}} v(t)^{1-\overline{p}'} dt \right\}^{\frac{1}{r}}.$$

(iii) If $1 < \overline{p} \le p < \infty$,

$$J_n = \sup_{0\le\alpha<1} \|g\chi_{(n+\alpha,n+1)}\|_{p,\infty,u} \left(\int_{n}^{n+\alpha} v^{1-\overline{p}'} \right)^{\frac{1}{\overline{p}'}}.$$

(iv) If $1 < p < \overline{p} < \infty$,

$$\Phi_n(x) = \sup_{n<a<x<b<n+1} \left[\inf_{y\in(a,b)} g(y) \left(\int_{a}^{b} u \right)^{\frac{1}{\overline{p}}} \left(\int_{n}^{a} v^{1-\overline{p}'} \right)^{\frac{1}{\overline{p}'}} \right].$$

(v) If $1 < p, \overline{p} < \infty$, $1 < q < \overline{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$,

$$S = \sum_{n\in\mathbf{Z}} \left(\sum_{k=n}^{\infty} \left(\int_k^{k+1} u \right)^{\frac{q}{p}} \right)^{\frac{s}{q}} \left(\sum_{k=-\infty}^{n} \left(\int_{k-1}^{k} v^{1-\overline{p}'} \right)^{\frac{\overline{q}'}{\overline{p}'}} \right)^{\frac{s}{q'}} \left(\int_{n-1}^{n} v^{1-\overline{p}'} \right)^{\frac{\overline{q}'}{\overline{p}'}}.$$

(vi) If $1 < p, \overline{p} < \infty$ and $1 < \overline{q} \le q < \infty$,

$$B = \sup_{m\in\mathbf{Z}} \left(\sum_{n=m}^{\infty} \|g\chi_{(n,n+1)}\|_{p,\infty,u}^q \right)^{\frac{1}{q}} \left(\sum_{n=-\infty}^{m} \left(\int_{n-1}^{n} v^{1-\overline{p}'} \right)^{\frac{\overline{q}'}{\overline{p}'}} \right)^{\frac{1}{\overline{q}'}}.$$

(vii) If $1 < p, \overline{p} < \infty$, $1 < q < \overline{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$,

$$\overline{B} = \sum_{n\in\mathbf{Z}} \left(\sum_{k=n}^{\infty} \|g\chi_{(k,k+1)}\|_{p,\infty;u}^q \right)^{\frac{s}{q}} \left(\sum_{k=-\infty}^{n} \left(\int_{k-1}^{k} v^{1-\overline{p}'} \right)^{\frac{\overline{q}'}{\overline{p}'}} \right)^{\frac{s}{q'}} \left(\int_{n-1}^{n} v^{1-\overline{p}'} \right)^{\frac{\overline{q}'}{\overline{p}'}}.$$

We will apply the following well known theorems concerning the characterizations of the weighted weak and strong type inequalities for the Hardy operator and the modified Hardy operators, as well as the characterization of the weighted strong type inequalities for the discrete Hardy operator.

Theorem A. ([?], [?], [?]) *Let $-\infty \le a < b \le \infty$ and let $P_{a,b}$ be the operator defined for nonnegative functions on (a,b) by*

$$P_{a,b}f(x) = \int_a^x f.$$

Suppose that $1 \le p, \overline{p} < \infty$ and u, v are positive measurable functions on (a,b). Then the inequality

$$\|P_{a,b}f\|_{p;u} \le C\|f\|_{\overline{p},v} \tag{2.1}$$

holds for all f with a constant C independent of f if and only if

(i) *in the case $1 \le \overline{p} \le p$,*

$$D_1 \equiv \sup_{a<x<b} \left(\int_x^b u \right)^{\frac{1}{p}} \left(\int_a^x v^{1-\overline{p}'} \right)^{\frac{1}{\overline{p}'}} < \infty;$$

(ii) *in the case $1 < p < \overline{p}$,*

$$D_2 \equiv \int_a^b \left(\int_t^b u \right)^{\frac{r}{p}} \left(\int_a^t v^{1-\overline{p}'} \right)^{\frac{r}{p'}} v(t)^{1-\overline{p}'} dt < \infty,$$

where $\frac{1}{r} = \frac{1}{p} - \frac{1}{\overline{p}}$.

*Moreover, the best constant C in (**??**) satisfies $D_1 \le C \le \overline{p}' D_1$ in the case $1 \le \overline{p} \le p$ and $p\overline{p}^{-\frac{1}{r}} (\overline{p}')^{\frac{1}{p'}} r^{-\frac{1}{r'}} 2^{-\frac{1}{p}} D_2 \le C \le p^{\frac{1}{p}} \overline{p}' D_2$ in the case $1 < p < \overline{p}$.*

Theorem B. ([?], [?], [?]) *Let $-\infty \le a < b \le \infty$ and let $H_{a,b}$ be the operator defined for functions on (a,b) by*

$$H_{a,b}f(x) = g(x)\int_a^x f.$$

Let $1 \le p, \overline{p} < \infty$ and u, v positive measurable functions on (a,b). Then the weighted weak type inequality

$$\|H_{a,b}f\|_{p,\infty;u} \le C\|f\|_{\overline{p};v} \tag{2.2}$$

holds for all f with a constant C independent of f if and only if

(i) *in the case $1 \le \overline{p} \le p$,*

$$E \equiv \sup_{a<x<b} \|g\chi_{(x,b)}\|_{p,\infty;u}\left(\int_a^x v^{1-\overline{p}'}\right)^{\frac{1}{\overline{p}'}} < \infty;$$

(ii) *in the case $1 < p < \overline{p}$ and g monotone, the function Φ defined on (a,b) by*

$$\Phi(x) = \sup_{a<c<x<d<b}\left[\inf_{y\in(c,d)} g(y)\left(\int_c^d u\right)^{\frac{1}{p}}\left(\int_a^c v^{1-\overline{p}'}\right)^{\frac{1}{\overline{p}'}}\right]$$

belongs to $L^{r,\infty}(u)$, where $\frac{1}{r} = \frac{1}{p} - \frac{1}{\overline{p}}$.

*Moreover, the best constant C in (**??**) satisfies $E \le C \le 4E$ in the case $1 \le \overline{p} \le p$ and $2^{-\frac{1}{p}}\|\Phi\|_{r,\infty;u} \le C \le 2^{\frac{1}{p}}(1+4^{\frac{\overline{p}}{p}})\|\Phi\|_{r,\infty;u}$.*

Theorem C. ([?]) *Let $1 \le q, \overline{q} < \infty$ and let $\{u_n\}$ and $\{v_n\}$ be sequences of positive numbers. Then the inequality*

$$\left\{\sum_{n\in\mathbf{Z}}\left(\sum_{k=-\infty}^{n} a_k\right)^q u_n\right\}^{\frac{1}{q}} \le C\left\{\sum_{n\in\mathbf{Z}} a_n^{\overline{q}} v_n\right\}^{\frac{1}{\overline{q}}} \tag{2.3}$$

holds for all sequences $\{a_n\}$ of nonnegative numbers with a constant C independent of $\{a_n\}$ if and only if

(i) *in the case $1 \le \overline{q} \le q < \infty$,*

$$F_1 \equiv \sup_{m\in\mathbf{Z}}\left(\sum_{n=m}^{\infty} u_n\right)^{\frac{1}{q}}\left(\sum_{n=-\infty}^{m} v_n^{1-\overline{q}'}\right)^{\frac{1}{\overline{q}'}} < \infty;$$

(ii) *in the case $1 < q < \overline{q} < \infty$,*

$$F_2 \equiv \sum_{m\in\mathbf{Z}}\left(\sum_{n=m}^{\infty} u_n\right)^{\frac{s}{q}}\left(\sum_{n=-\infty}^{m} v_n^{1-\overline{q}'}\right)^{\frac{s}{q'}} v_m^{1-\overline{q}'} < \infty,$$

where $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$.

*Moreover, the best constant C in (**??**) satisfies $F_1 \le C \le \overline{q}'F_1$ in the case $1 \le \overline{q} \le q$ and $q\overline{q}^{-\frac{1}{s}}(\overline{q}')^{\frac{1}{q'}}s^{-\frac{1}{s'}}2^{-\frac{1}{q}}F_2 \le C \le q^{\frac{1}{q}}\overline{q}'F_2$ in the case $1 < q < \overline{q}$.*

Finally, we will need the straightforward characterization of the embedding of $\ell^{\overline{q}}(\{v_n^{\overline{q}}\})$ into $\ell^q(\{u_n^q\})$ for $1 < q < \overline{q}$. The result reads as follows:

Lemma 1. *Let $1 < q < \overline{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences of positive real numbers. The following statements are equivalent:*

(i) *There exists $C > 0$ such that the inequality*

$$\left\{\sum_{n\in\mathbf{Z}}(|a_n|u_n)^q\right\}^{\frac{1}{q}} \le C\left\{\sum_{n\in\mathbf{Z}}(|a_n|v_n)^{\overline{q}}\right\}^{\frac{1}{\overline{q}}}$$

holds for all sequences $\{a_n\}$ of real numbers.

(ii) *The sequence $\{u_n v_n^{-1}\}$ belongs to the space ℓ^s.*

Moreover, the best constant C in (i) *is $\|\{u_n v_n^{-1}\}\|_{\ell^s}$.*

3. Strong type inequalities for the Hardy operator on weighted amalgams

The result which characterizes (**??**) in the case $1 < q < \overline{q} < \infty$ is the following one:

Theorem 1. *Let $1 < \overline{p}, p < \infty$, $1 < q < \overline{q} < \infty$ and $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$. Suppose that u, v are positive measurable functions. Then there exists a positive constant C such that the inequality (**??**) holds for all nonnegative measurable functions f if and only if*

(i) *in the case $1 < \overline{p} \le p < \infty$, $S < \infty$ and $\{S_n\} \in \ell^s$;*

(ii) *in the case $1 < p < \overline{p} < \infty$, $S < \infty$ and $\{S'_n\} \in \ell^s$.*

Proof. The sufficiency of the conditions and the necessity of the condition $S < \infty$ were proved in [**?**]. We only prove the necessity of the condition $\{S_n\} \in \ell^s$ in the case $1 < \overline{p} \le p < \infty$ and the necessity of the condition $\{S'_n\} \in \ell^s$ in the case $1 < p < \overline{p} < \infty$.

Suppose that (**??**) holds. Fix $n \in \mathbf{Z}$ and let f be a nonnegative function supported in $(n, n+1)$. Then

$$\|Pf\|_{p,u,q} \ge \left(\int_n^{n+1}\left(\int_n^x f\right)^p u(x)dx\right)^{\frac{1}{p}}$$

and (**??**) gives

$$\left(\int_n^{n+1}\left(\int_n^x f\right)^p u(x)dx\right)^{\frac{1}{p}} \le C\left(\int_n^{n+1} f^{\overline{p}} v\right)^{\frac{1}{\overline{p}}}. \tag{3.1}$$

(i) If $1 < \overline{p} \le p < \infty$, (**??**) implies, by Theorem A, that $\sup_{n\in\mathbf{Z}} S_n < \infty$. For each $n \in \mathbf{Z}$, let $\alpha_n \in (0,1)$ such that

$$S_n - \left(\int_{n+\alpha_n}^{n+1} u\right)^{\frac{1}{p}}\left(\int_n^{n+\alpha_n} v^{1-\overline{p}'}\right)^{\frac{1}{\overline{p}'}} < \frac{1}{2^{|n|}}.$$

Since we have to show that $\{S_n\} \in \ell^s$, it suffices to prove that

$$\left\{ \left(\int_{n+\alpha_n}^{n+1} u \right)^{\frac{1}{p}} \left(\int_{n}^{n+\alpha_n} v^{1-\overline{p}'} \right)^{\frac{1}{\overline{p}'}} \right\} \in \ell^s.$$

Let $\{a_n\}$ be a sequence of nonnegative numbers and let f be the function defined by $f = \sum_{n\in\mathbf{Z}} a_n v^{1-\overline{p}'} \chi_{(n,n+\alpha_n)}$. For $n \in \mathbf{Z}$ and $x \in (n+\alpha_n, n+1)$ we have

$$Pf(x) \geq a_n \int_{n}^{n+\alpha_n} v^{1-\overline{p}'}.$$

Then

$$\|Pf\|_{p,u,q} \geq \left\{ \sum_{n\in\mathbf{Z}} a_n^q \left(\int_{n}^{n+\alpha_n} v^{1-\overline{p}'} \right)^{q} \left(\int_{n+\alpha_n}^{n+1} u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}},$$

$$\|f\|_{\overline{p},v,\overline{q}} = \left\{ \sum_{n\in\mathbf{Z}} a_n^{\overline{q}} \left(\int_{n}^{n+\alpha_n} v^{1-\overline{p}'} \right)^{\frac{\overline{q}}{\overline{p}}} \right\}^{\frac{1}{\overline{q}}}$$

and (**??**) yields

$$\left\{ \sum_{n\in\mathbf{Z}} a_n^q \left(\int_{n}^{n+\alpha_n} v^{1-\overline{p}'} \right)^{q} \left(\int_{n+\alpha_n}^{n+1} u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \leq C \left\{ \sum_{n\in\mathbf{Z}} a_n^{\overline{q}} \left(\int_{n}^{n+\alpha_n} v^{1-\overline{p}'} \right)^{\frac{\overline{q}}{\overline{p}}} \right\}^{\frac{1}{\overline{q}}}.$$

Since the previous inequality holds for all sequences $\{a_n\}$ of nonnegative numbers, Lemma 1 gives

$$\left\{ \left(\int_{n}^{n+\alpha_n} v^{1-\overline{p}'} \right)^{\frac{1}{\overline{p}'}} \left(\int_{n+\alpha_n}^{n+1} u \right)^{\frac{1}{p}} \right\} \in \ell^s,$$

as we wished to prove.

(ii) If $1 < p < \overline{p} < \infty$, then (**??**) implies, by Theorem A, that $\sup_{n\in\mathbf{Z}} S_n' < \infty$.

Let $\{a_n\}$ be a sequence of nonnegative numbers and let

$$f(x) = \sum_{n\in\mathbf{Z}} a_n \chi_{(n,n+1)}(x) \left(\int_{x}^{n+1} u \right)^{\frac{r}{p\overline{p}}} \left(\int_{n}^{x} v^{1-\overline{p}'} \right)^{\frac{r}{p'\overline{p}}} v^{1-\overline{p}'}(x).$$

If $x \in (n, n+1)$,

$$\begin{aligned} \int_{n}^{x} f(t)dt &= a_n \int_{n}^{x} \left(\int_{t}^{n+1} u \right)^{\frac{r}{p\overline{p}}} \left(\int_{n}^{t} v^{1-\overline{p}'} \right)^{\frac{r}{p'\overline{p}}} v^{1-\overline{p}'}(t)dt \\ &\geq a_n \left(\int_{x}^{n+1} u \right)^{\frac{r}{p\overline{p}}} \int_{n}^{x} \left(\int_{n}^{t} v^{1-\overline{p}'} \right)^{\frac{r}{p'\overline{p}}} v^{1-\overline{p}'}(t)dt \\ &= a_n \frac{p\overline{p}'}{r} \left(\int_{x}^{n+1} u \right)^{\frac{r}{p\overline{p}}} \left(\int_{n}^{x} v^{1-\overline{p}'} \right)^{\frac{r}{p\overline{p}'}}. \end{aligned} \tag{3.2}$$

On the other hand,

$$\begin{aligned}\int_n^{n+1}(Pf(x))^p u(x)dx &\geq \int_n^{n+1}\left(\int_n^x f\right)^p u(x)dx\\ &= \int_n^{n+1}\left(\int_n^x\left[\left(\int_n^y f\right)^p\right]' dy\right)u(x)dx \qquad (3.3)\\ &= p\int_n^{n+1}\left(\int_n^y f\right)^{p-1} f(y)\left(\int_y^{n+1}u(x)dx\right)dy.\end{aligned}$$

Transporting (**??**) to (**??**) we find

$$\begin{aligned}\int_n^{n+1}(Pf(x))^p u(x)dx &\geq p\int_n^{n+1}\left(a_n\frac{p\overline{p}'}{r}\left(\int_y^{n+1}u\right)^{\frac{r}{p\overline{p}}}\left(\int_n^y v^{1-\overline{p}'}\right)^{\frac{r}{p\overline{p}'}}\right)^{p-1}\\ &\times a_n\left(\int_y^{n+1}u\right)^{\frac{r}{p\overline{p}}}\left(\int_n^y v^{1-\overline{p}'}\right)^{\frac{r}{p'\overline{p}}} v^{1-\overline{p}'}(y)\left(\int_y^{n+1}u\right)dy\\ &= a_n^p p\left(\frac{p\overline{p}'}{r}\right)^{p-1}\int_n^{n+1}\left(\int_y^{n+1}u\right)^{\frac{r}{p}}\left(\int_n^y v^{1-\overline{p}'}\right)^{\frac{r}{p'}} v^{1-\overline{p}'}(y)dy\\ &= a_n^p p\left(\frac{p\overline{p}'}{r}\right)^{p-1}(S_n')^r.\end{aligned}$$

Then

$$\|Pf\|_{p,u,q} \geq p^{\frac{1}{p}}\left(\frac{p\overline{p}'}{r}\right)^{\frac{1}{p'}}\left\{\sum_{n\in\mathbf{Z}} a_n^q (S_n')^{\frac{qr}{p}}\right\}^{\frac{1}{q}}.$$

Since $\|f\|_{\overline{p},v,\overline{q}} = \left\{\sum_{n\in\mathbf{Z}} a_n^{\overline{q}}(S_n')^{\frac{r\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}$, (**??**) yields

$$\left\{\sum_{n\in\mathbf{Z}} a_n^q (S_n')^{\frac{rq}{p}}\right\}^{\frac{1}{q}} \leq C\left\{\sum_{n\in\mathbf{Z}} a_n^{\overline{q}}(S_n')^{\frac{r\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}$$

and applying Lemma 1 we obtain $\{S_n'\} \in \ell^s$.

□

4. Weak type inequalities for modified Hardy operators on weighted amalgams

As in the previous results, the characterizing condition for the weak type inequality is double, with a local part and a global one. The result which characterizes the inequality (**??**) is the following one:

Theorem 2. *Let $1 < \overline{p}, p < \infty$ and $1 < \overline{q}, q < \infty$. Suppose that u, v are positive measurable functions on $\mathbf{R}$. Assume also that g is monotone in each interval $(n, n+1)$ when $p < \overline{p}$. Then there exists a constant $C > 0$ such that (**??**) holds for all nonnegative measurable functions f if and only if*

(i) *in the case $1 < \overline{p} \leq p < \infty$ and $1 < \overline{q} \leq q < \infty$, $J = \sup_{n\in\mathbf{Z}} J_n < \infty$ and $B < \infty$;*

(ii) *in the case* $1 < p < \overline{p} < \infty$ *and* $1 < \overline{q} \le q < \infty$, $F = \sup_{n\in\mathbf{Z}} \|\Phi_n\|_{r,\infty;u} < \infty$ *and* $B < \infty$, *where* $\frac{1}{r} = \frac{1}{p} - \frac{1}{\overline{p}}$;

(iii) *in the case* $1 < \overline{p} \le p < \infty$ *and* $1 < q < \overline{q} < \infty$, $\overline{B} < \infty$ *and* $\{J_n\} \in \ell^s$, *where* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$;

(iv) *in the case* $1 < p < \overline{p} < \infty$ *and* $1 < q < \overline{q} < \infty$, $\overline{B} < \infty$ *and* $\{\|\Phi_n\|_{r,\infty;u}\} \in \ell^s$, *where* $\frac{1}{r} = \frac{1}{p} - \frac{1}{\overline{p}}$ *and* $\frac{1}{s} = \frac{1}{q} - \frac{1}{\overline{q}}$.

Proof. Suppose that (**??**) holds. If $m \in \mathbf{Z}$ and f is a nonnegative measurable function supported in $(m, m+1)$, then

$$\|T_g f\|_{p,\infty,u,q} \ge \sup_{\lambda>0} \lambda \left(u \left\{ x \in (m, m+1) : g(x) \int_m^x f > \lambda \right\} \right)^{\frac{1}{p}}$$

and (**??**) yields

$$\sup_{\lambda>0} \lambda \left(u \left\{ x \in (m, m+1) : g(x) \int_m^x f > \lambda \right\} \right)^{\frac{1}{p}} \le C \left(\int_m^{m+1} f^{\overline{p}} v \right)^{\frac{1}{\overline{p}}}, \tag{4.1}$$

where C is a positive constant independent of m.

On the other hand, let $\{a_k\}$ be a sequence of nonnegative numbers and let f be the function defined by $f = \sum_{k\in\mathbf{Z}} a_k \left(\int_{k-1}^k v^{1-\overline{p}'} \right)^{-1} v^{1-\overline{p}'} \chi_{(k-1,k)}$. If $n \in \mathbf{Z}$ and $x \in (n, n+1)$, we have

$$\begin{aligned} T_g f(x) &= g(x) \int_{-\infty}^x f = g(x) \int_{-\infty}^x \left(\sum_{k\in\mathbf{Z}} a_k \left(\int_{k-1}^k v^{1-\overline{p}'} \right)^{-1} v^{1-\overline{p}'} \chi_{(k-1,k)} \right) \\ &= g(x) \left(\sum_{k=-\infty}^n a_k + \left(\int_n^x a_{n+1} v^{1-\overline{p}'} \left(\int_n^{n+1} v^{1-\overline{p}'} \right)^{-1} \right) \right) \ge g(x) \sum_{k=-\infty}^n a_k. \end{aligned}$$

Then,

$$\|T_g f\|_{p,\infty,u,q} = \left\{ \sum_{n\in\mathbf{Z}} \|(T_g f)\chi_{(n,n+1)}\|_{p,\infty,u}^q \right\}^{\frac{1}{q}} \ge \left\{ \sum_{n\in\mathbf{Z}} \left(\sum_{k=-\infty}^n a_k \right)^q \|g\chi_{(n,n+1)}\|_{p,\infty,u}^q \right\}^{\frac{1}{q}},$$

$$\|f\|_{\overline{p},v,\overline{q}} = \left\{ \sum_{n\in\mathbf{Z}} a_n^{\overline{q}} \left(\int_{n-1}^n v^{1-\overline{p}'} \right)^{-\frac{\overline{q}}{\overline{p}'}} \right\}^{\frac{1}{\overline{q}}}$$

and from (**??**) it follows that the inequality

$$\left\{ \sum_{n\in\mathbf{Z}} \left(\sum_{k=-\infty}^n a_k \right)^q \|g\chi_{(n,n+1)}\|_{p,\infty,u}^q \right\}^{\frac{1}{q}} \le C \left\{ \sum_{n\in\mathbf{Z}} a_n^{\overline{q}} \left(\int_{n-1}^n v^{1-\overline{p}'} \right)^{-\frac{\overline{q}}{\overline{p}'}} \right\}^{\frac{1}{\overline{q}}} \tag{4.2}$$

holds for all sequences $\{a_k\}$ of nonnegative numbers. This means that the weighted strong type $(\overline{q}, q)$ discrete Hardy inequality with weights $u_n = \|g\chi_{(n,n+1)}\|_{p,\infty,u}^q$ and $v_n = \left(\int_{n-1}^n v^{1-\overline{p}'} \right)^{-\frac{\overline{q}}{\overline{p}'}}$ holds.

(i) If $1 < \overline{p} \le p < \infty$ and $1 < \overline{q} \le q < \infty$, (**??**) implies, by Theorem B, that $J = \sup_n J_n < \infty$ and (**??**) gives, by Theorem C, that $B < \infty$.

(ii) If $1<p<\overline{p}<\infty$ and $1<\overline{q}\le q<\infty$, from (**??**) we obtain, applying Theorem B, that $\sup_n\|\Phi_n\|_{r,\infty;u}<\infty$ and (**??**) implies, by Theorem C, that $B<\infty$.

(iii) If $1<\overline{p}\le p<\infty$ and $1<q<\overline{q}<\infty$, then (**??**) implies $\overline{B}<\infty$ by Theorem C. As in *(i)*, from (**??**) we have $J=\sup_n J_n<\infty$. For each $n\in\mathbf{Z}$, let α_n be a real number in the interval $(0,1)$ such that

$$J_n-\|g\chi_{(n+\alpha_n,n+1)}\|_{p,\infty,u}\left(\int_n^{n+\alpha_n}v^{1-\overline{p}'}\right)^{\frac{1}{\overline{p}'}}<\frac{1}{2^{|n|}}.$$

Since we have to show that $\{J_n\}\in\ell^s$, it suffices to prove that

$$\left\{\|g\chi_{(n+\alpha_n,n+1)}\|_{p,\infty,u}\left(\int_n^{n+\alpha_n}v^{1-\overline{p}'}\right)^{\frac{1}{\overline{p}'}}\right\}\in\ell^s.$$

Let $\{a_k\}$ be a sequence of nonnegative numbers and $f=\sum_{k\in\mathbf{Z}}a_kv^{1-\overline{p}'}\chi_{(k,k+\alpha_k)}$. If $n\in\mathbf{Z}$ and $x\in(n+\alpha_n,n+1)$, we have

$$T_gf(x)\ge g(x)a_n\int_n^{n+\alpha_n}v^{1-\overline{p}'}.$$

Then,

$$\begin{aligned}\|T_gf\|_{p,\infty,u,q}&\ge\left\{\sum_{n\in\mathbf{Z}}\|\chi_{(n+\alpha_n,n+1)}T_gf\|^q_{p,\infty;u}\right\}^{\frac{1}{q}}\\&\ge\left\{\sum_{n\in\mathbf{Z}}a_n^q\|g\chi_{(n+\alpha_n,n+1)}\|^q_{p,\infty;u}\left(\int_n^{n+\alpha_n}v^{1-\overline{p}'}\right)^q\right\}^{\frac{1}{q}}\end{aligned}$$

and (**??**) yields

$$\left\{\sum_{n\in\mathbf{Z}}a_n^q\|g\chi_{(n+\alpha_n,n+1)}\|^q_{p,\infty;u}\left(\int_n^{n+\alpha_n}v^{1-\overline{p}'}\right)^q\right\}^{\frac{1}{q}}\le C\left\{\sum_{n\in\mathbf{Z}}a_n^{\overline{q}}\left(\int_n^{n+\alpha_n}v^{1-\overline{p}'}\right)^{\frac{\overline{q}}{\overline{p}}}\right\}^{\frac{1}{\overline{q}}}.$$

Now, Lemma 1 gives $\{J_n\}\in\ell^s$.

(iv) If $1<p<\overline{p}<\infty$ and $1<q<\overline{q}<\infty$, from (**??**) and Theorem C we obtain that $\overline{B}<\infty$. As in *(ii)*, we show that $\sup_n\|\Phi_n\|_{r,\infty;u}<\infty$. Let us see now that $\{\|\Phi_n\|_{r,\infty;u}\}\in\ell^s$. For each n there exist $\lambda_n>0$ and a compact set $K_n\subset\{x\in(n,n+1):\Phi_n(x)>\lambda_n\}$ such that

$$\|\Phi_n\|_{r,\infty;u}-\lambda_n\left(\int_{K_n}u\right)^{\frac{1}{r}}<\frac{1}{2^{|n|}}.$$

It suffices to show that $\{\lambda_n\left(\int_{K_n}u\right)^{\frac{1}{r}}\}\in\ell^s$. For fixed n and $x\in K_n$, there exist a_n^x and b_n^x such that $n<a_n^x<x<b_n^x<n+1$ and

$$\inf_{y\in(a_n^x,b_n^x)}g(y)\left(\int_{a_n^x}^{b_n^x}u\right)^{\frac{1}{p}}\left(\int_n^{a_n^x}v^{1-\overline{p}'}\right)^{\frac{1}{\overline{p}'}}>\lambda_n.\tag{4.3}$$

Since the intervals (a_n^x, b_n^x) cover K_n, there exist $x_1, x_2, \ldots, x_{s_n}$ such that $K_n \subset \cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})$. We may suppose that $\sum_{j=1}^{s_n} \chi_{(a_n^{x_j}, b_n^{x_j})} \le 2\chi_{\cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})}$.

Let $\{A_k\}$ be a sequence of nonnegative numbers and

$$f = \sum_{n\in\mathbf{Z}} A_n \left(\sum_{j=1}^{s_n} \chi_{(n, a_n^{x_j})} \left(\inf_{y\in(a_n^{x_j}, b_n^{x_j})} g(y) \int_n^{a_n^{x_j}} v^{1-\overline{p}'} \right)^{-\overline{p}} \right)^{\frac{1}{\overline{p}}}.$$

If $n \in \mathbf{Z}$ and $x \in (a_n^{x_j}, b_n^{x_j})$, then $T_g f(x) \ge A_n$ and

$$\|T_g f\|_{p,\infty,u,q} \ge \left\{ \sum_{n\in\mathbf{Z}} A_n^q \|\chi_{\cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})}\|_{p,\infty;u}^q \right\}^{\frac{1}{q}} = \left\{ \sum_{n\in\mathbf{Z}} A_n^q \left(\int_{\cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})} u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}}.$$

On the other hand, by (**??**), we have for each n

$$\begin{aligned} \|f\chi_{(n,n+1)}\|_{\overline{p};v} &= A_n \left(\sum_{j=1}^{s_n} \left(\inf_{y\in(a_n^{x_j}, b_n^{x_j})} g(y) \right)^{-\overline{p}} \left(\int_n^{a_n^{x_j}} v^{1-\overline{p}'} \right)^{1-\overline{p}} \right)^{\frac{1}{\overline{p}}} \\ &\le A_n \left(\sum_{j=1}^{s_n} \lambda_n^{-\overline{p}} \int_{a_n^{x_j}}^{b_n^{x_j}} u \right)^{\frac{1}{\overline{p}}} \le 2^{\frac{1}{\overline{p}}} \frac{A_n}{\lambda_n} \left(\int_{\cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})} u \right)^{\frac{1}{\overline{p}}}. \end{aligned}$$

Therefore,

$$\|f\|_{\overline{p},v,\overline{q}} \le C \left\{ \sum_{n\in\mathbf{Z}} A_n^{\overline{q}} \frac{1}{\lambda_n^{\overline{q}}} \left(\int_{\cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})} u \right)^{\frac{\overline{q}}{\overline{p}}} \right\}^{\frac{1}{\overline{q}}}$$

and, by (**??**),

$$\left\{ \sum_{n\in\mathbf{Z}} A_n^q \left(\int_{\cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})} u \right)^{\frac{q}{p}} \right\}^{\frac{1}{q}} \le C \left\{ \sum_{n\in\mathbf{Z}} A_n^{\overline{q}} \frac{1}{\lambda_n^{\overline{q}}} \left(\int_{\cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})} u \right)^{\frac{\overline{q}}{\overline{p}}} \right\}^{\frac{1}{\overline{q}}}.$$

Applying Lemma 1, we obtain

$$\left\{ \lambda_n \left(\int_{\cup_{j=1}^{s_n}(a_n^{x_j}, b_n^{x_j})} u \right)^{\frac{1}{r}} \right\} \in \ell^s,$$

which implies $\left\{ \lambda_n \left(\int_{K_n} u \right)^{\frac{1}{r}} \right\} \in \ell^s$.

Conversely, let us prove the inequality (**??**) from each one of the conditions corresponding to the different cases of values of $p, \overline{p}, q, \overline{q}$.

Let $\lambda > 0$ and $n \in \mathbf{Z}$. Let us consider the sets

$$O_{\lambda,n} = \{x \in (n, n+1) : T_g f(x) > \lambda\},$$

$$A_{\lambda,n} = \left\{ x \in (n, n+1) : g(x) \int_{-\infty}^n f > \frac{\lambda}{2} \right\}$$

and

$$B_{\lambda,n} = \left\{ x \in (n, n+1) : g(x) \int_n^x f > \frac{\lambda}{2} \right\}.$$

It is clear that $O_{\lambda,n} \subset A_{\lambda,n} \cup B_{\lambda,n}$. Then,

$$\sup_{\lambda>0} \lambda \left(\int_{O_{\lambda,n}} u \right)^{\frac{1}{p}} \leq \sup_{\lambda>0} \lambda \left(\int_{A_{\lambda,n}} u \right)^{\frac{1}{p}} + \sup_{\lambda>0} \lambda \left(\int_{B_{\lambda,n}} u \right)^{\frac{1}{p}}.$$

By Minkowski's inequality in ℓ^q, we have

$$\begin{aligned} &\left\{ \sum_{n\in\mathbf{Z}} \left[\sup_{\lambda>0} \lambda \left(\int_{O_{\lambda,n}} u \right)^{\frac{1}{p}} \right]^q \right\}^{\frac{1}{q}} \\ &\leq \left\{ \sum_{n\in\mathbf{Z}} \left[\sup_{\lambda>0} \lambda \left(\int_{A_{\lambda,n}} u \right)^{\frac{1}{p}} \right]^q \right\}^{\frac{1}{q}} + \left\{ \sum_{n\in\mathbf{Z}} \left[\sup_{\lambda>0} \lambda \left(\int_{B_{\lambda,n}} u \right)^{\frac{1}{p}} \right]^q \right\}^{\frac{1}{q}} \\ &= I_1 + I_2. \end{aligned}$$

Let us see first that $I_1 \leq C\|f\|_{\overline{p},v,\overline{q}}$ in all cases. If $n \in \mathbf{Z}$ and $x \in (n, n+1)$,

$$g(x) \int_{-\infty}^n f = \sum_{k=-\infty}^n g(x) \int_{k-1}^k f = g(x) \sum_{k=-\infty}^n a_k,$$

where $a_k = \int_{k-1}^k f$. Then

$$I_1 = \left\{ \sum_{n\in\mathbf{Z}} \left(\sum_{k=-\infty}^n a_k \right)^q \|g\chi_{(n,n+1)}\|_{p,\infty,u}^q \right\}^{\frac{1}{q}}. \tag{4.4}$$

If $1 < \overline{q} \leq q < \infty$, the characterization of the strong type inequality for the discrete Hardy operator contained in Theorem C gives

$$\left\{ \sum_{n\in\mathbf{Z}} \left(\sum_{k=-\infty}^n a_k \right)^q \|g\chi_{(n,n+1)}\|_{p,\infty,u}^q \right\}^{\frac{1}{q}} \leq C \left\{ \sum_{n\in\mathbf{Z}} a_n^{\overline{q}} v_n \right\}^{\frac{1}{\overline{q}}}, \tag{4.5}$$

for all sequences $\{v_n\}$ verifying

$$D \equiv \sup_{m\in\mathbf{Z}} \left(\sum_{n=m}^\infty \|g\chi_{(n,n+1)}\|_{p,\infty,ud\mu}^q \right)^{\frac{1}{q}} \left(\sum_{n=-\infty}^m v_n^{1-\overline{q}'} \right)^{\frac{1}{\overline{q}'}} < \infty.$$

If $v_n = \left(\int_{n-1}^n v^{1-\overline{p}'} \right)^{-\frac{\overline{q}}{\overline{p}'}}$, we have $D \equiv B < \infty$, and therefore,

$$I_1 \leq C \left\{ \sum_{n\in\mathbf{Z}} a_n^{\overline{q}} v_n \right\}^{\frac{1}{\overline{q}}} = C \left\{ \sum_{n\in\mathbf{Z}} a_n^{\overline{q}} \left(\int_{n-1}^n v^{1-\overline{p}'} \right)^{-\frac{\overline{q}}{\overline{p}'}} \right\}^{\frac{1}{\overline{q}}}.$$

Applying Hölder inequality with exponents $\overline{p}$ and $\overline{p}'$ we obtain for each n

$$a_n \le \left(\int_{n-1}^{n} f^{\overline{p}} v\right)^{\frac{1}{\overline{p}}} \left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{\frac{1}{\overline{p}'}}$$

and the estimation of I_1 ends as follows

$$I_1 \le C\left\{\sum_{n\in\mathbf{Z}} \left(\int_{n-1}^{n} f^{\overline{p}} v\right)^{\frac{\overline{q}}{\overline{p}}} \left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{\frac{\overline{q}}{\overline{p}'}} \left(\int_{n-1}^{n} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}\right\}^{\frac{1}{\overline{q}}} = C\|f\|_{\overline{p},v,\overline{q}}.$$

If $1 < q < \overline{q} < \infty$, by Theorem C, the inequality (**??**) holds for all sequences $\{v_n\}$ verifying

$$\overline{B}' \equiv \sum_{n\in\mathbf{Z}} \left(\sum_{k=n}^{\infty} \|g\chi_{(k,k+1)}\|^{q}_{p,\infty;u}\right)^{\frac{s}{q}} \left(\sum_{k=-\infty}^{n} v_k^{1-\overline{q}'}\right)^{\frac{s}{q'}} v_n^{1-\overline{q}'} < \infty.$$

If $v_k = \left(\int_{k-1}^{k} v^{1-\overline{p}'}\right)^{-\frac{\overline{q}}{\overline{p}'}}$, we have that $\overline{B}' \equiv \overline{B}$ and therefore, $\overline{B}' < \infty$. Working from now on as in the previous case, we arrive to $I_1 \le C\|f\|_{\overline{p},v,\overline{q}}$.

Finally, let us estimate I_2.

(i) If $1 < \overline{p} \le p < \infty$ and $1 < \overline{q} \le q < \infty$, for each $n \in \mathbf{Z}$ we have $J_n < \infty$. We may apply Theorem B and obtain

$$\begin{aligned}&\sup_{\lambda>0} \lambda \left(u\left\{x \in (n, n+1) : g(x)\int_{n}^{x} f > \frac{\lambda}{2}\right\}\right)^{\frac{1}{p}} \\ &\le CJ_n \left(\int_{n}^{n+1} f^{\overline{p}} v\right)^{\frac{1}{\overline{p}}} \le CJ\left(\int_{n}^{n+1} f^{\overline{p}} v\right)^{\frac{1}{\overline{p}}}.\end{aligned}$$

Taking ℓ^q-norms, we arrive to $I_2 \le C\|f\|_{\overline{p},v,q}$ and since $\overline{q} \le q$, we get $I_2 \le C\|f\|_{\overline{p},v,\overline{q}}$.

(ii) If $1 < p < \overline{p} < \infty$ and $1 < \overline{q} \le q < \infty$, for each $n \in \mathbf{Z}$ we have $\Phi_n \in L^{r,\infty}(u)$ and Theorem B gives

$$\sup_{\lambda>0} \lambda \left(\int_{B_{\lambda,n}} u\right)^{\frac{1}{p}} \le C\|\Phi_n\|_{r,\infty;u} \left(\int_{n}^{n+1} f^{\overline{p}} v\right)^{\frac{1}{\overline{p}}} \le CF\left(\int_{n}^{n+1} f^{\overline{p}} v\right)^{\frac{1}{\overline{p}}}.$$

Taking ℓ^q-norms, we obtain $I_2 \le C\|f\|_{\overline{p},v,q}$. Since $\overline{q} \le q$, we finally have $I_2 \le C\|f\|_{\overline{p},v,\overline{q}}$.

(iii) If $1 < \overline{p} \le p < \infty$ and $1 < q < \overline{q} < \infty$, as in *(i)* we obtain

$$I_2 \le C\left\{\sum_{n\in\mathbf{Z}} J_n^q \left(\int_{n}^{n+1} f^{\overline{p}} v\right)^{\frac{q}{\overline{p}}}\right\}^{\frac{1}{q}}.$$

Since $\{J_n\} \in \ell^s$, Hölder inequality with exponents $\gamma = \frac{\overline{q}}{q}$ and γ' gives

$$I_2 \le C\left(\sum_{n\in\mathbf{Z}} J_n^{q\gamma'}\right)^{\frac{1}{q\gamma'}} \left(\sum_{n\in\mathbf{Z}} \left(\int_{n}^{n+1} f^{\overline{p}} v\right)^{\frac{q}{\overline{p}}\gamma}\right)^{\frac{1}{q\gamma}} = C\left(\sum_{n\in\mathbf{Z}} J_n^s\right)^{\frac{1}{s}} \|f\|_{\overline{p},v,\overline{q}}.$$

(iv) If $1 < p < \overline{p} < \infty$ and $1 < q < \overline{q} < \infty$, we have

$$I_2 \leq C \left\{ \sum_{n \in \mathbf{Z}} \|\Phi_n\|^q_{r,\infty;u} \left(\int_n^{n+1} f^{\overline{p}} v \right)^{\frac{q}{\overline{p}}} \right\}^{\frac{1}{q}}$$

and as in the previous case we obtain

$$I_2 \leq C \left(\sum_{n \in \mathbf{Z}} \|\Phi_n\|^s_{r,\infty;u} \right)^{\frac{1}{s}} \|f\|_{\overline{p},v,\overline{q}}.$$

□

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Pedro Ortega Salvador, Análisis Matemático, Facultad de Ciencias, Universidad de Málaga, 29071 Málaga, Spain
Email address: `portega@uma.es`

Consuelo Ramírez Torreblanca, Departamento de Matemáticas, Escuela Politécnica Superior, Universidad de Córdoba, Córdoba, Spain
Email address: `ma1ratoc@uco.es`