A course on Leavitt path algebras

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Chapter 1

Definitions, examples. Finite dimensional Leavitt path algebras

Introduction

Leavitt path algebras are a specific type of path K-algebras associated to a graph E, modulo some relations. Its appearance, for row-finite graphs, took place in [2] and [15]. They can be considered, on the one hand, natural generalizations of Leavitt algebras $L(1, n)$ of type $(1, n)$, introduced and investigated by Leavitt in [41] in order to give examples of algebras not satisfying the IBN property.

On the other hand, they are the algebraic version of Cuntz-Krieger graph C*-algebras, a class of algebras intensively investigated by analysts for more than two decades. For a complete explanation of the history of Leavitt path algebras see [20].

In this chapter we will relate Leavitt path algebras to the work by Leavitt and will give some important and interesting examples. Our main concern will be to classify the finite dimensional Leavitt path algebras.

1.1 The IBN property and the type of a ring.

Let R be a unital ring. We say that R satisfies the *invariant basis number (IBN) property* if any two bases (i.e., linearly independent spanning sets) for a free left R-module have the same number of elements. In words, the IBN property says that if m and n are integers with the property that the free left modules ${}_RR^m$ and ${}_RR^n$ are isomorphic, then $m = n$.

Noetherian rings and commutative rings are included among the many classes of rings having this property. But the IBN property does not hold for all rings, as the following example shows.

Example 1.1.1 For a field K, let $V = K^{(\mathbb{N})}$, which is a countably infinite dimensional vector space over K, and let $R = \mathrm{End}_K(V)$. It is not difficult to see that $R \cong \mathrm{RFM}_{\mathbb{N}}(K)$, the *countable row-finite matrices over* K (use the standard basis for V, view the elements of V as row-vectors, and apply transformations on the right hand side). Then ${}_RR^m \cong {}_RR^n$ for all $m, n \in \mathbb{N}$:

The first step is to show ${}_RR^1 \cong {}_RR^2$; such an isomorphism is given by the map which associates $X \in R$ with the pair of matrices (X_1, X_2), where X_1 (resp. X_2) is built from the odd-numbered (resp. even numbered) columns of X. But then ${}_RR^1 \cong {}_RR^2$ gives ${}_RR^1 \oplus {}_RR^1 \cong {}_RR^2 \oplus {}_RR^1$, so ${}_RR^2 \cong {}_RR^3$, and the result follows by continuing in this way.

It is easy to determine, algebraically, whether or not for a ring R we have ${}_RR^1 \cong {}_RR^n$ for some $n > 1$. Note that such an isomorphism exists if and only if there is a set of $2n$ elements in R which produce the appropriate isomorphisms as matrix multiplications by an n-row vector and an n-column vector with entries in R. Specifically, it is easy to show that ${}_RR^1 \cong {}_RR^n$ for some $n > 1$ if and only if there exist elements $x_1, ..., x_n, y_1, ..., y_n \in R$ for which $x_i y_j = \delta_{ij} 1_R$ for all i, j, and $\sum_{i=1}^n y_i x_i = 1_R$.

Suppose that a unital ring R does not have IBN. Let $m \in \mathbb{N}$ be minimal with ${}_RR^m \cong {}_RR^n$ for some $n > m$ and find the minimal such n for m. Then it is said that R has *module type* (m, n). (Warning: some authors call the module type of such a ring $(m, n-m)$.) For example, $\mathrm{RFM}_{\mathbb{N}}(K)$ has module type (1,2).

In his paper, Leavitt proved that for each pair of positive integers $n > m$ and any field K there exists a K-algebra of module type (m, n). To do this, observe that, as it has been used before, isomorphisms between free modules can be realized as matrix multiplications by matrices having coefficients in R. So we need only construct algebras which contain elements which behave "correctly". Do this as a quotient of a free associative K-algebra in the appropriate number of variables satisfying the appropriate relations.

For example, to get an algebra of type (1,3) we need an algebra containing elements $x_1, x_2, x_3, y_1, y_2, y_3$ for which $x_i y_j = \delta_{ij} 1_R$ for all i, j, and $\sum_{i=1}^3 y_i x_i = 1_R$. Consider the polynomial algebra over a field K in 6 non-commuting variables. Then factor by the ideal generated by the appropriate relations. It is not difficult to show that this quotient is not zero if $m \geq 2$, but this is much more difficult to show (directly) if $m = 1$.

The quotient algebra described above is denoted $L_K(m,n)$, and called the *Leavitt K-algebra of type* (m,n).

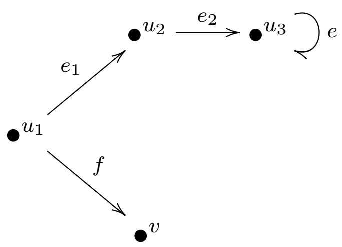
1.2 Path algebras and Leavitt path algebras

Definitions 1.2.1 A *(directed) graph* $E = (E^0, E^1, r, s)$ consists of two countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*.

If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called *row-finite*. If E^0 is finite then, by the row-finite hypothesis, E^1 must necessarily be finite as well; in this case we simply say that E is *finite*. A vertex which emits no edges is called a *sink*. A *path* μ in a graph E is a sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $s(\mu) := s(e_1)$ is the *source* of μ, $r(\mu) := r(e_n)$ is the *range* of μ, and n is the *length* of μ, i.e, $l(\mu) = n$. We denote by μ^0 the set of its vertices, that is: $\mu^0 = \{s(e_1), r(e_i) : i = 1, \dots, n\}$.

Although much work has been done on arbitrary graphs, we will be concerned only with row-finite graphs.

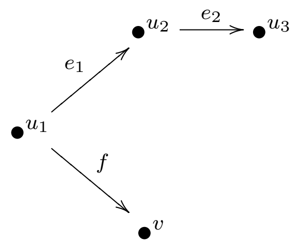
Example 1.2.2 Consider the following graph:



Then $E^0 = \{u_1, u_2, u_3, v\}$, $E^1 = \{e_1, e_2, ef\}$, $r(e_1) = u_1$, $s(e_1) = u_2$, etc. The vertex v is a sink. Some paths are, for example, v, f, e_1e_2, e, ee, e_2eee, etc. For $\mu = e_1e_2e^3$, $\mu^0 = \{u_1, u_2, u_3\}$.

Definition 1.2.3 Now let K be a field and let KE denote the K-vector space which has as a basis the set of paths. It is possible to define an algebra structure on KE as follows: for any two paths $\mu = e_1 \dots e_m, \nu = f_1 \dots f_n$, we define $\mu\nu$ as zero if $r(\mu) \neq s(\nu)$ and as $e_1 \dots e_m f_1 \dots f_n$ otherwise. This K-algebra is called the *path algebra* of E over K.

Example 1.2.4 Consider a field K and the following graph E:



Then the path algebra KE has, as a vector space over K, dimension 8, while the path K-algebra asociated to the graph in example 1.2.2 is infinite dimensional.

There are several ways of defining Leavitt path algebras.

Definition 1.2.5 Given a graph E we define the *extended graph of* E as the new graph $\widehat{E} = (E^0, E^1 \cup (E^1)^*, r', s')$, where $(E^1)^* = \{e_i^* : e_i \in E^1\}$ and the functions r' and s' are defined as

$$r'|_{E^1} = r,\ s'|_{E^1} = s,\ r'(e_i^*) = s(e_i) \text{ and } s'(e_i^*) = r(e_i).$$

Definition 1.2.6 Let K be a field and E be a row-finite graph. The *Leavitt path algebra of* E *with coefficients in* K is defined as the path algebra over the extended graph $\widehat{E}$, with relations:

(CK1) $e_i^* e_j = \delta_{ij} r(e_j)$ for every $e_j \in E^1$ and $e_i^* \in (E^1)^*$.

(CK2) $v_i = \sum_{\{e_j \in E^1 : s(e_j) = v_i\}} e_j e_j^*$ for every $v_i \in E^0$ which is not a sink.

This algebra is denoted by $L_K(E)$ (or by $L(E)$ if there is no risk of confusion with the field K).

The conditions (CK1) and (CK2) are called the *Cuntz-Krieger relations.* In particular condition (CK2) is the *Cuntz-Krieger relation at* v_i. If v_i is a sink, we do not have a (CK2) relation at v_i. Note that the condition of row-finiteness is needed in order to define the equation (CK2).

There exists a natural inclusion of the path algebra KE into the Leavitt path algebra $L_K(E)$ sending vertices to vertices and edges to edges. We will use this monomorphism without any explicit mention to it.

Another way of introducing Leavitt path algebras is as follows.

Definition 1.2.7 For a field K and a row-finite graph E, the *L*eavitt path K-algebra $L_K(E)$ is defined as the universal K-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set of variables $\{e, e^* : e \in E^1\}$, which satisfy the following relations:

(1) $s(e)e = er(e) = e$ for all $e \in E^1$.
(2) $r(e)e^* = e^*s(e) = e^*$ for all $e \in E^1$.
(3) $e^*e' = \delta_{e,e'}r(e)$ for all $e, e' \in E^1$.
(4) $v = \sum_{\{e \in E^1 | s(e)=v\}} ee^*$ for every $v \in E^0$ that emits edges.

Universal means that if A is a K-algebra containing a set of pairwise orthogonal idempotents $\{a_v : v \in E^0\}$ and a set of elements $\{b_e, b_{e^*} : e \in E^1\}$ satisfying the relations (1)-(4), then there exists an algebra homomorphism $\Phi : L_K(E) \to A$ satisfying $\Phi(v) = a_v$ for all $v \in E^0$, $\Phi(e) = b_e$ and $\Phi(e^*) = b_{e^*}$ for all $e \in E^1$.

The uniqueness of the Leavitt path algebra associated to a graph E and to a field K follows from the universal property.

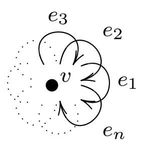
Again the identities (3) and (4) are called the *Cuntz-Krieger relations*.

The elements of E^1 are called *(real) edges*, while for $e \in E^1$ we call e^* a *ghost edge*. The set $\{e^* \mid e \in E^1\}$ will be denoted by $(E^1)^*$. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path, then we denote by μ^* the element $e_n^* \dots e_1^*$ of $L_K(E)$.

1.3 Examples of Leavitt path algebras

Many well-known examples of K- algebras can be seen as Leavitt path K-algebras over concrete graphs.

Example 1.3.1 (Leavitt algebras of type $(1, n)$, $n > 1$). The Leavitt path K-algebra of the following graph

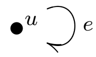


is the Leavitt algebra of type $(1, n)$, for $n > 1$.

Example 1.3.2 (Matrix algebras). Consider the graph

Then $M_{n+1}(K) \cong L_K(E)$, via the map $u_i \mapsto e_{ii}, e_i \mapsto e_{i+1i}$, and $e_i^* \mapsto e_{ii+1}$, where e_{ij} denotes the matrix unit in $M_n(K)$ with all entries equal zero except that in row i and column j.

Example 1.3.3 (The Laurent polynomial ring). Consider the graph



Then the map $\varphi : L_K(E) \to K[x, x^{-1}]$ given on generators (as a K- vector space) by $\varphi(e^n) = x^n$ and $\varphi((e^*)^n) = x^{-n}$ produces an isomorphism between $L_K(E)$ and $K[x, x^{-1}]$.

A path μ is called a *cycle* if $s(\mu) = r(\mu)$ and $s(\mu_i) \neq s(\mu_j)$ for every $i \neq j$. A graph E without cycles is said to be *acyclic*.

1.4 Finite dimensional Leavitt path algebras

We finish this chapter by showing that the Leavitt path K- algebra of every finite and acyclic graph is a direct sum of matrices of finite size over K. In particular, they are semisimple and artinian. The converse is also true, that is, every semisimple and artinian Leavitt path algebra is associated to a finite and acyclic graph.

First we need to state that Leavitt path algebras are $\mathbb{Z}$-graded algebras, and that every set of paths is linearly independent.

Lemma 1.4.1 *Every monomial in $L_K(E)$ is of the following form:*

(i) $k_i v_i$ *with* $k_i \in K$ *and* $v_i \in E^0$*, or*

(ii) $k e_{i_1} \dots e_{i_\sigma} e^*_{j_1} \dots e^*_{j_\tau}$ *where* $k \in K$*;* $\sigma,\ \tau \geq 0,\ \sigma + \tau > 0,\ e_{i_s} \in E^1$ *and* $e^*_{j_t} \in (E^1)^*$ *for* $0 \leq s \leq \sigma, 0 \leq t \leq \tau$*.*

Proof. Follow the proof of [47, Corollary 1.15], a straightforward induction argument on the length of the monomial $kx_1 \dots x_n$ with $x_i \in E^0 \cup E^1 \cup (E^1)^*$.
□

Although not every Leavitt path algebra is unital (this happens, for example, when the number of vertices is infinite), they are “nearly unital”, concretely, they are algebras with *local units*, i.e., for E a graph, K a field and $L_K(E)$ the associated Leavitt path algebra, there exists a set of idempotentes $\{u_n\}_{n\in\mathbb{N}}$ in $L_K(E)$ satisfying the following properties:

(1) $u_n \in u_{n+1} L_K(E) u_{n+1}$,

(2) for every finite subset $X \subseteq L_K(E)$ there exists $m \in \mathbb{N}$ such that $X \subseteq u_m L_K(E) u_m$.

This statement is proved in the following result.

Lemma 1.4.2 *If E^0 is finite then $L(E)$ is a unital K-algebra. If E^0 is infinite, then $L(E)$ is an algebra with local units (specifically, the set generated by finite sums of distinct elements of E^0).*

Proof. First assume that E^0 is finite: We will show that $\sum_{i=1}^n v_i$ is the unit element of the algebra. First we compute $(\sum_{i=1}^n v_i)v_j = \sum_{i=1}^n \delta_{ij}v_j = v_j$. Now if we take $e_j \in E^1$ we may use the equations (2) in the definition of path algebra together with the previous computation to get $(\sum_{i=1}^n v_i)e_j = (\sum_{i=1}^n v_i)r(e_j)e_j = r(e_j)e_j = e_j$. In a similar manner we see that $(\sum_{i=1}^n v_i)e_j^* = e_j^*$. Since $L(E)$ is generated by $E^0 \cup E^1 \cup (E^1)^*$, then it is clear that $(\sum_{i=1}^n v_i)\alpha = \alpha$ for every $\alpha \in L(E)$, and analogously $\alpha(\sum_{i=1}^n v_i) = \alpha$ for every $\alpha \in L(E)$. Now suppose that E^0 is infinite. Consider a finite subset $\{a_i\}_{i=1}^t$ of $L(E)$ and use 1.4.1 to write $a_i = \sum_{s=1}^{n_i} k_s^i v_{j_s{}^i} + \sum_{l=1}^{m_i} k_l'^i p_l^i$ where $k_s^i, k_l'^i \in K - \{0\}$, and p_l^i are monomials of type (b). Then with the same ideas as above it is not difficult to prove that

$$\alpha = \sum_{i=1}^{t} \left(\sum_{s=1}^{n_i} v_{j_s{}^i} + \sum_{l=1}^{m_i} r(p_l^i) + \sum_{l=1}^{m_i} s(p_l^i) \right)$$

is a finite sum of vertices such that $\alpha a_i = a_i \alpha = a_i$ for every i. □

Definitions 1.4.3 An algebra A is said to be $\mathbb{Z}$-graded if there exists a family $\{A_n\}_{n\in\mathbb{Z}}$ of subspaces of A such that $A = \oplus_{n\in\mathbb{Z}} A_n$ and $A_n A_m \subseteq A_{n+m}$ for every $n, m \in \mathbb{Z}$.

Every element in $\cup_{n\in\mathbb{Z}} A_n$ is called a *homogeneous element*, and A_n is called the *homogeneous component of degree* n.

Example 1.4.4 The algebra $K[x]$ of polynomials over a field K over an indeterminate x is a $\mathbb{Z}$-graded algebra, where for each $n \in \mathbb{Z}$, the homogeneous component of degree n is the vector space generated by x^n.

Now we see that every Leavitt path algebra is a $\mathbb{Z}$-graded algebra and describe the grading.

Lemma 1.4.5 *$L(E)$ is a $\mathbb{Z}$-graded algebra, with grading induced by*

$$deg(v_i) = 0 \text{ for all } v_i \in E^0; deg(e_i) = 1 \text{ and } deg(e_i^*) = -1 \text{ for all } e_i \in E^1.$$

That is, $L(E) = \bigoplus_{n\in\mathbb{Z}} L(E)_n$, where $L(E)_0 = KE^0 + A_0$, $L(E)_n = A_n$ for $n \neq 0$ where

$$A_n = \sum\{ke_{i_1} \dots e_{i_\sigma} e_{j_1}^* \dots e_{j_\tau}^* : \sigma + \tau > 0,\ e_{i_s} \in E^1,\ e_{i_t} \in (E^1)^*,\ k \in K,\ \sigma - \tau = n\}.$$

Proof. The fact that $L(E) = \sum_{n\in\mathbb{Z}} L(E)_n$ follows from 1.4.1. The grading on $L(E)$ follows directly from the fact that $A(\widehat{E})$ is $\mathbb{Z}$-graded, and that the relations (CK1) and (CK2) are homogeneous in this grading. □

Note that the natural monomorphism from the path algebra KE into the Leavitt path algebra $L_K(E)$ is graded, hence KE is a $\mathbb{Z}$-graded subalgebra of $L_K(E)$.

The following result appears in [52].

Lemma 1.4.6 *Let E be a graph and K a field. Any set of different paths is K-linearly independent in $L_K(E)$.*

Proof. Let $\mu_1, \dots, \mu_n$ be different paths. Write $\sum_i k_i\mu_i = 0$, for $k_i \in K$. Applying that $L_K(E)$ is $\mathbb{Z}$-graded we may suppose that all the paths have the same length. Since $\mu_j^*\mu_i = \delta_{ij} r(\mu_j)$ then $0 = \sum_i k_i\mu_j^*\mu_i = k_j r(\mu_j)$; this implies $k_j = 0$. □

The results that follow appear in [6].

Lemma 1.4.7 *Let E be a finite and acyclic graph and $v \in E^0$ a sink. Then*

$$I_v := \sum\{k\alpha\beta^* : \alpha, \beta \in E^*, r(\alpha) = v = r(\beta),\ k \in K\}$$

is an ideal of $L(E)$ (in fact, it is the ideal generated by v), and $I_v \cong \mathbb{M}_{n(v)}(K)$, where $n(v)$ denotes the number of paths ending at v.

Proof. Consider $\alpha\beta^* \in I_v$ and a nonzero monomial $e_{i_1} \dots e_{i_n} e_{j_1}^* \dots e_{j_m}^* = \gamma\delta^* \in L(E)$. If $\gamma\delta^*\alpha\beta^* \neq 0$ we have two possibilities: Either $\alpha = \delta p$ or $\delta = \alpha q$ for some paths $p, q \in E^*$.

In the latter case $deg(q) \geq 1$ cannot happen, since v is a sink.

Therefore we are in the first case (possibly with $deg(p) = 0$), and then

$$\gamma\delta^*\alpha\beta^* = (\gamma p)\beta^* \in I_v$$

because $r(\gamma p) = r(p) = v$. This shows that I_v is a left ideal. Similarly we can show that I_v is a right ideal as well.

Let $n = n(v)$ (which is clearly finite because the graph is acyclic, finite and row-finite), and rename $\{\alpha \in E^* : r(\alpha) = v\}$ as $\{p_1, \dots, p_n\}$ so that

$$I_v := \sum\{kp_ip_j^* : i, j = 1, \dots, n; k \in K\}.$$

Take $j \neq t$. If $(p_ip_j^*)(p_tp_l^*) \neq 0$, then as above, $p_t = p_jq$ with $deg(q) > 0$ (since $j \neq t$), which contradicts that v is a sink.

Thus, $(p_i p_j^*)(p_t p_l^*) = 0$ for $j \neq t$. It is clear that

$$(p_i p_j^*)(p_j p_l^*) = p_i v p_l^* = p_i p_l^*.$$

We have shown that $\{p_i p_j^* : i, j = 1, \ldots, n\}$ is a set of matrix units for I_v, and the result now follows. □

Proposition 1.4.8 *Let E be a finite and acyclic graph. Let $\{v_1, \ldots, v_t\}$ be the sinks. Then*

$$L(E) \cong \bigoplus_{i=1}^{t} \mathbb{M}_{n(v_i)}(K).$$

Proof. We will show that $L(E) \cong \bigoplus_{i=1}^{t} I_{v_i}$, where I_{v_i} are the sets defined in Lemma 1.4.7.

Consider $0 \neq \alpha\beta^*$ with $\alpha, \beta \in E^*$. If $r(\alpha) = v_i$ for some i, then $\alpha\beta^* \in I_{v_i}$. If $r(\alpha) \neq v_i$ for every i, then $r(\alpha)$ is not a sink, and the relation (4) in the definition of $L_K(E)$ applies to yield:

$$\alpha\beta^* = \alpha \left(\sum_{\substack{e \in E^1 \\ s(e) = r(\alpha)}} ee^* \right) \beta^* = \sum_{\substack{e \in E^1 \\ s(e) = r(\alpha)}} \alpha e (\beta e)^*.$$

Now since the graph is finite and there are no cycles, for every summand in the expression above, either the summand is already in some I_{v_i}, or we can repeat the process (expanding as many times as necessary) until reaching sinks. In this way $\alpha\beta^*$ can be written as a sum of terms of the form $\alpha\gamma(\beta\gamma)^*$ with $r(\alpha\gamma) = v_i$ for some i. Thus $L(E) = \sum_{i=1}^{t} I_{v_i}$.

Consider now $i \neq j$, $\alpha\beta^* \in I_{v_i}$ and $\gamma\delta^* \in I_{v_j}$. Since v_i and v_j are sinks, we know as in Lemma 1.4.7 that there are no paths of the form $\beta\gamma'$ or $\gamma\beta'$, and hence $(\alpha\beta^*)(\gamma\delta^*) = 0$. This shows that $I_{v_i} I_{v_j} = 0$, which together with the facts that $L(E)$ is unital and $L(E) = \sum_{i=1}^{t} I_{v_i}$, implies that the sum is direct. Finally, Lemma 1.4.7 gives the result. □

We now get as corollaries to Proposition 1.4.8 the two results mentioned.

Theorem 1.4.9 *The Leavitt path algebra $L_K(E)$ is a finite dimensional K-algebra if and only if E is a finite and acyclic graph.*

Proof. If E is finite and acyclic, then Proposition 1.4.8 immediately yields that $L_K(E)$ is finite dimensional.

Suppose on the other hand that E is not finite; in other words, the set E^0 of vertices is infinite. But then $\{v \mid v \in E^0\}$ is a linearly independent set in $L_K(E)$. Furthermore, if E is not acyclic, then there is a vertex v and a closed path μ based at v. But then $\{\mu^n \mid n \geq 1\}$ is a linearly independent set in $L_K(E)$. □

Combining Proposition 1.4.8 with Theorem 1.4.9 immediately yields

Corollary 1.4.10 *The only finite dimensional K-algebras which arise as $L_K(E)$ for a graph E are of the form $A = \bigoplus_{i=1}^{t} \mathbb{M}_{n_i}(K)$.*

Chapter 2

Uniqueness Theorems. Simple Leavitt path algebras

Introduction

We start this chapter by studying semiprimeness of path algebras and of Leavitt path algebras, property that differs in each context. Our following goal will be to describe graded ideals in Leavitt path algebras.

2.1 Semiprimeness in path algebras and in Leavitt path algebras

We first study the semiprimeness of the path algebra associated to a graph E. Recall that an algebra A is said to be *semiprime* if it has no nonzero ideals of zero square, equivalently, if $aAa = 0$ for $a \in A$ implies $a = 0$ (an algebra A that satisfies this last condition is called in the literature *nondegenerate*).

Proposition 2.1.1 *([52, Proposition 2.1]). For a graph E and a field K the path algebra KE is semiprime if and only if for every path μ there exists a path μ' such that $s(\mu') = r(\mu)$ and $r(\mu') = s(\mu)$.*

Proof. Suppose first that KE is semiprime. Given a path μ, since $\mu(KE)\mu \neq 0$, there exists a path $\nu \in KE$ such that $\mu\nu\mu \neq 0$. This means that $s(\nu) = r(\mu)$ and $r(\nu) = s(\mu)$.

Now, let us prove the converse. Note that by [44, Proposition II.1.4 (1)], a $\mathbb{Z}$-graded algebra is semiprime if and only if it is graded semiprime. Hence, and taking into account that being graded semiprime and graded nondegenerate are equivalent, it suffices to show that if x is any nonzero homogeneous element of KE, then $x(KE)x \neq 0$. Write $x = \sum_{i=1}^{n} k_i\alpha_i$, with $0 \neq k_i \in K$ and $\alpha_1, \ldots, \alpha_n$ different paths of the same degree (i.e. of the same length). Denote the source and range of α_1 by u_1 and v_1, respectively. Then, by (3), ${\alpha_1}^*x = k_1{\alpha_1}^*\alpha_1 = k_1v_1$. By the hypothesis, there exists a path ${\alpha_1}'$ such that $s({\alpha_1}') = v_1$ and $r({\alpha_1}') = u_1$. Observe that ${\alpha_1}'x \neq 0$; otherwise $0 = ({\alpha_1}')^*{\alpha_1}'x = u_1x$, a contradiction since a set of different paths is always linearly independent over K (Lemma 1.4.6) and $\alpha_1 = u_1\alpha_1 \neq 0$. Therefore $0 \neq k_1{\alpha_1}'x = k_1v_1{\alpha_1}'x = {\alpha_1}^*x{\alpha_1}'x \in {\alpha_1}^*x(KE)x$. □

Lemma 2.1.2 *([18, Lemma 1.5]). Let E be an arbitrary graph. Let v be a vertex in E^0 such that there exists a cycle without exits c based at v. Then:*

$$vL_K(E)v = \left\{ \sum_{i=-m}^{n} k_ic^i \mid k_i \in K;\ m, n \in \mathbb{N} \right\} \cong K[x, x^{-1}],$$

where $\cong$ denotes a graded isomorphism of K-algebras, and considering (by abuse of notation) $c^0 = w$ and $c^{-t} = (c^*)^t$, for any $t \geq 1$.

Proof. First, it is easy to see that if $c = e_1 \ldots e_n$ is a cycle without exits based at v and $u \in T(v)$, then $s(f) = s(g) = u$, for $f, g \in E^1$, implies $f = g$. Moreover, if $r(h) = r(j) = w \in T(v)$, with $h, j \in E^1$, and $s(h),\ s(j) \in T(v)$ then $h = j$. We have also that if $\mu \in E^*$ and $s(\mu) = u \in T(v)$ then there exists $k \in \mathbb{N},\ 1 \leq k \leq n$ verifying $\mu = e_k\mu'$ and $s(e_k) = u$.

Let $x \in vL_K(E)v$ be given by $x = \sum_{i=1}^{p} k_i\alpha_i\beta_i^* + \delta v$, with $s(\alpha_i) = r(\beta_i^*) = s(\beta_i) = v$ and $\alpha_i,\ \beta_i \in E^*$. Consider $A = \{\alpha \in E^* \colon s(\alpha) = v\}$; we prove now that if $\alpha \in A$, $deg(\alpha) = mn + q,\ m,\ q \in \mathbb{N}$ with $0 \leq q < n$, then $\alpha = c^me_1 \ldots e_q$. We proceed by induction on $deg(\alpha)$. If $deg(\alpha) = 1$ and $s(\alpha) = s(e_1)$ then $\alpha = e_1$. Suppose now that the result holds for any $\beta \in A$ with $deg(\beta) \leq sn + t$ and consider any $\alpha \in A$, with $deg(\alpha) = sn + t + 1$. We can write $\alpha = \alpha'f$ with $\alpha' \in A,\ f \in E^1$ and $deg(\alpha') = sn + t$, so by the induction hypothesis $\alpha' = c^se_1 \ldots e_t$. Since $s(f) = r(e_t) = s(e_{t+1})$ implies $f = e_{t+1}$, then $\alpha = \alpha'f = c^se_1 \ldots e_{t+1}$.

We shall show that the elements $\alpha_i\beta_i^*$ are in the desired form, i.e., c^d with $d \in \mathbb{Z}$. Indeed, if $deg(\alpha_i) = deg(\beta_i)$ and $\alpha_i\beta_i^* \neq 0$, we have $\alpha_i\beta_i^* = c^pe_1 \ldots e_ke_k^* \ldots e_1^*c^{-p} = v$ by (4). On the other hand $deg(\alpha_i) > deg(\beta_i)$ and $\alpha_i\beta_i^* \neq 0$ imply $\alpha_i\beta_i^* = c^{d+q}e_1 \ldots e_ke_k^* \ldots e_1^*c^{-q} = c^d,\ d \in \mathbb{N}^*$. In a similar way, from $deg(\alpha_i) < deg(\beta_i)$ and $\alpha_i\beta_i^* \neq 0$ it follows that $\alpha_i\beta_i^* = c^qe_1 \ldots e_ke_k^* \ldots e_1^*c^{-q-d} = c^{-d},\ d \in \mathbb{N}^*$. Define $\varphi \colon K[x, x^{-1}] \to L_K(E)$ by $\varphi(1) = v,\ \varphi(x) = c$ and $\varphi(x^{-1}) = c^*$. It is a straightforward routine to check that φ is a graded monomorphism with image $vL_K(E)v$, so that $vL_K(E)v$ is graded isomorphic to $K[x, x^{-1}]$ as a graded K-algebra. □

For a not necessarily associative K-algebra A, and fixed $x, y \in A$, the left and right *multiplication operators* $L_x, R_y\colon A \to A$ are defined by $L_x(y) := xy$ and $R_y(x) := xy$. Denoting by $\mathrm{End}_K(A)$ the K-algebra of K-linear maps $f\colon A \to A$, the *multiplication algebra* of A (denoted $\mathcal{M}(A)$) is the subalgebra of $\mathrm{End}_K(A)$ generated by the unit and all left and right multiplication operators $L_a, R_a : A \to A$. There is a natural action of $\mathcal{M}(A)$ on A such that A is an $\mathcal{M}(A)$-module whose submodules are just the ideals of A. This is given by $\mathcal{M}(A) \times A \longrightarrow A$, where $f \cdot a := f(a)$ for any $(f, a) \in \mathcal{M}(A) \times A$. Given $x, y \in A$ we shall say that x *is linked to* y if there is some $f \in \mathcal{M}(A)$ such that $y = f(x)$. This fact will be denoted by $x \vdash y$.

The result that follows states that any nonzero element in a Leavitt path algebra is linked to either a vertex or to a nonzero polynomial in a cycle with no exits. So it gives a full account of the action of $\mathcal{M}(L_K(E))$ on $L_K(E)$. This result is very powerful as the main ingredient to show that the socle of a Leavitt path algebra of a row-finite graph is the ideal generated by the line points. Other interesting results are also obtained as a consequence of it.

Proposition 2.1.3 *([17, Proposition 3.1]). Let E be an arbitrary graph. Then, for every nonzero element $x \in L_K(E)$, there exist $\mu_1, \ldots, \mu_r, \nu_1, \ldots, \nu_s \in E^0 \cup E^1 \cup (E^1)^*$ such that:*

1. *$\mu_1 \ldots \mu_r x \nu_1 \ldots \nu_s$ is a nonzero element in Kv, for some $v \in E^0$, or*
2. *there exist a vertex $w \in E^0$ and a cycle without exits c based at w such that $\mu_1 \ldots \mu_r x \nu_1 \ldots \nu_s$ is a nonzero element in $wL_K(E)w$.*

Both cases are not mutually exclusive.

Proof. Show first that for a nonzero element $x \in L(E)$, there exists a path $\mu \in L(E)$ such that $x\mu$ is nonzero and in only real edges.

Consider a vertex $v \in E^0$ such that $xv \neq 0$. Write $xv = \sum_{i=1}^{m} \beta_i e_i^* + \beta$, with $e_i \in E^1$, $e_i \neq e_j$ for $i \neq j$ and $\beta_i, \beta \in L(E)$, β in only real edges and such that this is a minimal representation of xv in ghost edges.

If $xve_i = 0$ for every $i \in \{1, \ldots, m\}$, then $0 = xve_i = \beta_i + \beta e_i$, hence $\beta_i = -\beta e_i$, and $xv = \sum_{i=1}^{m} -\beta e_i e_i^* + \beta = \beta(\sum_{i=1}^{m} -e_i e_i^* + v) \neq 0$. This implies that $\sum_{i=1}^{m} -e_i e_i^* + v \neq 0$ and since $s(e_i) = v$ for every i, this means that there exists $f \in E^1$, $f \neq e_i$ for every i, with $s(f) = v$. In this case, $xvf = \beta f \neq 0$ (because β is in only real edges), with βf in only real edges, which would conclude our discussion.

If $xve_i \neq 0$ for some i, say for $i = 1$, then $0 \neq xve_1 = \beta_1 + \beta e_1$, with $\beta_1 + \beta e_1$ having strictly less degree in ghost edges than x.

Repeating this argument, in a finite number of steps we prove our first statement.

Now, assume $x = xv$ for some $v \in E^0$ and x in only real edges. Let $0 \neq x = \sum_{i=1}^{r} k_i\alpha_i$ be a linear combination of different paths α_i with $k_i \neq 0$ for any i. We prove by induction on r that after multiplication on the left and/or the right we get a vertex or a polynomial in a cycle with no exit. For $r = 1$ if α_1 has degree 0 then it is a vertex and we have finished. Otherwise we have $x = k_1\alpha_1 = k_1 f_1 \cdots f_n$ so that $k_1^{-1} f_n^* \cdots f_1^* x = v$ where $v = r(f_n) \in E^0$.

Suppose now that the property is true for any nonzero element which is a sum of less than r paths in the conditions above. Let $0 \neq x = \sum_{i=1}^{r} k_i\alpha_i$ such that $\deg(\alpha_i) \leq \deg(\alpha_{i+1})$ for any i. If for some i we have $\deg(\alpha_i) = \deg(\alpha_{i+1})$ then, since $\alpha_i \neq \alpha_{i+1}$, there is some path μ such that $\alpha_i = \mu f \nu$ and $\alpha_{i+1} = \mu f' \nu'$ where $f, f' \in E^1$ are different and ν, ν' are paths. Thus $0 \neq f^*\mu^* x$ and we can apply the induction hypothesis to this element. So we can go on supposing that $\deg(\alpha_i) < \deg(\alpha_{i+1})$ for each i.

We have $0 \neq \alpha_1^* x = k_1 v + \sum_i k_i\beta_i$, where $v = r(\alpha_1)$ and $\beta_i = \alpha_1^*\alpha_i$. If some β_i is null then apply the induction hypothesis to $\alpha_1^* x$ and we are done. Otherwise if some β_i does not start (or finish) in v we apply the induction hypothesis to $v\alpha_1^* x \neq 0$ (or $\alpha_1^* x v \neq 0$). Thus we have

$$0 \neq z := \alpha_1^* x = k_1 v + \sum_{i=1}^{r} k_i\beta_i,$$

where $0 < \deg(\beta_1) < \cdots < \deg(\beta_r)$ and all the paths β_i start and finish in v.

Now, if there is a path τ such that $\tau^*\beta_i = 0$ for some β_i but not for all of them, then we apply our inductive hypothesis to $0 \neq \tau^* z \tau$. Otherwise for any path τ such that $\tau^*\beta_j = 0$ for some β_j, we have $\tau^*\beta_i = 0$ for all β_i. Thus $\beta_{i+1} = \beta_i r_i$ for some path r_i and z can be written as

$$z = k_1 v + k_2\gamma_1 + k_3\gamma_1\gamma_2 + \cdots + k_r\gamma_1\cdots\gamma_{r-1},$$

where each path γ_i starts and finishes in v. If the paths γ_i are not identical we have $\gamma_1 \neq \gamma_i$ for some i, then $0 \neq \gamma_i^* z \gamma_i = k_1 v$ proving our thesis. If the paths are identical then z is a polynomial in the cycle $c = \gamma_1$ with independent term $k_1 v$, that is, an element in $vL(E)v$.

If the cycle has an exit, it can be proved that there is a path η such that $\eta^* c = 0$, in the following way: Suppose that there is a vertex $w \in T(v)$, and two edges e, f, with $e \neq f$, $s(e) = s(f) = w$, and such that $c = aweb = aeb$, for a and b paths in $L(E)$. Then $\eta = af$ gives $\eta^* c = f^* a^* aeb = f^* eb = 0$. Therefore, $\eta^* z \eta$ is a nonzero scalar multiple of a vertex.

Moreover, if c is a cycle without exits, by Lemma 2.1.2,

$$vL(E)v = \left\{ \sum_{i=-m}^{n} l_i c^i, \text{ with } l_i \in K \text{ and } m, n \in \mathbb{N} \right\},$$

where we understand $c^{-m} = (c^*)^m$ for $m \in \mathbb{N}$ and $c^0 = v$.

Finally, consider the graph E consisting of one vertex and one loop based at the vertex to see that both cases can happen at the same time. This completes the proof. □

Corollary 2.1.4 *For any nonzero $x \in L$ we have $x \vdash v$ for some $v \in E^0$ or $x \vdash p(c, c^*)$ where c is a cycle with no exits and p a nonzero polynomial in c and c^*.*

Proof. Use Lemma 2.1.2 together with Proposition 2.1.3. □

Proposition 2.1.5 *Let E be an arbitrary graph. Then $L_K(E)$ is semiprime.*

Proof. Take a nonzero ideal I such that $I^2 = 0$. If I contains a vertex we are done. On the contrary there is a nonzero element $p(c, c^*) \in I$ by Corollary 2.1.4. If we consider the (nonzero) coefficient of maximum degree in c and write $p(c, c^*)^2 = 0$ we immediately see that this scalar must be zero, a contradiction. □

2.2 Uniqueness theorems

An edge e is an *exit* for a path $\mu = e_1 \dots e_n$ if there exists i such that $s(e) = s(e_i)$ and $e \neq e_i$. A graph is said to satisfy *Condition* (L) if every cycle in the graph has an exit.

For any K-algebra A the $\mathcal{M}(A)$-submodules of A are just the ideals of A and the cyclic $\mathcal{M}(A)$-submodules of A are the ideals generated by one element (*principal ideals* in the sequel), so Corollary 2.1.4 states that the nonzero principal ideals of any Leavitt path algebra contain either vertices or nonzero elements of the form $p(c, c^*)$. Therefore, for graphs in which every cycle has an exit, each nonzero ideal contains a vertex.

Definition 2.2.1 Let $A = \sum_{n \in \mathbb{Z}} A_n$ be a $\mathbb{Z}$-graded algebra. An ideal I of A is said to be a *graded ideal* if $I \cap A_n \subseteq I$.

Definition 2.2.2 We say that a graph E satisfies *Condition* (L) if every cycle has an exit.

The following result is a consequence of Proposition 2.1.3.

Corollary 2.2.3 *Let E be an arbitrary graph.*

(i) Every $\mathbb{Z}$-graded nonzero ideal of $L_K(E)$ contains a vertex.

(ii) Suppose that E satisfies Condition (L). *Then every nonzero ideal of $L_K(E)$ contains a vertex.*

Proof. The second assertion has been proved above. So assume that I is a graded ideal of Ł which contains no vertices. Let $0 \neq x \in I$ and use Corollary 2.1.4 to find elements $y, z \in L_K(E)$ such that $yxz = \sum_{i=-m}^{n} k_i c^i \neq 0$. But I being a graded ideal implies that every summand is in I. In particular, for $t \in \{-m, \dots, n\}$ such that $k_t c^t \neq 0$ we have $0 \neq (k_t)^{-1} c^{-t} k_t c^t = w \in I$, which is absurd. □

Theorem 2.2.4 *Let E be an arbitrary graph, and let $L_K(E)$ be the associated Leavitt path algebra.*

(1) Graded Uniqueness Theorem.

If A is a $\mathbb{Z}$-graded ring and $\pi : L_K(E) \to A$ is a graded ring homomorphism with $\pi(v) \neq 0$ for every vertex $v \in E^0$, then π is injective.

(2) Cuntz-Krieger Uniqueness Theorem.

Suppose that E satisfies Condition (L). *If $\pi : L_K(E) \to A$ is a ring homomorphism with $\pi(v) \neq 0$, for every vertex $v \in E^0$, then π is injective.*

Proof. In both cases, the kernel of the ring homomorphism π is an algebra ideal (a graded ideal in the first one). By Corollary 2.2.3, Ker(π) must be zero because otherwise it would contain a vertex (apply (i) in the corollary to (1) and (ii) to the other case), which is not possible by the hypotheses. □

2.3 Simple Leavitt path algebras

In this section we use Proposition 2.1.3 to proof a characterization of simple Leavitt path algebras (see [2, Theorem 3.11] and [18, Corollary 3.8]).

Recall that an algebra A is said to be *simple* if $A^2 \neq 0$ and it has no nonzero proper ideals. If the algebra is graded by a group G, write $A = \sum_{\sigma \in G} A_\sigma$, it is called *graded simple* if $A^2 \neq 0$ and it has no nonzero proper graded ideals (an ideal I of A is *graded* if whenever $y = \sum_\sigma (y_\sigma)$, every $y_\sigma \in I$). In general, and in the particular case of Leavitt path algebras, simplicity and graded-simplicity are not equivalent, as we shall see.

For $n \geq 2$ we write E^n to denote the set of paths of length n, and $E^* = \bigcup_{n \geq 0} E^n$ the set of all paths. We define a relation $\geq$ on E^0 by setting $v \geq w$ if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. A subset H

of E^0 is called *hereditary* if $v \geq w$ and $v \in H$ imply $w \in H$. A hereditary set is *saturated* if every vertex which feeds into H and only into H is again in H, that is, if $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \subseteq H$ imply $v \in H$. Denote by $\mathcal{H}$ (or by $\mathcal{H}_E$ when it is necessary to emphasize the dependence on E) the set of hereditary saturated subsets of E^0. For a graph E, the empty set and E^0 are hereditary and saturated subsets of E^0.

The set $T(v) = \{w \in E^0 \mid v \geq w\}$ is the *tree* of v, and it is the smallest hereditary subset of E^0 containing v. We extend this definition for an arbitrary set $X \subseteq E^0$ by $T(X) = \bigcup_{x \in X} T(x)$. The *hereditary saturated closure* of a set X is defined as the smallest hereditary and saturated subset of E^0 containing X. It is shown in [15] that the hereditary saturated closure of a set X is $\overline{X} = \bigcup_{n=0}^{\infty} \Lambda_n(X)$, where

$\Lambda_0(X) = T(X)$, and

$\Lambda_n(X) = \{y \in E^0 \mid s^{-1}(y) \neq \emptyset \text{ and } r(s^{-1}(y)) \subseteq \Lambda_{n-1}(X)\} \cup \Lambda_{n-1}(X)$, for $n \geq 1$.

One way of constructing graded ideals in a Leavitt path algebra is the following:

Lemma 2.3.1 *Let H be a hereditary subset of E^0, for a graph E. Then*

$$I(H) = \left\{ \sum k\alpha\beta^*, \text{ with } k \in K, \alpha, \beta \text{ paths such that } r(\alpha) = r(\beta) \in H \right\}.$$

In fact, $I(H)$ is a graded ideal.

Proof. Denote by J the following set:

$$J = \left\{ \sum k\alpha\beta^* \mid k \in K, \alpha, \beta \text{ are paths and } r(\alpha) = r(\beta) \in H \right\}.$$

The containment $J \subseteq I(H)$ is clear. For the converse, consider μ, ν, α, β paths in $L(E)$, and $u \in H$ such that $\mu\nu^* u\alpha\beta^* \neq 0$. By [53, Lemma 3.1], $\mu\nu^* u\alpha\beta^*$ is $\mu\alpha'\beta^*$ if $\alpha = \nu\alpha'$ or $\mu\nu'^*\beta^*$ if $\nu = \alpha\nu'$. Note that $\alpha = \nu\alpha'$, $u = s(\alpha)$ and H hereditary imply $r(\alpha') \in H$, hence $\mu\alpha'\beta^* \in J$. In the second case, $\nu = \alpha\nu'$, $u = s(\alpha)$ and H hereditary imply $s(\nu'^*) = r(\nu') \in H$, hence $\mu r(\nu')\nu'^*\beta^* \in J$, therefore $I(H) \subseteq J$.

The last statement follows immediately by the form the elements of $I(H)$ have.

□

Moreover, it was proved in [2, Lemma 3.9]

Lemma 2.3.2 *For every ideal I of a Leavitt path algebra $L_K(E)$, $I \cap E^0$ is a hereditary and saturated subset of E^0.*

In fact, all graded ideals of a Leavitt path algebra come from hereditary and saturated subsets of vertices.

Remark 2.3.3 An ideal J of $L(E)$ is graded if and only if it is generated by idempotents; in fact, $J = I(H)$, where $H = J \cap E^0 \in \mathcal{H}_E$. (See the proofs of [15, Proposition 4.2 and Theorem 4.3].)

Now, the question that arises is if every graded ideal of a Leavitt path algebra is again a Leavitt path algebra (note that this question has sense only for graded ideals as every Leavitt path algebra is graded). Other natural question is if the quotient of a Leavitt path algebra by an ideal is a Leavitt path algebra too.

In both cases, as we shall see, the answer is yes.

For a graph E and a hereditary subset H of E^0, we denote by E/H the *quotient graph*

$$(E^0 \setminus H, \{e \in E^1 \mid r(e) \notin H\}, r|_{(E/H)^1}, s|_{(E/H)^1}),$$

and by E_H the *restriction graph*

$$(H, \{e \in E^1 \mid s(e) \in H\}, r|_{(E_H)^1}, s|_{(E_H)^1}).$$

Observe that while $L(E_H)$ can be seen as a subalgebra of $L(E)$, the same cannot be said about $L(E/H)$.

Lemma 2.3.4 *([19, Lemma 2.3]) Let E be a graph and consider a proper $H \in \mathcal{H}_E$. Define $\Psi : L(E) \to L(E/H)$ by setting $\Psi(v) = \chi_{(E/H)^0}(v)v$, $\Psi(e) = \chi_{(E/H)^1}(e)e$ and $\Psi(e^*) = \chi_{((E/H)^1)^*}(e^*)e^*$ for every vertex v and every edge e, where $\chi_{(E/H)^0} : E^0 \to K$ and $\chi_{(E/H)^1} : E^1 \to K$ denote the characteristic functions. Then:*

1. *The map Ψ extends to a K-algebra epimorphism of $\mathbb{Z}$-graded algebras with Ker$(\Psi) = I(H)$ and therefore $L(E)/I(H) \cong L(E/H)$.*
2. *If X is hereditary in E, then $\Psi(X) \cap (E/H)^0$ is hereditary in E/H.*
3. *For $X \supseteq H$, $X \in \mathcal{H}_E$ if and only if $\Psi(X) \cap (E/H)^0 \in \mathcal{H}_{(E/H)}$.*
4. *For every $X \supseteq H$, $\overline{\Psi(X) \cap (E/H)^0} = \Psi(\overline{X}) \cap (E/H)^0$.*

Proof. (1) It was shown in [2, Proof of Theorem 3.11] that Ψ extends to a K-algebra morphism. By definition, Ψ is $\mathbb{Z}$-graded and onto. Moreover, $I(H) \subseteq \mathrm{Ker}(\Psi)$.

Since Ψ is a graded morphism, $\mathrm{Ker}(\Psi) \in \mathcal{L}_{gr}(L(E))$. By [15, Theorem 4.3], there exists $X \in \mathcal{H}_E$ such that $\mathrm{Ker}(\Psi) = I(X)$. By Lemma **??**, $H = I(H) \cap E^0 \subseteq I(X) \cap E^0 = X$. Hence, $I(H) \neq \mathrm{Ker}(\Psi)$ if and only if there exists $v \in X \setminus H$. But then $\Psi(v) = v \neq 0$ and $v \in \mathrm{Ker}(\Psi)$, which is impossible.

(2) It is clear by the definition of Ψ.

(3) Since Ψ is a graded epimorphism, there is a bijection between graded ideals of $L(E/H)$ and graded ideals of $L(E)$ containing $I(H)$. Thus, the result holds by [15, Theorem 4.3].

(4) It is immediate by part (3). □

Lemma 2.3.5 *Let E be a graph. For every hereditary and saturated subset H of E, the ideal $I(H)$ is isomorphic to $L({}_HE)$.*

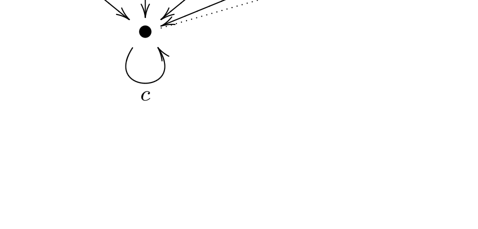
Corollary 2.3.6 *Every graded ideal of a Leavitt path algebra is again a Leavitt path algebra.*

One interesting property for Leavitt path algebras is that cycles without exists behaves in a similar way to sinks, so, roughly speaking, for a graph having no cycles with exits, and such that every vertex connects to the cycle (the so called C_n-comet), the corresponding Leavitt path algebra is a direct sum of matrices over something appropriate.

The notion of C_n-comet was introduced in [7] to describe the locally finite Leavitt path algebras. The role of the cycle C_n within a C_n-comet is similar to that played by sinks in more general graphs. If a graph E is a C_n-comet, then its associated Leavitt path algebra is isomorphic to $\mathbb{M}_n(K[x, x^{-1}])$. Since C_n-comets have a finite number of vertices, it is natural to generalize this concept to the case of an infinite (numerable) set of vertices.

Definition 2.3.7 We say that a graph E is a *comet* if it has exactly one cycle c, $T(v) \cap c^0 \neq \emptyset$ for every vertex $v \in E^0$, and every infinite path ends in the cycle c.

Remark 2.3.8 The following is not an example of a C_n-comet:



and the reason is that the infinite path $\gamma = e_1 e_2 e_3 \ldots$ does not end either in a sink or in a cycle.

Proposition 2.3.9 *Let E be a graph which is a comet. Then the Leavitt path algebra $L(E)$ is isomorphic to $\mathbb{M}_n(K[x,x^{-1}])$, where $n \in \mathbb{N}$ if E is finite, or $n = \infty$ otherwise.*

Proof. We can adapt [7, Theorem 3.3] to our situation. Concretely, let c be the cycle in E, v a vertex at which the cycle c is based and consider $\{p_i\}$ the (perhaps infinite) family of all paths in E which end in v but do not contain the cycle c. Let $n \in \mathbb{N} \cup \{\infty\}$ be the number of all such paths. Denote by N the set $\{1, \ldots, n\}$ when n is finite and $N = \mathbb{N}$ when $n = \infty$. Consider the family $\mathcal{B} := \{p_i c^k p_j^*\}_{i,j \in N, k \in \mathbb{N}}$ of monomials in $L(E)$ where we understand $c^0 = v$ and $c^n = (c^*)^{-n}$ for negative n.

As in [7, Theorem 3.3], we can show that $\mathcal{B}$ is a linearly independent set. We will prove that $\mathcal{B}$ generates $L(E)$ as a K-vector space. First, note that since E is a comet, then $T(v)$ is a finite set for every $v \in c^0$. Not only is this true for any vertex on the cycle c but also for any vertex in E as follows: Suppose on the contrary that there exists $w \in E$ with $|T(w)| = \infty$. In particular, w does not lie on the cycle. As E is row-finite, we are able to find and edge e_1 in E with $s(e_1) = w$ and $v_1 := r(e_1)$ such that $|T(v_1)| = \infty$. Again v_1 does not lie on the cycle. Repeating this process, we find an infinite path such that none of its vertices lie on c, which contradicts the fact that every infinite path in E ends in the cycle c.

Take an arbitrary element $\sum_i k_i \alpha_i \beta_i^*$ of $L(E)$, where α_i, β_i are paths in E and $k_i \in K$. Consider the set $\{r(\alpha_i)\}$. Some of these vertices could lie on the cycle c, in which case we leave the corresponding monomial as is. For those monomials $\alpha_k \beta_k^*$ whose $\{r(\alpha_k)\}$ is not on c, we proceed as in [6, Proof of Proposition 3.5] by using relation (4) to expand it as

$$\alpha_k \beta_k^* = \sum_{\{e \in E^1 : s(e) = r(\alpha_k)\}} \alpha_k e e^* \beta_k^* = \sum_{\{e \in E^1 : s(e) = r(\alpha_k)\}} (\alpha_k e)(\beta_k e)^*.$$

As we have just proved that the tree of any vertex is finite, so will be this process of expanding these monomials until reaching vertices of c.

Consider now a monomial $\alpha_k \beta_k^*$ with $r(\alpha_k) \in c^0$. Let t be the subpath of c with $s(t) = r(\alpha_k)$ and $r(t) = v$. Since c does not have exits then $\alpha_k \beta_k^* = \alpha_k t t^* \beta_k^* = (\alpha_k t)(\beta_k t)^* = \alpha \beta^*$, where α and β are paths in E that end in v. Finally, since E is a comet, we can always factor some powers of c out of α and β so that there exist integers m, n such that $\alpha = p_i c^m$ and $\beta = p_j c^n$ for some paths p_i, p_j which do not contain the path c. Hence, we obtain that $\alpha_k \beta_k^* = p_i c^{m-n} p_j^* \in \mathcal{B}$. This proves that $\mathcal{B}$ is a K-generator of $L(E)$.

Now, by defining $\phi : L(E) \to \mathbb{M}_n(K[x,x^{-1}])$ on the basis by setting $\phi(p_i c^k p_j^*) = x^k e_{ij}$ for e_{ij} the (i,j)-matrix unit, then again one easily checks that ϕ is a K-algebra isomorphism. □

For a graph E, denote by $P_c(E)$ the set of vertices in the cycles without exits of E.

Proposition 2.3.10 *Let E be a graph. Then:*

(i) $I(P_c(E)) = \bigoplus_{j\in\Upsilon} I(P_{c_j}(E))$, where Υ is a countable set and $\{c_j\}_{j\in\Upsilon}$ is the set of all different cycles without exits of E (and by abuse of notation we identify two cycles that have the same vertices).

(ii) $P_c(E)$ is hereditary and if H denotes the saturated closure of $P_c(E)$, we have that

$$I(P_c(E)) = I(H) \cong L({}_HE) \cong \bigoplus_{i\in\Upsilon_1} \mathbb{M}_{n_i}(K[x,x^{-1}]) \oplus \bigoplus_{j\in\Upsilon_2} \mathbb{M}_{m_j}(K[x,x^{-1}]),$$

where Υ_1 and Υ_2 are countable sets, $n_i \in \mathbb{N}$ and $m_j = \infty$.

Proof. We will use Lemma 2.3.1 implicitly. This can be done because $P_c(E)$ is, clearly, a hereditary set.

(i). To shorten the notation, write: $\mathcal{J} = I(P_c(E))$ and $\mathcal{J}_j = I(P_{c_j}(E))$.

Consider monomials $\gamma\delta^*$ with $r(\delta) \in (c_j)^0$ and $\sigma\tau^* \in \mathcal{J}$. Since the cycles c_j have no exits, they are disjoint and then, similar arguments to that of the previous paragraph show that $\gamma\delta^*\sigma\tau^*, \sigma\tau^*\gamma\delta^* \in \mathcal{J}_{c_j}$. Moreover, these arguments also yield that if $\sigma\tau^* \in \mathcal{J}_{c_k}$ with $j \neq k$, then $\gamma\delta^*\sigma\tau^* = \sigma\tau^*\gamma\delta^* = 0$. Thus, $\{\mathcal{J}_{c_j}\}$ is indeed a family of orthogonal ideals of $\mathcal{J}$.

To show that $\mathcal{J} = \sum_j \mathcal{J}_j$ apply Lemma 2.3.1 to $H = \cup_j c_j^0$, which is a hereditary set since the considered cycles have no exits.

(ii). $I(P_c(E)) = I(H)$ follows by [19, Lemma 2.1] and $I(H) \cong L({}_HE)$ by [16, Lemma 1.2] The same results applied to c_j instead of c imply $I(P_{c_j}(E)) = I(H_j) \cong L({}_{H_j}E)$, for H_j the saturated closure of P_{c_j}. By the definition of H_j, and since c_j has no exits, every vertex in H_j connects to c_j. The same can be said about ${}_{H_j}E$, where c_j can be seen as its only cycle. Now suppose that γ is an infinite path in ${}_{H_j}E$. Again, by the way this graph is constructed, there must exist a finite path p and an infinite path β such that $\gamma = p\beta$, with β being completely contained in E_{H_j}. Suppose that β does not end in the cycle c_j. This, together with the fact that c_j does not have exits, yield that $\beta^0 \cap c_j^0 = \emptyset$. On the other hand, because $\beta^0 \subseteq H_j$ we can consider m to be the minimum n such that $\Lambda_n(c_j^0) \cap \beta^0 \neq \emptyset$. Now, $\beta^0 \cap c_j^0 = \emptyset$ implies that $m > 0$ so that there exists $w \in \{v \in (E_{H_j})^0 \mid \emptyset \neq r(s^{-1}(v)) \subseteq \Lambda_{m-1}(c_j^0)\} \cap \beta^0$.

As β is infinite, there is an edge e in β such that $s(e) = w$ and $r(e) \in \beta^0$. This contradicts the minimality of m. Therefore β ends in the cycle c, and consequently γ. Hence, ${}_{H_j}E$ is a comet. Apply Proposition 2.3.9 and (i) to obtain the result. □

Theorem 2.3.11 *Let E be an arbitrary graph. Then $L_K(E)$ is simple if and only if E satisfies Condition* (L) *and the only hereditary and saturated subsets of E^0 are the trivial ones.*

Proof. Suppose first that $L_K(E)$ is simple. If there exists some cycle without exits in E, the simplicity of $L_K(E)$ and Proposition 2.3.10 imply that the Leavitt path algebra coincides with the ideal generated by this cycle, which is isomorphic to a matrix ring over the Laurent polynomial ring. But this rings are not simple, so this cannot happen, that is, every cycle in E must have an exit.

Now, if H were a hereditary and saturated subset of E^0, then the ideal it generates would be a proper nonzero ideal of $L_K(E)$, contradicting the hypothesis of simplicity.

For the converse take into account that Condition (L) implies that any nonzero element in $L_K(E)$ is linked to a vertex (see Proposition 2.2.3). Thus, there is a vertex in any nonzero ideal I of $L_K(E)$. But on the other hand $\emptyset \neq I \cap E^0$ is hereditary and saturated ([4, Lemma 2.3]), therefore it coincides with E^0 and so $I = L_K(E)$. □

Recall that a *matricial algebra* is a finite direct product of full matrix algebras over K, while a *locally matricial algebra* is a direct limit of matricial algebras. At this point we have shown that finite and acyclic graphs produce matricial algebras. Our following target will be to show that acyclic Leavitt path algebras are locally matricial. The following results can be found in [3].

Lemma 2.3.12 *Let E be a finite acyclic graph. Then $L(E)$ is finite dimensional.*

Proof. Since the graph is row-finite, the given condition on E is equivalent to the condition that E^* is finite. The result now follows from the previous observation that $L(E)$ is spanned as a K-vector space by $\{pq^* : p, q \text{ are paths in } E\}$. □

This lemma is just what we need to the following

Proposition 2.3.13 *Let E be a graph. Then E is acyclic if and only if $L_K(E)$ is a union of a chain of finite dimensional subalgebras. Concretely, it is a locally matricial K-algebra.*

Proof. Assume first that E is acyclic. If E is finite, then Lemma 3 gives the result. So now suppose E is infinite, and rename the vertices of E^0 as a sequence $\{v_i\}_{i=1}^{\infty}$. We now define a sequence $\{F_i\}_{i=1}^{\infty}$ of subgraphs of E. Let $F_i = (F_i^0, F_i^1, r, s)$ where $F_i^0 := \{v_1, \dots, v_i\} \cup r(s^{-1}(\{v_1, \dots, v_i\})$, $F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$, and r, s are induced from E. In particular, $F_i \subseteq F_{i+1}$ for all i. For any $i > 0$, $L(F_i)$ is a subalgebra of $L(E)$ as follows. First note that we can construct $\phi : L(F_i) \to L(E)$ a K-algebra homomorphism because the Cuntz-Krieger relations in $L(F_i)$ are consistent with those in $L(E)$, in the following way: Consider v a sink in F_i (which need not be a sink in E), then we do not have CK2 at v in $L(F_i)$. If v is not a sink in F_i, then there exists $e \in F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$ such that $s(e) = v$. But $s(e) \in \{v_1, \dots, v_i\}$ and therefore $v = v_j$ for some j, and then $F_i^1 := s^{-1}(\{v_1, \dots, v_i\})$ ensures that all the edges coming to v are in F_i, so CK2 at v is the same in $L(F_i)$ as in $L(E)$. The other relations offer no difficulty. Now, with a similar construction and argument to that used in [2, Proof of Theorem 3.11] we find $\psi : L(E) \to L(F_i)$ a K-algebra homomorphism such that $\psi\phi = Id|_{L(F_i)}$, so that ϕ is a monomorphism, which we view as the inclusion map. By construction, each vertex in E^0 is in F_i for some i; furthermore, the edge e has $e \in F_j^1$, where $s(e) = v_j$. Thus we conclude that $L(E) = \cup_{i=1}^{\infty} L(F_i)$. (We note here that the embedding of graphs $j : F_i \hookrightarrow E$ is a complete graph homomorphism in the sense of [15], so that the conclusion $L(E) = \cup_{i=1}^{\infty} L(F_i)$ can also be achieved by invoking [15, Lemma 2.1].)

Since E is acyclic, so is each F_i. Moreover, each F_i is finite since, by the row-finiteness of E, in each step we add only finitely many vertices. Thus, by Lemma 2.3.12, $L(F_i)$ is finite dimensional, so that $L(E)$ is indeed a union of a chain of finite dimensional subalgebras.

For the converse, let $p \in E^*$ be a cycle in E. Then $\{p^m\}_{m=1}^{\infty}$ is a linearly independent infinite set, so that p is not contained in any finite dimensional subalgebra of $L(E)$. □

A different proof derives from Corollary 1.4.8 and Lemma 4.3.3.

2.4 Purely infinite Leavitt path algebras

The concept of purely infinite simple C*-algebra was introduced by Cuntz in 1981 [28] and implied a significant advance in the development of the theory of C*-algebras. It was in 2002 that Ara, Goodearl and Pardo gave the definition of purely infinite (unital) simple ring (see [14]). Both definitions agree when considering C*-algebras. An idempotent e in a ring R is called

infinite if eR is isomorphic as a right R-module to a proper direct summand of itself. The ring R is called *purely infinite* in case every nonzero right ideal of R contains an infinite idempotent.

The following characterization of purely infinite simple rings can be found in [3].

Proposition 2.4.1 *For a ring R with local units, the following are equivalent conditions:*

(i) R is purely infinite simple.

(ii) R is simple, and for each nonzero finitely generated projective right A-module P, every nonzero submodule C of P contains a direct summand T of P for which T is directly infinite. (In particular, the property 'purely infinite simple' is a Morita invariant of the ring.)

(iii) wRw is purely infinite simple for every nonzero idempotent $w \in R$.

(iv) R is simple, and there exists a nonzero idempotent w in R for which wRw is purely infinite simple.

(v) R is not a division ring, and A has the property that for every pair of nonzero elements α, β in R there exist elements a, b in R such that $a\alpha b = \beta$.

As in the case of simplicity for Leavitt path algebras, being purely infinite and simple can be characterized in terms of the graph.

Most of the results of this section belong to the paper [3], although some proofs differ from the original ones.

Lemma 2.4.2 *Suppose A is a union of finite dimensional subalgebras. Then A is not purely infinite. In fact, A contains no infinite idempotents.*

Proof. It suffices to show the second statement. So just suppose $e = e^2 \in A$ is infinite. Then eA contains a proper direct summand isomorphic to eA, which in turn, by definition and a standard argument, is equivalent to the existence of elements $g, h, x, y \in A$ such that $g^2 = g, h^2 = h, gh = hg = 0, e = g + h, h \neq 0, x \in eAg, y \in gAe$ with $xy = e$ and $yx = g$. But by hypothesis the five elements e, g, h, x, y are contained in a finite dimensional subalgebra B of A, which would yield that B contains an infinite idempotent, and thus contains a non-artinian right ideal, which is impossible. □

A *closed simple path based at v_{i_0}* is a path $\mu = \mu_1 \dots \mu_n$, with $\mu_j \in E^1$, $n \geq 1$ such that $s(\mu_j) \neq v_{i_0}$ for every $j > 1$ and $s(\mu) = r(\mu) = v_{i_0}$. Denote by $CSP(v_{i_0})$ the set of all such paths. We note that a cycle is a closed simple path based at any of its vertices, but not every closed simple path based at v_{i_0} is a cycle. We define the following subsets of E^0:

$$V_0 = \{v \in E^0 : CSP(v) = \emptyset\}$$

$$V_1 = \{v \in E^0 : |CSP(v)| = 1\}$$

$$V_2 = E^0 - (V_0 \cup V_1)$$

Proposition 2.4.3 *Let E be a graph. Suppose that $w \in E^0$ has the property that, for every $v \in E^0$, $w \geq v$ implies $v \in V_0$. Then the corner algebra $wL(E)w$ is not purely infinite.*

Proof. Consider the graph $H = (H^0, H^1, r, s)$ defined by $H^0 := \{v : w \geq v\}$, $H^1 := s^{-1}(H^0)$, and r, s induced by E. The only nontrivial part of showing that H is a well defined graph is verifying that $r(s^{-1}(H^0)) \subseteq H^0$. Take $z \in H^0$ and $e \in E^1$ such that $s(e) = z$. But we have $w \geq z$ and thus $w \geq r(e)$ as well, that is, $r(e) \in H^0$.

Using that H is acyclic, along with the same argument as given in Proposition 2.3.13, we have that $L(H)$ is a subalgebra of $L(E)$. Thus Proposition 2.3.13 applies, which yields that $L(H)$ is the union of finite dimensional subalgebras, and therefore contains no infinite idempotents by Lemma 2.4.2. As $wL(H)w$ is a subalgebra of $L(H)$, it too contains no infinite idempotents, and thus is not purely infinite.

We claim that $wL(H)w = wL(E)w$. To see this, given $\alpha = \sum p_i q_i^* \in L(E)$, then $w\alpha w = \sum p_{i_j} q_{i_j}^*$ with $s(p_{i_j}) = w = s(q_{i_j})$ and therefore $p_{i_j}, q_{i_j} \in L(H)$. Thus $wL(E)w$ is not purely infinite as desired. □

Lemma 2.4.4 *Let E be a graph. If $L(E)$ is simple, then $V_1 = \emptyset$.*

Proof. For any subset $X \subseteq E^0$ we define the following subsets. $H(X)$ is the set of all vertices that can be obtained by one application of the hereditary condition at any of the vertices of X; that is, $H(X) := r(s^{-1}(X))$. Similarly, $S(X)$ is the set of all vertices obtained by applying the saturated condition among elements of X, that is, $S(X) := \{v \in E^0 : \emptyset \neq \{r(e) : s(e) = v\} \subseteq X\}$. We now define $G_0 := X$, and for $n \geq 0$ we define inductively $G_{n+1} := H(G_n) \cup S(G_n) \cup G_n$. It is not difficult to show that the smallest hereditary and saturated subset of E^0 containing X is the set $G(X) := \bigcup_{n \geq 0} G_n$.

Suppose now that $v \in V_1$, so that $CSP(v) = \{p\}$. In this case p is clearly a cycle. By Theorem 2.3.11 we can find an edge e which is an exit for p. Let A be the set of all vertices in the cycle. Since p is the only cycle based at v, and e is an exit for p, we conclude that $r(e) \notin A$. Consider then the set $X = \{r(e)\}$, and construct $G(X)$ as described above. Then $G(X)$ is nonempty and, by construction, hereditary and saturated.

Now Theorem 2.3.11 implies that $G(X) = E^0$, so we can find $n = min\{m : A \cap G_m \neq \emptyset\}$. Take $w \in A \cap G_n$. We are going to show that $w \geq r(e)$. First, since $r(e) \notin A$, then $n > 0$ and therefore $w \in H(G_{n-1}) \cup S(G_{n-1}) \cup G_{n-1}$. Here, $w \in G_{n-1}$ cannot happen by the minimality of n. If $w \in S(G_{n-1})$ then $\emptyset \neq \{r(e) : s(e) = w\} \subseteq G_{n-1}$. Since w is in the cycle p, there exists $f \in E^1$ such that $r(f) \in A$ and $s(f) = w$. In that case $r(f) \in A \cup G_{n-1}$ again contradicts the minimality of n. So the only possibility is $w \in H(G_{n-1})$, which means that there exists $e_{i_1} \in E^1$ such that $r(e_{i_1}) = w$ and $s(e_{i_1}) \in G_{n-1}$.

We now repeat the process with the vertex $w' = s(e_{i_1})$. If $w' \in G_{n-2}$ then we would have $w \in G_{n-1}$, again contradicting the minimality of n. If $w' \in S(G_{n-2})$ then, as above, $\{r(e) : s(e) = w'\} \subseteq G_{n-2}$, so in particular would give $w = r(e_{i_1}) \in G_{n-2}$, which is absurd. So therefore $w' \in H(G_{n-2})$ and we can find $e_{i_2} \in E^1$ such that $r(e_{i_2}) = w'$ and $s(e_{i_2}) \in G_{n-2}$.

After n steps we will have found a path $q = e_{i_n} \dots e_{i_1}$ with $r(q) = w$ and $s(q) = r(e)$. In particular we have $w \geq s(e)$, and therefore there exists a cycle based at w containing the edge e. Since e is not in p we get $|CSP(w)| \geq 2$. Since w is a vertex contained in the cycle p, we then get $|CSP(v)| \geq 2$, contrary to the definition of the set V_1. □

Theorem 2.4.5 *Let E be a graph. Then $L(E)$ is purely infinite simple if and only if E has the following properties.*

(i) The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

(ii) Every cycle in E has an exit.

(iii) Every vertex connects to a cycle.

Proof. First, assume (i), (ii) and (iii) hold. By Theorem 2.3.11 we have that $L(E)$ is simple. By Proposition 2.4.1 it suffices to show that $L(E)$ is not a division ring, and that for every pair of elements α, β in $L(E)$ there exist elements a, b in $L(E)$ such that $a\alpha b = \beta$. Conditions (ii) and (iii) easily imply that $|E^1| > 1$, so that $L(E)$ has zero divisors, and thus is not a division ring. We now apply Proposition 2.1.3 to find $\overline{a}, \overline{b} \in L(E)$ such that $\overline{a}\alpha\overline{b} = w \in E^0$. By condition (iii), w connects to a vertex $v \notin V_0$. Either $w = v$ or there exists a path p such that $r(p) = v$ and $s(p) = w$. By choosing $a' = b' = v$ in the former case, and $a' = p^*, b' = p$ in the latter, we have produced elements $a', b' \in L(E)$ such that $a'wb' = v$.

An application of Lemma 2.4.4 yields that $v \in V_2$, so there exist $p, q \in CSP(v)$ with $p \neq q$. For any $m > 0$ let c_m denote the closed path $p^{m-1}q$. Using [2, Lemma 2.2], it is not difficult to show that $c_m^* c_n = \delta_{mn} v$ for every $m, n > 0$.

Now consider any vertex $v_l \in E^0$. Since $L(E)$ is simple, there exist $\{a_i, b_i \in L(E) \mid 1 \leq i \leq t\}$ such that $v_l = \sum_{i=1}^t a_i v b_i$. But by defining $a_l = \sum_{i=1}^t a_i c_i^*$ and $b_l = \sum_{j=1}^t c_j b_j$, we get

$$a_l v b_l = \left(\sum_{i=1}^t a_i c_i^*\right) v \left(\sum_{j=1}^t c_j b_j\right) = \sum_{i=1}^t a_i c_i^* v c_i b_i = v_l.$$

Now let s be a left local unit for β (i.e., $s\beta = \beta$), and write $s = \sum_{v_l \in S} v_l$ for some finite subset of vertices S. By letting $\widetilde{a} = \sum_{v_l \in S} a_l c_l^*$ and $\widetilde{b} = \sum_{v_l \in S} c_l b_l$, we get

$$\widetilde{a} v \widetilde{b} = \sum_{v_l \in S} a_l c_l^* v c_l b_l = \sum_{v_l \in S} v_l = s.$$

Finally, letting $a = \widetilde{a} a' \overline{a}$ and $b = \overline{b} b' \widetilde{b} \beta$, we have that $a\alpha b = \beta$ as desired.

For the converse, suppose that $L(E)$ is purely infinite simple. By Theorem 2.3.11 we have (i) and (ii). If (iii) does not hold, then there exists a vertex $w \in E^0$ such that $w \leq v$ implies $v \in V_0$. Applying Proposition 2.4.3 we get that $wL(E)w$ is not purely infinite. But then Proposition 2.4.1 implies that $L(E)$ is not purely infinite, contrary to hypothesis. □

2.5 The dichotomy principle for simple Leavitt path algebras

We are now in a position to show that every simple Leavitt path algebra is locally matricial or purely infinite.

We denote by E^∞ the set of infinite paths $\gamma = (\gamma_n)_{n=1}^\infty$ of the graph E and by $E^{\leq\infty}$ the set E^∞ together with the set of finite paths in E whose end vertex is a sink. We say that a vertex v in a graph E is *cofinal* if for every $\gamma \in E^{\leq\infty}$ there is a vertex w in the path γ such that $v \geq w$. We say that a graph E is *cofinal* if so are all the vertices of E.

Lemma 2.5.1 *([19, Lemma 2.7]). A graph E is cofinal if and only if $\mathcal{H} = \{\emptyset, E^0\}$.*

Proof. Suppose E to be cofinal. Let $H \in \mathcal{H}$ with $\emptyset \neq H \neq E^0$. Fix $v \in E^0 \setminus H$ and build a path $\gamma \in E^{\leq\infty}$ such that $\gamma^0 \cap H = \emptyset$: If v is a sink, take $\gamma = v$. If not, then $s^{-1}(v) \neq \emptyset$ and $r(s^{-1}(v)) \nsubseteq H$; otherwise, H saturated implies $v \in H$, which is impossible. Hence, there exists $e_1 \in s^{-1}(v)$ such that $r(e_1) \notin H$. Let $\gamma_1 = e_1$ and repeat this process with $r(e_1) \notin H$. By recurrence either we reach a sink or we have an infinite path γ whose vertices are not in H, as desired. Now consider $w \in H$. By the hypothesis, there exists $z \in \gamma$ such that $w \geq z$, and by hereditariness of H we get $z \in H$, contradicting the definition of γ.

Conversely, suppose that $\mathcal{H} = \{\emptyset, E^0\}$. Take $v \in E^0$ and $\gamma \in E^{\leq\infty}$, with $v \notin \gamma^0$ (the case $v \in \gamma^0$ is obvious). By hypothesis the hereditary saturated subset generated by v is E^0, i.e., $E^0 = \bigcup_{n\geq 0} \Lambda_n(v)$. Consider m, the minimum n such that $\Lambda_n(v) \cap \gamma^0 \neq \emptyset$, and let $w \in \Lambda_m(v) \cap \gamma^0$. If $m > 0$, then by minimality of m it must be $s^{-1}(w) \neq \emptyset$ and $r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$. The first condition implies that w is not a sink and since $\gamma = (\gamma_n) \in E^{\leq\infty}$, there exists $i \geq 1$ such that $s(\gamma_i) = w$ and $r(\gamma_i) = w' \in \gamma^0$, the latter meaning that $w' \in r(s^{-1}(w)) \subseteq \Lambda_{m-1}(v)$, contradicting the minimality of m. Therefore $m = 0$ and then $w \in \Lambda_0(v) = T(v)$, as we needed. □

Theorem 2.5.2 *Every simple Leavitt path algebra is locally matricial or purely infinite.*

Proof. Let E be a graph and K a field, and suppose that $L_K(E)$ is a simple algebra. If E has no cycles, then $L_K(E)$ is a locally matricial agebra, by Proposition 2.3.13.

Now, suposse that E has cycles. As it is simple, by Theorem 2.3.11 and Lemma 2.5.1, every vertex connects to a cycle, hence, by Theorem 2.4.5, the Leavitt path algebra $L_K(E)$ is purely infinite and simple. □

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