A new approach to the study of lightlike manifolds

Francisco J. Palomo
Department of Applied Mathematics
UNIVERSIDAD DE MÁLAGA

First Joint Meeting Brazil-Spain in Mathematics
Fortaleza, December 2015

INTRODUCTION

Let (M, g) be a $(n+2)$-dimensional Lorentz manifold and a hypersurface $\psi : \mathcal{L} \to M$

Lightlike hypersurface

$\psi : \mathcal{L} \to M$ is said to be a lightlike hypersurface whenever

- $\psi^*(g)$ is degenerate at every point, i.e., $\mathrm{Rad}(T_p\mathcal{L}) \neq 0$

$$\Downarrow$$

$$\mathrm{Rad}(T_p\mathcal{L}) = \{v \in T_p\mathcal{L} : g(v, u) = 0 \text{ for all } u \in T_p\mathcal{L}\} \neq 0 \Rightarrow \dim(\mathrm{Rad}(T_p\mathcal{L})) = 1.$$

$$\mathcal{D}(p) = \mathrm{Rad}(T_p\mathcal{L}) = (T_p\mathcal{L})^{\perp} \subset T_{\psi(p)}M, \quad p \in M$$

$\mathcal{D}$ defines a 1-dimensional distribution (and the characteristic foliation) on $\mathcal{L}$.

- The normal fiber bundle satisfies $\left(T\mathcal{L}\right)^{\perp} = \mathcal{D} \subset T\mathcal{L}$ and therefore *is not transverse* to $\mathcal{L}$.

The classical submanifold theory cannot be applied!!

Motivations...

1. Cauchy Horizons.
2. Characteristic hypersurfaces of the wave equation.
3. Degenerate orbits of Lorentz isometric actions.
4. Lightlike cones on Lorentz manifolds.

An ad hoc technique for lightlike hypersurfaces $\psi : \mathcal{L} \to M^{n+2}$

The following technique was introduced by Duggal and Bejancu in 1996.

$$\boxed{\text{Screen distribution } S(\mathcal{L})}$$

Fix an arbitrary n-distribution $S(\mathcal{L})$ on $\mathcal{L}$ such that $T\mathcal{L} = \mathcal{D} \oplus S(\mathcal{L})$. Then,

1. $S(\mathcal{L})$ inherits a Riemannian metric.
2. There exists a unique lightlike transverse fiber bundle $tr(\mathcal{L})$ such that

$$\boxed{TM \mid_{\mathcal{L}} = T\mathcal{L} \oplus tr(\mathcal{L}).}$$

Let $\overline{\nabla}$ be the Levi-Civita connection of M and $X, Y \in \mathfrak{X}(\mathcal{L}), N \in \Gamma(tr(\mathcal{L}))$. The following decompositions strongly depend on $S(\mathcal{L})$.

$$\overline{\nabla}_X Y = \nabla_X Y + \mathrm{II}(X, Y), \quad \text{Gauss equation}$$

$$\overline{\nabla}_X N = -A_N Y + \nabla^{\perp}_X N, \quad \text{Weingarten equation}$$

and the usual terminology is introduced: induced connection, second fundamental form, Weingarten operator...

Several issues to be mentioned

Let $(\mathcal{L}, g, S(\mathcal{L}))$ be a lightlike hypersurface of (M, g).

- The notion of totally geodesic lightlike hypersurface $(\mathrm{II} = 0)$ does not depend on the screen distribution.
- There exists also a notion of totally umbilical lightlike hypersurface which is independent on the screen distribution. But as far as we know, this notion has no clear geometric meaning.

The intrinsic geometry of $\mathcal{L}$ strongly depends on $S(\mathcal{L})$.

Intrinsic geometry of $\mathcal{L} \subset M$

- The induced connection ∇ is independent of $S(\mathcal{L})$ if and only if $\mathrm{II} = 0$.
- $\nabla g = 0$ if and only if $\mathrm{II} = 0$.

It will be useful an intrinsic definition of lightlike manifold $\mathcal{L}$:

Lightlike manifolds

A lightlike manifold is a pair $(\mathcal{L}^{n+1}, g)$ where

- $g \in \mathcal{T}_{0,2}(\mathcal{L})$ is a symmetric tensor (to simplify we assume g is positive semidefinite)
- $\mathrm{Rad}(g)$ is a 1-dimensional distribution on $\mathcal{L}$.

For a lightlike manifold $(\mathcal{L}, g)$, the distribution $\mathrm{Rad}(g)$ is Killing when every vector field $\xi \in \mathrm{Rad}(g)$ is Killing. That is, the Lie derivative of g satisfies $L_\xi g = 0$.

Duggal-Jin, 2007

A torsion free linear connection on a lightlike manifold $\mathcal{L}^{n+1}$ compatible with g exists if and only if $\mathrm{Rad}(g)$ is a Killing distribution. In this case, there is an infinitude of connections with none distinguished.

$\mathrm{Rad}(g)$ is a Killing distribution.

$$\Updownarrow$$

There are coordinates systems $(r, x_1, ..., x_n)$ with $\frac{\partial}{\partial r}$ spans $\mathrm{Rad}(g)$ and $\frac{\partial g_{ij}}{\partial r} = 0$.

Summing up... from my point of view:

1. The Screen distribution construction could be a good approach to study extrinsic properties but not for intrinsic geometry.
2. The lightlike hypersurfaces are conformal invariants. It would be desirable certain conformal invariance.

Natural questions

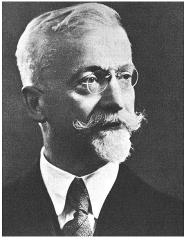
- Is it possible to construct an *intrinsic geometric structure* on $\mathcal{L}$?
- This *intrinsic geometric structure* should be independent of any arbitrary election...
- ... and should provide local invariants which permit to distinguish locally two lightlike manifolds.
- Where can we look for such *geometric structures* ?

Although I do not have a definitive answer to the above questions, let me introduce you the tangle of ideas I have developed hoping to find these geometric structures.

The main ideas come from the notion of Cartan geometry.

KLEIN AND CARTAN GEOMETRIES

In the early 1920s, Elie Cartan found a common generalization for the Klein's Erlangen program and Riemann geometry. He called *Espaces généralizés* and now we call Cartan Geometries.



Elie Cartan (Wikimedia Commons)

From a Klein Geometry G/H

$$\Downarrow$$

A Cartan Geometry with model (G, H) on M is a differential geometric structure on M whose objects may be thought as curved analogs of the homogeneous model.

Around 1950, Charles Ehresmann gave for the first time a rigorous global definition of a Cartan connection as a particular case of a more general notion now called Ehresmann connection.

Charles Ehresmann

This is the point of view of the influential book *Foundations of Differential Geometry, Volumes I and II* by S. Kobayashi and K. Nomizu.

The models of Cartan geometries

Klein Geometry

A pair (G, H), where G is a Lie group and H a closed subgroup of G, is said to be a Klein Geometry when $M = G/H$ is connected and is considered with a *geometric structure* such that

1. The left translations ℓ_g for $g \in G$.

$$\ell_g : M \to M, \quad x \mapsto g \cdot x.$$

   are automorphisms of the *geometric structure*.

2. For every automorphism $f : M \to M$ of the *geometric structure*, there is $g \in G$ such that $f = \ell_g$.

Examples of Klein geometries

Semi-Euclidean Geometry $\mathbb{R}^n_{(p,q)} = (\mathbb{R}^n, \langle\ ,\ \rangle_{(p,q)}), \quad p+q=n$

$$G = \left\{ \begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix} : x \in \mathbb{R}^n, A \in O(p,q) \right\}$$

Consider the transitive action

$$G \times \mathbb{R}^n \to \mathbb{R}^n, \quad \begin{pmatrix} 1 & 0 \\ x & A \end{pmatrix} \cdot \begin{pmatrix} 1 \\ p \end{pmatrix} = \begin{pmatrix} 1 \\ x + Ap \end{pmatrix}$$

Thus, the isotropy group at the origin is $H \cong O(p,q)$ and therefore

$$\boxed{\mathbb{R}^n = G/O(p,q), \quad \mathrm{Iso}(\mathbb{R}^n_{(p,q)}) = G}$$

- For $q = 0 \Rightarrow \mathbb{E}^n, \quad G = \mathrm{Euc}(n), \quad H = O(n) \Rightarrow$ Riemann Geometry.
- For $q = 1 \Rightarrow \mathbb{L}^n, \quad G = \mathrm{Lor}(n), \quad H = O(n,1) \Rightarrow$ Lorentz Geometry.

The Möbius sphere $(\mathbb{S}^{(n)}, [g]), \quad n \geq 3$

- Let $C^{n+1} \subset \mathbb{L}^{n+2}$ be the cone of the (nonzero) lightlike vectors and consider the space of lines in C^{n+1} (the Möbius sphere)

$$\mathbb{S}^{(n)} \subset \mathbb{R}P^{n+1}.$$

- From the obvious projection

$$\Pi : C^{n+1} \to \mathbb{S}^{(n)}, \quad v \mapsto \mathbb{R} \cdot v,$$

  $\mathbb{S}^{(n)}$ carries a natural Riemannian conformal structure $[g]$
  As a manifold $\mathbb{S}^{(n)} = \mathbb{S}^n$ and $[g]$ corresponds to the conformal class of the round metric of radius one.

How can we see $(\mathbb{S}^{(n)}, [g])$ as a Klein geometry?

- Consider the transitive (and natural) action,

$$O(n+1,1) \times \mathbb{S}^{(n)} \to \mathbb{S}^{(n)}$$

  This action is not effective and it is better to use the group

$$G := PO(n+1,1) = O(n+1,1)/\{\pm \mathrm{Id}\}$$

- Thus, we can identify

$$\mathbb{S}^{(n)} = G/P$$

  where $P \subset G$ is the isotropy group of a lightlike line.

  P is called the Poincaré conformal group.

$$\boxed{\mathrm{Conf}(\mathbb{S}^{(n)}) = O(n+1,1)/\{\pm \mathrm{Id}\}}$$

- $(G, P) \Rightarrow$ Riemannian Conformal Geometry.

Several types of Klein Geometries

Let (G, H) be a Klein Geometry and $M = G/H$

- The Klein geometry (G, H) is said to be of first order when the representation of H given by

$$\underline{\mathrm{Ad}} : H \to \mathrm{Gl}(\mathfrak{g}/\mathfrak{h}), \quad h \mapsto \underline{\mathrm{Ad}}(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)(X) + \mathfrak{h}.$$

  is injective.
  Otherwise, (G, H) is a higher-order geometry.
- For a Klein geometry (G, H) of first order, we can see $G \subset L(M)$ (as a fiber bundle of frames over M).

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively.
A Klein geometry (G, H) is said to be...

- Reductive (with complement fixed $\mathfrak{m}$)

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \text{ as vector spaces and}$$

$$\mathrm{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}.$$

Under the assumption H is connected this condition is equivalent to

$$[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}.$$

Semi-Euclidean geometries are reductive Klein geometries of first order

- $\mid k \mid$-graded ($k \geq 1$ and the grading is assumed to be fixed)

$$\mathfrak{g} = \mathfrak{g}_{-k} \oplus \ldots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k,$$

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \quad \mathfrak{h} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k.$$

where $\mathfrak{g}_{-k} \neq \{0\}$, $\mathfrak{g}_k \neq \{0\}$ and $\mathfrak{g}_i = \{0\}$ for $\mid i \mid > k$.

The Möbius sphere $(\mathbb{S}^{(n)}, [g])$ is a $|\, 1 \,|$-graded Klein geometry

It is better to consider

$$\boxed{\mathbb{L}^{n+2} = \mathbb{E}^n \oplus^{\perp} \mathbb{L}^2}$$

This yields to write the usual scalar Lorentzian product as the corresponding to the matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{I}_n & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Recall

$$G = O(n+1,1)/\{\pm \mathrm{Id}\}$$

$$P = \left\{ \begin{bmatrix} \lambda & -\lambda w^t C & -\frac{\lambda}{2}\langle w, w\rangle \\ 0 & C & w \\ 0 & 0 & \lambda^{-1} \end{bmatrix} : \lambda \in \mathbb{R}^*, w \in \mathbb{R}^n, C \in O(n) \right\}$$

Thus,

$$\mathfrak{g} = \left\{ \begin{pmatrix} a & z & 0 \\ x & A & -z^t \\ 0 & -x^t & -a \end{pmatrix} : a \in \mathbb{R}, x \in \mathbb{R}^n, z \in (\mathbb{R}^n)^*, A \in \mathfrak{o}(n) \right\}$$

$$\boxed{\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1} \quad \text{and} \quad \boxed{\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1}$$

Looking for the model of lightlike manifolds $\mathcal{L} = G/H$

1. The isometries of the degenerate tensor g on $\mathcal{L}$ should be exactly the left multiplications by elements of G.
2. (Liouville Theorem) Suppose that $\mathcal{L} = G/H$ is connected. Then any isometry between two connected open subset of $\mathcal{L}$ uniquely globalizes to an isometry of $\mathcal{L}$

Why not $\mathcal{L} = \mathbb{R}^{n+1}$?

Why don't we use $(\mathbb{R}^{n+1} = \mathbb{R} \oplus \mathbb{E}^n, 0 \oplus \langle\ ,\ \rangle)$ as the model for lightlike manifolds?

- The group of isometric tranformations is infinite dimensional in this case!!!

What could it be the model for lightlike manifolds?

$\mathcal{L}$: The upper lightlike cone $Q^{n+1} \subset \mathbb{L}^{n+2}$

Recall that $\boxed{\mathbb{L}^{n+2} = \mathbb{E}^n \oplus^{\perp} \mathbb{L}^2}$

Fix $\ell, \eta \in \mathbb{L}^2$ lightlike vectors such that $g(\ell,\eta) = 1$ and take $T = \frac{1}{\sqrt{2}}(\ell - \eta)$.

$$Q^{n+1} = \Big\{ v \in \mathbb{L}^{n+2} : g(v,v) = 0, \quad g(v,T) < 0 \Big\} \approx C^{n+1}/\mathbb{Z}_2$$

is a lightlike hypersurface (g is the Lorentz metric).

- The natural action, $O(n+1,1) \times C^{n+1} \to C^{n+1}$ induces a transitive action
$$PO(n+1,1) \times Q^{n+1} \to Q^{n+1}$$
- Thus, we can identify $Q^{n+1} = PO(n+1,1)/H$ where $H \subset PO(n+1,1)$ is the isotropy group of the first basis vector ℓ.

$$H = \left\{ \begin{bmatrix} 1 & -w^t C & -\frac{1}{2}\langle w, w\rangle \\ 0 & C & w \\ 0 & 0 & 1 \end{bmatrix} : w \in \mathbb{R}^n, C \in O(n) \right\} \equiv \mathrm{Euc}(n).$$

(here $\langle \ , \ \rangle$ is the Euclidean scalar product)

We have the above required properties [1],

1. For $n \geq 3$

$$\boxed{\mathrm{Iso}(Q^{n+1}) = PO(n+1,1)} \approx O^{+}(n+1,1)$$

2. This is true even locally: any isometry between two connected open subset of Q^{n+1} is the restriction of an element in $PO(n+1,1)$.

[1] Bekkara, Frances and Zeghib, 2009

Taking a look at the model $Q^{n+1} = G/H$ at an algebraical level

- The Klein geometry (G, H) is of first order.

  In particular observe that $G \subset L(Q^{n+1})$ as follows

$$\left(v \mid e_1 \mid e_2 \mid \ldots \mid e_n \mid w \right) \mapsto (e_1, \ldots, e_n, w) \subset T_v Q^{n+1}.$$

- The Klein geometry $Q^{n+1} = G/H$ is not reductive.

Summing-up:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \ \ \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \quad \text{conformal model}$$
$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1, \ \ \mathfrak{h} = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{g}_1 \leq \mathfrak{p} \quad \text{lightlike model?}$$

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0 & z & 0 \\ 0 & A & -z^t \\ 0 & 0 & 0 \end{pmatrix} : z \in (\mathbb{R}^n)^*, A \in \mathfrak{o}(n) \right\} = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathfrak{g}_1 \leq \mathfrak{p} \leq \mathfrak{g}$$

Now we would like to introduce the meaning on the sentence:

The Klein geometry (G, H) is a model for a curved geometric structure.

Cartan Geometry of type (G,H) on M

- A principal fiber bundle $p:\mathcal{P}\to M$ with structure group H.
- A $\mathfrak{g}$-valuated one form $\omega\in\Omega^1(\mathcal{P},\mathfrak{g})$, called the Cartan connection such that:
  1. $\omega(u):T_u\mathcal{P}\to\mathfrak{g}$ is a linear isomorphism for all $u\in\mathcal{P}$.
  2. For ξ_X, the fundamental vector field corresponding to $X\in\mathfrak{h}$,
  $$\omega(\xi_X)=X \qquad \left(\xi_X(u):=\frac{d}{dt}|_0\,(u\cdot\exp(tX))\right)$$
  3. For every $h\in H$, let r^h be the corresponding right multiplication on $\mathcal{P}$. Then
  $$(r^h)^*\omega=\mathrm{Ad}(h^{-1})\circ\omega$$
  That is, the following diagram commutes.

$$\begin{array}{ccc} T_u\mathcal{P} & \xrightarrow{\omega(u)} & \mathfrak{g} \\ {\scriptstyle T_u r^h}\downarrow & & \downarrow{\scriptstyle \mathrm{Ad}(h^{-1})} \\ T_{uh}\mathcal{P} & \xrightarrow[\omega(uh)]{} & \mathfrak{g} \end{array}$$

$$\boxed{\dim(\mathcal{P})=\dim(G),\quad \dim(M)=\dim(G/H)}$$

The curvature of a Cartan connection

Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry of type (G, H) on M.
The curvature form $K \in \Omega^2(\mathcal{P}, \mathfrak{g})$ is given, for every $\xi, \eta \in \mathfrak{X}(\mathcal{P})$, by:

$$K(\xi, \eta) := d\omega(\xi, \eta) + [\omega(\xi), \omega(\eta)].$$

The Klein Geometry (G, H) as a Cartan geometry

- The principal fiber bundle: $\pi : G \to M \cong G/H,\ g \mapsto gH$;
- The Cartan connection: the Maurer-Cartan form $\omega_G \in \Omega^1(G, \mathfrak{g})$:

$$\omega_G(g) : T_g G \to \mathfrak{g}, \quad \xi \in T_g G \mapsto \omega_G(g)(\xi) := T_g \lambda_{g^{-1}} \cdot \xi \in T_e G \cong \mathfrak{g},$$

where $\lambda_g : G \to G, \quad \sigma \mapsto g\sigma$.

This is called the homogeneous model of a Cartan geometry of type (G, H).

The homogeneous models are always flat (Maurer-Cartan equation gives $K = 0$) and the converse is essentially true.

The tangent bundle of a Cartan geometry of type (G, H)

Consider the representation of H given by

$$\underline{\mathrm{Ad}} : H \to \mathrm{Gl}(\mathfrak{g}/\mathfrak{h}), \quad h \mapsto \underline{\mathrm{Ad}}(h)(X + \mathfrak{h}) = \mathrm{Ad}(h)(X) + \mathfrak{h}.$$

For each $u \in \mathcal{P}$ with $p(u) = x \in M$, there is a canonical linear isomorphism

$$\phi_u : T_x M \to \mathfrak{g}/\mathfrak{h}$$

such that the following diagram conmutes

$$\begin{array}{ccc} T_u\mathcal{P} & \xrightarrow{\omega(u)} & \mathfrak{g} \\ {\scriptstyle T_u p}\downarrow & & \downarrow{\scriptstyle \rho} \\ T_x M & \xrightarrow[\phi_u \cong]{} & \mathfrak{g}/\mathfrak{h} \end{array} \quad \text{with } \phi_{uh} = \underline{\mathrm{Ad}}(h^{-1})\phi_u \text{ for all } h \in H.$$

- $TM \cong \mathcal{P} \times_H (\mathfrak{g}/\mathfrak{h}), \quad (x, v) \in TM \mapsto [u, \phi_u(v)]$, where $p(u) = x$.

LIGHTLIKE MANIFOLDS

From now on, we return to the upper lightlike cone $Q^{n+1} \subset \mathbb{L}^{n+2}$ and consider our chain of Lie groups.

$$H \subset P \subset G \quad \Rightarrow \quad Q^{n+1} = G/H \longrightarrow \mathbb{S}^{(n)} = G/P$$

$$G = \mathrm{Iso}(Q^{n+1}), \quad G = \mathrm{Conf}(\mathbb{S}^{(n)}).$$

Proposition

Let $(\pi : \mathcal{P} \to \mathcal{L}, \omega)$ be a Cartan geometry with model (G, H).
Then, $\mathcal{L}$ is a lightlike manifold with tensor g and there is a global vector field $\xi \in \mathfrak{X}(\mathcal{L})$ which spans $\mathrm{Rad}(g)$.

- For every $u \in \mathcal{P}$ with $\pi(u) = x$, consider $\phi_u : T_x\mathcal{L} \to \mathfrak{g}/\mathfrak{h} \cong \mathbb{R}^n \oplus \mathbb{R}$.
- $\underline{\mathrm{Ad}} : H \to \mathrm{Gl}(\mathfrak{g}/\mathfrak{h}), \quad h \mapsto \underline{\mathrm{Ad}}(h)(X, a) = (CX, a - \langle C^{-1}w, X\rangle)$, where $h \leftrightarrow (C, w) \in O(n) \times \mathbb{R}^n$.

Note that the *natural* hyperplane of $\mathbb{R}^n \oplus \mathbb{R}$ is not invariant by $\underline{\mathrm{Ad}}(H)$.

In order to get a first approach to the converse...

Correspondence spaces

Let $(p : \mathcal{P} \to M, \omega)$ be a Cartan geometry with arbitrary model (G, P).
We define the correspondence space $\mathcal{C}(M)$ of M for $H \subset P$ to be the quotient space $\mathcal{P}/H$:

$$\begin{array}{ccc} & \mathcal{P} & \\ & \downarrow & \searrow \\ \mathcal{P}/H = \mathcal{C}(M) & \xrightarrow{\Pi} & M = \mathcal{P}/P \end{array}$$

The projection $\Pi : \mathcal{C}(M) \to M$ is a fiber bundle with fiber the homogeneous space P/H and

$$(\pi : \mathcal{P} \to \mathcal{C}(M), \omega) \text{ is a Cartan geometry of type } (G, H).$$

Return to ours fixed Lie groups $H \subset P \subset G$...

Which are the correspondence spaces for $H \subset P \subset G$?

Conformal Riemannian structures on M

$\Updownarrow$ 1-1

Cartan connections on M with model (G, P)

(satisfying certain curvature properties)

Proposition

Let $(M, [g])$ be a conformal Riemannian manifold and $(p : \mathcal{P} \to M, \omega)$ the corresponding Cartan geometry with model (G, P).
Then the correspondence space $\mathcal{C}(M)$ for $H \subset P$ is

$$\mathcal{E}(M) = \cup_{x \in M}\{c \cdot g_x : c > 0\}$$

the bundle of scales associated to $(M, [g])$.
In particular, $\mathcal{E}(M)$ is a lightlike manifold and admits a distinguished Cartan connection of type (G, H).

Which are the lightlike manifolds which can be seen as bundles of scales of Riemann conformal manifolds?

(these will admit a distinguished Cartan connection of type (G, H))

Assume $(\mathcal{L}, g)$ is a lightlike manifold with $\mathrm{Rad}(g) = \mathrm{Span}(\xi)$ for $\xi \in \mathfrak{X}(\mathcal{L})$. The vector field ξ is said to be conformal when there is a smooth function ρ such that the Lie derivative $L_\xi g = 2\rho\, g$.

Lemma

For $(\mathcal{L}, g)$ a lightlike manifold with $\mathrm{Rad}(g) = \mathrm{Span}(\xi)$ are equivalent:

1. ξ is conformal.
2. For every point $p \in \mathcal{L}$ there is a coordinate system $(r, x_1, ..., x_n)$ around p such that, for all $i, j = 1, .., n$,

$$\xi = \frac{\partial}{\partial r} \quad \text{and} \quad g_{ij}(r, x_1, ..., x_n) = q_{ij}(x_1, ...x_n)e^{\Phi(r, x_1, \ldots, x_n)}.$$

Let $\Pi : \mathcal{L} \to \mathcal{L}/\mathrm{Rad}(g)$ be the natural projection on the space of orbits.

Theorem

Let $(\mathcal{L}, g)$ be a lightlike manifold with $\mathrm{Rad}(g) = \mathrm{Span}(\xi)$ and ξ a conformal vector field.
Assume that around every point $p \in \mathcal{L}$ there is a regular coordinate system $(U, \psi = (r, x_1, ..., x_n))$ respect to the distribution $\mathrm{Rad}(g)$. That is,

1. $\psi(U)$ is an open cube.
2. $\xi \mid_U = \frac{\partial}{\partial r}$.
3. Every integral curve of ξ intersects U in at most one slice $x_1 = c_1, ..., x_n = c_n$.

Then, the *local orbits space* $\Pi(U)$ can be endowed with a conformal class of Riemannian metrics $[\sigma]$. This conformal class does not change if we replace g by any $e^f g$.

... moreover, the correspondence space $\mathcal{C}(\Pi(U))$ for $H \subset P$ is an *extension* of U

$$U \hookrightarrow \mathcal{C}(\Pi(U))?$$

Thus, $(\mathcal{L}, g)$ could be covered by open subsets $U's$ which are base of a distinguished Cartan geometry with model (G, H).

Theorem

Let $\psi : \mathcal{L} \to M$ be lightlike hypersurface in a strongly causal Lorentzian manifold (M, g). Assume the radical of $g_{|\mathcal{L}}$ is spanned by a conformal vector field.
Then every point $p \in \mathcal{L}$ is in an open subset $U \subset \mathcal{L}$ such that $\Pi(U)$ admits a conformal class of Riemannian metric $[\sigma]$ which depends only on the conformal class of g

... in a such way that the correspondence space $\mathcal{C}(\Pi(U))$ for $H \subset P$ is an *extension* of U?
Thus, $(\mathcal{L}, g)$ could be covered by open subsets $U's$ which are base of a distinguished Cartan geometry with model (G, H).

EXAMPLES

1. Every integral lightlike hypersurface of $\xi^{\perp}$ in a Brinkmann space $(\nabla\xi = 0)$.
   - Cahen-Wallach spaces.
   - 2-symmetric Lorentzian spaces (which are not 1-symmetric).
   - Plane-fronted waves (as introduced by Candela, Flores and Sánchez).
2. Every integral lightlike hypersurface of $\xi^{\perp}$ in a Lorentz manifold with $\nabla\xi = \omega \otimes \xi$.