An equivalence result for (Logit) QRE and Level-k

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April 15th, 2016

IMEBESS (Rome)

Introduction

- This paper is part of a long running research programme on boundedly rational models, joint with Enrique and David.
  - It all started with a “numerical coincidence” for the Minimum Effort Game (EconLett, 2013)
  - It has continued with “this paper”
  - It all will end with an paper on an “intuitive contradiction”

The numerical coincidence

- Fatas and Morales (EconLett, 2013) for the Minimum Effort Game, Goeree and Holt (2005)

$$\pi_i(x_i, x_j) = \min\{x_i, x_j\} - cx_i$$

Table 1
Minimum-effort games.

Effort in last 3 periods	Minimum-effort games predicted and actual average efforts			
	$n = 2$		$n = 3$	
	$c = 0.25$	$c = 0.75$	$c = 0.1$	$c = 0.5$
Actual	159	126	170	127
Logit	153	127	154	129

For the logit predictions, estimation of a noise parameter is required. The objective is to match actual behaviour

The numerical coincidence

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Logit	153	127	154	129

For Step Thinking, no estimation is needed. level 1 play = Level 2 = Level 3 …

The numerical coincidence

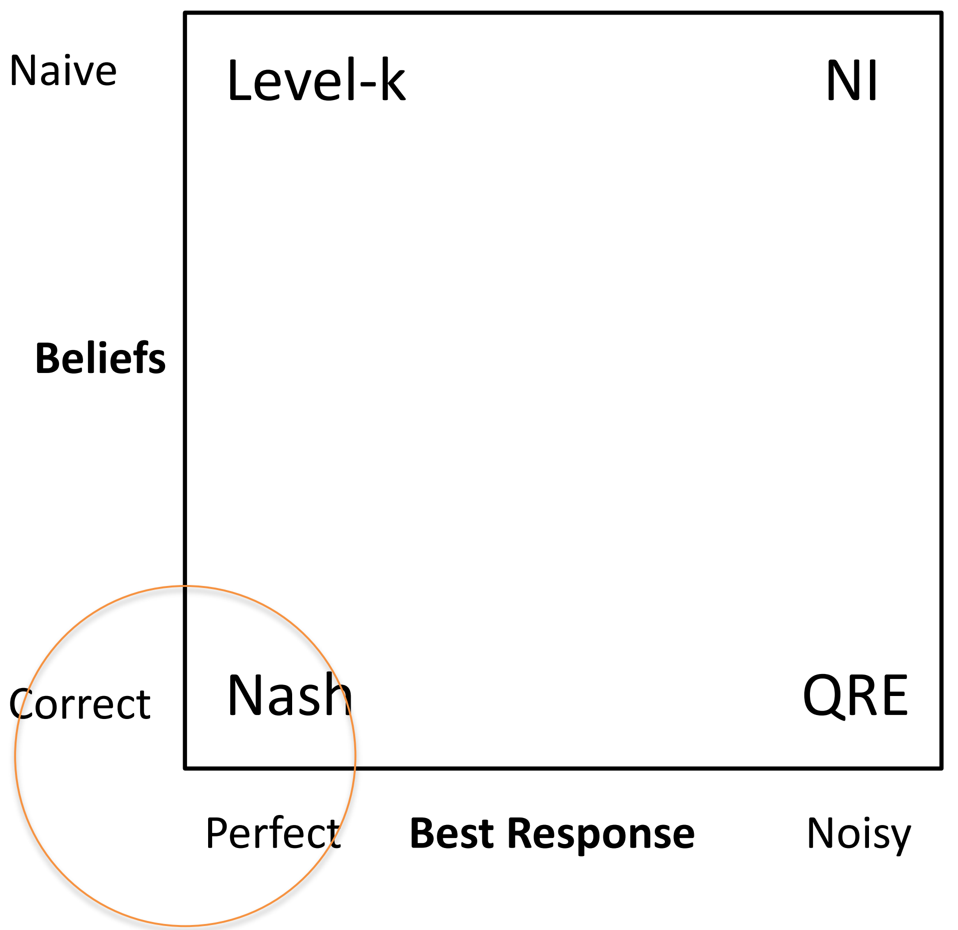
- Out of these results, there were two research questions

1. Why is the Logit best estimation so close to level k?
2. Why is actual behaviour so close to predictions?

- This paper is about Research Question #1

Was it just a coincidence??

Nash Play = Perfect best response + Correct beliefs



Step-Thinking

- A “simplification” of NE
- Models based on best responses without mutual consistent beliefs
  - Level-k (Nagel, 1995 and Stahl and Wilson, 1994, 1995)
  - Cognitive Hierarchy (Camerer et al, 2004)
- Types of players:
  - L0 players: Typically choose *randomly*
  - L1 type best responds to L0
  - Lk types best respond to lower types

μ

Quantal Response Equilibrium

- A "generalization" of NE
- Choices are positively but imperfectly (errors) related to expected payoffs…
  - QRE (McKelvey and Palfrey, 1995)
  - Logit version (Anderson, Goeree and Holt, 2002)
- Fixed point (equilibrium approach)

$$f_i(x) = \frac{\exp(\pi_i^e(x)/\mu)}{\int_0^\omega \exp(\pi_i^e(s)/\mu)\,\mathrm{d}s}$$

- μ is a noise parameter

The conventional wisdom

- Their different nature...
  - QRE is an equilibrium model that requires “learning” -> Long run
  - Level-k simplifies others’ decisions-> initial stage
- … suggests a differential time horizon
- Experimental literature focuses on their differences:
  - Logit QRE: Holt et al…(Goeree and Holt, AER 2001)
  - Level-k: Crawford et al (JEL 2014)

The conventional wisdom

- There is also a “fight” as to which model better captures human behaviour within the same time horizon (usually in the initial round)
  - Crawford et al (2009) on VHBB coordination games
  - Arad and Rubinstein’s experiment on “11-20 Game” (2013) and Goeree et al’s reaction (2013)

- In this paper, we explore the connections between level k and logit quantal response

This paper

- We explore the connections between Level-k and Logit QRE
  - Schmutzler (GEB 2011): A unified approach to *comparative statics* puzzles in experiments (over a wide class of games)
- We focus on point predictions
- The key insight was to realise that the minimum game belongs to a particular class of games
  - Games with the local payoff property (Anderson et al, 2002)

Local Payoff Property

- Consider (2-player) games with actions in the interval $[\underline{x}, \overline{x}]$ with **rank-based payoffs**, e.g. the payoff to a player depends on whether his action is the higher or lower action (Anderson et al, 2002)
  - Examples are price competition, minimum coordination game, traveller's dilemma, auctions …

Local Payoff Property

- Suppose the expected payoff to player i when he expects player j to play according to “mixed strategy” f_j can be written as follows where the α() and β() are additively separable and continuous

$$\pi_i^e(x) = \underbrace{\int_{\underline{x}}^{x} \left[\alpha_H(x) + \beta_H(y)\right] f_j(y)\,dy}_{\text{i's action is the higher}} + \underbrace{\int_{x}^{\bar{x}} \left[\alpha_L(x) + \beta_L(y)\right] f_j(y)\,dy}_{\text{i's action is the lower}}$$

- Such a game has **the local payoff property**
  - The expected payoff derivative $\pi_i^{e\prime}(x)$ only depends on the player’s own decision, on the distribution function F(x), on the density function f(x) evaluated at x and on a “shift” parameter c_i.
  - Example: Minimum effort game

$$\pi_i\left(x_i, x_j\right) = min\left\{x_i, x_j\right\} - c_i x_i$$

$$\pi_i^e{}'(x) = 1 - F(x) - c_i$$

Logit QRE and Level-k

- Proposition 4 in Anderson et al (2002) is about the comparative statics of the logit equilibrium prediction wrt parameter α
- The conditions required in Prop 4 imply the same comparative statics for the level k prediction

Logit QRE and Level-k

- Logit QRE is the solution to the following differential equation:

$$\mu f_i'(x) = f_i(x)\pi_i^{e\,\prime}(x) \quad \begin{cases} \mu \to 0 : \text{Nash equilibrium} \\ \mu \to \infty : \text{Uniform play} \end{cases}$$

Derivative of expected payoffs

- Level-1 solves the simpler equation:

$$\pi_i^{e\,\prime}(x) = 0$$

f(.) logit distribution

Uniform distribution

The Equivalence Result

Proposition 1. *If the symmetric game satisfying the local payoff condition is such that the marginal expected payoff is (i) not increasing in x and (ii) strictly decreasing in F, then the comparative statics of the optimal behaviour of a type-1 player and the logit equilibrium with respect to the exogenous parameter α coincide.*

$$\pi^{e\,\prime}\left(\underset{(-)}{F_j(x)}, f_j(x), \underset{\substack{(-)\\(0)}}{x}, \alpha\right)$$

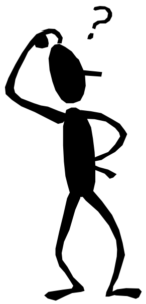
- The most notorious applications of the logit QRE comply with these conditions
- In many applications, the logit equation cannot be explicitly solved. But the simpler “level-k equation” can be easily solved

Table 1. LQRE and L1 predictions for some games with the local payoff property

Game: Expected payoff derivative $\pi^{e\prime}(x)$	Comparative statics LQRE
Travellers' Dilemma $1 - F_j - 2Rf_j$	$R\ (-)$
Minimum Effort game $\left(1 - F_j\right)^{N-1} - c$	$c\ (-)$ $N\ (-)$
Imperfect Price Competition $-(1-\alpha)xf_j + 1 - F_j$	$\alpha\ (+)$
Bertrand Game $\left(1 - F_j\right)^{N-1} - x(N-1)\left(1 - F_j\right)^{N-2} f_j$	$N\ (-)$
Spatial Competition $\frac{-F_j}{2} + \frac{1 - F_j}{2} + f_j(1-2x) - c$	$c\ (-)$

Table 1. LQRE and L1 predictions for some games with the local payoff property

Game:		Comparative statics	
Expected payoff derivative $\pi^{e\prime}(x)$	Level 1 prediction	Level 1	LQRE
Travellers' Dilemma $1 - F_j - 2Rf_j$	$\overline{x} - 2R$	$R\ (-)$	$R\ (-)$
Minimum Effort game $(1 - F_j)^{N-1} - c$	$\underline{x} + (\overline{x} - \underline{x})(1 - \sqrt[N-1]{c})$	$c\ (-)$ $N\ (-)$	$c\ (-)$ $N\ (-)$
Imperfect Price Competition $-(1-\alpha)xf_j + 1 - F_j$	$\dfrac{\overline{x}}{2-\alpha}$	$\alpha\ (+)$	$\alpha\ (+)$
Bertrand Game $(1 - F_j)^{N-1} - x(N-1)(1 - F_j)^{N-2} f_j$	$\dfrac{\overline{x}}{N}$	$N\ (-)$	$N\ (-)$
Spatial Competition $\dfrac{-F_j}{2} + \dfrac{1-F_j}{2} + f_j(1-2x) - c$	$\dfrac{2 + \underline{x} + \overline{x} - 2c(\overline{x} - \underline{x})}{6}$	$c\ (-)$	$c\ (-)$



Level-k predictions within games with the local payoff property

- A "nice" property of level k predictions within the class of games with the local payoff property is that level k predictions do not depend on the distribution of types in the population
  - In games with continuous strategy space, the optimal behaviour of higher types in these auction-like games will be arbitrarily close the optimal choice of a type-1 player
  - There is no freedom of degree

Pure coordination game (Goeree and Holt, 2005) with n=2

Effort interval [110, 170]

	Predicted efforts	
Model	$c = 0.25$	$c = 0.75$
LQRE ($\mu = 10$)	150	129
Step Thinking	155	125

Travellers' Dilemma (Basu, 1994)

- Two travellers have lost their baggage, which happen to be identical
- The airline announces that they will be reimbursed the lowest of their claims but that a reward *R* will be transferred from the highest to the lowest claimant
- <u>Unique NE for all values of R</u>: lower bound of the claim interval (undercutting argument)

Capra et al (1999)

Claim interval [80, 200]

	Predicted claims					
Model	R=5	R=10	R=20	R=25	R=50	R=80
LQRE ($\mu = 10$)	181	173	151	136	112	82
Step thinking	190	180	160	150	100	80

Imperfect price competition

- Two firms
- The market price is the minimum price
- The low price firm gets her price (the minimum price)
- The high price firm gets a fraction α of the minimum price (In Bertrand, $\alpha=0$)

Capra et al, (2002)

Price interval [60, 160]

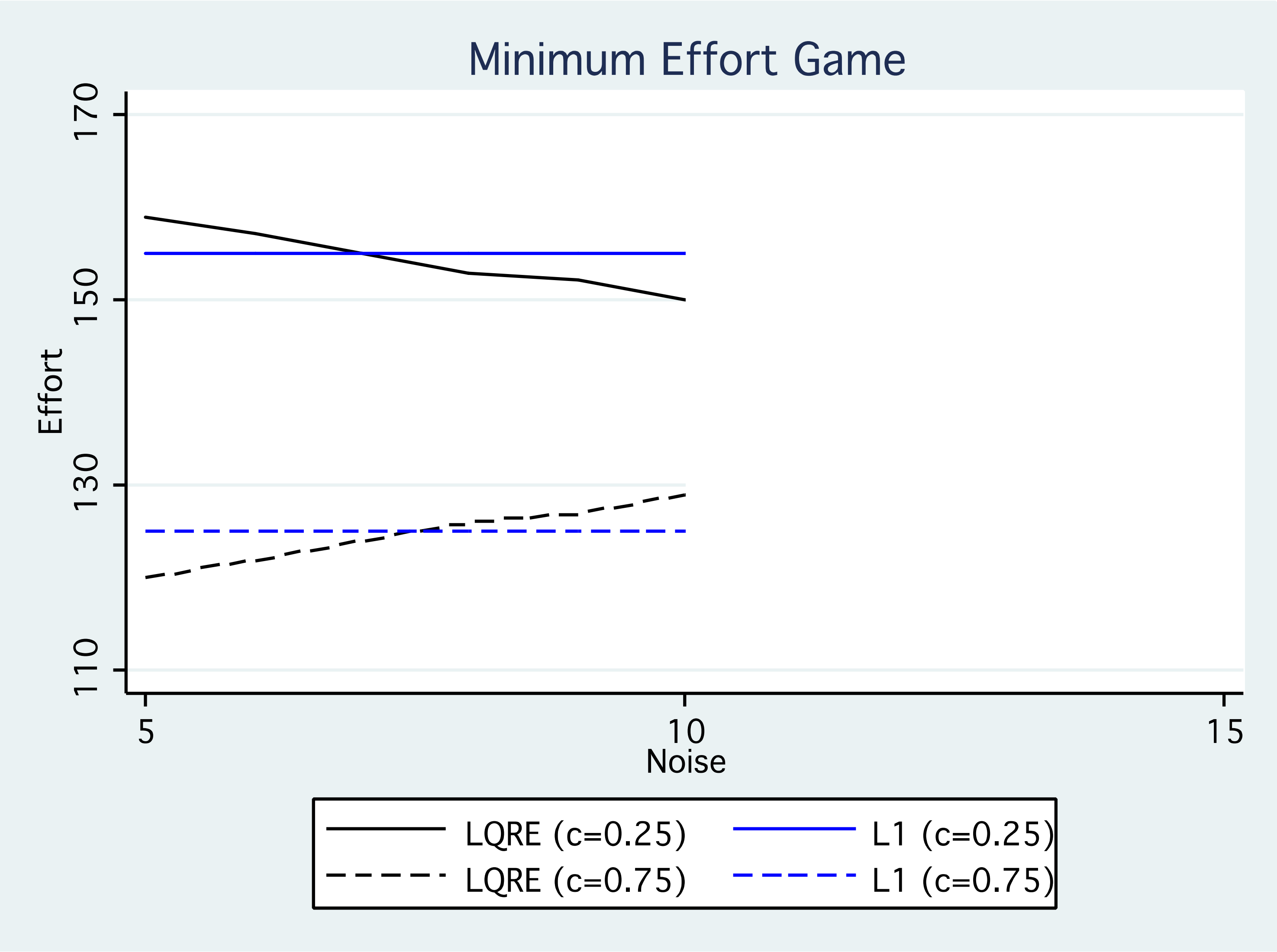
	Predicted prices	
Model	$\alpha = 0.2$	$\alpha = 0.8$
LQRE ($\mu = 10$)	78	127
Step Thinking	88	133

Why LQRE is so close to level-k?

- Two points here:

1. Logit QRE predictions are quite insensitive to the noise parameter (Within the class of games with the local payoff property) within reasonable values
   – There is a footnote on this in Anderson et al (2002)
   – Within this class, there is no degree of freedom (remember Haile et al (AER, 2008))
2. What seems to matter is that some kind of randomness is added to the decision making process, not the logit structure of LQRE

LQRE is "insensitive" to noise



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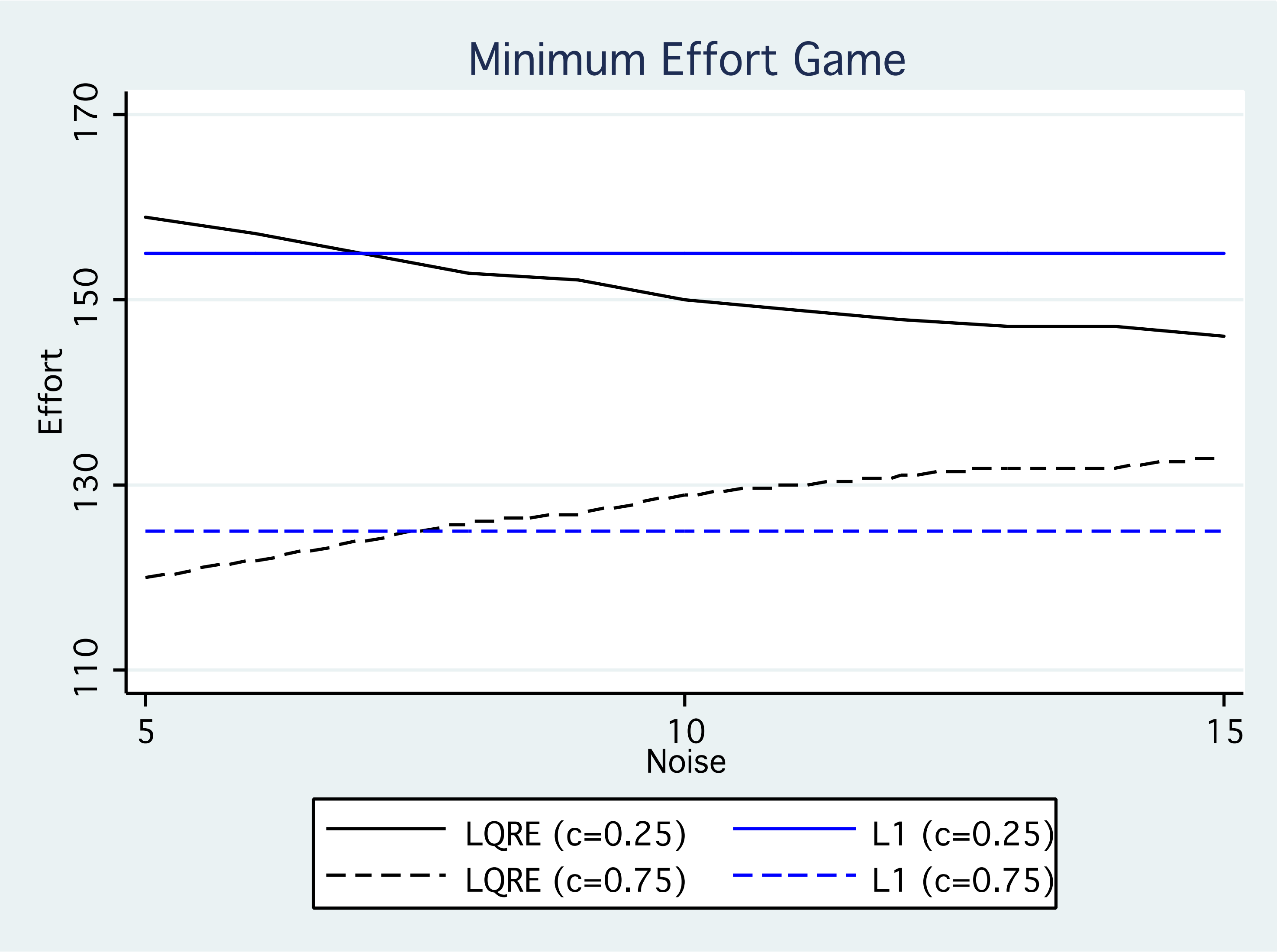

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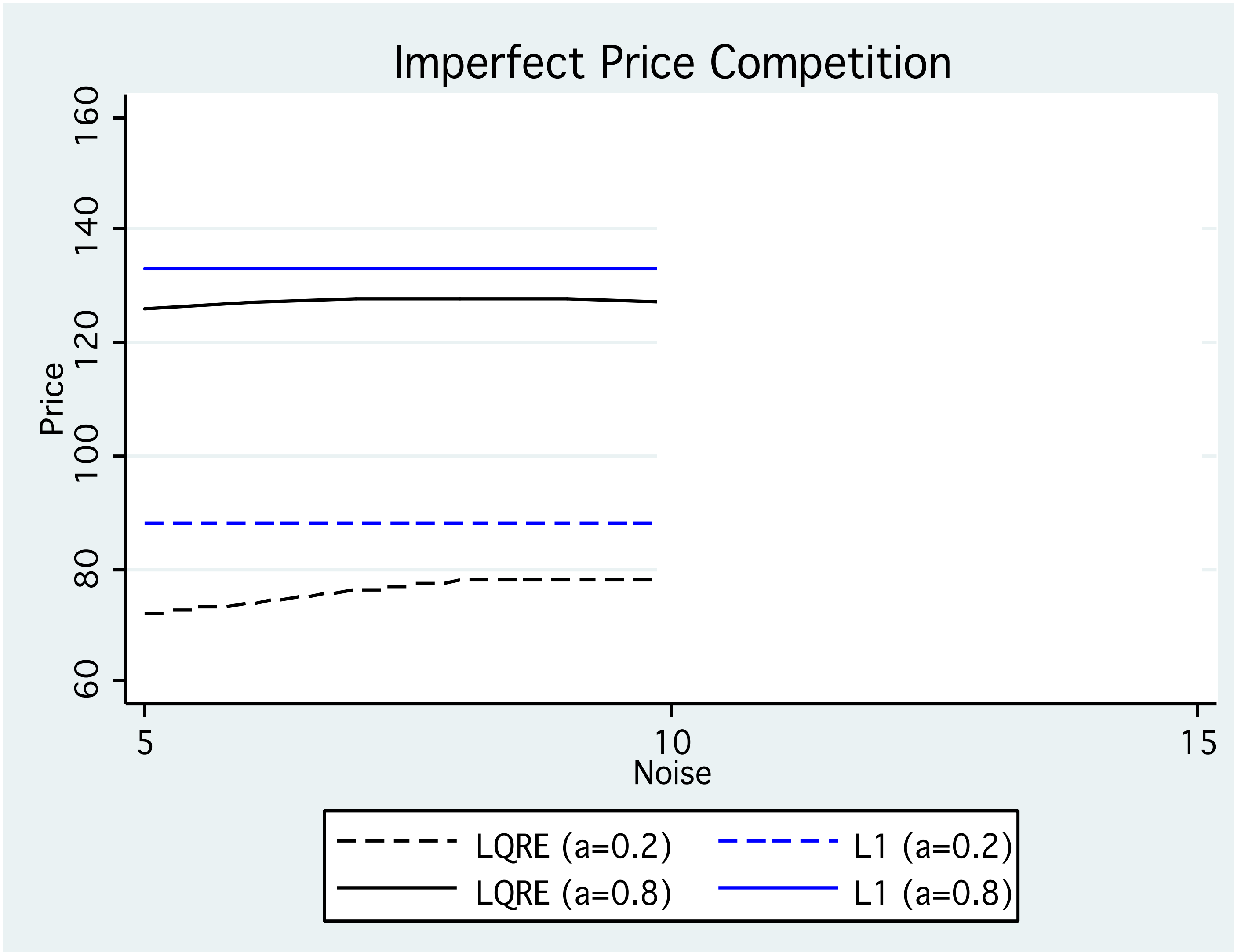

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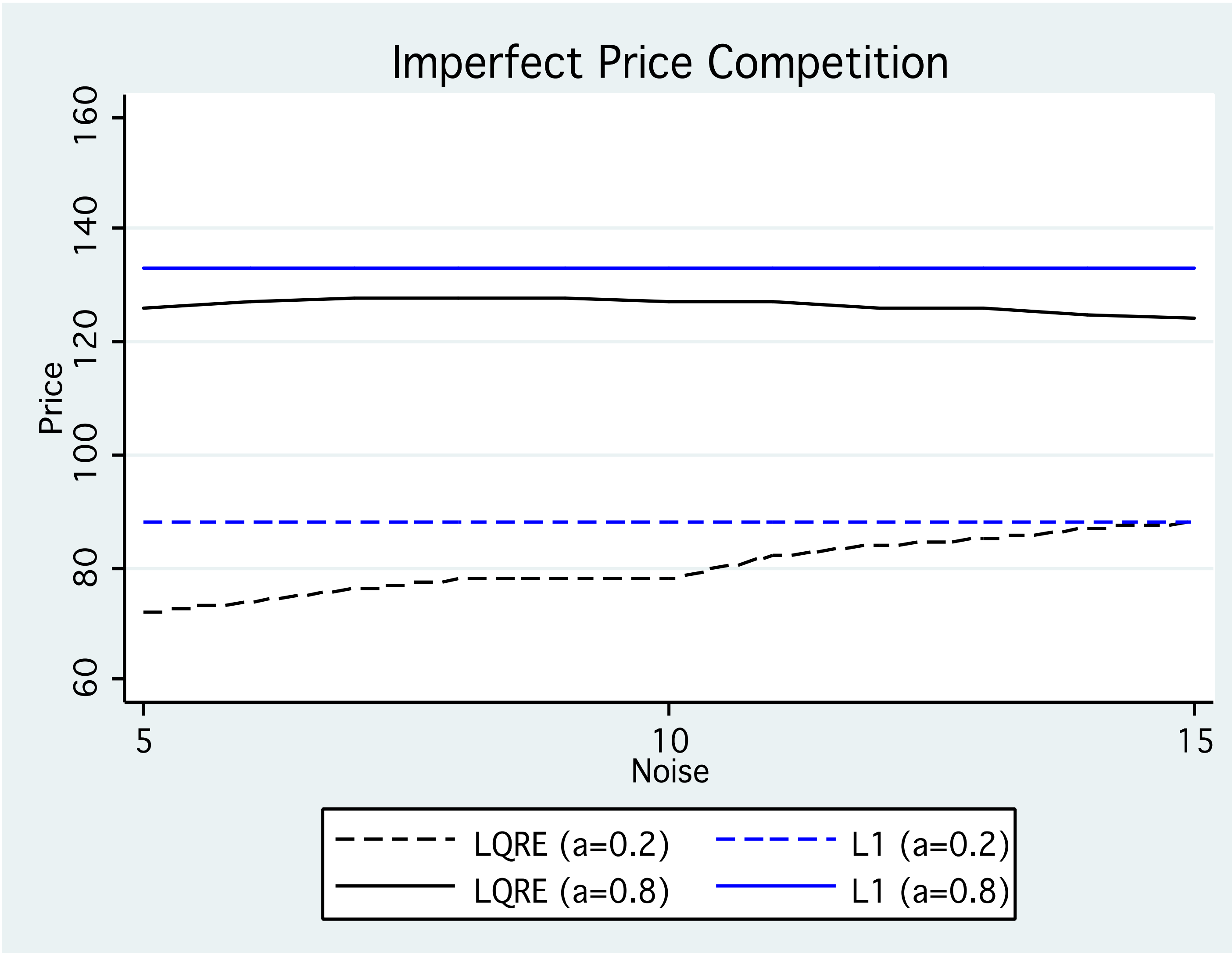

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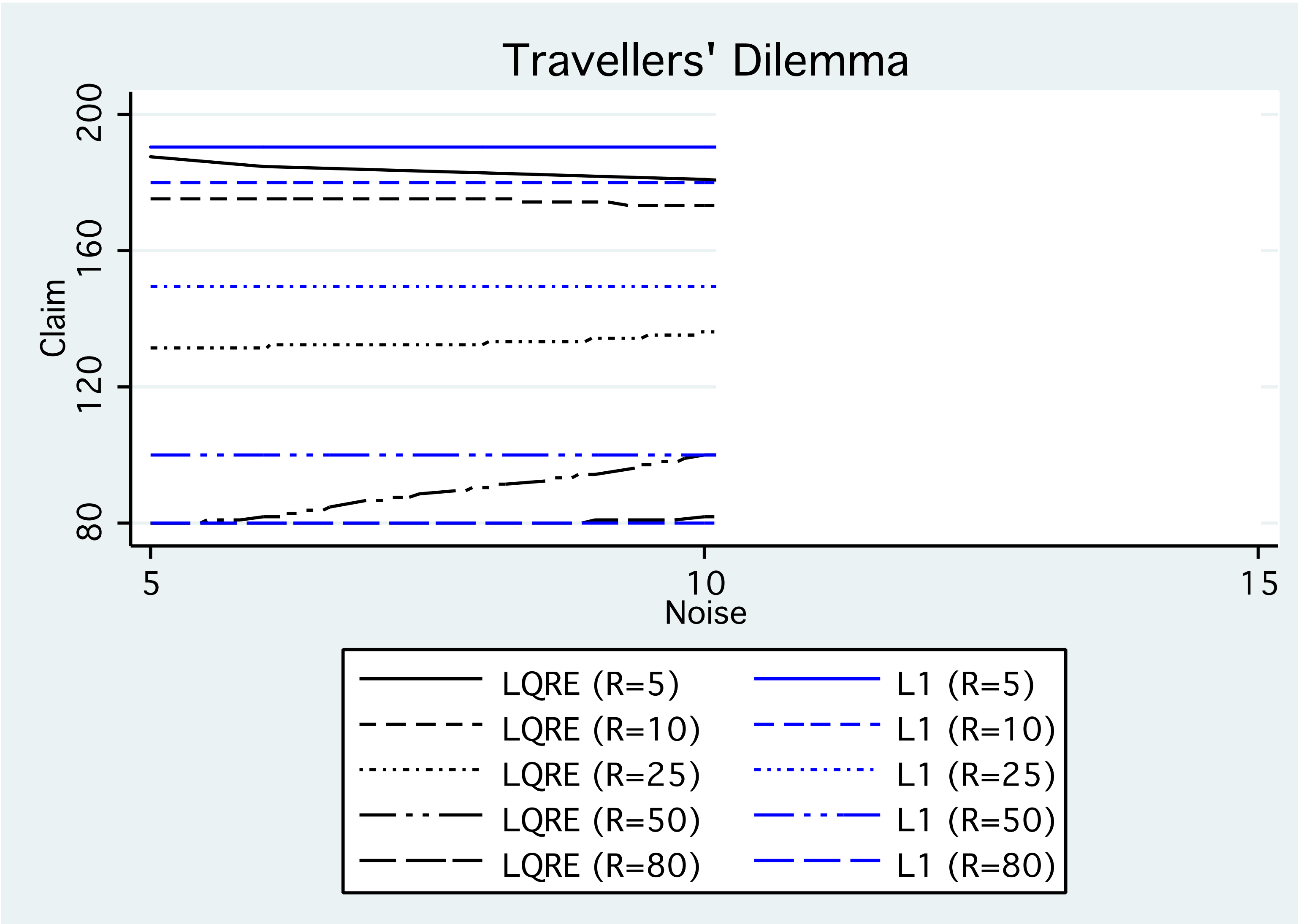

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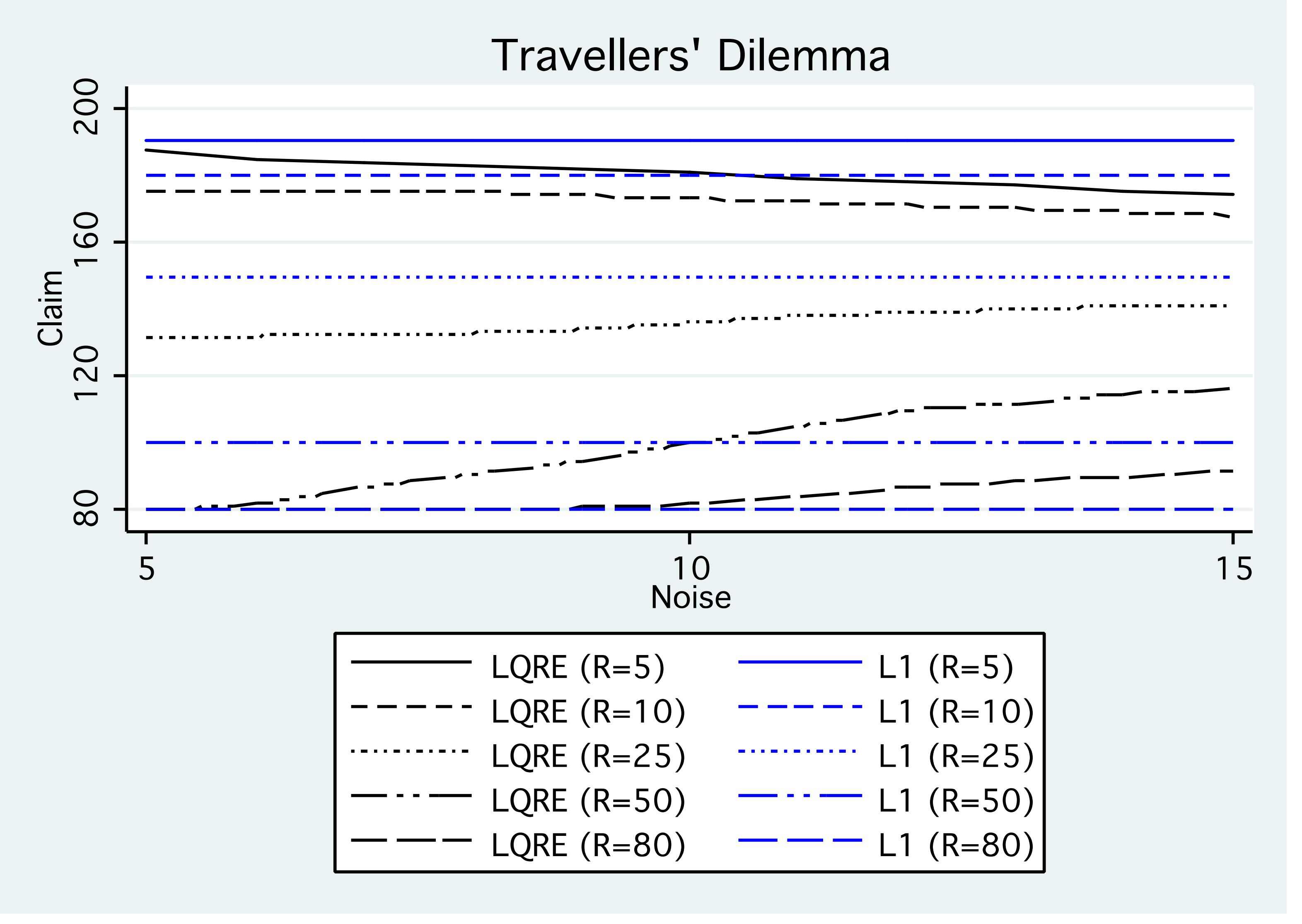

Randomness, which randomness??

- The level-k predictions are based on the uniform distribution
- Level-k was “conceived” for one shot games (or initial play) while Logit QRE is for later rounds
- I have shown data on last periods of play and compare it with level-k

Recall the numerical coincidence

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Randomness, which randomness??

- When subjects play a game several times, they may learn about two things:
  1. About others’ play
  2. About the game

- Can we replicate the treatment effects within the class of games with the local payoff property in one shot games?
  – If so, logit beliefs run into problems…

New experiments - Experimental design

- One-shot games
- Discrete strategy space: (110, 120, 130, … 190, 200)
- Within subject analysis: Subjects made 20 decisions
  - They played 5 games (imperfect price competition, minimum coordination, travellers' dilemma, "11-20" game and all-pay auction).
  - They played 4 variations for each game:

Player 2 (α_2)

Player 1 (α_1)		
	Low, Low	Low, High
	High, Low	High, High

LOW: 20
HIGH: 80

New experiments - procedures

- Experiments conducted at U. of Valencia
- ≈ 1½ hours, 18 – 20 euros
- Four sessions, 224 subjects
- All subjects play **all 5 classes of games**
  - Minimum game, Travelers' dilemma and Imperfect price competition rotated across sessions
  - 11 – 20 and All-Pay always last two classes
- **No feedback, subjects paid for one randomly selected game**
- All games were played with paper & pencil
- **Initial Instructions (on how to read payoff matrices)**
- Each class of games was handed out as a separate packet
- Each class had a set of instructions, comprehension questions, and four payoff tables.

B1-D1

Participante 2 (20%)

Participante 1 (20%)

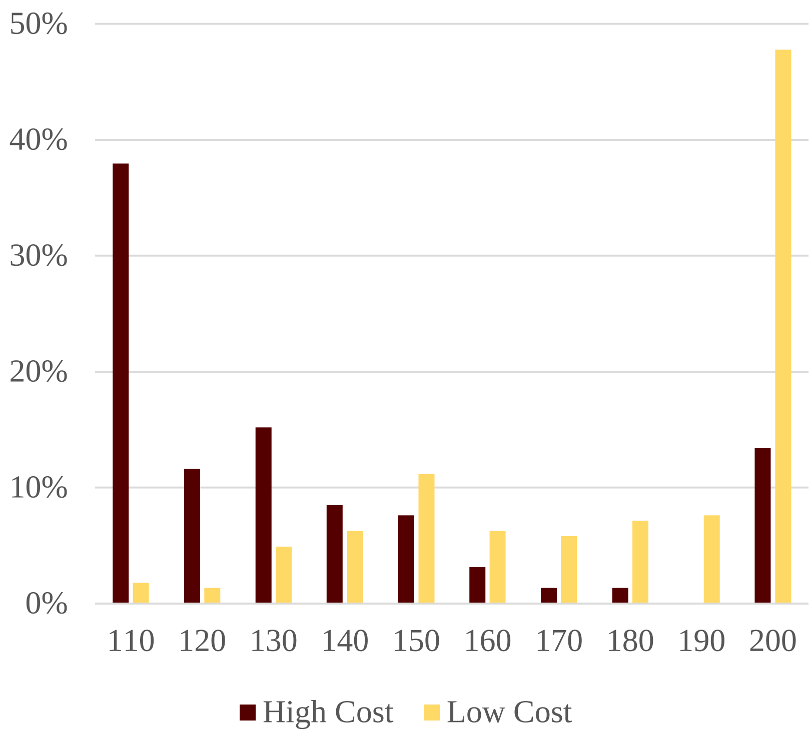
	110	120	130	140	150	160	170	180	190	200
110	66	110	110	110	110	110	110	110	110	110
	66	22	22	22	22	22	22	22	22	22
120	22	72	120	120	120	120	120	120	120	120
	110	72	24	24	24	24	24	24	24	24
130	22	24	78	130	130	130	130	130	130	130
	110	120	78	26	26	26	26	26	26	26
140	22	24	26	84	140	140	140	140	140	140
	110	120	130	84	28	28	28	28	28	28
150	22	24	26	28	90	150	150	150	150	150
	110	120	130	140	90	30	30	30	30	30
160	22	24	26	28	30	96	160	160	160	160
	110	120	130	140	150	96	32	32	32	32
170	22	24	26	28	30	32	102	170	170	170
	110	120	130	140	150	160	102	34	34	34
180	22	24	26	28	30	32	34	108	180	180
	110	120	130	140	150	160	170	108	36	36
190	22	24	26	28	30	32	34	36	114	190
	110	120	130	140	150	160	170	180	114	38
200	22	24	26	28	30	32	34	36	38	120
	110	120	130	140	150	160	170	180	190	120

Descriptive results

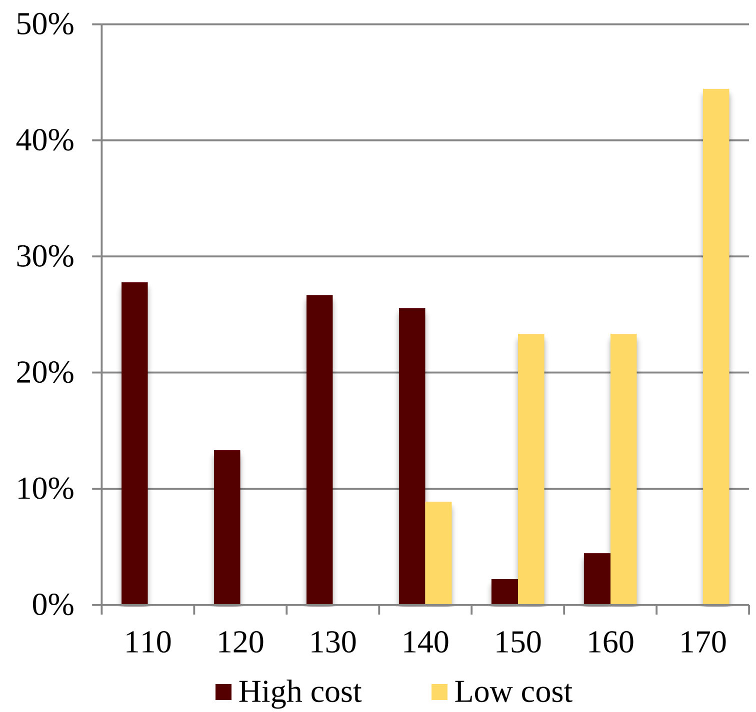
- Our data –remember, one shot game- looks similar to earlier datasets gathered for these games –remember, last periods-.

Minimum Game

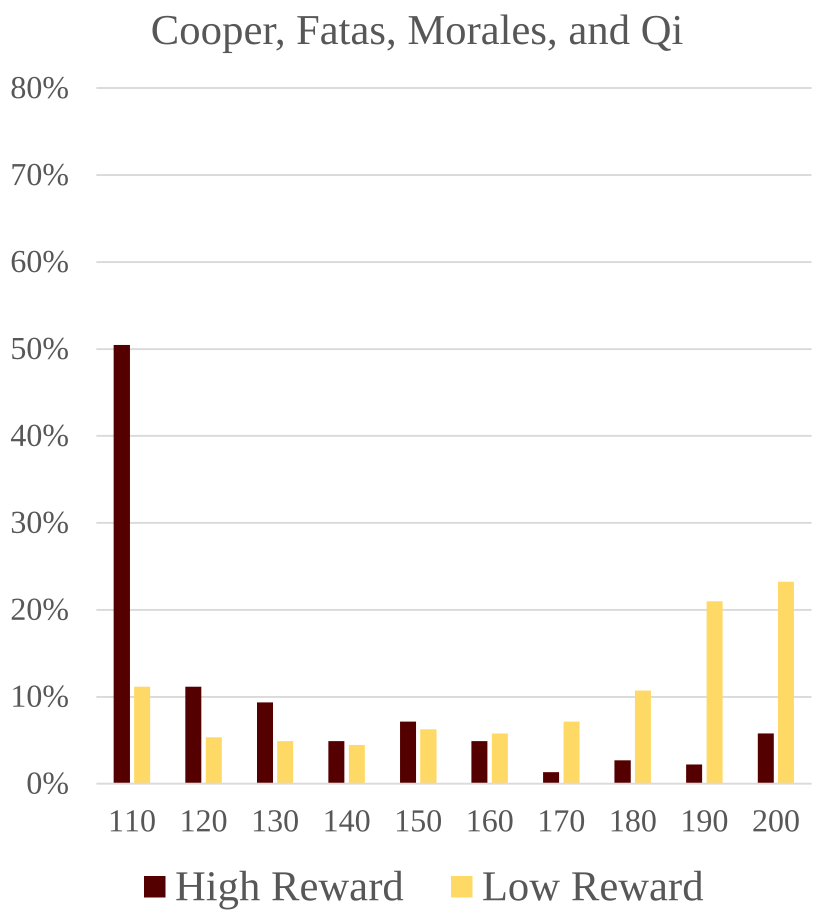
Cooper, Fatas, Morales, and Qi
One shot

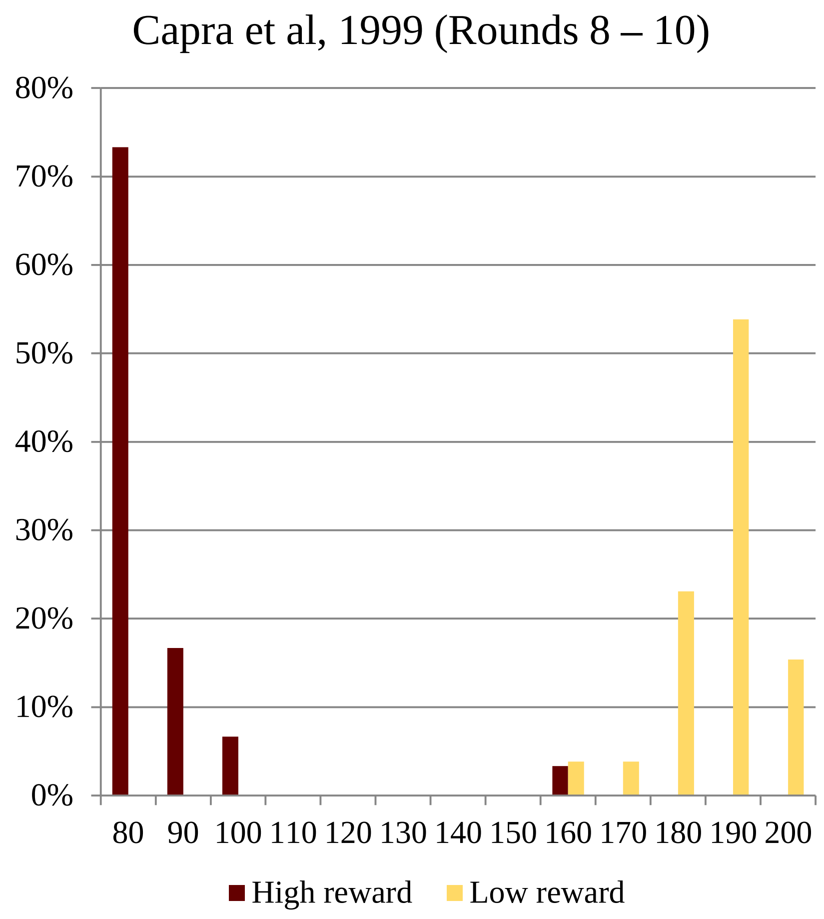


Goeree and Holt (2005)
Rounds 8-10

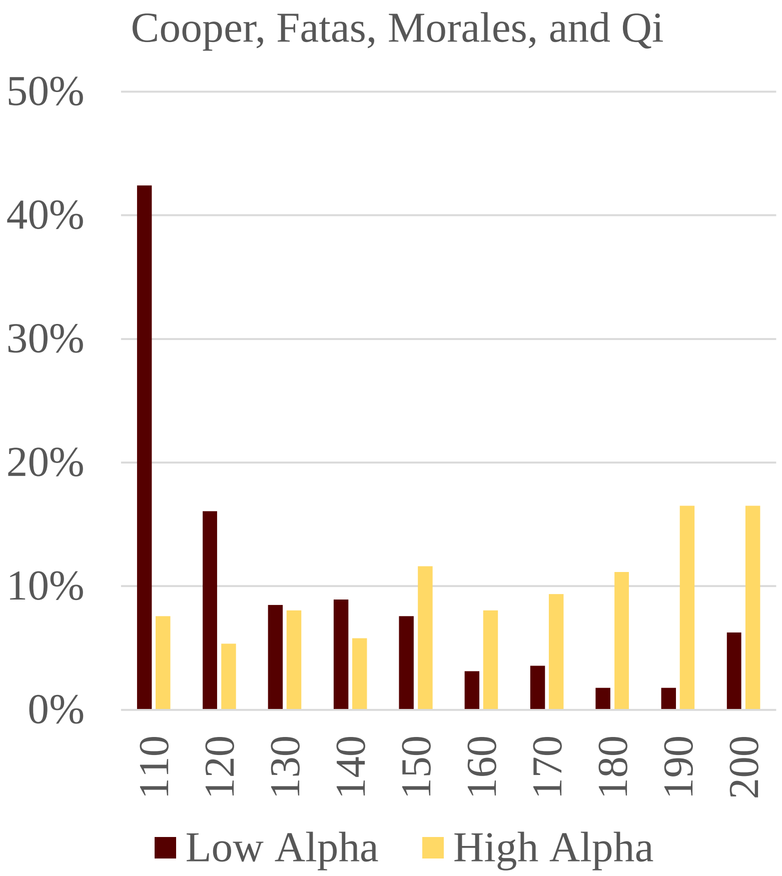


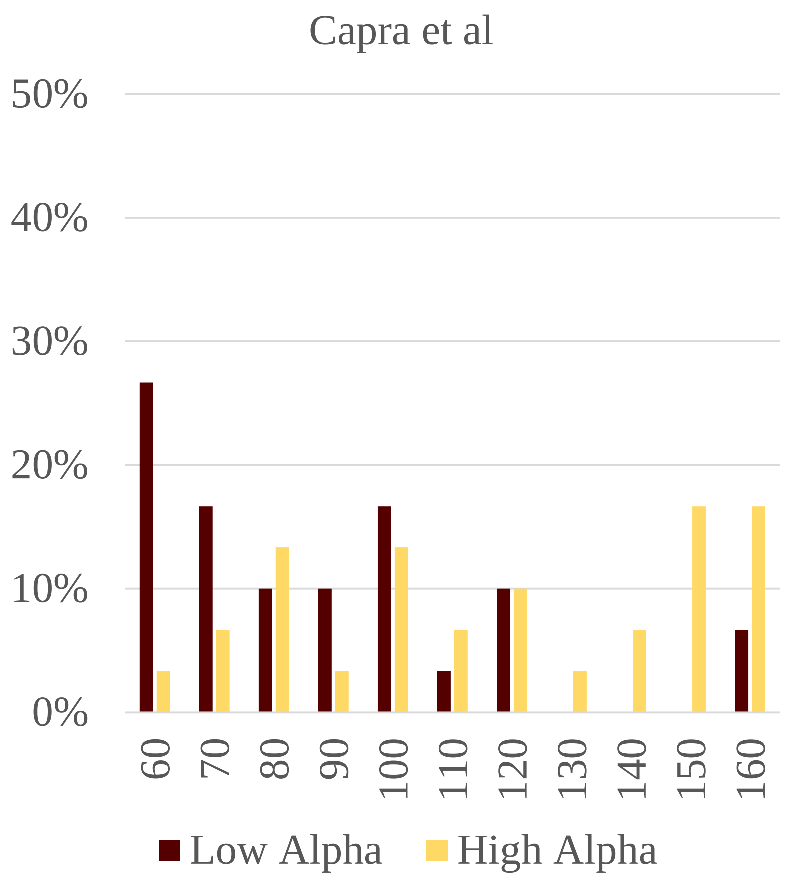
Travelers' dilemma





Imperfect Competition





Conclusions

- As Holt and coauthors state: "**Randomness is key to capture *intuitive* deviations**"
- Both level-k and logit QRE include randomness
- Within the class of games with the local payoff property
  - Both models share the same comparative statics
  - Both models provide similar point predictions
  - This suggests that equilibrium beliefs are not needed -> in repeated versions of the games, subjects learn about the game, not about others' play

Conclusions

- Thank you very much for your attention (and comments ☺)