Fuzzy adjunctions revisited

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Extended abstract

In this work we are aiming at obtaining the weakest notion of adjunction between fuzzy structures. This topic continues our research line on the study and construction of adjunctions [2,5,6].

We will focus on the notion of fuzzy relation which, in some sense, can be interpreted as a fuzzy function. A number of papers dealing with this problem can be found in the literature [3,4,7] and among all these approaches, we will work with the notion of *functional* fuzzy relation, which is equivalent to what Ćirić calls partial fuzzy function (but actually does not coincide with the original notion of partial fuzzy function given by Demirci).

Definition 1 *A fuzzy relation* $\mu \in L^{A\times B}$ *is said to be* functional *if for each* $a_1, a_2 \in A$ *and* $b_1, b_2 \in B$ *the following inequality holds*

$$\mu(a_1,b_1) \otimes \mu(a_2,b_1) \otimes \mu(a_1,b_2) \leq \mu(a_2,b_2) \tag{1}$$

A functional fuzzy relation μ induces two fuzzy equivalence relations, $\approx_\mu$ on B and $\approx_{\mu^{-1}}$ on A, defined, respectively by

$$(b_1 \approx_\mu b_2) = (b_1 = b_2) \vee \bigvee_{a\in A} \big(\mu(a,b_1) \otimes \mu(a,b_2)\big)$$

$$(a_1 \approx_{\mu^{-1}} a_2) = (a_1 = a_2) \vee \bigvee_{b\in B} \big(\mu(a_1,b) \otimes \mu(a_2,b)\big)$$

for all $b_1, b_2 \in B$ and $a_1, a_2 \in A$. The two fuzzy equivalences above allow to consider A and B as fuzzy structures $\langle A, \approx_A\rangle$ and $\langle B, \approx_B\rangle$ where the fuzzy equivalences $\approx_A$ and $\approx_B$ are the induced by μ.

In order to consider an adequate notion of fuzzy adjunction between fuzzy structures, it is necessary to fix fuzzy preorders on both of them. A fuzzy relation $\rho_A\colon A\times A \to L$ is said to be a *fuzzy preorder compatible with* $\langle A, \approx_A\rangle$ if and only if the two following conditions hold

$$(a_1 \approx_A a_2) \leq \rho_A(a_1,a_2) \quad \text{and} \quad \rho_A(a_1,a_2) \otimes \rho_A(a_2,a_3) \leq \rho_A(a_1,a_3)$$

In this abstract setting, it makes sense to study suitable notions of isotonicity for a functional fuzzy relation which generalize the well-known definitions of isotonicity for compatible crisp mappings and for perfect fuzzy functions. A possible definition could be

$$\mu(a_1,b_1)\otimes\mu(a_2,b_2)\otimes\rho_A(a_1,a_2)\leq\rho_B(b_1,b_2) \tag{2}$$

for all $b_1,b_2\in B$ and $a_1,a_2\in A$.

Similarly, another related notion which has to be extended is the inflationary property, whose definition for an internal functional fuzzy relation $\varphi\colon A\times A\to L$ in this setting could be $\varphi(a_1,a_2)\leq\rho_A(a_1,a_2)$.

What would be a desirable notion of adjunction in this extended framework? An adjunction between fuzzy preorders $\langle A,\rho_A\rangle$ and $\langle B,\rho_B\rangle$ can be defined as a pair (μ,ν) of functional fuzzy relations. In principle, we can find the three possibilities below:

- $\mu(a_1,b_1)\otimes\nu(b_2,a_2)\leq\rho_B(b_1,b_2)\leftrightarrow\rho_A(a_1,a_2)$.
- $\nu(b_2,a_2)\otimes\rho_A(a_2,a_2)=\mu(a_1,b_1)\otimes\rho_B(b_1,b_2)$.
- μ and ν are isotone, $\nu\circ\mu$ inflationary and $\mu\circ\nu$ deflationary.

It has to be studied which of the definitions above behaves as expected in relation to closure operators and closure systems.

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