8:05

Philipps Universität Marburg

G_2 : THE LIE ALGEBRA AND THE LIE GROUP.

Aim: provide descriptions of g_2

Contents
- ① Starting point ✡ ↰
- ② Linear construction of $L \subset gl_7(\mathbb{F})$ $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$
- ③ What is preserving L ? $\overset{14}{L} \subset so(\overset{21}{V}, n)$
  $\overset{\shortparallel}{Der}(\underset{\mathbb{F}^7}{\underset{\shortparallel}{V}}, x) = Der(V, \langle \, , \, \rangle) \cong Der\, Oct.$
  $\wr\!\wr$ ⑥
- ④ Engel's results about G_2 and 3-forms on dim 7 $\{ d \in spin(V, -n), d(\varphi) = 0 \}$
- ⑤ G_2 & S^6 G_2 & S^7

8:10 ① SHORT INTRO TO ROOT SYSTEMS.

(L, [,]) simple Lie algebra over $\mathbb{C}$

Classification Killing 1887 :

adjoint operator $x \in L \rightsquigarrow adx : L \to L$ $\Rightarrow L \cong adL$
$y \to [x, y]$

Killing form $\kappa : L \times L \to \mathbb{C}$ <u>NON DEG</u>
$(x, y) \to tr(adx \circ ady)$

Def: $h \leq L$ is a Cartan <u>subalgebra</u> if $\begin{cases} adh \text{ diagonalizable } \forall h \in h \\ \text{maximal} \end{cases}$ $\Rightarrow h$ abelian

The simultaneous diagonalization

$$L = \bigoplus_{\alpha \in h^*} L_\alpha \quad \uparrow = \underbrace{L_0}_{h} \oplus \Big(\bigoplus_{\alpha \in \Phi} L_\alpha \Big)$$

$L_\alpha = \{ x \in L \, / \, adh(x) = \alpha(h)x \ \forall h \in h \}$ (rootspace)

$\Phi = \{ \alpha \in h^* \, / \, L_\alpha \neq 0 \}$ (Root system)

<u>Properties of Φ</u> : $k|_h$ nondeg $\Rightarrow$ identifies h with h^*
$\Rightarrow h$ induces $(\, , \,) : h^* \times h^* \to \mathbb{C}$ s.t $\begin{cases} (\overset{\in \Phi}{\widehat{\alpha, \beta}}) \in \underline{\mathbb{Q}} \\ (\alpha, \alpha) > 0 \end{cases}$

Hence $(E = \sum \mathbb{R}\alpha, (\, . \,))$ EUCLIDEAN SPACE

Def : root system $\Phi \subset E$ $\begin{cases} \text{finite generating set} \\ + \text{extra properties} \end{cases}$

Simple Lie algebra $\longrightarrow$ Root system $\left\{ \begin{array}{l} \text{Classified} \\ A_\ell, B_\ell, C_\ell, D_\ell \\ \underbrace{G_2}_{\downarrow}, F_4, E_6, E_7, E_8 \end{array} \right.$

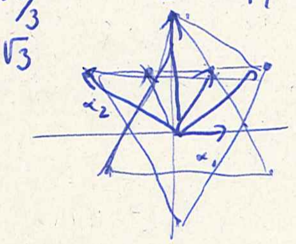
$\longleftarrow$ Q1 Existence

$\longrightarrow$ Q2 Uniqueness

$\Phi_{G_2} = \pm\{\alpha_1, \alpha_2, \alpha_2+\alpha_1, \alpha_2+2\alpha_1, \alpha_2+3\alpha_1, 2\alpha_2+3\alpha_1\}$

$\measuredangle(\alpha_1,\alpha_2) = 2\pi/3$

$\|\alpha_1\| / \|\alpha_2\| = \sqrt{3}$



Q2 is achieved $\left\{ \begin{array}{l} \dim L_\alpha = 1 \\ [L_\alpha, L_\beta] \subset \underset{\underset{R}{\uparrow}}{L_{\alpha+\beta}} \end{array} \right.$.

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① One (1st) ALGEBRA OF TYPE G_2.

Let $\{e_i\}_1^3$ the canonical basis of $\mathbb{F}^3$

If $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \sum x_i e_i \in \mathbb{F}^3$, the matrix of $\begin{array}{l} F^3 \to F^3 \\ y \to x \times y \end{array}$ is denoted by $l_x = \begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}$

Take

$$L = \left\{ \underset{\substack{1 \quad\;\; 3 \quad\;\; 3}}{\begin{pmatrix} 0 & -2y^t & -2x^t \\ x & a & l_y \\ y & l_x & -a^t \end{pmatrix}} \overset{\shortparallel}{} M_{(a,x,y)} \;\Big/\; \begin{array}{l} x, y \in F^3 \\ a \in sl_3(F) \end{array} \right\} \subset gl_7(F) = (Mat_7(F), [\,,\,])$$

★ $\dim L = 3 + 3 + 8 = 14$

★ $[L, L] \subset L$ $\left\{ \begin{array}{l} [M_{(a,0,0)}, M_{(b,0,0)}] = M_{([a,b],0,0)} \in L \\ [\;\;" \qquad, M_{(0,u,0)}] = M_{(0,au,0)} \\ [\;\;" \qquad, M_{(0,0,v)}] = M_{(0,0,-a^t v)} \\ [M_{(0,x,0)}, M_{(0,u,0)}] = M_{(0,0,2x\times u)} \\ [\;\;" \qquad, M_{(0,0,v)}] = M_{(-3\,pr_{sl_3}(vx^t),0,0)} \end{array} \right.$

$[M_{(0,0,y)}, M_{(0,0,v)}] = M_{(0,2y\times v,0)}$

★ L is simple with root system Φ_{G_2}

$\mathfrak{h}$ = diagonal elements in L

$$h \in \mathfrak{h} \rightsquigarrow h = \begin{pmatrix} 0 & & 0 \\ & a = \begin{pmatrix} s_1 & & \\ & s_2 & \\ & & s_3 \end{pmatrix} & \\ 0 & & -a^t \end{pmatrix} \qquad \underline{\sum s_i = 0}$$

Diagonalize $\mathrm{ad}\,h$:

$E_{ij} = \left(\overset{}{\underset{j}{-\!\!+\!\!-}} \right)$:

$$[h, M(E_{ij}, 0, 0)] = M(\underbrace{[(s_1 \; s_2 \; s_3), E_{ij}]}_{(s_i - s_j)E_{ij}}, 0, 0)$$

$$[h, M(0, e_i, 0)] = M(0, \underbrace{(s_1 \; s_2 \; s_3)e_i}_{s_i e_i}, 0)$$

$$[h, M(0, 0, e_i)] = M(0, 0, \underbrace{-(s_1 \; s_2 \; s_3) e_i}_{-s_i e_i})$$

$\Rightarrow$ ~~Def~~ $\varepsilon_i : \mathfrak{h} \to \mathbb{F}$

$h \to s_i$

$L_{\varepsilon_i - \varepsilon_j} = \langle M(E_{ij}, 0, 0) \rangle$

$L_{\varepsilon_i} = \langle M(0, e_i, 0) \rangle$

$L_{-\varepsilon_i} = \langle M(0, 0, e_i) \rangle$

$\Rightarrow \Phi = \{ \varepsilon_i - \varepsilon_j, \pm \varepsilon_i \,/\, i \neq j \}$

$\overset{\uparrow}{=} \pm \{ \alpha_2, \alpha_1, \alpha_2 + \alpha_1, \alpha_2 + 2\alpha_1, \dots \} = \Phi_{G_2}$.

$\alpha_1 = \varepsilon_2$

$\alpha_2 = \varepsilon_1 - \varepsilon_2$

(Simplicity decay).

8'35 ③ What is preserving L? $\underline{V = \mathbb{F}^7}$

Take $n : V \Rightarrow F$ $\rightsquigarrow \begin{pmatrix} -1 & & \\ & 0 & -2I_3 \\ & -2I_3 & 0 \end{pmatrix}$ ($\mathbb{R}$, signature (4, 3))

$$so(V, n) = \{ A \,/\, A^t N + NA = 0 \} = \left\{ \begin{pmatrix} 0 & -2y^t & -2x^t \\ x & a & b \\ y & c & -a^t \end{pmatrix} \middle/ \begin{matrix} x, y \in F^3 \\ b, c \\ \text{skew} \\ a \in gl_3 \end{matrix} \right\}$$

$\cup$

L

1 3 3

3 + 3 + 3 + 3 + 9 = 21

$E_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $E_i = \begin{pmatrix} 0 \\ e_i \\ 0 \end{pmatrix}$ $F_i = \begin{pmatrix} 0 \\ 0 \\ e_i \end{pmatrix}$

$\longrightarrow \begin{cases} n(E_0, E_0) = -1 \\ n(E_i, F_i) = -2 \end{cases}$

Def : a linear ~~p~~ map $\times : W \times W \to W$ is a CROSS PRODUCT if

$\exists\, n : W \to F$ nondg quad form st $\begin{cases} u \times v \perp u, v \\ n(u \times v) = \begin{vmatrix} n(u,u) & n(u,v) \\ n(v,u) & n(v,v) \end{vmatrix} \end{cases}$

Ex : $(V, \times)$

$$\begin{array}{c} 1 \\ 3 \\ 3 \end{array} \underset{\underset{\mathbb{F}^7}{\uparrow}}{\begin{pmatrix} s \\ u \\ v \end{pmatrix}} \times \begin{pmatrix} t \\ x \\ y \end{pmatrix} = \begin{pmatrix} 2u \cdot y - 2x \cdot v \\ sx - tu - 2v \times y \\ -sy + tv + 2u \times x \end{pmatrix}$$

$E_0 \times E_i = E_i$

$\quad '' \times F_i = -F_i$

$E_i \times E_{i+1} = 2F_{i+2}$

$F_i \times F_{i+1} = -2E_{i+2}$

$E_i \times F_i = 2E_0$

It is a cross product and $L \subset \underset{\overset{\shortparallel}{\{ d \in V \to V \,/\, d(X \times Y) = d(X) \times Y + X \times d(Y) \}}}{\mathrm{Der}(V, \times)}$

$\downarrow$

$=$

Cross products & octonions

Def : Octonion algebra $\begin{cases} 1 \in \mathcal{C},\ \dim_F \mathcal{C} = 8 \\ n : \mathcal{C} \to F \text{ non dg quad. form MULTIPLICATIVE } n(xy) = n(x)n(y) \end{cases}$

Ex. $\mathcal{C} = \mathbb{F} \oplus V$ product $\begin{cases} 1 \text{ unit} \\ uv = -n(u,v)1 + u \times v \end{cases}$

norm $\begin{cases} n(1) = 1 \\ 1 \perp V \end{cases}$ $(n|_V = n)$

$\Rightarrow L = \mathrm{Der}(V, \times) \cong \mathrm{Der}\, \mathcal{C}$

$d \rightsquigarrow \tilde{d}$ $\quad \tilde{d}(1) = 0$, $\tilde{d}|_V = d$

$\longrightarrow \{E_0, E_i, F_i\} = -2$

$\{E_1 E_2 E_3\} = -4$

$= \{F_1 F_2 F_3\}$

More objects preserved by L

$\{\ ,\ ,\ \} : V \times V \times V \longrightarrow \mathbb{F}$, $\{x, y, z\} = n(x \times y, z)$

$\Rightarrow L = \mathrm{Der}(V, \{\ ,\ ,\ \}) = \{ d : V \to V \,/\, \{d(X), Y, Z\} + \{X, d(Y), Z\} + \{X, Y, d(Z)\} = 0 \}$

$\subset$) obvious $L = \mathrm{Der}(V, \times) \subset \mathfrak{so}(V, n)$

$\supset$) we would need to prove that $\{\ ,\ ,\ \}$ determines n

p'45 Who is $\{\ ,\ ,\ \}$ as a 3-form? $\Omega_0 \overset{-2}{=} E_0^* \wedge (E_1^* \wedge F_1^* + E_2^* \wedge F_2^* + E_3^* \wedge F_3^*) - 4 E_1^* \wedge E_2^* \wedge E_3^* + 4 F_1^* \wedge F_2^* \wedge F_3^*$

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In terms of groups

Lie group G_2 : $\mathrm{Aut}(V, \times) = \{ f \in Gl(V) \ / \ f(X \times Y) = f(X) \times f(Y) \}$

$\mathrm{Aut}(V, \{\ ,\ ,\ \}_+) = \{ f \in Gl(V) \ / \ \{f(X), f(Y), f(Z)\} = \{X, Y, Z\} \}$

$\simeq \mathrm{Aut}\, \mathbb{O}$

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④ SOME G_2 - HISTORY

- 1887 , Killing
- 1893 Engel & Cartan : infinitesimal symmetries of a Pfaffian system SPLIT
- 1900 Engel DESCRIPTION OF BOTH SPLIT AND <u>COMPACT</u> stabilizer of a generic 3-form
- 1914 Cartan

etc

$$\begin{tikzcd} Gl(V) \times \Lambda^3 V^* \arrow[r] & \Lambda^3 V^* \end{tikzcd}$$

$(f, \Omega) \rightsquigarrow f \cdot \Omega = f^* \Omega : (X, Y, Z) \to \Omega(f^{-1}X, f^{-1}Y, f^{-1}Z)$

All above $\Rightarrow$ the isotropy group ~~of~~ $G_{\Omega_0} = \{ g \in Gl(V) \ / \ g \cdot \Omega_0 = \Omega_0 \}$ is G_2!!

Note that : $\dim Gl(V) = 49$, $\dim \Lambda^3 V^* = \binom{7}{3} = 35$ $\Big\} \Rightarrow \dim G_\Omega = 49 - \dim \text{orbit} \geq 14$??

Def : A 3-form Ω is said <u>generic</u> if its orbit is open

$\mathcal{O}_\Omega = Gl(V) / G_\Omega$

$\dim \mathcal{O}_\Omega = 35$ $\Updownarrow$ $\underline{\dim G_\Omega = 14}$

<u>Engel's th</u>

$F = \mathbb{C}$: there is only $\underbrace{\text{one orbit}}_{\Omega_0}$ of generic 3-forms in $\Lambda^3 V^*$

$F = \mathbb{R}$ There are TWO orbits $\begin{cases} \Omega_0 \\ \Omega_1 \text{ "the Compact!"} \end{cases}$

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Proofs of Engel results (Sketch)

ℂ) $\Omega \in \Lambda^3 V^* \xrightarrow[\text{Fix } B=\{b_1..b_7\}]{} \text{def } n_{B,\Omega} : V \times V \longrightarrow \mathbb{C}$

$$n_{B,\Omega}(X,Y) = \sum_{\sigma\in S^7} (-1)^\sigma \, \Omega(X, b_{\sigma(1)}, b_{\sigma(2)}) \, \Omega(Y\, b_{\sigma(3)}\, b_{\sigma(4)}) \, \Omega(b_{\sigma(5)} b_{\sigma(6)}, b_{\sigma(7)})$$

(only depends on B up to scale)

Ω generic ⇒ n is nondeg !

⇝ This defines a $\overset{\text{skew sym product}}{\wedge} : V \times V \to V$

$$n(x \times y, z) = \Omega(x, y, z)$$

Aim : $\wedge$ is a cross product (suitable multiple of n)

Key: Lagrange identity $u \wedge (u \wedge v) = -n(u)\, v$ if $v \perp u$

Begin with nonisotropic u ⇝ $V = \langle u \rangle \oplus u^\perp$

⇝ $f = l_u : u^\perp \to u^\perp$
$v \to u \times v$

skewadjoint (Ω 3-form)

f^2 selfadjoint & $f^2 = -n(u)\,\mathrm{id}$?

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ℝ) $\Omega \in \Lambda^3 (\mathbb{R}^7)^* \to n, \wedge \overset{-\otimes_{\mathbb{R}}\mathbb{C}}{\rightsquigarrow}$ Now starting point is $\wedge$ cross product

Case n is not pos def | Take $n(X) = -1 \to f = l_x : X^\perp \to X^\perp$ satisfies $f^2 = I$

E_0

⇝ $X^\perp = \mathrm{Ker}(f+I) \oplus (\mathrm{Ker}(f-I)$

E_1, E_2, E_3
$\overset{\shortparallel}{E_1 \wedge E_2}$

$F_1 F_2 F_3$
(and +-)

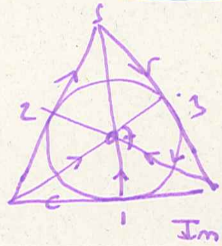
Ω_0

Case n is pos def | Take $n(\overset{e_7}{X}) = 1 \Rightarrow f = J : l_x : X^\perp \to X^\perp \quad J^2 = -I$

⇝ in some orthogonal basis

$$J = \left(\begin{array}{cccccc} e_1 & e_4 & e_2 & e_5 & e_3 & e_6 \\ 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{array}\right)$$

$e_7 \times e_1 = e_4$



$\mathrm{Im}(\mathbb{O}) \geq \mathbb{Q}$

$$\Omega_1 = e^{714} + e^{725} + e^{736} + e^{123} - e^{156} - e^{426} - e^{453}$$

$e^{ijk} = e_i^* \wedge e_j^* \wedge e_k^*$

9'10.

G_2 and $\mathbb{S}^6$

n pos. def | $V = \mathbb{R}^7$, $\Omega_1 \in \Lambda^3 V^*$

$$\underbrace{G_{\Omega_1}}_{G_{2,-14}} \times V \to V$$

Take $X \in V$, $n(X) = 1 \Rightarrow$ orbit $\mathcal{O}_x \subset \{Y \in V \,/\, n(Y) = 1\} \equiv \mathbb{S}^6$

The action of G_2 on $\mathbb{S}^6$ $\begin{cases} 1) \text{ TRANSITIVE} \\ 2) \text{ ISOTROPY GROUP } SU(3) \end{cases}$ $\Rightarrow G_{2,-14}/SU(3) \simeq \mathbb{S}^6$

Proof 1) $Y, Y' \in \mathbb{R}^7$, $n(Y) = n(Y') = 1$.

¿$\exists\, g \in Gl(V) \,/\, \begin{matrix} g \cdot \Omega_1 = \Omega_1 \\ g(Y) = Y' \end{matrix}$?

Yes: Stronger:

$\overset{}{\underset{e_7}{\cancel{X}}}, e_1, e_2$ orthonormal $\Omega_1(X, e_1, e_2) = 0$ $\underset{\text{determines } \Omega_1}{\rightsquigarrow}$ $X' e_1' e_2'$ similar conditions.

2) • A complex 3-dim space:

recall $J = l_x : X^\perp \to X^\perp \equiv T_x \mathbb{S}^6$

$J^2 = -id$

$$\rightsquigarrow \begin{matrix} \mathbb{C} \times X^\perp \to X^\perp \\ (i, X) \to J(Y) \end{matrix}$$

• A nondegenerate hermitian form

$$\sigma : \cancel{X}^\perp \times X^\perp \to \mathbb{C}$$
$$(Y, Z) \rightsquigarrow n(Y,Z) - i\, n(J(Y), Z))$$

• $\{g \in G_{\Omega_1} \,/\, g(X) = X\} \xrightarrow{\cong} SU(X^\perp, \sigma)$

$$g \rightsquigarrow g|_{\cancel{X}^\perp}$$

- inj $(\det g|_{X^\perp} = 1)$
- surj $\dim H_x \leq \dim SU(X^\perp) = 8$

$$\overset{\shortparallel}{\dim \underset{14}{G_{\Omega_1}} - \dim \underset{\text{orbit}\,(\cong S^6)}{}} \geq 8$$

$\rightsquigarrow$ $G_{2,2}/SL(3) \cong \{ Y \in \mathbb{R}^7_4 \ / \ n(Y) = -1 \} = H^6_3$

Both are
hom red spaces
$\mathfrak{g}_{2,-14} = \mathfrak{su}(3) \oplus \underbrace{\mathfrak{m}}_{\substack{\mathfrak{su}(3)\text{-module.} \\ \shortparallel \\ T_x S^6}}$

9'18.

PARENTHESIS : $\overset{\text{linear}}{\text{Model}}$ for Marcus' talk.

~~$\mathfrak{g}_{2,-14} = \{ su(W) \}$~~

$W = \mathbb{C}^3, \quad \sigma(u,v) = \sum u_i \bar{v}_i ,$

$\mathfrak{g}_{2,-14} = \mathfrak{su}(W) \oplus W$

$$\left\{ \begin{array}{ll} [\phi, \phi'] & = \phi\phi' - \phi'\phi \\ [\phi, u] & = \phi(u) \\ [u, v] & = 3\,\mathrm{pr}\,\underbrace{\sigma_{u,v}} + \underbrace{2u \times v}_{\hat{W}} \end{array} \right.$$

$\phi, \phi' \in \mathfrak{su}(W)$
$u, v \in W$.

$\sigma_{u,v} = \sigma(u,-)\,v - \sigma(v,-)\,u$

$(3\,\mathrm{pr}\,\sigma_{u,v} = \sigma_{u,v} - \sigma(u,v) I_3 .)$

Action on $\mathbb{R}^7 = \mathbb{R} \oplus W$.

$$\phi \left\{ \begin{array}{ll} \phi \cdot 1 & = 0 \\ \phi \cdot w & = \phi(v) \end{array} \right.$$

$$u \left\{ \begin{array}{ll} u \cdot 1 & = 2u \\ u \cdot v & = -2\,\mathrm{Re}\,\sigma(u,v) + i\,\overline{(u \times v)} \end{array} \right.$$

9'22.

⑥ SPINORIAL APPROACH

Goal: $G_{2,-14} = \{ g \in \underset{Spin(3,4)}{Spin\,7} \ / \ g \cdot \psi = \psi \}$
$\quad\;\; {}_{2}$

$$\left(\begin{array}{l} Spin\,7/G_{2,-14} \simeq S^7 \\ Spin(3,4)/G_{2,2} \simeq S^7_4 \end{array} \right)$$

If $n: V \longrightarrow \mathbb{R}$ non deg quad form

$$Cl(V,n) = \frac{T(V)}{id\langle v\otimes v - n(v)1\rangle} = \mathbb{R}\oplus V\oplus (V\otimes V)\oplus(V\otimes V\otimes V)+\dots.$$

$$v^2 = n(v)1$$

<u>Ex</u> $V=\mathbb{R}^7$, $n=-I_7$.

$\downarrow$

$\{e_i\}$ can basis $\left\{\begin{array}{l} e_i^2 = n(e_i) = -1 \\ e_ie_j + e_je_i = 0 \end{array}\right.$

$\rightsquigarrow$ basis of $Cl(V,n) = \langle 1, e_1 \dots e_7, e_1e_2, e_1e_3 \dots e_6e_7,$
$e_1e_2e_3 \cdots\cdots$

dim $1+7+\binom{7}{2}+ \dots +\binom{7}{7} = 2^7$

$$Cl(V,n) = Cl_{\bar{0}}(V,n)\oplus Cl_{\bar{1}}(V,n)$$

$\uparrow$
even number of $\otimes$.

Υ

$Spin(V,n) = \{\pm a_1\dots a_{2n} \ / \ a_i\in V \ \& \ \prod_1^{2n} n(a_i)=1\}$

$\downarrow \qquad\qquad \}$

$SO(V,n) \qquad \pm\tau_{a_1}\dots\tau_{a_n}$

- $a\in V, n(a)\neq 0 \rightsquigarrow a^2=n(a)1$
  $\rightsquigarrow a$ invertible
- $\tau_a : V \longrightarrow V$ reflection relative $a^\perp$
  $v \longrightarrow -v + \frac{2n(a,v)}{n(a)}a$

$$\tau_a(v) = ava^{-1}$$

Spinorial representation $(V=\mathbb{R}^7)$

$$\begin{array}{rcl} \rho:\mathbb{R}^7 & \longrightarrow & End\,\mathcal{O} \\ x & \longrightarrow & L_x : \mathcal{O}\to\mathcal{O} \\ & & \quad y \to xy \\ & & L_x^2 = -n(x)id \end{array} \quad \Rightarrow \quad \tilde{\rho}: Cl(\mathbb{R}^7,-n) \to End\,\mathcal{O}$$

$\Downarrow$

$$\tilde{\rho}: \underset{\text{rigl } 2^6}{Cl_{\bar{0}}(\mathbb{R}^7,-n)} \xrightarrow{\cong} End\,\underset{2^6}{\boxed{\mathcal{O}}}$$

$\uparrow$ irreducible rep for $Cl_{\bar{0}}$

$\Rightarrow Spin(V,-n)$ acts on $\overset{\text{elementary spinors}}{\mathcal{O}}$ by means of $\tilde{\rho}$

$$\begin{array}{rcl} Spin(V,-n)\times\mathcal{O} & \longrightarrow & \mathcal{O} \\ (\pm a_1\dots a_{2n}, \ \Psi) & \longrightarrow & \pm L_{a_1}\dots L_{a_{2n}}\Psi \end{array}$$

Consider $1 \in \mathcal{C}$

$$\begin{cases} \text{Orbit } \{x \in \mathcal{C} \,/\, n(x) = 1\} = \begin{cases} S^7 & n \text{ posit def} \\ S_4^7 & \text{otherwise} \end{cases} \\ \text{Isotropy subgroup } \{g \in Spin(V, -n) \,/\, g \cdot 1 = 1\} = \begin{cases} G_{2,-14} \\ G_{2,2} \end{cases} \end{cases}$$

$g \in Spin(V, -n)$ preserves $\langle x, y, z \rangle = (x\bar{y})z$

$g \cdot 1 = 1$ $\Rightarrow$ '' $\langle x, 1, z \rangle = xz$

$$\Rightarrow \begin{cases} Spin7 / G_{2,-14} = S^7 \\ Spin(3,4) / G_{2,2} \cong S_4^7 . \end{cases}$$

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1:30 exacto ----

Sin barra!!