Evolution Algebras

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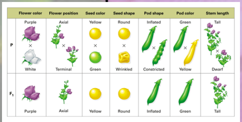
HISTORY OF ADVANCES

1856

Gregor Mendel, "the father of genetics", begins detailed experiments breeding pea plants.

1865



Mendel publishes his work describing the basic laws of inheritance.

1934

Serebrowsky gave an algebraic interpretation of sign "x".

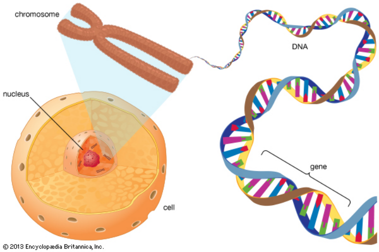
1936

Glivenkov introduced the so-called Mendelian algebras.

1939

Etherington provided a precise mathematical formulation of Mendel's laws in terms of nonassociative algebras.

Gene: Molecular unit of hereditary information.



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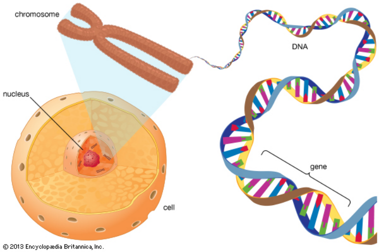
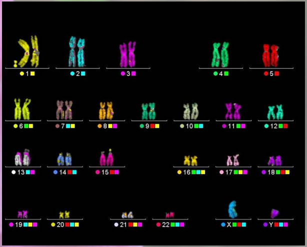
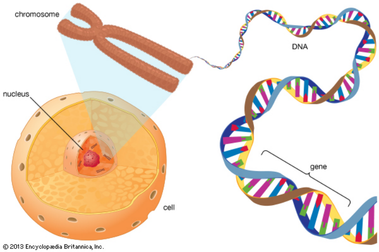


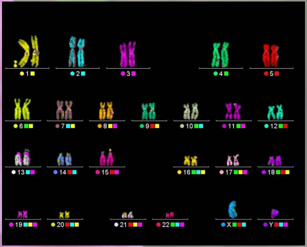


Chromosomes: Long strands of DNA formed by ordered sequences of genes. In the process of reproduction, the attributes of the offspring are inherited from alleles contained in the chromosomes of the parents.

Gene: Molecular unit of hereditary information.

Allele: Distinct forms of genes to an attribute. For example, the gene for eye color has three alleles: brown, green and blue.





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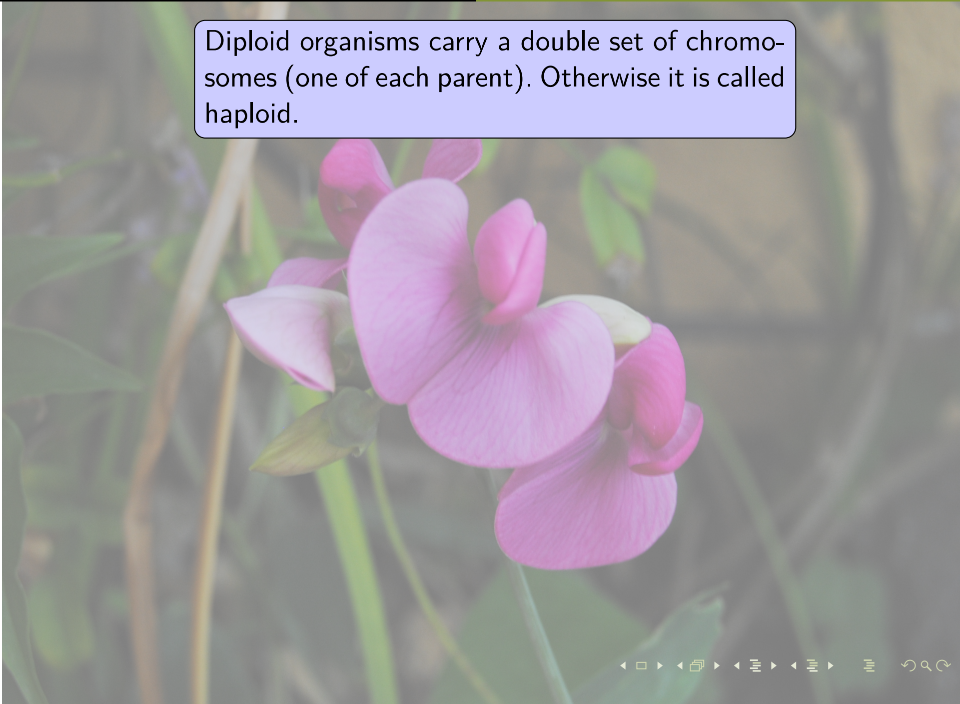

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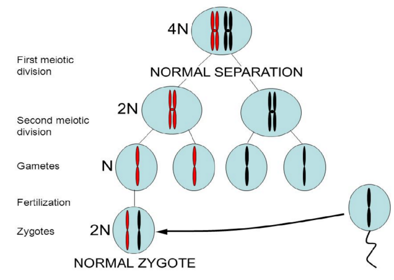

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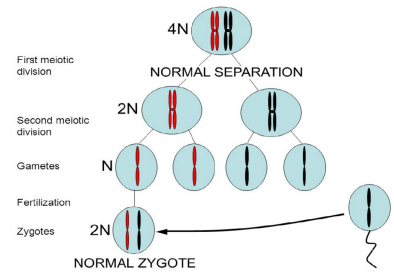

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The fusion of two gametes of opposite sex gives rise to a **zygote**, which contains a double set of chromosomes.

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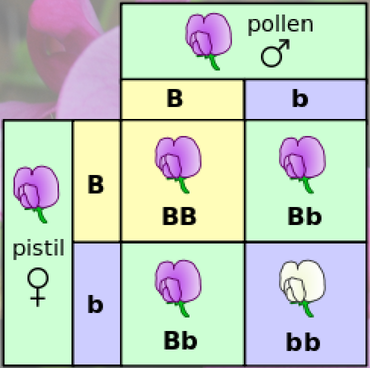

Gametic Algebra

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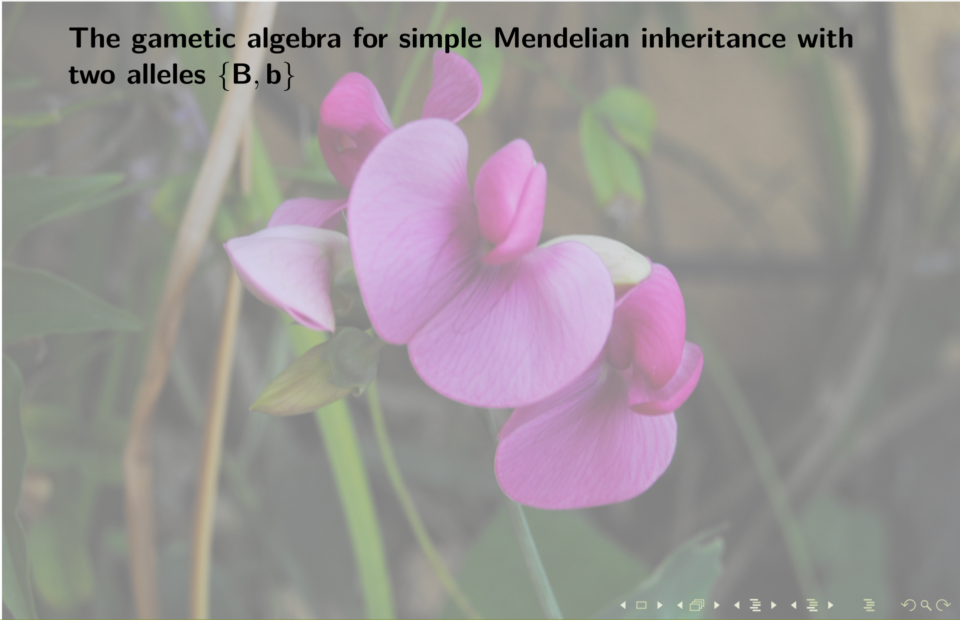
The gametic algebra for simple Mendelian inheritance with two alleles $\{B, b\}$



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Consider the set of gametes $B = \{a_1, \ldots, a_n\}$ as abstract elements. Define the n dimensional algebra over $\mathbb{R}$ with basis B and multiplication

$$a_i a_j = \sum_{k=1}^{n} \gamma_{ijk} a_k \text{ such that } \sum_{k=1}^{n} \gamma_{ijk} = 1.$$

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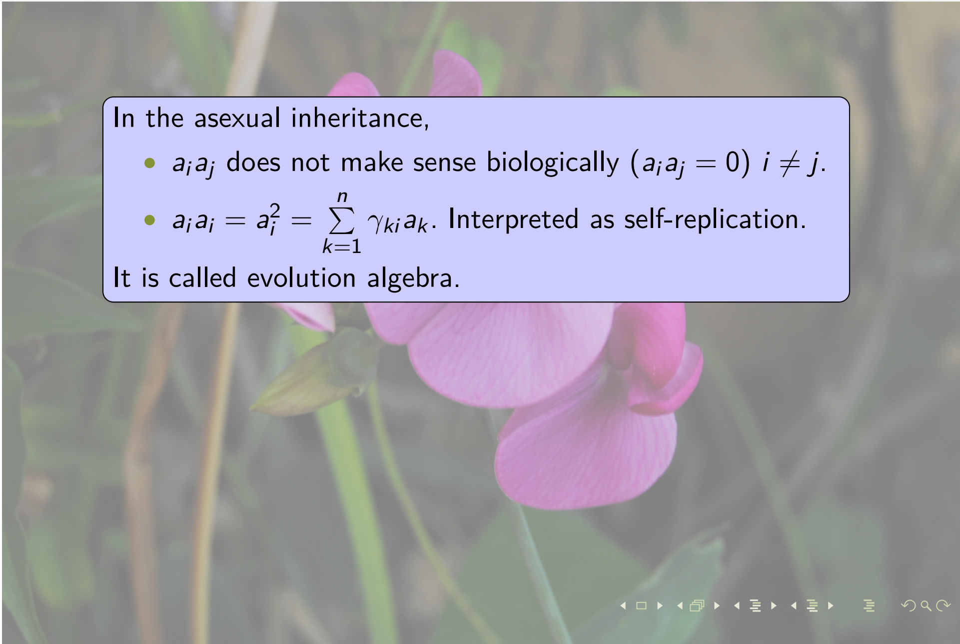

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- $a_i a_j$ does not make sense biologically $(a_i a_j = 0)$ $i \neq j$.
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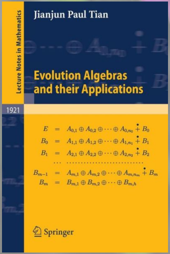

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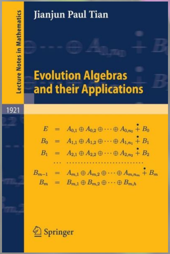

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- $\mathrm{CFM}_\Lambda(\mathbb{K}) = (M_\Lambda(\mathbb{K}), +, \cdot)$ such that for which every column has at most a finite number of non-zero entries.

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$$M_B(\xi_B(f_i) \bullet_B \xi_B(f_j)) = 0$$

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1. If $B' = \{f_i \mid i \in \Lambda\}$ is a natural basis of A with $P_{B'B} = (p_{ij})$ the change of basis matrices, then

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Evolution subalgebras. Evolution ideals

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Definitions

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1. A subalgebra of an evolution algebra does not need to be an evolution algebra.
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3. Not every ideal of an evolution algebra has a natural basis.

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Remark

Does non-degeneracy depend on the considered natural basis?

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4. $|\Lambda_0(B)| = |\Lambda_0(B')|$ for every natural basis B' of A.

Consequently, the definition of non-degenerate evolution algebra does not depend on the considered natural basis.

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2. $A/\mathrm{ann}(A)$ is not necessarily a non-degenerate evolution algebra.

Absorption property. Properties

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Lemma

- An ideal I of an evolution algebra A has the absorption property if and only if $\mathrm{ann}(A/I) = \overline{0}$.
- If I is a non-zero ideal which it has the absorption property, then I is an evolution ideal and has the extension property.

Absorption radical

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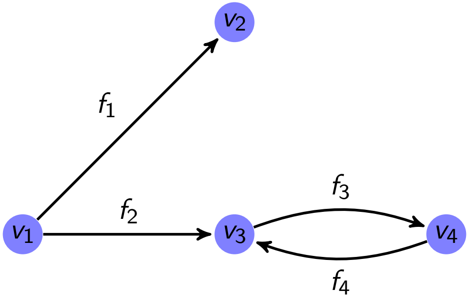
Corollary

Let I be an ideal of an evolution algebra A. Then $A/\mathrm{rad}(A)$ is a non-degenerate evolution algebra.

Consider the following graph E:



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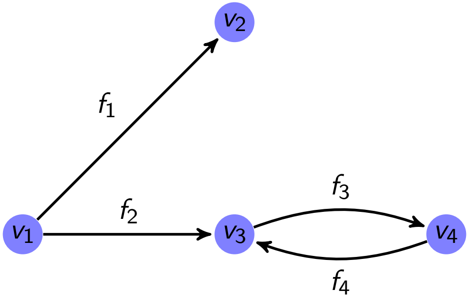

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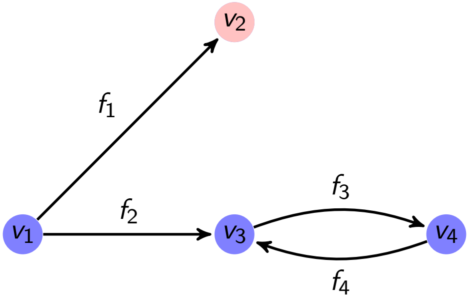

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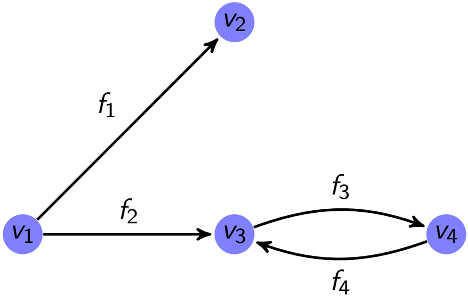

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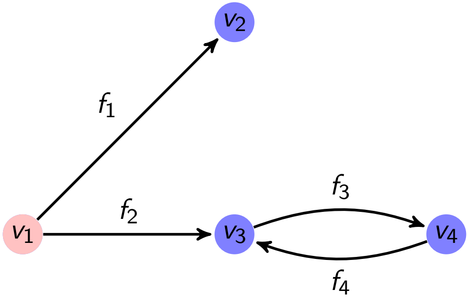

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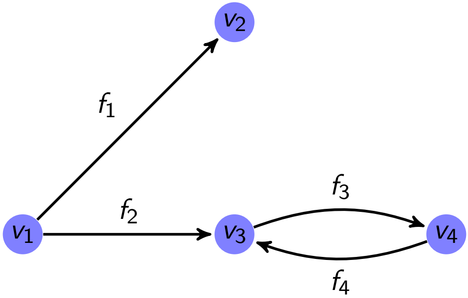

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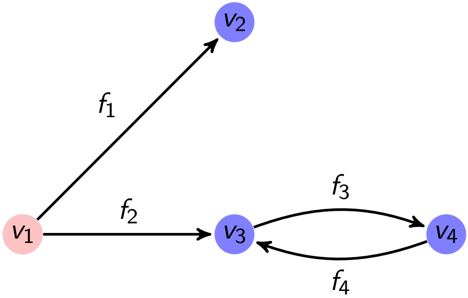

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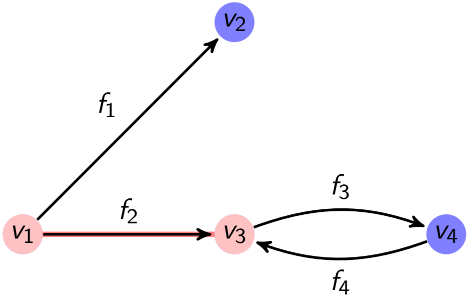

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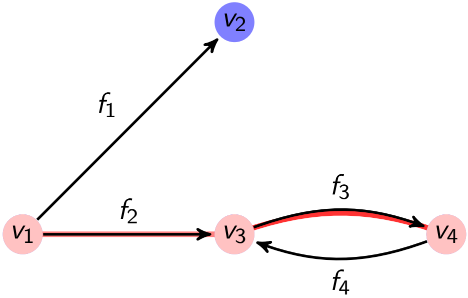

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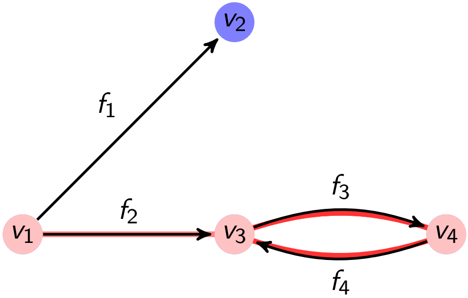

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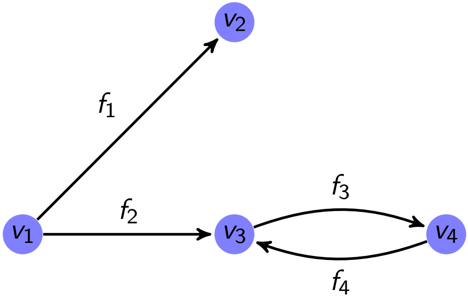

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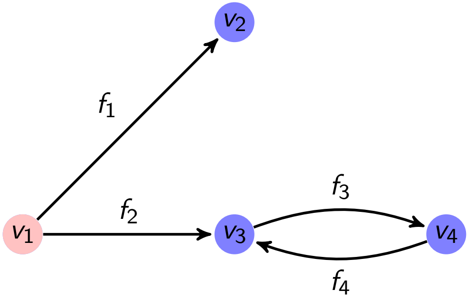

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- Path $(E) = \{f_1, f_2, f_3, f_4, f_2f_3, f_3f_4, f_2f_3f_4\}$.

- Adjacency matrix

$$\begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

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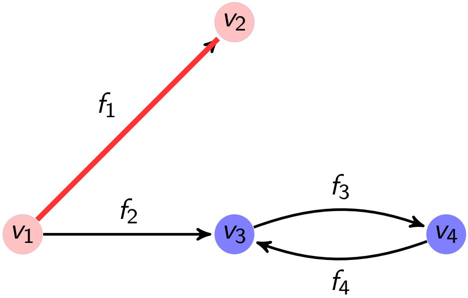

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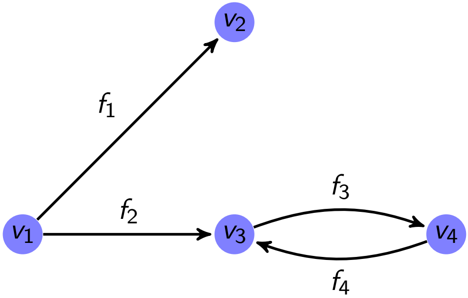
- Path $(E) = \{f_1, f_2, f_3, f_4, f_2 f_3, f_3 f_4, f_2 f_3 f_4\}$.

- Adjacency matrix

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{pmatrix} 0 & {\color{red}1} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

Basic concepts about graphs

Consider the following graph E:



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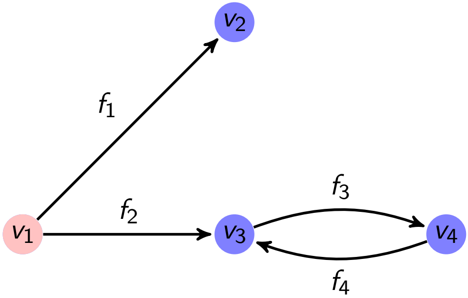

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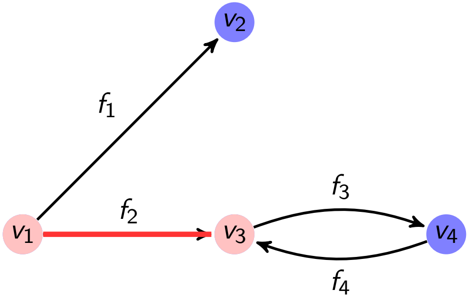

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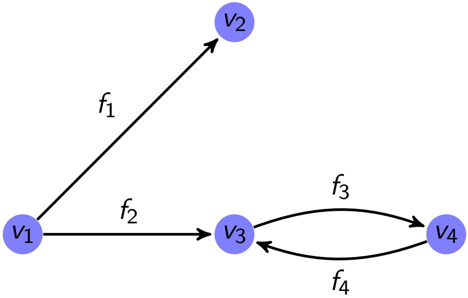

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\begin{array}{c c}
 & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\
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\end{array}
$$

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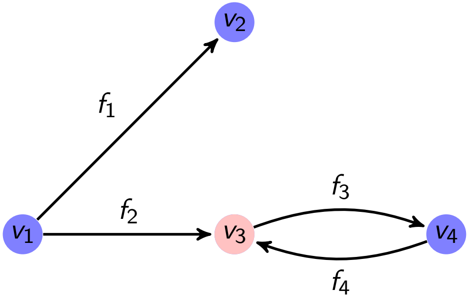

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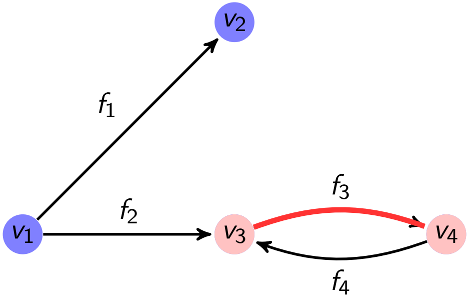

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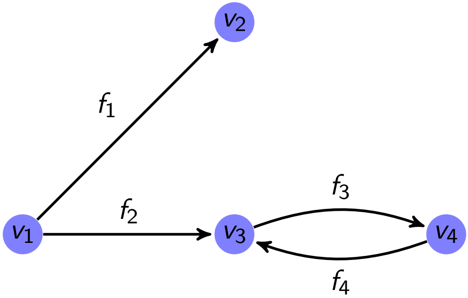

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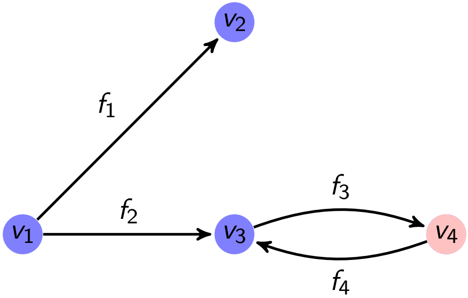

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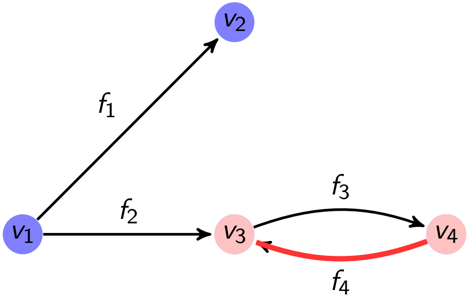

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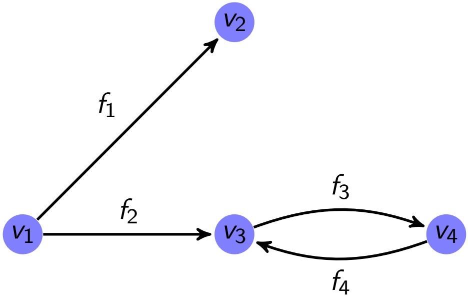

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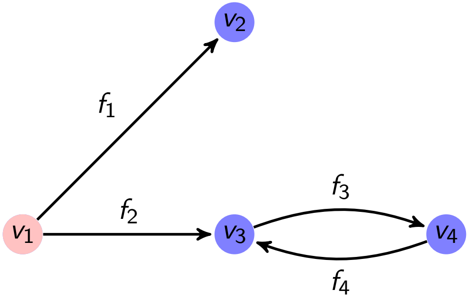

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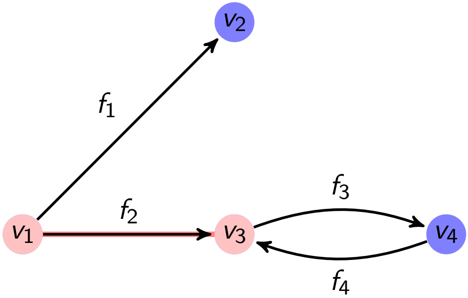

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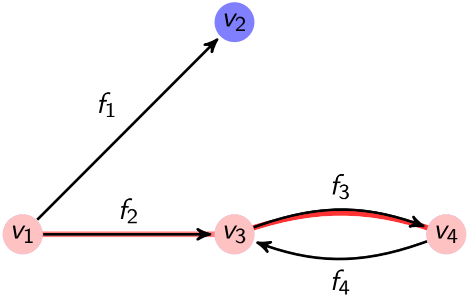

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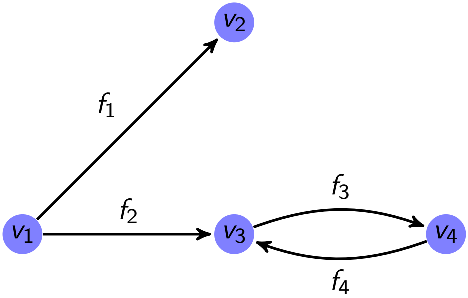

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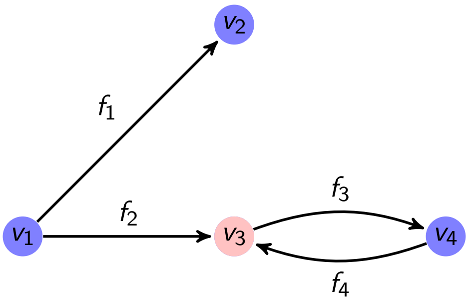

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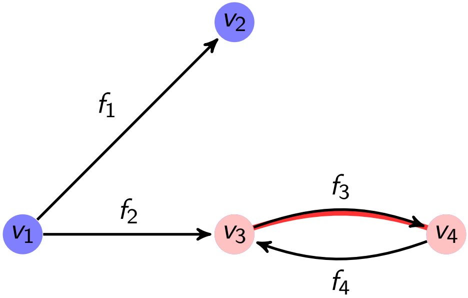

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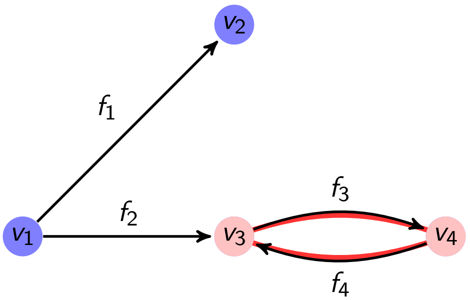

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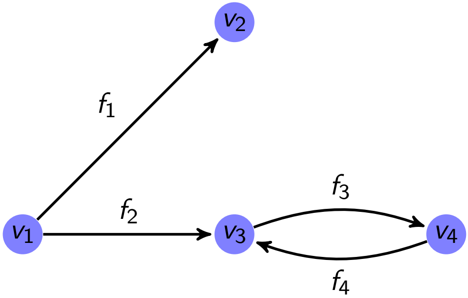

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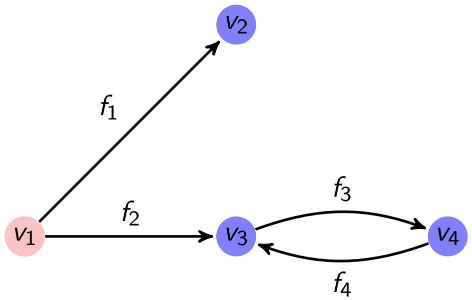

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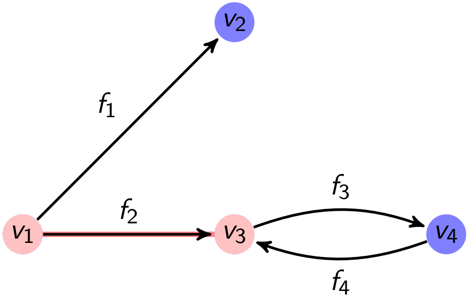

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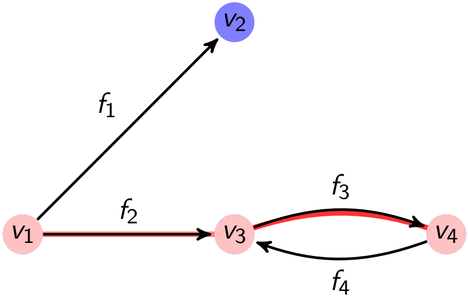

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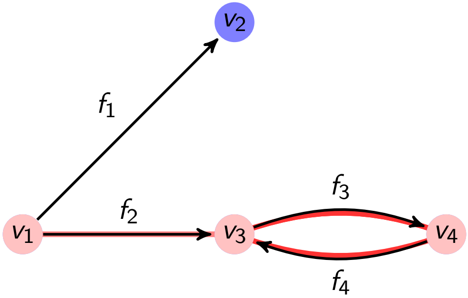

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Graph associated

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Remark

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$$M_B = \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \\ \end{array} \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & -2 & 0 \end{pmatrix} \\ \begin{array}{cccc} e_1^2 & e_2^2 & e_3^2 & e_4^2 \end{array} \end{array} \qquad P = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 \end{array}$$

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$$M_B = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ \\ \end{array} \begin{array}{c} \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 5 \\ 0 & 0 & -2 & 0 \end{array}\right) \\ \begin{array}{cccc} e_1^2 & e_2^2 & e_3^2 & e_4^2 \end{array} \end{array} \qquad P = \begin{array}{c|cccc} & v_1 & v_2 & v_3 & v_4 \\ \hline v_1 & 0 & 1 & 1 & 0 \\ v_2 & 0 & 0 & 0 & 0 \\ v_3 & 0 & 0 & 0 & 1 \\ v_4 & 0 & 0 & 1 & 0 \end{array}$$

Graph associated

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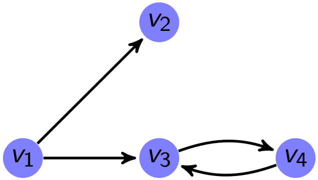

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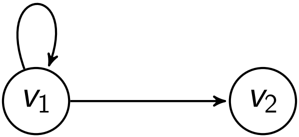
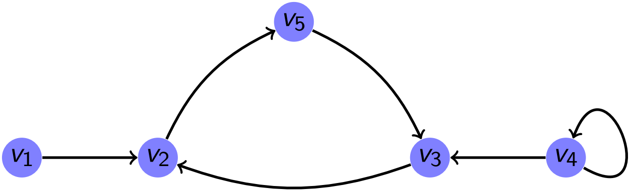
E:



F:

Outline

Let A evolution algebra with structure matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$



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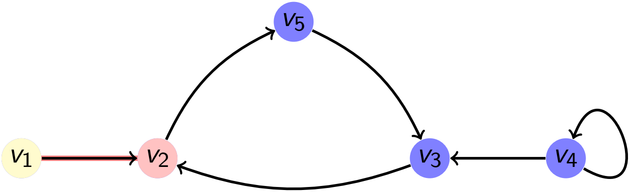

- $D^1(1) = \{k \in \Lambda \mid e_1^2 = \sum_k \omega_{k1} e_k \text{ with } \omega_{k1} \neq 0\} = \{2\}.$

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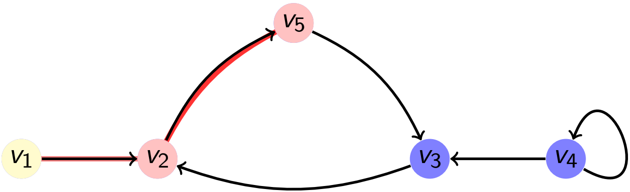
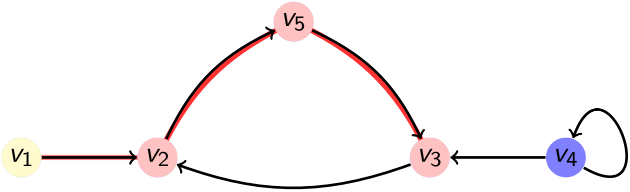


- $D^2(1) = \bigcup_{k \in D^1(1)} D^1(k) = \{5\}.$

Let A evolution algebra with structure matrix $\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$



- $D^3(1) = \{3\}$.

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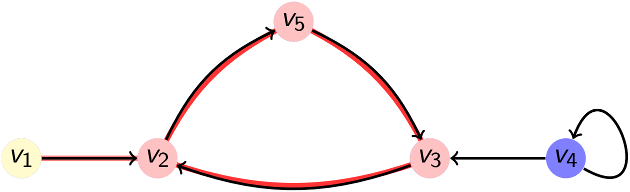

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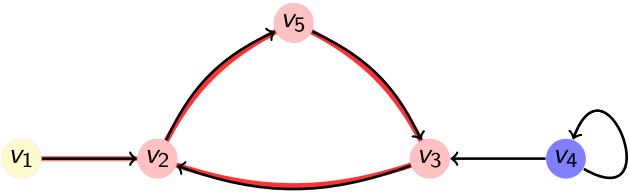

- $D(1) = \bigcup_{m \in \mathbb{N}} D^m(1) = \{2, 5, 3\}$.

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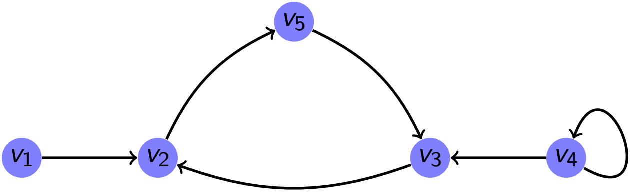

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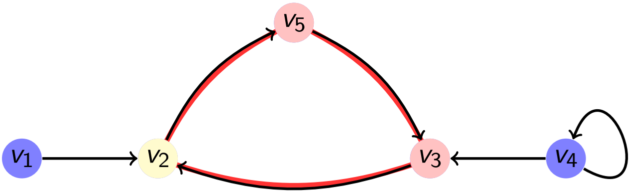

- $C(2) = \{j \in \Lambda \mid j \in D(2) \text{ and } 2 \in D(j)\} = \{2, 5, 3\}$.

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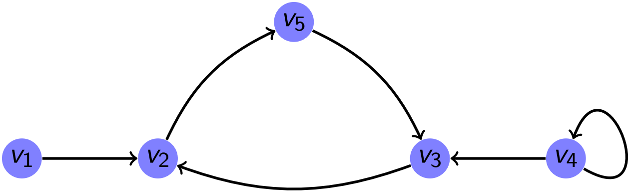

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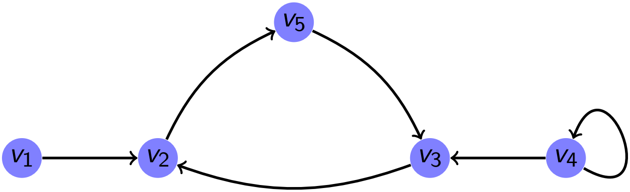

- A cyclic index i_0 is a **principal cyclic index** if $j \in D(i_0)$ for every $j \in \Lambda$ with $i_0 \in D(j)$.

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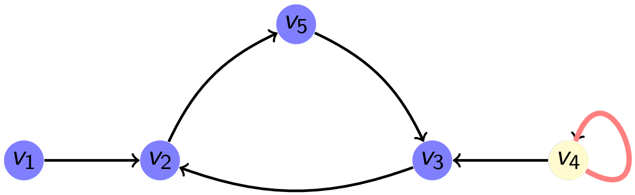

- 4 is a principal cyclic index.

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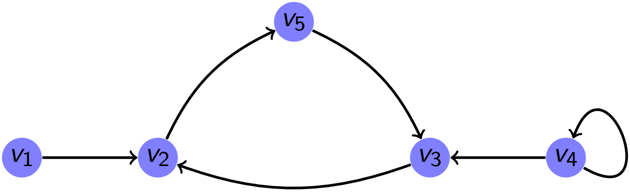

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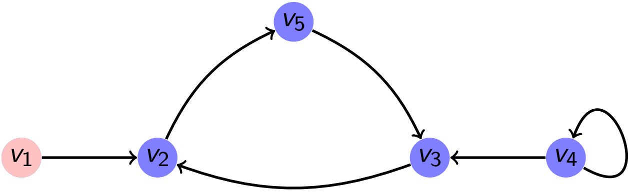

- 1 is a chain-start index because 1 has no ascendents.

Ideal generated by one element

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Corollary

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Let A be an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis.

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Corollary

Let A be an evolution algebra. For any element $x \in A$,

$\dim \langle x \rangle$ is at most countable.

Simple evolution algebras

Simple evolution algebras

Definition

Simple evolution algebras

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An algebra A is **simple**

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An algebra A is **simple** if $A^2 \neq 0$ and 0 is the only proper ideal.

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Moreover, the dimension of A is at most countable and if $|\Lambda| < \infty$ then the converse is true.

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 & \vdots & & & & \vdots &
\end{array}$$

Then A satisfies the conditions 1, 2 and 3 but A is not simple as $\langle e_1^2 \rangle$ and $\langle e_2^2 \rangle$ are two non-zero proper ideals.

Characterization for finite dimension

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Let A an evolution algebra with dim(A)=n and $B = \{e_i \mid i \in \Lambda\}$ a natural basis of A.

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for some $m \in \mathbb{N}$ with $m < n$ and matrices $W_{m\times m}$, $U_{m\times(n-m)}$ *and* $Y_{(n-m)\times(n-m)}$.

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Characterizations

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A non-degenerate finite dimensional evolution algebra A with natural basis $B = \{e_i \mid i = 1, \ldots, n\}$ is reducible

Characterizations

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A non-degenerate finite dimensional evolution algebra A with natural basis $B = \{e_i \mid i = 1, \ldots, n\}$ is reducible if and only if there exists $B' = \{e_{\sigma(i)} \mid i = 1, \ldots, n\}$ with $\sigma \in S_n$ such that

Characterizations

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A non-degenerate finite dimensional evolution algebra A with natural basis $B = \{e_i \mid i = 1, \dots, n\}$ is reducible if and only if there exists $B' = \{e_{\sigma(i)} \mid i = 1, \dots, n\}$ with $\sigma \in S_n$ such that

$$M_{B'} = \begin{pmatrix} W_{m\times m} & 0_{(n-m)\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix}$$

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Corollary

Let A be a non-degenerate evolution algebra, $B = \{e_i \mid i \in \Lambda\}$ a natural basis and let E be its associated graph.

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Let A be a non-degenerate evolution algebra, $B = \{e_i \mid i \in \Lambda\}$ a natural basis and let E be its associated graph. Then A es irreducible if and only if E is a connected graph.

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Theorem

Let A be a non-degenerate evolution algebra. Then A admits an optimal direct-sum decomposition. Moreover, it is unique.

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If for every $i \in \{1, \ldots, k\}$ the index set $\Lambda_i = \cup_{j \in S_i} \Upsilon_j$ is not fragmentable then we say that $\Lambda = \cup_{i=1}^k \Lambda_i$ is an optimal fragmentation.

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$$(\dagger) \quad \Lambda = \Lambda(C_1) \cup \cdots \cup \Lambda(C_k) \cup \Lambda(i_1) \cup \cdots \cup \Lambda(i_m).$$

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Moreover, if A is non-degenerate, then $A = \oplus_{\gamma\in\Gamma} I_\gamma$ is the optimal direct-sum decomposition of A.

Optimal Fragmentation

PROGRAM 1

Program A

In[1]:=

```
P = ( 0 0 0 0 1 0 0 1
      0 0 1 0 0 0 1 0
      0 0 0 0 0 1 0 0
      0 0 1 0 0 0 0 0
      0 0 0 0 1 0 0 0
      0 0 0 0 0 0 0 0
      0 0 0 0 0 0 0 1
      1 0 0 0 0 0 1 1 );

D1[i_, P_] := Select[Table[j, {j, Length[P]}], P[[#, i]] ≠ 0 &];
D1[5, P]
Dm_[i_, P_] := Module[{a, k, s}, a = {}; s = Length[Dm-1[i, P]];
    If[m == 1, D1[i, P],
     Union[Flatten[
       Table[D1[Dm-1[i, P][[t]], P], {t, Length[Dm-1[i, P]]}]]]]]];

CycleQ[P_] := Module[{n, x}, n = Length[P];
    x = Union[Flatten[Table[
         Diagonal[MatrixPower[P, i]], {i, 1, n}]]]];
    MemberQ[x, 1]];
CycleQ[P]

DYesCycle[i_, P_] := Module[{b, j},
    b = {};
    For[j = 1, j ≤ Length[P], j ++, AppendTo[b, Dj[i, P]]];
    Apply[Union, b]];

DNotCycle[i_, P_] := Module[{t, b},
    b = {D1[i, P]};
     For[t = 1, Dt[i, P] ≠ Dt+1[i, P], t ++,
      AppendTo[b, Dt+1[i, P]]
    ];
    Apply[Union, b]];

DP[i_, P_] := If[CycleQ[P], DYesCycle[i, P], DNotCycle[i, P]]
```

Out[3]= {1, 5}

Out[6]= True

In[10]:=

```
DP[5, P]
```

Out[10]= {1, 2, 5, 7, 8}

Program B

In[11]:=
```
CyclicQ[i_, P_] := If[MemberQ[DP[i], i],
   Print[i "is a cyclic index"], Print[i " is not a cyclic index"]];
CycleAssociated[i_, P_] := Select[Table[j, {j, Length[P]}],
   MemberQ[DP[i, P], #] && MemberQ [DP[#, P], i] &];
Ascendents[i_, P_] := Module[{j, b}, b = {};
   For[j = 1, j ≤ Length[P], j++, If[MemberQ[DP[j, P], i], AppendTo[b, j]]]; b];
Subset[A_, B_] := (Union[A, B] == Union[B]);
PrincipalCycleQ[i_, P_] :=
  If[Subset[Ascendents[i, P], CycleAssociated[i, P]],
   Print[i "is a principal cyclic-index"],
   Print[i "is not a principal cyclic-index"]];
ElementsNotNoneRow[P_] := Module[{j},
   Select[Table[j, {j, Length[P]}], P[[#]] == 0 P[[1]] &]];
(* La función incad me devuelve los índices i tales que la fila i es nula*)
ChainStartQ[i_, P_] :=
  If[MemberQ[ElementsNotNoneRow[P], i], Print[i "is a chain-start index"],
   Print[i " is not a chain-start index"]];
CycleAssociated[5, P]
Ascendents[5, P]
```

Out[18]= {5}

Out[19]= {5}

Program C

In[20]:=
```
LambdaPrincipalCycle[P_] := Module[{j, b},
   b = {};
   For[j = 1, j ≤ Length[P], j++,
    If[
     Subset[Ascendents[j, P], CycleAssociated[j, P]], AppendTo[b, DP[j, P]]]]
   ; b];
Λ[i_, P_] := Union[{i}, DP[i, P]];
LambdaChainStart[P_] :=
 Table[Λ[ElementsNotNoneRow[P][[i]], P], {i, Length[ElementsNotNoneRow[P]]}]
CanonicalDecomposition[P_] :=
  Join[LambdaChainStart[P], LambdaPrincipalCycle[P]];
CanonicalDecomposition[P]
```

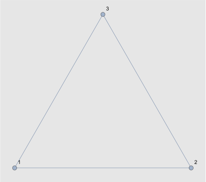
Out[24]= {{2, 3, 4, 6}, {1, 2, 5, 7, 8}, {2, 3, 4}}

PROGRAM 2

In[25]:=

```
f[i_, j_, P_] := If[i == j, 0, If[Intersection[Part[CanonicalDecomposition[P], i],
      Part[CanonicalDecomposition[P], j]] ≠ {}, 1, 0]];
Matr[P_] := Table[f[i, j, P], {i, Length[CanonicalDecomposition[P]]},
   {j, Length[CanonicalDecomposition[P]]}];
AdjacencyGraph[Matr[P], VertexLabels → "Name"]
OptimalFragmentation[P_] :=
 ConnectedComponents[AdjacencyGraph[Matr[P], VertexLabels → "Name"]]
OptimalFragmentation[P]
```

Out[27]=



Out[29]=

(1 2 3)

Outline

Classification of 2-dimensional complex evolution algebras

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Theorem (Casas J.M., Ladra M., Omirov B.A., and Rozikov U.A., 2014)

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Any 2-dimensional complex evolution algebra E is isomorphic to one of the following pairwise non isomorphic algebras:

1. dim E^2=1
   - E_1: $e_1e_1 = e_1$,
   - E_2: $e_1e_1 = e_1$, $e_2e_2 = e_1$,
   - E_3: $e_1e_1 = e_1 + e_2$, $e_2e_2 = -e_1 - e_2$,
   - E_4: $e_1e_1 = e_2$.

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   - E_3: $e_1 e_1 = e_1 + e_2$, $e_2 e_2 = -e_1 - e_2$,
   - E_4: $e_1 e_1 = e_2$.
2. dim E^2=2
   - E_5: $e_1 e_1 = e_1 + a_2 e_2$, $e_2 e_2 = a_3 e_1 + e_2$, $1 - a_2 a_3 \neq 0$, where $E_5(a_2, a_3) \cong E_5'(a_3, a_2)$,
   - E_6: $e_1 e_1 = e_2$, $e_2 e_2 = e_1 + a_4 e_2$, $a_4 \neq 0$, where $E_6(a_4) \cong E_6(a_4')$ $\Leftrightarrow \frac{a_4'}{a_4} = \cos \frac{2\pi k}{3} + \imath \sin \frac{2\pi k}{3}$ for some $k = 0, 1, 2$.

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If $\dim(A^2) = 0$ then $M_B = 0$ for any natural basis B of A.

If $\dim(A^2) = 1$ then M_B is one of the following four matrices:

1. $M_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,
2. $M_B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$,
3. $M_B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,
4. $M_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

   They are mutually non-isomorphic.

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If $\dim(A^2) = 2$ then M_B is one of the following three types of matrices:

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5. $M_B(\alpha, \beta) = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$ for some $\alpha, \beta \in \mathbb{K}^\times$ and $1 - \alpha\beta \neq 0$.
6. $M_B(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ for some $\alpha \in \mathbb{K}^\times$.
7. $M_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
8. $M_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
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Classification of 2-dimensional evolution algebras

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9. $M_B(\gamma) = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix}$ for some $\gamma \in \mathbb{K}^\times$.

They are mutually non-isomorphic except in the case $\{M_B(\gamma) \mid \gamma \in \mathbb{K}\}$ when $\dfrac{\gamma}{\gamma'}$ is a $3rd$ root of unity.

$\dim(A^2) = 1$

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Fix a two-dimensional evolution algebra A and a natural basis $B = \{e_1, e_2\}$. Let

$$M_B = \begin{pmatrix} \omega_1 & \omega_3 \\ \omega_2 & \omega_4 \end{pmatrix}$$

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- Suppose that $\{e_1^2\}$ is a basis of A^2. Since $e_2^2 \in A^2$, there exists $c_1 \in \mathbb{K}$ such that $e_2^2 = c_1 e_1^2 = c_1(\omega_1 e_1 + \omega_2 e_2)$.

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- Extend the basis $\{e_1^2\}$ of A^2 to a new basis $B' = \{e_1', e_2'\}$ of A (B' is not necessarily a natural basis) with change of basis matrix $P_{B'B}$

$$P_{B'B} = \begin{pmatrix} \omega_1 & p_1 \\ \omega_2 & p_2 \end{pmatrix}.$$

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  - $e_2'^2 = p_1^2 e_1^2 + p_2^2 e_2^2 = (p_1^2 + p_2^2 c_2) e_1^2.$

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  - $e_1^2 e_2' = (\omega_1 e_1 + \omega_2 e_2)(p_1 e_1 + p_2 e_2) = (\omega_1 p_1 + \omega_2 p_2 c_2) e_1^2 = 0$.

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- Distinguish different cases depending on $\omega_1 p_1 + \omega_2 p_2 c_2$ is zero or not.

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$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$			
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- We study if the parametric families of evolution algebras are isomorphic when we change of parameters.

Outline

Action of $S_3 \ltimes (\mathbb{K}^{\times})^3$ on $\mathrm{M}_3(\mathbb{K})$

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1. $G = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{K}^\times \right\} = \left\{ (\alpha, \beta, \gamma) \mid \alpha, \beta, \gamma \in \mathbb{K}^\times \right\}.$

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3. $H = \{\sigma(\alpha, \beta, \gamma) \mid \sigma \in S_3,\ (\alpha, \beta, \gamma) \in (\mathbb{K}^\times)^3\} =$
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4. $\sigma(\alpha_1, \alpha_2, \alpha_3)\tau(\beta_1, \beta_2, \beta_3) = \sigma\tau(\alpha_{\tau(1)}\beta_1, \alpha_{\tau(2)}\beta_2, \alpha_{\tau(3)}\beta_3).$

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5. Semidirect product of S_3 and $(\mathbb{K}^\times)^3$ is denoted by $S_3 \ltimes (\mathbb{K}^\times)^3$.
6. The action of P on M can be formulated as follows:

$$P \cdot M := P^{-1} M P^{(2)}.$$

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2. The number of zero entries in the main diagonal of M coincides with the number of zero entries in the main diagonal of $P \cdot M$.
3. The rank of M and the rank of $P \cdot M$ coincide.
4. Let M_B be the structure matrix of an evolution algebra A such that $A^2 = A$. If N is the structure matrix of A relative to a natural basis B' then there exists $Q \in S_3 \rtimes (\mathbb{K}^\times)^3$ such that $N = Q \cdot M_B$.

Three-dimensional evolution algebras

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✓ If $\dim(A^2) = 1$. Let $M_B = (\omega_{ij})$ be the structure matrix.

- We may assume $e_1^2 \neq 0$.

$$
\begin{aligned}
e_1^2 &= \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \\
e_2^2 &= c_1 e_1^2 = c_1(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3) \\
e_3^2 &= c_2 e_1^2 = c_2(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3).
\end{aligned}
$$

- We analyze when A^2 has the extension property, i.e., if there exists a natural basis $B' = \{e_1^2, e_2', e_3'\}$ of A with

$$P_{B'B} = \begin{pmatrix} \omega_1 & \alpha & \delta \\ \omega_2 & \beta & \nu \\ \omega_3 & \gamma & \eta \end{pmatrix}$$

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- The conditions are as follows:

$$\begin{aligned} \alpha\omega_1 + \beta\omega_2 c_1 + \gamma\omega_3 c_2 &= 0 \\ \delta\omega_1 + \nu\omega_2 c_1 + \eta\omega_3 c_2 &= 0 \\ \alpha\delta + \beta\nu c_1 + \gamma\eta c_2 &= 0 \\ |P_{B'B}| &\neq 0 \end{aligned}$$

- We analyze when A^2 has the extension property, i.e., if there exists a natural basis $B' = \{e_1^2, e_2', e_3'\}$ of A with

$$P_{B'B} = \begin{pmatrix} \omega_1 & \alpha & \delta \\ \omega_2 & \beta & \nu \\ \omega_3 & \gamma & \eta \end{pmatrix}$$

- The conditions are as follows:

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- A^2 has the extension property if and only if

$$\omega_1^2 + \omega_2^2 c_1 + \omega_3^2 c_2 \neq 0$$

.

$\dim(A^2) = 1$

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Type			
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$			
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$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

$\dim(A^2) = 1$

Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

$\dim(A^2) = 1$

Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$			

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Type	A^2 has the extension property		
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No		
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes		

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Type	A^2 has the extension property	dimension of ann(A)	
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

$\dim(A^2) = 1$

Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	No
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	No
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	$I =< e_3 >$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	

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Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	No
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	$I =< e_3 >$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	$I =< e_3 >$

$\dim(A^2) = 2$

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✓ If $\dim(A^2) = 2$. We may assume that there exists a natural basis $B = \{e_1, e_2, e_3\}$ such that

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$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & c_1\omega_{11} + c_2\omega_{12} \\ \omega_{21} & \omega_{22} & c_1\omega_{21} + c_2\omega_{22} \\ \omega_{31} & \omega_{32} & c_1\omega_{31} + c_2\omega_{32} \end{pmatrix}$$

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$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & c_1\omega_{11} + c_2\omega_{12} \\ \omega_{21} & \omega_{22} & c_1\omega_{21} + c_2\omega_{22} \\ \omega_{31} & \omega_{32} & c_1\omega_{31} + c_2\omega_{32} \end{pmatrix}$$

for some $c_1, c_2 \in \mathbb{K}$ with $\omega_{11}\omega_{22} - \omega_{12}\omega_{21} \neq 0$.

$\dim(A^2) = 2$

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- We compute the possible change of basis matrices $P_{B'B}$ for another natural basis B'.

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$$\begin{cases} \omega_{11}p_{11}p_{12} + \omega_{12}p_{21}p_{22} + (\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{32} = & 0 \\ \omega_{21}p_{11}p_{12} + \omega_{22}p_{21}p_{22} + (\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{32} = & 0 \\ \omega_{31}p_{11}p_{12} + \omega_{32}p_{21}p_{22} + (\omega_{31}c_1 + \omega_{32}c_2)p_{31}p_{32} = & 0 \end{cases}$$

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\omega_{31}p_{11}p_{12} + \omega_{32}p_{21}p_{22} + (\omega_{31}c_1 + \omega_{32}c_2)p_{31}p_{32} = & 0
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- We distinguish several cases depending on c_1 and c_2.

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- We study what happen when we change the parameters. Are the corresponding evolution algebras isomorphic?

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	

Four non-zero entries

- There are 3(15-6-3)=18 cases.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$				

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$			

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$\begin{pmatrix}0&0&0\\0&1&1\\1&0&c_1\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&0\\0&1&c_1\end{pmatrix}$	$\begin{pmatrix}c_1&0&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}c_1&1&0\\0&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&c_1&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\0&1&c_2\\1&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&c_2\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\c_2&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\0&c_2&1\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\c_2&0&1\end{pmatrix}$	$\begin{pmatrix}1&c_2&0\\1&0&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\1&0&c_1\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&c_1\\0&0&0\\1&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\c_1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&1\\1&c_1&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\c_1&1&0\end{pmatrix}$	$\begin{pmatrix}0&c_1&1\\1&0&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\1&1&0\\1&0&c_1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&0&0\\0&1&c_1\end{pmatrix}$	$\begin{pmatrix}c_1&0&1\\0&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1&0\\1&0&1\end{pmatrix}$		

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
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Mutually isomorphic evolution algebras

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M_B	$P_{B'B}$	$M_{B'}$

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	

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$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -c_1 \end{pmatrix}$

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$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	

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$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{1}{c_1} \\ 1 & 1 & 0 \end{pmatrix}$

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$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{1}{c_1} \\ 1 & 1 & 0 \end{pmatrix}$
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$$
\begin{aligned}
p_{11}p_{12} &= 0; \quad & p_{21}p_{22} &= -c_2 p_{31}p_{32};\\
p_{11}p_{13} &= 0; \quad & p_{21}p_{23} &= -c_2 p_{31}p_{33};\\
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$$\left\{ \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{pmatrix}, \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & 0 & p_{23} \\ 0 & p_{32} & 0 \end{pmatrix}, \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \frac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix} \right\}.$$

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- We see if their evolution algebras are mutually isomorphic.

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- Are the parametric families evolution algebras isomorphic?.

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Outline

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- The different methods use to obtain the classification can be generalized to arbitrary finite-dimensional evolution algebras.

“The lack of real contact between mathematics and biology is either a tragedy, a scandal or a challenge, it is hard to decide which.”

Gian Carlo Rota

Thanks!