UNIVERSIDAD DE MÁLAGA

FACULTAD DE CIENCIAS

Programa de Doctorado en Matemáticas

PhD thesis (Tesis doctoral)

Evolution Algebras

Yolanda Cabrera Casado

Advisors (Directoras):

Mercedes Siles Molina

M. Victoria Velasco Collado

Diciembre 2016

AUTOR: Yolanda Cabrera Casado

http://orcid.org/0000-0003-4299-4392

EDITA: Publicaciones y Divulgación Científica. Universidad de Málaga

Mercedes Siles Molina, Catedrática de Álgebra de la Universidad de Málaga y María Victoria Velasco Collado, Profesora Titular de Análisis Matemático de la Universidad de Granada,

Informan:

Que han dirigido la Tesis Doctoral titulada: "Álgebras de evolución", realizada por la Licenciada Dña. Yolanda Cabrera Casado.

Asimismo, se hace constar que el mencionado trabajo es totalmente original, y ha dado lugar a la siguiente publicación que lo avala:

Yolanda Cabrera Casado, Mercedes Siles Molina, M. Victoria Velasco, Evolution algebras of arbitrary dimension and their decompositions. *Linear Algebra Appl*. **495** (2016), 122-162.

Se hace constar por último que ningún contenido ni resultado de la anterior publicación ha sido utilizado en ninguna otra Tesis Doctoral o trabajo de investigación.

Finalizada la investigación que ha llevado a la conclusión de la citada Tesis Doctoral, y de acuerdo con el Real Decreto 99/2011, autorizan su presentación por considerar que reúne los requisitos formales y científicos legalmente establecidos para la obtención del título de Doctor en Matemáticas.

Y para que así conste y surta los efectos oportunos expiden y firman el presente informe en Málaga a 27 de septiembre del 2016.

Fdo.: Mercedes Siles Molina

Fdo.: María Victoria Velasco Collado

A mi familia,

por los momentos robados.

Contents

Introduction III

Resumen en español (Spanish abstract) XIII

Agradecimientos (Acknowledgements) XXXVII

1. Basic facts about evolution algebra 1

1.1. Preliminaries 1

1.2. Evolution algebras 6

1.3. Product and change of basis 11

1.4. Subalgebras and ideals of an evolution algebra 17

1.5. Non-degenerate evolution algebras 23

1.6. The graph associated to an evolution algebra 29

2. Decomposition of an evolution algebra 35

2.1. Ideals generated by one element 36

2.2. Simple evolution algebras 42

2.3. Reducible evolution algebras. 48

2.4. The optimal direct-sum decomposition of an evolution algebra 54

2.5. The fragmentation process 61

3. Classification two-dimensional evolution algebras 67

4. Classification of three-dimensional evolution algebras **81**

4.1. Action of $S_3 \rtimes (\mathbb{K}^\times)^3$ on $\mathrm{M}_3(\mathbb{K})$ 82

4.2. Three-dimensional evolution algebras 85

A. The optimal fragmentation computed with Mathematica **191**

Further works **195**

Bibliography **197**

Notation **201**

Index **202**

Introduction

The mathematical study of the genetic inheritance began in 1856 with the works of Gregor Mendel, who was a pioneer in using mathematical notation to express his genetics laws. After relevant contributions of authors as Jennings (1917), Serebrovskij (1934) and Glivenko (1936), to give an algebraic interpretation of the sign $\times$ of sexual reproduction, Etherington introduced the formal language of abstract algebra to the study of Genetics in his papers [17, 18]. This precise mathematical formulation of Mendel's laws in terms of non-associative algebras makes possible to contemplate the sexual reproduction and the system of inheritance of an organism by considering the combine of gamete cells to form the zygote as an algebraic multiplication whose structure constants determine the "frequency distribution.ºf the resulting gametes. Since then, many works pointed out that non-associative algebras are an appropriate mathematical framework for studying Mendelian Genetics [2, 25, 30, 33]. Thus, the term genetic algebra was coined to denote those algebras (most of them non-associative) used to model inheritance in Genetics.

Recently a new type of genetic algebras, denominated evolution algebras, has emerged to enlighten the study of non-Mendelian Genetics, which is the basic language of the molecular Biology. In particular, evolution algebras can be applied to the inheritance of organelle[1] genes, for instance,

[1]Specialized cellular structure in eukaryotic cells analogous to an organ in the

to predict all possible mechanisms to establish the homoplasmy of cell populations. The theory of evolution algebras was introduced by Tian in [30], a pioneering monograph where many connections of evolution algebras with other mathematical fields (such as graph theory, stochastic processes, group theory, dynamic systems, mathematical physics, etc) are established. In this book it is shown the close connection between evolution algebras, non-Mendelian Genetics and Markov chains, pointing out some further research topics. Algebraically, evolution algebras are non-associative algebras (which are not even power-associative), and dynamically they represent discrete dynamical systems. In this context, an evolution algebra is nothing but a finite-dimensional algebra A provided with a basis $B = \{e_i \mid i \in \Lambda\}$, such that $e_i e_j = 0$, whenever $i \neq j$ (such a basis is said to be natural). If $e_k^2 = \sum_{i \in \Lambda} \omega_{ik} e_i$, then the coefficients ω_{ij} define the named structure matrix M_B of A relative to B that codifies the dynamic structure of A.

In [30], evolution algebras are associated to free populations to give the explicit solutions of a nonlinear evolutionary equation in the absence of selection, as well as general theorems on convergence to equilibrium in the presence of selection. In the last years, many different aspects of the theory of evolution algebras have seen considered. For instance, in [27] evolution algebras are associated to function spaces defined by Gibbs measures on a graph, providing a natural introduction of thermodynamics in the study of several systems in biology, physics and mathematics. On the other hand, chains of evolution algebras (i.e. dynamical systems the state of which at each given time is an evolution algebra) are studied in [10, 29, 22, 23]. Also the derivations of some evolution algebras have been analyzed in [30, 7, 21]. In [21], the evolution algebras have been used to describe the inheritance

body. Examples of organelles are lysosomes, nucleus, mitochondria and the endoplasmic reticulum.

of a bisexual population and, in this setting, the existence of non-trivial homomorphisms onto the sex differentiation algebra have been studied in [20]. Algebraic notions as nilpotency and solvability may be interpreted biologically as the fact that some of the original gametes (or generators) become extinct after a certain number of generations, and these algebraic properties have been studied in [11, 8, 28, 32, 12, 19, 14].

We describe now the organization of this thesis, namely, the content of the chapters and their sections. The first and the second chapter is devoted to the study of evolution algebras of arbitrary dimension; the original results can be found in the paper [4]. We start by introducing, in Section 1.2, the essential definitions that will be needed throughout the chapter and even the thesis. Moreover we study some properties of evolution algebras in Remark 1.2.2 such as commutativity, associativity, etc.

Since evolution algebras appear after Mendelian algebras, it is natural to ask if these are evolution algebras. The answer is no, as we show in Example 1.2.3. The fact that evolution algebras are not Mendelian algebras is known.

The product of an arbitrary algebra has been studied in the Section 1.3. Fix a basis $B = \{e_i \mid i = 1, \ldots, n\}$ this product, relative to the basis B is determined by the matrices of the multiplication operators, $M_B(\lambda_{e_i})$. The relationship under change of basis is also established. In the particular case of evolution algebras Theorem 1.3.2 shows this connection. We can find these results for finite dimensional algebra in [5].

In Section 1.4 we study the notions of subalgebra and ideal and explore when they have natural bases and when their natural bases can be extended to the whole algebra (we call this the extension property) and provide examples in different situations showing the relationship between these concepts. We also show that the class of evolution algebras is closed under quotients and under homomorphic images, but not under subalgebras or ideals and give an example of a homomorphism of evolution algebras whose kernel is not an evolution ideal.

The aim of the Section 1.5 is the study of non-degeneracy. We show that this notion, which is given in terms of a fixed natural basis of the evolution algebra, does not depend on the election of the natural basis (Corollary 1.5.4). A absortion radical is introduced (the intersection of all the absorption ideals) which is zero if and only if the evolution algebra is non-degenerate (Proposition 1.5.13). Moreover, the quotient algebra by this ideal is a non-degenerate evolution algebra (Corollary 1.5.14).

The classical notions of semiprimeness and nondegeneracy are also studied and compared to that of non-degeneracy (see Proposition 1.5.15 and the paragraph before).

The bulk of Section 1.6 is to associate a graph to any evolution algebra (relative to a natural basis) will play a fundamental role to describe the structure of the algebra. This has been done yet in the literature, although for finite dimensional evolution algebras. The use of the graph will allow to see in a more visual way properties of the evolution algebra. For example, we can detect the annihilator of an evolution algebra by looking at its graph (concretely determining its sinks) and we can say when a non-degenerate evolution algebra is irreducible (as we explain below).

Our main objective in Chapter 2 is to prove the existence and unicity of a direct sum decomposition into irreducible components for every non-degenerate evolution algebra. When the algebra is degenerate, the uniqueness cannot be assured.

In Section 2.1 we describe the ideals generated by one element and we use the graph representation and the notion of descendent to describe these type of ideals in an evolution algebra (Proposition 2.1.11) and show that its dimension as a vector space is at most countable (Corollary 2.1.12). This implies that any simple evolution algebra has dimension at most countable.

Section 2.2 is devoted to the study and characterization of simple evolution algebras (Proposition 2.2.1 and Theorem 2.2.7). We also provide examples to show that the conditions in the characterizations cannot be dropped. We finish the section with the characterization of finite dimensional simple evolution algebras (Corollary 2.2.10).

The direct sum of a certain number of evolution algebras is an evolution algebra in a canonical way. In Section 2.3 we deal with the question of when a non-zero evolution algebra A is the direct sum of non-zero evolution subalgebras. In particular, an evolution algebra with an associated graph (relative to a certain natural basis) which is not connected is reducible (see Proposition 2.3.4). Next, Theorem 2.3.6 characterizes the decomposition of a non-degenerate evolution algebra into subalgebras (equivalently ideals) in terms of the elements of any natural basis. We also are interested in determining when every component in a direct sum is irreducible. In Corollary 2.3.8 we prove that a non-degenerate evolution algebra is irreducible if and only if the associated graph (relative to any natural basis) is connected.

The main objective of the Section 2.4 is to get a decomposition of an evolution algebra in terms of irreducible evolution subalgebras. The decomposition $A = \oplus_{\gamma\in\Gamma} I_\gamma$ of an evolution algebra into irreducible ideals (called an optimal direct-sum decomposition) exists and is unique whenever the algebra is non-degenerate (Theorem 2.4.2). To assure the uniqueness, this hypothesis cannot be eliminated (Example 2.4.3).

To get a direct-sum decomposition of a finite dimensional evolution algebra we identify in the associated graph (relative to a natural basis) the principal cycles and the chain-start indices through the fragmentation process (Proposition 2.5.4) in the Section 2.5. This provides an optimal direct-sum decomposition, which is unique, when the algebra is non-degenerate, as shown in Theorem 2.5.5.

In the appendix of this work we provide a routine with Mathematica to obtain the optimal fragmentation of a natural basis. From this we get a direct-sum decomposition of a reducible evolution algebra starting from its structure matrix.

As we pointed before non-Mendelian Genetics is a basic language of molecular Genetics and so we have tried to translate the mathematical concepts relative to evolution algebras into biological meaning in both first and second chapters.

The following two chapters are devoted to classification of two and three dimensional evolution algebras, respectively. Our main purpose is to get the classification of evolution algebras of dimension three having in mind to apply this classification in a near future in a biological setting and to detect possible tools to implement in wider classifications.

It should be pointed out that evolution algebras of de dimension two over the complex numbers were described in [12]. However, we note that the evolution algebra A with natural basis $\{e_1, e_2\}$ such that $e_1^2 = e_2$ and $e_2^2 = e_1$ is a two-dimensional evolution algebra not isomorphic to any of the six types in [12]. We realized of this fact when classifying the three-dimensional evolution algebras A such that the dimension of A^2 is 2 and having annihilator of dimension 1 (see Chapter 4, Tables 14-16). In our case, the resulting classification is done over a field $\mathbb{K}$ where for every $k \in \mathbb{K}$ the polynomial $x^n - k$ has a root whenever $n = 2, 3$. In order to simplify resulting expressions we denote by ϕ a seventh root of the unit and by ζ a third root of the unit.

As can be appreciated on observing the length of the Chapter 4, the three dimensional case is much more complicated than the two dimensional case. This classification can be found in the paper [5]. Just while we were doing this work we found the article [16], where one of the aims of the authors is to classify indecomposable nilpotent evolution algebras up to dimension five over algebraically closed fields of characteristic not two. The three-dimensional ones can be localized in our classification and for these, it is not necessary to consider algebraically closed fields.

For this classification we deal with evolution algebras over a field $\mathbb{K}$ of characteristic different from 2 and in which every polynomial of the form $x^n - k$, for $n = 2, 3, 7$ and $k \in \mathbb{K}$ has a root in the field.

We will demonstrate that there are 116 types of three-dimensional evolution algebras. All of them are classified in Tables 1-24. Structure matrices appearing in different tables are not isomorphic (in the meaning that they do not generate the same evolution algebra). Structure matrices in different rows of a same table neither are isomorphic. In general, different values of the parameters appearing in the structure matrices give non-isomorphic evolution algebras, but in some case this is not true. These cases are displayed in Tables $2'$-$23'$.

We begin Section 4.2 by introducing the notion of the action of the group $S_3 \rtimes (\mathbb{K}^\times)^3$ on $\mathrm{M}_3(\mathbb{K})$. This result will play an important role in the process of classification. The orbits of this action will completely determine the non-isomorphic evolution algebras A when $\dim(A^2) = 3$ and in some cases when $\dim(A^2) = 2$.

We have divided our study into four cases depending on the dimension of A^2, which can be 0, 1, 2 or 3. The first case is trivial. The study of the third and of the fourth ones is made by taking into account which are the possible matrices P that appear as change of basis matrices. It happens that for dimension 3, as we have said, the only matrices are those in $S_3 \rtimes (\mathbb{K}^\times)^3$.

When the dimension of A^2 is 2, there exists three groups of cases (four in fact, but two of them are essentially the same). Let $B = \{e_1, e_2, e_3\}$ be a natural basis of A such that $\{e_1^2, e_2^2\}$ is a basis of A^2 and $e_3^2 = c_1 e_1^2 + c_2 e_2^2$ for some $c_1, c_2 \in \mathbb{K}$. The first case happens when $c_1 c_2 \neq 0$. Then, $P \in S_3 \rtimes (\mathbb{K}^\times)^3$. The second group of cases arises when $c_1 = 0$ and $c_2 \neq 0$. Then, the matrix P is id_3, $(2,3)$,[2] or the matrix Q given by (4.25). The third one appears when $c_1, c_2 = 0$. In this case the matrix P is id_3 or the matrices Q' and Q'' given by (4.27) and (4.29) respectively.

For $P \in S_3 \rtimes (\mathbb{K}^\times)^3$, we classify taking into account: the dimension of the annihilator of A, the number of non-zero entries in the structure matrix (which remains invariant, as it is proved in Proposition 4.1.2), and if the evolution algebra A satisfies Property (2LI)[3].

For $P \in \{\mathrm{id}_3, (2,3), Q\}$, we obtain a first classification, given in the different Figures. Then we compare which structure matrices produce isomorphic algebras and eliminating redundancies we get the structure matrices given in the set S that appears in Theorem 4.2.2. Again, some of these structure matrices give isomorphic evolution algebras. In order to classify them, we take into account that the number of non-zero entries of the structure matrices in S remains invariant under the action of the matrix P (see Remark 4.2.3). Note that the resulting structure matrices correspond to evolution algebras with zero annihilator and do not satisfy Property (2LI).

For $P \in \{\mathrm{id}_3, Q', Q''\}$ we classify taking into account that the third column of the structure matrix has three zero entries (the dimension of the annihilator

[2]The matrix obtained from the identity matrix, id_3, when exchanging the second and the third rows

[3]For any basis $\{e_1, e_2, e_3\}$ the ideal A^2 has dimension two and it is generated by $\{e_i^2, e_j^2\}$, for every $i, j \in \{1, 2, 3\}$ with $i \neq j$.

is one and, consequently, they do not satisfy Property (2LI)) and the number of zeros in the first and the second rows remains invariant under change of basis matrices (see Remark 4.2.4).

For $\dim(A^2) = 3$ we classify by the number of non-zero entries in the structure matrix.

In the case $\dim(A^2) = 1$ it is not efficient to tackle the problem of the classification by obtaining the possible change of basis matrices, although for completeness we have determined them in Remark 4.2.5. This is because we follow a different pattern. The key point for this study will be the extension property (EP for short). We have classified taking into account the following properties: if A^2 has the extension property, the dimension of the annihilator of A, and if the evolution algebra A has a principal two-dimensional evolution ideal which is degenerate as an evolution algebra (PD2EI for short).

The classification of three-dimensional evolution algebras is achieved in Theorem 4.2.2. We summarize the cases in the tables that follow.

A^2 has EP	dim(ann(A))	A has a PD2EI	Number
No	0	Yes	1
No	1	Yes	1
Yes	2	No	1
Yes	1	No	1
Yes	0	No	1
Yes	2	Yes	1
Yes	1	Yes	1

$\dim(A^2) = 1$

dim(ann(A))	Non-zero entries * Non-zero entries in S ** Non-zero entries in rows 1 and 2	A has Property (2LI)	Number
1	1**	No	2
1	2**	No	4
1	3**	No	2
1	4**	No	3
0	4*	No	3
0	5*	No	6
0	6*	No	3
0	7*	No	6
0	8*	No	3
0	9*	No	3
0	4	Yes	4
0	5	Yes	3
0	6	Yes	7
0	7	Yes	6
0	8	Yes	2
0	9	Yes	1

$\dim(A^2) = 2$

Non-zero entries	Number
3	3
4	6
5	16
6	15
7	8
8	2
9	1

$\dim(A^2) = 3$

Resumen en español
Spanish abstract

El estudio matemático de la herencia genética comenzó en la segunda mitad del siglo XIX con los trabajos de Gregor Mendel, quien fue el pionero en utilizar el lenguaje matemático para expresar sus leyes sobre genética. Después de las importantes contribuciones de autores como Jenings, Serebrowskij y Glivenko para formular algebraicamente el significado del signo $\times$ usado por Mendel en la reproducción sexual, se llegó a una formulación matemática de las leyes de Mendel en los conocidos trabajos de Etherington [17, 18]. Llegados a este punto, muchas contribuciones posteriores corroboraron que las álgebras no asociativas son el marco matemático apropiado para el estudio de las genéticas mendelianas [4] [2, 25, 30, 33]. Por ello, se acuñó el término álgebra genética para hacer referencia a aquellas álgebras -por lo general no asociativas- utilizadas para modelar la herencia en el campo de la genética.

Recientemente, un nueva clase de álgebras genéticas, denominadas álgebras de evolución, emergieron para formular la Genética no mendeliana, que puede considerarse como el lenguaje de la Biología Molecular. En particular, las álgebras de evolución pueden aplicarse para describir la

[4]Conjunto de teorías que intenta explicar la herencia y la diversidad biológica según los principios de Gregor Mendel en cuanto a la transmisión de carácteres genéticos de organismos paternales a su descendiente basado en el análisis estadístico y en los experimentos científicos con las plantas de guisantes. Sus importantes contribuciones son la piedra angular de la genética.

herencia genética de los genes extranucleares presentes en determinados orgánulos (mitocondrias y cloroplastos) y con ello, por ejemplo, predecir todos los posibles mecanismos de la homoplasmia [5] de las células. La teoría de las álgebras de evolución fue introducida por J. P. Tian en una monografía [30] donde se establecen numerosas conexiones entre las álgebras de evolución y otros campos de las Matemáticas, tales como la teoría de grafos, procesos estocásticos, teoría de grupos, sistemas dinámicos y física matemática entre otros. Concretamente, se muestra la estrecha relación existente entre las álgebras de evolución, la genética no mendeliana y las cadenas de Markov, que no son más que álgebras de evolución cuya matriz de coeficientes es una matriz estocástica, abriendo así una vía original e interesante de futuras investigaciones al respecto. Algebraicamente, las álgebras de evolución son álgebras no asociativas, ni siquiera son de potencias asociativas, y dinámicamente representan sistemas dinámicos discretos, de nuevo es la matriz de estructura la que determina la naturaleza dinámica del álgebra.

A pesar de que las álgebras de evolución han sido introducidas recientemente, ya son muchos los aspectos de las mismas que han sido estudiados. Por ejemplo, en [30] las álgebras de evolución se han usado en relación con poblaciones libres cuya evolución viene descrita por una ecuación de evolución no lineal en ausencia de selección, para dar las soluciones explícitas de la misma, así como los teoremas generales sobre la convergencia al equilibrio en presencia de selección. Por ejemplo, en [27] las álgebras de evolución están asociadas a espacios de funciones determinados por medidas de Gibbs sobre un gráfico, introduciendo con ello de manera natural la termodinámica en el estudio de diferentes sistemas de la Biología, la Física

[5] Presencia dentro de una célula u organismo de mitocondrias genéticamente idénticas. Esto es la condición común para la mayor parte de grupos de organismos, aunque haya algunas excepciones (heteroplasmia)

y las Matemáticas. Por otra parte, las cadenas de álgebras de evolución -sistemas dinámicos continuos en el que cada momento del estado es un álgebra de la evolución- se estudian en [10, 29, 22, 23]. También se han analizado las derivaciones de algunas álgebras de evolución en [30, 7, 21]. En [21], las álgebras de evolución se han utilizado para describir la herencia genética en poblaciones bisexuales y, en este contexto, la existencia de homomorfismos no triviales sobre el álgebra dada por la diferenciación sexual se ha estudiado en [20]. Nociones algebraicas como nilpotencia, por ejemplo, se pueden interpretar biológicamente como el hecho de que algunos de los gametos originales (o generadores) se extinguen después de un cierto número de generaciones, y otras propiedades algebraicas de este han sido estudiadas en [11, 8, 28, 32, 12, 19, 14]. Sirvan los trabajos mencionados (sin menoscabo de otros) de muestra para justificar el interés científico que las álgebras de evolución han suscitado en la comunidad matemática a pesar de tratarse de un campo de investigación emergente en el que hay muchos aspectos básicos todavía por explorar. De hecho, en la presente Memoria vamos a analizar muchos de ellos (como los ideales, la simplicidad, la descomponibilidad, o la estructura de las álgebras de evolución de dimensión pequeña) por tratarse de partes todavía inéditos de teoría de las álgebras de evolución.

A continuación, describiremos cómo está organizada la tesis, es decir, el contenido de los capítulos y de las secciones. El primer y segundo capítulos están dedicados al estudio de las álgebras de evolución de dimensión arbiraria; los resultados originales pueden encontrarse en el artículo [4]. Empezamos introduciendo, en la sección 1.2 las definiciones básicas que necesitaremos a lo largo del capítulo y de la tesis. En primer lugar, empezaremos recordando definiciones básicas del álgebra para tener un punto de partida común:

- Un **álgebra** es un espacio vectorial A sobre un cuerpo $\mathbb{K}$, provisto de una aplicación bilineal $A \times A \to A$ dada por $(a, b) \mapsto ab$, llamada **multiplicación** o **producto** de A.

- Un álgebra A se dirá **commutativa** si $ab = ba$ para cada $a, b \in A$.
- Si $(ab)c = a(bc)$ para cada $a, b, c \in A$, entonces decimos que A es **asociativa**.
- Un álgebra A es **flexible** si $a(ba) = (ab)a$ para cada $a, b \in A$.
- Las álgebras **de potencia asociativa** son aquellas tales que cada subálgebra generada por un elemento es asociativa.

En segundo lugar, definimos el concepto de **álgebra de evolución** sobre un cuerpo $\mathbb{K}$ como aquella $\mathbb{K}$-álgebra A dotada de una base $B = \{e_i \mid i \in \Lambda\}$ tal que $e_i e_j = 0$ para todo $i \neq j$. Tal base B recibe el nombre de **base natural**. Fijada una base natural B en A, los escalares $\omega_{ki} \in \mathbb{K}$ tal que $e_i^2 := e_i e_i = \sum_{k \in \Lambda} \omega_{ki} e_k$ se denominan **constantes de estructura** de A **relativa a** B, y la matriz $M_B := (\omega_{ki})$ se dice que es la **matriz de estructura de** A **relativa a** B. Escribiremos $M_B(A)$ cuando queramos indicar el álgebra de evolución a la que se refiere. Obsérvese que $|\{k \in \Lambda |\ \omega_{ki} \neq 0\}| < \infty$ para cada i. Entonces, M_B es una matriz en $\mathrm{CFM}_\Lambda(\mathbb{K})$, donde $\mathrm{CFM}_\Lambda(\mathbb{K})$ es el espacio vectorial de aquellas matrices (infinitas o no) sobre $\mathbb{K}$ de orden $\Lambda \times \Lambda$ en la cual cada columna tiene a lo sumo un número finito de entradas no nulas.

Para el caso particular en el que el álgebra A es finita, podemos decir que A es de evolución si y solo si existe una base $B = \{e_1, ..., e_n\}$ tal que la tabla de multiplicación en dicha base es diagonal. En consecuencia, el producto del álgebra ueda determinado por los coeficientes que determinan el cuadrado de los elementos de la base que se listan por columnas en la llamada matriz de estructura:

$$M_B = \begin{pmatrix} \omega_{11} & \dots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \in \mathrm{M_n}(\mathbb{K}).$$

En este punto, ya podemos observar qué lejos están las álgebras de evolución de las álgebras asociativas. De hecho, proporcionamos un ejemplo de álgebra de evolución que no es ni siquiera de potencia asociativa y por tanto no es asociativa. Además, damos las condiciones que tiene que cumplir la matriz de estructura para que haya asociatividad de las potencias.

Por tanto, las álgebras de evolución no son, en general, de Jordan o alternativas. Además, tampoco pertenecen al grupo de las álgebras de Lie. Sin embargo, de la definición se deduce que son conmutativas y consiguientemente flexibles.

Como ya hemos mencionado, las álgebras de evolución aparecieron después de las álgebras mendelianas; por tanto, una pregunta natural que podemos hacernos es si verdaderamente son modelos distintos. Por ello estudiamos si las álgebras mendelianas cumplen la condición de álgebras de evolución. La respuesta es no como muestra el Ejemplo 1.2.3, en el que se ve que el álgebra cigótica que define la herencia mendeliana de un gen con dos alelos E y e no es un álgebra de evolución. Los tres posibles genotipos serían EE, Ee y ee. Esto nos lleva a considerar el espacio vectorial generado por la base $B = \{EE, Ee, ee\}$ y de este modo la tabla de multiplicación que viene dada por las reglas de la herencia basadas en las leyes de Mendel sería la siguiente:

	EE	Ee	ee
EE	EE	$\frac{1}{2}EE + \frac{1}{2}Ee$	Ee
Ee	$\frac{1}{2}EE + \frac{1}{2}Ee$	$\frac{1}{4}EE + \frac{1}{4}ee + \frac{1}{2}Ee$	$\frac{1}{2}ee + \frac{1}{2}Ee$
ee	Ee	$\frac{1}{2}ee + \frac{1}{2}Ee$	ee

En el sentido contrario, es conocido el hecho de que las álgebras de evolución no son álgebras mendelianas.

La Sección 1.3 está enfocada a describir del producto de un álgebra arbitraria. Fijada una base $B = \{e_i \mid i = 1, \ldots, n\}$, este producto relativo a la base B, está determinado por las matrices de los operadores de multiplicación, $M_B(\lambda_{e_i})$. Se establece la relación entre dichas matrices bajo un cambio de base y, por último, el siguiente teorema establece la relación entre las matrices de estructura relativas a dos bases naturales arbitrarias en el caso de que las álgebras sean de evolución:

Teorema 1.3.2

Sea A un álgebra de evolución y sea $B = \{e_i \mid i \in \Lambda\}$ una base natural de A con matriz de estructura $M_B = (\omega_{ij})$. Entonces:

(I) *Si $B' = \{f_i \mid i \in \Lambda\}$ es una base natural de A tal que $P_{B'B} = (p_{ij})$ y $P_{BB'} = (q_{ij})$ son las matrices de cambio de base, entonces:*

$$M_B(\xi_B(f_i) \bullet_B \xi_B(f_j)) = 0$$

para cada $i \neq j$ con $i, j \in \Lambda$.

Además,

$$M_{B'} = P_{BB'} M_B P^{(2)}_{B'B},$$

donde $P^{(2)}_{B'B} = (p^2_{ij})$.

(II) *Suponemos que $P = (p_{ij}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$ es inversible o regular y satisface las relaciones (1.7). Se define $B' = \{f_i \mid i \in \Lambda\}$, donde $f_i = \sum_{j \in \Lambda} p_{ji} e_j$ para cada $i \in \Lambda$. Entonces, B' es una base natural y se cumple (1.8).*

La demostración para el caso finito dimensional se puede encontrar en [5].

La expresión (1.7) puede reescribirse de forma más condensada ([30]). Concretamente,

$$M_B(P_{B'B} * P_{B'B}) = 0,$$

donde $P_{B'B} * P_{B'B} = (c_{k(i,j)}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$, siendo $c_{k(i,j)} = p_{ki}p_{kj}$ para cada par (i, j) con $i < j$ y $i, j \in \Lambda$

Estudiamos las nociones de subálgebra e ideal en la Sección 1.4 y analizamos por un lado, cuándo admiten una base natural, en cuyo caso tenemos la definiciones de subálgebra e ideal de evolución: Una **subálgebra de evolución** de un álgebra de evolución A es una subálgebra $A' \subseteq A$ tal que A' es un álgebra de evolución, esto es, A' admite una base natural. Una subálgebra de evolución I tal que I es un ideal de A ($IA \subseteq I$ en un álgebra A conmutativa) es un **ideal de evolución**.

Por otro lado, la pregunta que nos planteamos es cuándo dichas bases se pueden extender a una base del álgebra de evolución considerada; es lo que llamamos propiedad de extensión. En definitiva, los conceptos de subálgebra de evolución e ideal de evolución se definen de forma natural al observar con diferentes ejemplos que no toda subálgebra de un álgebra de evolución admite una base natural y que no toda subálgebra de evolución de un álgebra de evolución cumple la propiedad de ser ideal como se puede ver en el Ejemplo 1.4.5. Además, mostramos que un ideal de un álgebra de evolución no es necesariamente un ideal de evolución (véase Ejemplo 1.4.6).

Llegados a este punto, queremos señalar que una subálgebra de evolución en el sentido de [30] es una subálgebra de evolución según las definiciones anteriores (1.4.3) que cumple la propiedad de extensión. Por tanto, la definiciones que hemos dado en este texto de subálgebra e ideal de evolución son menos restrictiva que la dada por Tian en [30] como muestra el Ejemplo

1.4.10, el cual proporciona un ideal de evolución I que no tiene la propiedad de extensión. Por otro lado, en [30, Proposition 2, p. 24] se prueba que cada subálgebra de evolución es un ideal de evolución (ambos conceptos en el sentido de [30]). A diferencia de lo que ocurre con nuestras definiciones tal que y como se puede ver en los Ejemplos 1.4.5 y 1.4.10).

También mostramos que la clase de las álgebras de evolución es cerrada bajo cocientes y bajo imágenes homomórficas. En este sentido damos un ejemplo de un homomorfismo de álgebras de evolución cuyo núcleo no es un ideal de evolución.

El objetivo de la Sección 1.5 es estudiar la propiedad de que un álgebra de evolución sea no degenerada en el sentido siguiente: Un álgebra de evolución A es **no degenerada** si admite una base natural $B = \{e_i \mid i \in \Lambda\}$ tal que $e_i^2 \neq 0$ para cada $i \in \Lambda$.

Vemos que este concepto, el cuál está dado en términos de una base natural prefijada del álgebra de evolución, no depende de la base elegida. En la demostración utilizamos el concepto de anulador. Recordamos que para un álgebra conmutativa, su anulador, denotado por $\operatorname{ann}(A)$ se define como

$$\operatorname{ann}(A) := \{x \in A \mid xA = 0\}.$$

Por la importancia que tendrán las álgebras de evolución no degeneradas en la descomposición en suma directa, buscamos, por un lado, una caracterización de dichas álgebras y, por otro lado, queremos encontrar un ideal tal que el cociente por este ideal sea un álgebra de evolución no degenerada. Por este motivo se introduce la nociones de propiedad de absorción y de radical de absorción: Sea I un ideal de un álgebra de evolución A. Diremos que I tiene la **propiedad de absorción** si $xA \subseteq I$ implica $x \in I$. Teniendo en cuenta que la intersección de todos los ideales de A que tienen la propiedad de absorción sigue siendo un ideal con la propiedad de absorción, se define de forma natural lo que llamamos **radical de absorción** como el ideal formado por dicha intersección. Lo denotamos por $\mathrm{rad}(A)$. Se observa que el radical es el ideal más pequeño de A con la propiedad de absorción.

El siguiente resultado proporciona una caracterización en términos del radical de absorción de un álgebra de evolución no degenerada:

Proposición 1.5.13

Sea A un álgebra de evolución. Entonces, $\mathrm{rad}(A) = 0$ si y solo si A es no degenerada.

Finalmente, relacionado con este concepto tenemos el siguiente corolario que proporciona una forma de encontrar un álgebra de evolución no degenerada a partir de un álgebra de evolución arbitraria:

Corolario 1.5.14

Sea I un ideal de un álgebra de evolución A. Entonces, I tiene la propiedad de absorción si y solamente si $\mathrm{rad}(A/I) = \overline{0}$. En particular $\mathrm{rad}(A/\mathrm{rad}(A)) = \overline{0}$, esto es, $A/\mathrm{rad}(A)$ es un álgebra de evolución no degenerada.

Terminamos esta sección comparando, en el ambiente que nos ocupa, los conceptos clásicos de semiprimidad y no degeneración (en el sentido: Si $a(Aa) = 0$ para algún $a \in A$, entonces $a = 0$ con A un álgebra), escribiremos degenerado* para distinguirlo de la Definición 1.5.1 con la noción de álgebra de evolución no degenerada (Definición 1.5.1):

Proposición 1.5.15

Sea A un álgebra de evolución con producto no cero. Consideremos las siguientes aserciones:

(I) *A es no degenerada*.*

(II) *A es semiprimo.*

(III) *A no tiene ideales no triviales de cuadrado cero.*

(IV) *A es no degenerada.*

Entonces: (I) $\Rightarrow$ (II) $\Leftrightarrow$ (III) $\Rightarrow$ (IV).

En la última sección de este capítulo asociamos un grafo a un álgebra de evolución (relativa a una base natural) y viceversa de la siguiente forma: Sea $B = \{e_i \mid i \in \Lambda\}$ una base natural de un álgebra de evolución A y sea $M_B = (\omega_{ji}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$ su matriz de estructura. Consideramos la matriz $P^t = (p_{ji}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$ tal que $p_{ji} = 0$ si $\omega_{ji} = 0$ y $p_{ji} = 1$ si $\omega_{ji} \neq 0$. El **grafo asociado al álgebra de evolución** A (relativa a la base B), denotado por E_A^B (o simplemente por E si el álgebra A y la base B se sobreentienden) es el grafo cuya matriz de adyacencia es $P = (p_{ij})$.

Esta relación existente entre los grafos y las álgebras de evolución jugará un papel fundamental para describir la estructura de dichas álgebras. Este hecho ya ha sido tratado en la literatura, aunque para álgebras de evolución de dimensión finita. El trabajar con grafos nos permitirá observar de forma más visual distintas propiedades de las álgebras de evolución. Por ejemplo, podemos hallar el anulador de un álgebra de evolución observando su grafo, concretamente determinando sus sumideros o podemos decir cuándo un álgebra de evolución no degenerada es irreducible, como se explica más adelante.

La principal finalidad del Capítulo 2 es probar la existencia y unicidad de una descomposición en suma directa de componentes irreducibles para un álgebra de evolución no degenerada. Cuando el álgebra es degenerada, la unicidad no está asegurada.

En la Sección 2.1 comenzamos introduciendo las definiciones de descendientes y ascendientes: Sea $B = \{e_i \mid i \in \Lambda\}$ una base natural de un álgebra de evolución A y sea $i_0 \in \Lambda$. Los **descedientes de primera generación** de i_0 son los elementos del subconjunto $D^1(i_0)$ dado por:

$$D^1(i_0) := \left\{ k \in \Lambda \mid e_{i_0}^2 = \sum_k \omega_{ki_0} e_k \text{ con } \omega_{ki_0} \neq 0 \right\}.$$

De una forma abreviada, $D^1(i_0) := \{j \in \Lambda \mid \omega_{ji_0} \neq 0\}$. Obsérvese que $j \in D^1(i_0)$ si y solo si, $\pi_j(e_{i_0}^2) \neq 0$ (donde π_j es la proyección canónica de A sobre $\mathbb{K}e_j$).

Análogamente, decimos que j es un **descendiente de segunda generación** de i_0 siempre que $j \in D^1(k)$ para algún $k \in D^1(i_0)$. Entonces,

$$D^2(i_0) = \bigcup_{k \in D^1(i_0)} D^1(k).$$

Por recurrencia definimos el conjunto de **descendientes de la generación emésima** de i_0 como

$$D^m(i_0) = \bigcup_{k \in D^{m-1}(i_0)} D^1(k).$$

Finalmente, el **conjunto de descendientes** de i_0 se define como el subconjunto de Λ dado por

$$D(i_0) = \bigcup_{m \in \mathbb{N}} D^m(i_0).$$

Por otro lado, decimos que $j \in \Lambda$ es un **ascendiente** de i_0 si $i_0 \in D(j)$, esto es, i_0 es un descendiente de j.

En términos de la teoría de grafos, estas definiciones se puden interpretar como: Sea E un grafo. Para un vértice $j \in E^0$ definimos:

$$D^m(j) := \{v \in E^0 \mid \text{ existe un camino } \mu \text{ tal que } |\mu| = m, s(\mu) = v_j, r(\mu) = v\}.$$

Dicho con palabras, los elementos de $D^m(j)$ son aquellos vértices v_j que conecta mediante un camino de longitud m. Se define también

$$D(j) = \bigcup_{m\in\mathbb{N}} D^m(j) = \{v \in E^0 \mid \text{ existe un camino } \mu \text{ tal que } s(\mu) = v_j, r(\mu) = v\}.$$

Cuando queramos recalcar el grafo E escribimos $D^m_E(j)$ y $D_E(j)$, respectivamente.

En el siguiente paso y haciendo uso de la representación de grafos y de la noción de descendiente, describimos los ideales generados por un elemento en un álgebra de evolución en la siguiente proposición:

Proposición 2.1.11

Sea A un álgebra de evolución con base natural $B = \{e_i \mid i \in \Lambda\}$.

(I) *Sea $k \in \Lambda$ tal que $e_k^2 \neq 0$.*

(a) $\mu^n_A(e_k^2) = \mathrm{lin}\{e_j^2 \mid j \in D^n(k)\}$, *para cada* $n \in \mathbb{N}$.

(b) $\langle e_k^2 \rangle = \mathrm{lin} \bigcup_{n=0}^{\infty} \mu^n_A(e_k^2)$.

(c) $\langle e_k^2 \rangle = \mathrm{lin}\{e_j^2 \mid j \in D(k) \cup \{k\}\}$.

(II) *Para cada $x \in A$,*

(a) $\mu^1_A(x) = \mathrm{lin}\{e_i^2 \mid i \in \Lambda^x\}$ *y para cada* $n \geq 2$, $\mu^n_A(x) = \mathrm{lin} \bigcup_{i\in\Lambda^x} \{e_j^2 \mid j \in D^{n-1}(i)\}$.

(b) $\langle x \rangle = \mathrm{lin} \bigcup_{n=0}^{\infty} \mu^n_A(x)$.

En este contexto, demostramos que la dimensión de este tipo de ideales como espacio vectorial es a lo sumo numerable:

Corolario 2.1.12

Sea A un álgebra de evolución. Entonces, para cada elementos $x \in A$ la dimensión del ideal generado por x es a lo sumo numerable.

Esto implica que cualquier álgebra de evolución simple tiene dimensión a lo sumo numerable como veremos en la siguiente sección.

La Sección 2.2 está dedicada al estudio y la caracterización de las álgebras de evolución simples. Recordamos que un álgebra A es **simple** si $A^2 \neq 0$ y 0 es el único ideal propio. En primer lugar, probamos dos resultados que caracterizan las álgebras de evolución simples, que, como dijimos, tienen dimensión a lo sumo numerable:

Proposición 2.2.1

Sea A un álgebra de evolución y sea $B = \{e_i \mid i \in \Lambda\}$ una base natural de A. Consideramos las siguientes afirmaciones:

(i) *A es simple.*

(ii) *A satisface las siguientes propiedades:*

  (a) *A es no degenerada.*

  (b) $A = \mathrm{lin}\{e_i^2 \mid i \in \Lambda\}$.

  (c) *Si $\mathrm{lin}\{e_i^2 \mid i \in \Lambda'\}$ es un ideal de A distinto de cero para un subconjunto $\Lambda' \subseteq \Lambda$ no vacío, entonces $|\Lambda'| = |\Lambda|$.*

Entonces: (i) $\Rightarrow$ (ii) *y* (ii) $\Rightarrow$ (i) *si $|\Lambda| < \infty$. Además, si A es un álgebra de evolución simple, entonces la dimensión de A es a lo sumo numerable.*

Se observa mediante un ejemplo que la hipótesis de que A tenga dimensión finita es necesaria (véase 2.2.2).

Teorema 2.2.7

Sea A un álgebra de evolución distinta de cero y sea $B = \{e_i \mid i \in \Lambda\}$ una base natural. Las siguientes condiciones son equivalentes:

(i) *A es simple.*

(II) *Si* $\text{lin}\{e_i^2 \mid i \in \Lambda'\}$ *es un ideal para un subconjunto* $\Lambda' \subseteq \Lambda$ *no vacío, entonces* $A = \text{lin}\{e_i^2 \mid i \in \Lambda'\}$.

(III) $A = \langle e_i^2 \rangle = \text{lin}\{e_j^2 \mid j \in D(i)\}$ *para cada* $i \in \Lambda$.

(IV) $A = \text{lin}\{e_i^2 \mid i \in \Lambda\}$ *y* $\Lambda = D(i)$ *para cada* $i \in \Lambda$.

También proporcionamos ejemplos que muestren que las condiciones de dicha caracterización son necesarias. Terminamos la sección con el estudio del caso particular de las álgebras de evolución simples finito dimensionales:

Corolario 2.2.6

Sea A *un álgebra de evolución de dimensión* n *y sea* $B = \{e_i \mid i \in \Lambda\}$ *una base natural de* A*. Entonces,* A *es simple si y solo si el determinante de la matriz de estructura* $M_B(A)$ *es distinto de cero y* B *no puede reordenarse de forma que la matriz de estructura correspondiente es como sigue:*

$$\begin{pmatrix} W_{m\times m} & U_{m\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix},$$

para algún $m \in \mathbb{N}$ *con* $m < n$ *y matrices* $W_{m\times m}$, $U_{m\times(n-m)}$ *y* $Y_{(n-m)\times(n-m)}$.

Partiendo de la observación de que la suma directa de álgebras de evolución es un álgebra de evolución, en la Sección 2.3 tratamos la cuestión de cuándo un álgebra de evolución distinta de cero A es **reducible**, es decir, se puede expresar como suma directa de subálgebras de evolución distintas de cero. En particular, un álgebra de evolución con un grafo asociado, el cual no es conexo, es reducible:

Proposición 2.3.4

Sea A *un álgebra de evolución distinta de cero y sea* E *su grafo asociado relativo a la base natural* $B = \{e_i \mid i \in \Lambda\}$.

(I) *Supongamos* $E = E_1 \sqcup E_2$, *donde* E_1 *y* E_2 *son subgrafos no vacíos de* E. *Escribimos* $E_k^0 = \{v_i \mid i \in \Lambda_k\}$, *para* $k = 1, 2$, *donde* $\Lambda_k \subseteq \Lambda$ *y* $\Lambda = \Lambda_1 \sqcup \Lambda_2$. *Entonces, existen ideales de evolución distintos de cero* I_1, I_2 *de* A *tal que* $A = I_1 \oplus I_2$ *son* E_1, E_2 *son los grafos asociados a las álgebras de evolución* I_1 *y* I_2, *respectivamente, relativa a su base natural* $B_k = \{e_i \mid i \in \Lambda_k\}$ *(para* $k = 1, 2$*). Además,* $B = B_1 \sqcup B_2$.

(II) *Sea* $E = \sqcup_{\gamma\in\Gamma} E_\gamma$ *la descomposición de* E *en sus componentes conexas. Para cada* $\gamma \in \Gamma$, *escribimos* $E_\gamma^0 = \{v_i \mid i \in \Lambda_\gamma\}$, *donde* $\Lambda_\gamma \subseteq \Lambda$ *y* $\Lambda = \sqcup_{\gamma\in\Gamma}\Lambda_\gamma$. *Entonces, existen ideales de evolucion* $\{I_\gamma\}_{\gamma\in\Gamma}$ *de* A, *tal que* $A = \oplus_{\gamma\in\Gamma} I_\gamma$ *y* E_γ *es el grafo asociado al álgebra de evolución* I_γ *relativa a la base natural* B_γ *descrita abajo. Además:*

(a) $B = \sqcup_{\gamma\in\Gamma} B_\gamma$, *donde* $B_\gamma = \{e_i \mid i \in \Lambda_\gamma\}$ *es una base natural de* I_γ, *para cada* $\gamma \in \Gamma$.

(b) I_γ *es un álgebra de evolución simple y sólo si* $I_\gamma = \mathrm{lin}\{e_i^2 \mid i \in \Lambda_\gamma\}$ *y* $D(i) = \Lambda_\gamma$ *para cada* $i \in \Lambda_\gamma$.

(c) A *es no degenerada si y solamente si cada* I_γ *es un álgebra de evolución no degenerada.*

En el siguiente teorema, caracterizamos la descomposición de un álgebra de evolución no degenerada en subálgebras (equivalentemente ideales) en términos de los elementos de una base natural cualquiera.

Teorema 2.3.6

Sea A *un álgebra de evolución no degenerada con* $B = \{e_i \mid i \in \Lambda\}$ *una base natural y supongamos que* $A = \oplus_{\gamma\in\Gamma} I_\gamma$, *donde cada* I_γ *es un ideal de* A. *Entonces:*

(I) *Para cada* $e_i \in B$ *existe un único* $\mu \in \Gamma$ *tal que* $e_i \in I_\mu$. *Además,* $e_i \in I_\mu$ *si y solo si* $e_i^2 \in I_\mu$.

(ii) *Existe una descomposición disjunta de* Λ, $\Lambda = \sqcup_{\gamma\in\Gamma}\Lambda_\gamma$, *tal que*

$$I_\gamma = \text{lin}\{e_i \mid i \in \Lambda_\gamma\}.$$

Teniendo en mente la estrecha relación entre teoría de grafos y álgebras de evolución, en el siguiente corolario probamos que un álgebra de evolución no degenerada es irreducible si el grafo asociado (relativo a una base natural) es conexo.

Corolario 2.3.8

Sea A *un álgebra de evolución no degenerada,* $B = \{e_i \mid i \in \Lambda\}$ *una base natural, y sea* E *su grafo asociado. Entonces,* A *es irreducible si y solo si* E *es un grafo conexo.*

Se observa en el Ejemplo 2.3.9 que la hipótesis de no degenerada no se puede eliminar.

Por supuesto, nuestro interés radica en determinar también cuándo cada componente de la descomposición en suma directa es irreducible. Éste va a ser el objetivo principal de la Sección 2.4. Es decir, tratamos de obtener una descomposición del álgebra de evolución en términos de suálgebras de evolución irreducibles, es lo que llamamos **óptima descomposición en suma directa**.

La descomposición $A = \oplus_{\gamma\in\Gamma} I_\gamma$ de un álgebra de evolución en ideales irreducibles existe y es única si el álgebra es no degenerada:

Teorema 2.4.2

Sea A *un álgebra de evolución no degenerada. Entonces,* A *admite una óptima descomposición en suma directa. Ademas, es única.*

Al igual que ocurría en otras ocasiones, si queremos asegurar la unicidad no podemos eliminar la hipótesis de no degenerada.

Para comprender mejor la estructura interna del álgebra de evolución empezamos definiendo algunos conceptos fundamentales como índice cíclico, lazo, ciclo asociado y ciclo: Sea $B = \{e_i \mid i \in \Lambda\}$ una base natural de un álgebra de evolución A. Decimos que $i_0 \in \Lambda$ es **cíclico** si $i_0 \in D(i_0)$. Esto significa que i_0 es descendiente (y por tanto ascendiente) de sí mismo.

En particular, si $D(i_0) = \{i_0\}$ (en cuyo caso $e_{i_0}^2 = \omega_{i_0 i_0} e_{i_0}$ para algún $\omega_{i_0 i_0} \in \mathbb{K} \setminus \{0\}$), entonces decimos que el índice cíclico i_0 es un **lazo**.

Si $i_0 \in \Lambda$ es cíclico, entonces el **ciclo asociado a** i_0 se define como el conjunto:

$$C(i_0) = \{j \in \Lambda \mid j \in D(i_0) \text{ and } i_0 \in D(j)\}.$$

Obsérvese que si i_0 es cíclico, entonces $C(i_0)$ es no vacío porque en particular contiene a i_0. Además, i_0 es un lazo si y solo si $C(i_0) = \{i_0\}$.

Decimos que un subconjunto $C \subseteq \Lambda$ es un **ciclo** si $C = C(i_0)$, para algún índice cíclico $i_0 \in \Lambda$.

A continuación, clasificamos los ciclos en dos tipos, si tienen o no ascendientes fuera del ciclo de la siguiente forma: Sea $B = \{e_i \mid i \in \Lambda\}$ una base natural de un álgebra de evolución A, y sea $i_0 \in \Lambda$ un índice cíclico. Decimos que i_0 es un **índice cíclico principal** si el conjunto de ascendientes de i_0 está contenido en $C(i_0)$, el ciclo asociado a i_0. Esto implica que $i_0 \in \Lambda$ es un índice cíclico principal si $i_0 \in D(i_0)$ y $j \in D(i_0)$ para cada $j \in \Lambda$ con $i_0 \in D(j)$.

Decimos que un subconjunto C de Λ es un **ciclo principal** si $C = C(i_0)$, para algún índice cíclico principal $i_0 \in \Lambda$.

Y ahora, distinguimos entre aquellos ciclos que tengan descendientes propios de aquellos que no lo tengan: Sea $B = \{e_i \mid i \in \Lambda\}$ una base natural de un álgebra de evolución A y sea S un subconjunto de Λ. Definimos el

conjunto de índices derivado de S como el conjunto dado por

$$\Lambda(S) := S \cup_{i \in S} D(i).$$

Por ejemplo, si $i \in \Lambda$, entonces el conjunto de índices derivado de $\{i\}$ es

$$\Lambda(\{i\}) := \{i\} \cup D(i),$$

donde $D(i)$ es el conjunto de descendientes de i.

Y, por último, definimos índice de inicio de cadena: Sea $B = \{e_i \mid i \in \Lambda\}$ una base natural de un álgebra de evolución A. Decimos que $i_0 \in \Lambda$ es un **índice inicio de cadena** si i_0 no tiene ascendientes, es decir, $i_0 \notin D(j)$, para cada $j \in \Lambda$. Equivalentemente, i_0 es un índice inicio de cadena si y solo si todos los elementos de la fila correspondiente a i_0 de la matriz de estructura $M_B(A)$ son cero.

Si consideramos el conjunto $\{C_1, \ldots, C_k\}$ de los ciclos principales de Λ y el conjunto $\{i_1, \ldots, i_m\}$ de todos los índices inicio de cadena de Λ, entonces dado un índice cualquiera que no es inicio de cadena $i \in \Lambda$ se tiene que existe $j \in \Lambda$ tal que $i \in D(j)$, y o j es un índice inicio de cadena o j pertenece a un ciclo principal. Por tanto, podemos escribir Λ como sigue:

$$\Lambda = \Lambda(C_1) \cup \cdots \cup \Lambda(C_k) \cup \Lambda(i_1) \cup \cdots \cup \Lambda(i_m).$$

A esta descomposición la llamamos **descomposición canónica** de Λ asociada a B.

Una vez asentadas las bases con los conceptos básicos, en la Sección 2.5 damos un proceso que permita descomponer un álgebra de evolución de dimensión finita en suma directa de subálgebras de evolución. Añadimos siguiente párrafo Para ello, necesitamos definir algunos conceptos relacionados con la expresión de un conjunto como unión de subconjuntos con ciertas características. Dado un conjunto finito Λ y dados $\Upsilon_1, \ldots, \Upsilon_n$ subconjuntos no vacíos de Λ tales que $\Lambda = \cup_{i=1}^n \Upsilon_i$. Decimos que $\Lambda = \cup_{i=1}^n \Upsilon_i$ es una **unión fragmentable** si existen subconjuntos disjuntos distintos del vacío Λ_1, Λ_2 de Λ satisfaciendo

$$\Lambda = \cup_{i=1}^n \Upsilon_i = \Lambda_1 \cup \Lambda_2,$$

y tal que para cada $i = 1, \ldots, n$, se tiene que o bien $\Upsilon_i \subseteq \Lambda_1$ o bien $\Upsilon_i \subseteq \Lambda_2$. Si $\Upsilon_i \cap \Upsilon_j \neq \emptyset$ para cada $i \neq j$, entonces se dice que la unión $\Lambda = \cup_{i=1}^n \Upsilon_i$ es no fragmentable.

Por otro lado, se tiene que una **fragmentación** de Λ es una unión $\Lambda = \cup_{i=1}^k \Lambda_i$ tal que:

(i) Si $i \in \{1, \ldots, k\}$ entonces $\Lambda_i = \cup_{j \in S_i} \Upsilon_j$ para S_i un subconjunto no vacío de $\{1, \ldots, n\}$.

(ii) $\Lambda_i \cap \Lambda_j = \emptyset$, para cada $i, j \in \{1, \ldots, k\}$, con $i \neq j$.

Si en dicha fragmentación $\Lambda = \cup_{i=1}^k \Lambda_i$ se verifica que para cada $i \in \{1, \ldots, k\}$ el conjunto de índices $\Lambda_i = \cup_{j \in S_i} \Upsilon_j$ es no fragmentable entonces se dice que es una **fragmentación óptima**.

Con el objetivo de conseguir la descomposición en suma directa de un álgebra de evolución de dimensión finita a través del proceso de fragmentación (Proposición 2.5.4), identificamos los ciclos principales y los índices de inicio de cadena. Esto proporcionará una óptima descomposición en suma directa cuando el álgebra de evolución sea no degenerada, tal y como muestra el siguiente teorema:

Teorema 2.5.5

Sea A un álgebra de evolución de dimensión finita con $B = \{e_i \mid i \in \Lambda\}$ una base natural. Sea $\{C_1, \ldots, C_k\}$ el conjunto de los ciclos principales de

Λ, $\{i_1, \ldots, i_m\}$, el conjunto de todos los índices de inicio de cadena de Λ y consideremos la descomposición canónica

$$(\dagger) \quad \Lambda = \Lambda(C_1) \cup \cdots \cup \Lambda(C_k) \cup \Lambda(i_1) \cup \cdots \cup \Lambda(i_m).$$

Sea $\Lambda = \sqcup_{\gamma\in\Gamma}\Lambda_\gamma$ la fragmentación óptima de $(\dagger)$ y descomponemos $B = \sqcup_{\gamma\in\Gamma}B_\gamma$, donde $B_\gamma = \{e_i \mid i \in \Lambda_\gamma\}$. Entonces, $A = \oplus_{\gamma\in\Gamma}I_\gamma$, para $I_\gamma = \operatorname{lin} B_\gamma$, el cual es un ideal de evolución de A. Además, si A es no degenerado, entonces $A = \oplus_{\gamma\in\Gamma}I_\gamma$ es la óptima descomposición en suma directa de A.

Como la óptima descomposición en suma directa de un álgebra de evolución no degenerada A es única, concluimos que, en el caso no degenerado, la descomposición obtenida en el Teorema 2.4.2 no depende de la base natural B prefijada.

Hemos añadido un apéndice a este trabajo en el que se muestra una rutina con Mathematica para obtener la fragmentación óptima de una base natural B. A partir de la cual conseguimos la descomposición en suma directa de un álgebra de evolución partiendo de su matriz de estructura, M_B. En este proceso, también calculamos (en el caso de que existan) valores como: los descendientes de la primera generación de un índice i, los descendientes de i de la generación enésima, ciclos de la matriz de estructura M_B, el conjunto $D(i)$, los índices cíclicos de M_B, el ciclo asociado a i, los ascendientes de i, los índices cíclicos principales, los índices de inicio de cadena, la descomposición canónica asociada a M_B y la fragmentación óptima asociada a M_B.

Como señalamos anteriormente, ya que las álgebras de evolución están estrechamente relacionadas con la Genética no Mendeliana, tanto en el primero como en el segundo capítulo hemos intentado interpretar algunos de los conceptos matemáticos tratados desde el punto de vista de la Genética.

Los siguientes dos capítulos están dedicados a la clasificación de las álgebras de evolución de dimensión dos y tres respectivamente. El propósito de obtener esta clasificación radica en poder aplicar los resultados conseguidos, en un futuro no muy lejano, al campo de la Biología y detectar posibles generalizaciones para álgebras de evolución de dimensión mayor.

Hay que destacar que las álgebras de evolución de dimensión dos sobre el cuerpo de los complejos están descritas en [12]. Sin embargo, observamos que el álgebra de evolución A con base natural $\{e_1, e_2\}$ dada por el producto $e_1^2 = e_2$ y $e_2^2 = e_1$ es un álgebra de evolución de dimensión dos y no isomorfa a ninguna de las que aparecen en [12]. En nuestro caso, la clasificación resultante está realizada sobre un cuerpo $\mathbb{K}$ donde para cada $k \in \mathbb{K}$ el polinomio $x^n - k$ tiene una raíz donde $n = 2, 3$.

El caso de dimensión tres es mucho más complicado que el caso de dimensión dos tal y como como se puede comprobar observando la extensión del Capítulo 4. Esta clasificación puede encontrarse en el artículo [5].

Nos gustaría reseñar que mientras elaborábamos este trabajo, se publicó en el artículo [16] la clasificación de las álgebras de evolución indescomponibles nilpotentes hasta dimensión cinco definidas sobre cuerpos algebraicamente cerrados de característica distinta de dos. El caso de dimensión tres puede localizarse en nuestra clasificación donde no es necesario considerar cuerpos álgebraicamente cerrados.

Para llevar a cabo esta clasificación trataremos con álgebras de evolución sobre un cuerpo $\mathbb{K}$ de característica distinta de dos y en el cual cada polinomio de la forma $x^n - \alpha$, para $n = 2, 3, 7$ y $\alpha \in \mathbb{K}$ tiene una raíz en el cuerpo.

Demostramos que existen 116 tipos de álgebras de evolución de dimensión tres. Todas ellas están clasificadas en las Tablas 1-24. Las matrices de estructura que aparecen en las diferetes tablas son no isomorfas (en el sentido de que ellas no generan la misma álgebra de evolución). Matrices de estructura colocadas en diferentes filas de una misma tabla tampoco son isomorfas. En general, para distintos valores de los parámetros que aparecen en las matrices de estructura dan álgebras de evolución no isomorfas, aunque en algunos casos esto no es verdad como se refleja en las Tablas $1'$-$23'$.

Comenzamos la Sección 4.2 introduciendo la noción de la acción del grupo $S_3 \rtimes (\mathbb{K}^\times)^3$ en $\mathrm{M}_3(\mathbb{K})$. Este resultado jugará un papel muy importante en el proceso de clasificación. Las órbitas de esta acción determinarán completamente las álgebras de evolución no isomorfas A cuando $\dim(A^2) = 3$ y en casos muy concretos cuando $\dim(A^2) = 2$.

Nuestro estudio está dividido en cuatro casos según la dimensión de A^2, que puede ser 0, 1, 2 o 3. El primer caso es trivial. El tercer y cuarto caso se han realizado teniendo en cuentas cuáles son las posibles matrices P de cambio de base. Ocurre que para dimension 3, como ya dijimos, las únicas matrices son las que pertenecen al grupo $S_3 \rtimes (\mathbb{K}^\times)^3$.

Cuando la dimensión de A^2 es 2, hay que distinguir tres grupo de casos (cuatro en realidad, pero dos de ellos son esencialmente el mismo). Sea $B = \{e_1, e_2, e_3\}$ una base natural de A tal que $\{e_1^2, e_2^2\}$ es una base de A^2 y $e_3^2 = c_1 e_1^2 + c_2 e_2^2$ para algún $c_1, c_2 \in \mathbb{K}$. En el primer grupo de casos $c_1 c_2 \neq 0$. Entonces, $P \in S_3 \rtimes (\mathbb{K}^\times)^3$. Si $c_1 = 0$ y $c_2 \neq 0$, entonces estamos en el segundo grupo de casos y la matriz P es id_3, $(2,3)$,[6] o bien la matriz Q dada por (4.25). El tercer grupo de casos se tiene cuando $c_1, c_2 = 0$. Entonces, la matriz P es id_3 o las matrices Q' y Q'' dadas por (4.27) y (4.29) respectivamente.

Para $P \in S_3 \rtimes (\mathbb{K}^\times)^3$, clasificamos teniendo en cuenta: la dimensión del anulador de A, el número de entradas distintas de cero en la matriz de estructura (el cual permanece invariante, como se prueba en la Proposición

[6]La matriz obtenida de la matriz identidad, id_3, intercambiando las filas segunda y tercera

4.1.2), y si el álgebra de evolución A satisface la Propiedad (2LI)[7].

Para $P \in \{\mathrm{id}_3, (2,3), Q\}$, obtenemos una primera clasificación dada por las diferentes Figuras. Entonces, comparamos qué matrices de estructura producen álgebras isomorfas y, eliminando redundancias, obtenemos las matrices dadas en el conjunto S que aparecen en el Teorema 4.2.2. De nuevo, alguna de estas matrices dan álgebras de evolución isomorfas. Para clasificarlas, consideramos que el número de entradas no nulas de las matrices en S es un invariante bajo la acción de la matriz P (véase Observación 4.2.3). Obsérvese que las matrices de estructura corresponden a álgebras de evolución con anulador cero y no satisfaciendo la Propiedad (2LI).

Para $P \in \{\mathrm{id}_3, Q', Q''\}$ realizamos la clasificación teniendo en cuenta que la tercerca columna de la matriz de estructura tiene tres entradas nulas (la dimension del anulador es una y, consecuentemente, no satisfacen la Propiedad (2LI)) y el número de ceros en la primera y la segunda filas permanece invariante bajo cambio de base (véase Observación 4.2.4).

Para $\dim(A^2) = 3$ la clasificación se lleva a cabo dependiendo del número de entradas no nulas en la matriz de estructura.

En el caso $\dim(A^2) = 1$ no es lo más eficiente plantear el problema de la clasificación calculando las posibles matrices de cambio de base, aunque por completar el estudio, las hemos determinado en la Observación 4.2.5. Por ello, seguimos un método diferente. La particularidad sobre la que centraremos el estudio será la Propiedad de Extensión [8] (lo denotamos como EP).

Por tanto, hemos clasificado teniendo en cuenta las siguientes propiedades: si A^2 tiene la propiedad de extensión, la dimensión del anulador de A, y si el álgebra de evolución tiene un ideal de evolución principal [9] de dimensión dos

[7]Para cualquier base $\{e_1, e_2, e_3\}$ el ideal A^2 tiene dimensión dos y está generado por $\{e_i^2, e_j^2\}$, para cada $i, j \in \{1,2,3\}$ con $i \neq j$.

[8]Existe una base natural de A^2 que puede extenderse a una base natural de A

[9]Principal signigica que está generado como ideal por un elemento.

degenerado como álgebra de evolución (lo denotamos por PD2EI).

La clasificación de las álgebras de evolución de dimensión tres está desarrollada en el Teorema 4.2.2. Por último, recogemos en las siguientes tablas el número de matrices de estructura que aparecen en los diferentes casos.

A^2 tiene EP	dim(ann(A))	A tiene un PD2EI	Número
No	0	Si	1
No	1	Si	1
Si	2	No	1
Si	1	No	1
Si	0	No	1
Si	2	Si	1
Si	1	Si	1

$\dim(A^2) = 1$

dim(ann(A))	Entradas no nulas * Entradas no nulas en S ** Entradas no nulas en las filas 1 y 2	A tiene la Propiedad (2LI)	Número
1	1**	No	2
1	2**	No	4
1	3**	No	2
1	4**	No	3
0	4*	No	3
0	5*	No	6
0	6*	No	3
0	7*	No	6
0	8*	No	3
0	9*	No	3
0	4	Si	4
0	5	Si	3
0	6	Si	7
0	7	Si	6
0	8	Si	2
0	9	Si	1

$\dim(A^2) = 2$

Entradas no nulas	Número
3	3
4	6
5	16
6	15
7	8
8	2
9	1

$\dim(A^2) = 3$

Acknowledgements
Agradecimientos

Ün sueño cumplido”podría ser el título de este trabajo. Todavía no me creo que hayan pasado casi cuatro años desde aquel momento en que Mercedes Siles Molina me propuso seguir con la investigación que habíamos comenzado en el Máster de Matemáticas. Han sido muchos momentos, muchas experiencias, muchas vivencias surgidas en este camino. Un camino que no ha estado exento de dificultades, miedos, desesperanzas y como no de alegrías. Al fin y al cabo un camino lleno de muchas emociones y mucho, mucho trabajo.

Resulta muy convincente cuando mi sobrino de 6 años me pregunta: Pero, tú vienes conmigo, ¿no? Gracias a mis directoras, yo no he tenido que hacer esta pregunta. Muchas gracias en primer lugar y por orden alfabético, a Mercedes. Por su estímulo, confianza, motivación y entusiasmo, porque se ha forjado una amistad que espero dure por muchos años. .Encontramos la horma de nuestro zapato”, como diría su marido, Juan Luis. Gracias a los dos por abrirme las puertas de vuestra casa, de vuestras vidas. Nunca olvidaré aquella celebración improvisada en el día de mi cumpleaños.

Y cómo no, quiero agradecer también a M. Victoria Velasco mi otro gran apoyo en este camino. Aunque en la distancia, sus palabras, aportaciones y su ánimo han jugado un papel muy importante. Un aire fresco y nuevo necesario para ir creciendo en la buena dirección. En este sentido, también quisiera dar las gracias por haberme dado la oportunidad de participar en el seminario

de los jueves, especialmente por las inestimables aportaciones de Cándido y Loli. Su alegría, cercanía y su ayuda desinteresada han enriquecido esta investigación. Y por último pero no menos importante, quiero acordarme de aquellos otros que andan por esta aventura de descubrimientos y conquistas matemáticas: aquellos mis compañeros al otro lado del charco. ¡Qué diferente hubieran sido esos tres meses de estancia sin vuestro ánimo, vuestra generosa ayuda y cómo no, las risas y los buenos momentos que compartimos! Mucha fortaleza para seguir en este barco a pesar de las inclemencias que puedan ocasionar la distancia y los desalientos.

Chapter 1

Basic facts about evolution algebra

The goal of this chapter is to introduce the notion of evolution algebra. We will show how far evolution algebras are from being associative. Actually we will prove that evolution algebras are not even power associative (unless they satisfy some very restrictive additional conditions that we will determine). We compute the formula for product and change of basis of an arbitrary algebra and in particular we obtain the expressions to the case of evolution algebras. We analyze in deep the notions of evolution subalgebras, ideals and non-degeneracy. In the last part of this chapter we associate a graph to any evolution algebra. The use of these graphs will allow to see in a more visual way properties of the evolution algebras.

1.1. Preliminaries

In order to understand the interaction history between evolution algebras and genetics (in fact, this new algebras have arisen from study of non-Mendelian inheritance) it is necessary to briefly present some preliminary concepts of molecular genetics.

In every living thing there exists a substance referred to as the genetic material. Except in certain viruses, this material is composed of the nucleic acid, **DNA**. A molecule of DNA is organized into units called **genes**, which direct all metabolic activities of cells and we will define as an inherited factor that determines a characteristic. Genes control the characteristics that an offspring will have and they are organized into **chromosomes**, structures that serve as the vehicle for transmission of genetic information. The particular location of a gene on a chromosome is referred to as the gene's **locus** (plural loci). The cells of each species have a characteristic number of chromosomes; for example, bacterial cells normally posses a single chromosome; human somatic (non-sexuals) cells possess 46. There appears to be no special relation between the complexity of an organism and its number of chromosomes per cell. There exists two types of eukaryote cells depend on the number of chromosome sets found in the nucleus: Haploid cells and diploid cells. **Haploid cells** are cells that contain only one complete set of chromosomes. The most common type of haploid cells is **gametes**, or sex cells. Haploid cells are produced by meiosis. They are genetically diverse cells that are used in **sexual reproduction**. When the haploid gametes, one from the male parent and the other from the female parent fuse and are fertilized, the offspring has a complete set of chromosomes and becomes a **diploid cell** called **zygote**. Thus, in diploid cells, the chromosomes come in pairs called **homologous chromosomes**. These are same in length, genes and position of the centromere, the point of spindle fiber attachment during division. In humans, the haploid cells have 23 chromosomes, versus the 46 in the diploid cells (23 pairs of homologous chromosomes). The new cell then divides over and over again by mitosis. This creates the many cells that eventually form a new individual.

Since we have two copies of each chromosome, we have two copies of each gene. Our genes give rise to traits, or observable characteristics, like height, eye color and so forth. Alternative forms of the same gene are called **alleles**. Alleles are commonly represented by letters: for example, for a gene related to the albinism trait, the alleles could be called A and a. In a population of members of the same species, many different alleles of the same gene may exist. The set of alleles that an individual organism possesses are called genotype. This genetic constitution of an individual influences but is not solely responsible for many of its traits. The phenotype is the visible or expressed trait, such as hair color. The phenotype depends upon the genotype but can also be influenced by environmental factors, for example, human height is affected by many genes as well as by factors such as nutrition. If an organism has two copies of the same allele, for example AA or aa, it is homozygous for that trait. If the organism has one copy of two different alleles, for example Aa, it is heterozygous. Alleles can be dominant or recessive. A dominant allele takes precedence over a recessive allele. For example, the allele that gives rise to albinism is recessive, and the allele for normal pigment production is dominant. That means that a heterozygous individual with the genotype Aa, who has one copy of the normal allele and one copy of the albinism allele, would not be an albino. This is because since A is dominant, one copy of A is enough to give the normal phenotype. An obvious important concept is that only the genotype is inherited. Although the phenotype is determined, at least to some extent, by genotype, organisms do not transmit their phenotypes to the next generation.

Algebras in genetics originate from the work of I.M.H. Etherington [17], who put the Mendelian laws into an algebraic form. Consider an infinitely large, random mating population of diploid individuals which differ genetically at one or several loci. Let $a_1, \ldots, a_n$ be the genetically distinct gametes produced by this population. The state of the population can be described by a vector of gamete frequencies. The union of gametes a_i and a_j forms a zygote $a_i a_j$, which can produce a gamete a_k. In absence of selection, one can define the segregation rates. The segregation rate γ_{ijk} is the probability that the zygote $a_i a_j$ produces a gamete a_k. By definition the γ_{ijk}'s satisfy the following relations

$$0 \leq \gamma_{ijk} \leq 1 \;\; i, j, k = 1, \ldots, n.$$

$$\sum_{k=1}^{n} \gamma_{ijk} = 1 \;\; i, j = 1, \ldots, n.$$

Consider the gametes $a_1, \ldots, a_n$ as abstract elements, free over the field $\mathbb{R}$. They span an n-dimensional vector space

$$F := \left\{ \sum_{k=1}^{n} \alpha_i a_i \,|\alpha_i \in \mathbb{R} \;\; i = 1, \ldots, n \right\}$$

Using the segregation rates γ_{ijk} one can define a multiplication in F by

$$a_i a_j = \sum_{k=1}^{n} \gamma_{ijk} a_k \;\; i, j = 1, \ldots, n$$

extended bilinearly onto $F \times F$. Like this, F is equipped with a commutative algebra structure. It is called the **gametic algebra**.

A special case of gametic algebras is the case the gametic algebra for simple Mendelian inheritance. Assume that all zygotes have equal fertility and that there is no mutation, i.e., homozygotes $a_i a_i$ produce gametes a_i only and heterozygotes $a_i a_j$ $(i \neq j)$ produce gametes a_i and a_j in equal proportions. So we can write the segregation rates as

$$\gamma_{ijk} = \frac{1}{2}(\delta_{ik} + \delta_{jk}) \;\; i, j, k = 1, \ldots, n$$

where δ_{rs} is the Kronecker delta.

As a natural example, we consider simple Mendelian inheritance for a single gene with two alleles E and e. So, multiplication table of the gametic algebra for simple Mendelian inheritance is the following

	E	e
E	E	$\frac{1}{2}(E+e)$
e	$\frac{1}{2}(e+E)$	e

However, when we consider the asexual inheritance, the interpretation $a_i a_j$ as a zygote does not make sense biologically if $a_i \neq a_j$. But, $a_i a_i = a_i^2$ can still be interpreted as self-replication. Therefore, in asexual inheritance, we can use the following relations to define an algebra

$$\begin{cases} a_i a_i = \sum\limits_{k=1}^{n} \gamma i k a_k \\ a_i a_j = 0, i \neq j. \end{cases}$$

where ω_{ki} is a positive number that can be interpreted as the rate of the genotype e_k produced by the genotype e_i (see [30, pp. 9, 10]). Therefore, as pointed out in [30], evolution algebra theory models all the non-Mendelian inheritance phenomena.

This is the case, for example, of the bacterial species Escherichia coli because their reproduction is asexual. In particular, evolution algebras model population Genetics (which is the study of the frequency and interaction of alleles and genes in populations) of organelles (specialized subunits within a cell that have a specific function) as well as organisms such as the Phytophthora infestans (an oomycete that causes the serious potato disease known as late blight or potato blight, and which also infects tomatoes and some other members of the Solanaceae).

1.2. Evolution algebras

Before introducing evolution algebras we establish in a precise way what we mean by an algebra. An **algebra** is a vector space A over a field $\mathbb{K}$, provided with a bilinear map $A \times A \to A$ given by $(a, b) \mapsto ab$, called the **multiplication** or the **product** of A. An algebra A such that $ab = ba$ for every $a, b \in A$ will be called **commutative**. If $(ab)c = a(bc)$ for every $a, b, c \in A$, then we say that A is **associative**. Therefore in contrast with texts like [3], [13], [24], [26] and many others we do not assume that an algebra is associative, unless it be specifically stated. We recall that an algebra A is **flexible** if $a(ba) = (ab)a$ for every $a, b \in A$. **Power associative** algebras are those such that the subalgebra generated by an element is associative. Particular cases of flexible algebras are the commutative and also the associative ones. It is known (see [6]) that an algebra A over a field $\mathbb{K}$ of characteristic different from zero is power associative if and only if

$$\begin{aligned} x^2x &= xx^2 \\ x^3x &= x^2x^2. \end{aligned} \tag{1.1}$$

for every $x \in A$.

Let us start by introducing the basic facts about evolution algebras. As already mentioned, evolution algebras has been used to describe the non-Mendelian inheritance. For this reason, we have tried to translate the mathematical concepts of this section into genetic meaning.

Definitions 1.2.1

An **evolution algebra** over a field $\mathbb{K}$ is a $\mathbb{K}$-algebra A provided with a basis $B = \{e_i \mid i \in \Lambda\}$ such that $e_ie_j = 0$ whenever $i \neq j$. Such a basis B is called a **natural basis**. Fixed a natural basis B in A, the scalars $\omega_{ki} \in \mathbb{K}$ such that $e_i^2 := e_ie_i = \sum_{k\in\Lambda} \omega_{ki}e_k$ will be called the **structure constants** of A **relative**

to B, and the matrix $M_B := (\omega_{ki})$ is said to be the **structure matrix of** A **relative to** B. We will write $M_B(A)$ to emphasize the evolution algebra we refer to. Observe that $|\{k \in \Lambda | \ \omega_{ki} \neq 0\}| < \infty$ for every i. Therefore M_B is a matrix in $\mathrm{CFM}_\Lambda(\mathbb{K})$, where $\mathrm{CFM}_\Lambda(\mathbb{K})$ is the vector space of those matrices (infinite or not) over $\mathbb{K}$ of size $\Lambda \times \Lambda$ for which every column has at most a finite number of non-zero entries.

Note that an n-finite dimensional algebra A is an evolution algebra if and only if there is a basis $B = \{e_1, ..., e_n\}$ relative to which the multiplication table is diagonal:

	e_1	...	e_n
e_1	$\sum_{k=1}^{n} \omega_{k1} e_k$	0	0
$\vdots$	0	$\sum_{k=1}^{n} \omega_{ki} e_k$	0
e_n	0	0	$\sum_{k=1}^{n} \omega_{kn} e_k$

In this case, the structure matrix of the evolution algebra A relative to the natural basis B is the following one:

$$M_B = \begin{pmatrix} \omega_{11} & \cdots & \omega_{1n} \\ \vdots & \ddots & \vdots \\ \omega_{n1} & \cdots & \omega_{nn} \end{pmatrix} \in \mathrm{M_n}(\mathbb{K}).$$

Every evolution algebra is uniquely determined by its structure matrix: if A is an evolution algebra and B a natural basis of A, there is a matrix, M_B, associated to B which represents the product of the elements in this basis. Conversely, fixed a basis $B = \{e_i \mid i \in \Lambda\}$ of a $\mathbb{K}$ vector space A, each matrix in $\mathrm{CFM}_\Lambda(\mathbb{K})$ defines a product in A under which A is an evolution algebra and B is a natural basis.

Remark 1.2.2

Note that by definition, every evolution algebra is commutative and, hence, *flexible. However, evolution algebras are not associative in general. Further, evolution algebras are not power associative in general. Indeed, we provide with an example of evolution algebra which it is not power associative. Let A the evolution algebra with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_1$ and $e_2^2 = e_1$. Then $e_2^2 e_2^2 = e_1 e_1 = e_1$ and on the other hand $(e_2^2 e_2)e_2 = (e_1 e_2)e_2 = 0$.*

Hence, evolution algebras are not, in general, Jordan or alternative. Evolution algebras are not Lie algebras either.

Now, we are going to study which are the conditions for an evolution algebra A over a field $\mathbb{K}$ with characteristic different from zero in order to be power associative. We will use the characterization (1.1). On the one hand we have

$$e_i^2 e_i^2 = \left(\sum_{k\in\Lambda} \omega_{ki} e_k\right) \sum_{j\in\Lambda} \omega_{ji} e_j = \sum_{k\in\Lambda} \omega_{ki}^2 e_k^2 = \sum_{k\in\Lambda} \omega_{ki}^2 \sum_{j\in\Lambda} \omega_{jk} e_j = \sum_{j,k\in\Lambda} \omega_{ki}^2 \omega_{jk} e_j.$$

and on the other hand

$$(e_i^2 e_i)e_i = \left(\left(\sum_{k\in\Lambda} \omega_{ki} e_k\right) e_i\right) e_i = \omega_{ii} e_i^2 e_i = \omega_{ii} \left(\sum_{k\in\Lambda} \omega_{ki} e_k\right) e_i = \omega_{ii}^2 e_i^2 = \sum_{j\in\Lambda} \omega_{ii}^2 \omega_{ji} e_j.$$

Thus the only evolution algebras over a field $\mathbb{K}$ with characteristic different from zero which are power associative are those such that $\sum_{k\in\Lambda} \omega_{jk}\omega_{ki}^2 = \omega_{ii}^2 \omega_{ji}$ for every $i, j, k \in \mathbb{K}$. Consequently in terms of matrices, we obtain that an evolution algebra over a field $\mathbb{K}$ with characteristic different from zero is power associative if and only if $M_B M_B^{(2)} = M_B D$ where $M_B^{(2)} = (\omega_{ij}^2)$ and $D = (d_{ij})$ with $d_{ij} = 0$ if $i \neq j$ and $d_{ii} = \omega_{ii}^2$ for every $i \in \mathbb{K}$.

We said at the beginning of this chapter that evolution algebras are the language of non-Mendelian Genetics. In this terms, according to [30], the product e_ie_i, where $B = \{e_1, \dots, e_n\}$ is natural basis, mimics the self-reproduction of alleles in these type of Genetics.

Since genetics algebras models non-Mendelian Genetics, a natural question that arises is if the class of algebras modeling Mendel's laws are included in the class of evolution algebras. The answer is no, as the next example shows. More precisely we will see that the zygotic algebra for simple Mendelian inheritance for one gene with two alleles, E and e, is not an evolution algebra. For this algebra, according to Mendel laws, zygotes have three possible genotypes, namely: EE, Ee and ee. The rules of simple Mendelian inheritance are expressed in the multiplication table included in the example that follows (see [25] for details and similar examples of algebras following Mendel's laws).

Example 1.2.3

Consider the vector space generated by the basis $B = \{EE, Ee, ee\}$ provided with the multiplication table given by:

	EE	Ee	ee
EE	EE	$\frac{1}{2}EE + \frac{1}{2}Ee$	Ee
Ee	$\frac{1}{2}EE + \frac{1}{2}Ee$	$\frac{1}{4}EE + \frac{1}{4}ee + \frac{1}{2}Ee$	$\frac{1}{2}ee + \frac{1}{2}Ee$
ee	Ee	$\frac{1}{2}ee + \frac{1}{2}Ee$	ee

We claim that this is not an evolution algebra.

Proof of the claim. We will see that this algebra does not have a natural basis. Suppose on the contrary that there exists a natural basis $B' = \{e_i \mid i \in \{1,2,3\}\}$. For each $i \in \{1,2,3\}$ we may write $e_i = \alpha_{1i}EE + \alpha_{2i}Ee + \alpha_{3i}ee$.

Since $e_ie_j = 0$ for each $i, j \in \{1,2,3\}$, with $i \neq j$, we have that:

$$\begin{aligned}
4\alpha_{1j}\alpha_{1i} + 2\alpha_{1j}\alpha_{2i} + 2\alpha_{2j}\alpha_{1i} + \alpha_{2j}\alpha_{2i} &= 0\\
\alpha_{1i}\alpha_{2j} + 2\alpha_{1i}\alpha_{3j} + \alpha_{2i}\alpha_{1j} + \alpha_{2i}\alpha_{2j} + \alpha_{2i}\alpha_{3j} + 2\alpha_{3i}\alpha_{1j} + \alpha_{3i}\alpha_{2j} &= 0\\
4\alpha_{3i}\alpha_{3j} + 2\alpha_{3i}\alpha_{2j} + 2\alpha_{2i}\alpha_{3j} + \alpha_{2i}\alpha_{2j} &= 0
\end{aligned}$$

for every $i, j \in \{1, 2, 3\}$.

We can express these three equations as:

$$\begin{aligned} (2\alpha_{1i} + \alpha_{2i})(2\alpha_{1j} + \alpha_{2j}) &= 0 \\ (2\alpha_{3i} + \alpha_{2i})(2\alpha_{3j} + \alpha_{2j}) &= 0 \\ \alpha_{1i}\alpha_{2j} + 2\alpha_{1i}\alpha_{3j} + \alpha_{2i}\alpha_{1j} + \alpha_{2i}\alpha_{2j} + \alpha_{2i}\alpha_{3j} + 2\alpha_{3i}\alpha_{1j} + \alpha_{3i}\alpha_{2j} &= 0 \end{aligned}$$

Since these identities hold for every $i, j \in \{1, 2, 3\}$, the only option is that there exist m, n, $s \in \{1, 2, 3\}$, with $m \neq n$ and $m \neq s$, such that:

$$\left\{ \begin{aligned} 2\alpha_{1m} + \alpha_{2m} &= 0 \\ 2\alpha_{1n} + \alpha_{2n} &= 0 \\ 2\alpha_{3m} + \alpha_{2m} &= 0 \\ 2\alpha_{3s} + \alpha_{2s} &= 0 \end{aligned} \right. \tag{1.2}$$

Now, we distinguish two cases:

Case 1 $n \neq s$.

From (1.2) we obtain that:

$$\begin{aligned} \alpha_{1m} &= \alpha_{3m} \\ \alpha_{2m} &= -2\alpha_{1m} \\ \alpha_{2n} &= -2\alpha_{1n} \\ \alpha_{2s} &= -2\alpha_{3s} \end{aligned}$$

It follows:

$$\begin{aligned} v_m &= \alpha_{1m}e_1 - 2\alpha_{1m}e_2 + \alpha_{3m}e_3 \\ v_n &= \alpha_{1n}e_1 - 2\alpha_{1n}e_2 + \alpha_{3n}e_3 \\ v_s &= \alpha_{1s}e_1 - 2\alpha_{3s}e_2 + \alpha_{3s}e_3 \end{aligned}$$

On the other hand, if we take $i = n$ and $j = s$ in (1), then

$$\alpha_{1n}\alpha_{2s} + 2\alpha_{1n}\alpha_{3s} + \alpha_{2n}\alpha_{1s} + \alpha_{2n}\alpha_{2s} + \alpha_{2n}\alpha_{3s} + 2\alpha_{3n}\alpha_{1s} + \alpha_{3n}\alpha_{2s} = 0;$$

this means that $\alpha_{1n} = \alpha_{3n}$ or $\alpha_{3s} = \alpha_{1s}$. In any case, this is impossible due to the fact that v_m, v_n and v_s are linearly independent.

Case 2 $n = s$.

Then,

$$\begin{aligned} \alpha_{1m} &= \alpha_{3m} \\ \alpha_{1n} &= \alpha_{3n} \\ \alpha_{2m} &= -2\alpha_{1m} \\ \alpha_{2n} &= -2\alpha_{1n} \end{aligned}$$

which is impossible because v_m and v_n are linearly independent.

□

1.3. Product and change of basis

In this section we study the product in an arbitrary algebra (which does not need to be an evolution algebra) by considering the matrices associated to the product by any element in a fixed basis. We specialize to the case of evolution algebras and obtain the relationship for two structure matrices of the same evolution algebra relative to different basis.

In [4] this study is already done for an finite dimensional evolution algebra.

The product of an algebra

Let A be a $\mathbb{K}$-algebra. Let $B = \{e_i \mid i \in \Lambda\}$ be a basis of A. We may assume that for any $x \in A$ there exists a finite subset $\Lambda_x \subseteq \Lambda$ such that $x = \sum_{i\in\Lambda} \alpha_i e_i = \sum_{i\in\Lambda_x} \alpha_i e_i$ with $\alpha_i \in \mathbb{K}$, that is $\alpha_i = 0$ for every $i \in \Lambda \setminus \Lambda_x$. In what follows we will use indistinctly $x = \sum_{i\in\Lambda} \alpha_i e_i$ and $x = \sum_{i\in\Lambda_x} \alpha_i e_i$ according with our interest.

Let $\{\omega_{kij}\}_{i,j,k\in\Lambda} \subseteq \mathbb{K}$ be the **structure constants**, i.e., for each pair $(i,j) \in \Lambda \times \Lambda$, $e_i e_j = \sum_{k\in\Lambda} \omega_{kij} e_k$. Note that for every $i, j \in \Lambda$, we have that $\omega_{kij} = 0$, for every $k \in \Lambda \setminus \Lambda_{e_i e_j}$.

For any element $a \in A$ the following linear map

$$\begin{aligned} \lambda_a : \ A &\to A \\ x &\mapsto ax \end{aligned}$$

is called the **left multiplication operator** by a.

For any linear map any linear map $T : A \to A$ we write $M_B(T) = (\omega_{ij})$ to denote the matrix in $\mathrm{CFM}_\Lambda(\mathbb{K})$ associated to T relative to the basis B i.e., $T(e_i) = \sum_{j\in\Lambda} \omega_{ji} e_j$. Therefore, for every $i \in \Lambda$ we have $M_B(\lambda_{e_i}) = (\omega_{kij})$.

Now, we will compute the product of two arbitrary elements of A.

Let $x = \sum_{i\in\Lambda_x} \alpha_i e_i$ and $y = \sum_{i\in\Lambda_y} \beta_i e_i$ be two any elements of A. Denote by $\xi_B(x)$ and $\xi_B(y)$ the vector $\Lambda \times 1$ give by the coordinates of x and y respect to B respectively. Note that the matrix product $M_B(\lambda_{e_i})\xi_B(x)$ makes sense because the columns of $M_B(\lambda_{e_i})$ as well as that of $\xi_B(x)$ have only a finite number of non-zero entries. Then:

$$xy = \left(\sum_{i\in\Lambda_x} \alpha_i e_i\right)\left(\sum_{j\in\Lambda_y} \beta_j e_j\right) = \sum_{(i,j)\in\Lambda_x\times\Lambda_y} \alpha_i\beta_j e_i e_j = \sum_{(i,j)\in\Lambda_x\times\Lambda_y} \left(\alpha_i\beta_j \sum_{k\in\Lambda_{e_ie_j}} \omega_{kij} e_k\right)$$
$$= \sum_{(i,j,k)\in\Lambda_x\times\Lambda_y\times\Lambda_{e_ie_j}} \alpha_i\beta_j\omega_{kij} e_k.$$

Thus

$$\xi_B(xy) = \sum_{i\in\Lambda} M_B(\lambda_{e_i})\alpha_i\xi_B(y) = \sum_{i,j\in\Lambda} (\omega_{kij})_{\Lambda\times\Lambda} (\alpha_i\beta_j)_{\Lambda\times 1}. \tag{1.3}$$

In case of A being an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis of A, the structure constants satisfy that $\omega_{kij} = 0$ for every $k, i, j \in \Lambda$ with $i \neq j$. If we denote $\omega_{kii} = \omega_{ki}$ we obtain that $(\omega_{kij})_{(k,j)\in\Lambda\times\Lambda}$ is a matrix such that all its columns consist in zero entries unless the i-th one whose entries are ω_{ki} for $k \in \Lambda$. Because of this $(\alpha_i\beta_j)_{\Lambda\times 1}$ can be replaced by a column vector where all its entries are zero except the i-th one whose value is $\alpha_i\beta_j$. But by considering a matrix $\Lambda\times\Lambda$ whose i-th column be $(\omega_{ki})_{k\in\Lambda}$ and a matrix $\Lambda\times 1$ whose j-th file be $\alpha_j\beta_j$ we obtain that

$$\xi_B(xy) = (\omega_{ki})_{\Lambda\times\Lambda} (\alpha_i\beta_i)_{\Lambda\times 1}. \tag{1.4}$$

This fact justify the following definition.

Definition 1.3.1

Let A be an algebra and $B = \{e_i \mid i \in \Lambda\}$ a basis of A. For arbitrary elements $x = \sum_{i \in \Lambda_x} \alpha_i e_i$ and $y = \sum_{i \in \Lambda_y} \beta_i e_i$ in A for certain $\Lambda_x, \Lambda_y \subseteq \Lambda$, we define

$$x \bullet_B y = \left(\sum_{i \in \Lambda_x} \alpha_i e_i \right) \bullet_B \left(\sum_{i \in \Lambda_y} \beta_i e_i \right) := \sum_{i \in \Lambda_x \cap \Lambda_y} \alpha_i \beta_i e_i.$$

Now, in the case of an evolution algebra we may write (1.4) as follows.

$$\xi_B(xy) = M_B \left(\xi_B(x) \bullet_B \xi_B(y) \right), \tag{1.5}$$

where, by abuse of notation, we write $\bullet_B$ to multiply two matrices, by identifying the matrices with the corresponding vectors and multiplying them as in Definition 1.3.1.

Change of basis

First, we study the relationship between two structure matrices of the same algebra relative to different basis. Then we specialize in the case of evolution algebras.

Let $B = \{e_i \mid i \in \Lambda\}$ and $B' = \{f_j \mid j \in \Lambda\}$ be two bases of an algebra A. Suppose that the relation between these bases is given by

$$f_i = \sum_{k \in \Lambda_{f_i}} p_{ki} e_k \quad \text{and} \quad e_i = \sum_{k \in \Lambda_{e_i}} q_{ki} f_k,$$

where for every i, $\Lambda_{f_i}, \Lambda_{e_i} \subseteq \Lambda$ so that $P_{B'B} := (p_{ki})$ and $P_{BB'} := (q_{ki})$ are the change of basis matrices. Assume that the structure constants of A relative to B and to B' are, respectively, $\{\varpi_{kij}\}_{i,j,k \in \Lambda}$ and $\{\omega_{kij}\}_{i,j,k \in \Lambda}$. Then, for every

$i, j \in \Lambda$:

$$f_i f_j = \left(\sum_{k\in\Lambda} p_{ki} e_k\right)\left(\sum_{t\in\Lambda} p_{tj} e_t\right) = \sum_{k,t\in\Lambda} p_{ki}p_{tj} e_k e_t = \sum_{m,k,t\in\Lambda} p_{ki}p_{tj}\varpi_{mkt} e_m$$
$$= \sum_{m,k,t,l\in\Lambda} p_{ki}p_{tj}\varpi_{mkt}q_{lm} f_l = \sum_{l\in\Lambda}\left(\sum_{m,k,t\in\Lambda}(p_{ki}p_{tj}\varpi_{mkt}q_{lm})\right) f_l = \sum_{l\in\Lambda}\omega_{lij} f_l.$$

Therefore, $\sum_{m,k,t\in\Lambda}(p_{ki}p_{tj}\varpi_{mkt}q_{lm}) = \omega_{lij}$.

Our next aim is to express every ω_{lij} in terms of certain matrices. To find such matrices, we assume that Λ is at most countable. Then, we write:

$$\begin{aligned}\omega_{lij} &= p_{1i}p_{1j}\varpi_{111}q_{l1} + \ldots + p_{1i}p_{1j}\varpi_{m11}q_{mn} + \ldots\\ &\vdots\\ &+ p_{1i}p_{tj}\varpi_{11t}q_{l1} + \ldots + p_{1i}p_{nj}\varpi_{m1t}q_{lm} + \ldots\\ &\vdots\\ &+ p_{ki}p_{1j}\varpi_{1k1}q_{l1} + \ldots + p_{ki}p_{1j}\varpi_{mk1}q_{lm} + \ldots\\ &\vdots\\ &+ p_{ki}p_{tj}\varpi_{1kt}q_{l1} + \ldots + p_{ki}p_{tj}\varpi_{mkt}q_{lm} + \ldots\\ &\vdots\end{aligned}$$

In terms of matrices,

$$\begin{aligned}\omega_{lij} = {} & \begin{pmatrix} q_{l1} & \cdots & q_{lm} & \cdots \end{pmatrix}\begin{pmatrix}\varpi_{111} & \cdots & \varpi_{11t} & \cdots\\ \vdots & \ddots & \vdots & \ddots\\ \varpi_{m11} & \cdots & \varpi_{m1t} & \cdots\\ \vdots & \ddots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}p_{1i}p_{1j}\\ \vdots\\ p_{1i}p_{tj}\\ \vdots\end{pmatrix}\\ & + \cdots\\ & + \begin{pmatrix} q_{l1} & \cdots & q_{lm} & \cdots \end{pmatrix}\begin{pmatrix}\varpi_{1k1} & \cdots & \varpi_{1kt} & \cdots\\ \vdots & \ddots & \vdots & \ddots\\ \varpi_{mk1} & \cdots & \varpi_{mkt} & \cdots\\ \vdots & \ddots & \vdots & \ddots\end{pmatrix}\begin{pmatrix}p_{ki}p_{1j}\\ \vdots\\ p_{ki}p_{tj}\\ \vdots\end{pmatrix}\\ & + \cdots\end{aligned}$$

This is equivalent to:

$$
\begin{aligned}
M_{B'}(\lambda_{f_i}) = & \begin{pmatrix} q_{11} & \cdots & q_{1m} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ q_{l1} & \cdots & q_{lm} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \varpi_{111} & \cdots & \varpi_{11t} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \varpi_{m11} & \cdots & \varpi_{m1t} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{1i}p_{11} & \cdots & p_{1i}p_{1j} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ p_{1i}p_{t1} & \cdots & p_{1i}p_{tj} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix} \\
& + \cdots \\
& + \begin{pmatrix} q_{11} & \cdots & q_{1m} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ q_{l1} & \cdots & q_{lm} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} \varpi_{1k1} & \cdots & \varpi_{1kt} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ \varpi_{mk1} & \cdots & \varpi_{mkt} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_{ki}p_{11} & \cdots & p_{ki}p_{1j} & \cdots \\ \vdots & \ddots & \vdots & \ddots \\ p_{ki}p_{t1} & \cdots & p_{ki}p_{tj} & \cdots \\ \vdots & \ddots & \vdots & \ddots \end{pmatrix} \\
& + \cdots \\
& = P_{B'B} \left(\sum_k M_B(\lambda_{e_k}) p_{ki} \right) P_{BB'}, \qquad (1.6)
\end{aligned}
$$

Observe that this matrices are well defined because on the one hand if we fix $i, j \in \Lambda$, then $p_{ki}p_{tj} = 0$ for every $(k, t) \in (\Lambda \times \Lambda) \setminus (\Lambda_{f_i} \times \Lambda_{f_j})$ and on the other hand if we fix $(k, t) \in (\Lambda \times \Lambda) \setminus (\Lambda_{f_i} \times \Lambda_{f_j})$, $\varpi_{mkt} = 0$ for every $m \in \Lambda \setminus \Lambda_{f_i f_j}$.

As we said before, it is easy to check that the formula (1.6) is verified for an evolution algebra of arbitrary dimension.

We finish the section by asserting the relationship among two structure matrices associated to the same evolution algebra relative to different bases. We include the proof of Theorem 1.3.2 for completeness. The ideas we have used can be found in [30, Section 3.2.2.].

Theorem 1.3.2

Let A be an evolution algebra and let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of A with structure matrix $M_B = (\omega_{ij})$. Then:

(I) *If $B' = \{f_i \mid i \in \Lambda\}$ is a natural basis of A such that $P_{B'B} = (p_{ij})$ and $P_{BB'} = (q_{ij})$ are the change of basis matrices, then:*

$$M_B(\xi_B(f_i) \bullet_B \xi_B(f_j)) = 0 \qquad (1.7)$$

for every $i \neq j$ *with* $i, j \in \Lambda$.

Moreover,

$$M_{B'} = P_{BB'} M_B P^{(2)}_{B'B},$$

where $P^{(2)}_{B'B} = (p^2_{ij})$.

(II) *Assume that* $P = (p_{ij}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$ *is invertible and satisfies the relations in* (1.7). *Define* $B' = \{f_i \mid i \in \Lambda\}$, *where* $f_i = \sum_{j\in\Lambda} p_{ji} e_j$ *for every* $i \in \Lambda$. *Then* B' *is a natural basis and* (1.8) *is satisfied.*

Demostración. (i). Clearly, since B and B' are two bases of A then $P_{B'B}$ is invertible. Besides, since B and B' are natural bases, by (1.5) we have:

$$\xi_B(f_i f_j) = M_B(\xi_B(f_i) \bullet_B \xi_B(f_j)) = 0$$

and

$$\xi_B(f_i^2) = M_B(\xi_B(f_i) \bullet_B \xi_B(f_i))$$

for every $i, j \in \Lambda$, being $i \neq j$.

On the other hand, recall that $\xi_{B'}(x) = P_{BB'}\xi_B(x)$ for every $x \in A$. Then:

$$\xi'_B(f_i^2) = P_{BB'} M_B(\xi_B(f_i) \bullet_B \xi_B(f_i))$$

Note that if $M_{B'} = (\varpi_{ij})$, then $\xi_{B'}(f_i^2)$ is the ith column of the structure matrix $M_{B'}$ and consequently

$$M_{B'} = P_{BB'} M_B P^{(2)}_{B'B}.$$

(ii). Assume that $P = (p_{ij})$ is invertible. Then B', defined as in the statement, is a basis of A. Moreover, if (1.7) is satisfied, then B' is a natural basis as follows by (1.5). □

The formula (1.7) can be rewritten in a more condensed way ([30]). Concretely,

$$M_B(P_{B'B} * P_{B'B}) = 0 \tag{1.8}$$

where $P_{B'B} * P_{B'B} = (c_{k(i,j)}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$, being $c_{k(i,j)} = p_{ki}p_{kj}$ for every pair (i, j) with $i < j$ and $i, j \in \Lambda$

1.4. Subalgebras and ideals of an evolution algebra

In this section we study the notions of evolution subalgebra and evolution ideal. We will see that the class of evolution algebras is not closed neither under subalgebras (Example 1.4.1) nor under ideals (Example 1.4.6). This last example also shows that the kernel of a homomorphism between evolution algebras is not necessarily an evolution ideal (contradicting [30, Theorem 2, p. 25]). The next example is [31, Example 1.2].

Example 1.4.1

Let A be the evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ and multiplication table given by $e_1^2 = e_1 + e_2 = -e_2^2$ and $e_3^2 = -e_2 + e_3$. Define $u_1 := e_1 + e_2$ and $u_2 := e_1 + e_3$. Then the subalgebra generated by u_1 and u_2 is not an evolution algebra as follows. Suppose on the contrary that there exist $\alpha, \beta, \gamma, \delta \in \mathbb{K}$ such that $v_1 = \alpha u_1 + \beta u_2$ and $v_2 = \gamma u_1 + \delta u_2$ determine a natural basis of the considered subalgebra. Since $v_1 v_2 = (\alpha u_1 + \beta u_2)(\gamma u_1 + \delta u_2) = (\alpha\delta + \beta\gamma)u_1 + \beta\delta u_2$, the identity $v_1 v_2 = 0$

and the linear independency of u_1 and u_2 imply that v_1 and v_2 are linearly dependent, a contradiction.

Because a subalgebra of an evolution algebra does not need to be an evolution algebra it is natural to introduce the notion of evolution subalgebra.

Remark 1.4.2

In [30, Definition 4, p. 23] (and also in [31]), an evolution subalgebra of an evolution algebra A is defined as a subspace A', closed under the product of A and endowed with a natural basis $\{e_i \mid i \in \Lambda'\}$ which can be extended to a natural basis $B = \{e_i \mid i \in \Lambda\}$ of A with $\Lambda' \subseteq \Lambda$. Nevertheless, we prefer to introduce the following new definition of evolution subalgebra that will be the only one that we will consider from now on.

Definitions 1.4.3

An **evolution subalgebra** of an evolution algebra A is a subalgebra $A' \subseteq A$ such that A' is an evolution algebra, i.e. A' has a natural basis.

We say that A' has the **extension property** if there exists a natural basis B' of A' which can be extended to a natural basis of A.

Remark 1.4.4

Let A be an evolution algebra with basis $\{e_i \mid i \in \Lambda\}$. As it was said before every element e_i can be interpreted as a genotype. A linear combination $\sum_{i\in\Lambda} \alpha_i e_i$ can be seen as a single individual such that the frequency of having genotype e_i is α_i.

A subalgebra A' of A is a population consisting of single individuals, each of which has a certain frequency of having genotype e_i and such that its reproduction (i.e. its product) remains in A'.

An evolution subalgebra A' will have the extension property if there exist genotype sets B' and B'' of A such that B' is a natural basis of A' and $B' \cup B''$

is a natural basis of A.

Note that an evolution subalgebra in the meaning of [30] is an evolution subalgebra in the sense of Definitions 1.4.3 having the extension property. Thus, this last definition of evolution subalgebra is natural and less restrictive as Example 1.4.10 below proves (where we give an ideal I which is an evolution algebra but has not the extension property). First, we introduce the notion of evolution ideal.

Recall that a subspace I of a commutative algebra A is said to be an **ideal** if $IA \subseteq I$. Next we show that an evolution subalgebra does not need to be an ideal. This contrast with the fact stated in [30] asserting that every evolution subalgebra is an ideal (according with the definition there). This is not the case with the definition of evolution subalgebra given in Definitions 1.4.3 as the following example shows.

Example 1.4.5

Let A be an evolution algebra with natural basis $B = \{e_{1,}e_2, e_3\}$ and multiplication given by $e_1^2 = e_2$, $e_2^2 = e_1$ and $e_3^2 = e_3$. Then, the subalgebra A' generated by $e_1 + e_2$ and e_3 is an evolution subalgebra with natural basis $B' = \{e_1 + e_2, e_3\}$ but it is not an ideal as $e_1(e_1 + e_2) \notin A'$.

On the other hand, *not every ideal of an evolution algebra has a natural basis.*

Example 1.4.6

Let A be the evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ and product given by $e_1^2 = e_2 + e_3$, $e_2^2 = e_1 + e_2$ and $e_3^2 = -(e_1 + e_2)$. Define $u_1 := e_1^2$ and $u_2 := e_2^2$. It is easy to check that u_1 and u_2 are linearly independent and that the subspace they generate is

$$I := \{\alpha e_1 + (\alpha + \beta)e_2 + \beta e_3 \mid \alpha, \beta \in \mathbb{K}\}.$$

Since $e_1u_1 = 0$, $e_2u_1 = u_2$, $e_3u_1 = -u_2$, $e_1u_2 = u_1$, $e_2u_2 = u_2$ and $e_3u_2 = 0$, we have that I is an ideal of A. Nevertheless, I has not a natural basis because if v_1 and v_2 are elements of I such that $v_1v_2 = 0$, then v_1 and v_2 are not linearly independent. Indeed, if $v_1 = \alpha e_1 + (\alpha + \beta)e_2 + \beta e_3$ and $v_2 = \lambda e_1 + (\lambda + \mu)e_2 + \mu e_3$, for $\alpha, \beta, \lambda, \mu \in \mathbb{K}$, then

$$v_1v_2 = \alpha\lambda u_1 + [(\alpha + \beta)(\lambda + \mu) - \beta\mu]u_2.$$

Consequently, if $v_1v_2 = 0$ then $\alpha\lambda = 0$ and $(\alpha + \beta)(\lambda + \mu) = \beta\mu$. It follows that $\alpha = \lambda = 0$, or $\alpha = \beta = 0$, or $\lambda = \mu = 0$, and hence v_1 and v_2 are not linearly independent.

This justifies the introduction of the following definition.

Definition 1.4.7
An **evolution ideal** of an evolution algebra A is an ideal I of A such that I has a natural basis.

Remark 1.4.8
Biologically, an ideal I of an evolution algebra A is a subalgebra such that the reproduction (multiplication) of genotypes of A by single individuals of I produces single individuals in I.

Clearly, evolution ideals are evolution subalgebras but the converse is not true as Example 1.4.5 proves (because an evolution subalgebra does not need to be an ideal). Also Example 1.4.6 shows that an ideal of an evolution algebra does not need to be an evolution ideal.

Remark 1.4.9
In [30, Definition 4, p. 23], the evolution ideals of an evolution algebra A are defined as those ideals I of A having a natural basis that can be extended to a natural basis of A. It is shown in [30, Proposition 2, p. 24] that every evolution

subalgebra is an evolution ideal (in the sense of [30]), that is, evolution ideals and evolution subalgebras are the same mathematical concept. This contrasts with our approach (Definitions 1.4.3 and 1.4.7, and Examples 1.4.5 and 1.4.10).

We finally show that there are examples of evolution ideals for which any natural basis can be extended to a natural basis of the whole evolution algebra. In other words, our definition of evolution ideal is more general than the corresponding definition given in [30].

Example 1.4.10

Let A be an evolution algebra with natural basis $B = \{e_{1,}e_2, e_3\}$ and multiplication given by $e_1^2 = e_3$, $e_2^2 = e_1 + e_2$ and $e_3^2 = e_3$. Let I be the ideal generated by $e_1 + e_2$ and e_3. Then I is an evolution ideal with natural basis $B_0 = \{e_1 + e_2, e_3\}$. However no natural basis of I can be extended to a natural basis of A. Indeed, if $u = \alpha(e_1 + e_2) + \beta e_3$ $v = \gamma(e_1 + e_2) + \delta e_3$ and $w = \lambda e_1 + \mu e_2 + \rho e_3$ is such that the set $\{u, v, w\}$ is a natural basis of A, then $uv = 0$, $uw = 0$ and $vw = 0$. This implies the following conditions: $\alpha\gamma = 0$, $\beta\delta = 0$, $\alpha\mu = 0$, $\alpha\lambda + \beta\rho = 0$, $\gamma\mu = 0$ and $\gamma\lambda + \delta\rho = 0$. Therefore, the only possibilities are $\alpha = \delta = \rho = \mu = 0$ or $\gamma = \beta = \mu = \lambda = \rho = 0$, a contradiction because $\{u, v, w\}$ is a basis. This means that I has not the extension property.

We finish this subsection with the result stating that the class of evolution algebras is closed under quotients by ideals (see also [15, Lemma 2.9]). The proof is straightforward.

Lemma 1.4.11

Let A be an evolution algebra and I an ideal of A. Then A/I with the natural product is an evolution algebra.

Remark 1.4.12

Let $B := \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A and let I be an ideal of A. Then $B_{A/I} := \{\overline{e_i} \mid i \in \Lambda, e_i \notin I\}$ is not necessarily a natural basis of A/I. For an example, consider A and I as in Example 1.4.6. Then $e_1, e_2, e_3 \notin I$ and hence $B_{A/I} = \{\overline{e_1}, \overline{e_2}, \overline{e_3}\}$, which is not a basis of A/I as the dimension of A/I (as a vector space) is one. Nevertheless, the set $B_{A/I}$ always contains a natural basis of A/I.

Given two algebras A and A', we recall that a linear map $f : A \to A'$ is said to be an **homomorphism of algebras** (**homomorphism** for short) if $f(xy) = f(x)f(y)$ for every $x, y \in A$.

Remark 1.4.13

[30, Theorem 2, p. 25] is not valid in general. Let I be an ideal of an evolution algebra. Then the map $\pi : A \to A/I$ given by $\pi(a) = \overline{a}$ is a homomorphism of evolution algebras (indeed, A/I is an evolution algebra by Lemma 1.4.11) whose kernel is I. By [30, Theorem 2, p. 25], $\mathrm{Ker}(\pi) = I$ is an evolution subalgebra in the sense of [30] and, in particular, I has a natural basis. But this is not always true. For example, take A and I as in Example 1.4.6. Then I is not an evolution ideal (i.e. has not a natural basis), as it is shown in that example.

We finish by showing that the class of evolution algebras is closed under homomorphic images.

Corollary 1.4.14

Let $f : A \to A'$ be a homomorphism between the evolution algebras A and A'. Then $\mathrm{Im}(f)$ is an evolution subalgebra of A'.

Demostración. By Lemma 1.4.11, $A/\mathrm{Ker}(f)$ is an evolution algebra. Apply that $\mathrm{Ker}(f)$ is an ideal of A and $\mathrm{Im}(f)$ is isomorphic to the evolution algebra

$A/\mathrm{Ker}(f)$. □

In the chapter 2 we will come back to the study of ideals. Concretely, we will determine the ideals generated by one single element.

1.5. Non-degenerate evolution algebras

In what follows we study the notion of non-degenerate evolution algebra and introduce a radical for an arbitrary evolution algebra such that the quotient algebra by this ideal is a non-degenerate evolution algebra.

Definition 1.5.1

An evolution algebra A is **non-degenerate** if it has a natural basis $B = \{e_i \mid i \in \Lambda\}$ such that $e_i^2 \neq 0$ for every $i \in \Lambda$.

Remark 1.5.2

That a genotype e_i in an evolution algebra A satisfies $e_i^2 = 0$ means biologically that is not able to have descendents. By Corollary 1.5.4 the evolution algebra A will be non-degenerate if all of its genotypes can reproduce.

In Corollary 1.5.4 we will show that non-degeneracy does not depend on the considered natural basis. Our proof will rely on the well-known notion of annihilator.

We recall that, for a commutative algebra A its **annihilator**, denoted by $\mathrm{ann}(A)$, is defined as

$$\mathrm{ann}(A) := \{x \in A \mid xA = 0\}.$$

Let X a subset of a vector space. We denote by $\mathrm{lin}\{X\}$ to subspace generated by elements of X.

Proposition 1.5.3

Let A be an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis. Denote by $\Lambda_0(B) := \{i \ \in \Lambda \mid e_i^2 = 0\}$. Then:

(I) $\operatorname{ann}(A) = \operatorname{lin}\{e_i \in B \mid i \in \Lambda_0(B)\}$.

(II) $\operatorname{ann}(A) = 0$ *if and only if* $\Lambda_0 = \emptyset$.

(III) $\operatorname{ann}(A)$ *is an evolution ideal of* A.

(IV) $|\Lambda_0(B)| = |\Lambda_0(B')|$ *for every natural basis* B' *of* A.

Demostración. By [15, Lemma 2.7] we have that $\operatorname{ann}(A) = \operatorname{lin}\{e_i \in B \mid i \in \Lambda_0\}$. This implies (i) and (iii). Item (ii) is obvious from (i) and (iv) follows from the fact that $|\Lambda_0(B)| = \dim(\operatorname{ann}(A))$. □

From now on, for simplicity, we will write Λ_0 instead of $\Lambda_0(B)$.

Corollary 1.5.4

An evolution algebra A is non-degenerate if and only if ann(A) $= 0$. *Consequently, the definition of non-degenerate evolution algebra does not depend on the considered natural basis.*

Demostración. Since A is non-degenerate if and only if $\Lambda_0 = \emptyset$, the result follows directly from Proposition 1.5.3 (ii). □

Remark 1.5.5

Let A be an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis. Denote by $\Lambda_1 := \{i \ \in \Lambda \mid e_i^2 \neq 0\}$. Then:

(i) $A_1 := \operatorname{lin}\{e_i \in B \mid i \in \Lambda_1\}$ is not necessarily a subalgebra of A.

(ii) $A/\operatorname{ann}(A)$ is not necessarily a non-degenerate evolution algebra.

Indeed, for an example concerning (i), consider the evolution algebra A with natural basis $\{e_1, e_2\}$ and product given by: $e_1^2 = 0$, $e_2^2 = e_1 + e_2$. Then $\text{ann}(A) = \text{lin}\{e_1\}$ and $A_1 = \text{lin}\{e_2\}$, which is not a subalgebra of A as $e_2^2 = e_1 + e_2 \notin A_1$.

To see (ii), let A be an evolution algebra with natural basis $B = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ such that $e_1^2 = e_2^2 = e_3^2 = 0$, $e_4^2 = e_1 + e_2$, $e_5^2 = e_2$ and $e_6^2 = e_2 + e_5$. Then $\text{ann}(A) = \text{lin}\{e_1, e_2, e_3\}$ and $A/\text{ann}(A)$ is an evolution algebra generated (as a vector space) by $\overline{e_4}$, $\overline{e_5}$ and $\overline{e_6}$ and has non-zero annihilator. In fact, $\text{ann}(A/\text{ann}(A)) = \text{lin}\{\overline{e_4}, \overline{e_5}\}$.

To get non-degenerate evolution algebras, we introduce a radical for an evolution algebra A, denoted by $\text{rad}(A)$, in such a way that $\text{rad}(A/\text{rad}(A)) = \overline{0}$, and so $A/\text{rad}(A)$ is non-degenerate.

Definition 1.5.6

Let I be an ideal of an evolution algebra A. We will say that I has the **absorption property** if $xA \subseteq I$ implies $x \in I$.

Remark 1.5.7

Biologically, an ideal I has the absorption property if whenever we consider one single individual x of A such that its descendence produces only individuals inside I, then the initial individual x belongs to I.

Example 1.5.8

Consider the evolution algebra A with natural basis $\{e_1, e_2, e_3\}$ and product given by: $e_1^2 = e_2$, $e_2^2 = e_1$ and $e_3^2 = e_3$. Let I be the ideal of A with basis $\{e_1, e_2\}$. It is not difficult to see that I has the absorption property.

Lemma 1.5.9

An ideal I of an evolution algebra A has the absorption property if and only if $\text{ann}(A/I) = \overline{0}$.

Demostración. Assume first that I has the absorption property. Take $\overline{a} \in \operatorname{ann}(A/I)$. Then $\overline{a}\ A/I = \overline{0}$, so that $aA \subseteq I$. This implies $a \in I$, that is, $\overline{a} = \overline{0}$. For the converse, use that $aA \subseteq I$ implies $\overline{a}\ A/I = \overline{0}$, that is, $\overline{a} \in \operatorname{ann}(A/I) = \overline{0}$ and hence $a \in I$. □

Lemma 1.5.10

Let I be a non-zero ideal of an evolution algebra A. Denote by $B = \{e_i \mid i \in \Lambda\}$ a natural basis of A. If I has the absorption property, then there exists $B_1 \subseteq B$ such that B_1 is a natural basis of I. In particular, I is an evolution ideal and has the extension property.

Demostración. By Lemma 1.4.11, we have that A/I is an evolution algebra. Let $\Lambda_2 \subseteq \Lambda$ be such that $\overline{B} := \{\overline{e}_i \mid i \in \Lambda_2\}$ is a natural basis of A/I. Denote by $\Lambda_1 = \Lambda \setminus \Lambda_2$, and let $B' = \{e_i \mid i \in \Lambda_1\}$. We claim that B' is a natural basis of I.

Take $e_i \in B'$. Then $\overline{e_i}\ A/I = \overline{0}$; this means $\overline{e_i} \in \operatorname{ann}(A/I)$, which is zero by Lemma 1.5.9. This implies $e_i \in I$. To see that I is generated by B', take $y \in I$ and write $y = \sum_{i\in\Lambda_1} k_i e_i + \sum_{i\in\Lambda_2} k_i e_i$ for some $k_i \in \mathbb{K}$. Taking classes in this identity we get $\overline{0} = \overline{y} = \sum_{i\in\Lambda_2} k_i \overline{e_i} \in \operatorname{lin} \overline{B}$. Since $\overline{B}$ is a basis, all the k_i (with $i \in \Lambda_2$) must be zero, implying $y = \sum_{i\in\Lambda_1} k_i e_i \in \operatorname{lin} B'$. □

Remark 1.5.11

The converse of Lemma 1.5.10 is not true. If we take the evolution algebra A with natural basis $\{e_1, e_2\}$ and product given by $e_1^2 = e_1$ and $e_2^2 = e_1$, then $I = \mathbb{K}e_1$ is an evolution ideal having the extension property but it has not the absorption property because $e_2 A \subseteq I$ and $e_2 \notin I$.

It is not difficult to prove that the intersection of any family of ideals with the absorption property is again an ideal with the absorption property.

Definition 1.5.12

We define the **absorption radical** of an evolution algebra A as the intersection of all the ideals of A having the absorption property. Denote it by $\operatorname{rad}(A)$. It is clear that the radical is the smallest ideal of A with the absorption property.

Proposition 1.5.13

Let A be an evolution algebra. Then $\operatorname{rad}(A) = 0$ if and only if $\operatorname{ann}(A) = 0$ if and only if A is non-degenerate.

Demostración. Note that $\operatorname{ann}(A) \subseteq \operatorname{rad}(A)$, hence $\operatorname{rad}(A) = 0$ implies $\operatorname{ann}(A) = 0$. On the other hand, if $\operatorname{ann}(A) = 0$, then 0 is an ideal having the absorption property. This implies $\operatorname{rad}(A) = 0$ as the radical of A is the intersection of all ideals having the absorption property. Finally, the assertion $\operatorname{ann}(A) = 0$ if and only if A is non-degenerate follows from Corollary 1.5.4. □

Corollary 1.5.14

Let I be an ideal of an evolution algebra A. Then I has the absorption property if and only if $\operatorname{rad}(A/I) = \overline{0}$. In particular $\operatorname{rad}(A/\operatorname{rad}(A)) = \overline{0}$, that is, $A/\operatorname{rad}(A)$ is a non-degenerate evolution algebra.

Demostración. By Lemmas 1.4.11 and 1.5.9, and by Proposition 1.5.13 it follows that I has the absorption property if and only if $\operatorname{ann}(A/I) = 0$ (and hence A/I is a non-degenerate evolution algebra), equivalently $\operatorname{rad}(A/I) = \overline{0}$. Since $\operatorname{rad}(A)$ is an ideal with the absortion property, the particular case about $A/\operatorname{rad}(A)$ follows immediately. □

We recall that an arbitrary algebra A is **semiprime** if there are no non-zero ideals I of A such that $I^2 = 0$, and is **nondegenerate** if $a(Aa) = 0$ for some $a \in A$ implies $a = 0$. Note that this is a different definition than that of non-degenerate (given in Definition 1.5.1). Although these definitions (in spite of the hyphen) can be confused, they appear with those names in the literature, and this is the reason because of which we compare them.

In the associative case, semiprimeness and nondegeneracy are equivalent concepts. We close this subsection by relating non-degenerate evolution algebras (in the meaning of Definition 1.5.1) with semiprime and nondegenerate evolution algebras. In fact, we obtain the following additional information.

Proposition 1.5.15

Let A be an evolution algebra with non-zero product. Consider the following conditions:

(I) *A is nondegenerate.*

(II) *A is semiprime.*

(III) *A has no non trivial evolution ideals of zero square.*

(IV) *A is non-degenerate.*

Then: (I) $\Rightarrow$ (II) $\Leftrightarrow$ (III) $\Rightarrow$ (IV).

Demostración. (I) $\Rightarrow$ (II) is well-known for any (evolution or not) algebra.

(II) $\Rightarrow$ (III) is a tautology.

(III) $\Rightarrow$ (II) follows because every ideal I such that $I^2 = 0$ is an evolution ideal.

(III) $\Rightarrow$ (IV). By Proposition 1.5.3, the annihilator of A is an evolution ideal. Since it has zero square, by the hypothesis, it must be zero. By Proposition 1.5.13 □

Remark 1.5.16

The implications (II) $\Rightarrow$ (I) and (IV) $\Rightarrow$ (III) in Proposition 1.5.15 do not hold in general.

To see that (II) $\not\Rightarrow$ (I), consider the evolution algebra A with natural basis $\{e_1, e_2\}$ and product given by $e_1^2 = e_2$ and $e_2^2 = e_1 + e_2$. Note that $e_1(Ae_1) = 0$. Suppose that I is a non-zero ideal such that $I^2 = 0$. Then it has to be proper and one dimensional because the dimension of A is 2. Therefore I has to be generated (as a vector space) by one element, say, $u = \alpha e_1 + \beta e_2$ for some $\alpha, \beta \in \mathbb{K}$. Since $0 = u^2 = \alpha^2 e_2 + \beta^2(e_1 + e_2) = \beta^2 e_1 + (\alpha^2 + \beta^2)e_2$ it follows that $\alpha = \beta = 0$, a contradiction.

To show that (IV) $\not\Rightarrow$ (III), let A be the evolution algebra with natural basis $B = \{e_1, e_2, e_3\}$ and product given by $e_1^2 = e_2 + e_3 = e_2^2$ and $e_3^2 = -e_2 - e_3$. Then the ideal I generated by $e_2 + e_3$ is such that $I^2 = 0$ and nevertheless A is non-degenerate.

1.6. The graph associated to an evolution algebra

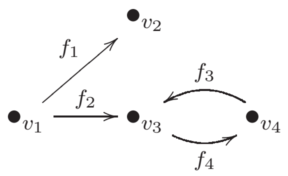
We conclude this chapter by associating a graph to every evolution algebra after fixing a natural basis. This will be very useful because it will allow to visualize when an evolution algebra is reducible or not as well as the results in Section 2.4 to get the optimal direct-sum decomposition.

A **directed graph** is a 4-tuple $E = (E^0, E^1, r_E, s_E)$ consisting of two disjoint sets E^0, E^1 and two maps $r_E, s_E : E^1 \to E^0$. The elements of E^0 are called the **vertices** of E and the elements of E^1 the edges of E while for $f \in E^1$ the vertices $r_E(f)$ and $s_E(f)$ are called the **range** and the **source** of f, respectively. If there is no confusion with respect to the graph we are considering, we simply write $r(f)$ and $s(f)$.

If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called **row-finite**. If E^0 is finite and E is row-finite, then E^1 must necessarily be finite as well; in this case we say simply that E is **finite**.

Example 1.6.1

Consider the following graph E:



Then $E^0 = \{v_1, v_2, v_3, v_4\}$ and $E^1 = \{f_1, f_2, f_3, f_4\}$. Examples of source and range are: $s(f_3) = v_4 = r(f_4)$.

A vertex which emits no edges is called a **sink**. A vertex which does not receive any vertex is called a **source**. A **path** μ in a graph E is a finite sequence of edges $\mu = f_1 \ldots f_n$ such that $r(f_i) = s(f_{i+1})$ for $i = 1, \ldots, n-1$. In this case, $s(\mu) := s(f_1)$ and $r(\mu) := r(f_n)$ are the **source** and **range** of μ, respectively, and n is the **length** of μ. This fact will be denoted by $|\mu| = n$. We also say that μ is **a path from** $s(f_1)$ **to** $r(f_n)$ and denote by μ^0 the set of its vertices, i.e., $\mu^0 := \{s(f_1), r(f_1), \ldots, r(f_n)\}$. On the other hand, by μ^1 we denote the set of edges appearing in μ, i.e., $\mu^1 := \{f_1, \ldots, f_n\}$. We view the elements of E^0 as paths of length 0. The set of all paths of a graph E is denoted by Path(E). Let $\mu = f_1 f_2 \cdots f_n \in$ Path(E). If $n = |\mu| \geq 1$, and if $v = s(\mu) = r(\mu)$, then μ is called a **closed path based at** v. If $\mu = f_1 f_2 \cdots f_n$ is a closed path based at v and $s(f_i) \neq s(f_j)$ for every $i \neq j$, then μ is called a **cycle based at** v or simply a **cycle**.

Given a graph E for which every vertex is a finite emitter, the **adjacency matrix** is the matrix $Ad_E = (a_{ij}) \in \mathbb{Z}^{(E^0 \times E^0)}$ given by $a_{ij} = |\{\text{edges from } i \text{ to } j\}|$.

A graph E is said to satisfy **Condition Sing** if among two vertices of E^0 there is at most one edge.

There are different ways in which a graph can be associated to an evolution algebra. For instance, we could have considered weighted evolution graphs (these are graphs for which every edge has associated a weight ω_{ij}, determined by the corresponding structure constant). In this way every evolution algebra (jointly with a fixed natural basis) has associated a unique weighted graph, and viceversa. However, for our purposes we don't need to pay attention to the weights; we only need to take into account if two vertices are connected or not (and in which direction). This is the reason because of which, in order to simplify our approach, it is enough to consider graphs as we do in the following definition.

Definition 1.6.2

Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A and $M_B = (\omega_{ji}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$ be its structure matrix. Consider the matrix $P^t = (p_{ji}) \in \mathrm{CFM}_\Lambda(\mathbb{K})$ such that $p_{ji} = 0$ if $\omega_{ji} = 0$ and $p_{ji} = 1$ if $\omega_{ji} \neq 0$. The **graph associated to the evolution algebra** A (relative to the basis B), denoted by E_A^B (or simply by E if the algebra A and the basis B are understood) is the graph whose adjacency matrix is $P = (p_{ij})$.

Note that the graph associated to an evolution algebra depends on the selected basis. In order to simplify the notation, and if there is no confusion, we will avoid to refer to such a basis.

Example 1.6.3

Let A be the evolution algebra with natural basis $B = \{e_1, e_2\}$ and product given by $e_1^2 = e_1 + e_2$ and $e_2^2 = 0$. Consider the natural basis $B' = \{e_1 + e_2, e_2\}$.

Then the graphs associated to the bases B and B' are, respectively:

Example 1.6.4

Let A be the evolution algebra with natural basis $B = \{e_1, e_2, e_3, e_4\}$ and product given by: $e_1^2 = e_2 + e_3$, $e_2^2 = 0$, $e_3^2 = -2e_4$ and $e_4^2 = 5e_3$. Then the adjacency matrix of the graph associated to the basis B is:

$$P = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and E is the graph given in Example 1.6.1.

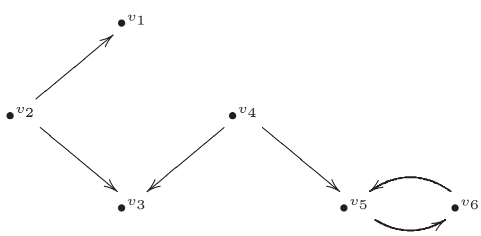
Now, conversely, to every row-finite graph satisfying Condition (Sing) we associate an evolution algebra whose corresponding structure matrix consists of 0 and 1, as follows.

Definition 1.6.5

Let E be a row-finite graph satisfying Condition (Sing) and $P = (p_{ij})$ be its adjacency matrix. Assume $E^0 = \{v_i\}_{i\in\Lambda}$. For every field $\mathbb{K}$ the **evolution $\mathbb{K}$-algebra associated to the graph** E, denoted by A_E, is the free algebra whose underlined vector space has a natural basis $B = \{e_i\}_{i\in\Lambda}$ and with structure matrix relative to B given by $P^t = (p_{ji})$.

Example 1.6.6

Let E be the following graph:



Its adjacency matrix is

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

and the corresponding evolution algebra is the algebra A having a natural basis $B = \{e_1, \ldots, e_6\}$ and product determined by: $e_1^2 = 0$, $e_2^2 = e_1 + e_3$, $e_3^2 = 0$, $e_4^2 = e_3 + e_5$, $e_5^2 = e_6$ and $e_6^2 = e_5$.

Remark 1.6.7

It is easy to determine the annihilator of an evolution algebra A by looking at the sinks of the graph associated to a basis. By Proposition 1.5.3, the annihilator of A consists of the linear span of the elements of the basis whose square is zero (these are, precisely, the sinks of the corresponding graph). For instance, in Example 1.6.6, $\text{ann}(A) = \text{lin}\{e_1, e_3\}$.

Chapter 2

Decomposition of an evolution algebra

In this chapter we characterize the decomposition of any non-degenerate evolution algebra into direct summands of ideals as well as the non-degenerate irreducible (indecomposable in [15]) evolution algebras in many ways, one of then in terms of the associated graph (relative to a natural basis). When the graph associated to an evolution algebra and to a natural basis is non-connected then it gives a decomposition of the algebra into direct summands. We define the optimal direct-sum decomposition of an evolution algebra and prove its existence and unicity when the algebra is non-degenerate.

When the algebra is finite dimensional we determine those elements in the associated graph relative to a natural basis (respectively in the algebra) which generate a decomposition into direct summands by means of the ideals generated by them.

A decomposition of an evolution algebra can be seen, biologically, as a disjoint union of families of genotypes, each of these families reproduces only with the single individuals of the proper family.

2.1. Ideals generated by one element

In order to characterize those ideals generated by one element, we introduce the following useful definitions.

Definitions 2.1.1

Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A and let $i_0 \in \Lambda$. The **first-generation descendents** of i_0 are the elements of the subset $D^1(i_0)$ given by:

$$D^1(i_0) := \left\{ k \in \Lambda \mid e_{i_0}^2 = \sum_k \omega_{ki_0} e_k \text{ with } \omega_{ki_0} \neq 0 \right\}.$$

In an abbreviated form, $D^1(i_0) := \{j \in \Lambda \mid \omega_{ji_0} \neq 0\}$. Note that $j \in D^1(i_0)$ if and only if, $\pi_j(e_{i_0}^2) \neq 0$ (where π_j is the canonical projection of A over $\mathbb{K}e_j$).

Similarly, we say that j is a **second-generation descendent** of i_0 whenever $j \in D^1(k)$ for some $k \in D^1(i_0)$. Therefore,

$$D^2(i_0) = \bigcup_{k \in D^1(i_0)} D^1(k).$$

By recurrency, we define the set of ***m*th-generation descendents** of i_0 as

$$D^m(i_0) = \bigcup_{k \in D^{m-1}(i_0)} D^1(k).$$

Finally, the **set of descendents** of i_0 is defined as the subset of Λ given by

$$D(i_0) = \bigcup_{m \in \mathbb{N}} D^m(i_0).$$

On the other hand, we say that $j \in \Lambda$ is an **ascendent** of i_0 if $i_0 \in D(j)$; that is, i_0 is a descendent of j.

Remark 2.1.2

From a biological point of view, the first-generation descendents of (the genotype) i are the genotypes appearing in e_i^2 (note that here we are identifying e_i and i).

The second-generation descendents of (the genotype) i are the genotypes appearing in the reproduction of the first-generation descendents of e_i.

In general, the mth-generation descendents of (the genotype) i are the genotypes appearing in the reproduction of the $(m-1)$th-generation descendents of e_i.

The set of descendents of i are the genotypes appearing in the nth-generation descendents of i for an arbitrary generation n.

We illustrate the definitions just introduced in terms of the underlying graph associated to an evolution algebra (relative to a natural basis). We will abuse of the notation for simplicity.

Definitions 2.1.3

Let E be a graph. For a vertex $j \in E^0$ we define:

$D^m(j) := \{v \in E^0 \mid \text{ there is a path } \mu \text{ such that } |\mu| = m, s(\mu) = v_j, r(\mu) = v\}.$

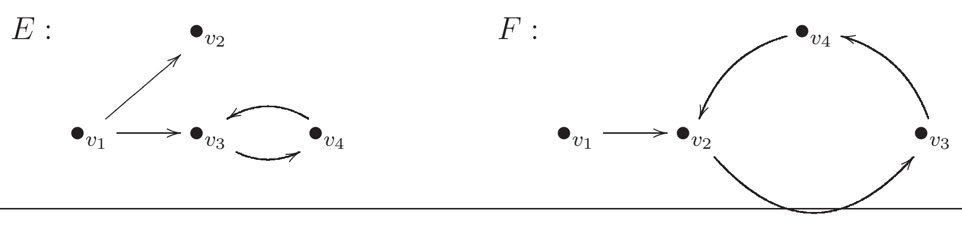
In words, the elements of $D^m(j)$ are those vertices to which v_j connects via a path of length m. We also define

$$D(j) = \bigcup_{m \in \mathbb{N}} D^m(j) = \{v \in E^0 \mid \text{ there is a path } \mu \text{ such that } s(\mu) = v_j, r(\mu) = v\}.$$

When we want to emphasize the graph E we will write $D_E^m(j)$ and $D_E(j)$, respectively.

Examples 2.1.4

Let E and F be the following graphs:



Some examples of the sets of the nth-generation descendents and of the set of descendents of some indexes are the following.

$D_E^1(3) = \{v_4\} = D_E^{1+2m}(3)$; $D_E^2(3) = \{v_3\} = D_E^{2m}(3)$ for every $m \in \mathbb{N}$, and so $D_E(3) = D_E^1(3) \cup D_E^2(3) = \{v_3, v_4\}$.

$D_F^1(2) = \{v_3\} = D_F^{1+3m}(2)$; $D_F^2(2) = \{v_4\} = D_F^{2+3m}(2)$; $D_F^3(2) = \{v_2\} = D_F^{3m}(2)$ for every $m \in \mathbb{N}$, and so $D_E(3) = D_F^1(2) \cup D_F^2(2) \cup D_F^3(2) = \{v_2, v_3, v_4\}$.

Next we characterize the descendents (and hence the ascendents) of every index $i_0 \in \Lambda$. More precisely, we describe the set $D^m(i_0)$.

Proposition 2.1.5

Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A. Consider $i_0, j \in \Lambda$ and $m \geq 2$.

(I) *If $j \in D^1(i_0)$ (if and only if $\omega_{ji_0} \neq 0$), then*

$$e_j e_{i_0}^2 = \omega_{ji_0} e_j^2.$$

(II) *$j \in D^m(i_0)$ if and only if there exist $k_1, k_2, \ldots, k_{m-1} \in \Lambda$ such that*

$$\omega_{jk_{m-1}} \omega_{k_{m-1}k_{m-2}} \cdots \omega_{k_2k_1} \omega_{k_1i_0} \neq 0,$$

in which case,

$$e_j^2 = \left(\omega_{jk_{m-1}} \omega_{k_{m-1}k_{m-2}} \cdots \omega_{k_2k_1} \omega_{k_1i_0}\right)^{-1} e_j e_{k_{m-1}} e_{k_{m-2}} \ldots e_{k_2} e_{k_1} e_{i_0}^2.$$

Demostración. Note that $j \in D^1(i_0)$ if and only if $\omega_{ji_0} \neq 0$, in which case $e_j e_{i_0}^2 = \omega_{ji_0} e_j^2$ so that

$$e_j^2 = \omega_{ji_0}^{-1} e_j e_{i_0}^2.$$

Suppose that the result holds for $m-1$. Thus, $k \in D^{m-1}(i_0)$ if and only if there exist $k_1, k_2, \ldots, k_{m-2} \in \Lambda$ such that

$$e_k^2 = (\omega_{kk_{m-2}}\omega_{k_{m-2}k_{m-3}} \cdots \omega_{k_2k_1}\omega_{k_1i_0})^{-1} e_k e_{k_{m-2}} \cdots e_{k_2} e_{k_1} e_{i_0}^2.$$

Let $j \in D^m(i_0)$. This means that $j \in D^1(k)$ for some $k \in D^{m-1}(i_0)$, so that $\omega_{jk} \neq 0$, and hence $e_j^2 = (\omega_{jk})^{-1} e_j e_k^2$. Consequently,

$$e_j^2 = (\omega_{jk}\omega_{kk_{m-2}} \cdots \omega_{k_2k_1}\omega_{k_1i_0})^{-1} e_j e_k e_{k_{m-2}} \cdots e_{k_2} e_{k_1} e_{i_0}^2,$$

as desired. □

From Proposition 2.1.5 we deduce that if i is a descendent of j, and if j is a descendent of k, then i is a descendent of k.

Another direct consequence of the mentioned proposition is the corollary that follows. From now on, if S is a subset of an algebra A then we will denote by $\langle S \rangle$ the ideal of A generated by S.

Corollary 2.1.6

Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A. If $j \in \Lambda$ is a descendent of $i_0 \in \Lambda$, then $\langle e_j^2 \rangle \subseteq \langle e_{i_0}^2 \rangle$.

Proposition 2.1.5 will allow to describe easily the ideal generated by an element in a natural basis, as well as the ideal generated by its square.

Corollary 2.1.7

Let A be an evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis. Then, for every $k \in \Lambda$,

$$\langle e_k^2 \rangle = \mathrm{lin}\{e_j^2 \mid j \in D(k) \cup \{k\}\} \quad \textit{and} \quad \langle e_k \rangle = \mathbb{K}e_k + \langle e_k^2 \rangle .$$

Demostración. Since $D^1(k) = \{j \in \Lambda \mid \omega_{jk} \neq 0\}$, by Proposition 2.1.5 we have

$$Ae_k^2 = \mathrm{lin}\{e_j^2 \mid j \in D^1(k)\}.$$

Consequently, $A(Ae_k^2) = \text{lin}\{e_j^2 \mid j \in D^2(k)\}$, and, therefore, $\langle e_k^2 \rangle = \text{lin}\{e_j^2 \mid j \in D(k) \cup \{k\}\}$. The rest is clear. □

Another proof of Corollary 2.1.7 will be obtained in Proposition 2.1.11.

Remark 2.1.8

Since $\langle e_k \rangle = \mathbb{K}e_k + \langle e_k^2 \rangle$, it is clear that $\langle e_k \rangle = \langle e_k^2 \rangle$ if and only if $e_k \in \langle e_k^2 \rangle$. On the other hand, because $D(k)$ is at most countable, by definition, the dimension of $\langle e_k \rangle$ is, at most, countable.

We can also describe the ideal generated by any element in a natural basis of an evolution algebra in terms of multiplication operators. This result will be very useful in order to characterize simple evolution algebras.

Definitions 2.1.9

Let A be an evolution $\mathbb{K}$-algebra. For any element $a \in A$ we define the multiplication operator by a, denoted by μ_a, as the following map:

$$\begin{aligned} \mu_a : \ & A \to A \\ & x \mapsto ax \end{aligned}$$

By μ_A we will mean the linear span of the set $\{\mu_a \mid a \in A\}$. For an arbitrary $n \in \mathbb{N}$, denote by μ_A^n:

$$\mu_A^n := \text{lin}\{\mu_{a_1} \dots \mu_{a_n} \mid a_1, \dots, a_n \in A\}.$$

For $n = 0$ we define μ_a^0 as the identity map $i_A : A \to A$, while μ_A^0 denotes $\mathbb{K}i_A$. Now, for $x \in A$, the notation $\mu_A^n(x)$ will stand for the following linear span:

$$\begin{aligned} \mu_A^n(x) : &= \text{lin}\{\mu_{a_1}\mu_{a_2} \dots \mu_{a_{n-1}}\mu_{a_n}(x) \mid a_1, \dots, a_n \in A\} \\ &= \text{lin}\{a_1(a_2(\dots(a_{n-1}(a_n x))\dots) \mid a_1, \dots, a_n \in A\}. \end{aligned}$$

For example, $\mu_A^3(x) = \text{lin}\{a_1(a_2(a_3 x))) \mid a_1, a_2, a_3 \in A\}$.

Definition 2.1.10

Let A be an evolution algebra with a natural basis $B = \{e_i \mid i \in \Lambda\}$. For any $x \in A$, we define

$$\Lambda^x := \{i \in \Lambda \mid e_i x \neq 0\}.$$

Proposition 2.1.11

Let A be an evolution algebra with a natural basis $B = \{e_i \mid i \in \Lambda\}$.

(I) *Let $k \in \Lambda$ be such that $e_k^2 \neq 0$.*

(a) $\mu_A^n(e_k^2) = \text{lin}\{e_j^2 \mid j \in D^n(k)\}$, *for every $n \in \mathbb{N}$.*

(b) $\langle e_k^2 \rangle = \text{lin} \bigcup_{n=0}^{\infty} \mu_A^n(e_k^2)$.

(c) $\langle e_k^2 \rangle = \text{lin}\{e_j^2 \mid j \in D(k) \cup \{k\}\}$.

(II) *For any $x \in A$,*

(a) $\mu_A^1(x) = \text{lin}\{e_i^2 \mid i \in \Lambda^x\}$ *and for any* $n \geq 2$, $\mu_A^n(x) = \text{lin} \bigcup_{i \in \Lambda^x} \{e_j^2 \mid j \in D^{n-1}(i)\}$.

(b) $\langle x \rangle = \text{lin} \bigcup_{n=0}^{\infty} \mu_A^n(x)$.

Demostración. We prove (a) in item (i) by induction. Suppose first $n = 1$. Note that $e_k^2 = \sum_{i \in D^1(k)} \omega_{ik} e_i$ with $\omega_{ik} \in \mathbb{K} \setminus \{0\}$. For an arbitrary $e_l \in B$ we have $e_k^2 e_l = \sum_{i \in D^1(k)} \omega_{ik} e_i e_l$. This sum is zero, if $l \neq i$ for every $i \in D^1(k)$, or it coincides with $\omega_{ik} e_i^2$ if $l = i$ for some i. Therefore,

$$\mu_A^1(e_k^2) \subseteq \text{lin}\{e_i^2 \mid i \in D^1(k)\}.$$

To show $\text{lin}\{e_i^2 \mid i \in D^1(k)\} \subseteq \mu_A^1(e_k^2)$, take any e_i with $i \in D^1(k)$. By Proposition 2.1.5 (i) we have $e_i^2 = {\omega_{ik}}^{-1} e_k^2 e_i \subseteq \mu_A^1(e_k^2)$. This finishes the first step in the induction process.

Assume we have the result for $n-1$. Using the induction hypothesis we get:

$$\begin{aligned}\mu_A^n(e_k^2) &= A\,\mu_A^{n-1}(e_k^2) = A\,\left(\mathrm{lin}\{e_i^2 \mid i\in D^{n-1}(k)\}\right) = \mathrm{lin}\bigcup_{i\in D^{n-1}(k)}\mu_A^1(e_i^2)\\ &= \mathrm{lin}\bigcup_{i\in D^{n-1}(k)}\{e_j^2 \mid j\in D^1(i)\} = \mathrm{lin}\{e_j^2 \mid j\in D^n(k)\}.\end{aligned}$$

This proves (a) in (i). Item (b) in (i) follows immediately from (a) and item (c) can be obtained from (a) and (b).

Now we prove (ii). Note that $\mu_A^1(x) = \mathrm{lin}\{e_i^2 \mid i\in\Lambda^x\}$. It is not difficult to see that, for $n>1$,

$$\mu_A^n(x) = \mathrm{lin}\bigcup_{i\in\Lambda^x}\mu_A^{n-1}(e_i^2).$$

Apply condition (b) in item (i) to finish the proof of (a) in (ii). Finally, item (b) in (ii) is easy to check. □

Corollary 2.1.12

Let A be an evolution algebra. Then for any element $x\in A$ the dimension of the ideal generated by x is at most countable.

Demostración. By (ii) in Proposition 2.1.11 the dimension of the ideal generated by x is the dimension of $\cup_{n=0}^{\infty}\mu_A^n(x)$. Since any $\mu_A^n(x)$ is finite dimensional, for every $n\in\mathbb{N}\cup\{0\}$, we are done. □

2.2. Simple evolution algebras

This section is addressed to the study and characterization of simple evolution algebras. We recall that an algebra A is **simple** if $A^2\neq 0$ and 0 is the only proper ideal.

Proposition 2.2.1

Let A be an evolution algebra and let $B=\{e_i \mid i\in\Lambda\}$ be a natural basis of A. Consider the following conditions:

(I) *A is simple.*

(II) *A satisfies the following properties:*

(a) *A is non-degenerate.*

(b) $A = \text{lin}\{e_i^2 \mid i \in \Lambda\}$.

(c) *If* $\text{lin}\{e_i^2 \mid i \in \Lambda'\}$ *is a non-zero ideal of* A *for a non-empty* $\Lambda' \subseteq \Lambda$ *then* $|\Lambda'| = |\Lambda|$.

Then: (i) $\Rightarrow$ (ii) *and* (ii) $\Rightarrow$ (i) *if* $|\Lambda| < \infty$. *Moreover, if* A *is a simple evolution algebra, then the dimension of* A *is at most countable.*

Demostración. (i) $\Rightarrow$ (ii). Suppose first that A is a simple evolution algebra. If A is degenerate, then $e^2 = 0$ for some element e in a natural basis B of A. Then $\text{lin}\{e\}$ is a nonzero ideal of A. The simplicity implies $\text{lin}\{e\} = A$, but then $A^2 = 0$, a contradiction. This shows (a).

Note that $A^2 = \text{lin}\{e_i^2 \mid i \in \Lambda\}$ is an ideal of A. Since $A^2 \neq 0$ and A is simple, we have $A = A^2$, which is (b).

If $\text{lin}\{e_i^2 \mid i \in \Lambda'\}$ is a non-zero ideal of A, the simplicity of A implies $\text{lin}\{e_i^2 \mid i \in \Lambda'\} = A = \text{lin}\{e_i^2 \mid i \in \Lambda\} = \text{lin}\{e_i \mid i \in \Lambda\}$. This gives $|\Lambda'| = |\Lambda|$.

(ii) $\Rightarrow$ (i). Assume that the dimension of A is finite, say n. Since A satisfies (a), $A^2 \neq 0$. To prove that A is simple, suppose that this is not the case. Then, there exists $u \in A$ such that $\langle u \rangle$ is a non-zero proper ideal of A. Let $k \in \Lambda$ be such that $\pi_k(u) \neq 0$. Then $\langle e_k^2 \rangle$ is a non-zero ideal of A contained in $\langle u \rangle$, so that $\langle e_k^2 \rangle$ is proper. Proposition 2.1.11 implies that $\langle e_k^2 \rangle = \text{lin}\{e_j^2 \mid j \in D(k) \cup \{k\}\}$, which, by (a), is a non-zero ideal of A. Use (c) to get $|\Lambda| = |D(k) \cup \{k\}|$. Since $D(k) \cup \{k\} \subseteq \Lambda$, we have $\Lambda = D(k) \cup \{k\}$. Now using (b),

$$\langle e_k^2 \rangle = \text{lin}\{e_j^2 \mid j \in D(k) \cup \{k\}\} = \text{lin}\{e_j^2 \mid j \in \Lambda\} = A,$$

a contradiction as $\langle e_k^2\rangle$ is a proper ideal of A.

The dimension of A is at most countable when A is simple by Corollary 2.1.12. □

Although every simple evolution algebra is non-degenerate, at most countable dimensional and coincides with the linear span of the square of the elements of any natural basis, as Proposition 2.2.1 says, the converse is not true because the hypothesis of finite dimension is necessary as the following example shows.

Example 2.2.2

Let A be an evolution algebra with natural basis $\{e_i \mid i \in \mathbb{N}\}$ and product given by:

$$\begin{array}{rclcrcl} e_1^2 &=& e_3+e_5 & \quad & e_2^2 &=& e_4+e_6 \\ e_3^2 &=& e_1+e_3+e_5 & & e_4^2 &=& e_2+e_4+e_6 \\ e_5^2 &=& e_5+e_7 & & e_6^2 &=& e_6+e_8 \\ e_7^2 &=& e_3+e_5+e_7 & & e_8^2 &=& e_4+e_6+e_8 \\ & \vdots & & & & \vdots & \end{array}$$

Then A satisfies the conditions (a), (b) and (c) in Proposition 2.2.1 (ii) but A is not simple as $\langle e_1^2\rangle$ and $\langle e_2^2\rangle$ are two nonzero proper ideals.

Example 2.2.3

Consider the evolution algebra A having a natural basis $\{e_1, e_2\}$ and product given by $e_i^2 = e_i$ for $i = 1, 2$. Then $\langle e_i\rangle = \mathbb{K}e_i$ is a non-zero proper ideal of A. This means that the condition (c) in Proposition 2.2.1 (ii) cannot be dropped.

We show now that there exist simple evolution algebras of infinite dimension.

Example 2.2.4

Let A be the evolution algebra with natural basis $\{e_i \mid i \in \mathbb{N}\}$ and product given by:

$$\begin{aligned} e_{2n-1}^2 &= \qquad e_{n+1} + e_{n+2} \\ e_{2n}^2 &= e_n + e_{n+1} + e_{n+2} \end{aligned}$$

Then A is simple.

Remark 2.2.5

An evolution algebra A whose associated graph (relative to a natural basis) has sinks cannot be simple. The reason is that a sink corresponds to an element in a natural basis of zero square, hence to an element in the annihilator of A. By Proposition 2.2.1, every simple evolution algebra has to be non-degenerate.

Corollary 2.2.6

Let A be a finite-dimensional evolution algebra of dimension n and $B = \{e_i \mid i \in \Lambda\}$ a natural basis of A. Then A is simple if and only if the determinant of the structure matrix $M_B(A)$ is non-zero and B cannot be reordered in such a way that the corresponding structure matrix is as follows:

$$\begin{pmatrix} W_{m\times m} & U_{m\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix}, \qquad (2.1)$$

for some $m \in \mathbb{N}$ with $m < n$ and matrices $W_{m\times m}$, $U_{m\times(n-m)}$ and $Y_{(n-m)\times(n-m)}$.

Demostración. If A is simple then, by Proposition 2.2.1, $A = \text{lin}\{e_i^2 \mid i \in \Lambda\}$. This means that the determinant of $M_B(A)$ is non-zero. To see the other condition, take into account that a reordering of the basis B producing a matrix of type (2.1) would imply that A has a proper ideal of dimension $m \geq 1$, a contradiction as we are assuming that A is simple.

Conversely, if $|M_B(A)| \neq 0$, then A is generated by the linear span of $\{e^2 \mid e \in B\}$. On the other hand, A cannot be degenerate as, otherwise,

$\operatorname{ann}(A) = \operatorname{lin}\{e \in B \mid e^2 = 0\}$ (see Proposition 1.5.3). Decompose B as $B = B_0 \sqcup B_1$, where $B_0 = \{e \in B \mid e^2 = 0\}$ and $B_1 = B \setminus B_0$ and let B' be a reordering of B in such a way that the first elements correspond to the elements of B_0 and the rest to the elements of B_1. Then $M_{B'}(A)$ is as matrix (2.1) in the statement, a contradiction. We have shown that A satisfies conditions (a) and (b) in Proposition 2.2.1 (ii). Now we see that condition (c) is also satisfied. Assume that $\Lambda' \subseteq \Lambda$ is such that $\operatorname{lin}\{e_i \mid i \in \Lambda'\}$ is a non-zero ideal of A. If we reorder B in such a way that the first elements are in $\{e_i \mid i \in \Lambda'\}$, then the corresponding structure matrix is of type (2.1) in the statement, a contradiction. Now use Proposition 2.2.1 to prove that A is simple. □

Next we characterize simple evolution algebras of arbitrary dimension (at most countable).

Theorem 2.2.7

Let A be a non-zero evolution algebra and $B = \{e_i \mid i \in \Lambda\}$ a natural basis. The following conditions are equivalent:

(I) *A is simple.*

(II) *If $\operatorname{lin}\{e_i^2 \mid i \in \Lambda'\}$ is an ideal for a non-empty subset $\Lambda' \subseteq \Lambda$, then $A = \operatorname{lin}\{e_i^2 \mid i \in \Lambda'\}$.*

(III) *$A = \langle e_i^2 \rangle = \operatorname{lin}\{e_j^2 \mid j \in D(i)\}$ for every $i \in \Lambda$.*

(IV) *$A = \operatorname{lin}\{e_i^2 \mid i \in \Lambda\}$ and $\Lambda = D(i)$ for every $i \in \Lambda$.*

Demostración. (I) $\Rightarrow$ (II). If A is simple, then it is non-degenerate by Proposition 2.2.1 and hence, $\operatorname{lin}\{e_i^2 \mid i \in \Lambda'\}$ is a nonzero ideal of A, so that the result follows.

(II) $\Rightarrow$ (III). By Proposition 2.1.11 (ii) we have $\langle e_i^2 \rangle = \text{lin}\{e_j^2 \mid j \in D(i) \cup \{i\}\}$. By (II), this set is A. Since $e_i \in A = \langle e_i^2 \rangle$ we have $i \in D(i)$ and (III) has being proved.

(III) $\Rightarrow$ (IV). Since $D(i) \subseteq \Lambda$, we have $A = \text{lin}\{e_i^2 \mid i \in \Lambda\}$. Now, take $j \in \Lambda$. Then $e_j \in A = \langle e_i^2 \rangle = \text{lin}\{e_k^2 \mid k \in D(i)\}$ (by (ii)). It follows that $j \in D(i)$ and therefore $\Lambda \subseteq D(i)$.

(IV) $\Rightarrow$ (I). Let I be a nonzero ideal of A. Since $Ie_i \neq 0$ for some $i \in \Lambda$, then $e_i^2 \in I$ and so $I \supseteq \langle e_i^2 \rangle = A$ (by (IV)). □

The two conditions in Theorem 2.2.7 (IV) are not redundant as we see in the next examples.

Examples 2.2.8

Consider the evolution algebra A with natural basis $\{e_1, e_2, e_3\}$ and product given by $e_1^2 = e_3^2 = e_1 + e_2$; $e_2^2 = e_3$. Then $\{1, 2, 3\} = D(i)$ for every $i \in \{1, 2, 3\}$ but $\text{lin}\{e_i^2 \mid i = 1, 2, 3\} = \text{lin}\{e_1 + e_2, e_3\} \neq A$.

On the other hand, consider the evolution algebra A given in Example 2.2.2. Then $A = \text{lin}\{e_i^2 \mid i \in \mathbb{N}\}$ but $\mathbb{N} \neq D(i)$ for every $i \in \mathbb{N}$.

Now we show that in Theorem 2.2.7 (III) the hypothesis "for every $i \in \Lambda$" cannot be eliminated.

Example 2.2.9

Let A be the evolution algebra with natural basis $B = \{e_n \mid n \in \mathbb{N}\}$ and product given by:

$$\begin{array}{rclrcl} e_1^2 &=& 0 & & & \\ e_2^2 &=& e_3 + e_5 & e_3^2 &=& e_4 + e_6 \\ e_4^2 &=& e_1 + e_3 + e_5 & e_5^2 &=& e_2 + e_4 + e_6 \\ e_6^2 &=& e_5 + e_7 & e_7^2 &=& e_6 + e_8 \\ e_8^2 &=& e_3 + e_5 + e_7 & e_9^2 &=& e_4 + e_6 + e_8 \\ & \vdots & & & \vdots & \end{array}$$

Then $A = \text{lin}\{e_i^2\}$ for every $i \in \mathbb{N}$, but A is not simple as it is not non-degenerate.

Another characterization of simplicity for finite dimensional evolution algebras is the following.

Corollary 2.2.10

If A is a finite dimensional evolution algebra and B a natural basis, then A is simple if and only if $|M_B(A)| \neq 0$ and $\Lambda = D(i)$ for every $i \in \Lambda$.

Demostración. Apply Theorem 2.2.7 (iv) taking into account that finite dimensionality of A implies that $A = \text{lin}\{e_i^2 \mid i \in \Lambda\}$ if and only if $|M_B(A)| \neq 0$. □

Remark 2.2.11

In terms of graphs, the condition "$\Lambda = D(i)$ïn Theorem 2.2.7 (iv) means that the graph associated to A relative to a natural basis B is cyclic, in the sense that given two vertices there is always a path from one to the other one.

The following remark shows how to get ideals in non-simple evolution algebras.

Remark 2.2.12

If A is a non-degenerate evolution algebra having a natural basis $B = \{e_i \mid i \in \Lambda\}$ such that every element $i \in \Lambda$ is a descendent of every $j \in \Lambda$, then A is not simple if and only if $\text{lin}\{e_i^2 \mid i \in \Lambda\}$ is a proper ideal of A.

2.3. Reducible evolution algebras.

In this subsection we are interested in the study of those evolution algebras which can be written as direct sums of (evolution) ideals.

Definition 2.3.1

Let $\{A_\gamma\}_{\gamma\in\Gamma}$ be a non-empty family of evolution $\mathbb{K}$-algebras. We define the **direct sum** of these evolution algebras and denote it by $A := \oplus_{\gamma\in\Gamma} A_\gamma$ with the following operations: given $a = \sum\limits_{\gamma\in\Gamma} a_\gamma$, $b = \sum\limits_{\gamma\in\Gamma} b_\gamma \in A$ and $\alpha \in \mathbb{K}$ (note that a_γ and b_γ are zero for almost every $\gamma \in \Gamma$), define

$$a + b := \sum_{\gamma\in\Gamma} (a_\gamma + b_\gamma)\,, \qquad \alpha a := \sum_{\gamma\in\Gamma} (\alpha a_\gamma)\,, \qquad ab := \sum_{\gamma\in\Gamma} (a_\gamma b_\gamma)\,.$$

Note that A is an evolution algebra as, if B_γ is a natural basis of A_γ for every $\gamma \in \Gamma$, then $B := \cup_{\gamma\in\Gamma} B_\gamma$ is a natural basis of A. Here, by abuse of notation, we understand $A_\gamma \subseteq A$ so that every A_γ can be regarded as an (evolution) ideal of A. Moreover, for $\gamma \neq \mu$, the ideals A_γ and A_μ are orthogonal, in the sense that $A_\gamma A_\mu = 0$.

Lemma 2.3.2

Let A be an evolution algebra. The following assertions are equivalent:

(I) *There exists a family of evolution subalgebras $\{A_\gamma\}_{\gamma\in\Gamma}$ such that $A = \oplus_{\gamma\in\Gamma} A_\gamma$.*

(II) *There exists a family of evolution ideals $\{I_\gamma\}_{\gamma\in\Gamma}$ such that $A = \oplus_{\gamma\in\Gamma} I_\gamma$.*

(III) *There exists a family of ideals $\{I_\gamma\}_{\gamma\in\Gamma}$ such that $A = \oplus_{\gamma\in\Gamma} I_\gamma$.*

Demostración. (i) $\Rightarrow$ (ii). By the definition of direct sum of evolution algebras (see Definition 2.3.1), every A_γ is, in fact, an evolution ideal.

(ii) $\Rightarrow$ (iii) is a tautology.

(iii) $\Rightarrow$ (i). Suppose $A = \oplus_{\gamma\in\Gamma} I_\gamma$, where each I_γ is an ideal of A. For $\mu \in \Gamma$ we have:

$$I_\mu \cong A/\left(\oplus_{\gamma\in\Gamma\setminus\{\mu\}} I_\gamma\right).$$

By Lemma 1.4.11 we obtain that $A/\left(\oplus_{\gamma\in\Gamma\setminus\{\mu\}} I_\gamma\right)$ is an evolution algebra, and hence I_μ is an evolution algebra by Corollary 1.4.14. □

Definition 2.3.3

A **reducible evolution algebra** is an evolution algebra A which can be decomposed as the direct sum (in the sense of Definition 2.3.1) of two non-zero evolution algebras, equivalently, of two non-zero evolution ideals, equivalently, of two non-zero ideals, as shown in Lemma 2.3.2. An evolution algebra which is not reducible will be called **irreducible**.

Reducibility of an evolution algebra is related to the connection of the underlying graphs, as we show next. For the description of the (existent) connected components of a graph see, for example, [1, Definitions 1.2.13].

Proposition 2.3.4

Let A be a non-zero evolution algebra and E its associated graph relative to a natural basis $B = \{e_i \mid i \in \Lambda\}$.

(I) *Assume $E = E_1 \sqcup E_2$, where E_1 and E_2 are non-empty subgraphs of E. Write $E_k^0 = \{v_i \mid i \in \Lambda_k\}$, for $k = 1, 2$, where $\Lambda_k \subseteq \Lambda$ and $\Lambda = \Lambda_1 \sqcup \Lambda_2$. Then there exist non-zero evolution ideals I_1, I_2 of A such that $A = I_1 \oplus I_2$ and E_1, E_2 are the graphs associated to the evolution algebras I_1 and I_2, respectively, relative to their natural basis $B_k = \{e_i \mid i \in \Lambda_k\}$ (for $k = 1, 2$). Moreover, $B = B_1 \sqcup B_2$.*

(II) *Let $E = \sqcup_{\gamma\in\Gamma} E_\gamma$ be the decomposition of E into its connected components. For every $\gamma \in \Gamma$, write $E_\gamma^0 = \{v_i \mid i \in \Lambda_\gamma\}$, where $\Lambda_\gamma \subseteq \Lambda$ and $\Lambda = \sqcup_{\gamma\in\Gamma}\Lambda_\gamma$. Then there exist $\{I_\gamma\}_{\gamma\in\Gamma}$, evolution ideals of A, such that $A = \oplus_{\gamma\in\Gamma} I_\gamma$ and E_γ is the associated graph to the evolution algebra I_γ relative to the natural basis B_γ described below. Moreover:*

(a) $B = \sqcup_{\gamma\in\Gamma} B_\gamma$, *where* $B_\gamma = \{e_i \mid i \in \Lambda_\gamma\}$ *is a natural basis of* I_γ, *for every* $\gamma \in \Gamma$.

(b) I_γ *is a simple evolution algebra if and only if* $I_\gamma = \mathrm{lin}\{e_i^2 \mid i \in \Lambda_\gamma\}$ *and* $D(i) = \Lambda_\gamma$ *for every* $i \in \Lambda_\gamma$.

(c) A *is non-degenerate if and only if every* I_γ *is a non-degenerate evolution algebra.*

Demostración. (i). Let $B = \{e_i \mid i \in \Lambda\}$. The decomposition $E = E_1 \sqcup E_2$ of E into two non empty components provides a decomposition of A into ideals as follows. Denote by v_i, with $i \in \Lambda$, the vertices of E. Write $E^0 = E_1^0 \sqcup E_2^0$, and let $\Lambda_k \subseteq \Lambda$ be such that $\Lambda_k = \{i \in \Lambda \mid v_i \in E_k^0\}$, for $k = 1, 2$. Define $I_k = \mathrm{lin}\{e_i \mid i \in \Lambda_k\}$. Then $A = I_1 \oplus I_2$. The moreover part follows easily.

(ii). The first part can be proved as (i). Item (a) follows immediately. As for (b), apply Theorem 2.2.7 (IV). To prove (c) use Proposition 1.5.3 and Corollary 1.5.4. □

Remark 2.3.5

Once we have defined what an optimal direct-sum decomposition is (see Definition 2.4.1) we can say that if A is non-degenerate then $A = \oplus_{\gamma\in\Gamma} I_\gamma$ in Proposition 2.3.4 (ii) is the optimal direct sum decomposition of A, as will follow from Theorem 2.4.2.

In the next result we characterize when a non-degenerate evolution algebra A is reducible, giving an answer to one of our main questions in this work.

Theorem 2.3.6

Let A *be a non-degenerate evolution algebra with a natural basis* $B = \{e_i \mid i \in \Lambda\}$ *and assume that* $A = \oplus_{\gamma\in\Gamma} I_\gamma$, *where each* I_γ *is an ideal of* A. *Then:*

(i) *For every* $e_i \in B$ *there exists a unique* $\mu \in \Gamma$ *such that* $e_i \in I_\mu$*. Moreover,* $e_i \in I_\mu$ *if and only if* $e_i^2 \in I_\mu$*.*

(ii) *There exists a disjoint decomposition of* Λ*, say* $\Lambda = \sqcup_{\gamma\in\Gamma}\Lambda_\gamma$*, such that*

$$I_\gamma = \mathrm{lin}\{e_i \mid i \in \Lambda_\gamma\}.$$

Demostración. We show both statements at the same time. Let π_i be the linear projection of A over $\mathbb{K}e_i$. We show first that $\pi_i(I_\gamma) \neq 0$ implies $e_i^2 \in I_\gamma$, and hence $\pi_i(I_\mu) = 0$ for every $\mu \in \Gamma \setminus \{\gamma\}$. Indeed, if $\pi_i(I_\gamma) \neq 0$, then there exists $y \in I_\gamma$ such that $\pi_i(y) = \alpha e_i \neq 0$ for some $\alpha \in \mathbb{K}$. Multiplying by e_i we get $e_i y = e_i \pi_i(y) = \alpha e_i^2 \in I_\gamma$ and, therefore, $e_i^2 \in I_\gamma$. If $\pi_i(I_\mu) \neq 0$ for some $\mu \in \Gamma$, reasoning as before, we get $e_i^2 \in I_\mu$, and so $e_i^2 \in I_\gamma \cap I_\mu = 0$, a contradiction because we are assuming that A is non-degenerate.

Define $\Lambda_\gamma := \{i \in \Lambda \mid \pi_i(I_\gamma) \neq 0\}$. It is easy to see that $\cup_{\gamma\in\Gamma}\Lambda_\gamma = \Lambda$. Moreover, the first paragraph of the proof shows that this is a disjoint union, as claimed in (ii).

Now, it is easy to see that for every $\gamma \in \Gamma$ we have that $I_\gamma \subseteq \mathrm{lin}\{e_i \mid i \in \Lambda_\gamma\}$. To show that $\mathrm{lin}\{e_i \mid i \in \Lambda_\gamma\} \subseteq I_\gamma$, consider $e_j \in B$, with $j \in \Lambda_\gamma$ and denote $J = \oplus_{\mu\in\Gamma\setminus\{\gamma\}} I_\mu$. Because $A = I_\gamma \oplus J$ we may write $e_j = u + v$, with $u \in I_\gamma$ and $v \in J$. Then, $v = e_j - u \in \mathrm{lin}\{e_i \mid i \in \Lambda_\gamma\}$ because e_j and u are in $\mathrm{lin}\{e_i \mid i \in \Lambda_\gamma\}$. Since $v \in J \subseteq \mathrm{lin}\{e_i \mid i \in \cup_{\mu\in\Gamma\setminus\{\gamma\}}\Lambda_\mu\}$ we deduce that v must be zero. □

Remark 2.3.7

Theorem 2.3.6 gives another proof, for non-degenerate evolution algebras, of the fact that if an evolution algebra is a direct sum of ideals, then such ideals are evolution algebras (and, consequently, evolution ideals). This is the assertion (ii) ⇔ (iii) established in Lemma 2.3.2.

Another application of Theorem 2.3.6 allows us to recognize easily when a non-degenerate finite dimensional evolution algebra A is reducible: if $B = \{e_i \mid i = 1, ..., n\}$ is a natural basis of A then A is the direct sum of two (evolution) ideals if and only if there is a permutation $\sigma \in S_n$ such that, if $B' := \{e_{\sigma(i)} \mid i = 1, ..., n\}$, then the corresponding structure matrix is

$$M_{B'} = \begin{pmatrix} W_{m\times m} & 0_{(n-m)\times(n-m)} \\ 0_{(n-m)\times m} & Y_{(n-m)\times(n-m)} \end{pmatrix},$$

for some $m \in \mathbb{N}$, $m < n$ and some matrices $W_{m\times m}$ and $Y_{(n-m)\times(n-m)}$ with entries in $\mathbb{K}$. In this case $A = I \oplus J$, where $I = \text{lin}\{e_{\sigma(1)}, ..., e_{\sigma(m)}\}$ and $J = \text{lin}\{e_{\sigma(m+1)}, ..., e_{\sigma(n)}\}$. The basis B' is what we will called a **reordering** of B.

Corollary 2.3.8

Let A be a non-degenerate evolution algebra, $B = \{e_i \mid i \in \Lambda\}$ a natural basis, and let E be its associated graph. Then A is irreducible if and only if E is a connected graph.

Demostración. Suppose first that E is connected. To show that A is irreducible suppose, on the contrary, that there exist I and J, non-zero ideals of A, such that $A = I \oplus J$. By Theorem 2.3.6 there exists a decomposition $\Lambda = \Lambda_I \sqcup \Lambda_J$ such that $I = \text{lin}\{e_i \mid i \in \Lambda_I\}$ and $J = \text{lin}\{e_i \mid i \in \Lambda_J\}$. Then $E = E_I \sqcup E_J$, a contradiction since we are assuming that E is connected.

The converse follows easily: by Proposition 2.3.4 (i), a decomposition $E = E_1 \sqcup E_2$ into two non empty components provides a decomposition $A = I_1 \oplus I_2$, for I_1 and I_2 non-zero ideals of A, contradicting that A is irreducible. □

In [15, Proposition 2.8] the authors show the result above for finite-dimensional evolution algebras using a different approach.

The hypothesis of non-degeneracy cannot be eliminated in Corollary 2.3.8.

Example 2.3.9

Consider the evolution algebra given in Example 1.6.3, which is not non-degenerate. Then the graph E, associated to the basis B is connected while the graph F, associated to the basis B' is not.

2.4. The optimal direct-sum decomposition of an evolution algebra

The aim of this subsection is to obtain a decomposition of an evolution algebra in terms of irreducible evolution ideals.

Definition 2.4.1

Let A be a non-zero evolution algebra and assume that $A = \oplus_{\gamma\in\Gamma} I_\gamma$ is a direct sum of non-zero ideals. If every I_γ is an irreducible evolution algebra, then we say that $A = \oplus_{\gamma\in\Gamma} I_\gamma$ is an **optimal direct-sum decomposition** of A.

We show that the optimal direct sum decomposition of an evolution algebra A with a natural basis $B = \{e_i \mid i \in \Lambda\}$ does exist and it is unique whenever the algebra is non-degenerate. Moreover, for finite dimensional evolution algebras (degenerated or not), we will describe how to get an optimal decomposition of Λ through the fragmentation process. This will be done in Subsection 2.5

Theorem 2.4.2

Let A be a non-degenerate evolution algebra. Then A admits an optimal direct-sum decomposition. Moreover, it is unique.

Demostración. We start by showing the existence. Let E be the graph associated to A relative to a natural basis B and decompose it in its connected components, say $E = \sqcup_{\gamma\in\Gamma} E_\gamma$. By Proposition 2.3.4 (ii) we have $A = \oplus_{\gamma\in\Gamma} I_\gamma$, where every I_γ is an ideal of A. Note that, by construction (see the proof of Proposition 2.3.4), every I_γ has a natural basis, say B_γ, consisting of elements of the basis $B = \{e_i \mid i \in \Lambda\}$ of A. Because A is non-degenerate, $e_i^2 \neq 0$, by Corollary 1.5.4 and Proposition 1.5.3. Using again these results we have that every I_γ is a non-degenerate evolution algebra. Since E_γ is the graph associated to I_γ relative to the basis B_γ and E_γ is connected, by Corollary 2.3.8 every I_γ is an irreducible evolution algebra.

Now we prove the uniqueness. Fix a natural basis $B = \{e_i \mid i \in \Lambda\}$. Suppose that there are two optimal direct-sum decompositions of A, say $A = \oplus_{\gamma\in\Gamma} I_\gamma$ and $A = \oplus_{\omega\in\Omega} J_\omega$. By Theorem 2.3.6 there exist two decompositions $\Lambda = \sqcup_{\gamma\in\Gamma}\Lambda_\gamma$ and $\Lambda = \sqcup_{\omega\in\Omega}\Lambda_\omega$ such that

$$I_\gamma = \operatorname{lin}\{e_i \mid i \in \Lambda_\gamma\} \quad \text{and} \quad J_\omega = \operatorname{lin}\{e_i \mid i \in \Lambda_\omega\}.$$

Take $i \in \Lambda_\gamma$ for an arbitrary $\gamma \in \Gamma$. Then there is an $\omega \in \Omega$ such that $e_i \in J_\omega$. This means $I_\gamma \cap J_\omega \neq 0$. Decompose

$$I_\gamma = (I_\gamma \cap J_\omega) \oplus (I_\gamma \cap (\oplus_{\omega\neq\omega'\in\Omega} J_{\omega'}))\,.$$

Since I_γ is irreducible and $I_\gamma \cap J_\omega \neq 0$, necessarily $(I_\gamma \cap (\oplus_{\omega\neq\omega'\in\Omega} J_{\omega'})) = 0$. Therefore $I_\gamma = I_\gamma \cap J_\omega$ and so $I_\gamma \subseteq J_\omega$. Changing the roles of I_γ and J_ω we get $J_\omega \subseteq I_\gamma$, implying $I_\gamma = J_\omega$ and, consequently, that each decomposition is nothing but a reordering of the other one. □

The hypothesis of non-degeneracy cannot be eliminated in order to assure the unicity of the optimal direct sum decomposition in Theorem 2.4.2, as the following example shows. It is also an example which illustrates that in Theorem 2.3.6 non-degeneracy is also required.

Example 2.4.3

Let A be the evolution $\mathbb{K}$-algebra with natural basis $B = \{e_1, e_2, e_3, e_4, e_5\}$ and multiplication given by: $e_1^2 = e_2^2 = e_1$, $e_3^2 = e_3 + e_5$ and $e_4^2 = e_5^2 = 0$. Then $A = I_1 \oplus I_2 \oplus I_3 \oplus I_4$, where $I_1 := \text{lin}\{e_1, e_2 + e_4\}$, $I_2 := \text{lin}\{e_3 + e_5\}$, $I_3 := \text{lin}\{e_4\}$ and $I_4 := \text{lin}\{e_5\}$ are irreducible ideals, as we are going to show.

The ideals I_2, I_3 and I_4 are irreducible because their dimension is one. Now we prove that I_1 is also irreducible. Assume, on the contrary, $I_1 = J_1 \oplus J_2$, with J_1 and J_2 non-zero ideals. Then $\dim J_1 = \dim J_2 = 1$, so that $J_1 = \mathbb{K}u_1$ and $J_2 = \mathbb{K}u_2$ for some $u_1 = \alpha_1 e_1 + \beta_1(e_2 + e_4)$ and $u_2 = \alpha_2 e_1 + \beta_2(e_2 + e_4)$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K}$. Then, $u_1 u_2 = 0$ implies $(\beta_1\beta_2 + \alpha_1\alpha_2)e_1 = 0$. On the other hand, $u_1 e_1 = \alpha_1 e_1 \in J_1$ and $u_2 e_1 = \alpha_2 e_1 \in J_2$. Since $J_1 \cap J_2 = 0$, then $\alpha_1 = 0$ or $\alpha_2 = 0$. Assume, for example, $\alpha_1 = 0$. Then $J_1 = \mathbb{K}(e_2 + e_4)$, but this is not an ideal as $(e_2 + e_4)^2 = e_1$. The case $\alpha_2 = 0$ is similar.

Now we give another decomposition of A into irreducible ideals. Consider $A = J \oplus I_2 \oplus I_3 \oplus I_4$, where $J := \text{lin}\{e_1, e_2\}$. We claim that J is an irreducible ideal of A. Indeed, if $J = M_1 \oplus M_2$, for M_1 and M_2 non-zero ideals, then $M_1 = \mathbb{K}u_1$ and $M_2 = \mathbb{K}u_2$ for some $u_1 = \alpha_1 e_1 + \beta_1 e_2$ and $u_2 = \alpha_2 e_1 + \beta_2 e_2$, where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{K}$. Then, $u_1 e_1 = \alpha_1 e_1$ and $u_2 e_1 = \alpha_2 e_1$. Since $M_1 \cap M_2 = 0$, then $\alpha_1 = 0$ or $\alpha_2 = 0$. Assume $\alpha_1 = 0$. This implies $u_1 = \beta_1 e_2$ and $M_1 = \mathbb{K}e_2$, but this is not an ideal because $e_2^2 = e_1$. The case $\alpha_2 = 0$ is similar.

Note that we have two different decompositions of A as a direct sum of irreducible ideals.

As we have seen (Remark 2.2.3), non-degenerate evolution algebras are not necessarily simple. On the other hand, concerning reducibility, the next example shows that there exists irreducible evolution algebras which are not simple (while, obviously, simple evolution algebras are irreducible).

Example 2.4.4

Let A be an evolution algebra with a natural basis $B = \{e_1, e_2\}$ such that $e_1^2 = e_2^2 = e_2$. Then $\mathbb{K}e_2$ is a proper ideal of A. However, A is irreducible because if $A = I \oplus J$, for some ideals I and J of A. Then, by Theorem 2.3.6, we have either $e_1 \in I$, in which case $A = I$, or $e_1 \in J$, in which case, $A = J$. In any case I or J is zero.

The next definition will be helpful to understand the inner structure of an evolution algebra.

Definition 2.4.5

Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A. We say that $i_0 \in \Lambda$ is **cyclic** if $i_0 \in D(i_0)$. This means that i_0 is descendent (and hence ascendent) of itself.

In particular, if $D(i_0) = \{i_0\}$ (in which case $e_{i_0}^2 = \omega_{i_0 i_0} e_{i_0}$ for some $\omega_{i_0 i_0} \in \mathbb{K} \setminus \{0\}$), then we say that the cyclic index i_0 is a **loop**.

If $i_0 \in \Lambda$ is cyclic, then the **cycle associated to** i_0 is defined as the set:

$$C(i_0) = \{j \in \Lambda \mid j \in D(i_0) \text{ and } i_0 \in D(j)\}.$$

Note that if i_0 is cyclic then $C(i_0)$ is non-empty because it contains i_0 in particular. Moreover, i_0 is a loop if and only if $C(i_0) = \{i_0\}$.

We say that a subset $C \subseteq \Lambda$ is a **cycle** if $C = C(i_0)$, for some cyclic index $i_0 \in \Lambda$.

Remark 2.4.6

By identifying an index i with the genotype e_i, biologically, an index is cyclic if it is a descendent of its descendents. The cycle associated to an index i is the set of all its descendents j such that i is a descendent of j.

In the same context as in Definition 2.4.5, consider $i_0 \in \Lambda$, and let $\omega_{i_0 i_0}$ be the corresponding element in the structure matrix for the evolution algebra A. If $\omega_{i_0 i_0} \neq 0$ then we have that i_0 is cyclic, independently of the value of the other elements in the structure matrix. If $\omega_{i_0 i_0} = 0$, then i_0 is cyclic if and only if it is a descendent of some of its own descendents.

On the other hand, if $B = \{e_i \mid i \in \Lambda\}$ is a natural basis of A, and if $i_1, i_2 \in \Lambda$ are cyclic, then we have either $C(i_1) = C(i_2)$ or $C(i_1) \cap C(i_2) = \emptyset$.

These facts can be understood more easily looking at the corresponding graphical concepts, as we will do below.

Now we classify the cycles into two types, depending on if they have or not ascendents outside the cycle.

Definition 2.4.7

Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A and let $i_0 \in \Lambda$ be a cyclic index. We say that i_0 is a **principal cyclic index** if the set of ascendents of i_0 is contained in $C(i_0)$, the cycle associated to i_0. Thus, $i_0 \in \Lambda$ is a principal cyclic index if $i_0 \in D(i_0)$ and $j \in D(i_0)$ for every $j \in \Lambda$ with $i_0 \in D(j)$.

We say that a subset C of Λ is a **principal cycle** if $C = C(i_0)$, for some principal cyclic index $i_0 \in \Lambda$.

It is clear that if $i_0 \in \Lambda$ is a principal cyclic index then every $j \in C(i_0)$ is also a principal cyclic index. Moreover, if $i_0 \in \Lambda$ is a cyclic index, then $C(i_0)$ is not principal if and only if there exists $j \in \Lambda \setminus C(i_0)$ such that $i_0 \in D(j)$.

On the other hand, a non-empty subset $C \subseteq \Lambda$ is a principal cycle if and only if it satisfies the following properties:

(I) For every $i, j \in C$ we have that $i \in D(j)$ and $j \in D(i)$.

(II) If $D(k) \cap C \neq \emptyset$ then $k \in C$.

Note that if i_0 is a loop, then $\{i_0\}$ is a principal cycle if and only if i_0 has no other ascendents than i_0. Moreover, if C is a principal cycle, then $C = C(i) = C(j)$ for every $i, j \in C$ and, hence, $D(i) = D(j)$ for very $i, j \in C$.

Now we will distinguish between cycles that have proper descendents from those that do not have them.

Definition 2.4.8

Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A, and let S be a subset of Λ. We define the **index-set derived from** S as the set given by

$$\Lambda(S) := S \cup_{i \in S} D(i).$$

For instance, if $i \in \Lambda$, then the index set derived from $\{i\}$ is $\Lambda(\{i\}) := \{i\} \cup D(i)$, where $D(i)$ is the set of descendents of i. The index set derived from a principal cycle is obtained next.

Remark 2.4.9

Let C be a principal cycle. Then, $C = C(i) = C(j)$ and $D(i) = D(j)$, for every $i, j \in C$. Moreover, $C(i) \subseteq D(i)$, for every $i \in C$ (the inclusion may or not be strict). Thus, $C \subseteq \Lambda(C) = D(i)$ for every $i \in C$.

The Definition 2.4.5 in terms of graphs gives rise to the following definitions.

Definitions 2.4.10

Let E be a graph with vertices $\{v_i \mid i \in \Lambda\}$ satisfying Condition (Sing). An index $j \in \Lambda$ is said to be **cyclic** if $j \in D^m(j)$ for some $m \in \mathbb{N}$ (see Definitions 2.1.3). Equivalently, if the graph E has a cycle c such that $v_j \in c^0$.

If j is a cyclic index, we define

$$C(j) := \{v_k \in E^0 \mid k \in D(j) \text{ and } j \in D(k)\},$$

that is, $C(j)$ are those vertices connected to v_j such that v_j is also connected to them.

A cyclic index j is called **principal** (see Definition 2.4.7) if

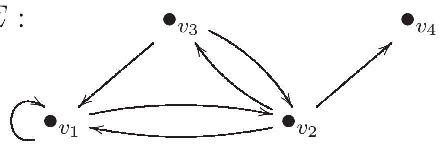
$$\{v_k \in E^0 \mid j \in D(k)\} \subseteq C(j).$$

Therefore, a cyclic index j is principal if and only if it belongs to a cycle without entries or such that every entry comes from a path starting at the cycle. By extension, we will also say that $C(j)$ is a **principal cycle**.

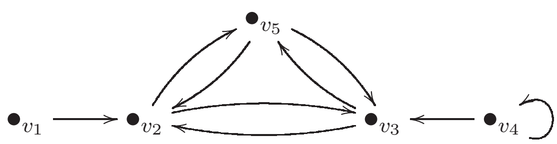
Examples 2.4.11

Consider the following graphs.

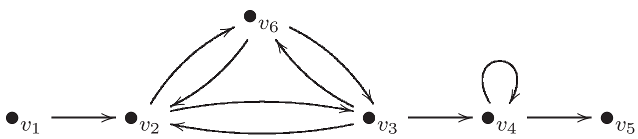
E :



F :



G :



Concerning E, the indices $1, 2$ and 3 are cyclic and $C(1) = C(2) = C(3) = \{v_1, v_2, v_3\}$. Also, $1, 2$ and 3 are principal indices.

For the graph F, the cyclic indices are $2, 3, 4$ and 5. Moreover, $C(2) = C(3) = C(5) = \{v_2, v_3, v_5\}$ and $C(4) = \{v_4\}$. The only index which is principal is 4.

As to the graph G, its cyclic indices are $2, 3, 4$ and 6. None of them is principal.

Definition 2.4.12

Let $B = \{e_i \mid i \in \Lambda\}$ be a natural basis of an evolution algebra A. We say that $i_0 \in \Lambda$ is a **chain-start index** if i_0 has no ascendents, i.e., $i_0 \notin D(j)$, for every $j \in \Lambda$. Equivalently, i_0 is a chain-start index if and only if all the elements of the i_0-th row of the structure matrix $M_B(A)$ are zero.

Remark 2.4.13

In terms of graphs, an index i_0 is a chain-start index if and only if the vertex v_{i_0} is a source. In the graphs of Examples 2.4.11, the only chain-start index is the vertex v_1 in F and the vertex v_1 in G.

In the case of finite-dimensional evolution algebras, if the determinant of the structure matrix $M_B(A)$ is non-zero then Λ has no chain-start indices. The (graphical) reason is that $|M_B(A)| \neq 0$ implies that $M_B(A)$ has no zero rows, hence the associated graph relative to B has no sources.

The lack of chain-start indices is a necessary condition for A to be simple. The graphical reason is that if a vertex v_i is a source, then the ideal generated by e_i^2 does not contains e_i, hence $\langle e_i^2 \rangle$ is a nonzero proper ideal.

2.5. The fragmentation process

In this subsection we will consider only finite dimensional evolution algebras, degenerated or not. We give a process that allow to decompose an evolution algebra into direct sums of evolution algebras (the optimal decomposition when the algebra is non-degenerate).

Definition 2.5.1

Let A be a finite dimensional evolution algebra, and fix a natural basis $B = \{e_i \mid i \in \Lambda\}$. Consider the set $\{C_1, \ldots, C_k\}$ of the principal cycles of Λ and the set $\{i_1, \ldots, i_m\}$ of all chain-start indices of Λ.

Given any $i \in \Lambda$ which is not a chain-start index, there exists $j \in \Lambda$ such that $i \in D(j)$, and either j is a chain-start index or j belong to a principal cycle (because Λ is finite). Therefore, according to Definition 2.4.8,

$$\Lambda = \Lambda(C_1) \cup \cdots \cup \Lambda(C_k) \cup \Lambda(i_1) \cup \cdots \cup \Lambda(i_m).$$

This decomposition will be called the **canonical decomposition of** Λ **associated to** B.

Note that the sets in the canonical decomposition are not necessarily disjoint. This is the case, for example, when two different principal cycles, or two chain-start indices, have common descendents.

Definition 2.5.2

Let Λ be a finite set and let $\Upsilon_1, \ldots, \Upsilon_n$ be non-empty subsets of Λ such that $\Lambda = \cup_{i=1}^n \Upsilon_i$. We say that $\Lambda = \cup_{i=1}^n \Upsilon_i$ is a **fragmentable union** if there exists disjoint non-empty subsets Λ_1, Λ_2 of Λ satisfying

$$\Lambda = \cup_{i=1}^n \Upsilon_i = \Lambda_1 \cup \Lambda_2,$$

and such that for every $i = 1, \ldots, n$, either

$\Upsilon_i \subseteq \Lambda_1$ or $\Upsilon_i \subseteq \Lambda_2$.

For instance, if the sets Υ_i are disjoint then $\Lambda = \cup_{i=1}^n \Upsilon_i$ is fragmentable. Note that a fragmentable union may admit different fragmentations.

On the other hand, if $\Upsilon_i \cap \Upsilon_j \neq \emptyset$ for every $i \neq j$, then the union $\Lambda = \cup_{i=1}^n \Upsilon_i$ is not fragmentable.

Definitions 2.5.3

Let Λ be a finite set and let $\Upsilon_1, \ldots, \Upsilon_n$ be non-empty subsets of Λ such that $\Lambda = \cup_{i=1}^n \Upsilon_i$ is a fragmentable union. A **fragmentation of** $\Lambda = \cup_{i=1}^n \Upsilon_i$ is a union $\Lambda = \cup_{i=1}^k \Lambda_i$ such that:

(I) If $i \in \{1, \ldots, k\}$ then $\Lambda_i = \cup_{j \in S_i} \Upsilon_j$ for S_i a non-empty subset of $\{1, \ldots, n\}$.

(II) $\Lambda_i \cap \Lambda_j = \emptyset$, for every $i, j \in \{1, \ldots, k\}$, with $i \neq j$.

Note that conditions (i) and (ii) imply that for every $j \in \{1, \ldots, n\}$ there exists a unique $i \in \{1, \ldots, k\}$ such that $\Upsilon_j \subseteq \Lambda_i$.

An **optimal fragmentation** of a fragmentable union $\Lambda = \cup_{i=1}^n \Upsilon_i$ is a fragmentation $\Lambda = \cup_{i=1}^k \Lambda_i$ such that for every $i \in \{1, \ldots, k\}$ the index set $\Lambda_i = \cup_{j \in S_i} \Upsilon_j$ is not fragmentable.

In what follows we build the optimal fragmentation for any $\Lambda = \cup_{i=1}^n \Upsilon_i$.

Let Λ be a finite set and consider $\Upsilon_1, \ldots, \Upsilon_n$, non-empty subsets of Λ such that $\Lambda = \cup_{i=1}^n \Upsilon_i$ is a fragmentable union. To obtain an optimal fragmentation of this union we define the following equivalence relation in the set $\{\Upsilon_1, \ldots, \Upsilon_n\}$.

We say that $\Upsilon_i \sim \Upsilon_j$ if there exist $m_1, \ldots, m_k \in \{1, \ldots, n\}$ such that

$$\Upsilon_i \cap \Upsilon_{m_1} \neq \emptyset,\ \Upsilon_{m_1} \cap \Upsilon_{m_2} \neq \emptyset, \ldots,\ \Upsilon_{m_{k-1}} \cap \Upsilon_{m_k} \neq \emptyset,\ \Upsilon_{m_k} \cap \Upsilon_j \neq \emptyset.$$

Let $S_1 := \{i \in \{1, \ldots, n\} \mid \Upsilon_i \sim \Upsilon_1\}$; define $\Lambda_1 := \cup_{i \in S_1} \Upsilon_i$. Set $S_2 = \{1, \ldots, n\} \setminus S_1$ and $\widetilde{\Lambda}_2 := \cup_{i \in S_2} \Upsilon_i$. Then $\Lambda = \Lambda_1 \cup \widetilde{\Lambda}_2$ with Λ_1 non-fragmentable. If $\widetilde{\Lambda}_2 = \cup_{i \in S_2} \Upsilon_i$ is non-fragmentable then by defining

$\widetilde{\Lambda}_2 = \Lambda_2$ we have that $\Lambda = \Lambda_1 \cup \Lambda_2$ is the optimal fragmentation of $\Lambda = \cup_{i=1}^n \Upsilon_i$. Otherwise, $\widetilde{\Lambda}_2 = \cup_{i \in S_2} \Upsilon_i$ is fragmentable and, as before, we may decompose $\widetilde{\Lambda}_2 := \Lambda_2 \cup \widetilde{\Lambda}_3$, with Λ_2 non-fragmentable. By reiterating the process we obtain a decomposition $\Lambda = \cup_{i=1}^k \Lambda_i$, where every Λ_i is non-fragmentable. This produces an optimal fragmentation $\Lambda = \cup_{i=1}^k \Lambda_i$ of the initial decomposition $\Lambda = \cup_{i=1}^n \Upsilon_i$.

Proposition 2.5.4

Let Λ be a finite set and let $\Upsilon_1, \dots, \Upsilon_n$ be non-empty subsets of Λ such that $\Lambda = \bigcup_{i=1}^{n} \Upsilon_i$ is a fragmentable union. Then the optimal fragmentation $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_k$ of $\Lambda = \bigcup_{i=1}^{n} \Upsilon_i$ is unique (unless reordering).

Demostración. Suppose that $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_k$ and $\Lambda = \widetilde{\Lambda}_1 \cup \dots \cup \widetilde{\Lambda}_m$ are two optimal fragmentations. Take $i \in \{1, \dots, k\}$. If there exist $j, k \in \{1, \dots, m\}$ such that $\Lambda_i \cap \widetilde{\Lambda}_j \neq \emptyset$ and $\Lambda_i \cap \widetilde{\Lambda}_k \neq \emptyset$, then $j = k$ because, otherwise, Λ_i is fragmentable. It follows that for every $i \in \{1, \dots, k\}$ there is a unique $j \in \{1, \dots, m\}$ such that $\Lambda_i \subseteq \widetilde{\Lambda}_j$. We claim that $\Lambda_i = \widetilde{\Lambda}_j$ because otherwise $\widetilde{\Lambda}_j$ would be fragmentable, a contradiction. We conclude that $m = k$ and that $\Lambda = \widetilde{\Lambda}_1 \cup \dots \cup \widetilde{\Lambda}_m$ is a reordering of $\Lambda = \Lambda_1 \cup \dots \cup \Lambda_k$. □

By combining Theorems 2.2.7 and 2.3.6 with the optimal fragmentation process we obtain the following result.

Theorem 2.5.5

Let A be a finite-dimensional evolution algebra with natural basis $B = \{e_i \mid i \in \Lambda\}$. Let $\{C_1, \dots, C_k\}$ be the set of principal cycles of Λ, $\{i_1, \dots, i_m\}$ the set of all chain-start indices of Λ and consider the canonical decomposition

$$(\dagger) \quad \Lambda = \Lambda(C_1) \cup \dots \cup \Lambda(C_k) \cup \Lambda(i_1) \cup \dots \cup \Lambda(i_m).$$

Let $\Lambda = \sqcup_{\gamma\in\Gamma}\Lambda_\gamma$ *be the optimal fragmentation of* (†) *and decompose* $B = \sqcup_{\gamma\in\Gamma}B_\gamma$*, where* $B_\gamma = \{e_i \mid i \in \Lambda_\gamma\}$*. Then* $A = \oplus_{\gamma\in\Gamma}I_\gamma$*, for* $I_\gamma = \text{lin } B_\gamma$*, which is an evolution ideal of* A*. Moreover, if* A *is non-degenerate, then* $A = \oplus_{\gamma\in\Gamma}I_\gamma$ *is the optimal direct-sum decomposition of* A*.*

Since the optimal direct-sum decomposition of a non-degenerate evolution algebra A is unique, we conclude that, in the non-degenerate case, the decomposition given in Theorem 2.5.5 does not depend on the prefixed natural basis B (i.e. any other natural basis leads to the same optimal direct sum decomposition).

Remark 2.5.6

Every finite dimensional evolution algebra A (non-degenerated or not) is the direct sum of a finite number of irreducible evolution algebras. Indeed, if A is irreducible, then we are done. Otherwise, decompose $A = I_1 \oplus I_2$, for I_1, I_2 ideals of A. If I_1 and I_2 are irreducible, then we have finished. If this is not the case, we decompose them. Since the dimension of A is finite, proceeding in this way in a finite number of steps we finish.

For A an evolution algebra of arbitrary dimension such that $A = \oplus_{\gamma\in\Gamma}I_\gamma$ is the optimal direct-sum decomposition of A, the study of A can be reduced to the study of the irreducible evolution algebras I_γ separately.

The last result in this section is a consequence of Proposition 2.2.1 and Theorem 2.5.5.

Corollary 2.5.7

Let A *be a non-degenerate finite dimensional evolution algebra with a natural basis* $B = \{e_i \mid i \in \Lambda\}$*. Then* $A = I_1 \oplus \cdots \oplus I_k$*, where* I_i *is an ideal, simple as an algebra, if and only if* Λ *has the following property:* $\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_k$*,* *where every* Λ_j *is non-empty and* $I_j = \mathrm{lin}\{e_i \mid i \in \Lambda_j\} = \mathrm{lin}\{e_i^2 \mid i \in \Lambda_j\}$ *and* $D(i) = \Lambda_j$, *for every* $i \in \Lambda_j$.

Chapter 3

Classification two-dimensional evolution algebras

The aim of this chapter is to determine the two-dimensional evolution algebras over a field $\mathbb{K}$ where for every $k \in \mathbb{K}$ the polynomial $x^n - k$ has a root whenever $n = 2, 3$. By $\mathbb{K}^\times$ we denote $\mathbb{K} \setminus \{0\}$.

Two-dimensional evolution algebras over the complex numbers were determined in [12, Theorem 4.1], although there is a case which is mising: the algebra A with natural basis $\{e_1, e_2\}$ such that $e_1^2 = e_2$ and $e_2^2 = e_1$ (Case 6 in the Theorem 3.3). This is a two-dimensional evolution algebra not isomorphic to any of the six types in [12, Theorem 4.1]. Some of ideas we have used here to prove the theorem below can be found in [12, Theorem 4.1].

Given two structure matrices M_B and M'_B of two evolution algebras A and A', by abuse of notation we will say that M_B and M'_B are isomorphic if the evolution algebras A and A' are isomorphic.

Lemma 3.1

Let A be an evolution algebra, let $B = \{e_1, e_2\}$ be a natural basis, and

$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$$

the corresponding coefficient matrix. If $\det M_B \neq 0$ then, $B' = \{f_1, f_2\}$ is a natural basis of A if and only if there exists r and $s \in \mathbb{K}^\times$ such that B' is one *of the following types:*

(i) $B' = \{re_1, se_2\}$ *in whose case* $M_{B'} = \begin{pmatrix} r\omega_{11} & \frac{s^2}{r}\omega_{12} \\ \frac{r^2}{s}\omega_{21} & s\omega_{22} \end{pmatrix}$

(ii) $B' = \{se_2, re_1\}$, *in whose case* $M_{B'} = \begin{pmatrix} s\omega_{22} & r^2\frac{\omega_{21}}{s} \\ s^2\frac{\omega_{12}}{r} & r\omega_{11} \end{pmatrix}$

Demostración. Since $\det M_B \neq 0$ it follows that e_1^2 and e_2^2 are linearly independents independent vectors. Therefore if $f_1 = p_{11}e_1 + p_{21}e_2$ and $f_2 = p_{12}e_1 + p_{22}e_2$ we have that $f_1f_2 = p_{11}p_{12}e_1^2 + p_{21}p_{22}e_2^2 = 0$ if and only if $p_{11}p_{12} = 0$ and $p_{21}p_{22} = 0$. If f_1 and f_2 are non-zero then it follows that either $p_{11} = 0 = p_{22}$ or $p_{12} = 0 = p_{21}$. Therefore, either $B' = \{re_1, se_2\}$ or $B' = \{se_2, re_1\}$ for certain r and $s \in \mathbb{K}^{times}$. In the first case, the structure matrix follows from the following equalities:

$$\begin{aligned} f_1^2 &= (re_1)^2 = r^2\left(\omega_{11}e_1 + \omega_{21}e_2\right) = r\omega_{11}f_1 + \frac{r^2}{s}\omega_{21}f_2. \\ f_2^2 &= (se_2)^2 = s^2(\omega_{12}e_1 + \omega_{22}e_2) = \frac{s^2}{r}\omega_{12}f_1 + s\omega_{22}f_2. \end{aligned}$$

In other words,

$$M_{B'} = \begin{pmatrix} r\omega_{11} & \frac{s^2}{r}\omega_{12} \\ \frac{r^2}{s}\omega_{21} & s\omega_{22} \end{pmatrix}$$

In the second case a similar straightforward calculation shows that

$$M_{B'} = \begin{pmatrix} s\omega_{22} & r^2\frac{\omega_{21}}{s} \\ s^2\frac{\omega_{12}}{r} & r\omega_{11} \end{pmatrix}$$

□

Remark 3.2

Note that the number of zeros of the structure matrix relative to a natural basis is invariant under change of basis. What is more, the number of zeros in the main diagonal of the structure matrix is invariant too.

Theorem 3.3

Let A be a two-dimensional evolution algebra over a field $\mathbb{K}$ where for every $k \in \mathbb{K}$ the polynomial $x^n - k$ has a root whenever $n = 2, 3$.

(I) *If* $\dim(A^2) = 0$ *then* $M_B = 0$ *for any natural basis* B *of* A.

(II) *If* $\dim(A^2) = 1$ *then there exists a natural basis* $B = \{e_1, e_2\}$ *of* A *such that* M_B *is one of the following four matrices:*

(a) $M_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$,

(b) $M_B = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$,

(c) $M_B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$,

(d) $M_B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Moreover they are mutually non-isomorphic.

(III) *If* $\dim(A^2) = 2$ *then there exists a natural basis* $B = \{e_1, e_2\}$ *of* A *such that* M_B *is one of the following three types of matrices:*

(e) $M_B(\alpha, \beta) = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$ *for some* $\alpha, \beta \in \mathbb{K}^\times$ *such that* $1 - \alpha\beta \neq 0$.

(f) $M_B(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$ *for some* $\alpha \in \mathbb{K}^\times$.

(g) $M_B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(h) $M_B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

(i) $M_B(\gamma) = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix}$ *for some* $\gamma \in \mathbb{K}^\times$.

$\{M_B(\alpha,\beta),\ M_B(\alpha),\ \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, M_B(\gamma)\mid \alpha,\beta,\gamma\in\mathbb{K}^\times$ *such that* $1-\alpha\beta\neq 0\}$ *is a set of mutually non-isomorphic evolution algebras except when* $\gamma,\gamma'\in\mathbb{K}^\times$ *are such that* $\dfrac{\gamma}{\gamma'}$ *is a third root of unity.*

Demostración. Fix a two-dimensional evolution algebra A and a natural basis $B=\{e_1,e_2\}$. Let $M_B=(\omega_{ij})$ be the structure matrix of A relative to B.

Case $\dim(A^2)=0$**.**

Then $M_B=0$.

Case $\dim(A^2)=1$**.**

We may assume without loss of generality that $\{e_1^2\}$ is a basis of A^2 ($(\omega_1,\omega_2)\neq(0,0)$). Since $e_2^2\in A^2$, there exists $c_1\in\mathbb{K}$ such that $e_2^2=c_1e_1^2=c_1(\omega_1e_1+\omega_2e_2)$. Then,

$$M_B=\begin{pmatrix}\omega_1 & c_1\omega_1\\ \omega_2 & c_1\omega_2\end{pmatrix}.$$

Extend the basis $\{e_1^2\}$ of A^2 to a new basis $B'=\{e'_1,e'_2\}$ of A (B' is not necessarily a natural basis) where $e'_1=e_1^2$ and $e'_2=p_1e_1+p_2e_2$ for some $p_1,p_2\in\mathbb{K}$. Then the change of basis matrix $P_{B'B}$ is

$$P_{B'B}=\begin{pmatrix}\omega_1 & p_1\\ \omega_2 & p_2\end{pmatrix}.$$

Our aim is to know when B' is a natural basis or, equivalently, when $\{e_1^2\}$ has the extension property. For this, if we apply Theorem 1.3.2, B' is a natural basis if and only if it verifies:

$$M_{B'}=\begin{pmatrix}\dfrac{(\omega_1p_2-\omega_2p_1)(\omega_1^2+c_1\omega_2^2)}{\omega_1p_2-\omega_2p_1} & \dfrac{(\omega_1p_2-\omega_2p_1)(p_1^2+c_1p_2^2)}{\omega_1p_2-\omega_2p_1}\\ 0 & 0\end{pmatrix}$$

and

$$M_{B'}(P_{B'B} * P_{B'B}) = \begin{pmatrix} \omega_1(\omega_1 p_1 + c_1\omega_2 p_2) \\ \omega_2(\omega_1 p_1 + c_1\omega_2 p_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.1}$$

Since $\omega_1 p_2 - \omega_2 p_1 \neq 0$ the structure matrix $M_{B'}$ can be written as follows

$$M_{B'} = \begin{pmatrix} \omega_1^2 + c_1\omega_2^2 & p_1^2 + c_1 p_2^2 \\ 0 & 0 \end{pmatrix}.$$

In other words, as $(\omega_1, \omega_2) \neq (0,0)$ then $\{e_1^2\}$ has the extension property if and only if $\omega_1 p_1 + c_1\omega_2 p_2 = 0$.

Case 1 Suppose that $\omega_1 p_1 + c_1\omega_2 p_2 = 0$.

So $\{e_1^2\}$ has the extension property.

Case 1.1 If $\omega_1 \neq 0$.

Then $p_1 = \dfrac{-c_1\omega_2 p_2}{\omega_1}$ and replacing this value into $M_{B'}$ we obtain:

$$M_{B'} = \begin{pmatrix} \omega_1^2 + c_1\omega_2^2 & \dfrac{c_1 p_2^2}{\omega_1^2}(\omega_1^2 + c_1\omega_2^2) \\ 0 & 0 \end{pmatrix}.$$

Note that necessarily $\omega_1^2 + c_1\omega_2^2 \neq 0$ because $M_{B'}$ has rank one.

Case 1.1.1 Assume that $c_1 = 0$.

The structure matrix relative to B' is

$$M_{B'} = \begin{pmatrix} \omega_1^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Now, if we consider the new natural basis $B'' = \{e_1'', e_2'\}$ such that $e_1'' = \dfrac{e_1'}{\omega_1^2}$ or, equivalently, the change of basis matrix

$$P_{B''B'} = \begin{pmatrix} \dfrac{1}{\omega_1^2} & 0 \\ 0 & 1 \end{pmatrix},$$

we have that

$$M_{B''} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Case 1.1.2 If that $c_1 \neq 0$.

Taking $p_2 = \dfrac{\omega_1}{\sqrt{c_1}}$ and the change of basis matrix

$$P_{B''B'} = \begin{pmatrix} \dfrac{1}{\omega_1^2 + c_1\omega_2^2} & 0 \\ 0 & \dfrac{1}{\omega_1^2 + c_1\omega_2^2} \end{pmatrix}$$

we obtain that

$$M_{B''} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Case 1.2 Suppose that $\omega_1 = 0$.

Necessarily $\omega_2 \neq 0$. Now, we consider $p_2 = 0$ and $p_1 = 1$, then B' is a natural basis and

$$M_{B'} = \begin{pmatrix} c_1\omega_2^2 & 1 \\ 0 & 0 \end{pmatrix}.$$

Case 1.2.1 Assume that $c_1 = 0$.

Then the structure matrix is

$$M_{B'} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Case 1.2.2 If $c_1 \neq 0$.

Considering the change of basis matrix

$$P_{B''B'} = \begin{pmatrix} \dfrac{1}{c_1\omega_2^2} & 0 \\ & \\ 0 & \dfrac{1}{\sqrt{c_1}\omega_2} \end{pmatrix}$$

we have that the new structure matrix is

$$M_{B''} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}.$$

Case 2 Suppose that $\omega_1 p_1 + c_1\omega_2 p_2 \neq 0$.

Therefore $\{e_1^2\}$ has not the extension property. Now we look for a natural basis B' such that $M_{B'}$ is as easy as possible.

Since $c_1 = \dfrac{-\omega_1^2}{\omega_2^2}$, then

$$M_B = \begin{pmatrix} \omega_1 & \dfrac{\omega_1^3}{\omega_2^2} \\ \\ \omega_2 & \dfrac{-\omega_1^2}{\omega_2} \end{pmatrix}.$$

In this case, considering the change of basis matrix

$$P_{B''B} = \begin{pmatrix} \dfrac{1}{\omega_1} & 0 \\ \\ 0 & \dfrac{\omega_2}{\omega_1^2} \end{pmatrix}$$

we get that

$$M_{B''} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}.$$

Now, in order to show that the evolution algebras given by the structure matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in certain natural basis $B = \{e_1, e_2\}$ are mutually non-isomorphic we will study some algebraic properties as dimension of annihilator, extension property and the existence of an one-dimension non-degenerate principal ideal. We denote this ideal as $\widehat{I}$. Recall that an ideal is principal if it is generated as an ideal by one element. Now, we are going to focus on the last property: analyzing if there exists an ideal like $\widehat{I}$.

In the first case the evolution algebra associated has an ideal like $\widehat{I}$. It is enough to consider the ideal generates by e_1.

The second evolution algebra which product is $e_1^2 = e_1 + e_2$ and $e_2^2 = -e_1 - e_2$ has not an ideal of the type $\widehat{I}$. Assume, on the contrary that there exist an ideal like $\widehat{I}$ generates by $pe_1 + qe_2 \in A$. Then,

$$e_1(pe_1 + qe_2) = pe_1^2 = p(e_1 + e_2) \in \widehat{I},$$
$$e_2(pe_1 + qe_2) = qe_2^2 = -q(+e_1 + e_2) \in \widehat{I}$$

and on the other hand

$$\begin{aligned}
&(pe_1 + qe_2)^2 = (p^2 - q^2)(e_1 + e_2) \in \widehat{I}, \\
&(pe_1 + qe_2)^3 := (pe_1 + qe_2)(pe_1 + qe_2)^2 = (p^2 - q^2)(p - q)(e_1 + e_2) \in \widehat{I} \\
&\qquad\qquad \vdots \\
&(pe_1 + qe_2)^n := (pe_1 + qe_2)(pe_1 + qe_2)^{n-1} = (p^2 - q^2)(p - q)^{n-2}(e_1 + e_2) \in \widehat{I}
\end{aligned}$$

for every $n \in \mathbb{N}$. We will analyze each case. If $pq \neq 0$ and $p \neq q$ then $\widehat{I}$ has dimension two. If $pq \neq 0$ and $p = q$ then $\widehat{I} =< e_1 + e_2 >$ but $\widehat{I}$ is degenerate. If $p = 0$ or $q = 0$, therefore $\widehat{I} =< e_2 >$ or $\widehat{I} =< e_1 >$ has dimension two. In any case, contradiction.

The third case, the evolution algebra which product is $e_1^2 = e_1$ and $e_2^2 = e_1$ has an ideal of the form $\widehat{I}$. Indeed, we may consider the ideal generates by e_1.

The last evolution algebra given by the structure matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has not an ideal like $\widehat{I}$. Indeed, suppose on the contrary that there exists an element $pe_1 + qe_2$ in A that generates $\widehat{I}$. Then,

$$e_1(pe_1 + qe_2) = 0 \in \widehat{I},$$
$$e_2(pe_1 + qe_2) = qe_2^2 = qe_1 \in \widehat{I}$$

and the other hand

$$\begin{aligned}
&(pe_1 + qe_2)^2 = q^2 e_1 \in \widehat{I}, \\
&\qquad\qquad \vdots \\
&(pe_1 + qe_2)^n := (pe_1 + qe_2)(pe_1 + qe_2)^{n-1} = 0 \in \widehat{I}
\end{aligned}$$

for every $n > 2$.

If $pq \neq 0$ then $\widehat{I}$ has dimension two. Contradiction. If $p = 0$ therefore $\widehat{I} =< e_2 >= \{e_2, e_1\}$ has dimension two and finally if $q = 0$ then $\widehat{I} =< e_1 >= \{e_1\}$ is a degenerate principal ideal. In both cases, contradiction.

Summarizing, we list the algebraic properties belong to each resulting evolution algebra in the following table:

Type	A^2 has the extension property	dimension of ann(A)	A has a one-dimension non-degenerate principal ideal
$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	Yes	1	$I =< e_1 >$
$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$	No	0	No
$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$	Yes	0	$I =< e_1 >$
$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$	Yes	1	No

Case $\dim(A^2) = 2$.

Recall that $M_B = \begin{pmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{pmatrix}$. By Lemma 3.1 and Remark 3.2 we will distinguish different cases depending on the number of non-zero entries in the structure matrix.

Case 1 If $\omega_{11}\omega_{22} \neq 0$.

Applying Lemma 3.1 and considering $r = \dfrac{1}{\omega_{11}}$ and $s = \dfrac{1}{\omega_{22}}$ we obtain that

$$M_{B'}(\alpha, \beta) = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix} \tag{3.2}$$

with $\alpha\beta \neq 1$.

Case 1.1 Assume that $\omega_{12}\omega_{21} \neq 0$.

In which case, we obtain the same structure matrix as case before but with $\alpha\beta \neq 0$:

Case 1.2 Suppose that $\omega_{12} = \omega_{21} = 0$.

Therefore we have the following structure matrix:

$$M_{B'} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Case 1.3 Assume that $\omega_{12} \neq 0$ and $\omega_{21} = 0$.

In this case, the structure matrix is as follows:

$$M_{B'}(\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

with $\alpha \neq 0$.

Case 1.4 Assume that $\omega_{12} = 0$ and $\omega_{21} \neq 0$.

If we consider the natural basis $B'' = \{e_2, e_1\}$ then we are in the same conditions as Case 1.3.

Case 2 If $\omega_{11} = 0$ and $\omega_{22} \neq 0$.

In this case, necessarily $\omega_{21}\omega_{12} \neq 0$. Now, we consider Lemma 3.1 for $r = \dfrac{1}{\sqrt[3]{\omega_{12}\omega_{21}^2}}$ and $s = \dfrac{1}{\sqrt[3]{\omega_{12}^2\omega_{21}}}$ and we obtain

$$M_{B'}(\gamma) = \begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix} \tag{3.4}$$

for some $\gamma \neq 0$.

Case 3 If $\omega_{11} = \omega_{22} = 0$.

This implies that $\omega_{21}\omega_{12} \neq 0$. Reasoning the same way as before, we have that the structure matrix is as follows:

$$M_{B'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Case 4 If $\omega_{11} \neq 0$ and $\omega_{22} = 0$.

Considering the natural basis $B'' = \{e_2, e_1\}$, we are in the same conditions as Case 2.

By Lemma 3.1 and Remark 3.2 we have that the resulting evolution algebras correspond to the structure matrices $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & \gamma \end{pmatrix}$ cannot be isomorphic because there are matrices where the number of non-zero entries is not the same and when is the same, the number of zeros in the main diagonal does not coincide.

In what follows we need to study on the one hand if two structure matrices $M_{B'}(\alpha)$ and $M_{B'}(\alpha')$ as in (3.3) are isomorphic with $\alpha, \alpha' \in \mathbb{K}^\times$ and on the other hand if two structure matrices $M_{B'}(\alpha, \beta)$ and $M_{B'}(\alpha', \beta')$ as in (3.2) are isomorphic given $\alpha, \alpha', \beta, \beta' \in \mathbb{K}^\times$ such that $(1 - \alpha\beta)(1 - \alpha'\beta') \neq 0$. But we can study both cases considering two structure matrices $M_{B'}(\alpha, \beta)$ and $M_{B'}(\alpha', \beta')$ as in (3.2) where $\alpha, \alpha', \beta, \beta' \in \mathbb{K}$ such that $(1-\alpha\beta)(1-\alpha'\beta') \neq 0$. Assume there exists a change of basis matrix $P_{B''B'} = (p_{ij})$. By Theorem 1.3.2 we obtain that

$$p_{11}p_{12} + p_{21}p_{22}\alpha = 0$$

$$p_{11}p_{12}\beta + p_{21}p_{22} = 0$$

Multiplying the first equation by $(-\beta)$ and adding the resulting equation to the second one we have $p_{21}p_{22}(1 - \alpha\beta) = 0$.

Now multiplying the second equation by $(-\alpha)$ and add it to the first one, we obtain $p_{11}p_{12}(1 - \alpha\beta) = 0$.

Since $(1 - \alpha\beta) \neq 0$ then $p_{11}p_{12} = p_{21}p_{22} = 0$.

Case 1 Suppose that $p_{21} = 0$.

Necessarily $p_{12} = 0$ since $p_{11}p_{22} \neq 0$. In this case the structure matrix is

$$M_{B''} = \begin{pmatrix} p_{11} & \dfrac{\alpha p_{22}^2}{p_{11}} \\ \dfrac{p_{11}^2\beta}{p_{22}} & p_{22} \end{pmatrix}. \tag{3.5}$$

In order for $M_{B''}$ to have the same form as in (3.2), necessarily $p_{11} = p_{22} = 1$ and the resulting structure matrix is

$$M_{B''} = \begin{pmatrix} 1 & \alpha \\ \beta & 1 \end{pmatrix}.$$

Case 2 If $p_{11} = 0$.

As $p_{21}p_{12} \neq 0$, then $p_{22} = 0$. In this case, the structure matrix is

$$M_{B''} = \begin{pmatrix} p_{21} & \dfrac{\beta p_{12}^2}{p_{21}} \\ \dfrac{p_{21}^2\alpha}{p_{12}} & p_{12} \end{pmatrix}. \tag{3.6}$$

In order to obtain a structure matrix as in (3.2), we consider $p_{21} = p_{12} = 1$. Then, we obtain that

$$M_{B''} = \begin{pmatrix} 1 & \beta \\ \alpha & 1 \end{pmatrix}.$$

Now, we will study if given $\gamma, \gamma' \in \mathbb{K}^\times$, two structure matrices $M_{B'}(\gamma)$ and $M_{B'}(\gamma')$ as in (3.4) are isomorphic.

Assume there exists a change of basis matrix $P_{B''B'} = (p_{ij})$. By Theorem 1.3.2 we deduce that

$$p_{21}p_{22} = 0$$
$$p_{11}p_{12} + p_{21}p_{22}\alpha = 0$$

This implies that $p_{21}p_{22} = 0$ and $p_{11}p_{12} = 0$. Without loss of generality, we may suppose that $p_{21} = 0$ and $p_{12} = 0$. Therefore, the structure matrix is

$$M_{B''} = \begin{pmatrix} 0 & \dfrac{p_{22}^2}{p_{11}} \\ \dfrac{p_{11}^2}{p_{22}} & \gamma p_{22} \end{pmatrix}. \tag{3.7}$$

In order for $M_{B''}$ to be of the same type as in (3.4), necessarily $\dfrac{p_{22}^2}{p_{11}} = 1$ and $\dfrac{p_{11}^2}{p_{22}} = 1$ we have that $p_{11} = p_{22}^2$, $p_{22} = p_{11}^2$. Equivalently $p_{22} = p_{22}^4$. This implies that $p_{22}^3 = 1$. Then, the structure matrix is as follows:

$$M_{B''} = \begin{pmatrix} 0 & 1 \\ 1 & \gamma' \end{pmatrix}$$

with $\dfrac{\gamma'}{\gamma} = p_{22} = \zeta$. Recall that ζ is a third root of the unit.

□

Chapter 4

Classification of three-dimensional evolution algebras

The aim of this chapter is to obtain the classification of three-dimensional evolution algebras.

We will prove that there are 116 types of three-dimensional evolution algebras. We organize all of them in Tables 1-24. The matrices appearing in different rows of a same table are not isomorphic(they do not generate evolution algebras isomorphic) and different tables give evolution algebras neither isomorphic. In some case different values of the parameter for matrices in the same row give isomorphic evolution algebras. These cases are displayed in Tables $1'$-$23'$.

Recall that we work in this chapter with evolution algebras over a field $\mathbb{K}$ of characteristic different from 2 and in which every polynomial of the form $x^n - \alpha$, for $n = 2, 3, 7$ and $\alpha \in \mathbb{K}$ has a root in the field.

We start by analyzing the action of the group $S_3 \rtimes (\mathbb{K}^\times)^3$ on $M_3(\mathbb{K})$. The orbits of this action will completely determine the non-isomorphic evolution algebras A when $\dim(A^2) = 3$ and in some cases when $\dim(A^2) = 2$.

4.1. Action of $S_3 \rtimes (\mathbb{K}^\times)^3$ on $\mathrm{M}_3(\mathbb{K})$

Let $\mathbb{K}$ be a field. For every $\alpha, \beta, \gamma \in \mathbb{K}^\times$, we define the matrices:

$$\Pi_1(\alpha) := \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \Pi_2(\beta) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ \Pi_3(\gamma) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \gamma \end{pmatrix}.$$

It is easy to prove that they commute each other. This implies that

$$G = \left\{\Pi_1(\alpha)\Pi_2(\beta)\Pi_3(\gamma) \mid \alpha, \beta, \gamma \in \mathbb{K}^\times\right\} = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{K}^\times \right\}$$

is an abelian subgroup of $\mathrm{GL}_3(\mathbb{K})$. We will denote the diagonal matrix $\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}$ by (α, β, γ). With this notation in mind, it is immediate to see that $G \cong \mathbb{K}^\times \times \mathbb{K}^\times \times \mathbb{K}^\times$ with product given by $(\alpha, \beta, \gamma)(\alpha', \beta', \gamma') := (\alpha\alpha', \beta\beta', \gamma\gamma')$.

Now, consider the symmetric group S_3 of all permutations of the set $\{1, 2, 3\}$. The standard notation for S_3 is:

$$S_3 = \{id, (1,2), (1,3), (2,3), (1,2,3), (1,3,2)\},$$

where id is the identity map, (i, j) is the permutation that sends the element i into the element j and (i, j, k) is the permutation sending i to j, j to k and k to i, for $\{i, j, k\} = \{1, 2, 3\}$.

We may identify S_3 with the set

$$\left\{ \mathrm{id}_3, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right\} \tag{4.1}$$

in the following way: *id* is identified with the identity matrix id_3, $(1,2)$ with the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

because this matrix appears when permuting the first and the second columns of id_3, etc. The matrices in (4.1) are called 3×3 **permutation matrices**.

From now on, we will consider that S_3 consists of the permutation matrices. This allows to see S_3 as a subgroup of $\mathrm{GL}_3(\mathbb{K})$. Denote by H the subgroup of $\mathrm{GL}_3(\mathbb{K})$ generated by S_3 and $(\mathbb{K}^\times)^3$.

It is not difficult to verify that for every $\sigma \in S_3$ and every $(\lambda_1, \lambda_2, \lambda_3) \in (\mathbb{K}^\times)^3$ its product is as follows:

$$(\lambda_1, \lambda_2, \lambda_3)\sigma = \sigma(\lambda_{\sigma(1)}, \lambda_{\sigma(2)}, \lambda_{\sigma(3)}).$$

Therefore, we may write

$$H = \{\sigma(\alpha, \beta, \gamma) \mid \sigma \in S_3,\ (\alpha, \beta, \gamma) \in (\mathbb{K}^\times)^3\}.$$

The multiplication in H is given by

$$\begin{aligned} \sigma(\alpha_1, \alpha_2, \alpha_3)\tau(\beta_1, \beta_2, \beta_3) &= \sigma\tau(\alpha_{\tau(1)}, \alpha_{\tau(2)}, \alpha_{\tau(3)})(\beta_1, \beta_2, \beta_3) \qquad (4.2) \\ &= \sigma\tau(\alpha_{\tau(1)}\beta_1, \alpha_{\tau(2)}\beta_2, \alpha_{\tau(3)}\beta_3). \end{aligned}$$

This is called a semidirect product of the group of S_3 and $(\mathbb{K}^\times)^3$ respect to the product (4.2) and it is denoted by $S_3 \rtimes (\mathbb{K}^\times)^3$. So, from now on, we use the notation $S_3 \rtimes (\mathbb{K}^\times)^3$ instead of H. Notice that the elements of $S_3 \rtimes (\mathbb{K}^\times)^3$ coincide with the elements of

$$\left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 \\ \beta & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha \\ 0 & \beta & 0 \\ \gamma & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & \beta \\ 0 & \gamma & 0 \end{pmatrix}, \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \gamma & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha \\ \beta & 0 & 0 \\ 0 & \gamma & 0 \end{pmatrix} \mid \alpha, \beta, \gamma \in \mathbb{K}^\times \right\} \qquad (4.3)$$

We define the action of $S_3 \rtimes (\mathbb{K}^\times)^3$ on the set $M_3(\mathbb{K})$ given by:

$$\sigma \cdot \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} := \begin{pmatrix} \omega_{\sigma(1)\sigma(1)} & \omega_{\sigma(1)\sigma(2)} & \omega_{\sigma(1)\sigma(3)} \\ \omega_{\sigma(2)\sigma(1)} & \omega_{\sigma(2)\sigma(2)} & \omega_{\sigma(2)\sigma(3)} \\ \omega_{\sigma(3)\sigma(1)} & \omega_{\sigma(3)\sigma(2)} & \omega_{\sigma(3)\sigma(3)} \end{pmatrix}, \tag{4.4}$$

$$(\alpha, \beta, \gamma) \cdot \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \omega_{31} & \omega_{32} & \omega_{33} \end{pmatrix} := \begin{pmatrix} \alpha\omega_{11} & \dfrac{\beta^2}{\alpha}\omega_{12} & \dfrac{\gamma^2}{\alpha}\omega_{13} \\ \frac{\alpha^2}{\beta}\omega_{21} & \beta\omega_{22} & \dfrac{\gamma^2}{\beta}\omega_{23} \\ \frac{\alpha^2}{\gamma}\omega_{31} & \dfrac{\beta^2}{\gamma}\omega_{32} & \gamma\omega_{33} \end{pmatrix} \tag{4.5}$$

for every $\sigma \in S_3$ and every $(\alpha, \beta, \gamma) \in (\mathbb{K}^\times)^3$.

For arbitrary $P \in S_3 \rtimes (\mathbb{K}^\times)^3$ and $M \in M_3(\mathbb{K})$, the action of P on M can be formulated as follows:

$$P \cdot M := P^{-1} M P^{(2)}. \tag{4.6}$$

Remark 4.1.1

The action given in (4.6) has been inspired by Condition (1.8) in Theorem 1.3.2. Notice that any matrix P in $S_3 \rtimes (\mathbb{K}^\times)^3$ is a change of basis matrix from a natural basis B into another natural basis B' and the relationship among the structure matrices M_B and $M_{B'}$ and the matrix P is as given in Condition (1.8), that is, $P^{-1} M_B P^{(2)} = M'_B$. This is the reason because we define the action of P on M_B by:

$$M_P \cdot M_B = P^{-1} M_B P^{(2)}.$$

The result that follows will be very useful in Theorem 4.2.2.

Proposition 4.1.2

For any $P \in S_3 \rtimes (\mathbb{K}^\times)^3$ and any $M \in M_3(\mathbb{K})$ we have:

(I) *The number of zero entries in M coincides with the number of zero entries in $P \cdot M$.*

(II) *The number of zero entries in the main diagonal of M coincides with the number of zero entries in the main diagonal of $P \cdot M$.*

(III) *The rank of M and the rank of $P \cdot M$ coincide.*

(IV) *Assume that M is the structure matrix of an evolution algebra A relative to a natural basis B. Assume that $A^2 = A$. If N is the structure matrix of A relative to a natural basis B' then there exists $Q \in S_3 \rtimes (\mathbb{K}^\times)^3$ such that $N = Q \cdot M$.*

Demostración. Fix an element P in $S_3 \rtimes (\mathbb{K}^\times)^3$. Then there exist $\sigma \in S_3$ and $(\alpha, \beta, \gamma) \in (\mathbb{K}^\times)^3$ such that $P = \sigma(\alpha, \beta, \gamma)$. Therefore $P \cdot M = (\sigma(\alpha, \beta, \gamma)) \cdot M = \sigma \cdot ((\alpha, \beta, \gamma) \cdot M)$. Item (I) and (II) follows by (4.4) and (4.5). Item (III) is easy to show because $P \cdot M = P^{-1} M P^{(2)}$ and P is an invertible matrix. Finally, (IV) follows from the definition of the action and [15, Theorem 4.4]. □

4.2. Three-dimensional evolution algebras

The aim of this section is to determine all the three-dimensional evolution algebras over a field $\mathbb{K}$ of characteristic different from 2 and such that for any $k \in \mathbb{K}$ and $n = 2$ or $n = 7$ or $n = 3$, every polynomial of the form $x^n - k$ has a root in the field. For our purposes, we divide our study in different cases, depending on the dimension of A^2.

Definition 4.2.1

A three dimensional evolution algebra A is said to have **Property** (2LI) if for any basis $\{e_1, e_2, e_3\}$ of A, the ideal A^2 has dimension two and it is generated by $\{e_i^2, e_j^2\}$, for $i, j \in \{1, 2, 3\}$ with $i \neq j$.

Theorem 4.2.2

Let A be a three-dimensional evolution $\mathbb{K}$-algebra where $\mathbb{K}$ is a field of characteristic different from 2 and such that for any $k \in \mathbb{K}$ the polynomial of the form $x^n - k$ has a root whenever $n = 2, 3, 7$.

(I) *If* $\dim(A^2) = 0$ *then* $M_B = 0$ *for any natural basis* B *of* A*.*

(II) *If* $\dim(A^2) = 1$ *then there exists a natural basis* B *such that* M_B *is one of the seven matrices given in Table 1. All of them produce mutually non-isomorphic evolution algebras. The algebras in this case are completely classified by the following properties: having* A^2 *the extension property,* $\dim(\mathrm{ann}(A))$*, and if* A *has a principal ideal of dimension two which is degenerate.*

(III) *If* $\dim(A^2) = 2$*, then then there exists a natural basis* B *such that* M_B *is one of the matrices given in the first column of Tables 2 to 17. They are all mutually non-isomorphic except in the cases shown in Tables 2' to 17'. There are 57 possible cases. Let* $B = \{e_1, e_2, e_3\}$ *be such that* $\{e_1^2, e_2^2\}$ *is a basis of* A^2 *and* $e_3^2 = c_1 e_1^2 + c_2 e_2^2$*, for* $c_1, c_2 \in \mathbb{K}$*.*

(a) *If* $c_1 c_2 \neq 0$*, then the evolution algebras have* $\dim(\mathrm{ann}(A)) = 0$*; the algebra* A *has Property* (2LI) *and the number of non-zero entries in* M_B *can be 4 to 9.*

(b) *If* $c_1 = 0$ *and* $c_2 \neq 0$ *(the case* $c_2 = 0$ *and* $c_1 \neq 0$ *is analogous), then the evolution algebras appearing have* $\dim(\mathrm{ann}(A)) = 0$*; the algebra* A *has not Property* (2LI) *and the number of non-zero entries in the set that follows can be from 4 to 9.*

$$S=\left\{\begin{pmatrix}1&0&0\\0&1&1\\0&\alpha&\alpha\end{pmatrix},\begin{pmatrix}0&1&1\\1&0&0\\\alpha&0&0\end{pmatrix},\begin{pmatrix}\alpha&1&1\\0&1&1\\0&\beta&\beta\end{pmatrix},\begin{pmatrix}0&0&0\\1&1&1\\\alpha&\beta&\beta\end{pmatrix},\right.$$
$$\left.\begin{pmatrix}1&1&1\\\alpha&0&0\\\beta&0&0\end{pmatrix},\begin{pmatrix}1&0&0\\\alpha&1&1\\\beta&\gamma&\gamma\end{pmatrix},\begin{pmatrix}0&1&1\\\alpha&1&1\\\beta&\gamma&\gamma\end{pmatrix},\begin{pmatrix}\alpha&1&1\\\beta&1&1\\\gamma&\lambda&\lambda\end{pmatrix}\right\}.$$

(c) *If* $c_1, c_2 = 0$*, then the evolution algebras appearing have* $\dim(\mathrm{ann}(A)) = 1$*; the algebra* A *has not Property* (2LI) *and the number of non-zero entries in rows one and two can be from 1 to 4.*

(IV) *If* $\dim(A^2) = 3$ *then there exists a natural basis* B *such that* M_B *is one of the matrices given in the first column of Tables 18 to 24. They are all mutually non-isomorphic except in the cases shown in Tables 19' to 23'. They are completely determined by the number of non-zero entries in* M_B*. There are 51 possible cases.*

Proof.

Fix a three-dimensional evolution algebra A and a natural basis $B = \{e_1, e_2, e_3\}$. Let M_B be the structure matrix of A relative to B:

$$M_B=\begin{pmatrix}\omega_{11}&\omega_{12}&\omega_{13}\\\omega_{21}&\omega_{22}&\omega_{23}\\\omega_{31}&\omega_{32}&\omega_{33}\end{pmatrix}.$$

In order to classify all the three dimensional evolution algebras we try to find a basis of A for which its structure matrix has an expression as easy as possible, where by 'easy' we mean with the maximum number of 0, 1 and -1 in the entries.

Case $\dim(A^2) = 0$**.**

Then $M_B = 0$ and there is a unique evolution algebra.

Case $\dim(A^2) = 1$**.**

Without loss of generality we may assume $e_1^2 \neq 0$. Write $e_1^2 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3$, where $\omega_i \in \mathbb{K}$ and $\omega_i \neq 0$ for some $i \in \{1,2,3\}$. Note that $\{e_1^2\}$ is a basis of A^2. Since $e_2^2, e_3^2 \in A^2$, there exist $c_1, c_2 \in \mathbb{K}$ such that

$$e_2^2 = c_1 e_1^2 = c_1(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3),$$
$$e_3^2 = c_2 e_1^2 = c_2(\omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3).$$

Then

$$M_B = \begin{pmatrix} \omega_1 & c_1\omega_1 & c_2\omega_1 \\ \omega_2 & c_1\omega_2 & c_2\omega_2 \\ \omega_3 & c_1\omega_3 & c_2\omega_3 \end{pmatrix}.$$

We start the study of this case by paying attention to the algebraic properties of the evolution algebras that we consider. To see which are the matrices that appear as change of basis matrices, we refer the reader to Remark 4.2.5.

We analyze when A^2 has the extension property. That is, if there exists a natural basis $B' = \{e_1', e_2', e_3'\}$ of A with

$$\begin{aligned} e_1' &= e_1^2 = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 \qquad (4.7)\\ e_2' &= \alpha e_1 + \beta e_2 + \gamma e_3 \\ e_3' &= \delta e_1 + \nu e_2 + \eta e_3, \end{aligned}$$

for some $\alpha, \beta, \gamma, \delta, \nu, \eta \in \mathbb{K}$ that we may choose satisfying $\nu(\beta - \gamma) \neq 0$. This condition will be useful below. Being B' a basis implies

$$|\, P_{B'B} \,| = \begin{vmatrix} \omega_1 & \alpha & \delta \\ \omega_2 & \beta & \nu \\ \omega_3 & \gamma & \eta \end{vmatrix} \neq 0. \qquad (4.8)$$

By Theorem 1.3.2, B' is a natural basis if and only if the following conditions are satisfied.

$$\alpha\omega_1 + \beta\omega_2 c_1 + \gamma\omega_3 c_2 = 0 \qquad (4.9)$$

$$\delta\omega_1 + \nu\omega_2 c_1 + \eta\omega_3 c_2 = 0 \qquad (4.10)$$

$$\alpha\delta + \beta\nu c_1 + \gamma\eta c_2 = 0$$

In these conditions, the structure matrix of A relative to B' is:

$$M_{B'} = \begin{pmatrix} \omega_1^2 + \omega_2^2 c_1 + \omega_3^2 c_2 & \alpha^2 + \beta^2 c_1 + \gamma^2 c_2 & \delta^2 + \nu^2 c_1 + \eta^2 c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To find the different mutually non-isomorphic evolution algebras it will be very useful to study if they have a two-dimensional evolution ideal generated by one element which is degenerate as an evolution algebra.

Now, we start with the analysis of the different cases.

Case 1 Suppose that $\omega_1 \neq 0$.

By changing the basis, we may consider $e_1^2 = e_1 + \omega_2 e_2 + \omega_3 e_3$. Using (4.9) we get $\alpha = -(\beta\omega_2 c_1 + \gamma\omega_3 c_2)$ and by (4.10), $\delta = -(\nu\omega_2 c_1 + \eta\omega_3 c_2)$. If we replace α and δ in (4.8) we obtain that:

$$| P_{B'B} | = (1 + \omega_2^2 c_1 + \omega_3^2 c_2)\nu(\beta - \gamma).$$

Now we distinguish if $| P_{B'B} |$ is zero or not. This happens depending on $1 + \omega_2^2 c_1 + \omega_3^2 c_2$ being zero or not since we had chosen $\beta, \gamma, \nu \in \mathbb{K}$ such that $\nu(\beta - \gamma) \neq 0$.

Case 1.1 Assume $1 + \omega_2^2 c_1 + \omega_3^2 c_2 = 0$.

In this case A^2 has not the extension property since $|P_{B'B}| = 0$. We will analyze what happens when $1 + \omega_3^2 c_2 \neq 0$ and when $1 + \omega_3^2 c_2 = 0$.

Case 1.1.1 If $1 + \omega_3^2 c_2 \neq 0$.

Note that $\omega_2^2 c_1 \neq 0$ since otherwise we get a contradiction. Then $c_1 = \dfrac{-1-\omega_3^2 c_2}{\omega_2^2}$. In this case, the structure matrix is:

$$M_B = \begin{pmatrix} 1 & \dfrac{-1-\omega_3^2 c_2}{\omega_2^2} & c_2 \\ \omega_2 & \dfrac{-1-\omega_3^2 c_2}{\omega_2} & c_2\omega_2 \\ \omega_3 & \dfrac{(-1-\omega_3^2 c_2)\omega_3}{\omega_2^2} & c_2\omega_3 \end{pmatrix}.$$

Case 1.1.1.1 Suppose that $\omega_3 \neq 0$.

If we take the natural basis $B'' = \{e_1, \omega_2 e_2, \omega_3 e_3\}$, then

$$M_{B''} = \begin{pmatrix} 1 & -1-\omega_3^2 c_2 & \omega_3^2 c_2 \\ 1 & -1-\omega_3^2 c_2 & \omega_3^2 c_2 \\ 1 & -1-\omega_3^2 c_2 & \omega_3^2 c_2 \end{pmatrix}. \tag{4.11}$$

We are going to distinguish two cases: $c_2 = 0$ and $c_2 \neq 0$.

Assume first $c_2 = 0$. Then

$$M_{B''} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

By considering another change of basis we find a structure matrix with more zeros. Concretely, let $B''' = \{e_2, e_1 + e_3, e_3\}$. Then

$$M_{B'''} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In what follows we will assume that $c_2 \neq 0$. We recall that we are considering the structure matrix given in (4.11).

Now, for B''' the natural basis given by

$$P_{B'''B''} = \begin{pmatrix} \frac{1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & \frac{-1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & -(\omega_3^2c_2) \\ \frac{-1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & \frac{1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & 0 \\ \frac{1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & \frac{-1+(\omega_3^2c_2)^3+2(\omega_3^2c_2)^2+(\omega_3^2c_2)}{2(1+\omega_3^2c_2)} & 1 \end{pmatrix}$$

we obtain:

$$M_{B'''} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that $|P_{B'''B''}| = -2(\omega_3^2c_2)(1+\omega_3^2c_2)^2 \neq 0$ because $\omega_3^2c_2 \neq 0$ and $\omega_3^2c_2 \neq -1$.

Case 1.1.1.2 Suppose that $\omega_3 = 0$.

Then $1+\omega_2^2c_1 = 0$ and necessarily $\omega_2^2c_1 \neq 0$. In this case,

$$M_B = \begin{pmatrix} 1 & \frac{-1}{\omega_2^2} & c_2 \\ \omega_2 & \frac{-1}{\omega_2} & c_2\omega_2 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.12}$$

Again we will distinguish two cases depending on c_2.

Assume $c_2 \neq 0$. Take $B'' = \{e_1, \omega_2 e_2, \frac{e_3}{\sqrt{c_2}}\}$. Then

$$M_{B''} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which has already appeared.

Suppose $c_2 = 0$. Then, for $B'' = \{e_1, \omega_2 e_2, e_3\}$ we have

$$M_{B''} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

matrix that has already appeared.

Case 1.1.2 Suppose that $1 + \omega_3^2 c_2 = 0$.

This implies that $\omega_3^2 c_2 \neq 0$ and $\omega_2^2 c_1 = 0$.

Case 1.1.2.1 Assume $c_1 \neq 0$.

This implies that $\omega_2 = 0$. Moreover, as $\omega_3 \neq 0$, necessarily $c_2 = \dfrac{-1}{\omega_3^2}$. If we take the natural basis $B'' = \{e_1, e_3, e_2\}$, then

$$M_{B''} = \begin{pmatrix} 1 & \frac{-1}{\omega_3^2} & c_1 \\ \omega_3 & \frac{-1}{\omega_3} & \omega_3 c_1 \\ 0 & 0 & 0 \end{pmatrix}$$

and we are as in Case 1.1.1.2.

Case 1.1.2.2 Suppose $c_1 = 0$ and $\omega_2 = 0$.

Take $B'' = \{e_1, \omega_3 e_3, e_2\}$. Then

$$M_{B''} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

again.

Case 1.1.2.3 Assume $c_1 = 0$ and $\omega_2 \neq 0$.

Taking $B'' = \{e_1, e_3, e_2\}$, we are in the same conditions as in Case 1.1.1.1 with $c_2 = 0$.

Case 1.2 Assume $1 + \omega_2^2 c_1 + \omega_3^2 c_2 \neq 0$.

We will prove that A^2 has the extension property. In any subcase we will provide with a natural basis for A one of which elements constitutes a natural basis of A^2.

Case 1.2.1 Suppose that $c_1 = c_2 = 0$.

Consider the natural basis $B' = \{e_1^2, e_2 + e_3, 2e_2 + e_3\}$. Then

$$M_{B'} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We claim that this evolution algebra does not have a two-dimensional evolution ideal generated by one element. To prove this, consider $f = me_1 + ne_2 + pe_3$. Then the ideal I that it generates is the linear span of $\{f\} \cup \{m^i e_1\}_{i\in\mathbb{N}}$. In order for I to have a natural basis with two elements, necessarily $m = 0$, implying that the dimension of I is one, a contradiction.

Case 1.2.2 Assume that $c_1 = 0$ and $c_2 \neq 0$.

Then $1 + c_2\omega_3^2 \neq 0$. For $B' = \{e_1 + \omega_2 e_2 + \omega_3 e_3, e_2, -\omega_3 c_2 e_1 + e_2 + e_3\}$ the structure matrix is

$$M_{B'} = \begin{pmatrix} 1 + c_2\omega_3^2 & 0 & c_2(1 + c_2\omega_3^2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that A^2 has the extension property because the first element in B' is e_1^2, which is a natural basis of A^2. Consider $B'' = \left\{\frac{e_1}{1 + c_2\omega_3^2}\ e_1, e_2, \frac{e_3}{\sqrt{c_2}(1 + c_2\omega_3^2)}\right\}$. Then

$$M_{B''} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We claim that this evolution algebra does not have a degenerate two-dimensional evolution ideal generated by one element. Let $f = me_1 + ne_2 + pe_3$. Then the ideal generated by f, say I, is the linear span of $\{f\} \cup \{pe_1\} \cup \{me_1\} \cup \{(m^2 + p^2)m^i e_1\}_{i\in\mathbb{N}\cup\{0\}} \cup \{(m^2 + p^2)^2 m^i e_1\}_{i\in\mathbb{N}\cup\{0\}}$. After some computations, in order for I to have dimension 2 and to be degenerate evolution ideal implies $m = 0$ or $p = 0$, a contradiction.

Case 1.2.3 If $c_1 \neq 0$ and $c_2 \neq 0$.

Case 1.2.3.1 Assume $1 + \omega_2^2 c_1 \neq 0$.

For B' the natural basis such that

$$P_{B'B} = \begin{pmatrix} 1 & -\omega_2 c_1 & \dfrac{-\omega_3 c_2}{1+c_1\omega_2^2} \\ \omega_2 & 1 & \dfrac{-\omega_3\omega_2 c_2}{1+c_1\omega_2^2} \\ \omega_3 & 0 & 1 \end{pmatrix},$$

we obtain that

$$M_{B'} = \begin{pmatrix} 1+\omega_2^2c_1+\omega_3^2c_2 & c_1\left(1+c_1\omega_2^2\right) & \dfrac{c_2\left(1+\omega_2^2c_1+\omega_3^2c_2\right)}{(1+c_1\omega_2^2)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, consider the natural basis $B'' = \{f_1, f_2, f_3\}$ such that

$$P_{B''B'} = \begin{pmatrix} \dfrac{1}{1+\omega_2^2c_1+\omega_3^2c_2} & 0 & 0 \\ 0 & \dfrac{1}{\sqrt{c_1\left(1+c_1\omega_2^2\right)\left(1+\omega_2^2c_1+\omega_3^2c_2\right)}} & 0 \\ 0 & 0 & \dfrac{\sqrt{1+c_1\omega_2^2}}{\sqrt{c_2}(1+c_1\omega_2^2+c_2\omega_3^2)} \end{pmatrix}$$

and the structure matrix is:

$$M_{B''} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we will see that this evolution algebra does not have a degenerate two-dimensional evolution ideal generated by one element. Indeed, let $f = me_1 + ne_2 + pe_3$ be an element of evolution algebra, then the ideal generated by f, say I, is the linear span of $\{f\}\cup\{me_1\}\cup\{ne_1\}\cup\{qe_1\}\cup\{(m^2+n^2+p^2)m^ie_1\}_{i\in\mathbb{N}\cup\{0\}}\cup\{(m^2+p^2)^2m^ie_1\}_{i\in\mathbb{N}\cup\{0\}}$. After some computations, in order for I to have dimension 2 and to be degenerate evolution ideal implies $m=0$ or $n=0$ or $p=0$, a contradiction.

Case 1.2.3.2 Assume $1+\omega_2^2c_1 = 0$.

Then $\omega_2\omega_3c_1c_2 \neq 0$ and so $c_1 = \dfrac{-1}{\omega_2^2}$. For B' such that

$$P_{B'B} = \begin{pmatrix} 1 & \frac{-c_2}{2} & \frac{2-\omega_3^2 c_2}{2} \\ \omega_2 & \frac{\omega_2 c_2}{2} & \omega_2(1+\frac{c_2\omega_3^2}{2}) \\ \omega_3 & \frac{1}{\omega_3} & \omega_3 \end{pmatrix}$$

we have

$$M_{B'} = \begin{pmatrix} \omega_3^2 c_2 & \frac{c_2}{\omega_3^2} & -\omega_3^2 c_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Now, we consider the natural basis B'' for which

$$P_{B''B'} = \begin{pmatrix} \frac{1}{\omega_3^2 c_2} & 0 & 0 \\ 0 & \frac{1}{c_2} & 0 \\ 0 & 0 & \frac{\sqrt{-1}}{\omega_3^2 c_2} \end{pmatrix}.$$

Then, the structure matrix is

$$M_{B''} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which has already appeared.

Case 1.2.4 Suppose that $c_1 \neq 0$ and $c_2 = 0$.

Considering the natural basis $B'' = \{e_1, e_3, e_2\}$ we obtain

$$M_{B''} = \begin{pmatrix} 1 & 0 & c_1 \\ \omega_3 & 0 & \omega_3 c_1 \\ \omega_2 & 0 & \omega_2 c_1 \end{pmatrix},$$

and we are in the same conditions as in Case 1.2.1.2.

Case 2 Suppose that $\omega_1 = 0$.

The structure matrix of the evolution algebra is

$$M_B = \begin{pmatrix} 0 & 0 & 0 \\ \omega_2 & \omega_2 c_1 & \omega_2 c_2 \\ \omega_3 & \omega_3 c_1 & \omega_3 c_2 \end{pmatrix}.$$

Necessarily there exists $i \in \{2, 3\}$ such that $\omega_i \neq 0$. Without loss of generality we assume $\omega_2 \neq 0$.

Case 2.1 Assume $c_1 \neq 0$.

Consider the natural basis $B'' = \{e_2, e_3, e_1\}$. Then

$$M_{B''} = \begin{pmatrix} \omega_2 c_1 & \omega_2 c_2 & 1 \\ \omega_3 c_1 & \omega_3 c_2 & \omega_3 \\ 0 & 0 & 0 \end{pmatrix}$$

and we are in the same conditions as in Case 1.

Case 2.2 If $c_1 = 0$.

Case 2.2.1 Assume $c_2\omega_3 \neq 0$.

Taking the natural basis $B'' = \{e_3, e_2, e_1\}$, then

$$M_{B''} = \begin{pmatrix} \omega_3 c_2 & 0 & \omega_3 \\ \omega_2 c_2 & 0 & \omega_2 \\ 0 & 0 & 0 \end{pmatrix}$$

and we are in the same conditions as in Case 1.

Case 2.2.2 Suppose that $c_2\omega_3 = 0$.

Case 2.2.2.1 Assume $c_2 = 0$.

Take the natural basis $B' = \{\omega_2 e_2 + \omega_3 e_3, \frac{e_3}{\omega_2}, e_1\}$. Then

$$M_{B'} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that A^2 has the extension property.

Case 2.2.2.2 Assume $c_2 \neq 0$.

Then $\omega_3 = 0$. For $B' = \{\omega_2 e_2, e_1, \frac{e_3}{\sqrt{c_2}}\}$ we have

$$M_{B'} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, A^2 has also the extension property.

We have completed the study of all the cases and will list them in a table. All of them produces evolution algebras A such that $\dim(A) = 3$ and $\dim(A^2) = 1$. They are mutually non-isomorphic, as will be clear from the table. We specify the following properties, already studied, that are invariant under isomorphisms of evolution algebras: If A^2 has the extension property, the dimension of the annihilator of A, and if A has a two-dimensional evolution ideal, which is degenerate as an evolution algebra, and which is principal. To compute the dimension of the annihilator we have used [4, Proposition 2.18].

Type	A^2 has the extension property	dimension of ann(A)	A has a principal degenerate two-dimensional evolution ideal
$\begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	No	0	$I =< e_3 >$
$\begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	No	1	$I =< e_1 + e_2 + e_3 >$
$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	0	No
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	No
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	No
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	1	$I =< e_3 >$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	Yes	2	$I =< e_3 >$

TABLE 1. $\dim(A^2) = 1$.

Case $\dim(A^2) = 2$**.**

The first step is to compute the possible change of basis matrices $P_{B'B}$ for natural basis B and B'. Without loss of generality, we may assume that there exists a natural basis $B = \{e_1, e_2, e_3\}$ such that

$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & c_1\omega_{11} + c_2\omega_{12} \\ \omega_{21} & \omega_{22} & c_1\omega_{21} + c_2\omega_{22} \\ \omega_{31} & \omega_{32} & c_1\omega_{31} + c_2\omega_{32} \end{pmatrix} \tag{4.13}$$

for some $c_1, c_2 \in \mathbb{K}$ with $\omega_{11}\omega_{22} - \omega_{12}\omega_{21} \neq 0$.

Let B' be another natural basis and let $P_{B'B}$ be the change of basis matrix. Write

$$P_{B'B} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Since B' is a natural basis, by (1.8) it verifies:

$$\left\{ \begin{array}{rl} \omega_{11}p_{11}p_{12} + \omega_{12}p_{21}p_{22} + (\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{32} = & 0 \\ \omega_{21}p_{11}p_{12} + \omega_{22}p_{21}p_{22} + (\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{32} = & 0 \\ \omega_{31}p_{11}p_{12} + \omega_{32}p_{21}p_{22} + (\omega_{31}c_1 + \omega_{32}c_2)p_{31}p_{32} = & 0 \end{array} \right. \tag{4.14}$$

$$\left\{ \begin{array}{rl} \omega_{11}p_{11}p_{13} + \omega_{12}p_{21}p_{23} + (\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{33} = & 0 \\ \omega_{21}p_{11}p_{13} + \omega_{22}p_{21}p_{23} + (\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{33} = & 0 \\ \omega_{31}p_{11}p_{13} + \omega_{32}p_{21}p_{23} + (\omega_{31}c_1 + \omega_{32}c_2)p_{31}p_{33} = & 0 \end{array} \right. \tag{4.15}$$

$$\left\{ \begin{array}{rl} \omega_{11}p_{12}p_{13} + \omega_{12}p_{22}p_{23} + (\omega_{11}c_1 + \omega_{12}c_2)p_{32}p_{33} = & 0 \\ \omega_{21}p_{12}p_{13} + \omega_{22}p_{22}p_{23} + (\omega_{21}c_1 + \omega_{22}c_2)p_{32}p_{33} = & 0 \\ \omega_{31}p_{12}p_{13} + \omega_{32}p_{22}p_{23} + (\omega_{31}c_1 + \omega_{32}c_2)p_{32}p_{33} = & 0 \end{array} \right. \tag{4.16}$$

We consider the homogeneous system (4.14) in the three variables $p_{11}p_{12}$, $p_{21}p_{22}$ and $p_{31}p_{32}$. Taking into account that the rank of this system is 2, we may compute its solutions as follows:

$$p_{11}p_{12} = \frac{\begin{vmatrix} -(\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{32} & \omega_{12} \\ -(\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{32} & \omega_{22} \end{vmatrix}}{\omega_{11}\omega_{22} - \omega_{21}\omega_{12}} = -c_1p_{31}p_{32} \tag{4.17}$$

$$p_{21}p_{22} = \frac{\begin{vmatrix} \omega_{11} & -(\omega_{11}c_1 + \omega_{12}c_2)p_{31}p_{32} \\ \omega_{21} & -(\omega_{21}c_1 + \omega_{22}c_2)p_{31}p_{32} \end{vmatrix}}{\omega_{11}\omega_{22} - \omega_{21}\omega_{12}} = -c_2p_{31}p_{32} \tag{4.18}$$

In an analogous way, we may consider the systems given in (4.15) and (4.16). Their solutions can be computed as follows:

$$p_{11}p_{13} = -c_1p_{31}p_{33}; \quad p_{21}p_{23} = -c_2p_{31}p_{33}; \tag{4.19}$$

and

$$p_{12}p_{13} = -c_1p_{32}p_{33}; \quad p_{22}p_{23} = -c_2p_{32}p_{33}. \tag{4.20}$$

Case 1 $c_1c_2 \neq 0$.

In this case the annihilator is zero because there cannot be a column of zeros (apply [4, Proposition 2.18]). It is clear to see that all the evolution algebras appearing in this case have Property (2LI).

In what follows we prove that $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$. Note that elements in $S_3 \rtimes (\mathbb{K}^\times)^3$ are those invertible matrices in $\mathrm{M}_3(\mathbb{K})$ having two zeros in every row and every column (see (4.3)). We will do in two steps: first we will show, by way of contradiction, that there can not exists a row (or column) with all entries different from zero and then we will prove that there are two zero entries in every row and column.
Assume, for example, that $p_{31}p_{32}p_{33} \neq 0$. By (4.19) and (4.20) we have that $p_{11}p_{12}p_{13} \neq 0$ and $p_{21}p_{22}p_{23} \neq 0$. If we replace $p_{11} = \frac{-c_1p_{31}p_{32}}{p_{12}}$ and $p_{21} = \frac{-c_2p_{31}p_{32}}{p_{22}}$ in (4.19) we obtain $p_{13} = \frac{p_{33}p_{12}}{p_{32}}$ and $p_{23} = \frac{p_{33}p_{22}}{p_{32}}$. Finally, if we replace these two values in (4.20), we get $p_{12}^2 = -c_1p_{32}^2$ and $p_{22}^2 = -c_2p_{32}^2$. Therefore,

$$p_{12} = \pm\sqrt{-c_1}\, p_{32}$$

$$p_{22} = \pm\sqrt{-c_2}\; p_{32}$$

and

$$p_{13} = \pm\sqrt{-c_1}\; p_{33}$$

$$p_{23} = \pm\sqrt{-c_2}\; p_{33}$$

$$p_{21} = \pm\sqrt{-c_2}\; p_{31}$$

$$p_{11} = \pm\sqrt{-c_1}\; p_{31}$$

Now,

$$|P_{B'B}| = p_{11}p_{22}p_{33}+p_{12}p_{23}p_{31}+p_{13}p_{21}p_{32}-p_{13}p_{22}p_{31}-p_{21}p_{12}p_{33}-p_{11}p_{32}p_{23} = 0.$$

This is a contradiction. Therefore $p_{31}p_{32}p_{33} = 0$, hence, there exists at least one $i \in \{1, 2, 3\}$ such that $p_{3i} = 0$. We may suppose without loss of generality that $p_{31} = 0$. This means that $p_{11}p_{12} = 0$, $p_{21}p_{22} = 0$, $p_{11}p_{13} = 0$, $p_{21}p_{23} = 0$ and, obviously, $p_{31}p_{32} = 0$ and $p_{31}p_{33} = 0$.
Assume $p_{11} = 0$. Since $|P_{B'B}| \neq 0$, necessarily $p_{21} \neq 0$. So, $p_{23} = p_{22} = 0$ and consequently, using (4.20), $p_{32}p_{33} = 0$ and $p_{12}p_{13} = 0$. We have $p_{22} = 0$ and $p_{23} = 0$.
In $p_{32}p_{33} = 0$ we distinguish two cases. First, assume $p_{32} = 0$. Then $p_{12} \neq 0$ (because $|P_{B'B}| \neq 0$). Since $p_{12}p_{13} = 0$ we get $p_{13} = 0$. Use again $|P_{B'B}| \neq 0$ to obtain $p_{33} \neq 0$ and we have proved that, in this case, $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$. Second, assume $p_{32} \neq 0$. Then $p_{33} = 0$ and $p_{13} \neq 0$ because $|P_{B'B}| \neq 0$. Use $p_{12}p_{13} = 0$ to get, reasoning as before, that $p_{12} = 0$ and $p_{32} \neq 0$. This proves again $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$.
First of all, we assume $p_{11} \neq 0$. Then, we get $p_{12} = p_{13} = 0$. So, by (4.17) $p_{32}p_{33} = 0$ and $p_{22}p_{23} = 0$. Now we are in the same conditions as before ($p_{32}p_{33} = 0$, $p_{12}p_{13} = 0$ and $p_{22}p_{23} = 0$) therefore $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$ as claimed.

Now that we know the possible change of basis matrices for $P_{B'B}$, we may look for all the possible structure matrices M_B. By Proposition 4.1.2 (I) all the structure matrices representing the same evolution algebra have the same number of zero entries. This is the reason for studying the classification depending on the number of non-zero entries in M_B (recall that M_B is the matrix given in (4.13)).

We claim that the first case to be considered is the one for which M_B has four non-zero entries (the maximum number of zero entries are five). This can happen in two different ways. Indeed, fix our attention in the first and second columns in M_B as given in (4.13). The maximum number of zero entries in that columns is four since rank of M_B is two. When this case happens, the third column have only one zero because c_1 and c_2 are non-zero. The other possibility is when we have three zero entries in the first and second columns and two zeros in the third column. One of them is obvious and the other happens if $c_1\omega_{i1} + c_2\omega_{i2} = 0$ for some $i \in \{1, 2, 3\}$ with $\omega_{i1}\omega_{i2} \neq 0$.

Taking into account (4.5), we may assume that three of non-zero entries are 1. In some cases, we will be able to place one more 1 in a fourth non-zero entry. The remaining non-zero entries will be parameters.

We explain now two types of tables we include, called “Table m.ªnd “Table m'”. For “Table m”, we list in the first row (starting by the second column) the five permutation matrices different from the identity. As for the second row we start with an arbitrary structure matrix under the case we are considering. Then we apply the action of an element of S_3 (listed at the beginning of each column) on that structure matrix and write in the corresponding row the obtaining structure matrix. Recall that this action was described in Section 4.1. We start the third row with a matrix under the case we are considering and not included in the second row, and continue in this way until we reach all the possibilities for this case. In order to make easier the understanding of the reader, we distinguish in color the different possibilities that we have. As for the second type of tables we include, the reason is the following: For given parameters (those appearing in the listed matrices), matrices in the same row of a concrete table produce isomorphic evolution algebras. Matrices appearing in different rows correspond to non-isomorphic evolution algebras. Now the question is: For matrices in the same row having different parameters, are the corresponding evolution algebras isomorphic? To answer this question we include the second type of tables, "Table m'".

Case 1.1 M_B has four non-zero entries.

Note that there is, necessarily, a row with all its entries equal zero, because there is no a column with all its entries equal zero (as $c_1c_2 \neq 0$). For each possible row with three zeros, there are $\binom{6}{2} = 15$ possible places where to put two zeros in the remaining rows. Because of A has Property (2LI) a row can not have two zeros. This happens 6 times. We have to eliminate the cases in which there is a zero column (three cases). Then we have $15-6-3 = 6$ cases for each possible row with three zeros. Therefore we obtain 18 cases.

There are only four families of mutually non-isomorphic evolution algebras: those whose structure matrix is in the first column of the table below.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & c_2 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ c_1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & c_1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

TABLE 2. $\dim(A^2) = 2$; $c_1 \neq 0$; $\dim(\text{ann}(A)) = 0$; A has Property (2LI); four non-zero entries.

Now, we study if every resulting family of evolution algebras contains isomorphic evolution algebras. The procedure we have used is the following: we start with one M_B and study if there are matrices $P_{B'B}$ such that $M_{B'}$ is in the same family. For the computations we have used Mathematica. This explanation serves for all the cases.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \sqrt{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -c_1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & c_1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \frac{1}{c_1} \\ 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & \frac{1}{c_1} \end{pmatrix}$

TABLE 2′.

Case 1.2 M_B has five non-zero entries.

The structure matrix must have a zero column. So, for each possible zero row, there exist $\binom{6}{3} = 6$ possible places where to write the remaining zero. Therefore, there are $6 \cdot 3 = 18$ cases. They appear in the list that follows. The parametric families of mutually non-isomorphic evolution algebras are three and appear in the first column of the table below. Immediately after we list the isomorphic evolution algebras when we change the parameters.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}0&0&0\\0&1&1\\1&\alpha&c_1+\alpha\end{pmatrix}$	$\begin{pmatrix}1&0&1\\0&0&0\\\alpha&1&c_1+\alpha\end{pmatrix}$	$\begin{pmatrix}c_1+\alpha&\alpha&1\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1+\alpha&\alpha\\0&1&1\end{pmatrix}$	$\begin{pmatrix}c_1+\alpha&1&\alpha\\0&0&0\\1&0&1\end{pmatrix}$	$\begin{pmatrix}1&1&0\\\alpha&c_1+\alpha&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\1&0&c_1\\1&1&c_1+c_2\end{pmatrix}$	$\begin{pmatrix}0&1&c_1\\0&0&0\\1&1&c_1+c_2\end{pmatrix}$	$\begin{pmatrix}c_1+c_2&1&1\\c_1&0&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1+c_2&1\\1&c_1&0\end{pmatrix}$	$\begin{pmatrix}c_1+c_2&1&1\\0&0&0\\c_1&1&0\end{pmatrix}$	$\begin{pmatrix}0&c_1&1\\1&c_1+c_2&1\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&0\\1&1&c_1+c_2\\1&0&c_1\end{pmatrix}$	$\begin{pmatrix}1&1&c_1+c_2\\0&0&0\\0&1&c_1\end{pmatrix}$	$\begin{pmatrix}c_1&0&1\\c_1+c_2&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&c_1&0\\1&c_1+c_2&1\end{pmatrix}$	$\begin{pmatrix}c_1&1&0\\0&0&0\\c_1+c_2&1&1\end{pmatrix}$	$\begin{pmatrix}1&c_1+c_2&1\\0&c_1&1\\0&0&0\end{pmatrix}$

TABLE 3. $\dim(A^2) = 2$; $\alpha, c_1, c_1+\alpha, c_1+c_2 \neq 0$; $\dim(\operatorname{ann}(A)) = 0$; A has Property (2LI). Five non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix}0&0&0\\0&1&1\\1&\alpha&c_1+\alpha\end{pmatrix}$	$\begin{pmatrix}\sqrt{-1}&0&0\\0&1&0\\0&0&-1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&1\\1&-\alpha&-c_1-\alpha\end{pmatrix}$

TABLE 3′.

Case 1.3 M_B has six non-zero entries.

There exists $\binom{9}{3} = 84$ possibilities to place three zeros. As the structure matrix has Property (2LI), it can have neither two zeros in a row nor a zero column. So, for each place, we eliminate the six cases where to write two zeros in a row. Then, we remove $9 \cdot 6 = 54$ cases. Therefore, we have 84-54-3=27 cases. The mutually non-isomorphic parametric families of evolution algebras are nine and are listed in the first column of the table below. Then, in the next table we study the cases in which the parametric families of evolution algebras are isomorphic for different parameters. In order to simplify expressions we denote by ϕ a seventh root of the unit.

In some cases, the parameters α, β and c_1 must satisfy certain conditions in order to rank of M_B is not three:

In the fourth row: $c_1\alpha\beta + 1 \neq 0$. In the fifth row: $c_1\alpha + \beta \neq 0$ and in the seventh row: $c_2\alpha + c_1 \neq 0$.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & c_1+c_2 \\ 1 & \alpha & c_1+\alpha c_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & c_1+c_2 \\ 0 & 0 & 0 \\ \alpha & 1 & c_1+\alpha c_2 \end{pmatrix}$	$\begin{pmatrix} c_1+\alpha c_2 & \alpha & 1 \\ c_1+c_2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & c_1+\alpha c_2 & \alpha \\ 1 & c_1+c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1+\alpha c_2 & 1 & \alpha \\ 0 & 0 & 0 \\ c_1+c_2 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_1+c_2 & 1 \\ \alpha & c_1+\alpha c_2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \alpha & 0 & \alpha c_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & \alpha & \alpha c_2 \end{pmatrix}$	$\begin{pmatrix} \alpha c_2 & 0 & \alpha \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & \alpha c_2 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha c_2 & \alpha & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ 0 & \alpha c_2 & \alpha \\ 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & \alpha c_2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 0 & \alpha c_2 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ \alpha c_2 & \alpha & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \alpha c_2 & \alpha \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ \alpha c_2 & 0 & \alpha \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \alpha & \alpha c_2 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ \alpha & 0 & 1 \\ 1 & \beta & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \beta \\ c_1 & 0 & 1 \\ 1 & \alpha & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \alpha \\ \beta & 0 & 1 \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \beta & 1 \\ 1 & 0 & \alpha \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ 1 & 0 & \beta \\ \alpha & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & \beta & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \beta & \alpha & 0 \\ 1 & 0 & 1 \\ 0 & 1 & c_1 \end{pmatrix}$	$\begin{pmatrix} c_1 & 0 & 1 \\ 0 & \beta & \alpha \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & c_1 & 0 \\ \alpha & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} c_1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & \alpha & \beta \end{pmatrix}$	$\begin{pmatrix} \beta & 0 & \alpha \\ 0 & c_1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & c_2 \\ 0 & 1 & c_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_2 \\ 0 & 1 & c_1 \\ 1 & 0 & c_2 \end{pmatrix}$	$\begin{pmatrix} c_2 & 1 & 0 \\ c_2 & 1 & 0 \\ c_1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_1 & 0 \\ 0 & c_2 & 1 \\ 0 & c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} c_2 & 0 & 1 \\ c_1 & 1 & 0 \\ c_2 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_2 & 0 \\ 1 & c_2 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & c_1 \\ 1 & \alpha & 0 \\ 0 & 1 & c_2 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 0 \\ 0 & 1 & c_1 \\ 1 & 0 & c_2 \end{pmatrix}$	$\begin{pmatrix} c_2 & 1 & 0 \\ 0 & \alpha & 1 \\ c_1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_1 & 0 \\ 0 & c_2 & 1 \\ 1 & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} c_2 & 0 & 1 \\ c_1 & 1 & 0 \\ 0 & 1 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 & 1 \\ 1 & c_2 & 0 \\ 0 & c_1 & 1 \end{pmatrix}$

TABLE 4. $\dim(A^2) = 2$; α, β, $c_1 + \alpha c_2$, c_1, $c_2, c_1 + c_2 \neq 0$; $\dim(\mathrm{ann}(A)) = 0$; A has Property (2LI); six non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & c_1+c_2 \\ 1 & \alpha & c_1+\alpha c_2 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{c_1+c_2\alpha}} & 0 & 0 \\ 0 & 0 & \frac{1}{c_1+\alpha c_2} \\ 0 & \frac{1}{c_1+c_2\alpha} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & \frac{\alpha}{c_1+c_2\alpha} \\ 1 & \frac{c_1+c_2}{c_1+c_2\alpha} & \frac{1}{c_1+c_2\alpha} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \alpha & 0 & \alpha c_2 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -\alpha & 0 & -c_2\alpha \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & c_2\alpha \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\alpha & -c_2\alpha \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\alpha & 1 \\ 1 & 0 & \phi^6 c_1 \\ -\phi^5\beta & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\alpha & 1 \\ 1 & 0 & -\phi^5 c_1 \\ -\phi^3\beta & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2\alpha & 1 \\ 1 & 0 & \phi^4 c_1 \\ -\phi^7\beta & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\alpha & 1 \\ 1 & 0 & -\phi^3 c_1 \\ \phi^6\beta & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\alpha & 1 \\ 1 & 0 & \phi^2 c_1 \\ \phi^4\beta & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\alpha & 1 \\ 1 & 0 & -\phi c_1 \\ \phi^2\beta & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \\ \phi^4 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\beta & 1 \\ 1 & 0 & -\phi^3\alpha \\ \phi^6 c_1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \\ -\phi & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\beta & 1 \\ 1 & 0 & \phi^6\alpha \\ -\phi^5 c_1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \\ -\phi^5 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\beta & 1 \\ 1 & 0 & \phi^2\alpha \\ \phi^4 c_1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \\ \phi^2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\beta & 1 \\ 1 & 0 & -\phi^5\alpha \\ -\phi^3 c_1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \\ \phi^6 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\beta & 1 \\ 1 & 0 & -\phi\alpha \\ \phi^2 c_1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \\ -\phi^3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2\beta & 1 \\ 1 & 0 & \phi^4\alpha \\ -\phi c_1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\phi \\ \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6 c_1 & 1 \\ 1 & 0 & -\phi^5\beta \\ -\phi^3\alpha & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \phi^2 \\ \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5 c_1 & 1 \\ 1 & 0 & -\phi^3\beta \\ \phi^6\alpha & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\phi^3 \\ \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4 c_1 & 1 \\ 1 & 0 & -\phi\beta \\ \phi^2\alpha & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \phi^4 \\ -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3 c_1 & 1 \\ 1 & 0 & \phi^6\beta \\ -\phi^5\alpha & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -\phi^5 \\ -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2 c_1 & 1 \\ 1 & 0 & \phi^4\beta \\ -\phi\alpha & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \phi^6 \\ -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi c_1 & 1 \\ 1 & 0 & \phi^2\beta \\ \phi^4\alpha & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt[7]{c_1^2\alpha\beta^4}} \\ 0 & \frac{1}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 \\ \frac{1}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} \\ \sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{\phi^2}{\sqrt[7]{c_1^2\alpha\beta^4}} \\ 0 & \frac{-\phi}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 \\ \frac{\phi^4}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 \\ 1 & 0 & -\phi^3\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} \\ \phi^6\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{\phi^4}{\sqrt[7]{c_1^2\alpha\beta^4}} \\ 0 & \frac{\phi^2}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 \\ \frac{-\phi}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 \\ 1 & 0 & \phi^6\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} \\ -\phi^5\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{\phi^6}{\sqrt[7]{c_1^2\alpha\beta^4}} \\ 0 & \frac{-\phi^3}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 \\ \frac{-\phi^5}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 \\ 1 & 0 & \phi^2\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} \\ \phi^4\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{-\phi}{\sqrt[7]{c_1^2\alpha\beta^4}} \\ 0 & \frac{\phi^4}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 \\ \frac{\phi^2}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 \\ 1 & 0 & -\phi^5\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} \\ -\phi^3\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{-\phi^3}{\sqrt[7]{c_1^2\alpha\beta^4}} \\ 0 & \frac{-\phi^5}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 \\ \frac{\phi^6}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 \\ 1 & 0 & -\phi\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} \\ \phi^2\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{-\phi^5}{\sqrt[7]{c_1^2\alpha\beta^4}} \\ 0 & \frac{\phi^6}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 \\ \frac{-\phi^3}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 \\ 1 & 0 & \phi^4\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} \\ -\phi\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 \\ \frac{1}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt[7]{c_1^4\alpha^2\beta}} \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} \\ \sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{\phi^2}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 \\ \frac{-\phi}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{\phi^4}{\sqrt[7]{c_1^4\alpha^2\beta}} \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 \\ 1 & 0 & \phi^6\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} \\ -\phi^5\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{\phi^4}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 \\ \frac{\phi^2}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{-\phi}{\sqrt[7]{c_1^4\alpha^2\beta}} \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 \\ 1 & 0 & -\phi^5\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} \\ -\phi^3\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{\phi^6}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 \\ \frac{-\phi^3}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{-\phi^5}{\sqrt[7]{c_1^4\alpha^2\beta}} \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 \\ 1 & 0 & \phi^4\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} \\ -\phi\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{-\phi}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 \\ \frac{\phi^4}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{\phi^2}{\sqrt[7]{c_1^4\alpha^2\beta}} \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 \\ 1 & 0 & -\phi^3\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} \\ \phi^6\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{-\phi^3}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 \\ \frac{-\phi^5}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{\phi^6}{\sqrt[7]{c_1^4\alpha^2\beta}} \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 \\ 1 & 0 & \phi^2\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} \\ \phi^4\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{-\phi^5}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 \\ \frac{\phi^6}{\sqrt[7]{c_1\alpha^4\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{-\phi^3}{\sqrt[7]{c_1^4\alpha^2\beta}} \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} & 1 \\ 1 & 0 & -\phi\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} \\ \phi^2\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt[7]{c_1\alpha^4\beta^2}} \\ 0 & \frac{1}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} \\ \sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{\phi^2}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 & 0 \\ 0 & 0 & \frac{-\phi}{\sqrt[7]{c_1\alpha^4\beta^2}} \\ 0 & \frac{\phi^4}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 \\ 1 & 0 & -\phi^5\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} \\ -\phi^3\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{\phi^4}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 & 0 \\ 0 & 0 & \frac{\phi^2}{\sqrt[7]{c_1\alpha^4\beta^2}} \\ 0 & \frac{-\phi}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 \\ 1 & 0 & -\phi^3\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} \\ \phi^6\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{\phi^6}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 & 0 \\ 0 & 0 & \frac{-\phi^3}{\sqrt[7]{c_1\alpha^4\beta^2}} \\ 0 & \frac{-\phi^5}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 \\ 1 & 0 & -\phi\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} \\ \phi^2\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{-\phi}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 & 0 \\ 0 & 0 & \frac{\phi^4}{\sqrt[7]{c_1\alpha^4\beta^2}} \\ 0 & \frac{\phi^2}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 \\ 1 & 0 & \phi^6\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} \\ -\phi^5\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{-\phi^3}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 & 0 \\ 0 & 0 & \frac{-\phi^5}{\sqrt[7]{c_1\alpha^4\beta^2}} \\ 0 & \frac{\phi^6}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 \\ 1 & 0 & \phi^4\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} \\ -\phi\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 1 \\ 1 & 0 & c_1 \\ \beta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{-\phi^5}{\sqrt[7]{c_1^2\alpha\beta^4}} & 0 & 0 \\ 0 & 0 & \frac{\phi^6}{\sqrt[7]{c_1\alpha^4\beta^2}} \\ 0 & \frac{-\phi^3}{\sqrt[7]{c_1^4\alpha^2\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\sqrt[7]{\frac{\beta^2}{c_1^6\alpha^3}} & 1 \\ 1 & 0 & \phi^2\sqrt[7]{\frac{c_1^2}{\alpha^6\beta^3}} \\ \phi^4\sqrt[7]{\frac{\alpha^2}{c_1^3\beta^6}} & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 1 & c_2 \\ \alpha & 1 & 0 \\ 1 & 0 & c_1 \end{pmatrix}$	$\begin{pmatrix} \frac{c_2}{{c_1}^2} & 0 & 0 \\ 0 & 0 & \frac{{c_2}^2\alpha}{{c_1}^4} \\ 0 & \frac{1}{c_1} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{c_2^3\alpha^2}{c_1^6} \\ \frac{c_2^2}{c_1^3} & 1 & 0 \\ 1 & 0 & \frac{{c_2}^2\alpha}{{c_1}^4} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & c_1 \\ 1 & \alpha & 0 \\ 0 & 1 & c_2 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{c_1}{\alpha^4} \\ \frac{1}{\alpha} & 0 & 0 \\ 0 & \frac{1}{\alpha^2} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & \frac{c_1^2}{\alpha^7} \\ 1 & \frac{c_2}{\alpha^2} & 0 \\ 0 & 1 & \frac{c_1}{\alpha^4} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & c_1 \\ 1 & \alpha & 0 \\ 0 & 1 & c_2 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{c_1}{c_2^2} & 0 \\ 0 & 0 & \frac{c_1^2}{c_2^4} \\ 0 & \frac{1}{\alpha^2} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & \frac{c_1^2}{\alpha^7} \\ 1 & \frac{c_2}{\alpha^2} & 0 \\ 0 & 1 & \frac{c_1}{\alpha^4} \end{pmatrix}$

TABLE 4′.

Case 1.4 M_B has seven non-zero entries.

There are $\binom{9}{2} = 36$ possibilities to place two zeros. But we have to eliminate the cases in which there are two zeros in the same row. So, there are $36-9 = 27$ cases. The parameters α, β and c_1 must satisfy certain conditions: in the third and fifth row $c_2\beta + c_1 \neq 0$. There are six mutually non-isomorphic evolution algebras, which are listed below. Then, we write their corresponding table about the study of the isomorphism by changing of parameters. Recall that we denote by ζ a third root of the unit.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ \beta & \alpha & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} c_1\alpha+\beta & \beta & \alpha \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & c_1\alpha+\beta & \beta \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1\alpha+\beta & \alpha & \beta \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ \beta & c_1\alpha+\beta & \alpha \\ 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & c_1 \\ 1 & 0 & 1 \\ \beta & \alpha & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} c_1\alpha+\beta & \beta & \alpha \\ c_1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & c_1\alpha+\beta & \beta \\ 1 & c_1 & 0 \end{pmatrix}$	$\begin{pmatrix} c_1\alpha+\beta & \alpha & \beta \\ 1 & 0 & 1 \\ c_1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_1 & 1 \\ \beta & c_1\alpha+\beta & \alpha \\ 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & c_2\alpha \\ 1 & \beta & 0 \\ 1 & 1 & c_1+c_2 \end{pmatrix}$	$\begin{pmatrix} \beta & 1 & 0 \\ \alpha & 0 & c_2\alpha \\ 1 & 1 & c_1+c_2 \end{pmatrix}$	$\begin{pmatrix} c_1+c_2 & 1 & 1 \\ 0 & \beta & 1 \\ c_2\alpha & \alpha & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & c_2\alpha & \alpha \\ 1 & c_1+c_2 & 1 \\ 1 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} c_1+c_2 & 1 & 1 \\ c_2\alpha & 0 & \alpha \\ 0 & 1 & \beta \end{pmatrix}$	$\begin{pmatrix} \beta & 0 & 1 \\ 1 & c_1+c_2 & 1 \\ \alpha & c_2\alpha & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & c_1 \\ \beta & \alpha & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} c_1\alpha+\beta & \beta & \alpha \\ 1 & 1 & 0 \\ c_1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & c_1 & 0 \\ \alpha & c_1\alpha+\beta & \beta \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} c_1\alpha+\beta & \alpha & \beta \\ c_1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ \beta & c_1\alpha+\beta & \alpha \\ 0 & c_1 & 1 \end{pmatrix}$
$\begin{pmatrix} \alpha & 0 & c_1\alpha \\ 1 & \beta & 0 \\ 1 & 1 & c_1+c_2 \end{pmatrix}$	$\begin{pmatrix} \beta & 1 & 0 \\ 0 & \alpha & {}_1\alpha \\ 1 & 1 & c_1+c_2 \end{pmatrix}$	$\begin{pmatrix} c_1+c_2 & 1 & 1 \\ 0 & \beta & 1 \\ c_1\alpha & 0 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & c_1\alpha & 0 \\ 1 & c_1+c_2 & 1 \\ 1 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} c_1+c_2 & 1 & 1 \\ c_1\alpha & \alpha & 0 \\ 0 & 1 & \beta \end{pmatrix}$	$\begin{pmatrix} \beta & 0 & 1 \\ 1 & c_1+c_2 & 1 \\ 1 & c_1\alpha & \alpha \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha+c_2 \\ \beta & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha & \alpha+c_2 \\ 0 & 1 & 1 \\ 0 & \beta & \beta \end{pmatrix}$	$\begin{pmatrix} \beta & 0 & \beta \\ \alpha+c_2 & 1 & \alpha \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ \beta & \beta & 0 \\ \alpha & \alpha+c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \beta & \beta & 0 \\ 1 & 1 & 0 \\ \alpha+c_2 & \alpha & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha+c_2 & \alpha \\ 0 & \beta & \beta \\ 0 & 1 & 1 \end{pmatrix}$

TABLE 5. $\dim(A^2) = 2$; $\alpha,\ \beta, c_1\alpha+\beta,\ \alpha+c_2, c_1, c_2, c_1+c_2 \neq 0$; $\dim(\mathrm{ann}(A)) = 0$; A has Property (2LI); seven non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ -\alpha & -\beta & -c_1\alpha-\beta \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ -\alpha & -\beta & -c_1\alpha-\beta \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} \zeta^2 & 0 & 0 \\ 0 & -\zeta & 0 \\ 0 & 0 & -\zeta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -\zeta c_1 \\ \alpha & -\zeta\beta & -\zeta(c_1\alpha+\beta) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} \zeta^2 & 0 & 0 \\ 0 & -\zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -\zeta c_1 \\ -\alpha & \zeta\beta & \zeta(c_1\alpha+\beta) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} -\zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & -\zeta^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \zeta^2 c_1 \\ -\alpha & -\zeta^2\beta & -\zeta^2(c_1\alpha+\beta) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} -\zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \zeta^2 c_1 \\ \alpha & \zeta^2\beta & \zeta^2(c_1\alpha+\beta) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{c_1}} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{1}{c_1} \\ \sqrt{c_1}\beta & \sqrt{c_1}\alpha & \frac{c_1\alpha+\beta}{\sqrt{c_1}} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt{c_1}} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{1}{c_1} \\ -\sqrt{c_1}\beta & -\sqrt{c_1}\alpha & \frac{-(c_1\alpha+\beta)}{\sqrt{c_1}} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & -\zeta & 0 \\ \zeta^2 & 0 & 0 \\ 0 & 0 & \frac{-\zeta}{\sqrt{c_1}} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{-\zeta}{c_1} \\ \sqrt{c_1}\beta & -\zeta\sqrt{c_1}\alpha & \frac{-\zeta(c_1\alpha+\beta)}{\sqrt{c_1}} \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & -\zeta & 0 \\ \zeta^2 & 0 & 0 \\ 0 & 0 & \frac{\zeta}{\sqrt{c_1}} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{-\zeta}{c_1} \\ -\sqrt{c_1}\beta & \zeta\sqrt{c_1}\alpha & \frac{\zeta(c_1\alpha+\beta)}{\sqrt{c_1}} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & \zeta^2 & 0 \\ -\zeta & 0 & 0 \\ 0 & 0 & \frac{-\zeta^2}{\sqrt{c_1}} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{\zeta^2}{c_1} \\ -\sqrt{c_1}\beta & -\zeta^2\sqrt{c_1}\alpha & \frac{-\zeta^2(c_1\alpha+\beta)}{\sqrt{c_1}} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & c_1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & \zeta^2 & 0 \\ -\zeta & 0 & 0 \\ 0 & 0 & \frac{\zeta^2}{\sqrt{c_1}} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{\zeta^2}{c_1} \\ \sqrt{c_1}\beta & \zeta^2\sqrt{c_1}\alpha & \frac{\zeta^2(c_1\alpha+\beta)}{\sqrt{c_1}} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 1 \\ -\alpha & -\beta & -(c_1\alpha+\beta) \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{c_1}}{c_1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & \frac{1}{c_1} \\ 0 & 1 & 1 \\ \sqrt{c_1}\beta & \sqrt{c_1}\alpha & \frac{c_1\alpha+\beta}{\sqrt{c_1}} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & c_1 \\ 0 & 1 & 1 \\ \alpha & \beta & c_1\alpha+\beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -\frac{\sqrt{c_1}}{c_1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & \frac{1}{c_1} \\ 0 & 1 & 1 \\ -\sqrt{c_1}\beta & -\sqrt{c_1}\alpha & -\frac{c_1\alpha+\beta}{\sqrt{c_1}} \end{pmatrix}$
$\begin{pmatrix} \alpha & 0 & c_1\alpha \\ 1 & \beta & 0 \\ 1 & 1 & c_1+c_2 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 0 & c_1\alpha \\ 1 & \beta & 0 \\ 1 & 1 & c_1+c_2 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha+c_2 \\ \beta & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{1}{\beta} \\ 0 & 1 & 0 \\ \frac{1}{\beta} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ \frac{\alpha+c_2}{\beta^2} & 1 & \frac{\alpha}{\beta^2} \\ \frac{1}{\beta} & 0 & \frac{1}{\beta} \end{pmatrix}$

TABLE 5′.

Case 1.5 M_B has eight non-zero entries.

There are nine possibilities to place a zero in the structure matrix. Therefore, there are two parametric families of evolution algebras listed in the following table. Then, we write their corresponding table about the study of the isomorphism by changing of parameters.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & c_1+\alpha \\ 1 & 0 & 1 \\ \gamma & \beta & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} c_1\beta+\gamma & \gamma & \beta \\ c_1+\alpha & \alpha & 1 \\ 1 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \beta & c_1\beta+\gamma & \gamma \\ 1 & c_1+\alpha & \alpha \end{pmatrix}$	$\begin{pmatrix} c_1\beta+\gamma & \beta & \gamma \\ 1 & 0 & 1 \\ c_1+\alpha & 1 & \alpha \end{pmatrix}$	$\begin{pmatrix} \alpha & c_1+\alpha & 1 \\ \gamma & c_1\beta+\gamma & \beta \\ 1 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha+c_2 \\ \beta & \gamma & \beta+c_2\gamma \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha & \alpha+c_2 \\ 0 & 1 & 1 \\ \gamma & \beta & \beta+c_2\gamma \end{pmatrix}$	$\begin{pmatrix} \beta+c_2\gamma & \gamma & \beta \\ \alpha+c_2 & 1 & \alpha \\ 1 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ \beta & \beta+c_2\gamma & \gamma \\ \alpha & \alpha+c_2 & 1 \end{pmatrix}$	$\begin{pmatrix} \beta+c_2\gamma & \beta & \gamma \\ 1 & 1 & 0 \\ \alpha+c_2 & \alpha & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha+c_2 & \alpha \\ \gamma & \beta+c_2\gamma & \beta \\ 0 & 1 & 1 \end{pmatrix}$

TABLE 6. $\dim(A^2) = 2$; α, β, γ, $c_1+\alpha$, $\alpha+c_2$, $c_1\beta+\gamma$, $\beta+c_2\gamma$, c_1, $c_2 \neq 0$; $\dim(\operatorname{ann}(A)) = 0$; A has Property (2LI); eight non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ -\beta & -\gamma & -(c_1\beta+\gamma) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \zeta^2 & 0 & 0 \\ 0 & -\zeta & 0 \\ 0 & 0 & -\zeta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -\zeta\alpha & -\zeta(c_1+\alpha) \\ \beta & -\zeta\gamma & -\zeta(c_1\beta+\gamma) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \zeta^2 & 0 & 0 \\ 0 & -\zeta & 0 \\ 0 & 0 & \zeta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -\zeta\alpha & -\zeta(c_1+\alpha) \\ -\beta & \zeta\gamma & \zeta(c_1\beta+\gamma) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} -\zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & -\zeta^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \zeta^2\alpha & \zeta^2(c_1+\alpha) \\ -\beta & -\zeta^2\gamma & -\zeta^2(c_1\beta+\gamma) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} -\zeta & 0 & 0 \\ 0 & \zeta^2 & 0 \\ 0 & 0 & \zeta^2 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \zeta^2\alpha & \zeta^2(c_1+\alpha) \\ \beta & \zeta^2\gamma & \zeta^2(c_1\beta+\gamma) \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt[3]{\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt[3]{\beta}} \\ 0 & \frac{1}{\sqrt[3]{\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{c_1\beta+\gamma}{\sqrt[3]{\beta}} & \frac{\gamma}{\sqrt[3]{\beta}} \\ \frac{1}{\beta} & \frac{c_1+\alpha}{\sqrt[3]{\beta}} & \frac{\alpha}{\sqrt[3]{\beta}} \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt[3]{\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{-1}{\sqrt[3]{\beta}} \\ 0 & \frac{1}{\sqrt[3]{\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{c_1\beta+\gamma}{\sqrt[3]{\beta}} & \frac{\gamma}{\sqrt[3]{\beta}} \\ \frac{-1}{\beta} & \frac{-(c_1+\alpha)}{\sqrt[3]{\beta}} & \frac{-\alpha}{\sqrt[3]{\beta}} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \frac{\zeta^2}{\sqrt[3]{\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{-\zeta}{\sqrt[3]{\beta}} \\ 0 & \frac{-\zeta}{\sqrt[3]{\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{-\zeta(c_1\beta+\gamma)}{\sqrt[3]{\beta}} & \frac{-\zeta\gamma}{\sqrt[3]{\beta}} \\ \frac{1}{\beta} & \frac{-\zeta(c_1+\alpha)}{\sqrt[3]{\beta}} & \frac{-\zeta\alpha}{\sqrt[3]{\beta}} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \frac{\zeta^2}{\sqrt[3]{\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{\zeta}{\sqrt[3]{\beta}} \\ 0 & \frac{-\zeta}{\sqrt[3]{\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{-\zeta(c_1\beta+\gamma)}{\sqrt[3]{\beta}} & \frac{-\zeta\gamma}{\sqrt[3]{\beta}} \\ \frac{-1}{\beta} & \frac{\zeta(c_1+\alpha)}{\sqrt[3]{\beta}} & \frac{\zeta\alpha}{\sqrt[3]{\beta}} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \frac{-\zeta}{\sqrt[3]{\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{-\zeta^2}{\sqrt[3]{\beta}} \\ 0 & \frac{\zeta^2}{\sqrt[3]{\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{\zeta^2(c_1\beta+\gamma)}{\sqrt[3]{\beta}} & \frac{\zeta^2\gamma}{\sqrt[3]{\beta}} \\ \frac{-1}{\beta} & \frac{-\zeta^2(c_1+\alpha)}{\sqrt[3]{\beta}} & \frac{-\zeta^2\alpha}{\sqrt[3]{\beta}} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \alpha & c_1+\alpha \\ \beta & \gamma & c_1\beta+\gamma \end{pmatrix}$	$\begin{pmatrix} \frac{-\zeta}{\sqrt[3]{\beta^2}} & 0 & 0 \\ 0 & 0 & \frac{\zeta^2}{\sqrt[3]{\beta}} \\ 0 & \frac{\zeta^2}{\sqrt[3]{\beta}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & \frac{\zeta^2(c_1\beta+\gamma)}{\sqrt[3]{\beta}} & \frac{\zeta^2\gamma}{\sqrt[3]{\beta}} \\ \frac{1}{\beta} & \frac{\zeta^2(c_1+\alpha)}{\sqrt[3]{\beta}} & \frac{\zeta^2\alpha}{\sqrt[3]{\beta}} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha+c_2 \\ \beta & \gamma & \beta+c_2\gamma \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 1 \\ \alpha & 1 & \alpha+c_2 \\ -\beta & -\gamma & -(\beta+c_2\gamma) \end{pmatrix}$

TABLE 6′.

Case 1.6 M_B has nine non-zero entries.

The parameters α, β and γ have to verify that the three of them cannot be equal in order for M_B to have rank two. This produces only one parametric family of evolution algebras. Then, we write their corresponding table about the study of the isomorphism by changing of parameters.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}1 & \alpha & c_1+c_2\alpha\\ 1 & \beta & c_1+c_2\beta\\ 1 & \gamma & c_1+c_2\gamma\end{pmatrix}$	$\begin{pmatrix}\beta & 1 & c_1+c_2\beta\\ \alpha & 1 & c_1+c_2\alpha\\ \gamma & 1 & c_1+c_2\gamma\end{pmatrix}$	$\begin{pmatrix}c_1+c_2\gamma & \gamma & 1\\ c_1+c_2\beta & \beta & 1\\ c_1+c_2\alpha & \alpha & 1\end{pmatrix}$	$\begin{pmatrix}1 & c_1+c_2\alpha & \alpha\\ 1 & c_1+c_2\gamma & \gamma\\ 1 & c_1+c_2\beta & \beta\end{pmatrix}$	$\begin{pmatrix}c_1+c_2\gamma & 1 & \gamma\\ c_1+c_2\alpha & 1 & \alpha\\ c_1+c_2\beta & 1 & \beta\end{pmatrix}$	$\begin{pmatrix}\beta & c_1+c_2\beta & 1\\ \gamma & c_1+c_2\gamma & 1\\ \alpha & c_1+c_2\alpha & 1\end{pmatrix}$

TABLE 7. $\dim(A^2) = 2$; $\alpha, \beta, \gamma, c_1 + c_2\alpha, c_1 + c_2\beta, c_1 + c_2\gamma \neq 0$; $\dim(\operatorname{ann}(A)) = 0$; A has Property (2LI); nine non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix}1 & \alpha & c_1+c_2\alpha\\ 1 & \beta & c_1+c_2\beta\\ 1 & \gamma & c_1+c_2\gamma\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & c_1+c_2\alpha & \alpha\\ 1 & c_1+c_2\gamma & \gamma\\ 1 & c_1+c_2\beta & \beta\end{pmatrix}$
$\begin{pmatrix}1 & \alpha & c_1+c_2\alpha\\ 1 & \beta & c_1+c_2\beta\\ 1 & \gamma & c_1+c_2\gamma\end{pmatrix}$	$\begin{pmatrix}0 & 0 & \frac{\alpha}{\beta^2}\\ \frac{1}{\beta} & 0 & 0\\ 0 & \frac{\gamma}{\beta^2} & 0\end{pmatrix}$	$\begin{pmatrix}1 & \frac{\gamma^2(c_1+c_2\beta)}{\beta^3} & \frac{\alpha^2}{\beta^3}\\ 1 & \frac{\gamma(c_1+c_2\gamma)}{\beta^2} & \frac{\alpha^2}{\beta^2\gamma}\\ 1 & \frac{\gamma^2(c_1+c_2\alpha)}{\alpha\beta^2} & \frac{\alpha}{\beta^2}\end{pmatrix}$
$\begin{pmatrix}1 & \alpha & c_1+c_2\alpha\\ 1 & \beta & c_1+c_2\beta\\ 1 & \gamma & c_1+c_2\gamma\end{pmatrix}$	$\begin{pmatrix}0 & \frac{\alpha}{\beta^2} & 0\\ \frac{1}{\beta} & 0 & 0\\ 0 & 0 & \frac{\gamma}{\beta^2}\end{pmatrix}$	$\begin{pmatrix}1 & \frac{\alpha^2}{\beta^3} & \frac{\gamma^2(c_1+c_2\beta)}{\beta^3}\\ 1 & \frac{\alpha}{\beta^2} & \frac{\gamma^2(c_1+c_2\alpha)}{\alpha\beta^2}\\ 1 & \frac{\alpha^2}{\beta^2\gamma} & \frac{\gamma(c_1+c_2\gamma)}{\beta^2}\end{pmatrix}$
$\begin{pmatrix}1 & \alpha & c_1+c_2\alpha\\ 1 & \beta & c_1+c_2\beta\\ 1 & \gamma & c_1+c_2\gamma\end{pmatrix}$	$\begin{pmatrix}0 & 0 & \frac{c_1+c_2\alpha}{(c_1+c_2\gamma)^2}\\ 0 & \frac{c_1+c_2\beta}{(c_1+c_2\gamma)^2} & 0\\ \frac{1}{c_1+c_2\gamma} & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & \frac{\gamma(c_1+c_2\beta)^2}{(c_1+c_2\gamma)^3} & \frac{\gamma(c_1+c_2\alpha)^2}{(c_1+c_2\gamma)^3}\\ 1 & \frac{\beta(c_1+c_2\beta)}{(c_1+c_2\gamma)^2} & \frac{\gamma(c_1+c_2\alpha)^2}{(c_1+c_2\beta)(c_1+c_2\gamma)^2}\\ 1 & \frac{\alpha(c_1+c_2\beta)^2}{(c_1+c_2\alpha)(c_1+c_2\gamma)^2} & \frac{\gamma(c_1+c_2\alpha)}{(c_1+c_2\gamma)^2}\end{pmatrix}$
$\begin{pmatrix}1 & \alpha & c_1+c_2\alpha\\ 1 & \beta & c_1+c_2\beta\\ 1 & \gamma & c_1+c_2\gamma\end{pmatrix}$	$\begin{pmatrix}0 & \frac{c_1+c_2\alpha}{(c_1+c_2\gamma)^2} & 0\\ 0 & 0 & \frac{c_1+c_2\beta}{(c_1+c_2\gamma)^2}\\ \frac{1}{c_1+c_2\gamma} & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & \frac{\gamma(c_1+c_2\alpha)^2}{(c_1+c_2\gamma)^3} & \frac{\gamma(c_1+c_2\beta)^2}{(c_1+c_2\gamma)^3}\\ 1 & \frac{\beta(c_1+c_2\alpha)}{(c_1+c_2\gamma)^2} & \frac{\alpha(c_1+c_2\beta)^2}{(c_1+c_2\alpha)(c_1+c_2\gamma)^2}\\ 1 & \frac{(c_1+c_2\alpha)^2}{(c_1+c_2\beta)(c_1+c_2\gamma)^2} & \frac{\beta(c_1+c_2\beta)}{(c_1+c_2\gamma)^2}\end{pmatrix}$

TABLE 7′.

Case 2 $c_1 = 0$ and $c_2 \neq 0$.

In this case, the structure matrix is as follows:

$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & c_2\omega_{12} \\ \omega_{21} & \omega_{22} & c_2\omega_{22} \\ \omega_{31} & \omega_{32} & c_2\omega_{32} \end{pmatrix}$$

Note that if we consider the new natural basis $\{e'_1, e'_2, e'_3\}$ with $e'_1 = e_1$, $e'_2 = \sqrt{c_2}e_2 - e_3$ and $e'_3 = \sqrt{c_2}e_2 + e_3$ we obtain that $(e'_2)^2 = (e'_3)^2$. By abusing of notation, we may assume that the natural basis concerned is $\{e_1, e_2, e_3\}$ with $e_2^2 = e_3^2$.

Note that in this case the dimension of the annihilator of the evolution algebra is zero. We will see that the possible change of basis matrices are precisely two elements in $S_3 \rtimes (\mathbb{K}^\times)^3$ (we consider only those for which $e_3^2 = e_2^2$) and one more not in $S_3 \rtimes (\mathbb{K}^\times)^3$ that we will specify.

In this case, the equations (4.17), (4.18), (4.19) and (4.20) are as follows:

$$\begin{aligned} p_{11}p_{12} &= 0; & p_{21}p_{22} &= -p_{31}p_{32}; \\ p_{11}p_{13} &= 0; & p_{21}p_{23} &= -p_{31}p_{33}; \\ p_{12}p_{13} &= 0; & p_{22}p_{23} &= -p_{32}p_{33}. \end{aligned}$$

First of all, we suppose that $p_{11} = p_{12} = 0$.

Assume that $p_{31}p_{32}p_{33} \neq 0$. This implies that $p_{21}p_{22}p_{23} \neq 0$. As $p_{21} = \frac{-p_{31}p_{32}}{p_{22}}$, $p_{23} = \frac{p_{33}p_{22}}{p_{32}}$. Then $p_{22} = \pm\sqrt{-1}p_{32}$, $p_{23} = \pm\sqrt{-1}p_{33}$ and $p_{21} = \pm\sqrt{-1}p_{31}$. But, in these conditions $|P_{B'B}| = 0$. Therefore there exists at least one $i \in \{1, 2, 3\}$ such that $p_{3i} = 0$.

If $p_{31} = 0$, then $p_{21}p_{22} = 0$ and $p_{21}p_{23} = 0$. Since $p_{21} \neq 0$, necessarily $p_{22} = p_{23} = 0$, implying $p_{32}p_{33} = 0$. Consequently, $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$.

If $p_{31} \neq 0$ and $p_{32} = 0$, then $p_{21}p_{22} = 0$ and $p_{22}p_{23} = 0$. This implies $p_{21} = p_{23} = p_{33} = 0$ and again $P_{B'B} \in S_3 \rtimes (\mathbb{K}^\times)^3$.

If $p_{31}p_{32} \neq 0$ and $p_{33} = 0$, then $p_{22}p_{23} = 0$ and $p_{21}p_{23} = 0$. Necessarily $p_{23} = 0$. On the other hand, as $p_{31}p_{32} \neq 0$, $p_{21}p_{22} \neq 0$. So, $p_{22} = \frac{-p_{31}p_{32}}{p_{21}}$ and

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & \dfrac{-p_{31}p_{32}}{p_{21}} & 0 \\ p_{31} & p_{32} & 0 \end{pmatrix}, \tag{4.21}$$

with $p_{13}p_{22}p_{31}p_{32} \neq 0$ and $p_{31}^2 + p_{21}^2 \neq 0$ in order to have $|P_{B'B}| \neq 0$.

If we suppose that $p_{11} = p_{13} = 0$, reasoning in the same way as before, we obtain that the matrices $P_{B'B}$ are in $S_3 \rtimes (\mathbb{K}^\times)^3$ or they are as follows:

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & \dfrac{-p_{33}p_{31}}{p_{21}} \\ p_{31} & 0 & p_{33} \end{pmatrix}, \tag{4.22}$$

with $p_{12}p_{21}p_{31}p_{33} \neq 0$ and $p_{31}^2 + p_{21}^2 \neq 0$.

Finally, if $p_{12} = p_{13} = 0$, we obtain that the different matrices $P_{B'B}$ that appear are in $S_3 \rtimes (\mathbb{K}^\times)^3$ or are of the form:

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \dfrac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix}, \tag{4.23}$$

with $p_{11}p_{22}p_{32}p_{33} \neq 0$ and $p_{32}^2 + p_{22}^2 \neq 0$.

Now, we will see how the structure matrices are when we apply these change of basis matrices.

If $P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & \dfrac{-p_{31}p_{32}}{p_{21}} & 0 \\ p_{31} & p_{32} & 0 \end{pmatrix}$ then

$$M_{B'} = \begin{pmatrix} \omega_{22}p_{21} + \omega_{32}p_{31} & \dfrac{p_{32}^2(\omega_{22}p_{21} + \omega_{32}p_{31})}{p_{21}^2} & \dfrac{p_{13}^2(\omega_{21}p_{21} + \omega_{31}p_{31})}{p_{21}^2 + p_{31}^2} \\ \dfrac{p_{21}(\omega_{32}p_{21} - \omega_{22}p_{31})}{p_{32}} & \dfrac{p_{32}(\omega_{32}p_{21} - \omega_{22}p_{31})}{p_{21}} & \dfrac{p_{13}^2 p_{21}(\omega_{31}p_{21} - \omega_{21}p_{31})}{p_{32}(p_{21}^2 + p_{31}^2)} \\ \dfrac{\omega_{12}(p_{21}^2 + p_{31}^2)}{p_{13}} & \dfrac{\omega_{12}p_{32}^2(p_{21}^2 + p_{31}^2)}{p_{13}p_{21}^2} & \omega_{11}p_{13} \end{pmatrix}.$$

If $P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & \dfrac{-p_{31}p_{33}}{p_{21}} \\ p_{31} & 0 & p_{33} \end{pmatrix}$ then

$$M_{B'} = \begin{pmatrix} \omega_{22}p_{21} + \omega_{32}p_{31} & \dfrac{p_{12}^2(\omega_{21}p_{21} + \omega_{31}p_{31})}{p_{21}^2 + p_{31}^2} & \dfrac{p_{33}^2(\omega_{22}p_{21} + \omega_{32}p_{31})}{p_{21}^2} \\ \dfrac{\omega_{12}(p_{21}^2 + p_{31}^2)}{p_{12}} & \omega_{11}p_{12} & \dfrac{\omega_{12}p_{33}^2(p_{21}^2 + p_{31}^2)}{p_{12}p_{21}^2} \\ \dfrac{p_{21}(\omega_{32}p_{21} - \omega_{22}p_{31})}{p_{33}} & \dfrac{p_{12}^2 p_{21}(\omega_{31}p_{21} - \omega_{21}p_{31})}{p_{33}(p_{21}^2 + p_{31}^2)} & \dfrac{p_{33}(\omega_{32}p_{21} - \omega_{22}p_{31})}{p_{21}} \end{pmatrix}.$$

If $P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \dfrac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix}$ then

$$M_{B'} = \begin{pmatrix} \omega_{11}p_{11} & \dfrac{\omega_{12}(p_{22}^2 + p_{32}^2)}{p_{11}} & \dfrac{\omega_{12}p_{33}^2(p_{22}^2 + p_{32}^2)}{p_{11}p_{22}^2} \\ \dfrac{p_{11}^2(\omega_{21}p_{22} + \omega_{31}p_{32})}{p_{22}^2 + p_{32}^2} & \omega_{22}p_{22} + \omega_{32}p_{32} & \dfrac{p_{33}^2(\omega_{22}p_{22} + \omega_{32}p_{32})}{p_{22}^2} \\ \dfrac{p_{11}^2 p_{22}(\omega_{31}p_{22} - \omega_{21}p_{32})}{p_{33}(p_{22}^2 + p_{32}^2)} & \dfrac{p_{22}(\omega_{32}p_{22} - \omega_{22}p_{32})}{p_{33}} & \dfrac{p_{33}(\omega_{32}p_{22} - \omega_{22}p_{32})}{p_{22}} \end{pmatrix}.$$

Taking into account that we were assuming that the second and third column vector are linearly dependent then the possible change of basis matrices are the following:

$$\left\{ \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ 0 & 0 & p_{33} \end{pmatrix}, \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & 0 & p_{23} \\ 0 & p_{32} & 0 \end{pmatrix}, \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \dfrac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix} \mid p_{11}, p_{22}, p_{32}, p_{33} \in \mathbb{K}^\times \right\}. \tag{4.24}$$

In what follows we will classify in three steps: we start by taking into account the first two families of change of basis matrices of the set (4.24) which leave invariant the number of non-zero entries in the first and second columns (or first and third columns). Recall that the second column is the same as the third one. Then, we will analyze if the resulting families of evolution algebras are or not isomorphic under the action of one matrix of the third family in (4.24), i.e., we will see if some families of evolution algebras are included into other families when applying the change of basis matrices of the third type. Finally, we will analyze, for each of the resulting parametric families, if their algebras are mutually isomorphic.

We list the different matrices into tables taking in account the number of zeros in the first and second columns. Each of these tables will receive the name of "Figure m". According to (4.5) we will write as many 1 as possible and the others non-zero entries will be arbitrary parameters α, β, γ and λ under the restriction $e_2^2 = e_3^2$. We start by the first one and applying the action of the elements:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & \dfrac{-p_{32}p_{33}}{p_{22}} \\ 0 & p_{32} & p_{33} \end{pmatrix} \tag{4.25}$$

with $p_{11}, p_{22}, p_{32}, p_{33} \in \mathbb{K}^{\times}$ and $p_{32}^2 + p_{22}^2 \neq 0$.

Case 2.1 M_B has two non-zero entries in the first and second columns.

M_B has two non-zero entries in the first and second columns. There are $\binom{6}{4} = 15$ possible places where to put four zeros. Since some of the resulting matrices have rank 1, they must be removed from the 15 cases. This happens whenever the first or the second columns is zero (2 cases) and the remaining zeros can be settled in three different places. This produces 6 cases. We also eliminate the cases in which two different rows are zero (3 options). Therefore we have $15 - 6 - 3 = 6$ different matrices written in 3 types. Their structure matrices appear in the first column of the table that follows. We color the six cases that we have.

Type		(2,3)	Q
1	$\begin{pmatrix}1&0&0\\0&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}p_{11}&0&0\\0&p_{22}&\frac{p_{33}^2}{p_{22}}\\0&\frac{-p_{22}p_{32}}{p_{33}}&\frac{-p_{32}p_{33}}{p_{22}}\end{pmatrix}$
2	$\begin{pmatrix}0&1&1\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&\frac{p_{22}^2+p_{32}^2}{p_{11}}&\frac{(p_{22}^2+p_{32}^2)p_{33}^2}{p_{11}p_{22}^2}\\\frac{p_{11}^2p_{22}}{p_{22}^2+p_{32}^2}&0&0\\\frac{-p_{11}^2p_{22}p_{32}}{p_{33}(p_{22}^2+p_{32}^2)}&0&0\end{pmatrix}$
3	$\begin{pmatrix}0&0&0\\1&0&0\\0&1&1\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\\frac{p_{11}^2p_{22}}{p_{22}^2+p_{32}^2}&p_{32}&\frac{p_{32}p_{33}^2}{p_{22}^2}\\\frac{-p_{11}^2p_{22}p_{32}}{p_{33}(p_{22}^2+p_{32}^2)}&\frac{p_{22}^2}{p_{33}}&p_{33}\end{pmatrix}$

FIGURE 1. $\dim(A^2) = 2$; $\dim(\mathrm{ann}(A)) = 0$;
A has not Property (2LI); two non-zero entries in the first and second columns.

Case 2.2 M_B has three non-zero entries in the first and second columns.

There exist $\binom{6}{3} = 20$ possible places where to write three zeros. We remove the matrices which have rank 1. This happens 2 times: when the first or the second column is zero. Therefore we have $20 - 2 = 18$ cases that we color green. There are 10 types listed in the two tables below.

Type		(2,3)	Q
4	$\begin{pmatrix}1&0&0\\0&1&1\\0&\alpha&\alpha\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&\alpha&\alpha\\0&1&1\end{pmatrix}$	$\begin{pmatrix}p_{11}&0&0\\0&p_{22}+\alpha p_{32}&\frac{p_{33}^2(p_{22}+\alpha p_{32})}{p_{22}^2}\\0&\frac{p_{22}(\alpha p_{22}-p_{32})}{p_{33}}&\frac{p_{33}(\alpha p_{22}-p_{32})}{p_{22}}\end{pmatrix}$
5	$\begin{pmatrix}1&0&0\\\alpha&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&0\\\alpha&1&1\end{pmatrix}$	$\begin{pmatrix}p_{11}&0&0\\\frac{\alpha p_{11}^2p_{22}}{p_{22}^2+p_{32}^2}&p_{22}&\frac{p_{33}^2}{p_{22}}\\\frac{-\alpha p_{11}^2p_{22}p_{32}}{p_{33}(p_{22}^2+p_{32}^2)}&\frac{-p_{22}p_{32}}{p_{33}}&\frac{-p_{32}p_{33}}{p_{22}}\end{pmatrix}$
6	$\begin{pmatrix}0&1&1\\1&0&0\\0&\alpha&\alpha\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&\alpha&\alpha\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&\frac{p_{22}^2+p_{32}^2}{p_{11}}&\frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2}\\\frac{p_{11}^2p_{22}}{p_{22}^2+p_{32}^2}&\alpha p_{32}&\frac{\alpha p_{32}p_{33}^2}{p_{22}^2}\\\frac{-p_{11}^2p_{22}p_{32}}{p_{33}(p_{22}^2+p_{32}^2)}&\frac{\alpha p_{22}^2}{p_{33}}&\alpha p_{33}\end{pmatrix}$
7	$\begin{pmatrix}0&1&1\\\alpha&1&1\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&1\\0&0&0\\\alpha&1&1\end{pmatrix}$	$\begin{pmatrix}0&\frac{p_{22}^2+p_{32}^2}{p_{11}}&\frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2}\\\frac{\alpha p_{11}^2p_{22}}{p_{22}^2+p_{32}^2}&p_{22}&\frac{p_{33}^2}{p_{22}}\\\frac{-\alpha p_{11}^2p_{22}p_{32}}{p_{33}(p_{22}^2+p_{32}^2)}&\frac{-p_{22}p_{32}}{p_{33}}&\frac{-p_{32}p_{33}}{p_{22}}\end{pmatrix}$
8	$\begin{pmatrix}1&1&1\\0&0&0\\\alpha&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&1\\\alpha&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}p_{11}&\frac{p_{22}^2+p_{32}^2}{p_{11}}&\frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2}\\\frac{\alpha p_{11}^2p_{32}}{p_{22}^2+p_{32}^2}&0&0\\\frac{\alpha p_{11}^2p_{22}^2}{p_{33}(p_{22}^2+p_{32}^2)}&0&0\end{pmatrix}$
9	$\begin{pmatrix}1&1&1\\0&0&0\\0&\alpha&\alpha\end{pmatrix}$	$\begin{pmatrix}1&1&1\\0&\alpha&\alpha\\0&0&0\end{pmatrix}$	$\begin{pmatrix}p_{11}&\frac{p_{22}^2+p_{32}^2}{p_{11}}&\frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2}\\0&\alpha p_{32}&\frac{\alpha p_{33}^2p_{32}}{p_{22}^2}\\0&\frac{\alpha p_{22}^2}{p_{33}}&\alpha p_{33}\end{pmatrix}$

Type		(2,3)	Q
10	$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} p_{11} & 0 & 0 \\ \frac{\alpha p_{11}^2 p_{22}}{p_{22}^2+p_{32}^2} & p_{32} & \frac{p_{32}p_{33}^2}{p_{22}^2} \\ \frac{-\alpha p_{11}^2 p_{22} p_{32}}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}^2}{p_{33}} & p_{33} \end{pmatrix}$
11	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{p_{11}^2(p_{22}+\alpha p_{32})}{p_{22}^2+p_{32}^2} & 0 & 0 \\ \frac{p_{11}^2 p_{22}(\alpha p_{22}-p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & 0 & 0 \end{pmatrix}$
12	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & \alpha & \alpha \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & \alpha \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ \frac{p_{11}^2 p_{22}}{p_{22}^2+p_{32}^2} & p_{22}+\alpha p_{32} & \frac{p_{33}^2(p_{22}+\alpha p_{32})}{p_{22}^2} \\ \frac{-p_{11}^2 p_{22} p_{32}}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}(\alpha p_{22}-p_{32})}{p_{33}} & \frac{p_{33}(\alpha p_{22}-p_{32})}{p_{22}} \end{pmatrix}$
13	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ \frac{p_{11}^2(p_{22}+\alpha p_{32})}{p_{22}^2+p_{32}^2} & p_{22} & \frac{p_{33}^2}{p_{22}} \\ \frac{p_{11}^2 p_{22}(\alpha p_{22}-p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{-p_{22}p_{32}}{p_{33}} & \frac{-p_{32}p_{33}}{p_{22}} \end{pmatrix}$

FIGURE 2. $\dim(A^2) = 2$; $\alpha \neq 0$; $\dim(\mathrm{ann}(A)) = 0$;
A has not Property (2LI); three non-zero entries in the first and second columns.

Case 2.3 M_B has four non-zero entries in the first and second columns.

There exists $\binom{6}{3} = 15$ possible places where to write four zeros. The non-zero parameters α, β satisfy that $\alpha \neq \beta$ in the matrices appearing as types 14 and 20. This is because the rank of those matrices has to be two. There are nine different types. They are listed below.

Type		(2,3)	Q
14	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 0 & 0 \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha p_{11} & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{\beta p_{11}^2 p_{22}}{p_{22}^2+p_{32}^2} & p_{22} & \frac{p_{33}^2}{p_{22}} \\ \frac{-\beta p_{11}^2 p_{22}p_{32}}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{-p_{22}p_{32}}{p_{33}} & \frac{-p_{32}p_{33}}{p_{22}} \end{pmatrix}$
15	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ \beta & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha p_{11} & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{\beta p_{11}^2 p_{32}}{p_{22}^2+p_{32}^2} & p_{22} & \frac{p_{33}^2}{p_{22}} \\ \frac{\beta p_{11}^2 p_{22}^2}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{-p_{22}p_{32}}{p_{33}} & \frac{-p_{32}p_{33}}{p_{22}} \end{pmatrix}$
16	$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \beta & 0 & 0 \\ \alpha & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} p_{11} & 0 & 0 \\ \frac{p_{11}^2(\alpha p_{22}+\beta p_{32})}{p_{22}^2+p_{32}^2} & p_{22} & \frac{p_{33}^2}{p_{22}} \\ \frac{p_{11}^2 p_{22}(\beta p_{22}-\alpha p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{-p_{22}p_{32}}{p_{33}} & \frac{-p_{32}p_{33}}{p_{22}} \end{pmatrix}$
17	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ \alpha & \beta & \beta \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & \beta & \beta \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} p_{11} & 0 & 0 \\ \frac{\alpha p_{11}^2 p_{32}}{p_{22}^2+p_{32}^2} & p_{22}+\beta p_{32} & \frac{p_{33}^2(p_{22}+\beta p_{32})}{p_{22}^2} \\ \frac{\alpha p_{11}^2 p_{22}^2}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}(\beta p_{22}-p_{32})}{p_{33}} & \frac{p_{33}(\beta p_{22}-p_{32})}{p_{22}} \end{pmatrix}$
18	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \beta & \beta \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & \beta & \beta \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha p_{11} & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ 0 & p_{22}+\beta p_{32} & \frac{p_{33}^2(p_{22}+\beta p_{32})}{p_{22}^2} \\ 0 & \frac{p_{22}(\beta p_{22}-p_{32})}{p_{33}} & \frac{p_{33}(\beta p_{22}-p_{32})}{p_{22}} \end{pmatrix}$
19	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 0 & 0 \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \beta & 1 & 1 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{p_{11}^2(\alpha p_{22}+\beta p_{32})}{p_{22}^2+p_{32}^2} & p_{32} & \frac{p_{32}p_{33}^2}{p_{22}^2} \\ \frac{p_{11}^2 p_{22}(\beta p_{22}-\alpha p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}^2}{p_{33}} & p_{33} \end{pmatrix}$

Type		(2,3)	Q
20	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \beta & \beta \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & \beta & \beta \\ 1 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ \frac{p_{11}^2(p_{22}+\alpha p_{32})}{p_{22}^2+p_{32}^2} & p_{22}+\beta p_{32} & \frac{p_{33}^2(p_{22}+\beta p_{32})}{p_{22}^2} \\ \frac{p_{11}^2 p_{22}(\alpha p_{22}-p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}(\beta p_{22}-p_{32})}{p_{33}} & \frac{p_{33}(\beta p_{22}-p_{32})}{p_{22}} \end{pmatrix}$
21	$\begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ \beta & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} p_{11} & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{p_{11}^2(\alpha p_{22}+\beta p_{32})}{p_{22}^2+p_{32}^2} & 0 & 0 \\ \frac{p_{11}^2 p_{22}(\beta p_{22}-\alpha p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & 0 & 0 \end{pmatrix}$
22	$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ \alpha & \beta & \beta \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & \beta & \beta \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{\alpha p_{11}^2 p_{32}}{p_{22}^2+p_{32}^2} & p_{22}+\beta p_{32} & \frac{p_{33}^2(p_{22}+\beta p_{32})}{p_{22}^2} \\ \frac{\alpha p_{11}^2 p_{22}^2}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}(\beta p_{22}-p_{32})}{p_{33}} & \frac{p_{33}(\beta p_{22}-p_{32})}{p_{22}} \end{pmatrix}$

FIGURE 3. $\dim(A^2) = 2$; $\alpha, \beta \neq 0$; $\dim(\text{ann}(A)) = 0$;
A has not Property (2LI); four non-zero entries in the first and second columns.

Case 2.4 M_B has five non-zero entries in the first and second columns.

There are only 6 possibilities: those for which we place only one zero in one place of the first column or of the second column. There are four types which are listed below.

Type		(2,3)	Q
23	$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \beta & \gamma & \gamma \\ \alpha & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} p_{11} & 0 & 0 \\ \frac{p_{11}^2(\alpha p_{22}+\beta p_{32})}{p_{22}^2+p_{32}^2} & p_{22}+\gamma p_{32} & \frac{p_{33}^2(p_{22}+\gamma p_{32})}{p_{22}^2} \\ \frac{p_{11}^2 p_{22}(\beta p_{22}-\alpha p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}(\gamma p_{22}-p_{32})}{p_{33}} & \frac{p_{33}(\gamma p_{22}-p_{32})}{p_{22}} \end{pmatrix}$
24	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ 0 & \gamma & \gamma \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & \gamma & \gamma \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha p_{11} & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{\beta p_{22}p_{11}^2}{p_{22}^2+p_{32}^2} & \gamma p_{32}+p_{22} & \frac{p_{33}^2(p_{22}+\gamma p_{32})}{p_{22}^2} \\ \frac{-\beta p_{11}^2 p_{22}p_{32}}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}(\gamma p_{22}-p_{32})}{p_{33}} & \frac{p_{33}(\gamma p_{22}-p_{32})}{p_{22}} \end{pmatrix}$
25	$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \beta & \gamma & \gamma \\ \alpha & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{p_{11}^2(\alpha p_{22}+\beta p_{32})}{p_{22}^2+p_{32}^2} & p_{22}+\gamma p_{32} & \frac{p_{33}^2(p_{22}+\gamma p_{32})}{p_{22}^2} \\ \frac{-p_{11}^2 p_{22}(\beta p_{22}-\alpha p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}(\gamma p_{22}-\beta p_{32})}{p_{33}} & \frac{p_{33}(\gamma p_{22}-\beta p_{32})}{p_{22}} \end{pmatrix}$
26	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 0 & 0 \\ \gamma & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \gamma & 1 & 1 \\ \beta & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha p_{11} & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{p_{11}^2(\beta p_{22}+\gamma p_{32})}{p_{22}^2+p_{32}^2} & p_{32} & \frac{p_{32}p_{33}^2}{p_{22}^2} \\ \frac{p_{11}^2 p_{22}(\gamma p_{22}-\beta p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}^2}{p_{33}} & p_{33} \end{pmatrix}$

FIGURE 4. $\dim(A^2)=2$; $\alpha,\beta,\gamma\neq 0$; $\dim(\mathrm{ann}(A))=0$;
A has not Property (2LI); five non-zero entries in the first and second columns.

Case 2.5 M_B has six non-zero entries in the first and second columns.

The condition that the entries of the matrix must satisfy is one of the following: $\alpha\neq\beta$, or $\lambda\neq\gamma$ or $\alpha\lambda\neq\beta\gamma$. There is only one possibility.

Type		(2,3)	Q
27	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \lambda & \lambda \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \gamma & \lambda & \lambda \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha p_{11} & \frac{p_{22}^2+p_{32}^2}{p_{11}} & \frac{p_{33}^2(p_{22}^2+p_{32}^2)}{p_{11}p_{22}^2} \\ \frac{p_{11}^2(\gamma p_{32}-\beta p_{22})}{p_{22}^2+p_{32}^2} & \lambda p_{32}+p_{22} & \frac{p_{33}^2(\lambda p_{32}+p_{22})}{p_{22}^2} \\ \frac{p_{11}^2 p_{22}(\gamma p_{22}-\beta p_{32})}{p_{33}(p_{22}^2+p_{32}^2)} & \frac{p_{22}(\lambda p_{22}-p_{32})}{p_{33}} & \frac{p_{33}(\lambda p_{22}-p_{32})}{p_{22}} \end{pmatrix}$

FIGURE 5. $\dim(A^2)=2$; $\alpha,\beta,\gamma,\lambda\neq 0$; $\dim(\mathrm{ann}(A))=0$;
A has not Property (2LI); six non-zero entries in the first and second columns.

These tables give us a first classification, that can be redundant in some cases. Now we study if algebras having different types are isomorphic or not. The last step will be to study if algebras in the same type are isomorphic.

- The evolution algebra given in Type 1 is included in the parametric family of algebras of Type 4.

- The evolution algebra given in Type 2 is included in the parametric family of algebras of Type 11.

- The evolution algebra given in Types 3, 12 and 13 are included in the parametric family of algebras of Type 20.

- The parametric families of evolution algebras given in Types 5, 10, 16 and 17 are included in the parametric family of algebras of Type 23.

- The parametric families of evolution algebras given in Types 6, 7, 19 and 22 are included in the parametric family of algebras of Type 25.

- The parametric family of evolution algebras given in Type 8 is included in the parametric family of algebras of Type 21.

- The parametric family of evolution algebras given in Type 9 is included in the parametric family of algebras of Type 18.

- The parametric families of evolution algebras given in Types 14, 15, 24 and 26 are included in the parametric family of algebras of Type 27.

Therefore, there are eight subtypes of parametric families of evolution algebras, which are listed below.

$$S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \alpha & \alpha \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \beta & \beta \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \beta & \beta \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}, \begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \lambda & \lambda \end{pmatrix} \right\}.$$

Remark 4.2.3

Note that these matrices are precisely those appearing in the Figures for which the change of basis matrices of type Q leaves invariant the number of non-zero entries and its place in the structure matrix. This does not mean that the number of non-zero entries is preserved (see, for example, in Figure 2, that the first matrix of Type 5 has four non-zero entries while the third matrix in the same line has seven).

Now, we will analyze when the resulting parametric families of evolution algebras are mutually isomorphic. In some cases, we will reduce the number of parameters and some of these parametric families will be isomorphic to one of the known evolution algebras.

Every evolution algebra with structure matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & \alpha & \alpha \end{pmatrix}$$

satifying $\alpha^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Indeed, if $\alpha \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1+\alpha}{1+\alpha^2} & \frac{1-\alpha}{1+\alpha^2} \\ 0 & \frac{-1+\alpha}{1+\alpha^2} & \frac{1+\alpha}{1+\alpha^2} \end{pmatrix}.$$

In case of $\alpha = -1$, we assume the change of basis matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.26}$$

The evolution algebra with structure matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$$

satisfying $\alpha^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Indeed, if $\alpha \neq 1, -1$, we take the change of basis matrix

$$\begin{pmatrix} \sqrt[3]{\frac{2}{1+\alpha^2}} & 0 & 0 \\ 0 & \frac{1-\alpha}{\sqrt[3]{2(1+\alpha^2)^2}} & \frac{1+\alpha}{\sqrt[3]{2(1+\alpha^2)^2}} \\ 0 & \frac{1+\alpha}{\sqrt[3]{2(1+\alpha^2)^2}} & \frac{\alpha-1}{\sqrt[3]{2(1+\alpha^2)^2}} \end{pmatrix}.$$

If $\alpha = -1$, we consider again the change of basis matrix given in (4.26). Every evolution algebra with structure matrix

$$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \beta & \beta \end{pmatrix}$$

satisfying $\beta^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix

$$\begin{pmatrix} \alpha' & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

for some $\alpha' \in \mathbb{K}$. Indeed, if $\beta \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} \frac{2}{1+\beta^2} & 0 & 0 \\ 0 & \frac{1-\beta}{1+\beta^2} & \frac{1+\beta}{1+\beta^2} \\ 0 & \frac{1+\beta}{1+\beta^2} & \frac{\beta-1}{1+\beta^2} \end{pmatrix}.$$

In case of $\beta = -1$, we can also consider the change of basis matrix given in (4.26).

The evolution algebra with structure matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \beta & \beta \end{pmatrix}$$

satisfying $\beta^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha' & 1 & 1 \end{pmatrix}$$

for some $\alpha` \in \mathbb{K}$. Indeed, if $\beta \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} \frac{\sqrt{2}}{\sqrt{1+\alpha+\beta(-1+\alpha)}} & 0 & 0 \\ 0 & \frac{1-\beta}{1+\beta^2} & \frac{1+\beta}{1+\beta^2} \\ 0 & \frac{1+\beta}{1+\beta^2} & \frac{\beta-1}{1+\beta^2} \end{pmatrix}.$$

In case of $\beta = -1$, we take again the change of basis matrix given in (4.26).

Every evolution algebra with structure matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}$$

satisfying $\alpha^2 + \beta^2 \neq 0$ and $\beta^2 \neq 1$ is isomorphic to the evolution algebra having structure matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \beta' & 0 & 0 \end{pmatrix},$$

for some $\beta' \in \mathbb{K}$. Indeed, if $\alpha \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\alpha-\beta\mathrm{s}}{\alpha^2+\beta^2} & \dfrac{-(\beta+\alpha\mathrm{s})}{\alpha^2+\beta^2} \\ 0 & \dfrac{\beta+\alpha\mathrm{s}}{\alpha^2+\beta^2} & \dfrac{\alpha-\beta\mathrm{s}}{\alpha^2+\beta^2} \end{pmatrix},$$

where $\mathrm{s}=\sqrt{-1+\alpha^2+\beta^2}$.

For $\alpha = -1$, consider the change of basis matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

On the other hand, every evolution algebra with structure matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$\beta = 1$ and

$$\begin{pmatrix} 1 & 1 & 1 \\ \alpha & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$\beta = -1$ is isomorphic to the evolution algebra with structure matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \beta & 0 & 0 \end{pmatrix}.$$

Indeed, take the new natural bases $\{e_1, e_3, e_2\}$ and $\{e_1, -e_3, e_2\}$, respectively.

The evolution algebra with structure matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}$$

with $\gamma^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ \alpha' & 1 & 1 \\ \beta' & 1 & 1 \end{pmatrix}$$

for certain $\alpha', \beta' \in \mathbb{K}$. Indeed, if $\gamma \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1+\gamma}{1+\gamma^2} & \frac{1-\gamma}{1+\gamma^2} \\ 0 & \frac{\gamma-1}{1+\gamma^2} & \frac{1+\gamma}{1+\gamma^2} \end{pmatrix}.$$

If $\gamma = -1$, take again the change of basis matrix (4.26).

The evolution algebra with structure matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \gamma & \gamma \end{pmatrix}$$

with $\gamma^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix

$$\begin{pmatrix} 0 & 1 & 1 \\ \alpha' & 1 & 1 \\ \beta' & 1 & 1 \end{pmatrix}$$

for certain $\alpha', \beta' \in \mathbb{K}$. Indeed, if $\gamma \neq -1$, we take the change of basis matrix

$$\begin{pmatrix} \frac{2}{1+\gamma^2} & 0 & 0 \\ 0 & \frac{1+\gamma}{1+\gamma^2} & \frac{1-\gamma}{1+\gamma^2} \\ 0 & \frac{\gamma-1}{1+\gamma^2} & \frac{1+\gamma}{1+\gamma^2} \end{pmatrix}.$$

If $\gamma = -1$, also we take the change of basis matrix given in (4.26).

The evolution algebra with structure matrix

$$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \lambda & \lambda \end{pmatrix}$$

with $\lambda^2 + 1 \neq 0$ is isomorphic to the evolution algebra given by the structure matrix

$$\begin{pmatrix} \alpha' & 1 & 1 \\ \beta' & 1 & 1 \\ \gamma' & 1 & 1 \end{pmatrix}$$

for certain $\alpha', \beta', \gamma' \in \mathbb{K}$. Indeed, if $\lambda \neq -1$, we take the change of basis matrix:

$$\begin{pmatrix} \frac{2}{1+\lambda^2} & 0 & 0 \\ 0 & \frac{1+\lambda}{1+\lambda^2} & \frac{1-\lambda}{1+\lambda^2} \\ 0 & \frac{\lambda-1}{1+\lambda^2} & \frac{1+\lambda}{1+\lambda^2} \end{pmatrix}.$$

If $\gamma = -1$, we consider again the change of basis matrix determined in (4.26).

Summarizing, whenever $e_3^2 = e_2^2$ (or equivalently when the second and third column vector are linearly dependent) we obtain the following mutually non-isomorphic families of evolution algebras which are classified depending on the non-zero entries of the matrices in S. Also we include the study of the isomorphism when we change the corresponding parameters. We note that in the tables "Table m'" the elements p_{ij} have to satisfy the necessary conditions so that $|P_{B'B}| \neq 0$:

	(2,3)
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ \sqrt{-1} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \sqrt{-1} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -\sqrt{-1} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ -\sqrt{-1} & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

TABLE 8. $\dim(A^2) = 2$; $\dim(\operatorname{ann}(A)) = 0$;
A has not Property (2LI); four non-zero entries of the matrices in S.

	(2,3)
$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & 1 & 1\end{pmatrix}$
$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & \sqrt{-1} & \sqrt{-1}\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\0 & \sqrt{-1} & \sqrt{-1}\\0 & 1 & 1\end{pmatrix}$
$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 1\\0 & -\sqrt{-1} & -\sqrt{-1}\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\0 & -\sqrt{-1} & -\sqrt{-1}\\0 & 1 & 1\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 1\\1 & 0 & 0\\\alpha & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1\\\alpha & 0 & 0\\1 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 1\\\sqrt{-1}\alpha & 0 & 0\\\alpha & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1\\\alpha & 0 & 0\\\sqrt{-1}\alpha & 0 & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 1\\-\sqrt{-1}\alpha & 0 & 0\\\alpha & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 1\\\alpha & 0 & 0\\-\sqrt{-1}\alpha & 0 & 0\end{pmatrix}$

TABLE 9. $\dim(A^2) = 2$; $\alpha \neq 0$;$\dim(\mathrm{ann}(A)) = 0$;
A has not Property (2LI); five non-zero entries of the matrices in S.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -\alpha & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 1 \\ \sqrt{-1}\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p_{22} & -\sqrt{1-p_{22}^2} \\ 0 & \sqrt{1-p_{22}^2} & p_{22} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ \sqrt{-1}\alpha(p_{22}-\sqrt{-1+p_{22}^2}) & 0 & 0 \\ \alpha(p_{22}-\sqrt{-1+p_{22}^2}) & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 1 \\ \sqrt{-1}\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p_{22} & \sqrt{1-p_{22}^2} \\ 0 & -\sqrt{1-p_{22}^2} & p_{22} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ \sqrt{-1}\alpha(p_{22}+\sqrt{-1+p_{22}^2}) & 0 & 0 \\ \alpha(p_{22}+\sqrt{-1+p_{22}^2}) & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{-1}\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p_{22} & -\sqrt{1-p_{22}^2} \\ 0 & \sqrt{1-p_{22}^2} & p_{22} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{-1}\alpha(p_{22}+\sqrt{-1+p_{22}^2}) & 0 & 0 \\ \alpha(p_{22}+\sqrt{-1+p_{22}^2}) & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{-1}\alpha & 0 & 0 \\ \alpha & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p_{22} & \sqrt{1-p_{22}^2} \\ 0 & -\sqrt{1-p_{22}^2} & p_{22} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 1 \\ -\sqrt{-1}\alpha(p_{22}-\sqrt{-1+p_{22}^2}) & 0 & 0 \\ \alpha(p_{22}-\sqrt{-1+p_{22}^2}) & 0 & 0 \end{pmatrix}$

TABLE 9′.

	(2,3)
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & \sqrt{-1} & \sqrt{-1} \\ 1 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ \alpha & -\sqrt{-1} & -\sqrt{-1} \\ 1 & 1 & 1 \end{pmatrix}$

TABLE 10. $\dim(A^2) = 2$; $\alpha \neq 0$; $\dim(\mathrm{ann}(A)) = 0$;
A has not Property (2LI); six non-zero entries of the matrices in S.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt{\alpha}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \frac{1}{\alpha} & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} p_{11} & 0 & 0 \\ 0 & \frac{\sqrt{-1}+\alpha p_{11}^2}{2\sqrt{-1}+(-\sqrt{-1}+\alpha)p_{11}^2} & \frac{-1+p_{11}^2}{2\sqrt{-1}+(-\sqrt{-1}+\alpha)p_{11}^2} \\ 0 & \frac{1-p_{11}^2}{2\sqrt{-1}+(-\sqrt{-1}+\alpha)p_{11}^2} & \frac{\sqrt{-1}+\alpha p_{11}^2}{2\sqrt{-1}+(-\sqrt{-1}+\alpha)p_{11}^2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \sqrt{-1}+(-\sqrt{-1}+\alpha)p_{11}^2 & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ \alpha & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} p_{11} & 0 & 0 \\ 0 & \frac{-\sqrt{-1}+\alpha p_{11}^2}{-2\sqrt{-1}+(\sqrt{-1}+\alpha)p_{11}^2} & \frac{-1+p_{11}^2}{-2\sqrt{-1}+(\sqrt{-1}+\alpha)p_{11}^2} \\ 0 & \frac{1-p_{11}^2}{-2\sqrt{-1}+(\sqrt{-1}+\alpha)p_{11}^2} & \frac{-\sqrt{-1}+\alpha p_{11}^2}{-2\sqrt{-1}+(\sqrt{-1}+\alpha)p_{11}^2} \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -\sqrt{-1}+(\sqrt{-1}+\alpha)p_{11}^2 & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

TABLE 10′.

	(2,3)
$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & \sqrt{-1} & \sqrt{-1} \\ 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & -\sqrt{-1} & -\sqrt{-1} \\ 0 & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \beta & 1 & 1 \\ \alpha & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \beta & \sqrt{-1} & \sqrt{-1} \\ \alpha & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \\ \alpha & 1 & 1 \end{pmatrix}$

TABLE 11. $\dim(A^2) = 2$; $\alpha\beta \neq 0$; $\dim(\mathrm{ann}(A)) = 0$;
A has not Property (2LI); seven non-zero entries of the matrices in S.

$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

TABLE 11. $\dim(A^2) = 2$; $\alpha\beta \neq 0$; $\dim(\text{ann}(A)) = 0$;
A has not Property (2LI); seven non-zero entries of the matrices in S.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} -1+2p_{22} & 0 & 0 \\ 0 & p_{22} & \sqrt{-1}(1-p_{22}) \\ 0 & \sqrt{-1}(-1+p_{22}) & p_{22} \end{pmatrix}$	$\begin{pmatrix} \alpha(-1+2p_{22}) & 1 & 1 \\ 0 & 1 & 1 \\ 0 & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} \alpha & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} -1+2p_{22} & 0 & 0 \\ 0 & p_{22} & \sqrt{-1}(-1+p_{22}) \\ 0 & -\sqrt{-1}(-1+p_{22}) & p_{22} \end{pmatrix}$	$\begin{pmatrix} \alpha(-1+2p_{22}) & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p_{22} & -\sqrt{-1}(-1+p_{22}) \\ 0 & \sqrt{-1}(-1+p_{22}) & p_{22} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \frac{\sqrt{-1}\beta(-1+p_{22})+\alpha p_{22}}{-1+2p_{22}} & 1 & 1 \\ \frac{-\sqrt{-1}\alpha(-1+p_{22})+\beta p_{22}}{-1+2p_{22}} & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ \alpha & 1 & 1 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & p_{22} & \sqrt{-1}(-1+p_{22}) \\ 0 & -\sqrt{-1}(-1+p_{22}) & p_{22} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ \frac{-\sqrt{-1}\beta(-1+p_{22})+\alpha p_{22}}{-1+2p_{22}} & 1 & 1 \\ \frac{\sqrt{-1}\alpha(-1+p_{22})+\beta p_{22}}{-1+2p_{22}} & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$

TABLE 11′.

	(2,3)
$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \beta & 1 & 1 \\ \alpha & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \beta & \sqrt{-1} & \sqrt{-1} \\ \alpha & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \\ \alpha & 1 & 1 \end{pmatrix}$

TABLE 12. $\dim(A^2) = 2$; $\alpha\beta \neq 0$; $\dim(\text{ann}(A)) = 0$;
A has not Property (2LI); eight non-zero entries of the matrices in S.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1+2\sqrt{-1}p_{23} & 0 & 0 \\ 0 & 1+\sqrt{-1}p_{23} & p_{23} \\ 0 & -p_{23} & 1+\sqrt{-1}p_{23} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ (\sqrt{-1}-2p_{23})(\sqrt{-1}\beta p_{23}+\alpha(-\sqrt{-1}+p_{23})) & 1 & 1 \\ (1+2\sqrt{-1}p_{23})(\beta+\alpha p_{23}+\beta\sqrt{-1}p_{23}) & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} 1-2\sqrt{-1}p_{23} & 0 & 0 \\ 0 & 1-\sqrt{-1}p_{23} & p_{23} \\ 0 & -p_{23} & 1-\sqrt{-1}p_{23} \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 1 \\ (-1+2\sqrt{-1}p_{23})(\beta p_{23}+\alpha(-1+\sqrt{-1}p_{23})) & 1 & 1 \\ (1-2\sqrt{-1}p_{23})(\beta+\alpha p_{23}-\beta\sqrt{-1}p_{23}) & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

TABLE 12′.

	(2,3)
$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \gamma & 1 & 1 \\ \beta & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \gamma & \sqrt{-1} & \sqrt{-1} \\ \beta & 1 & 1 \end{pmatrix}$
$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \gamma & -\sqrt{-1} & -\sqrt{-1} \\ \beta & 1 & 1 \end{pmatrix}$

TABLE 13. $\dim(A^2) = 2$; $\alpha\beta\gamma \neq 0$; $\dim(\mathrm{ann}(A)) = 0$;
A has not Property (2LI); nine non-zero entries of the matrices in S.

$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & 1 & 1 \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

TABLE 13. $\dim(A^2) = 2$; $\alpha\beta\gamma \neq 0$; $\dim(\mathrm{ann}(A)) = 0$;
A has not Property (2LI); nine non-zero entries of the matrices in S.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} -1+2p_{22} & 0 & 0 \\ 0 & p_{22} & \sqrt{-1}(1-p_{22}) \\ 0 & -\sqrt{-1}(1-p_{22}) & p_{22} \end{pmatrix}$	$\begin{pmatrix} \alpha(-1+2p_{22}) & 1 & 1 \\ (-1+2p_{22})(\sqrt{-1}\gamma(-1+p_{22})+\beta p_{22}) & 1 & 1 \\ (-1+2p_{22})(-\sqrt{-1}\beta(-1+p_{22})+\gamma p_{22}) & \sqrt{-1} & \sqrt{-1} \end{pmatrix}$
$\begin{pmatrix} \alpha & 1 & 1 \\ \beta & 1 & 1 \\ \gamma & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$	$\begin{pmatrix} -1+2p_{22} & 0 & 0 \\ 0 & p_{22} & \sqrt{-1}(-1+p_{22}) \\ 0 & -\sqrt{-1}(-1+p_{22}) & p_{22} \end{pmatrix}$	$\begin{pmatrix} \alpha(-1+2p_{22}) & 1 & 1 \\ (-1+2p_{22})(-\sqrt{-1}\gamma(-1+p_{22})+\beta p_{22}) & 1 & 1 \\ (-1+2p_{22})(\sqrt{-1}\beta(-1+p_{22})+\gamma p_{22}) & -\sqrt{-1} & -\sqrt{-1} \end{pmatrix}$

TABLE 13′.

Case 3 Assume $c_2 = 0, c_1 \neq 0$.

Considering the natural basis $B' = \{e_2, e_1, e_3\}$ we obtain the following structure matrix:

$$M_{B'} = \begin{pmatrix} \omega_{22} & \omega_{21} & c_1\omega_{21} \\ \omega_{12} & \omega_{11} & c_1\omega_{11} \\ \omega_{32} & \omega_{31} & c_1\omega_{31} \end{pmatrix},$$

and now we are in the same conditions as in Case 2.

Case 4 Suppose $c_1 = c_2 = 0$.

Recall by (4.13) that the structure matrix is

$$M_B = \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}.$$

It is clear that in this case A has not Property (2LI) and $\dim(\text{ann}(A)) = 1$.

Remark 4.2.4

In what follows we are going to prove that the number of zero entries in the first and in the second rows in the structure matrix is preserved by any change of basis.

With the explained goal in mind, we study all the possible change of basis matrices. Let B' be another natural basis and consider the change of basis matrix $P_{B'B}$. The equations (4.17), (4.18), (4.19) and (4.20) give:

$$p_{11}p_{12} = 0; \quad p_{21}p_{22} = 0;$$
$$p_{11}p_{13} = 0; \quad p_{21}p_{23} = 0;$$
$$p_{12}p_{13} = 0; \quad p_{22}p_{23} = 0.$$

It is easy to check that $P_{B'B}$ has two zero entries in the first and the second rows. Moreover, since $|P_{B'B}| \neq 0$, necessarily $p_{1i}p_{2j} \neq 0$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$. We distinguish the six different cases that appear in order to study the structure matrix $M_{B'}$.

If
$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \tag{4.27}$$
with $p_{11}p_{22}p_{33} \neq 0$ then

$$M_{B'} = \begin{pmatrix} \omega_{11}p_{11} & \frac{\omega_{12}p_{22}^2}{p_{11}} & 0 \\ \frac{\omega_{21}p_{11}^2}{p_{22}} & \omega_{22}p_{22} & 0 \\ \frac{p_{11}(\omega_{31}p_{11}p_{22} - \omega_{11}p_{31}p_{22} - \omega_{21}p_{11}p_{32})}{p_{22}p_{33}} & \frac{p_{22}(\omega_{32}p_{11}p_{22} - \omega_{12}p_{31}p_{22} - \omega_{22}p_{11}p_{32})}{p_{11}p_{33}} & 0 \end{pmatrix}. \tag{4.28}$$

If
$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$
for $p_{11}p_{23}p_{32} \neq 0$ then

$$M_{B'} = \begin{pmatrix} \omega_{11}p_{11} & 0 & \frac{\omega_{12}p_{23}^2}{p_{11}} \\ \frac{p_{11}(\omega_{31}p_{11}p_{23} - \omega_{11}p_{31}p_{23} - \omega_{21}p_{11}p_{33})}{p_{23}p_{32}} & 0 & \frac{p_{23}(\omega_{32}p_{11}p_{23} - \omega_{12}p_{31}p_{23} - \omega_{22}p_{11}p_{33})}{p_{11}p_{32}} \\ \frac{\omega_{21}p_{11}^2}{p_{23}} & 0 & \omega_{22}p_{23} \end{pmatrix}.$$

If

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix} \tag{4.29}$$

where $p_{12}p_{21}p_{33} \neq 0$ then

$$M_{B'} = \begin{pmatrix} \omega_{22}p_{21} & \frac{\omega_{21}p_{12}^2}{p_{21}} & 0 \\ \frac{\omega_{12}p_{21}^2}{p_{12}} & \omega_{11}p_{12} & 0 \\ \frac{p_{21}(\omega_{32}p_{12}p_{21} - \omega_{12}p_{32}p_{21} - \omega_{22}p_{12}p_{31})}{p_{12}p_{33}} & \frac{p_{12}(\omega_{31}p_{12}p_{21} - \omega_{11}p_{32}p_{21} - \omega_{21}p_{12}p_{31})}{p_{21}p_{33}} & 0 \end{pmatrix}. \tag{4.30}$$

If

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

with $p_{12}p_{23}p_{31} \neq 0$ then

$$M_{B'} = \begin{pmatrix} 0 & \frac{p_{12}(\omega_{31}p_{12}p_{23} - \omega_{11}p_{32}p_{23} - \omega_{21}p_{12}p_{33})}{p_{23}p_{31}} & \frac{p_{23}(\omega_{32}p_{12}p_{23} - \omega_{12}p_{32}p_{23} - \omega_{22}p_{12}p_{33})}{p_{12}p_{31}} \\ 0 & \omega_{11}p_{12} & \frac{\omega_{12}p_{23}^2}{p_{12}} \\ 0 & \frac{\omega_{21}p_{12}^2}{p_{23}} & \omega_{22}p_{23} \end{pmatrix}.$$

If

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & 0 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

for $p_{13}p_{21}p_{32} \neq 0$ then

$$M_{B'} = \begin{pmatrix} \omega_{22}p_{21} & 0 & \frac{\omega_{21}p_{13}^2}{p_{21}} \\ \frac{p_{21}(\omega_{32}p_{13}p_{21} - \omega_{12}p_{33}p_{21} - \omega_{22}p_{13}p_{31})}{p_{13}p_{32}} & 0 & \frac{p_{13}(\omega_{31}p_{13}p_{21} - \omega_{11}p_{33}p_{21} - \omega_{21}p_{13}p_{31})}{p_{21}p_{32}} \\ \frac{\omega_{12}p_{21}^2}{p_{13}} & 0 & \omega_{11}p_{13} \end{pmatrix}.$$

If

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

where $p_{13}p_{21}p_{32} \neq 0$ then

$$M_{B'} = \begin{pmatrix} 0 & \frac{p_{22}(\omega_{32}p_{13}p_{22} - \omega_{12}p_{33}p_{22} - \omega_{22}p_{13}p_{32})}{p_{13}p_{31}} & \frac{p_{13}(\omega_{31}p_{13}p_{22} - \omega_{11}p_{33}p_{22} - \omega_{21}p_{13}p_{32})}{p_{22}p_{31}} \\ 0 & \omega_{22}p_{22} & \frac{\omega_{21}p_{13}^2}{p_{22}} \\ 0 & \frac{\omega_{12}p_{22}^2}{p_{13}} & \omega_{11}p_{13} \end{pmatrix}.$$

Note that we only have to take in to account the change of basis matrices which transform a structure matrix having the third column equals zero into another one of the same type. These are those $P_{B'B}$ appearing in the first and in the third cases. We denote them as follows:

$$Q' = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}; \quad Q'' = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}.$$

Looking at the different $M_{B'}$ that appear, we obtain the claim.

Then, if we omit the structure matrices which can be obtained from the permutation $(1, 2)$, the only possibilities are:

$$\left\{ \begin{pmatrix} \omega_{11} & 0 & 0 \\ 0 & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}, \begin{pmatrix} \omega_{11} & 0 & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}, \begin{pmatrix} 0 & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}, \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ \omega_{21} & \omega_{22} & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \right\}$$

$$\bigcup \left\{ \begin{pmatrix} \omega_{11} & 0 & 0 \\ 0 & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}, \begin{pmatrix} \omega_{11} & \omega_{12} & 0 \\ 0 & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix}, \begin{pmatrix} \omega_{11} & 0 & 0 \\ \omega_{21} & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_{12} & 0 \\ 0 & 0 & 0 \\ \omega_{31} & \omega_{32} & 0 \end{pmatrix} \right\}$$

with $\omega_{ij} \neq 0$ for $i, j \in \{1, 2\}$.

According to (4.28) and (4.30), we claim that we can remove the third row of the structure matrices of the first set and write 0 if and only if $\omega_{11}\omega_{22} - \omega_{12}\omega_{21} \neq 0$. For the matrix (4.28) we consider $p_{11} = p_{22} = 1$ and we have

$$\omega_{31}p_{11}p_{22} - \omega_{11}p_{31}p_{22} - \omega_{21}p_{11}p_{32} = \omega_{31} - \omega_{11}p_{31} - \omega_{21}p_{32} = 0;$$

$$\omega_{32}p_{11}p_{22} - \omega_{12}p_{31}p_{22} - \omega_{22}p_{11}p_{32} = \omega_{31} - \omega_{12}p_{31} - \omega_{22}p_{32} = 0.$$

So, this linear system has solution if $\omega_{11}\omega_{22} - \omega_{12}\omega_{21} \neq 0$.

If we take (4.30), we reason in the same way and our claim has been proved.

Now we can place 0 instead of ω_{31} in the first three matrices of the second set. Indeed, as in these structure matrices $\omega_{11} \neq 0$ and supposing $p_{11} = p_{22} = 1$ we have the equation $\omega_{31} - \omega_{11}p_{31} - \omega_{21}p_{32} = 0$ if $\omega_{21} \neq 0$ and $\omega_{31} - \omega_{11}p_{31} = 0$ if $\omega_{21} = 0$. In any case, the equations have always solution.

In the last structure matrix of the second set we can write 0 instead of ω_{32}. For this, it is enough to take $p_{22} = p_{11} = 0$ and $p_{31} = \frac{\omega_{32}}{\omega_{12}}$.

Finally, we can obtain the maximum number of entries equal 1 by using (4.5). When placing 1 is not possible we write the parameters α, β and γ. Therefore, there are eleven possibilities which are listed below. We only write the action of permutation matrix $(1, 2)$ because it is of type Q''.

	(1,2)
$\begin{pmatrix}1&0&0\\0&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\0&1&0\\1&0&0\end{pmatrix}$
$\begin{pmatrix}0&1&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$

TABLE 14. $\dim(A^2) = 2$; $\dim(\text{ann}(A)) = 1$;
A has not the Property (2LI); one non-zero entry in the first and second rows.

	(1,2)
$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}$
$\begin{pmatrix}1&1&0\\0&0&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&0\\1&1&0\\0&1&0\end{pmatrix}$
$\begin{pmatrix}1&0&0\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&1&0\\1&0&0\end{pmatrix}$

TABLE 15. $\dim(A^2) = 2$; $\dim(\text{ann}(A)) = 1$
A has not the Property (2LI); two non-zero entry in the first and second rows.

	(1,2)
$\begin{pmatrix}\alpha&0&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\0&\alpha&0\\0&0&0\end{pmatrix}$
$\begin{pmatrix}0&\alpha&0\\1&1&0\\0&0&0\end{pmatrix}$	$\begin{pmatrix}1&1&0\\\alpha&0&0\\0&0&0\end{pmatrix}$

TABLE 16. $\dim(A^2) = 2$; $\alpha \neq 0$; $\dim(\text{ann}(A)) = 1$;
A has not the Property (2LI); three non-zero entries in the first and second rows.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \alpha & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_{33} \end{pmatrix}$	$\begin{pmatrix} -\alpha & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \alpha & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & p_{33} \end{pmatrix}$	$\begin{pmatrix} 0 & -\alpha & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 0 \\ \alpha & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & p_{33} \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & 0 \\ -\alpha & -\beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$

TABLE 16′.

	(1,2)
$\begin{pmatrix} 1 & 1 & 0 \\ \alpha & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} \beta & \alpha & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} 1 & 1 & 0 \\ \alpha & \alpha & 0 \\ 1 & \beta & 0 \end{pmatrix}$	$\begin{pmatrix} \alpha & \alpha & 0 \\ 1 & 1 & 0 \\ \beta & 1 & 0 \end{pmatrix}$

TABLE 17. $\dim(A^2) = 2$; $\alpha(\beta - 1) \neq 0$; $\dim(\mathrm{ann}(A)) = 1$;
A has not the Property (2LI); four non-zero entries in the first and second rows.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \beta & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & p_{33}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ -\alpha & -\beta & 0\\ 0 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \beta & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \frac{\sqrt{\alpha\beta}}{\alpha\beta} & 0\\ \frac{1}{\beta} & 0 & 0\\ 0 & 0 & p_{33}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ \frac{\alpha\beta}{\beta^2} & \frac{\sqrt{\alpha\beta}}{\alpha\beta} & 0\\ 0 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \beta & 0\\ 0 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -\frac{\sqrt{\alpha\beta}}{\alpha\beta} & 0\\ \frac{1}{\beta} & 0 & 0\\ 0 & 0 & p_{33}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ -\frac{\alpha\beta}{\beta^2} & -\frac{\sqrt{\alpha\beta}}{\alpha\beta} & 0\\ 0 & 0 & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \beta & 0\\ 1 & 1 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ p_{31} & 0 & 1-p_{31}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ -\alpha & -\beta & 0\\ 1 & 1 & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \beta & 0\\ 1 & 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \frac{\sqrt{\alpha\beta}}{\alpha\beta} & 0\\ \frac{1}{\beta} & 0 & 0\\ p_{31} & \frac{\sqrt{\alpha\beta}}{\alpha\beta} & -p_{31}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ \frac{\alpha\beta}{\beta^2} & \frac{\sqrt{\alpha\beta}}{\alpha\beta} & 0\\ 1 & 1 & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \beta & 0\\ 1 & 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & -\frac{\sqrt{\alpha\beta}}{\alpha\beta} & 0\\ \frac{1}{\beta} & 0 & 0\\ p_{31} & -\frac{\sqrt{\alpha\beta}}{\alpha\beta} & -p_{31}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ -\frac{\alpha\beta}{\beta^2} & -\frac{\sqrt{\alpha\beta}}{\alpha\beta} & 0\\ 1 & 1 & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \alpha & 0\\ 1 & \beta & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 1+\alpha p_{32}-p_{33} & p_{32} & p_{33}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ -\alpha & -\alpha & 0\\ 1 & \frac{-1+\beta+p_{33}}{p_{33}} & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \alpha & 0\\ 1 & \beta & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 1-\alpha p_{32}-p_{33} & p_{32} & p_{33}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ \alpha & \alpha & 0\\ 1 & \frac{-1+\beta+p_{33}}{p_{33}} & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \alpha & 0\\ 1 & \beta & 0\end{pmatrix}$	$\begin{pmatrix}0 & -\frac{1}{\alpha} & 0\\ \frac{1}{\alpha} & 0 & 0\\ \frac{\beta+\alpha(p_{32}-\alpha p_{33})}{\alpha^2} & p_{32} & p_{33}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ -\frac{1}{\alpha} & -\frac{1}{\alpha} & 0\\ 1 & \frac{\alpha^2 p_{33}+1-\beta}{\alpha^2 p_{33}} & 0\end{pmatrix}$
$\begin{pmatrix}1 & 1 & 0\\ \alpha & \alpha & 0\\ 1 & \beta & 0\end{pmatrix}$	$\begin{pmatrix}0 & \frac{1}{\alpha} & 0\\ \frac{1}{\alpha} & 0 & 0\\ \frac{\beta-\alpha(p_{32}-\alpha p_{33})}{\alpha^2} & p_{32} & p_{33}\end{pmatrix}$	$\begin{pmatrix}1 & 1 & 0\\ \frac{1}{\alpha} & \frac{1}{\alpha} & 0\\ 1 & \frac{\alpha^2 p_{33}+1-\beta}{\alpha^2 p_{33}} & 0\end{pmatrix}$

TABLE 17′.

We remark that in all the tables “Table m'” the elements p_{ij} have to satisfy the necessary conditions in order for $P_{B'B}$ to have rank 3.

Case $\dim(A^2) = 3$**.**

In order to classify all the possible matrices corresponding to structure matrices of three-dimensional evolution algebras A such that $A^2 = A$ (equivalently $\dim(A^2) = 3$), we will use Proposition 4.1.2. Notice that in this case the number of zeros in all the structure matrices of a given evolution algebra is invariant (see Proposition 4.1.2 (i)). Equivalently, the number of non-zero entries is invariant. This is the reason because of which we will classify taking into account this last number. Note that the minimum number of non-zero entries in M_B is exactly three.

Case 1. M_B has three non-zero elements.

We compute the determinant of M_B.

$$|M_B| = \omega_{11}\omega_{22}\omega_{33}+\omega_{12}\omega_{23}\omega_{31}+\omega_{13}\omega_{21}\omega_{32}-\omega_{13}\omega_{22}\omega_{31}-\omega_{21}\omega_{12}\omega_{33}-\omega_{11}\omega_{32}\omega_{23}. \tag{4.31}$$

Since $|M_B| \neq 0$, only one of the six summands is non-zero. Assume, for example, $\omega_{12}\omega_{23}\omega_{31} \neq 0$. Take $\alpha = \frac{1}{\sqrt[7]{\omega_{12}\omega_{23}^2\omega_{31}^4}}$, $\beta = \alpha^4\omega_{23}\omega_{31}^2$ and $\gamma = \alpha^2\omega_{31}$. Then

$$(\alpha, \beta, \gamma) \cdot \begin{pmatrix} 0 & \omega_{12} & 0 \\ 0 & 0 & \omega_{23} \\ \omega_{31} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Reasoning in this way with $\omega_{1\sigma(1)}\omega_{2\sigma(2)}\omega_{3\sigma(3)}$ (where $\sigma \in S_3$) instead of with $\omega_{12}\omega_{23}\omega_{31}$, we obtain a natural basis B' such that $M_{B'} = (\varpi_{ij})$, with $\varpi_{i\sigma(i)} = 1$ and $\varpi_{ij} = 0$ for any $j \neq \sigma(i)$. This justifies that these are the only matrices we consider in order to get the classification. Notice that there are are only six. We again use the tables of type “Table m.ªnd “Table m'”to list all structure matrices appearing in each case.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$
$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$

TABLE 18. $\dim(A^2) = 3$; three non-zero entries.

Therefore, there are only three orbits and, consequently, only three non-isomorphic evolution algebras A in the case we are studying. Their structure matrices are:

$$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, \begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix} \text{ and } \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}.$$

Case 2. M_B has four non-zero elements.

Reasoning as in Case 1, we arrive at a natural basis B' of the evolution algebra A such that $M_{B'} = (\varpi_{ij})$, with $\varpi_{i\sigma(i)} = 1$, $\varpi_{ij} \neq 0$ for some $j \neq \sigma(i)$ and $\varpi_{ik} = 0$ for every $k \neq \sigma(i), j$ for every permutation $\sigma \in S_3$.

In order to describe the matrices producing non-isomorphic evolution algebras, first, we notice the following. Given a matrix as explained below, no matter where we put the four non-zero elements (three 1 and one arbitrary parameter μ which has to be non-zero) that the resulting matrices correspond to isomorphic evolution algebras. This is because we will not be worried about where to place the parameter. Then we explain which are the possible cases.

We have to put five 0 into nine places (the nine entries of the matrix). This can be done in $\binom{9}{5} = 126$ ways. But we must remove the cases in which $|M_{B'}| = 0$. This happens:

(a) When the entries of a row are zero.

(b) When the entries of a column are zero but there is no a row which consists of zeros.

(c) When the matrix has a 2×2 minor with every entry equals zero and it has not a row or a column of zeros.

These three cases are mutually exclusive.

(a) The cases in which there is a column of zeros are $3\binom{6}{2} = 45$ (3 corresponds to the three columns and $\binom{6}{2}$ corresponds to the different ways in which two zeros can be distributed in the six remaining places).

(b) For the rows the reasoning in similar: we have 45 cases. Now we have to take into account that there are cases which have been considered twice (just when there is a row and a column which are zero). This happens 9 times. Therefore, we have 45-9=36 options in this case.

(c) Once the matrix has a 2×2 minor with every entry equals zero, the fifth zero must be only in one place if we want to avoid the matrix having a row or column of zeros. There are 9 options to put a zero in a matrix. Once this happens, we remove the corresponding row and the corresponding column and there are four places where to put four zeros. Hence, there are 9 possibilities in this case.

Taking into account (a), (b) and (c), there are $126 - (45 + 36 + 9) = 36$ different matrices we can consider.

As in Case 1, we list all the options in a table. The elements that appear in every row correspond to the action of every element of S_3 on the matrix placed first. There are six mutually non-isomorphic parametric families of evolution algebras, which are listed below.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}1&\mu&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\mu&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&0&\mu\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&\mu\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\\mu&0&1\end{pmatrix}$
$\begin{pmatrix}\mu&1&0\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&\mu&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&0&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&\mu&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&0&\mu\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&\mu\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\mu&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&\mu&0\end{pmatrix}$
$\begin{pmatrix}0&1&0\\1&0&0\\\mu&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\0&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&0&\mu\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\\mu&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&\mu&0\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&\mu\\1&0&0\end{pmatrix}$
$\begin{pmatrix}\mu&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&\mu&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&\mu&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&\mu\end{pmatrix}$
$\begin{pmatrix}0&\mu&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\\mu&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&\mu&0\end{pmatrix}$	$\begin{pmatrix}0&1&\mu\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&\mu\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\\mu&1&0\end{pmatrix}$

TABLE 19. $\dim(A^2) = 3$; four non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} -\phi\mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} \phi^2\mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} -\phi^3\mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} \phi^4\mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} -\phi^5\mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} \phi^6\mu & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} 0\phi^2\mu & 1 & \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\mu & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \mu & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2}(-1+\sqrt{-3})\mu & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \mu & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2}(1+\sqrt{-3})\mu & 0 & 1 \end{pmatrix}$

TABLE 19′.

Case 3. M_B has five non-zero elements.

We proceed as in the cases above and obtain that in order to classify we need to consider only matrices with four zero entries and five non-zero entries. By changing the basis, we may assume that three of the elements are 1 and the other are arbitrary parameters λ and μ, with the only restriction of being non-zero and such that $\lambda\mu \neq 1$ (this condition is needed because the determinant must be non-zero).

The different matrices to be considered are those for which we place four zeros: $\binom{9}{4} = 126$. On the one hand, we must remove those for which there is a row or a column which are zero (because these matrices have zero determinant). If one row or column consists of zeros, then the fourth zero can be placed in six different positions. Since there are 3 rows and 3 columns, this happens 6 times. On the other hand, we must remove those for which there is a 2×2 minor with every entry equals zero. Consequently, we have $126-6^2-9 = 81$ cases that we display in the table that follows. The number of mutually non-isomorphic parametric families of evolution algebras is sixteen. We show them in two tables.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}1&\mu&\lambda\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\mu&1&\lambda\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\\lambda&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&\mu\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\lambda&1&\mu\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\\mu&\lambda&1\end{pmatrix}$
$\begin{pmatrix}1&\mu&0\\\lambda&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&0\\\mu&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&\lambda\\0&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&0&\mu\\0&1&0\\\lambda&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&\mu\\0&\lambda&1\end{pmatrix}$	$\begin{pmatrix}1&0&\lambda\\0&1&0\\\mu&0&1\end{pmatrix}$
$\begin{pmatrix}1&\mu&0\\0&1&\lambda\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&\lambda\\\mu&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\lambda&1&0\\0&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&0&\mu\\0&1&0\\0&\lambda&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&\mu\\\lambda&0&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&0\\0&1&0\\\mu&0&1\end{pmatrix}$
$\begin{pmatrix}1&\mu&0\\0&1&0\\0&\lambda&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\mu&1&0\\\lambda&0&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&0\\0&1&0\\0&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&0&\mu\\0&1&\lambda\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&\lambda\\0&1&\mu\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\lambda&1&0\\\mu&0&1\end{pmatrix}$
$\begin{pmatrix}\mu&1&\lambda\\1&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&\mu&\lambda\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\\lambda&1&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&\lambda&1\\0&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\lambda&\mu&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&0\\1&\lambda&\mu\end{pmatrix}$
$\begin{pmatrix}\mu&1&0\\1&0&\lambda\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&\lambda\\1&\mu&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\lambda&0&1\\0&1&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&0&1\\0&1&0\\1&\lambda&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&\mu&1\\\lambda&1&0\end{pmatrix}$	$\begin{pmatrix}0&\lambda&1\\0&1&0\\1&0&\mu\end{pmatrix}$
$\begin{pmatrix}\mu&1&0\\1&0&0\\\lambda&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&\mu&0\\0&\lambda&1\end{pmatrix}$	$\begin{pmatrix}1&0&\lambda\\0&0&1\\0&1&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&0&1\\\lambda&1&0\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&\lambda&0\\0&\mu&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&\lambda\\1&0&\mu\end{pmatrix}$
$\begin{pmatrix}\mu&1&0\\1&0&0\\0&\lambda&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&\mu&0\\\lambda&0&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&0\\0&0&1\\0&1&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&0&1\\0&1&\lambda\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&\lambda\\0&\mu&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\\lambda&1&0\\1&0&\mu\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&0&\lambda\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&\lambda\\1&0&\mu\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\lambda&0&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\0&1&0\\1&\lambda&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\mu&0&1\\\lambda&1&0\end{pmatrix}$	$\begin{pmatrix}0&\lambda&1\\0&1&0\\1&\mu&0\end{pmatrix}$

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}0&1&0\\1&0&\mu\\\lambda&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&\mu\\1&0&0\\0&\lambda&1\end{pmatrix}$	$\begin{pmatrix}1&0&\lambda\\\mu&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\\lambda&1&0\\1&\mu&0\end{pmatrix}$	$\begin{pmatrix}1&\lambda&0\\0&0&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\0&1&\lambda\\1&0&0\end{pmatrix}$
$\begin{pmatrix}0&1&0\\1&0&\mu\\0&\lambda&1\end{pmatrix}$	$\begin{pmatrix}0&1&\mu\\1&0&0\\\lambda&0&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&0\\\mu&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\0&1&\lambda\\1&\mu&0\end{pmatrix}$	$\begin{pmatrix}1&0&\lambda\\0&0&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\\lambda&1&0\\1&0&0\end{pmatrix}$
$\begin{pmatrix}0&1&0\\1&0&0\\\mu&\lambda&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&0\\\lambda&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&\mu\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\\mu&1&\lambda\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&\mu&\lambda\\0&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\\lambda&1&\mu\\1&0&0\end{pmatrix}$
$\begin{pmatrix}\mu&\lambda&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\\lambda&\mu&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&\lambda&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&1&\lambda\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&\mu&\lambda\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\\lambda&1&\mu\end{pmatrix}$
$\begin{pmatrix}\mu&0&1\\1&\lambda&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}\lambda&1&0\\0&\mu&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&\lambda&1\\1&0&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&1&0\\0&0&1\\1&0&\lambda\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&\mu&0\\0&1&\lambda\end{pmatrix}$	$\begin{pmatrix}\lambda&0&1\\1&0&0\\0&1&\mu\end{pmatrix}$
$\begin{pmatrix}\mu&0&1\\1&0&0\\\lambda&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&\mu&1\\1&\lambda&0\end{pmatrix}$	$\begin{pmatrix}0&1&\lambda\\0&0&1\\1&0&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&1&0\\\lambda&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&\lambda&1\\1&\mu&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&\lambda\\0&1&\mu\end{pmatrix}$
$\begin{pmatrix}0&\mu&1\\1&0&\lambda\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&\lambda\\\mu&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\\lambda&0&1\\1&\mu&0\end{pmatrix}$	$\begin{pmatrix}0&1&\mu\\0&0&1\\1&\lambda&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&\mu\\\lambda&1&0\end{pmatrix}$	$\begin{pmatrix}0&\lambda&1\\1&0&0\\\mu&1&0\end{pmatrix}$

TABLE 20. $\dim(A^2) = 3$; five non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ \lambda & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2}(-1+\sqrt{-3})\lambda & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ \lambda & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2}(1+\sqrt{-3})\lambda & 0 & 1 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \lambda & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3})\lambda & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ 0 & \lambda & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ \lambda & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & 0 \\ \frac{1}{2}(-1+\sqrt{-3})\lambda & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & 0 \\ 1 & 0 & 0 \\ \lambda & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & 0 \\ 1 & 0 & 0 \\ -\frac{1}{2}(1+\sqrt{-3})\lambda & 0 & 1 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \mu \\ \lambda & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ \frac{1}{2}(-1+\sqrt{-3})\lambda & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \mu \\ \lambda & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ -\frac{1}{2}(1+\sqrt{-3})\lambda & 0 & 1 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & \mu\\ 0 & \lambda & 1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}(1+\sqrt{-3}) & 0 & 0\\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\mu\\ 0 & -\frac{1}{2}(1+\sqrt{-3})\lambda & 1\end{pmatrix}$
$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & \mu\\ 0 & \lambda & 1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{2}(-1+\sqrt{-3}) & 0 & 0\\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\mu\\ 0 & \frac{1}{2}(-1+\sqrt{-3})\lambda & 1\end{pmatrix}$
$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ \mu & \lambda & 1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}(1+\sqrt{-3}) & 0 & 0\\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ \frac{1}{2}(-1+\sqrt{-3})\mu & -\frac{1}{2}(1+\sqrt{-3})\lambda & 1\end{pmatrix}$
$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ \mu & \lambda & 1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{2}(-1+\sqrt{-3}) & 0 & 0\\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ -\frac{1}{2}(1+\sqrt{-3})\mu & \frac{1}{2}(-1+\sqrt{-3})\lambda & 1\end{pmatrix}$
$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ \mu & \lambda & 1\end{pmatrix}$	$\begin{pmatrix}0 & -\frac{1}{2}(1+\sqrt{-3}) & 0\\ \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ -\frac{1}{2}(1+\sqrt{-3})\lambda & \frac{1}{2}(-1+\sqrt{-3})\mu & 1\end{pmatrix}$
$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ \mu & \lambda & 1\end{pmatrix}$	$\begin{pmatrix}0 & \frac{1}{2}(-1+\sqrt{-3}) & 0\\ -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ \frac{1}{2}(-1+\sqrt{-3})\lambda & -\frac{1}{2}(1+\sqrt{-3})\mu & 1\end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} -\phi\mu & -\phi^3\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} \phi^2\mu & \phi^6\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} -\phi^3\mu & \phi^2\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} \phi^4\mu & -\phi^5\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} -\phi^5\mu & -\phi\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} \phi^6\mu & \phi^4\lambda & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} -\phi\mu & 0 & 1 \\ 1 & \phi^2\lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} \phi^2\mu & 0 & 1 \\ 1 & \phi^4\lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} -\phi^3\mu & 0 & 1 \\ 1 & \phi^6\lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} \phi^4\mu & 0 & 1 \\ 1 & -\phi\lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} -\phi^5\mu & 0 & 1 \\ 1 & -\phi^3\lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & \lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} \phi^6\mu & 0 & 1 \\ 1 & -\phi^5\lambda & 0 \\ 0 & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ \lambda & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} -\phi\mu & 0 & 1 \\ 1 & 0 & 0 \\ -\phi^5\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ \lambda & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} \phi^2\mu & 0 & 1 \\ 1 & 0 & 0 \\ -\phi^3\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ \lambda & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} -\phi^3\mu & 0 & 1 \\ 1 & 0 & 0 \\ -\phi\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ \lambda & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} \phi^4\mu & 0 & 1 \\ 1 & 0 & 0 \\ \phi^6\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ \lambda & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} -\phi^5\mu & 0 & 1 \\ 1 & 0 & 0 \\ \phi^4\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & 0 & 1 \\ 1 & 0 & 0 \\ \lambda & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} \phi^6\mu & 0 & 1 \\ 1 & 0 & 0 \\ \phi^2\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\mu & 1 \\ 1 & 0 & \phi^6\lambda \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\mu & 1 \\ 1 & 0 & -\phi^5\lambda \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2\mu & 1 \\ 1 & 0 & \phi^4\lambda \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\mu & 1 \\ 1 & 0 & -\phi^3\lambda \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\mu & 1 \\ 1 & 0 & \phi^2\lambda \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\mu & 1 \\ 1 & 0 & -\phi\lambda \\ 0 & 1 & 0 \end{pmatrix}$

TABLE 20′.

Case 4. M_B has six non-zero elements.

Once again we reason in the same way and we can fix our attention in those matrices with three zeros and six non-zero entries.

The different possibilities are: $\binom{9}{3} - 6 = 78$. Note that $\binom{9}{3}$ are the different ways of placing 3 zeros in a 3×3 matrix while 6 corresponds to the cases in which there is a row or a column which is zero.

Making changes on the elements of the basis we may consider three entries equals 1. The only restrictions on the other three elements, say λ, μ and ρ, which must be non-zero, are the needed ones in order to not have zero determinant. This means $\mu\rho \neq 1$, $\lambda\rho \neq 1$, $\mu\lambda \neq 1$ and $\mu\rho\lambda \neq -1$.

There are fifteen mutually non-isomorphic parametric families of evolution algebras, which are listed in the table follows.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}1 & \mu & \lambda\\ \rho & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & \rho & 0\\ \mu & 1 & \lambda\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & \rho\\ \lambda & \mu & 1\end{pmatrix}$	$\begin{pmatrix}1 & \lambda & \mu\\ 0 & 1 & 0\\ \rho & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ \lambda & 1 & \mu\\ 0 & \rho & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \rho\\ 0 & 1 & 0\\ \mu & \lambda & 1\end{pmatrix}$
$\begin{pmatrix}1 & \mu & \lambda\\ 0 & 1 & \rho\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \rho\\ \mu & 1 & \lambda\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ \rho & 1 & 0\\ \lambda & \mu & 1\end{pmatrix}$	$\begin{pmatrix}1 & \lambda & \mu\\ 0 & 1 & 0\\ 0 & \rho & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ \lambda & 1 & \mu\\ \rho & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & \rho & 0\\ 0 & 1 & 0\\ \mu & \lambda & 1\end{pmatrix}$
$\begin{pmatrix}1 & \mu & 0\\ 0 & 1 & \lambda\\ \rho & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \lambda\\ \mu & 1 & 0\\ 0 & \rho & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \rho\\ \lambda & 1 & 0\\ 0 & \mu & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \mu\\ \rho & 1 & 0\\ 0 & \lambda & 1\end{pmatrix}$	$\begin{pmatrix}1 & \rho & 0\\ 0 & 1 & \mu\\ \lambda & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & \lambda & 0\\ 0 & 1 & \rho\\ \mu & 0 & 1\end{pmatrix}$
$\begin{pmatrix}1 & \mu & 0\\ \lambda & 1 & 0\\ \rho & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & \lambda & 0\\ \mu & 1 & 0\\ 0 & \rho & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \rho\\ 0 & 1 & \lambda\\ 0 & \mu & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \mu\\ \rho & 1 & 0\\ \lambda & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & \rho & 0\\ 0 & 1 & \mu\\ 0 & \lambda & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \lambda\\ 0 & 1 & \rho\\ \mu & 0 & 1\end{pmatrix}$
$\begin{pmatrix}\mu & 1 & \lambda\\ 1 & 0 & \rho\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & \rho\\ 1 & \mu & \lambda\\ 0 & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ \rho & 0 & 1\\ \lambda & 1 & \mu\end{pmatrix}$	$\begin{pmatrix}\mu & \lambda & 1\\ 0 & 1 & 0\\ 1 & \rho & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0 & 0\\ \lambda & \mu & 1\\ \rho & 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \rho & 1\\ 0 & 1 & 0\\ 1 & \lambda & \mu\end{pmatrix}$
$\begin{pmatrix}\mu & 1 & \lambda\\ 1 & 0 & 0\\ \rho & 0 & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & \mu & \lambda\\ 0 & \rho & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \rho\\ 0 & 0 & 1\\ \lambda & 1 & \mu\end{pmatrix}$	$\begin{pmatrix}\mu & \lambda & 1\\ \rho & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & \rho & 0\\ \lambda & \mu & 1\\ 0 & 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & \rho\\ 1 & \lambda & \mu\end{pmatrix}$
$\begin{pmatrix}\mu & 1 & \lambda\\ 1 & 0 & 0\\ 0 & \rho & 1\end{pmatrix}$	$\begin{pmatrix}0 & 1 & 0\\ 1 & \mu & \lambda\\ \rho & 0 & 1\end{pmatrix}$	$\begin{pmatrix}1 & \rho & 0\\ 0 & 0 & 1\\ \lambda & 1 & \mu\end{pmatrix}$	$\begin{pmatrix}\mu & \lambda & 1\\ 0 & 1 & \rho\\ 1 & 0 & 0\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \rho\\ \lambda & \mu & 1\\ 0 & 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & 0 & 1\\ \rho & 1 & 0\\ 1 & \lambda & \mu\end{pmatrix}$
$\begin{pmatrix}0 & 1 & \mu\\ 1 & \lambda & 0\\ \rho & 0 & 1\end{pmatrix}$	$\begin{pmatrix}\lambda & 1 & 0\\ 1 & 0 & \mu\\ 0 & \rho & 1\end{pmatrix}$	$\begin{pmatrix}1 & 0 & \rho\\ 0 & \lambda & 1\\ \mu & 1 & 0\end{pmatrix}$	$\begin{pmatrix}0 & \mu & 1\\ \rho & 1 & 0\\ 1 & 0 & \lambda\end{pmatrix}$	$\begin{pmatrix}1 & \rho & 0\\ \mu & 0 & 1\\ 0 & 1 & \lambda\end{pmatrix}$	$\begin{pmatrix}\lambda & 0 & 1\\ 0 & 1 & \rho\\ 1 & \mu & 0\end{pmatrix}$

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}0&1&\mu\\1&\lambda&0\\0&\rho&1\end{pmatrix}$	$\begin{pmatrix}\lambda&1&0\\1&0&\mu\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}1&\rho&0\\0&\lambda&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\0&1&\rho\\1&0&\lambda\end{pmatrix}$	$\begin{pmatrix}1&0&\rho\\\mu&0&1\\0&1&\lambda\end{pmatrix}$	$\begin{pmatrix}\lambda&0&1\\\rho&1&0\\1&\mu&0\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&0&\lambda\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&\lambda\\1&0&\mu\\0&\rho&1\end{pmatrix}$	$\begin{pmatrix}1&0&\rho\\\lambda&0&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\\rho&1&0\\1&\lambda&0\end{pmatrix}$	$\begin{pmatrix}1&\rho&0\\\mu&0&1\\\lambda&1&0\end{pmatrix}$	$\begin{pmatrix}0&\lambda&1\\0&1&\rho\\1&\mu&0\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&0&0\\\lambda&\rho&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&0&\mu\\\rho&\lambda&1\end{pmatrix}$	$\begin{pmatrix}1&\rho&\lambda\\0&0&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\\lambda&1&\rho\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&\lambda&\rho\\\mu&0&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\\rho&1&\lambda\\1&\mu&0\end{pmatrix}$
$\begin{pmatrix}0&1&0\\1&\mu&0\\\lambda&\rho&1\end{pmatrix}$	$\begin{pmatrix}\mu&1&0\\1&0&0\\\rho&\lambda&1\end{pmatrix}$	$\begin{pmatrix}1&\rho&\lambda\\0&\mu&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\\lambda&1&\rho\\1&0&\mu\end{pmatrix}$	$\begin{pmatrix}1&\lambda&\rho\\0&0&1\\0&1&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&0&1\\\rho&1&\lambda\\1&0&0\end{pmatrix}$
$\begin{pmatrix}\mu&\lambda&1\\1&\rho&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}\rho&1&0\\\lambda&\mu&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&\rho&1\\1&\lambda&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&1&\lambda\\0&0&1\\1&0&\rho\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&\mu&\lambda\\0&1&\rho\end{pmatrix}$	$\begin{pmatrix}\rho&0&1\\1&0&0\\\lambda&1&\mu\end{pmatrix}$
$\begin{pmatrix}\mu&\lambda&1\\1&0&0\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&0\\\lambda&\mu&1\\1&\rho&0\end{pmatrix}$	$\begin{pmatrix}0&1&\rho\\0&0&1\\1&\lambda&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&1&\lambda\\\rho&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}0&\rho&1\\1&\mu&\lambda\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&\rho\\\lambda&1&\mu\end{pmatrix}$
$\begin{pmatrix}0&\mu&1\\1&0&\lambda\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&1&\lambda\\\mu&0&1\\1&\rho&0\end{pmatrix}$	$\begin{pmatrix}0&1&\rho\\\lambda&0&1\\1&\mu&0\end{pmatrix}$	$\begin{pmatrix}0&1&\mu\\\rho&0&1\\1&\lambda&0\end{pmatrix}$	$\begin{pmatrix}0&\rho&1\\1&0&\mu\\\lambda&1&0\end{pmatrix}$	$\begin{pmatrix}0&\lambda&1\\1&0&\rho\\\mu&1&0\end{pmatrix}$

TABLE 21. $\dim(A^2) = 3$; six non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & \rho \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\rho \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & \rho \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\rho \\ 0 & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & 0 \\ \rho & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ \frac{1}{2}(-1+\sqrt{-3})\rho & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & 0 \\ \rho & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ -\frac{1}{2}(1+\sqrt{-3})\rho & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & 0 \\ 0 & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3})\rho & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & 0 \\ 0 & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3})\rho & 1 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix}0&1&\mu\\1&\lambda&0\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}(1+\sqrt{-3})&0&0\\0&\frac{1}{2}(-1+\sqrt{-3})&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&\frac{1}{2}(-1+\sqrt{-3})\mu\\1&\frac{1}{2}(-1+\sqrt{-3})\lambda&0\\\frac{1}{2}(-1+\sqrt{-3})\rho&0&1\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&\lambda&0\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{2}(-1+\sqrt{-3})&0&0\\0&-\frac{1}{2}(1+\sqrt{-3})&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&-\frac{1}{2}(1+\sqrt{-3})\mu\\1&-\frac{1}{2}(1+\sqrt{-3})\lambda&0\\-\frac{1}{2}(1+\sqrt{-3})\rho&0&1\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&\lambda&0\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}\frac{1}{\sqrt[3]{\mu\rho^2}}&0&0\\0&0&\frac{1}{\lambda}\\0&\frac{1}{\sqrt[3]{\mu^2\rho}}&0\end{pmatrix}$	$\begin{pmatrix}0&1&\frac{\sqrt[3]{\mu\rho^2}}{\lambda^2}\\1&\frac{1}{\sqrt[3]{\mu^2\rho}}&0\\\frac{\lambda}{\rho\sqrt[3]{\mu^2\rho}}&0&1\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&\lambda&0\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}\frac{-\zeta}{\sqrt[3]{\mu\rho^2}}&0&0\\0&0&\frac{1}{\lambda}\\0&\frac{\zeta^2}{\sqrt[3]{\mu^2\rho}}&0\end{pmatrix}$	$\begin{pmatrix}0&1&\frac{\zeta^2\sqrt[3]{\mu\rho^2}}{\lambda^2}\\1&\frac{\zeta^2}{\sqrt[3]{\mu^2\rho}}&0\\\frac{\zeta^2\lambda}{\rho\sqrt[3]{\mu^2\rho}}&0&1\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&\lambda&0\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}\frac{\zeta^2}{\sqrt[3]{\mu\rho^2}}&0&0\\0&0&\frac{1}{\lambda}\\0&\frac{-\zeta}{\sqrt[3]{\mu^2\rho}}&0\end{pmatrix}$	$\begin{pmatrix}0&1&\frac{-\zeta\sqrt[3]{\mu\rho^2}}{\lambda^2}\\1&\frac{-\zeta}{\sqrt[3]{\mu^2\rho}}&0\\\frac{-\zeta\lambda}{\rho\sqrt[3]{\mu^2\rho}}&0&1\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&\lambda&0\\0&\rho&1\end{pmatrix}$	$\begin{pmatrix}0&\frac{1}{2}(-1+\sqrt{-3})&0\\-\frac{1}{2}(1+\sqrt{-3})&0&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&-\frac{1}{2}(1+\sqrt{-3})\mu\\1&-\frac{1}{2}(1+\sqrt{-3})\lambda&0\\0&\frac{1}{2}(-1+\sqrt{-3})\rho&1\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&\lambda&0\\0&\rho&1\end{pmatrix}$	$\begin{pmatrix}-\frac{1}{2}(1+\sqrt{-3})&0&0\\0&\frac{1}{2}(-1+\sqrt{-3})&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&\frac{1}{2}(-1+\sqrt{-3})\mu\\1&\frac{1}{2}(-1+\sqrt{-3})\lambda&0\\0&-\frac{1}{2}(1+\sqrt{-3})\rho&1\end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ \rho & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ \frac{1}{2}(-1+\sqrt{-3})\rho & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ \rho & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ -\frac{1}{2}(1+\sqrt{-3})\rho & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & 0 \\ \lambda & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ 1 & 0 & 0 \\ \frac{1}{2}(-1+\sqrt{-3})\lambda & -\frac{1}{2}(1+\sqrt{-3})\rho & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & 0 \\ \lambda & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ 1 & 0 & 0 \\ -\frac{1}{2}(1+\sqrt{-3})\lambda & \frac{1}{2}(-1+\sqrt{-3})\rho & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & \mu & 0 \\ \lambda & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & \frac{1}{2}(-1+\sqrt{-3})\mu & 0 \\ \frac{1}{2}(-1+\sqrt{-3})\lambda & -\frac{1}{2}(1+\sqrt{-3})\rho & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & 0 \\ 1 & \mu & 0 \\ \lambda & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -\frac{1}{2}(1+\sqrt{-3})\mu & \\ -\frac{1}{2}(1+\sqrt{-3})\lambda & \frac{1}{2}(-1+\sqrt{-3})\rho & 1 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & \rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} -\phi\mu & -\phi^3\lambda & 1 \\ 1 & \phi^2\rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & \rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} \phi^2\mu & \phi^6\lambda & 1 \\ 1 & \phi^4\rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & \rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} -\phi^3\mu & \phi^2\lambda & 1 \\ 1 & \phi^6\rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & \rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} \phi^4\mu & -\phi^5\lambda & 1 \\ 1 & -\phi\rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & \rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} -\phi^5\mu & -\phi\lambda & 1 \\ 1 & -\phi^3\rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & \rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} \phi^6\mu & \phi^4\lambda & 1 \\ 1 & -\phi^5\rho & 0 \\ 0 & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} -\phi\mu & -\phi^3\lambda & 1 \\ 1 & 0 & 0 \\ -\phi^5\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} \phi^2\mu & \phi^6\lambda & 1 \\ 1 & 0 & 0 \\ -\phi^3\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} -\phi^3\mu & \phi^2\lambda & 1 \\ 1 & 0 & 0 \\ -\phi\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} \phi^4\mu & -\phi^5\lambda & 1 \\ 1 & 0 & 0 \\ \phi^6\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} -\phi^5\mu & -\phi\lambda & 1 \\ 1 & 0 & 0 \\ \phi^4\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ 1 & 0 & 0 \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} \phi^6\mu & \phi^4\lambda & 1 \\ 1 & 0 & 0 \\ \phi^2\lambda & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi & 0 & 0 \\ 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\mu & 1 \\ 1 & 0 & \phi^6\lambda \\ -\phi^5\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^2 & 0 & 0 \\ 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\mu & 1 \\ 1 & 0 & -\phi^5\lambda \\ -\phi^3\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^3 & 0 & 0 \\ 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2\mu & 1 \\ 1 & 0 & \phi^4\lambda \\ -\phi^7\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^4 & 0 & 0 \\ 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\mu & 1 \\ 1 & 0 & -\phi^3\lambda \\ \phi^6\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} -\phi^5 & 0 & 0 \\ 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\mu & 1 \\ 1 & 0 & \phi^2\lambda \\ \phi^4\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \phi^6 & 0 & 0 \\ 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\mu & 1 \\ 1 & 0 & -\phi\lambda \\ \phi^2\rho & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi & 0 \\ 0 & 0 & \phi^2 \\ \phi^4 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5\rho & 1 \\ 1 & 0 & -\phi^3\mu \\ \phi^6\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2 & 0 \\ 0 & 0 & \phi^4 \\ -\phi & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3\rho & 1 \\ 1 & 0 & \phi^6\mu \\ -\phi^5\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^3 & 0 \\ 0 & 0 & \phi^6 \\ -\phi^5 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi\rho & 1 \\ 1 & 0 & \phi^2\mu \\ \phi^4\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4 & 0 \\ 0 & 0 & -\phi \\ \phi^2 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6\rho & 1 \\ 1 & 0 & -\phi^5\mu \\ -\phi^3\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\phi^5 & 0 \\ 0 & 0 & -\phi^3 \\ \phi^6 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^4\rho & 1 \\ 1 & 0 & -\phi\mu \\ \phi^2\lambda & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^6 & 0 \\ 0 & 0 & -\phi^5 \\ -\phi^3 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \phi^2\rho & 1 \\ 1 & 0 & \phi^4\mu \\ -\phi\lambda & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{1}{\sqrt[7]{\mu\lambda^2\rho^4}} \\ 0 & \frac{1}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 \\ \frac{1}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\lambda^2}{\mu^6\rho^3}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\mu^2}{\lambda^3\rho^6}} \\ \sqrt[7]{\frac{\rho^2}{\mu^3\lambda^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{\phi^2}{\sqrt[7]{\mu\lambda^2\rho^4}} \\ 0 & \frac{-\phi}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 \\ \frac{\phi^4}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi^5\lambda^2}{\mu^6\rho^3}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi^3\mu^2}{\lambda^3\rho^6}} \\ \sqrt[7]{\frac{\phi^6\rho^2}{\mu^3\lambda^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{\phi^4}{\sqrt[7]{\mu\lambda^2\rho^4}} \\ 0 & \frac{\phi^2}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 \\ \frac{-\phi}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi^3\lambda^2}{\mu^6\rho^3}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^6\mu^2}{\lambda^3\rho^6}} \\ -\sqrt[7]{\frac{\phi^5\rho^2}{\mu^3\lambda^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{\phi^6}{\sqrt[7]{\mu\lambda^2\rho^4}} \\ 0 & \frac{-\phi^3}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 \\ \frac{-\phi^5}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi\lambda^2}{\mu^6\rho^3}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^2\mu^2}{\lambda^3\rho^6}} \\ \sqrt[7]{\frac{\phi^4\rho^2}{\mu^3\lambda^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{-\phi}{\sqrt[7]{\mu\lambda^2\rho^4}} \\ 0 & \frac{\phi^4}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 \\ \frac{\phi^2}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^6\lambda^2}{\mu^6\rho^3}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi^5\mu^2}{\lambda^3\rho^6}} \\ -\sqrt[7]{\frac{\phi^3\rho^2}{\mu^3\lambda^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{-\phi^3}{\sqrt[7]{\mu\lambda^2\rho^4}} \\ 0 & \frac{-\phi^5}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 \\ \frac{\phi^6}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^4\lambda^2}{\mu^6\rho^3}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi\mu^2}{\lambda^3\rho^6}} \\ \sqrt[7]{\frac{\phi^2\rho^2}{\mu^3\lambda^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \frac{-\phi^5}{\sqrt[7]{\mu\lambda^2\rho^4}} \\ 0 & \frac{\phi^6}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 \\ \frac{-\phi^3}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^2\lambda^2}{\mu^6\rho^3}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^4\mu^2}{\lambda^3\rho^6}} \\ -\sqrt[7]{\frac{\phi\rho^2}{\mu^3\lambda^6}} & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 \\ \frac{1}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt[7]{\mu^2\lambda^4\rho}} \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\mu^2}{\lambda^3\rho^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\rho^2}{\mu^3\lambda^6}} \\ \sqrt[7]{\frac{\lambda^2}{\mu^6\rho^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{\phi^2}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 \\ \frac{-\phi}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 & 0 \\ 0 & 0 & \frac{\phi^4}{\sqrt[7]{\mu^2\lambda^4\rho}} \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi^3\mu^2}{\lambda^3\rho^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^6\rho^2}{\mu^3\lambda^6}} \\ -\sqrt[7]{\frac{\phi^5\lambda^2}{\mu^6\rho^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{\phi^4}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 \\ \frac{\phi^2}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 & 0 \\ 0 & 0 & \frac{-\phi}{\sqrt[7]{\mu^2\lambda^4\rho}} \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^6\mu^2}{\lambda^3\rho^6}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi^5\rho^2}{\mu^3\lambda^6}} \\ -\sqrt[7]{\frac{\phi^3\lambda^2}{\mu^6\rho^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{\phi^6}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 \\ \frac{-\phi^3}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 & 0 \\ 0 & 0 & \frac{-\phi^5}{\sqrt[7]{\mu^2\lambda^4\rho}} \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^2\mu^2}{\lambda^3\rho^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^4\rho^2}{\mu^3\lambda^6}} \\ -\sqrt[7]{\frac{\phi\lambda^2}{\mu^6\rho^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{-\phi}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 \\ \frac{\phi^4}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 & 0 \\ 0 & 0 & \frac{\phi^2}{\sqrt[7]{\mu^2\lambda^4\rho}} \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi^5\mu^2}{\lambda^3\rho^6}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi^3\rho^2}{\mu^3\lambda^6}} \\ \sqrt[7]{\frac{\phi^6\lambda^2}{\mu^6\rho^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{-\phi^3}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 \\ \frac{-\phi^5}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 & 0 \\ 0 & 0 & \frac{\phi^6}{\sqrt[7]{\mu^2\lambda^4\rho}} \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi\mu^2}{\lambda^3\rho^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^2\rho^2}{\mu^3\lambda^6}} \\ \sqrt[7]{\frac{\phi^4\lambda^2}{\mu^6\rho^3}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{-\phi^5}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 \\ \frac{\phi^6}{\sqrt[7]{\mu^4\lambda\rho^2}} & 0 & 0 \\ 0 & 0 & \frac{-\phi^3}{\sqrt[7]{\mu^2\lambda^4\rho}} \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^4\lambda^2}{\mu^6\rho^3}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi\mu^2}{\lambda^3\rho^6}} \\ \sqrt[7]{\frac{\phi^2\rho^2}{\mu^3\lambda^6}} & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt[7]{\mu^4\lambda\rho^2}} \\ 0 & \frac{1}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\rho^2}{\mu^3\lambda^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\lambda^2}{\mu^6\rho^3}} \\ \sqrt[7]{\frac{\mu^2}{\lambda^3\rho^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{\phi^2}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 & 0 \\ 0 & 0 & \frac{-\phi}{\sqrt[7]{\mu^4\lambda\rho^2}} \\ 0 & \frac{\phi^4}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^6\rho^2}{\mu^3\lambda^6}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi^5\lambda^2}{\mu^6\rho^3}} \\ -\sqrt[7]{\frac{\phi^3\mu^2}{\lambda^3\rho^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{\phi^4}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 & 0 \\ 0 & 0 & \frac{\phi^2}{\sqrt[7]{\mu^4\lambda\rho^2}} \\ 0 & \frac{-\phi}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi^5\rho^2}{\mu^3\lambda^6}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi^3\lambda^2}{\mu^6\rho^3}} \\ \sqrt[7]{\frac{\phi^6\mu^2}{\lambda^3\rho^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{\phi^6}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 & 0 \\ 0 & 0 & \frac{-\phi^3}{\sqrt[7]{\mu^4\lambda\rho^2}} \\ 0 & \frac{-\phi^5}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^4\rho^2}{\mu^3\lambda^6}} & 1 \\ 1 & 0 & -\sqrt[7]{\frac{\phi\lambda^2}{\mu^6\rho^3}} \\ \sqrt[7]{\frac{\phi^2\mu^2}{\lambda^3\rho^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{-\phi}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 & 0 \\ 0 & 0 & \frac{\phi^4}{\sqrt[7]{\mu^4\lambda\rho^2}} \\ 0 & \frac{\phi^2}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi^3\rho^2}{\mu^3\lambda^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^6\lambda^2}{\mu^6\rho^3}} \\ -\sqrt[7]{\frac{\phi^5\mu^2}{\lambda^3\rho^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{-\phi^3}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 & 0 \\ 0 & 0 & \frac{-\phi^5}{\sqrt[7]{\mu^4\lambda\rho^2}} \\ 0 & \frac{\phi^6}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \sqrt[7]{\frac{\phi^2\rho^2}{\mu^3\lambda^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^4\lambda^2}{\mu^6\rho^3}} \\ -\sqrt[7]{\frac{\phi\mu^2}{\lambda^3\rho^6}} & 1 & 0 \end{pmatrix}$
$\begin{pmatrix} 0 & \mu & 1 \\ 1 & 0 & \lambda \\ \rho & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{-\phi^5}{\sqrt[7]{\mu\lambda^2\rho^4}} & 0 & 0 \\ 0 & 0 & \frac{\phi^6}{\sqrt[7]{\mu^4\lambda\rho^2}} \\ 0 & \frac{-\phi^3}{\sqrt[7]{\mu^2\lambda^4\rho}} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\sqrt[7]{\frac{\phi\rho^2}{\mu^3\lambda^6}} & 1 \\ 1 & 0 & \sqrt[7]{\frac{\phi^2\lambda^2}{\mu^6\rho^3}} \\ \sqrt[7]{\frac{\phi^4\mu^2}{\lambda^3\rho^6}} & 1 & 0 \end{pmatrix}$

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix}0&\mu&1\\1&0&\lambda\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&-\phi\\\phi^2&0&0\\0&\phi^4&0\end{pmatrix}$	$\begin{pmatrix}0&\phi^6\lambda&1\\1&0&-\phi^5\rho\\-\phi^3\mu&1&0\end{pmatrix}$
$\begin{pmatrix}0&\mu&1\\1&0&\lambda\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&\phi^2\\\phi^4&0&0\\0&-\phi&0\end{pmatrix}$	$\begin{pmatrix}0&-\phi^5\lambda&1\\1&0&-\phi^3\rho\\\phi^6\mu&1&0\end{pmatrix}$
$\begin{pmatrix}0&\mu&1\\1&0&\lambda\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&-\phi^3\\\phi^6&0&0\\0&-\phi^5&0\end{pmatrix}$	$\begin{pmatrix}0&\phi^4\lambda&1\\1&0&-\phi\rho\\\phi^2\mu&1&0\end{pmatrix}$
$\begin{pmatrix}0&\mu&1\\1&0&\lambda\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&\phi^4\\-\phi&0&0\\0&\phi^2&0\end{pmatrix}$	$\begin{pmatrix}0&-\phi^3\lambda&1\\1&0&\phi^6\rho\\-\phi^5\mu&1&0\end{pmatrix}$
$\begin{pmatrix}0&\mu&1\\1&0&\lambda\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&-\phi^5\\-\phi^3&0&0\\0&\phi^6&0\end{pmatrix}$	$\begin{pmatrix}0&\phi^2\lambda&1\\1&0&\phi^4\rho\\-\phi\mu&1&0\end{pmatrix}$
$\begin{pmatrix}0&\mu&1\\1&0&\lambda\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&\phi^6\\-\phi^5&0&0\\0&-\phi^3&0\end{pmatrix}$	$\begin{pmatrix}0&-\phi\lambda&1\\1&0&\phi^2\rho\\\phi^4\mu&1&0\end{pmatrix}$

TABLE 21′.

Case 5. M_B has seven non-zero elements.

The different cases that we must consider are $\binom{9}{2} = 36$. Every matrix has three entries which are 1 and four non-zero parameters $\delta, \lambda, \mu, \rho$, which must satisfy one of the following conditions, depending on the case we are considering, in order for the matrix to not have zero determinant: $\mu\rho \neq 1$; $\mu\rho + \delta\lambda \neq 1$; $\delta\mu \neq 1$; $\delta\mu + \lambda\rho \neq 1$; $\delta\lambda \neq 1$; $\delta\rho - \delta\lambda\mu \neq 1$; $\delta\rho \neq 1$; $\mu\rho - \delta\lambda\rho \neq 1$.

The number of mutually non-isomorphic parametric families of evolution algebras is eight, which are listed below.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix}1&\mu&\lambda\\\rho&1&\delta\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&\rho&\delta\\\mu&1&\lambda\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\delta&1&\rho\\\lambda&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&\mu\\0&1&0\\\rho&\delta&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\\lambda&1&\mu\\\delta&\rho&1\end{pmatrix}$	$\begin{pmatrix}1&\delta&\rho\\0&1&0\\\mu&\lambda&1\end{pmatrix}$
$\begin{pmatrix}1&\mu&\lambda\\\rho&1&0\\\delta&0&1\end{pmatrix}$	$\begin{pmatrix}1&\rho&0\\\mu&1&\lambda\\0&\delta&1\end{pmatrix}$	$\begin{pmatrix}1&0&\delta\\0&1&\rho\\\lambda&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&\mu\\\delta&1&0\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}1&\delta&0\\\lambda&1&\mu\\0&\rho&1\end{pmatrix}$	$\begin{pmatrix}1&0&\rho\\0&1&\delta\\\mu&\lambda&1\end{pmatrix}$
$\begin{pmatrix}1&\mu&\lambda\\\rho&1&0\\0&\delta&1\end{pmatrix}$	$\begin{pmatrix}1&\rho&0\\\mu&1&\lambda\\\delta&0&1\end{pmatrix}$	$\begin{pmatrix}1&\delta&0\\0&1&\rho\\\lambda&\mu&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&\mu\\0&1&\delta\\\rho&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&\delta\\\lambda&1&\mu\\0&\rho&1\end{pmatrix}$	$\begin{pmatrix}1&0&\rho\\\delta&1&0\\\mu&\lambda&1\end{pmatrix}$
$\begin{pmatrix}1&0&\mu\\\lambda&1&\rho\\\delta&0&1\end{pmatrix}$	$\begin{pmatrix}1&\lambda&\rho\\0&1&\mu\\0&\delta&1\end{pmatrix}$	$\begin{pmatrix}1&0&\delta\\\rho&1&\lambda\\\mu&0&1\end{pmatrix}$	$\begin{pmatrix}1&\mu&0\\\delta&1&0\\\lambda&\rho&1\end{pmatrix}$	$\begin{pmatrix}1&\delta&0\\\mu&1&0\\\rho&\lambda&1\end{pmatrix}$	$\begin{pmatrix}1&\rho&\lambda\\0&1&\delta\\0&\mu&1\end{pmatrix}$
$\begin{pmatrix}\mu&1&\lambda\\1&0&0\\\rho&\delta&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\1&\mu&\lambda\\\delta&\rho&1\end{pmatrix}$	$\begin{pmatrix}1&\delta&\rho\\0&0&1\\\lambda&1&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&\lambda&1\\\rho&1&\delta\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&\rho&\delta\\\lambda&\mu&1\\0&1&0\end{pmatrix}$	$\begin{pmatrix}0&0&1\\\delta&1&\rho\\1&\lambda&\mu\end{pmatrix}$
$\begin{pmatrix}\mu&1&\lambda\\1&0&\rho\\\delta&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&\rho\\1&\mu&\lambda\\0&\delta&1\end{pmatrix}$	$\begin{pmatrix}1&0&\delta\\\rho&0&1\\\lambda&1&\mu\end{pmatrix}$	$\begin{pmatrix}\mu&\lambda&1\\\delta&1&0\\1&\rho&0\end{pmatrix}$	$\begin{pmatrix}1&\delta&0\\\lambda&\mu&1\\\rho&1&0\end{pmatrix}$	$\begin{pmatrix}0&\rho&1\\0&1&\delta\\1&\lambda&\mu\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&\lambda&\rho\\\delta&0&1\end{pmatrix}$	$\begin{pmatrix}\lambda&1&\rho\\1&0&\mu\\0&\delta&1\end{pmatrix}$	$\begin{pmatrix}1&0&\delta\\\rho&\lambda&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\\delta&1&0\\1&\rho&\lambda\end{pmatrix}$	$\begin{pmatrix}1&\delta&0\\\mu&0&1\\\rho&1&\lambda\end{pmatrix}$	$\begin{pmatrix}\lambda&\rho&1\\0&1&\delta\\1&\mu&0\end{pmatrix}$
$\begin{pmatrix}0&1&\mu\\1&0&\lambda\\\rho&\delta&1\end{pmatrix}$	$\begin{pmatrix}0&1&\lambda\\1&0&\mu\\\delta&\rho&1\end{pmatrix}$	$\begin{pmatrix}1&\delta&\rho\\\lambda&0&1\\\mu&1&0\end{pmatrix}$	$\begin{pmatrix}0&\mu&1\\\rho&1&\delta\\1&\lambda&0\end{pmatrix}$	$\begin{pmatrix}1&\rho&\delta\\\mu&0&1\\\lambda&1&0\end{pmatrix}$	$\begin{pmatrix}0&\lambda&1\\\delta&1&\rho\\1&\mu&0\end{pmatrix}$

TABLE 22. $\dim(A^2) = 3$; seven non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & 0 \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ \frac{1}{2}(-1+\sqrt{-3})\rho & -\frac{1}{2}(1+\sqrt{-3})\delta & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & 0 \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 1 & 0 & 0 \\ -\frac{1}{2}(1+\sqrt{-3})\rho & \frac{1}{2}(-1+\sqrt{-3})\delta & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & \rho \\ \delta & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\rho \\ \frac{1}{2}(-1+\sqrt{-3})\delta & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & \rho \\ \delta & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & 1 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\rho \\ -\frac{1}{2}(1+\sqrt{-3})\delta & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & \lambda & \rho \\ \delta & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ 1 & \frac{1}{2}(-1+\sqrt{-3})\lambda & -\frac{1}{2}(1+\sqrt{-3})\rho \\ \frac{1}{2}(-1+\sqrt{-3})\delta & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & \lambda & \rho \\ \delta & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ 1 & -\frac{1}{2}(1+\sqrt{-3})\lambda & \frac{1}{2}(-1+\sqrt{-3})\rho \\ -\frac{1}{2}(1+\sqrt{-3})\delta & 0 & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ \frac{1}{2}(-1+\sqrt{-3})\rho & -\frac{1}{2}(1+\sqrt{-3})\delta & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ -\frac{1}{2}(1+\sqrt{-3})\rho & \frac{1}{2}(-1+\sqrt{-3})\delta & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & -\frac{1}{2}(1+\sqrt{-3}) & 0 \\ \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & -\frac{1}{2}(1+\sqrt{-3})\mu \\ 1 & 0 & \frac{1}{2}(-1+\sqrt{-3})\lambda \\ -\frac{1}{2}(1+\sqrt{-3})\delta & \frac{1}{2}(-1+\sqrt{-3})\rho & 1 \end{pmatrix}$
$\begin{pmatrix} 0 & 1 & \mu \\ 1 & 0 & \lambda \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \\ -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \frac{1}{2}(-1+\sqrt{-3})\mu \\ 1 & 0 & -\frac{1}{2}(1+\sqrt{-3})\lambda \\ \frac{1}{2}(-1+\sqrt{-3})\delta & -\frac{1}{2}(1+\sqrt{-3})\rho & 1 \end{pmatrix}$

TABLE 22′.

Case 6. There are eight non-zero elements in the matrix.

In this case there are only nine possibilities which appear in the table that follows. The condition that the entries of the matrix must satisfy is one of the following: $\eta\lambda + \mu\rho - \delta\eta\mu \neq 1$ or $\delta\mu + \eta\rho - \delta\eta\lambda \neq 1$, just to be sure that the determinant of the corresponding matrix is different from zero.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 1 & \mu & \lambda \\ \rho & 1 & \delta \\ \eta & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \rho & \delta \\ \mu & 1 & \lambda \\ 0 & \eta & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & \eta \\ \delta & 1 & \rho \\ \lambda & \mu & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \lambda & \mu \\ \eta & 1 & 0 \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \eta & 0 \\ \lambda & 1 & \mu \\ \delta & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \delta & \rho \\ 0 & 1 & \eta \\ \mu & \lambda & 1 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ \rho & 1 & \delta \\ 1 & \eta & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & \rho & \delta \\ \lambda & \mu & 1 \\ \eta & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \eta & 1 \\ \delta & 1 & \rho \\ 1 & \lambda & \mu \end{pmatrix}$	$\begin{pmatrix} \mu & 1 & \lambda \\ 1 & 0 & \eta \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & \eta \\ 1 & \mu & \lambda \\ \delta & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \delta & \rho \\ \eta & 0 & 1 \\ \lambda & 1 & \mu \end{pmatrix}$

TABLE 23. $\dim(A^2) = 3$; eight non-zero entries.

M_B	$P_{B'B}$	$M_{B'}$
$\begin{pmatrix} \mu & \lambda & 1 \\ \rho & 1 & \delta \\ 1 & \eta & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3}) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2}(-1+\sqrt{-3}) & 0 \end{pmatrix}$	$\begin{pmatrix} -\frac{1}{2}(1+\sqrt{-3})\mu & \frac{1}{2}(-1+\sqrt{-3})\lambda & 1 \\ \frac{1}{2}(-1+\sqrt{-3})\rho & 1 & -\frac{1}{2}(1+\sqrt{-3})\delta \\ 1 & -\frac{1}{2}(1+\sqrt{-3})\eta & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ \rho & 1 & \delta \\ 1 & \eta & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3}) & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{2}(1+\sqrt{-3}) \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2}(-1+\sqrt{-3})\mu & -\frac{1}{2}(1+\sqrt{-3})\lambda & 1 \\ -\frac{1}{2}(1+\sqrt{-3})\rho & 1 & \frac{1}{2}(-1+\sqrt{-3})\delta \\ 1 & \frac{1}{2}(-1+\sqrt{-3})\eta & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ \rho & 1 & \delta \\ 1 & \eta & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{\mu} & 0 \\ \frac{1}{\sqrt[3]{\delta\eta^2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt[3]{\delta^2\eta}} \end{pmatrix}$	$\begin{pmatrix} \frac{1}{\sqrt[3]{\delta\eta^2}} & \frac{\rho\sqrt[3]{\delta\eta^2}}{\mu^2} & 1 \\ \frac{\mu\lambda}{\eta\sqrt[3]{\delta^2\eta}} & 1 & \frac{\mu}{\delta\sqrt[3]{\delta\eta^2}} \\ 1 & \frac{\sqrt[3]{\delta^2\eta}}{\mu^2} & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ \rho & 1 & \delta \\ 1 & \eta & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{\mu} & 0 \\ -\frac{\phi}{\sqrt[3]{\delta\eta^2}} & 0 & 0 \\ 0 & 0 & \frac{\phi^2}{\sqrt[3]{\delta^2\eta}} \end{pmatrix}$	$\begin{pmatrix} -\frac{\phi}{\sqrt[3]{\delta\eta^2}} & \frac{\phi^2\rho\sqrt[3]{\delta\eta^2}}{\mu^2} & 1 \\ \frac{\phi^2\mu\lambda}{\eta\sqrt[3]{\delta^2\eta}} & 1 & -\frac{\phi\mu}{\delta\sqrt[3]{\delta\eta^2}} \\ 1 & -\frac{\phi\sqrt[3]{\delta^2\eta}}{\mu^2} & 0 \end{pmatrix}$
$\begin{pmatrix} \mu & \lambda & 1 \\ \rho & 1 & \delta \\ 1 & \eta & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & \frac{1}{\mu} & 0 \\ \frac{\phi^2}{\sqrt[3]{\delta\eta^2}} & 0 & 0 \\ 0 & 0 & -\frac{\phi}{\sqrt[3]{\delta^2\eta}} \end{pmatrix}$	$\begin{pmatrix} \frac{\phi^2}{\sqrt[3]{\delta\eta^2}} & -\frac{\phi\rho\sqrt[3]{\delta\eta^2}}{\mu^2} & 1 \\ -\frac{\phi\mu\lambda}{\eta\sqrt[3]{\delta^2\eta}} & 1 & \frac{\phi^2\mu}{\delta\sqrt[3]{\delta\eta^2}} \\ 1 & \frac{\phi^2\sqrt[3]{\delta^2\eta}}{\mu^2} & 0 \end{pmatrix}$

TABLE 23′.

Case 7 All the entries in the matrix are non-zero. In this case only one matrix appears.

	(1,2)	(1,3)	(2,3)	(1,2,3)	(1,3,2)
$\begin{pmatrix} 1 & \mu & \lambda \\ \rho & 1 & \delta \\ \eta & \tau & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \rho & \delta \\ \mu & 1 & \lambda \\ \tau & \eta & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \tau & \eta \\ \delta & 1 & \rho \\ \lambda & \mu & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \lambda & \mu \\ \eta & 1 & \tau \\ \rho & \delta & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \eta & \tau \\ \lambda & 1 & \mu \\ \delta & \rho & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \delta & \rho \\ \tau & 1 & \eta \\ \mu & \lambda & 1 \end{pmatrix}$

TABLE 24. $\dim(A^2) = 3$; nine non-zero entries.

and the condition that the parameters must satisfy is $\eta\rho + \delta\lambda + \mu\tau - \eta\lambda\tau - \delta\mu\rho \neq 1$.

Remark 4.2.5

In this remark we include the study of the different matrices that can appear as change of basis matrices for an evolution algebra A such that $\dim(A) = 3$ and $\dim(A^2) = 1$.

We have separated this piece from the proof of Theorem 4.2.2 in order to not enlarge it. We think that it can be of interest as we did a similar study when $\dim(A^2) = 2$ and when $\dim(A^2) = 3$. The notation we use is as in Case $\dim(A^2) = 1$.

Let B' be an arbitrary natural basis of A and let $P_{B'B} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$ be the change of basis matrix. By (1.8):

$$\begin{cases} p_{11}p_{12}a_{11} + p_{21}p_{22}c_1a_{11} + p_{31}p_{32}c_2a_{11} & = 0 \\ p_{11}p_{12}a_{21} + p_{21}p_{22}c_1a_{21} + p_{31}p_{32}c_2a_{21} & = 0 \\ p_{11}p_{12}a_{31} + p_{21}p_{22}c_1a_{31} + p_{31}p_{32}c_2a_{31} & = 0 \end{cases} \tag{4.32}$$

$$\begin{cases} p_{11}p_{13}a_{11} + p_{21}p_{23}c_1a_{11} + p_{31}p_{33}c_2a_{11} & = 0 \\ p_{11}p_{13}a_{21} + p_{21}p_{23}c_1a_{21} + p_{31}p_{33}c_2a_{21} & = 0 \\ p_{11}p_{13}a_{31} + p_{21}p_{23}c_1a_{31} + p_{31}p_{33}c_2a_{31} & = 0 \end{cases} \tag{4.33}$$

$$\begin{cases} p_{12}p_{13}a_{11} + p_{22}p_{23}c_1a_{11} + p_{32}p_{33}c_2a_{11} & = 0 \\ p_{12}p_{13}a_{21} + p_{22}p_{23}c_1a_{21} + p_{32}p_{33}c_2a_{21} & = 0 \\ p_{12}p_{13}a_{31} + p_{22}p_{23}c_1a_{31} + p_{32}p_{33}c_2a_{31} & = 0 \end{cases} \tag{4.34}$$

Since $e_1^2 \neq 0$, there exists $j \in \{1,2,3\}$ such that $a_{j1} \neq 0$. In any case, from (4.32), (4.33) and (4.34) we have:

$$p_{11}p_{12} = -(p_{21}p_{22}c_1 + p_{31}p_{32}c_2); \tag{4.35}$$

$$p_{11}p_{13} = -(p_{21}p_{23}c_1 + p_{31}p_{33}c_2); \tag{4.36}$$

$$p_{12}p_{13} = -(p_{22}p_{23}c_1 + p_{32}p_{33}c_2). \tag{4.37}$$

Case 1 Assume $p_{11}p_{12}p_{13} \neq 0$.

This implies that $p_{21}p_{22}c_1 + p_{31}p_{32}c_2 \neq 0$, $p_{21}p_{23}c_1 + p_{31}p_{33}c_2 \neq 0$ and $p_{22}p_{23}c_1 + p_{32}p_{33}c_2 \neq 0$. So, $p_{11} = \frac{-(p_{21}p_{22}c_1 + p_{31}p_{32}c_2)}{p_{12}}$. Replacing this value in (4.36), we get $p_{13} = \frac{(p_{21}p_{23}c_1 + p_{31}p_{33}c_2)p_{12}}{p_{21}p_{22}c_1 + p_{31}p_{32}c_2}$. Finally, we replace p_{13} in (4.37) and we have $p_{12} = \pm\sqrt{\frac{-(p_{21}p_{22}c_1 + p_{31}p_{32}c_2)(p_{22}p_{23}c_1 + p_{32}p_{33}c_2)}{p_{21}p_{23}c_1 + p_{31}p_{33}c_2}}$.

Therefore:

$$\begin{aligned} p_{11} &= -\sqrt{-\frac{(p_{21}p_{22}c_1 + p_{31}p_{32}c_2)(p_{21}p_{23}c_1 + p_{31}p_{33}c_2)}{p_{22}p_{23}c_1 + p_{32}p_{33}c_2}} \\ p_{12} &= \sqrt{-\frac{(p_{21}p_{22}c_1 + p_{31}p_{32}c_2)(p_{22}p_{23}c_1 + p_{32}p_{33}c_2)}{p_{21}p_{23}c_1 + p_{31}p_{33}c_2}} \\ p_{13} &= \sqrt{-\frac{(p_{21}p_{23}c_1 + p_{31}p_{33}c_2)(p_{22}p_{23}c_1 + p_{32}p_{33}c_2)}{p_{21}p_{22}c_1 + p_{31}p_{32}c_2}} \end{aligned}$$

or

$$\begin{aligned}
p_{11} &= \sqrt{-\frac{(p_{21}p_{22}c_1 + p_{31}p_{32}c_2)(p_{21}p_{23}c_1 + p_{31}p_{33}c_2)}{p_{22}p_{23}c_1 + p_{32}p_{33}c_2}} \\
p_{12} &= -\sqrt{-\frac{(p_{21}p_{22}c_1 + p_{31}p_{32}c_2)(p_{22}p_{23}c_1 + p_{32}p_{33}c_2)}{p_{21}p_{23}c_1 + p_{31}p_{33}c_2}} \\
p_{13} &= -\sqrt{-\frac{(p_{21}p_{23}c_1 + p_{31}p_{33}c_2)(p_{22}p_{23}c_1 + p_{32}p_{33}c_2)}{p_{21}p_{22}c_1 + p_{31}p_{32}c_2}}
\end{aligned}$$

Case 2 Suppose $p_{13} = 0$ and $p_{11}p_{12} \neq 0$.

We have the following equations:

$$p_{11}p_{12} = -(p_{21}p_{22}c_1 + p_{31}p_{32}c_2); \tag{4.38}$$

$$p_{21}p_{23}c_1 = -p_{31}p_{33}c_2; \tag{4.39}$$

$$p_{22}p_{23}c_1 = -p_{32}p_{33}c_2; \tag{4.40}$$

Case 2.1 $p_{31}c_2 \neq 0$.

Necessarily $p_{23} \neq 0$, since otherwise $p_{33} = 0$ contradicting the fact to $|P_{B'B}| \neq 0$. Moreover, $p_{32} \neq 0$. Indeed, if $p_{32} = 0$ then or $p_{22} = 0$ or $c_1 = 0$. But, on the other hand we have that $p_{21}p_{22}c_1 \neq 0$. Contradiction. Now, we distinguish between $c_1 = 0$ or not.

Case 2.1.1 $c_1 = 0$.

As $p_{31}p_{33}c_2 = 0$ necessarily $p_{33} = 0$. Then, the change of basis matrix is

$$P_{B'B} = \begin{pmatrix} p_{11} & -\dfrac{p_{31}p_{32}c_2}{p_{11}} & 0 \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & 0 \end{pmatrix}$$

with $p_{11}p_{12}p_{23}p_{31}p_{32}c_2 \neq 0$ and $p_{11}^2 + c_2p_{31}^2 \neq 0$.

Case 2.1.2 $c_1 \neq 0$.

Case 2.1.2.1 $p_{21} = 0$.

As $p_{31}p_{33}c_2 = 0$ necessarily $p_{33} = 0$. So $p_{22}p_{23}c_1 = 0$. Or equivalently $p_{22} = 0$. Then

$$P_{B'B} = \begin{pmatrix} p_{11} & -\frac{p_{31}p_{32}c_2}{p_{11}} & 0 \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & 0 \end{pmatrix}$$

with $p_{11}p_{23}p_{31}p_{32}c_2 \neq 0$ and $p_{11}^2 + c_2p_{31}^2 \neq 0$.

Case 2.1.2.2 $p_{21} \neq 0$.

By (4.39) $p_{33} = \frac{-p_{21}p_{23}c_1}{p_{31}c_2}$. If we remove p_{33} in (4.40) we get that $p_{22} = \frac{p_{32}p_{21}}{p_{31}}$. And finally if we replace p_{22} and p_{33} in (4.38) we have $p_{12} = -\frac{p_{32}(c_1p_{21}^2 + c_2p_{31}^2)}{p_{11}p_{31}}$. Then

$$P_{B'B} = \begin{pmatrix} p_{11} & -\frac{p_{32}(p_{21}^2c_1 + p_{31}^2c_2)}{p_{11}p_{31}} & 0 \\ p_{21} & \frac{p_{32}p_{21}}{p_{31}} & p_{23} \\ p_{31} & p_{32} & \frac{-p_{21}p_{23}c_1}{p_{31}c_2} \end{pmatrix},$$

where $p_{ij} \neq 0 \forall i,j \in 1,2,3$ except p_{13}. Furthermore, $p_{21}^2c_1 + p_{31}^2c_2 \neq 0$ and $p_{21}^2c_1 + p_{31}^2c_2 + p_{11}^2 \neq 0$.

Case 2.2 $c_2 \neq 0$ and $p_{31} = 0$.

The equations (4.38), (4.39) and (4.40) are as follows:

$$\begin{aligned} p_{11}p_{12} &= -p_{21}p_{22}c_1; \\ p_{21}p_{23}c_1 &= 0; \\ p_{22}p_{23}c_1 &= -p_{32}p_{33}c_2. \end{aligned}$$

As $p_{21}p_{22}c_1 \neq 0$ then necessarily $p_{23} = 0$ and so $p_{32} = 0$ ($p_{33} \neq 0$ because otherwise $|P| = 0$). Therefore

$$P_{B'B} = \begin{pmatrix} p_{11} & -\frac{p_{21}p_{22}c_1}{p_{12}} & 0 \\ p_{21} & p_{22} & 0 \\ 0 & 0 & p_{33} \end{pmatrix},$$

with $p_{11}p_{12}p_{21}p_{22}p_{33}c_1c_2 \neq 0$ and $p_{11}^2 + p_{21}^2c_1 \neq 0$.

Case 2.3 $c_2 = 0$.

The equations (4.38), (4.39) and (4.40) turn out

$$\begin{aligned} p_{11}p_{12} &= -p_{21}p_{22}c_1 \\ p_{21}p_{23}c_1 &= 0 \\ p_{22}p_{23}c_1 &= 0 \end{aligned}$$

As $p_{21}p_{22}c_1 \neq 0$ then $p_{23} = 0$. Therefore,

$$P_{B'B} = \begin{pmatrix} p_{11} & -\frac{p_{21}p_{22}c_1}{p_{11}} & 0 \\ p_{21} & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

with $p_{11}p_{21}p_{22}p_{33}c_1 \neq 0$ and $p_{11}^2 + p_{21}^2c_1 \neq 0$.

Case 3 $p_{12} = 0$ and $p_{11}p_{13} = 0$.

Reasoning in the same way as Case 1.2, we obtain the following results:

Case 3.1 $p_{31}c_2 \neq 0$.

Necessarily $p_{22}p_{33} \neq 0$.

Case 3.1.1 $c_1 \neq 0$.

Case 3.1.1.1 $p_{21} \neq 0$.

The change of basis matrix is as follows:

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & -\dfrac{p_{23}(p_{21}^2c_1 + p_{31}^2c_2)}{p_{11}p_{21}} \\ p_{21} & \dfrac{-p_{31}p_{32}c_2}{p_{21}c_1} & p_{23} \\ p_{31} & p_{32} & \dfrac{p_{23}p_{31}}{p_{21}} \end{pmatrix},$$

where $p_{ij} \neq 0 \forall i, j \in 1, 2, 3$ except p_{12}. Furthermore, $p_{21}^2c_1 + p_{31}^2c_2 \neq 0$ and $p_{21}^2c_1 + p_{31}^2c_2 + p_{11}^2 \neq 0$.

Case 3.1.1.2 $p_{21} = 0$.

So $p_{32} = 0$ and $p_{23} = 0$. And

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & -\dfrac{p_{31}p_{33}c_2}{p_{11}} \\ 0 & p_{22} & 0 \\ p_{31} & 0 & p_{33} \end{pmatrix}$$

with $p_{11}p_{22}p_{31}p_{33}c_2 \neq 0$ and $p_{11}^2 + p_{31}^2c_2 \neq 0$.

Case 3.1.2 $c_1 = 0$.

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & -\dfrac{p_{31}p_{33}c_2}{p_{11}} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & 0 & p_{33} \end{pmatrix},$$

where $p_{11}p_{31}p_{33}c_2 \neq 0$ and $p_{11}^2 + p_{31}^2c_2 \neq 0$.

Case 3.2 $p_{31} = 0$ and $c_2 \neq 0$.

Therefore $p_{22}p_{33} = 0$ and

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & -\dfrac{p_{21}p_{23}c_1}{p_{11}} \\ p_{21} & 0 & p_{23} \\ 0 & p_{32} & 0 \end{pmatrix}$$

with $p_{11}p_{13}p_{21}p_{23}p_{32}c_2 \neq 0$ and $p_{11}^2 + p_{21}^2c_1 \neq 0$.

Case 3.3 $c_2 = 0$.

Necessarily $p_{22} = 0$ and

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & -\dfrac{p_{21}p_{23}c_1}{p_{11}} \\ p_{21} & 0 & p_{23} \\ p_{31} & p_{32} & p_{32} \end{pmatrix},$$

where $p_{11}p_{21}p_{23}p_{32}c_1 \neq 0$ and $p_{11}^2 + c_1 p_{21}^2 \neq 0$.

Case 4 $p_{11} = 0$ and $p_{12}p_{13} = 0$.

In the same way as in Case 1.2 and Case 1.3 we obtain the following change of basis matrices:

Case 4.1 $p_{32}c_2 \neq 0$.

Then $p_{21}p_{33} \neq 0$.

Case 4.1.1 $c_1 \neq 0$.

Case 4.1.1.1 $p_{22} \neq 0$.

The change of basis matrix is as follows:

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & -\dfrac{p_{33}(p_{22}^2c_1 + p_{32}^2c_2)}{p_{32}p_{12}} \\ p_{21} & p_{22} & \dfrac{p_{22}p_{33}}{p_{32}} \\ -\dfrac{p_{21}p_{22}c_1}{p_{32}c_2} & p_{32} & p_{33} \end{pmatrix},$$

where $p_{ij} \neq 0 \forall i,j \in 1,2,3$ except p_{11}. Furthermore, $p_{22}^2c_1 + p_{32}^2c_2 \neq 0$ and $p_{22}^2c_1 + p_{32}^2c_2 + p_{12}^2 \neq 0$.

Case 4.1.1.2 $p_{22} = 0$.

Then $p_{31} = p_{23} = 0$. And

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & -\dfrac{p_{32}p_{33}c_2}{p_{12}} \\ p_{21} & 0 & 0 \\ 0 & p_{32} & p_{33} \end{pmatrix},$$

where $p_{12}p_{21}p_{32}p_{33}c_2 \neq 0$ and $p_{12}^2 + p_{32}^2c_2 \neq 0$.

Case 4.1.2 $c_1 = 0$.

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & -\dfrac{p_{32}p_{33}c_2}{p_{12}} \\ p_{21} & p_{22} & p_{23} \\ 0 & p_{32} & p_{33} \end{pmatrix}$$

with $(p_{12}p_{32}p_{33}c_2)(p_{12}^2 + p_{32}^2c_2) \neq 0$.

Case 4.2 $p_{32} = 0$ and $c_2 \neq 0$.

So $p_{21}p_{33} = 0$ and

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & -\dfrac{p_{22}p_{23}c_1}{p_{12}} \\ 0 & p_{22} & p_{23} \\ p_{31} & 0 & 0 \end{pmatrix},$$

where $p_{12}p_{13}p_{22}p_{23}p_{31}c_2(p_{12}^2 + p_{22}^2c_1) \neq 0$.

Case 4.3 $c_2 = 0$.

Then $p_{21} = 0$ and

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & -\dfrac{p_{22}p_{23}c_1}{p_{12}} \\ 0 & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

for $p_{12}p_{22}p_{23}p_{31}c_1 \neq 0$ and $p_{12}^2 + c_1p_{22}^2 \neq 0$.

Case 5 $p_{1i} = p_{1j} = 0$ for some $i, j \in 1, 2, 3\, i \neq j$.

The equations (4.38), (4.39) and (4.40) are as follows:

$$\begin{aligned} p_{21}p_{22}c_1 &= -p_{31}p_{32}c_2; \\ p_{21}p_{23}c_1 &= -p_{31}p_{33}c_2; \\ p_{22}p_{23}c_1 &= -p_{32}p_{33}c_2. \end{aligned}$$

Case 5.1 $c_1 = c_2 = 0$.

If, for example, $p_{12} = p_{13} = 0$, the change of basis matrix is

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

with $p_{11}(p_{22}p_{33} - p_{23}p_{32}) \neq 0$.

In case of $p_{11} = p_{12} = 0$, the change of basis matrix is

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

for $p_{13}(p_{21}p_{32} - p_{22}p_{31}) \neq 0$.

For $p_{11} = p_{13} = 0$

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix},$$

where $p_{12}(p_{23}p_{31} - p_{21}p_{33}) \neq 0$.

Case 5.2 $c_i = 0$ and $c_j \neq 0$ for $i, j \in \{1, 2\}$ $i \neq j$.

Necessarily $p_{sk} = p_{sm} = 0$ for some $k, m \in \{1, 2, 3\}$ $k \neq m$ and $s \in \{2, 3\}$ depending on $c_2 = 0$ or $c_1 = 0$ respectively. We have to take in account that $|P| \neq 0$ and so there are possibilities that it can not be (those in which the structure matrix has a zero minor of order two). For example $p_{11} = p_{12} = p_{31} = p_{32} = 0$ if $c_1 = 0$ is not available. Therefore there are nine possible change of basis matrices are the following:

For $c_1 = 0$

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & p_{33} \end{pmatrix}$$

with $p_{12}p_{21}p_{33} \neq 0$.

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & p_{22} & p_{23} \\ p_{31} & 0 & 0 \end{pmatrix}$$

with $p_{12}p_{23}p_{31} \neq 0$.

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & p_{22} & p_{23} \\ 0 & p_{32} & 0 \end{pmatrix}$$

with $p_{13}p_{21}p_{32} \neq 0$.

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & 0 & 0 \end{pmatrix}$$

with $p_{13}p_{22}p_{31} \neq 0$.

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ p_{21} & p_{22} & p_{23} \\ 0 & p_{32} & 0 \end{pmatrix}$$

with $p_{11}p_{23}p_{32} \neq 0$.

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ p_{21} & p_{22} & p_{23} \\ 0 & 0 & p_{33} \end{pmatrix}$$

with $p_{11}p_{22}p_{33} \neq 0$.

If $c_2 = 0$

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix},$$

where $p_{13}p_{22}p_{31} \neq 0$.

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & 0 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

with $p_{13}p_{21}p_{32} \neq 0$.

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

with $p_{12}p_{23}p_{31} \neq 0$.

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

for $p_{12}p_{21}p_{33} \neq 0$.

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & 0 \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

with $p_{11}p_{22}p_{33} \neq 0$.

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & 0 & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix}$$

with $p_{11}p_{23}p_{32} \neq 0$.

Case 5.3 $c_1c_2 \neq 0$.

Fix $i, j \in \{1, 2, 3\}$ with $i \neq j$ and such that $p_{1i} = p_{1j} = 0$.

Case 5.3.1 $p_{2i} = 0$ ($p_{2j} = 0$).

Then as p_{3i} (p_{3j}) can not be zero, necessarily $p_{3j} = p_{3k} = 0$ ($p_{3i} = p_{3k} = 0$) with $k \in \{1, 2, 3\}$ and $k \neq i, j$. Therefore we have that $p_{2k} = 0$ because p_{2j} (p_{2i}) is not possible to be zero. Therefore $p_{1i}p_{2j}p_{3k} \neq 0$ with $i, j, k \in \{1, 2, 3\}$ and $i \neq j \neq k$. So, in this case the elements of $S_3 \rtimes (\mathbb{K}^\times)^3$ are the change of basis matrices.

Case 5.3.2 $p_{2i}p_{2j} \neq 0$.

We claim that p_{2k} must be zero with $k \in \{1, 2, 3\}$ and $k \neq i \neq j$. Indeed, if $p_{2k} \neq 0$ then $p_{3s} \neq 0$ for every $s \in \{1, 2, 3\}$. Then as $p_{2i} = -\dfrac{p_{3i}p_{3j}c_2}{p_{2j}c_1}$ we have that $p_{3k} = \dfrac{p_{2k}p_{3j}}{p_{2j}}$. And finally we obtain that $p_{2j}^2c_1 + p_{3j}^2c_2 = 0$, contradicting $|P| \neq 0$.

So $p_{2k} = 0$. As $p_{2i}p_{2j} \neq 0$ necessarily $p_{3k} = 0$. There are three possible change of basis matrices.

$$P_{B'B} = \begin{pmatrix} 0 & 0 & p_{13} \\ p_{21} & -\dfrac{p_{31}p_{32}c_2}{p_{21}c_1} & 0 \\ p_{31} & p_{32} & 0 \end{pmatrix}$$

with $p_{13}p_{21}p_{31}p_{32} \neq 0$ and $p_{21}^2 c_1 + p_{31}^2 c_2 \neq 0$.

$$P_{B'B} = \begin{pmatrix} 0 & p_{12} & 0 \\ p_{21} & 0 & -\dfrac{p_{31}p_{33}c_2}{p_{21}c_1} \\ p_{31} & 0 & p_{33} \end{pmatrix}$$

for $p_{12}p_{21}p_{31}p_{33} \neq 0$ and $p_{21}^2 c_1 + p_{31}^2 c_2 \neq 0$.

$$P_{B'B} = \begin{pmatrix} p_{11} & 0 & 0 \\ 0 & p_{22} & -\dfrac{p_{32}p_{33}c_2}{p_{22}c_1} \\ 0 & p_{32} & p_{33} \end{pmatrix},$$

where $p_{11}p_{22}p_{32}p_{33} \neq 0$ and $p_{22}^2 c_1 + p_{32}^2 c_2 \neq 0$.

Appendix A

The optimal fragmentation computed with Mathematica

We have designed a routine that provides the optimal fragmentation of an evolution algebra when we introduce the structure matrix as input. Moreover, the code identifies if an index is a cyclic-index, a principal cyclic index or a chain-start index. On the other hand, it calculates the nth-generation descendents of any index for every n. We include the *Mathematica* codes needed for our computations. They consist on a list of functions written in the order they have been used. The computation of the invariants has been performed by the *Mathematica* software.

In order to compute the optimal fragmentation, we have used the proposition that follows.

Proposition A.1
Let Λ be a finite set and let $\Upsilon_1, ..., \Upsilon_n$ be non-empty subsets of Λ such that $\Lambda = \bigcup_{i=1}^{n} \Upsilon_i$. Let $(a_{ij}) \in M_n(\mathbb{K})$ be the matrix defined by: $a_{ii} = 0$ for every i, $a_{ij} = 1$ if $\Upsilon_i \cap \Upsilon_j \neq \emptyset$ and $a_{ij} = 0$ if $\Upsilon_i \cap \Upsilon_j = \emptyset$. Let E be the graph whose adjacency matrix is (a_{ij}). Then, E is connected if and only if $\Lambda = \bigcup_{i=1}^{n} \Upsilon_i$ is not a fragmentable union. Moreover, if E is not connected, then the connected components of E form an optimal fragmentation of Λ.

Demostración. Suppose that $\Lambda = \bigcup_{i=1}^{n} \Upsilon_i$ is a non fragmentable union. If E is not connected, let Ψ_i denote the connected components of E with $i \in \{1, 2, \ldots, m\}$ for some $m \in \mathbb{N}$. This means that we may write $\{1, 2, \ldots, n\} = \bigsqcup_{i=1}^{m} \Psi_i$ where $\Psi_i \subseteq \{1, 2, \ldots, n\}$. Now, we consider the sets: $\Lambda_i = \bigcup_{j \in \Psi_i} \Upsilon_j$. We will show that $\Lambda = \bigcup_{i=1}^{m} \Lambda_i$ is an optimal fragmentation. First, we have to prove that $\Lambda_i \cap \Lambda_j = \emptyset$ for every $i \neq j$, with $i, j \in \{1, \ldots, m\}$. If there exists $\omega \in \Lambda_i \cap \Lambda_j$, then there are $r \in \Psi_i$ and $s \in \Psi_j$ such that $\omega \in \Upsilon_r \cap \Upsilon_s$. This implies that $\Upsilon_r \cap \Upsilon_s \neq \emptyset$, i.e. r and s are connected. This is a contradiction because they belong to different connected components. Conversely, suppose that E is connected. If $\Lambda = \bigcup_{i=1}^{n} \Upsilon_i$ is a fragmentable union then there exist Λ_1 and Λ_2 disjoint subsets of Λ satisfying that $\Lambda = \Lambda_1 \cup \Lambda_2$ and such that for every $i = 1, \ldots, n$, either $\Upsilon_i \subseteq \Lambda_1$ or $\Upsilon_i \subseteq \Lambda_2$. Let $\alpha \in \Lambda_1$ and $\beta \in \Lambda_2$. This means that there exist $i, j \in \{1, \ldots, n\}$ such that $\alpha \in \Upsilon_i \subseteq \Lambda_1$ and $\beta \in \Upsilon_j \subseteq \Lambda_2$. As E is connected, there exists a path from α to β. This implies that there exist $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that

$$\Upsilon_i \cap \Upsilon_{i_1} \neq \emptyset, \Upsilon_{i_1} \cap \Upsilon_{i_2} \neq \emptyset, \ldots, \Upsilon_{i_k} \cap \Upsilon_j \neq \emptyset,$$

a contradiction because $\alpha \in \Upsilon_i \subseteq \Lambda_1$ and $\beta \in \Upsilon_j \subseteq \Lambda_2$. Furthermore, from this reasoning we deduce that the connected components of E make an optimal fragmentation of Λ. □

In what follows we provide a list with the routines that have been used together with a brief description of them.

- **D**$_1$: computes the first-generation descendents of i.
- **D**$_n$: computes the nth-generation descendents of i.
- **CycleQ**: checks if M has a cycle.

- **DP**: computes $D(i)$.
- **CyclicQ**: checks if the strucure matrix M has some cyclic index.
- **CycleAssociated**: computes the cycle associated to i.
- **Ascendents**: computes the ascendents of i.
- **PrincipalCycleQ**: checks if i is a principal cyclic-index.
- **ChainStartQ**: checks if i is a chain-start index.
- **CanonicalDecomposition**: computes a canonical decomposition associated to the structure matrix M.
- **OptimalFragmentation**: computes an optimal fragmentation associated to the structure matrix M.

Finally, we include the *Mathematica* code of all these functions.

```
l = Table[i, {i, n}];
    D1[i_, M_] := Select[l, M[[#, i]] ≠ 0&];

Dn_[i_, M_] := Module[{j, a, s},
    a = {}; s = Length[Dn−1[i, M]];
    If[n == 1, D1[i, M],
    Union[Flatten[Table[D1[Dn−1[i, M][[t]], M], {t, Length[Dn−1[i, M]]}]]]]]

CycleQ[M_] := Module[{n, a}, n = Length[M];
    a = Union[Flatten[Table[
    Diagonal[MatrixPower[M, i]], {i, 1, n}]]];
    MemberQ[a, 1];

DYesCycle[i_, M_] := Module[{j, a},
    a = {};
    For[j = 1, j <= Length[M], j ++,
    AppendTo[a, Dj[i, M]]];
    Apply[Union, a]]

DNotCycle[i_, M_] := Module[{j, a},
    a = {D1[i, M]};
    For[j = 1, Dj[i, M] ≠ Dj+1[i, M], j ++,
    AppendTo[a, Dj+1[i, M]]];
    Apply[Union, a]]

DP[i_, M_] := If[CycleQ[M], DYesCycle[i, M], DNotCycle[i, M]]


CyclicQ[i_, M_] := If[MemberQ[DP[i, M], i],
    Print[i "is a cyclic index"],Print[i "is not a cyclic index"]]
```

```
CycleAssociated[i_, M_] := Module[{j, a},
    a = {};
    For[j = 1, j <= Length[M], j + +,
    IfMemberQ[DP[i, M], j]&&MemberQ[DP[j, M], i],
    AppendTo[a, j]]];
    a]

Ascendents[i_, M_] := Module[{j, a},
    a = {};
    For[j = 1, j <= Length[M], j + +,
    IfMemberQ[DP[j, M], i],
    AppendTo[a, j]]];
    a]

Subset[A_, B_] := (Union[A, B] == Union[B])

PrincipalCycleQ[i_, M_] := If[Subset[Ascendents[i, M], CycleAssociated[i, M]],
    Print[i "is a principal cyclic-index"],
    Print[i "is not a principal cyclic-index"]]

ElementsNotNoneRow[M_] := Module[{j},
    Select[Table[j, {j, Length[M]}], M[[#]] == 0M[[1]]&]];

ChainStartQ[i_, M_] := If[MemberQ[ElementsNotNoneRow[M], i],
    Print[i " is a chain-start index"],
    Print[i "is not a chain-start index"]]

Λ[i_, M_] := Union[{i}, DP[i, M]];

LambdaChainStart[M_] := Table[Λ[ElementsNotNoneRow[M][[i]], M], {i, Length[ElementsNotNoneRow[M]]}]

LambdaPrincipalCycle[M_] := Module[{j, a},
    a = {};
    For[j = 1, j <= Length[M], j + +,
    If[Subset[Ascendents[j, M], CycleAssociated[j, M]],
    AppendTo[a, DP[j, M]]]];
    a]

CanonicalDecomposition[M_] := Join[LambdaChainStart[M], LambdaPrincipalCycle[M]]

f[i_, j_, M_] := If[[i == j, 0,
    If[Intersection[Part[CanonicalDecomposition[M], i],
    Part[CanonicalDecomposition[M], j]] ≠ 0, 1, 0]]

Matr[M_] := Table[
    f[i, j, M], {i, Length[CanonicalDecomposition[M]]}, {j
    Length[CanonicalDecomposition[M]]}]

OptimalFragmentation[M_] := ConnectedComponents[AdjacencyGraph[Matr[M]], VertexLabels → "Name"]
```

One concrete example showing how this program works can be found in https://www.dropbox.com/s/2mtdojjaj1o20m8/OptimalFragmentation.pdf?dl=0.

Further works

This manuscript deals with distinct aspects of the theory of evolution algebras. Particularly, we have focused on basic properties of this type of non-associative algebras: decomposition into direct sums of non-zero evolution subalgebras and the classification of two-dimensional and three-dimensional evolution algebras. Let us expose here further works that can be considered a follow up of the manuscript.

In Chapter 1 we have included a detailed study about concepts as evolution algebra, evolution subalgebra and evolution ideal, indicating the relation that exists between them. In this way, we have enabled the ground in order to continue to deepen the theory of evolution algebras. In this sense, we have established a strong relationship between graph theory and evolution algebras. Although, by the very definition of graph associated to an evolution algebra, the map sending an evolution algebra to its corresponding graph is not a bijection, a natural question is if there exist some (more) properties from the evolution algebra that can be read in terms of its underlying graph, and conversely. An example of such is decomposability: the evolution algebra is indecomposable if and only if the corresponding graph is connected. Taking into account the experience of our research group in characterizing algebraic properties in terms of graph properties (in the context of Leavitt path algebras), we hope that we will be able to study the relationship between the evolution ideals and the corresponding graph of the evolution algebra. Another interest we have started to work on is the classification of the alternative evolution algebras. In Chapter 2, once obtained the optimal direct-sum decomposition, the natural question that arises is the following. As evolution algebras have their origin in the non-Mendelian genetic, will this decomposition have an impact from the biological point of view? In Chapters 3 and 4 we have used different methods in order to obtain the classification of evolution algebras of dimension two and three. The following step will be to analyze if these approaches can be generalized to arbitrary finite-dimensional evolution algebras. Of course, this is a very ambitious project and not affordable in its full generality. However, the description of the three dimensional case has given us some clues on how to continue. In this line, we have started the study of the four-dimensional evolution algebras A such that the dimension of A and the dimension of A^2 coincide. Another question in which we are interested is in a biological application of the classification of evolution algebras. In this sense, we have contacted the team of Professor Enrique Viguera, geneticist at the University of Malaga, and other experts at the University of Granada, in order to get a common language that allows us to find some possible applications of our results in a biological context.

Bibliography

[1] Abrams, G., Ara, P., Siles Molina M., *Leavitt path algebras*, Springer (to appear).

[2] Bertrand, M., *Algèbres non associatives et algèbres génétiques*, Mémorial des Sciences Mathématiques, 162, Gauthier-Villars Éditeur, Paris (1966).

[3] Bonsall, F.F., Duncan, J., *Complete normed algebras*, Springer (1973).

[4] Cabrera Casado, Y., Siles Molina, M., Velasco, M.V., Evolution algebras of arbitrary dimension and their decompositions. *Linear Algebra Appl.* **495** (2016), 122-162.

[5] Cabrera Casado, Y., Siles Molina, M., Velasco, M.V., Classification of three-dimensional evolution algebras. Preprint (2016).

[6] Cabrera García, M., Rodríguez Palacios, Á., Encyclopedia of Mathematics and its Applications. Non-associative normed algebras. Vol. 1. The Vidav-Palmer and Gelfand-Naimark theorems. *Cambridge University Press, Cambridge* **154**(2014), xxii+712 pp.

[7] Camacho, L.M., Gómez, J. R., Omirov, B. A., Turdibaev, R. M., The derivations of some evolution algebras. *Linear Multilinear Algebra* **6** (1), (2013), 309–322.

[8] Camacho, L. M., Gómez, J. R., Omirov, B. A., Turdibaev, R. M., Some properties of evolution algebras. *Bull. Korean Math. Soc.* **50** (5), (2013), 1481–1494.

[9] Corrales García, M. G, Martín Barquero, D., Martín González, C., Siles Molina, M., Solanilla Hernández, J. F., Extreme cycles. The center of a Leavitt path algebra. *Pub. Mat.* (To appear).

[10] Casas, J.M., Ladra, M., Rozikov, U. A., A chain of evolution algebras. *Linear Algebra Appl.* **435** (2011), 852–870.

[11] Casas, J. M., Ladra, M., Omirov, B. A., Rozikov, U. A., On nilpotent index and dibaricity of evolution algebras, *Linear Algebra Appl.* **43** (9), (2013), 90–105.

[12] Casas, J. M., Ladra, M., Omirov, B. A., Rozikov, U. A., On evolution algebras. *Algebra Colloq.* **21** (2014), 331–342.

[13] Dales, H. G., *Banach algebras and automatic continuity*, London Math, Soc. Monographs, 24, Clarendon Press, Oxford (2000).

[14] Dzhumadil'daev, A., Omirov, B. A., Rozikov, U. A., On a class of evolution algebras of çhicken" population. *Internat. J. Math.* **25** (2014), 1450073, 19 pp.

[15] Elduque, A., Labra, A., Evolution algebras and graphs. *J. Algebra Appl.* **14** (7), (2015), 1550103, 10 pp.

[16] Elduque, A., Labra, A., On nilpotent evolution algebras. Preprint (2016) (arXiv:1602.01590v1).

[17] Etherington, I. M. H., Genetic algebras, *Proc. Royal Soc. Edinburgh* **59** (1939), 242–258.

[18] Etherington, I. M. H., Non-associative algebra and the symbolism of genetics, *Proc. Royal Soc. of Edinburgh* **61** (1941), 24–42.

[19] Labra, A., Ladra, M., Rozikov, U. A., An evolution algebra in population genetics. *Linear Algebra Appl.* **45** (7), (2014), 348–362.

[20] Ladra, M., Omirov, B. A., Rozikov, U. A., Dibaric and evolution algebras in biology. *Lobachevskii J. Math.* **35** (2014), 198–210.

[21] Ladra, M., Rozikov, U. A., Evolution algebra of a bisexual population. *J. Algebra* **378** (2013), 153–172.

[22] Murodov, Sh. N., Classification dynamics of two-dimensional chains of evolution algebras. *Internat. J. Math.* **25** (2014), 1450012, 23 pp.

[23] Omirov, B. A., Rozikov, U. A., Tulenbayev, K. M., On real chains of evolution algebras. *Linear Multilinear Algebra* **63** (2015), 586–600.

[24] Palmer, T. W., *Banach algebras and the general theory of *-algebras*, vol. I. Cambridge University Press (1994).

[25] Reed, M. L., Algebraic structure of genetics inheritance, *Bull. Amer. Math. Soc.* **34** (1997), 107–130.

[26] Rickart, C. E., *The General Theory of Banach Algebras*, Van Nostrand (1960).

[27] Rozikov, U.A., Tian, J. P., Evolution algebras generated by Gibbs measures. *Lobachevskii J. Math.* **32** (2011), 270–277.

[28] Rozikov, U. A., Murodov, Sh. N., Dynamics of two-dimensional evolution algebras. *Lobachevskii J. Math.* **3** (4), (2013), 344–358.

[29] Rozikov, U. A., Murodov, Sh. N., Chain of evolution algebras of çhicken" population. *Linear Algebra Appl.* **450** (2014), 186–201.

[30] Tian, J. P., *Evolution algebras and their applications*, Lecture Notes in Mathematics 1921, Springer (2008).

[31] Tian J. P., Vojtechovsky, P., Mathematical concepts of evolution algebras in non-Mendelian genetics, *Quasigroup and Related System* **14** (2006), 111–122.

[32] Tian, J. P and Zou, Y. M., Finitely generated nil but not nilpotent evolution algebras. *J. Algebra Appl.* **13** (1), (2014), 1350070, 10 pp.

[33] Wörz-Busekros, A., *Algebras in genetics*. Lecture Notes in Biomathematics 36, Springer-Verlag (1980).

Notation

$\mathbb{N}$	positive integers
$\mathbb{K}$	field
$\mathbb{K}^{\times}$	$\mathbb{K} \setminus 0$
S_3	Set of permutation matrices
$\mathrm{CFM}_{\Lambda}(\mathbb{K})$	vector space of matrices over $\mathbb{K}$ of size $\Lambda \times \Lambda$ for which every column has at most a finite number of non-zero entries.
$\mathrm{M_n}(\mathbb{K})$	square matrices over a field $\mathbb{K}$
$\mathrm{GL}_3(\mathbb{K})$	set of $n \times n$ invertible matrices with entries from $\mathbb{K}$
$S_3 \rtimes (\mathbb{K}^{\times})^3$	semidirect product of the group of S_3 and $(\mathbb{K}^{\times})^3$
$\xi_B(x)$	vector $\Lambda \times 1$ give by the coordinates of x respect to B
(i, j)	permutation that sends the element i into the element j
(i, j, k)	permutation sending i to j, j to k and k to i
$\cup$	union
$\sqcup$	disjoint union
$\cap$	intersection
$\subseteq \ \subset$	subset
ann(X)	annihilator of X
rad(X)	radical of X
$< X >$	ideal generated by X
lin{X}	subspace generated by elements of X
$\widehat{I}$	one-dimension non-degenerate evolution principal ideal
ϕ	a seventh root of the unit
ζ	a third root of the unit

Alphabetical Index

absorption
- property, 25
- radical, 27

algebra, 6
- annihilator, 23
- associative, 6
- commutative, 6
- flexible, 6
- homomorphism, 22
- nondegenerate, 27
- power associative, 6
- semiprime, 27
- simple, 42
- structure constants, 11

ascendent, 36

canonical decomposition, 62
cycle, 57

descendent, 36

evolution
- algebra, 6
  - graph associated, 31
  - irreducible, 50
  - non-degenerate, 23
  - quotient, 21
  - reducible, 50
- ideal, 20, 49
- subalgebra, 18

extension property, 18

fragmentable union, 62
fragmentation, 63
- optimal, 63

graph
- adjacency matrix, 30
- Condition Sing, 31
- cycle, 30
- directed, 29
- evolution algebra associated, 32
- index
  - cyclic, 59
  - principal, 60
- length, 30
- path, 30
- principal cycle, 60
- range, 29, 30
- row-finite, 30
- sink, 30
- source, 29
- vertices, 29

index
- chain-start, 61
- cycle associated, 57
- cyclic, 57
  - loop, 57
- principal cyclic, 58

index-set derived, 59

left multiplication operator, 11

natural basis, 6

optimal direct-sum decomposition, 54

permutation matrices, 83
principal cycle, 58
principal ideal, 73
property 2LI, 85

structure
- constants, 6
- matrix, 7