Toward the use of the contraposition law in multi-adjoint lattices

Nicolás Madrid

UNIVERSITY OF MÁLAGA
SPAIN

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Introduction

- In crisp logic, the connectives hold interesting relationships among them via tautologies.
- Some very well known examples are
  - $p \wedge q \quad \longleftrightarrow \quad \neg(\neg p \vee \neg q)$ (the Morgan's law)
  - $p \rightarrow q \quad \longleftrightarrow \quad \neg q \rightarrow \neg p$ (the contraposition law)
  - $p \rightarrow q \quad \longleftrightarrow \quad \neg p \vee q$ (the material implication)
  - $\neg(\neg p) \quad \longleftrightarrow \quad p$ (the double negation law)
- However, the satisfiability of such relationships in **Fuzzy logic** depends on how the connectives are interpreted.

Outline

In this talk I focus on the **double negation** and on the **Contraposition law** by:

- showing limitations of residuated structures to deal with both laws at the same time,
- presenting a structure based on multi-adjoint structures where, somehow, both laws hold,
- and providing some theoretical results of the mentioned structure.

Finally some future work are presented.

The residuated lattice

Let me begin by recalling the notion of residuated lattice.

Definition

A residuated lattice is a triple $\mathcal{L} = ((L, \leq), *, \rightarrow)$ such that:

1. $(L, \leq)$ is a complete and bounded lattice with largest element 1 and least element 0.
2. $(L, *, 1)$ is a commutative monoid unit element 1.
3. $*$ and $\rightarrow$ form an adjoint pair, i.e:

$$z \leq (y \rightarrow x) \text{ iff } y * z \leq x \quad \text{for all } x, y, z \in L.$$

Contraposition law in residuated lattices

Definition

A negation in a complete lattice $(L, \leq)$ is any antitonic mapping $n\colon L \to L$ such that $n(0) = 1$ and $n(1) = 0$.

The contraposition law in a logic based on a residuated lattice $((L, \leq), *, \to)$ and a negation n requires that

$$x \to y = n(y) \to n(x)$$

for all $x, y \in L$.

It is not hard to prove that if the equality above hold, the negation n must coincide with:

$$n(x) = x \to 0.$$

for all $x \in L$.

Double negation law vs contraposition law

On another hand, the Double negation law in a logic based on a residuated lattice $((L, \leq), *, \rightarrow)$ and a negation n requires that

$$n(n(x)) = x$$

for all $x \in L$; i.e., the negation n must be involutive

The problem with requiring both laws is that

- the negation defined by $x \rightarrow 0$ is seldom involutive,
- from a theoretical point of view, somehow the contraposition law is preferred,
- but from a practical point of view, somehow the double negation law is preferred.

Let's be shortly informal.

After checking the theory based on residuated lattices we can think that, in general,

- Contraposition Law
- and Double negation law

do not hold in fuzzy environments but ... informally the do!

> If I kick powerfully the ball, then it goes far.
>
> If the ball did not go far, then I did not kick the ball powerfully.

One option is to restrict ourselves in residuated lattices where both features hold; for instance in Łukasiewicz logic.

Adjoint Triples
The definition

The Multi-adjoint lattice structure is base on adjoint triples.

Definition

Let $(P_1, \leq_1)$, $(P_2, \leq_2)$, $(P_3, \leq_3)$ be three posets. The mappings $\&\colon P_1 \times P_2 \to P_3$, $\searrow\colon P_2 \times P_3 \to P_1$, and $\nearrow\colon P_1 \times P_3 \to P_2$ form an adjoint triple among P_1, P_2 and P_3 whenever:

$$\begin{aligned} x \leq_1 y \searrow z \quad &\text{if and only if} \\ x \,\&\, y \leq_3 z \quad &\text{if and only if} \\ y \leq_2 x \nearrow z \quad & \end{aligned}$$

for all $x \in P_1$, $y \in P_2$ and $z \in P_3$.

Adjoint Triples

Properties and one restriction

Lemma

If $(\&, \searrow, \nearrow)$ is an adjoint triple w.r.t. P_1, P_2, P_3, then

1. *$\&$ is order-preserving on both arguments,*
2. *$\searrow, \nearrow$ are order-preserving on the first argument and order-reversing on the second argument.*

In this paper I consider a simplified structure of adjoint triples:

$(P_1, \leq_1), (P_2, \leq_2)$ and $(P_3, \leq_3)$ **are equal to a lattice** $(L, \leq)$**.**

Multi-adjoint lattice

The notion of multi-adjoint lattice is defined as follows.

Definition

A multi-adjoint lattice is a tuple

$$(L, \leq, (\&_1, \swarrow_1, \nwarrow_1), (\&_2, \swarrow_2, \nwarrow_2), \ldots, (\&_k, \swarrow_k, \nwarrow_k))$$

where $(L, \leq)$ is a complete lattice and $(\&_i, \swarrow_i, \nwarrow_i)$ is an adjoint triple on $(L, \leq)$ for each $i \in \{1, \ldots, k\}$.

The idea behind the approach

Let me recall the issue of the talk:

- Both laws, contraposition and double negation, are natural in human reasonings.
- But we have seen that in residuated lattice we must be fairly restrictive to model them.

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- the structure of multi adjoint lattices allows to consider several conjunctions and implications.

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Let me recall the issue of the talk:

- Both laws, contraposition and double negation, are natural in human reasonings.
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We have now that

- the structure of multi adjoint lattices allows to consider several conjunctions and implications.

So, now, in contrast with residuated lattice:

> we do not need to use the same implication to model the contraposition law.

The $\searrow_n$-adjoint triple

Given an implication $\rightarrow$, I propose to build/find another implication $\rightarrow^*$ such that

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The $\searrow_n$-adjoint triple

Given an implication $\rightarrow$, I propose to build/find another implication $\rightarrow^*$ such that

$$(x \rightarrow y) \iff (n(y) \rightarrow^* n(x))$$

for all $x \in L$.

Definition

Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L with an involutive negation n.

The $\searrow_n$-adjoint triple $(\&_n, \searrow_n, \nearrow_n)$ of $(\&, \searrow, \nearrow)$ is given by the following operators:

- $x \nearrow_n y = n(x \,\&\, n(y))$ for all $x, y \in L$
- $x \,\&_n\, y = n(x \nearrow n(y))$ for all $x, y \in L$.
- $x \searrow_n y = n(y) \searrow n(x)$ for all $x, y \in L$

The $\nearrow_n$-adjoint triple

Similarly, we define the $\nearrow^n$-adjoint triple.

Definition

Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L with an involutive negation n.

The $\nearrow^n$-adjoint triple $(\&^n, \searrow^n, \nearrow^n)$ of $(\&, \searrow, \nearrow)$ is given by the following operators:

- $x \nearrow^n y = n(y) \nearrow n(x)$ for all $x, y \in L$
- $x \&^n y = n(y \searrow n(x))$ for all $x, y \in L$
- $x \searrow^n y = n(n(y) \& x)$ for all $x, y \in L$.

Hey! Are they adjoint triples?

The definition above requires a proof to justify the use of the term 'adjoint triple' in it.

Lemma

Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L with an involutive negation n. Then $(\&_n, \searrow_n, \nearrow_n)$ and $(\&^n, \searrow^n, \nearrow^n)$ are adjoint triples as well.

What about doing a reiterative construction?

By composing twice the constructions of adjoint triples, there are, a priori, four possible new adjoint triples, namely:

- The $\searrow_n$-adjoint triple of $(\&_n, \searrow_n, \nearrow_n)$:

$$((\&_n)_n, (\searrow_n)_n, (\nearrow_n)_n)$$

- The $\nearrow^n$-adjoint triple of $(\&_n, \searrow_n, \nearrow_n)$:

$$((\&_n)^n, (\searrow_n)^n, (\nearrow_n)^n)$$

- The $\nearrow^n$-adjoint triple of $(\&^n, \searrow^n, \nearrow^n)$:

$$((\&^n)^n, (\searrow^n)^n, (\nearrow^n)^n)$$

- And the $\searrow_n$-adjoint triple of $(\&^n, \searrow^n, \nearrow^n)$:

$$((\&^n)_n, (\searrow^n)_n, (\nearrow^n)_n)$$

The main result

Theorem

Let $(\&, \searrow, \nearrow)$ be an adjoint triple defined in a lattice L with an involutive negation n. Then the following equalities hold:

1. $x(\&_n)_n y = x(\&^n)^n y = x \,\&\, y.$
2. $x(\&^n)_n y = x(\&_n)^n y = y \,\&\, x.$
3. $x(\searrow^n)^n y = x \searrow y.$
4. $x(\nearrow^n)^n y = x \nearrow y.$
5. $x(\searrow_n)_n y = x \searrow y.$
6. $x(\nearrow_n)_n y = x \nearrow y.$
7. $x(\nearrow^n)_n y = x \searrow_n y.$
8. $x(\searrow^n)_n y = x \nearrow_n y.$
9. $x(\nearrow_n)^n y = x \searrow^n y.$
10. $x(\searrow_n)^n y = x \nearrow^n y.$

A closed framework for the contraposition law.

Given an adjoint triple $(\&, \swarrow, \nwarrow)$ defined on a lattice L with an involutive negation n, the multi-adjoint framework given by:

$$\big(L, \leq, (\&, \swarrow, \nwarrow), (\&_n, \swarrow_n, \nwarrow_n), (\&^n, \swarrow^n, \nwarrow^n)\big)$$

is a closed framework where it is possible to apply the contraposition rule an unlimited (numerable) number of times.

For instance, for every $x, y \in L$ we have:

$$x \nwarrow_n y = n(y)(\nwarrow_n)^n n(x) = n(y) \swarrow^n n(x).$$

That is: it is possible to apply the contraposition rule to the implication $\swarrow^n$ by using the implication $\nwarrow_n$

The case of a residuated lattice.

Before to end the talk, I study the construction above from a residuated pair $(\&, \to)$ and an involutive negation n.

By commutativity of $\&$, the multi-adjoint framework associated with the construction described above is:

$$(L, \leq, (\&, \to), (\&_n, \searrow_n, \nearrow_n))$$

Moreover, the following equalities hold:

- $x \to y = n(y) \searrow_n n(x)$,
- $x \searrow_n y = n(y) \to n(x)$,
- $x \nearrow_n y = n(y) \nearrow_n n(x)$

for all $x, y \in L$.

Conclusions

In this talk I have

- described some drawbacks about constructing a framework based on residuated lattice to hold both:

  the contraposition law and the double negation law.

- recalled the notion of multi-adjoint lattices

- and constructed a simple multi-adjoint framework where we can apply the contraposition rule an unlimited number of times.

Future Work

As future work I plan

- to study properties on the given framework for specific cases:
  - Gödel connectives with negation $n(x) = 1 - x$,
  - product t-norm and implication with negation $n(x) = 1 - x$
- As a long long term work, I would like to develop an algebras (formal logic systems) based on the multi-adjoint frameworks presented here.

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