UNIVERSITAS MALACITANA

ON THE STRUCTURE OF LEAVITT PATH ALGEBRAS AND KUMJIAN-PASK ALGEBRAS

DOCTORAL THESIS

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Gonzalo Aranda Pino, Profesor Titular del Área de Álgebra de la Universidad de Málaga informa: Que ha dirigido la Tesis Doctoral titulada "On the structure of Leavitt path algebras and Kumjian-Pask algebras" ("Sobre la estructura de las álgebras de caminos de Leavitt y de las álgebras de Kumjian-Pask"), realizada por el Licenciado Cristóbal Gil Canto.

Finalizada la investigación que ha llevado a la conclusión de la citada Tesis Doctoral, y de acuerdo con el artículo 21.1 del Real Decreto 1393/2007 de fecha 29 de octubre (BOE de 30 de octubre de 2007), autoriza su presentación por considerar que reúne los requisitos formales y científicos legalmente establecidos para la obtención del título de Doctor en Matemáticas.

Y para que así conste y surta los efectos oportunos, expide y firma el presente informe en Málaga a 16 de noviembre de 2016.

Gonzalo A. Pino

Fdo: Gonzalo Aranda Pino

Pilato le preguntó:

-¿Y qué es la verdad?

Juan 18, 38.

A Cristóbal *in memoriam* y a María,

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Cristóbal Gil Canto.
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Resumen en español
Spanish abstract

A comienzos de la década de los 60 fue W. G. Leavitt quien dio ejemplos de álgebras que no satisfacen la *propiedad (IBN)*, o también conocida como *propiedad del número invariante de la base*. Es sabido que un anillo unitario S se dice que tiene la propiedad del número invariante de la base en el caso en que, para cada par $i, j \in \mathbb{N}$, si los S-módulos por la izquierda S^i y S^j son isomorfos, entonces se tiene necesariamente que $i = j$. Se puede probar que una amplia variedad de anillos tienen la propiedad IBN, por ejemplo, los cuerpos K, el anillo de los enteros $\mathbb{Z}$, el anillo de los polinomios de Laurent $K[x, x^{-1}]$ o los anillos de matrices $M_n(K)$ para $n \in \mathbb{N}$.

W. Leavitt dio respuesta completa y satisfactoria a la siguiente pregunta: ¿existen anillos para los cuales ${}_SS^i \cong {}_SS^j$ para algunos, pero no todos, los pares $i \neq j$? Supongamos que existe tal isomorfismo entre ${}_SS^i \cong {}_SS^j$ para algún par $i \neq j$, entonces adjuntando k copias de S a este isomorfismo se tiene que ${}_SS^{i+k} \cong {}_SS^{j+k}$. Consideremos pues así m el menor entero para el cual ${}_SS^m \cong {}_SS^j$ para algún $j \neq m$; para este m en concreto, sea n el menor entero para el cual se cumple que ${}_SS^m \cong {}_SS^n$ y $n > m$. Llamamos entonces al par (m, n) el *tipo del módulo* de S. Ocurrió pues que W. Leavitt llegó a probar el siguiente resultado:

Teorema ([46, Teorema 8]). *Sean $m, n \in \mathbb{N}$ con $m > n$, y sea K un cuerpo cualquiera. Entonces existe una K-algebra $L_K(m, n)$ que tiene tipo del módulo (m, n).*

Nos referimos a $L_K(m, n)$ (o simplemente $L(m, n)$) como el *álgebra de*

Leavitt de tipo (m, n). En el caso específico en que $m = 1$, una descripción explícita del álgebra $L_K(1, n)$ dada en [46] da lugar a que $L_K(1, n)$ es en realidad la K-álgebra asociativa libre $K\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$ cociente por las siguientes relaciones:

$$y_i x_i = \delta_{ij} 1, \text{ y } \sum_{i=1}^{n} x_i y_i = 1.$$

Además W. Leavitt probó en [47] que las álgebras de la forma $L_K(1, n)$ son *simples*.

Por otro lado, después de una década más tarde, una de las cuestiones más importantes sobre la estructura de las *C^*-álgebras* fue resuelta: Cuntz dio, entre otros aspectos, la descripción explícita de una C^*-álgebra simple [36, Theorem 1.12]. Una C^*-álgebra es simple en caso de que no contenga ideales (biláteros) cerrados y no triviales. Nos estamos refiriendo a las conocidas como *álgebras de Cuntz* $\mathcal{O}_n$. De hecho cuando K es el cuerpo $\mathbb{C}$ de los números complejos, entonces $\mathcal{O}_n$ puede ser vista como la completación, en una apropiada norma, de $L_{\mathbb{C}}(1, n)$. Poco después serían numerosos investigadores en álgebras de operadores quienes se dedicaran a estudiar generalizaciones naturales de las C^*-álgebras de Cuntz $\mathcal{O}_n$ [37].

En la década de los 80 fueron estudiadas varias construcciones de C^*-álgebras correspondientes a *grafos dirigidos*. Después de esto, en [45] y en los siguientes y sucesivos artículos [44] y [21], la idea de construir una C^*-álgebra basándonos en un grafo dirigido se hizo clara. Sin más, para grafos finitos E, las álgebras de Cuntz-Krieger $C^*(E)$ que son simples y aquéllas que son *puramente infinita simples*, fueron identificadas en [44]. Para más información sobre *C^*-álgebras de grafo* el lector puede consultar la referencia [52] de la bibliografía.

No fue hasta comienzos de los años 2000 cuando esta conexión (antes comentada) entre $\mathcal{O}_n$ y $L_{\mathbb{C}}(1, n)$ fue por primera vez puesta de relieve en la literatura; P. Ara, K. Goodearl y E. Pardo comenzaron a estudiar la noción algebraica de anillos puramente infinito simples. Junto con M.A. González Barroso, los tres autores introdujeron en [12] las álgebras de Cuntz-Krieger (en su versión algebraica), motivados por el estudio ya existente del caso de las C^*-álgebras de grafo puramente infinitas simples.

Consideremos S un anillo; el conjunto de las *clases de isomorfía de los S-módulos proyectivos por la izquierda finitamente generados*, denotado por $\mathcal{V}(S)$ junto con la operación binaria $[P] \oplus [Q] = [P \oplus Q]$, es un *monoide* conmutativo. No podríamos seguir adelante sin antes mencionar los siguientes hechos destacables. Es típicamente dificultoso el hecho de dar una descripción explícita de este monoide $\mathcal{V}$; en primer lugar podemos resaltar que de nuevo P. Ara y E. Pardo, junto con M.A. Moreno, entendieron que mucha de la información sobre el $\mathcal{V}$-monoide para las CK álgebras (en su versión algebraica) podría ser vista directamente en términos de relaciones entre vértices y flechas en un grafo E asociado [14]. En segundo lugar se produjo este otro hecho. Simultáneamente en 2004, G. Aranda Pino estaba visitando a G. Abrams en la Universidad de Colorado en Colorado Springs (Estados Unidos); el interés del profesor G. Abrams en las álgebras de Leavitt (especialmente en el caso $L_K(1, n)$) junto con el trabajo que ya había sido desarrollado sobre el teorema de simplicidad para las C^*-álgebras de grafo motivaron que G. Abrams y G. Aranda Pino focalizaran su atención en lo que ellos entonces denominarían el *álgebra de caminos de Leavitt de un grafo* $L_K(E)$. En [4] ambos autores consiguen dar condiciones necesarias y suficientes sobre un grafo E de *filas finitas* (grafos numerables tales que cada vértice emite sólo un número finito de flechas) que implicaran que el álgebra $L_K(E)$ fuera simple. Habían nacido las álgebras de caminos de Leavitt. El resultado probado en [4] es el siguiente:

Teorema ([4, Teorema 3.11]). *Sea E un grafo de filas finitas y sea K un cuerpo cualquiera. El álgebra de caminos de Leavitt $L_K(E)$ es simple si y sólo si los subconjuntos hereditarios y saturados de E^0 son triviales y todo ciclo en E tiene una salida.*

Resumiendo podemos decir que las álgebras de caminos de Leavitt son un caso específico de *álgebras de caminos* asociadas a un grafo E cociente entre unas ciertas relaciones (las conocidas como *relaciones de Cuntz-Krieger*). Incluso en [55], cuya autora es la profesora M. Siles Molina, se prueba que las álgebras de caminos de Leavitt son en realidad *álgebras de cocientes* por la derecha de sus correspondientes álgebras de caminos. Las álgebras de caminos de Leavitt incluyen muchas álgebras muy conocidas como los anillos de matrices $\mathbb{M}_n(K)$ para $n \in \mathbb{N}$, el anillo de polinomios de Laurent $K[x, x^{-1}]$, o las álgebras de Leavitt clásicas $L_K(1, n)$ para $n \geq 2$.

Durante más de una década estas álgebras han atraído el especial interés y atención, no sólo de expertos en teoría de anillos, sino también de analistas trabajando en C^*-álgebras, ya que como hemos mencionado anteriormente, suponen la versión algebraica de las C^*-álgebras de grafo de Cuntz-Krieger. De hecho las teorías en sus versiones algebraica y analítica comparten importantes similitudes, pero también presentan diferencias destacables. La relación entre estas dos clases de álgebras de grafo ha sido mutuamente beneficiosa. En [60] se probó, de hecho, que para cualquier grafo E el álgebra de caminos de Leavitt $L_{\mathbb{C}}(E)$ es isomorfa a una $*$-subálgebra densa de $C^*(E)$.

Aparte del resultado sobre la simplicidad de las álgebras de caminos de Leavitt, un número importante de éstos se centran en traspasar información estructural desde el grafo dirigido E al álgebra de caminos de Leavitt $L_K(E)$ y viceversa, es decir, esquemáticamente:

$$E \text{ tiene una propiedad } \mathcal{P} \Longleftrightarrow L_K(E) \text{ tiene una propiedad } \mathcal{Q}.$$

Por ejemplo, éste es el caso también del análisis sobre las álgebras finito dimensionales de [8], las de *intercambio* en [19] o las álgebras de caminos de Leavitt puramente infinitas simples que describimos en la Sección 1.3. Es interesante observar que el cuerpo subyacente K no juega ningún papel en estos teoremas de caracterización. Sin embargo, cabe mencionar que por ejemplo en el trabajo [20] realizado por los autores G. Aranda Pino, K.M. Rangaswamy y L. Vaš, apareció el primer teorema de caracterización (sobre la **-regularidad o regularidad con involución propia* de un álgebra de caminos de Leavitt) que involucra además una propiedad como anillo sobre el cuerpo K, es decir, esquemáticamente en forma:

$$E \text{ tiene propiedad } \mathcal{P} \text{ y } K \text{ tiene propiedad } \mathcal{P}' \Leftrightarrow L_K(E) \text{ tiene propiedad } \mathcal{Q}.$$

En el área de las álgebras de caminos de Leavitt se han producido un cuantioso número de resultados. Muchísimos de ellos bastante interesantes y destacables; y han sido usados para dar algunas clases de álgebras teniendo ciertas propiedades teóricas de anillos. Además aparecieron la definición extendida a grafos numerables en [6] y a grafos arbitrarios en [41]. En la literatura también cabe destacar que la estructura de los *ideales primos* para álgebras de caminos de Leavitt de grafos arbitrarios es considerada en [53], el *zócalo* es descrito en [18] y el *centro* en [35]; entre otros muchos trabajos. M. Tomforde en [61] generaliza la construcción de las álgebras de caminos de Leavitt reemplazando el cuerpo K por un anillo R conmutativo y con unidad.

Esta tesis está dedicada a estas álgebras. Intentaremos obtener nuevos avances en el conocimiento de la estructura de las álgebras de caminos de Leavitt. Pasamos a describir ahora en más detalle los contenidos de los capítulos y sus secciones.

En el Capítulo 1 comenzaremos introduciendo la noción de álgebra de caminos de Leavitt sobre un anillo R conmutativo con unidad $L_R(E)$. Podríamos decir que el Capítulo 1 es la base de la tesis, donde presentamos algunos de los resultados importantes en la materia que serán necesarios para entender los sucesivos capítulos.

Después de algunas nociones y resultados introductorios en la primera sección, focalizaremos nuestra atención en los llamados *teoremas de unicidad* para álgebras de caminos de Leavitt, es decir, aquellos que imponen condiciones en el grafo E o en la aplicación Φ para asegurar así que una representación $\Phi : L_R(E) \to \mathcal{A}$ sea inyectiva (Sección 1.2). Después de analizar la estructura *graduada* de $L_R(E)$ presentamos, por un lado, el *teorema de unicidad graduado* y, por otro lado, el *teorema de unicidad de Cuntz-Krieger*. Estos resultados fueron originalmente dados en [60] y después generalizados en [61] para el caso de álgebras de caminos de Leavitt sobre un anillo conmutativo con unidad. Podemos decir que estos resultados son ambos versiones para álgebras de caminos de Leavitt de los dos teoremas de unicidad existentes de las C^*-álgebras de grafo: el *teorema de unicidad gauge invariante* y el *teorema* *de unicidad de Cuntz-Krieger*, los cuales dan condiciones sobre qué homomorfismos de $C^*(E)$ son inyectivos. Los teoremas de unicidad son resultados clave en el ámbito de las álgebras de caminos de Leavitt pues son frecuentemente usados para identificar clases de isomorfismos para un álgebra de caminos de Leavitt particular así como para deducir muchos resultados generales sobre la estructura de estas álgebras. Uno de nuestros principales objetivos será generalizar ambos resultados para las álgebras de caminos de Leavitt (Sección 2.5).

Por otra parte estamos especialmente interesados en explorar aquellas álgebras de caminos de Leavitt puramente infinitas simples. Los anillos puramente infinitos fueron introducidos por primera vez en [13]. En la Sección 1.3 consideraremos álgebras de caminos de Leavitt sobre un cuerpo K y veremos condiciones necesarias y suficientes sobre el grafo E de manera que $L_K(E)$ sea puramente infinita simple. Este resultado nos ofrece el análogo algebraico al resultado correspondiente para $C^*(E)$ descrito en [21]. Necesitamos caracterizar aquellas álgebras de caminos de Leavitt puramente infinitas simples para establecer posteriormente en la Sección 1.4 una de las principales herramientas que usaremos en los Capítulos 4 y 5 posteriores; nos estamos refiriendo al *teorema* (en su versión algebraica) *de Kirchberg y Phillips*.

No podemos continuar sin antes destacar lo siguiente. En [22, Teorema 6.2] G. Bergman describió de forma explícita una construcción general que parte de un determinado monoide y da lugar a un álgebra correspondiente que tiene la propiedad de que el monoide de módulos proyectivos finitamente generados para esta álgebra se comporta exactamente como el monoide dado. Esto motiva el llamado *teorema de realización* dado en [14, Theorem 3.5] por los autores P. Ara, M.A. Moreno y E. Pardo: sea E un grafo de filas finitas y K cualquier cuerpo; entonces existe un isomorfismo natural de monoides $\mathcal{V}(L_K(E)) \cong M_E$. Además ocurre que, para cualquier anillo S, el *grupo de Grothendieck* estándar $K_0(S)$ es el grupo universal correspondiente al monoide abeliano $\mathcal{V}(S)$. Sin embargo, si S es puramente infinito simple, entonces $\mathcal{V}(S) \setminus \{[0]\}$ es un grupo, y precisamente $K_0(S)$. De aquí que, teniendo en cuenta estas ideas en mente y empleando la *forma normal de Smith de una matriz* de valores enteros, seremos capaces de calcular el grupo K_0 de un álgebra de caminos de Leavitt puramente infinita simple sobre un cuerpo K (Proposición 1.4.6). Al final la información ofrecida por el K_0 nos da una respuesta natural de tipo clasificatorio en el contexto de las álgebras de caminos de Leavitt; en este sentido presentamos en esta sección el teorema algebraico de Kirchberg-Phillips.

Teorema ([10, Corolario 2.7]). *Supongamos que E y F son grafos finitos para los cuales las álgebras de caminos de Leavitt $L_K(E)$ y $L_K(F)$ son puramente infinitas simples. Supongamos que existe un isomorfismo $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ para el cual $\varphi([L_K(E)]) = [L_K(F)]$; y supongamos también que los dos enteros* $\det(I_{|E^0|} - A_E^t)$ *y* $\det(I_{|F^0|} - A_F^t)$ *tienen el mismo signo, donde $A_{(-)}$ denota la* matriz de adyacencia de un grafo *y $A_{(-)}^t$ su traspuesta. Entonces se tiene que $L_K(E) \cong L_K(F)$ como K-álgebras.*

La demostración de [10, Corolario 2.7] utiliza resultados y profundas ideas relacionados con la teoría de dinámica simbólica. Los nombres derivan de E. Kirchberg y N.C. Phillips, quienes (independientemente en el año 2000) probaron un resultado análogo para las C^*-álgebras de grafo.

En el Capítulo 2 encontramos una vez más otro ejemplo de la estrecha relación sobre la doble corriente algebraica y analítica que versan en este campo: el resultado en su versión analítica fue dado en [50] para las C^*-álgebras $C^*(E)$, y aquí expondremos el análogo algebraico para álgebras de caminos de Leavitt $L_R(E)$. Como idea general identificaremos el *corazón conmutativo* del álgebra de caminos de Leavitt de E con coeficientes en R, el cual es una subálgebra conmutativa maximal del álgebra de caminos de Leavitt. Por ir más lejos, seremos capaces de caracterizar la inyectividad de representaciones que dan una generalización del teorema de unicidad de Cuntz-Krieger, y por otro lado, generalizar y simplificar el resultado acerca de álgebras de caminos de Leavitt sobre cuerpos que son conmutativas. El trabajo desarrollado en este capítulo se corresponde con la referencia [40] que fue parcialmente llevado a cabo durante la visita del autor al Instituto para la Investigación en Ciencias Fundamentales (IPM) en Isfahán (Irán) para colaboración con el profesor A. Nasr-Isfahani.

Este Capítulo 2 se organiza de la siguiente forma. En la Sección 2.1 introducimos alguna información previa necesaria junto con la definición de los *sistemas de Cuntz-Krieger asociados* al grafo E, esto es, las relaciones análogas de las álgebras de caminos de Leavitt para una R-álgebra genérica. En la Sección 2.2 damos una representación particular del álgebra de caminos de Leavitt (Proposición 2.2.8) sobre una R-álgebra específica relacionada con el conjunto de todos los *senderos esencialmente aperiódicos* en el grafo; esto es, el conjunto de todos los caminos infinitos *aperiódicos*, además de aquellos infinitos que son *periódicos* (los que acaban en un ciclo sin salidas) junto con todos los caminos finitos cuyo rango es un *sumidero*.

En la Sección 2.3 introduciremos el corazón conmutativo $M_R(E)$ del álgebra de caminos de Leavitt $L_R(E)$ y probaremos uno de los resultados principales del Capítulo 2: el Teorema 2.3.12. Este teorema afirma que $M_R(E)$ es una subálgebra conmutativa maximal dentro de $L_R(E)$. Para este propósito, previamente probaremos el Teorema 2.3.11, donde mostramos la existencia de una aplicación conocida como la *expectativa condicional algebraica* sobre la subálgebra $M_R(E)$.

Teorema 2.3.12. *Sea E un grafo y R un anillo conmutativo y con unidad. Consideremos $M_R(E) \subseteq L_R(E)$. Entonces se tiene que*

$$M_R(E) = \{x \in L_R(E) : xd = dx \text{ para todo } d \in \Delta(E)\}.$$

Además, $M_R(E)$ es una $*$-subálgebra conmutativa maximal de $L_R(E)$.

Gracias a la existencia de la subálgebra $M_R(E)$, en el Corolario 2.4.4 de la Sección 2.4 generalizaremos el resultado para álgebras de caminos de Leavitt de grafos de filas finitas (sobre cuerpos) que son conmutativas dado originalmente en [17]. Dicho sea de paso, también analizaremos las *álgebras de caminos de Cohn* $C_R(E)$ sobre R que son conmutativas, ya que estas álgebras pueden verse como álgebras de caminos de Leavitt de ciertos grafos apropiados. La condición de Cuntz-Krieger (CK2) impuesta para cada vértice *regular* en la definición del álgebra de caminos de Leavitt puede ser eliminada: de este modo se define el álgebra de caminos de Cohn. La terminología "álgebra de caminos de Cohn" proviene del hecho de que para cada $n \in \mathbb{N}$ el álgebra de caminos de Cohn del grafo consistente en la *rosa de n pétalos* es precisamente el álgebra $U_{1,n}$ descrita e investigada por Cohn en [33]. En la literatura concerniente a las álgebras de caminos de Cohn podríamos resaltar que últimamente G. Abrams and M. Kanuni en su trabajo [9] han probado que estas álgebras satisfacen la propiedad del número invariante de la base; o en la referencia [34] sus autores han analizado el centro de las álgebras de caminos de Cohn que son primas.

Para la Sección 2.5 volveremos a los teoremas de unicidad de las álgebras de caminos de Leavitt. Análogamente a [50, Teorema 3.13] donde se prueba un teorema de unicidad al estilo del teorema de unicidad de Szymański para C^*-álgebras de grafo (véase [59, Teorema 1.2]), el Teorema 2.5.1 en particular dice que una representación de $L_R(E)$ es inyectiva si y sólo si es inyectiva cuando se restringe al corazón conmutativo $M_R(E)$. Este teorema también generaliza [18, Teorema 3.7] dado para álgebras de caminos de Leavitt sobre cuerpos y el teorema de unicidad de Cuntz-Krieger descrito en la Sección 1.2 para grafos en los que todo ciclo tiene una salida, la conocida como *Condición (L)*.

Trasladémonos ahora al Capítulo 3. Aquí estudiaremos los análogos a las álgebras de caminos de Leavitt asociados a *grafos de rango superior*; estas álgebras se denominan *álgebras de Kumjian-Pask*. Concretamente extenderemos los resultados dados en el Capítulo 2 a álgebras de Kumjian-Pask. Sea Λ un grafo de rango superior de filas finitas y que no tiene *fuentes*. Identificaremos una subálgebra conmutativa maximal $\mathcal{M}$ contenida en el álgebra de Kumjian-Pask $\mathrm{KP}_R(\Lambda)$. También probaremos un teorema de unicidad de Cuntz-Krieger generalizado para las álgebras de Kumjian-Pask que afirma que una representación de $\mathrm{KP}_R(\Lambda)$ es inyectiva si y sólo si es inyectiva cuando nos restrigimos a $\mathcal{M}$. En este caso el trabajo desarrollado en este capítulo se corresponde con [29], el cual fue llevado a cabo conjuntamente con L.O. Clark y A. Nasr-Isfahani.

Kumjian y Pask fueron los primeros en introducir la noción de grafo de rango superior o k-grafo Λ (en donde los caminos tienen grado k-dimensional y un 1-grafo se reduce a un grafo dirigido) y la C^*-álgebra asociada $C^*(\Lambda)$ en [43]. Estas C^*-álgebras nos dan un modelo visualizable para las versiones de rango superior de las álgebras de Cuntz-Krieger estudiadas por Robertson y Steger en [54]. El álgebra de Kumjian-Pask $\mathrm{KP}_R(\Lambda)$, definida y estudiada en [16], es el análogo algebraico de $C^*(\Lambda)$. Estas álgebras de Kumjian-Pask satisfacen una propiedad universal basadas en una familia de generadores que cumplen unas relaciones concretas. El estudio de sus ideales básicos y la simplicidad de $\mathrm{KP}_R(\Lambda)$ se lleva a cabo en [16], el zócalo y la semisimplicidad son consideradas en [24] y el centro es analizado en [23]. Las álgebras de Kumjian-Pask para grafos más generales son consideradas en [28] y [32].

Análogamente a la teoría de las álgebras de 1-grafo, uno de los temas centrales para las álgebras de k-grafos es determinar cuando un homomorfismo dado de $\mathrm{KP}_R(\Lambda)$ (o $C^*(\Lambda)$ en el caso analítico) es inyectivo; estamos tratando una vez más los teoremas de unicidad. En [16] un teorema de unicidad graduado y un teorema de unicidad de Cuntz-Krieger son probados para $\mathrm{KP}_R(\Lambda)$. Ambos requieren algunas condiciones: el primero considera solamente homomorfismos $\mathbb{Z}^k$-graduados, mientras que el segundo requiere de la hipótesis extra sobre el grafo, esto es, que Λ sea 'aperiódico'. En [25], se prueba una versión más general de los teoremas de unicidad de Cuntz-Krieger para el ámbito C^*-algebraico que no tiene hipótesis adicionales sobre el homomorfismo o el grafo. Aquí trasladaremos a álgebras de Kumjian-Pask el resultado analítico dado en [25]: una representación de $\mathrm{KP}_R(\Lambda)$ es inyectiva si y sólo si es inyectiva si se restringe a una subálgebra distinguida, conocida como *subálgebra ciclina*.

Al mismo tiempo probaremos una versión más general de los principales resultados dados en el Capítulo 2 en el contexto de los 1-grafos [40]. Recordemos que en el Capítulo 2 se prueba un teorema de unicidad para álgebras de caminos de Leavitt que establece que la inyectividad de una representación solamente depende de su inyectividad sobre cierta subálgebra conmutativa (Teorema 2.3.12). Tengamos en cuenta que estos resultados no son corolarios de los que obtendremos en este capítulo, ya que en el Capítulo 2 son considerados grafos arbitrarios mientras que aquí estamos suponiendo que los k-grafos son de filas finitas y sin fuentes.

De este modo el Capítulo 3 se organiza como sigue. Comenzamos con una sección donde damos todos los preliminares necesarios, incluyendo la definición de $\mathrm{KP}_R(\Lambda)$ y algunas propiedades básicas. En la Sección 3.2 establecemos algunas propiedades de la *subálgebra diagonal*. Para realizar este propósito usaremos una herramienta extremadamente útil: la existencia de un isomorfismo entre la subálgebra diagonal de un álgebra de Kumjian-Pask y la correspondiente diagonal de un *álgebra de Steinberg* ([32, Proposition 5.4]). Las álgebras de Steinberg, introducidas en [58], son álgebras asociadas a *grupoides* "ample": un grupoide es una generalización de un grupo en el cual la operación binaria es sólo parcialmente definida. En [27], se demuestra que cada álgebra de caminos de Leavitt $L_K(E)$ es isomorfa a un álgebra de Steinberg que se define como sigue: primero, uno construye un grupoide topológico G_E a partir del grafo dirigido E (que es un grupoide equipado con una topología en la que la composición y la inversión son continuas); luego, el álgebra de Steinberg asociada a E será el álgebra de convolución de funciones localmente constantes de soporte compacto de G_E en el cuerpo K. El modelo de álgebras de Steinberg ha sido usado recientemente por ejemplo en [30] y en [31] para estudiar propiedades del álgebra de caminos de Leavitt. Las álgebras de Steinberg también incluyen a las álgebras de Kumjian-Pask.

En la Sección 3.3 estudiamos la subálgebra ciclina. De manera ánaloga a la definición dada en [25], la subálgebra ciclina $\mathcal{M}$ está generada por elementos de la forma $s_\alpha s_{\beta^*}$ donde $(\alpha, \beta) \in \Lambda \times \Lambda$ es un *par ciclino* (Proposición 3.3.1). En el Teorema 3.3.6, probaremos que $\mathcal{M}$ es una subálgebra conmutativa maximal de $\mathrm{KP}_R(\Lambda)$. Se cumplirá además que en el caso de los 1-grafos, esta subálgebra ciclina $\mathcal{M}$ coincidirá con el corazón conmutativo $M_R(E)$.

Finalmente en la Sección 3.4 daremos uno de los principales resultados del capítulo: el Teorema 3.4.4. Previamente necesitaremos identificar un determinado subconjunto $\mathfrak{T}_\Lambda$ de caminos infinitos de Λ para poder probarlo.

Teorema 3.4.4. *Sea Λ un k-grafo de filas finitas y sin fuentes, R un anillo conmutativo y con 1 y $\mathcal{M}$ la subálgebra ciclina de* $\mathrm{KP}_R(\Lambda)$. *Si* $\Phi : \mathrm{KP}_R(\Lambda) \to A$ *es un homomorfismo de anillos, entonces Φ es inyectiva si y sólo si $\Phi|_{\mathcal{M}}$ es inyectiva.*

Para finalizar este trabajo doctoral pasamos a los dos últimos capítulos que están dedicados al estudio de las álgebras de caminos de Leavitt (sobre un cuerpo K) de unos grafos específicos: en concreto se trata del *grafo generalizado de Cayley* de la forma C_n^j donde $n \in \mathbb{N}$ y $0 \leq j \leq n-1$ correspondiente al grupo cíclico $\mathbb{Z}/n\mathbb{Z}$ con respecto al subconjunto $\{1, j\}$. Las álgebras de caminos de Leavitt de grafos de Cayley se iniciaron en [11], donde se daba una descripción de las álgebras de caminos de Leavitt en la forma $L_K(C_n^{n-1})$. Así se demostró en [11, Teorema 8] que existen exactamente cuatro clases de isomorfismos representadas en la colección $\{L_K(C_n^{n-1}) \mid n \in \mathbb{N}\}$. Como segundo trabajo sobre este tema encontramos la referencia [7] llevada a cabo por G. Abrams y G. Aranda Pino. En [7] en su sección segunda fue donde originalmente ya se calcularon los dos importantes (e íntimamente relacionados) enteros $|K_0(L_K(C_n^j))|$ y $\det(I - A^t_{C_n^j})$. En especial se probó que para un j fijado, los enteros $|K_0(L_K(C_n^j))|$ son (salvo consideraciones del signo) las entradas de la *j-ésima sucesión de Haselgrove* ([7, Definición 2.2]); sucesión investigada por Haselgrove en [42] con miras a establecer una conexión entre un método de resolución para el último teorema de Fermat (en su tiempo, por supuesto, última conjectura de Fermat) y ciertos enteros que comparten propiedades con los conocidos como *números de Mersenne*. El papel desempeñado por la sucesión de Haselgrove será fundamental en este estudio.

También en [7] están completamente descritas (salvo isomorfismo) la colección $\{L_K(C_n^0) \mid n \in \mathbb{N}\}$ (hay solamente una de estas tales álgebras), la colección $\{L_K(C_n^1) \mid n \in \mathbb{N}\}$ (existen infinitas de estas álgebras, cada una de ellas expresadas en términos de matrices sobre las álgebras de Leavitt estándar $L_K(1, m)$), y en [7, Proposición 4.14] se expone la colección $\{L_K(C_n^2) \mid n \in \mathbb{N}\}$:

los grupos $K_0(L_K(C_n^2))$ son explícitamente descritos en términos de los enteros que aparecen en la segunda sucesión de Haselgrove, junto con aquellos enteros que aparecen en la *sucesión clásica de Fibonacci*. Las descripciones de todas estas álgebras se siguen gracias a la aplicación inmediata de nuestra principal herramienta en este campo: el teorema algebraico de Kirchberg-Phillips (descrito en la Sección 1.4).

Comenzamos en el Capítulo 4 dando toda la información previa necesaria sobre el tema. En la Sección 4.1 introduciremos los grafos de Cayley C_n^j y sus correspondientes *monoides de grafo*. En la segunda sección de este capítulo volveremos al caso para el grafo C_n^2 originalmente estudiado en [7, Section 4]. En estas circunstacias proponemos otra forma de hallar el grupo de Grothendieck del álgebra de caminos de Leavitt para el grafo C_n^2 de un modo totalmente distinto al empleado en el trabajo llevado a cabo en [7]. Seremos capaces de reducir el cálculo de la forma normal de Smith de la matriz $I_n - A^t_{C_n^2}$ (que supone ser la clave fundamental a la hora de hallar el grupo K_0) al caso de calcular la forma normal de Smith de una matriz más pequeña (de tamaño 2×2), que está íntimamente relacionada con la sucesión de Fibonacci: nos estamos refiriendo a la matriz $(M_2^n)^t - I_2$, donde M_2 es la *matriz de acompañamiento* del polinomio característico asociado a la sucesión de Fibonacci. No quisiera el autor dejar de pasar por alto que esta nueva aproximación empleada en usar la forma normal de Smith de la matriz $(M_2^n)^t - I_2$ a la hora de calcular el grupo K_0 del álgebra de caminos de Leavitt $L_K(C_n^2)$, vino tras un muy fructífero debate con M. Iovanov conjuntamente con G. Abrams, durante la celebración del "33 Ohio State-Denison" congreso matemático de Columbus, Ohio (Estados Unidos).

Mientras tanto, en el Capítulo 5 continuamos con la investigación de las álgebras de caminos de Leavitt asociadas a grafos de Cayley concretamente ya en el grafo específico C_n^3. Para este caso particular procedemos de manera similar a la nueva aproximación dada en el capítulo previo para el caso de las álgebras de caminos de Leavitt $L_K(C_n^2)$: de nuevo reduciremos el cómputo de la forma normal de Smith a la de la matriz $(M_3^n)^t - I_3$ donde M_3 es la matriz de acompañamiento del polinomio característico asociado a la conocida como *sucesión de las vacas de Narayana* (Sección 5.1). El nombre de esta sucesión, denotada de ahora en adelante por G, es empleado en la Enciclopedia Online de Sucesiones Enteras [51, Sucesión A000930] y se define imponiendo las condiciones iniciales $G(1) = 1$, $G(2) = 1$, $G(3) = 1$ y

$$G(n) = G(n-1) + G(n-3) \text{ para todo } n \geq 4.$$

Narayana fue un destacado matemático indio perteneciente al siglo XIV; él estaba interesado en sumatorios de series aritméticas y cuadrados mágicos, y aplicó sus técnicas al problema de una manada de vacas y novillos (al igual que haría Fibonacci con sus conejos). Estos números jugarán un papel muy importante en el desarrollo de este tema.

En las dos siguientes secciones trabajaremos en el cálculo de los *divisores determinantales* según aparecen en la definición de la forma normal de Smith de la matriz $(M_3^n)^t - I_3$; para llevar a cabo dicho proceso nos veremos involucrados en un análisis teórico-númerico que nos llevará a establecer ciertas relaciones quizás sorprendentes y aparentemente no triviales entre la tercera sucesión Haselgrove H_3 y la sucesión de las vacas de Narayana G. Para este propósito se ha colaborado estrechamente con S. Erickson (dedicado al campo de teoría de números) junto con G. Abrams. Además para trabajar con estos números se ha empleado el software *Magma*. En concreto con un poco de esfuerzo, se prueba el Teorema 5.3.6 donde viene dada una fórmula para $H_3(n)$ ($n \in \mathbb{N}$) en términos de los números de Narayana $G(n-1)$, $G(n-2)-1$ y $G(n-3)$; lo que da lugar al principal resultado de la Sección 5.3.

Finalmente en la Sección 5.4 expondremos el resultado principal de este capítulo. Explícitamente describiremos el grupo K_0 para las álgebras de caminos de Leavitt de grafos de la forma C_n^3 en términos de los enteros que aparecen en la tercera sucesión de Haselgrove, junto con los enteros procedentes de la sucesión de las vacas de Narayana. Además extraeremos información adicional sobre la estructura de estos grupos de Grothendieck mediante la aplicación inmediata del resultado tipo Kirchberg-Phillips para indentificar las álgebras $L_K(C_n^3)$ como las álgebras de caminos de Leavitt para grafos teniendo a lo sumo cuatro vértices.

Teorema 5.4.1. *Sea* $n \in \mathbb{N}$. *Entonces*

$$K_0(L_K(C_n^3)) \cong \mathbb{Z}_{d_3(n)} \times \mathbb{Z}_{\frac{d_3'(n)}{d_3(n)}} \times \mathbb{Z}_{\frac{H_3(n)}{d_3'(n)}},$$

donde

$$\begin{aligned} d_3(n) &= \gcd\{G(n-1), G(n-2)-1, G(n-3)\} \text{ y},\\ d_3'(n) &= \gcd\{G(n-1)G(n-3) - (G(n-2)-1)^2,\\ &\qquad G(n)G(n-3) - G(n-1)(G(n-2)-1),\\ &\qquad G(n-1)^2 - G(n)(G(n-2)-1)\}. \end{aligned}$$

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Para finalizar este trabajo doctoral, concluiremos con un último apartado denominado “Further Work” donde presentamos algunos de las cuestiones actuales no resueltas así como el trabajo futuro que podríamos llevar a cabo en relación a los temas descritos en esta tesis doctoral.

Introduction

In the early 1960's W. G. Leavitt gave examples of algebras not satisfying the *Invariant Basis Number (IBN)* property. It is known that a unital ring S has the Invariant Basis Number (IBN) property in case, for each pair $i, j \in \mathbb{N}$, if the left S-modules S^i and S^j are isomorphic, then $i = j$. A wide class of rings can be shown to have the IBN property, for instance, fields K, the ring of integers $\mathbb{Z}$, the Laurent polynomial ring $K[x, x^{-1}]$ or the matrix rings $M_n(K)$ for $n \in \mathbb{N}$.

W. Leavitt completely answered the following question: do there exist rings for which ${}_SS^i \cong {}_SS^j$ for some, but not all, pairs $i \neq j$? Let us suppose that it exists an isomorphism between ${}_SS^i \cong {}_SS^j$ for some pair $i \neq j$, then by appending k copies of S to this isomorphism we have that ${}_SS^{i+k} \cong {}_SS^{j+k}$. Therefore let us consider m the least integer for which ${}_SS^m \cong {}_SS^j$ for some $j \neq m$; for this m, let n denote the least integer for which ${}_SS^m \cong {}_SS^n$ and $n > m$. We call the pair (m, n) the *module type* of S. W. Leavitt proved the following result:

Theorem ([46, Theorem 8]). *Let $m, n \in \mathbb{N}$ with $m > n$, and let K be any field. Then there exists a K-algebra $L_K(m, n)$ having module type (m, n).*

We refer to $L_K(m, n)$ (or simply $L(m, n)$) as the *Leavitt algebra of type* (m, n). In the specific case when $m = 1$, the explicit description of the algebra $L_K(1, n)$ given in [46] yields that $L_K(1, n)$ is the free associative K-algebra $K\langle x_1, \ldots, x_n, y_1, \ldots, y_n\rangle$ modulo the following relations:

$$y_i x_i = \delta_{ij} 1, \text{ and } \sum_{i=1}^{n} x_i y_i = 1.$$

Furthermore, W. Leavitt showed in [47] that the algebras of the form $L_K(1, n)$ are *simple*.

On the other hand, more than a decade later, one of the most important questions about the structure of *C^*-algebras* was resolved: Cuntz gave, among other things, an explicit characterization of a simple C^*-algebra [36, Theorem 1.12]. (A C^*-algebra is simple in case it contains no nontrivial closed two-sided ideals.) We are referring to the so-called *Cuntz algebras* $\mathcal{O}_n$. In fact when K is the field $\mathbb{C}$ of complex numbers, then $\mathcal{O}_n$ can be viewed as the completion, in an appropriate norm, of $L_{\mathbb{C}}(1, n)$. Soon after a number of researchers in operator algebras investigated natural generalizations of the Cuntz C^*-algebras $\mathcal{O}_n$ [37].

In the 1980's various constructions of C^*-algebras corresponding to directed graphs were studied. After that, in [45] and the subsequent followup articles [44] and [21], the idea of constructing a C^*-algebra based on a directed graph became clear. For finite graphs E, the Cuntz-Krieger algebras $C^*(E)$ which are simple and those which are *purely infinite simple*, were identified in [44]. For additional information about graph C^*-algebras the reader can see [52].

It was not until the early 2000's when this connection (before mentioned) between $\mathcal{O}_n$ and $L_{\mathbb{C}}(1, n)$ was first noted in the literature; P. Ara, K. Goodearl and E. Pardo started to study the algebraic notion of purely infinite simple rings [13]. Together with M.A. González Barroso, the three authors introduced in [12] the *algebraic Cuntz-Krieger (CK) algebras*, motivated by the case of purely infinite simple graph C^*-algebras.

For a ring S, the *set of isomorphism classes of finitely generated projective left S-modules*, denoted by $\mathcal{V}(S)$, with the binary operation $[P] \oplus [Q] = [P \oplus Q]$, is easily seen to be a commutative *monoid*. We should mention the following remarkable facts. It is typically hard to give an explicit description of $\mathcal{V}(S)$; but again P. Ara and E. Pardo, together with M.A. Moreno, understood that much of the information about the $\mathcal{V}$-monoid of the algebraic CK algebras could be seen directly in terms of relations between vertices and edges in an associated graph E [14]. Simultaneously in 2004, G. Aranda Pino was visiting G. Abrams in the University of Colorado at Colorado Springs (USA); G. Abrams's interest in Leavitt's algebras (specifically the case $L_K(1, n)$) jointly with the work developed about the simplicity theorem of graph C^*-algebras motivated that G. Abrams and G. Aranda Pino focused their attention on something which they would later denominate the *Leavitt path algebra of a graph* $L_K(E)$. In [4] both authors achieve to give necessary and sufficient conditions on the *row-finite* graph E (countable graphs such that every vertex emits only a finite number of edges) which imply that the algebra $L_K(E)$ is simple. Leavitt path algebras had been born. The result proved in [4] is the following:

Theorem ([4, Theorem 3.11]). *Let E be a row-finite graph and let K be any field. Then $L_K(E)$ is simple if and only if the only hereditary saturated subsets of E^0 are trivial and every cycle in E has an exit.*

In conclusion, Leavitt path algebras are a specific type of *path algebras* associated to a graph E modulo some relations (the so-called *Cuntz-Krieger relations*). Even in [55] by M. Siles Molina, Leavitt path algebras are shown to be *algebras of right quotients* of their corresponding path algebras. Leavitt path algebras include many well-known algebras such as matrix rings $\mathbb{M}_n(K)$ for $n \in \mathbb{N}$, the Laurent polynomial ring $K[x, x^{-1}]$, or the classical Leavitt algebras $L_K(1, n)$ for $n \geq 2$.

During more than a decade these algebras have attracted significant interest and attention, not only from ring theorists, but from analysts working in C^*-algebras as well. As we have mentioned before, they are also the algebraic version of Cuntz-Krieger graph C^*-algebras. In fact the algebraic and analytic theories share important similarities, but also present some remarkable differences. The relation between these two classes of graph algebras has been mutually beneficial. In [60] it was proved that, in fact, for any graph E, the Leavitt path algebra $L_{\mathbb{C}}(E)$ is isomorphic to a dense $*$-subalgebra of $C^*(E)$.

Apart from the result about the simplicity of Leavitt path algebras, an important number of them focus on passing structural information from the directed graph E to the Leavitt path algebra $L_K(E)$ and viceversa, that is,

$$E \text{ has graph-theoretic property } \mathcal{P} \Leftrightarrow L_K(E) \text{ has ring-theoretic property } \mathcal{Q}.$$

For instance, it is the case of the analysis of the *finite dimensional* [8], *exchange* [19] or purely infinite simple Leavitt path algebras that we describe in Section 1.3. It is interesting that the underlying field K does not play any role in these characterization theorems. However, we should mention that for example in the reference [20] by the authors G. Aranda Pino, K.M. Rangaswamy and L. Vaš, it appeared the first characterization theorem (about **-regularity or regularity with proper involution* of a Leavitt path algebra) that involves a ring-theoretic property of the field K as well, that is, in the form:

$$E \text{ has property } \mathcal{P} \text{ and } K \text{ has property } \mathcal{P}' \Leftrightarrow L_K(E) \text{ has property } \mathcal{Q}.$$

An important number of results have been produced in the subject of Leavitt path algebras. An important number of them are quite interesting and remarkable and they have been used to give classes of algebras having certain ring-theoretic properties. In addition, the extended definition to countable graphs in [6] and to arbitrary graphs in [41] appeared. In the literature we also mention that the *prime ideal structure* for Leavitt path algebras of arbitrary graphs is considered in [53], the *socle* is described in [18] and the *center* in [35]; among many others works. M. Tomforde in [61] generalizes the construction of Leavitt path algebras by replacing the field K with a commutative unital ring R.

This thesis is devoted to these algebras. We try to obtain new breakthroughs in the knowledge of the structure of Leavitt path algebras. We describe now in more detail the contents of the chapters and their sections.

In Chapter 1 we begin by introducing the notion of a Leavitt path algebra over a commutative ring R with unit $L_R(E)$. We could say that Chapter 1 is the base of the thesis, where we present some of the important results in the subject which will be necessary in order to understand the subsequent chapters.

After some preparatory notions and results in the first section, we focus our attention on the so-called *uniqueness theorems* for Leavitt path algebras, i.e., those which set conditions on the graph E or on the map Φ in order to ensure that a representation $\Phi : L_R(E) \to \mathcal{A}$ is injective (Section 1.2). After analyzing the *graded* structure of $L_R(E)$ we present, on the one hand, the *graded uniqueness theorem* and, on the other hand, the *Cuntz-Krieger uniqueness theorem*. These results were originally given in [60] and after generalized in [61] for the case of Leavitt path algebras over a commutative ring with unit. These theorems are both Leavitt path algebras versions of the two uniqueness theorems of graph C^*-algebras: the *gauge-invariant uniqueness theorem* and the *Cuntz-Krieger uniqueness theorem* which give conditions under which homomorphisms on $C^*(E)$ are injective. Uniqueness theorems are fundamental results in the subject of Leavitt path algebras: they are used extensively in identifying the isomorphism class of a particular Leavitt path algebra as well as deducing many general results about the structure of them. One of our main goal will be to generalize both results for Leavitt path algebras (Section 2.5).

On the other hand we are specially interested in exploring those purely infinite simple Leavitt path algebras. *Purely infinite* rings were first introduced in [13]. In Section 1.3 we will consider Leavitt path algebras over a field K and we will show necessary and sufficient conditions on a graph E so that $L_K(E)$ is purely infinite simple. This result provides the algebraic analog to the corresponding result for $C^*(E)$ in [21]. We need to characterize purely infinite simple Leavitt path algebras in order to establish in Section 1.4 one of the main tool needed for Chapters 4 and 5; we are talking about the *algebraic Kirchberg-Phillips theorem.*

In [22, Theorem 6.2] G. Bergman described an explicit general construction which starts with any appropriate monoid and produces a corresponding algebra which has the property that the monoid of finitely generated projective modules for this algebra behaves just like the given monoid. This motivates the so-called *realization theorem* given in [14, Theorem 3.5] by P.

Ara, M.A. Moreno and E. Pardo: let E be a row-finite graph and K any field; then there is a natural monoid isomorphism $\mathcal{V}(L_K(E)) \cong M_E$. Besides it happens that, for any ring S, the standard *Grothendieck group* $K_0(S)$ is the universal group corresponding to the abelian monoid $\mathcal{V}(S)$. However if S is purely infinite simple, then $\mathcal{V}(S) \setminus \{[0]\}$ is a group, precisely $K_0(S)$. Therefore, with these ideas in mind and by using the *Smith normal form* of an integer-valued matrix, we will be able to compute the group K_0 of a purely infinite simple Leavitt path algebra over a field K (Proposition 1.4.6). In the end the information provided by the K_0 gives us a natural classification-type answer in the context of Leavitt path algebras; in this sense we present in this section the algebraic Kirchberg-Phillips theorem:

Theorem ([10, Corollary 2.7]). *Suppose E and F are finite graphs for which the Leavitt path algebras $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose that there is an isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$; and suppose also that the two integers $\det(I_{|E^0|} - A_E^t)$ and $\det(I_{|F^0|} - A_F^t)$ have the same sign where $A_{(-)}$ denotes the incidence matrix of a graph and $A_{(-)}^t$ its transpose. Then $L_K(E) \cong L_K(F)$ as K-algebras.*

The proof of this [10, Corollary 2.7] utilizes deep results and ideas in the theory of symbolic dynamics. The names derive from E. Kirchberg and N.C. Phillips, who (independently in 2000) proved an analogous result for graph C^*-algebras.

In Chapter 2 we find yet another example of the relation between the algebraic and the analytic theories in the subject: the analytic result was given in [50] for the C^*-algebras $C^*(E)$, and we give here the algebraic analogue for the Leavitt path algebras $L_R(E)$. As a general idea we identify the *commutative core* of the Leavitt path algebra of E with coefficients in R which is a maximal commutative subalgebra of the Leavitt path algebra. Furthermore, we are able to characterize injectivity of representations which gives a generalization of the Cuntz-Krieger uniqueness theorem, and on the other hand, to generalize and simplify the result about commutative Leavitt path algebras over fields. The work developed in this chapter corresponds to [40] which was partly carried out during the visit of the author to the Institute for Research in Fundamental Sciences (IPM) in Isfahan (Iran), to collaborate with A. Nasr-Isfahani.

This Chapter 2 is organized as follows. In Section 2.1 we introduce some background information together with the definition of *Cuntz-Krieger E-systems*, the analog relations of Leavitt path algebras in a general R-algebra. In Section 2.2 we give a particular representation of the Leavitt path algebra (Proposition 2.2.8) on an specific R-algebra related to the set of all *essentially aperiodic trails* in the graph; that is, the set of all infinite *aperiodic* paths, as well as those infinite that are *periodic* (those that end in a cycle without exits) together with all finite paths whose range is a *sink*.

In Section 2.3 we introduce the commutative core $M_R(E)$ of the Leavitt path algebra $L_R(E)$ and we prove one of the main result of Chapter 2: Theorem 2.3.12. This theorem says that $M_R(E)$ is a maximal commutative subalgebra inside $L_R(E)$. For this purpose we previously show Theorem 2.3.11, where we prove the existence of an *algebraic conditional expectation* onto the subalgebra $M_R(E)$.

Theorem 2.3.12. *Let E be a graph and R a commutative ring with unit. Consider $M_R(E) \subseteq L_R(E)$. Then*

$$M_R(E) = \{x \in L_R(E) : xd = dx \text{ for every } d \in \Delta(E)\}.$$

Furthermore, $M_R(E)$ is a maximal commutative $*$-subalgebra of $L_R(E)$.

Thanks to the existence of $M_R(E)$, in Corollary 2.4.4 of Section 2.4 we generalize the result for commutative Leavitt path algebras of row-finite graphs over fields given in [17]. As a by-product, we also analyze the *Cohn path algebras* $C_R(E)$ over R which are commutative since these algebras can be realized as Leavitt path algebras of some appropriate graphs. The Cuntz-Krieger (CK2) condition imposed at any *regular* vertex in the definition of a Leavitt path algebra can be eliminated: in such a manner a Cohn path algebra is defined. The terminology “Cohn path algebra” comes from the fact that for each $n \in \mathbb{N}$ the Cohn path algebra of the rose with n petals graph is precisely the algebra $U_{1,n}$ described and investigated by Cohn in [33]. In the literature about Cohn path algebras we could remark that recently G. Abrams and M. Kanuni in [9] have proved that these algebras have the Invariant Basis Number property; or in [34] their authors have analyzed the center of prime Cohn path algebras.

For Section 2.5 we come back to the uniqueness theorems for Leavitt path algebras. Analogously to [50, Theorem 3.13] where it is proved a uniqueness theorem in the spirit of *Szymański's uniqueness theorem* for graph C^*-algebras (see [59, Theorem 1.2]), Theorem 2.5.1 says in particular that a representation of $L_R(E)$ is injective if and only if it is injective on the commutative core $M_R(E)$. This theorem also generalizes [18, Theorem 3.7] for fields and the Cuntz-Krieger uniqueness theorem given in Section 1.2 for graphs in which every cycle has an exit, the so-called *Condition (L)*.

Let us move now to Chapter 3. Here we study analogues of Leavitt path algebras associated to higher-rank graphs; these algebras are called *Kumjian-Pask algebras*. Concretely we extend the results given in Chapter 2 to Kumjian-Pask algebras. Let Λ be a row-finite *higher-rank graph* with no *sources*. We identify a maximal commutative subalgebra $\mathcal{M}$ inside the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$. We also prove a generalized Cuntz-Krieger uniqueness theorem for Kumjian-Pask algebras which says that a representation of $\mathrm{KP}_R(\Lambda)$ is injective if and only if it is injective on $\mathcal{M}$. In this case the work developed in this chapter corresponds to [29] which was done jointly with L.O. Clark and A. Nasr-Isfahani.

Kumjian and Pask first introduced the notion of a higher-rank graph or k-graph Λ (in which paths have a k-dimensional degree and a 1-graph reduces to a directed graph) and the associated C^*-algebras $C^*(\Lambda)$ in [43]. These C^*-algebras provide a visualisable model for higher-rank versions of the Cuntz-Krieger algebras studied by Robertson and Steger in [54]. The Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$, defined and studied in [16], is an algebraic version of $C^*(\Lambda)$. Kumjian-Pask algebras have a universal property based on a family of generators satisfying suitable relations. The study of basic ideals and simplicity of $\mathrm{KP}_R(\Lambda)$ is done in [16], the socle and semisimplicity are considered in [24] and the center is analysed in [23]. Kumjian-Pask algebras for more general graphs are considered in [28] and [32].

Analogously to the 1-graph algebra subject, a central topic in k-graphs algebras is to determine when a given homomorphism from $\mathrm{KP}_R(\Lambda)$ (or $C^*(\Lambda)$ in the analytic case) is injective; this is the content once again of the uniqueness theorems. In [16] a graded-uniqueness theorem and a Cuntz-Krieger uniqueness theorem are proved for $\mathrm{KP}_R(\Lambda)$. Both require some conditions: the first one considers only $\mathbb{Z}^k$-graded homomorphisms, while the second requires some extra hypothesis on the graph, that is, Λ is 'aperiodic' . In [25], a more general version of the Cuntz-Krieger uniqueness theorem is proved in the C^*-algebraic setting that has no additional hypotheses on the homomorphism or the graph. Here we translate to Kumjian-Pask algebras the analytic result given in [25]: a representation of $\mathrm{KP}_R(\Lambda)$ is injective if and only if it is injective on a distinguished subalgebra, called the *cycline subalgebra*.

At the same time we prove a more general version of the main results of Chapter 2 in the context of 1-graphs [40]. In Chapter 2 we prove a uniqueness theorem for Leavitt path algebras which establishes that the injectivity of a representation depends only on its injectivity on a certain commutative subalgebra (Theorem 2.3.12). Note that these results are not corollaries of the ones we will obtain in Chapter 3, since in Chapter 2 arbitrary graphs are considered and here we suppose that k-graphs are row-finite with no sources.

So then Chapter 3 is organized as follows. We begin with a section where we give the background material, including the definition of $\mathrm{KP}_R(\Lambda)$ and some basic properties. In Section 3.2 we establish some properties of the *diagonal subalgebra*. To this aim we will use an extremely useful tool: the existence of an isomorphism between the diagonal subalgebra of a Kumjian-Pask algebra and the corresponding diagonal of a *Steinberg algebra* ([32, Proposition 5.4]). Steinberg algebras, introduced in [58], are algebras associated to "ample" *groupoids*: a groupoid is a generalization of a group in which the binary operation is only partially defined. In [27], it is shown that each Leavitt path algebra $L_K(E)$ is isomorphic to a Steinberg algebra defined as follows: first, one constructs a topological groupoid G_E from the directed graph E (which is a groupoid equipped with a topology such that composition and inversion are continuous); next, the Steinberg algebra associated to E is the convolution algebra of locally constant, compactly supported functions from G_E into the field K. The Steinberg algebra model has recently been used for example in [30] and in [31] for studying properties of a Leavitt path algebra. Steinberg algebras include Kumjian-Pask algebras too.

In Section 3.3 we study the cycline subalgebra. Analogous to the definition given in [25], the cycline subalgebra $\mathcal{M}$ is generated by elements of the form $s_\alpha s_{\beta^*}$ where $(\alpha, \beta) \in \Lambda \times \Lambda$ is a *cycline pair* (Proposition 3.3.1). In Theorem 3.3.6, we prove that $\mathcal{M}$ is a maximal commutative subalgebra inside $\mathrm{KP}_R(\Lambda)$. It will be satisfied as well that for the 1-graph case, this cycline subalgebra $\mathcal{M}$ coincides with the commutative core $M_R(E)$.

Finally, in Section 3.4, we give our main result Theorem 3.4.4. Previously we need to identify a suitable subset of infinite paths $\mathfrak{T}_\Lambda$ of Λ in order to prove it.

Theorem 3.4.4. *Let Λ be a row-finite k-graph with no sources, R be a commutative ring with 1 and $\mathcal{M}$ be the cycline subalgebra of $\mathrm{KP}_R(\Lambda)$. If $\Phi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism, then Φ is injective if and only if $\Phi|_\mathcal{M}$ is injective.*

The last two chapters of the thesis are dedicated to the study of the Leavitt path algebras over a field K of some specific graphs: for $n \in \mathbb{N}$ and $0 \leq j \leq n-1$ the generalized *Cayley graphs* of the form C_n^j corresponding to the cyclic group $\mathbb{Z}/n\mathbb{Z}$ with respect to the subset $\{1, j\}$. Leavitt path algebras of Cayley graphs were initially studied in [11], where a description is given of the Leavitt path algebras of the form $L_K(C_n^{n-1})$. It is shown in [11, Theorem 8] that there are exactly four isomorphism classes represented by the collection $\{L_K(C_n^{n-1}) \mid n \in \mathbb{N}\}$. As a second step, [7, Section 2] contains computations of two important (and closely related) integers: $|K_0(L_K(C_n^j))|$ and $\det(I - A^t_{C_n^j})$, where $A_{(-)}$ denotes the incidence matrix of a graph and $A^t_{(-)}$ its transpose.

Specifically, it is shown that, for fixed j, the integers $|K_0(L_K(C_n^j))|$ are (up to a sign) the entries in the "j^{th} Haselgrove sequence" ([7, Definition 2.2]), a sequence investigated by Haselgrove in [42] with an eye towards establishing a connection between a resolution of *Fermat's last theorem* (at the time, of course, Fermat's last conjecture) and some integers which share properties of the *Mersenne numbers*.

Also in [7], it is completely described up to isomorphism the collection $\{L_K(C_n^0) \mid n \in \mathbb{N}\}$ (there is only one such algebra), the collection $\{L_K(C_n^1) \mid n \in \mathbb{N}\}$ (there are infinitely many such algebras, each of them subsequently realized in terms of matrices over the standard Leavitt algebras $L_K(1, m)$), and in [7, Proposition 4.14] the collection $\{L_K(C_n^2) \mid n \in \mathbb{N}\}$: the groups $K_0(L_K(C_n^2))$ are described explicitly in terms of the integers appearing in the second Haselgrove sequence, together with integers arising from the standard *Fibonacci sequence*. The descriptions of all these algebras follow from an application of our main tool: the algebraic Kirchberg-Phillips theorem (described in Section 1.4).

We begin in Chapter 4 giving all the background information about the topic. In Section 4.1 we introduce the Cayley graphs C_n^j and their corresponding *graphs monoids*. In the second section of the current chapter we will come back to the case C_n^2 originally studied in [7, Section 4]. In this case we propose another way of computing the Grothendieck group of the Leavitt path algebra $L_K(C_n^2)$ totally different from the work done in [7]. We are able to reduce the computation of the Smith normal form of the matrix $I_n - A^t_{C_n^2}$ (which is the key in order to obtain the K_0 group) to the case of calculating the Smith normal form of a smaller matrix (of size 2×2), which is very close related to the Fibonacci sequence: we refer to $(M_2^n)^t - I_2$ where M_2 is the *companion matrix* of the characteristic polynomial associated to the Fibonacci sequence F. The author would like to comment that this new approach using the Smith normal form of the matrix $(M_2^n)^t - I_2$ in order to compute the K_0 of the Leavitt path algebra $L_K(C_n^2)$, came from a fruitful discussion with M. Iovanov, together with G. Abrams, during the 33^{rd} Ohio State-Denison

Mathematics Conference in Columbus, Ohio (USA).

Meanwhile, in Chapter 5 we continue the investigation of Leavitt path algebras associated to Cayley graphs, concretely the case C_n^3. In this particular situation we proceed similarly to the new approach given in the previous chapter for the case $L_K(C_n^2)$: we reduce to the computation of the Smith normal form of $(M_3^n)^t - I_3$ where M_3 is the companion matrix of the characteristic polynomial associated to the so-called *Narayana's cows sequence* (Section 5.1). The name of this sequence, denoted henceforth by G, is used in the Online Encyclopedia of Integers Sequences [51, Sequence A000930] and it is defined by setting $G(1) = 1$, $G(2) = 1$, $G(3) = 1$ and

$$G(n) = G(n-1) + G(n-3) \text{ for all } n \geq 4.$$

Narayana was an outstanding Indian mathematician of the XIV century; he was interested in summation of arithmetic series and magic squares and he applied its rules to the problem of a herd of cows and heifers (similarly to what Fibonacci would do with their rabbits). These numbers will play a very important role in the sequel.

For the following two sections we deal with the computation of the *determinant divisors* appearing in the definition of the Smith normal form of the matrix $(M_3^n)^t - I_3$; in order to do that we become involved in a number-theoretic analysis which leads us to some perhaps surprising and apparently nontrivial connections between the third Haselgrove sequence H_3 and the Narayana's cows sequence G. For this purpose we have closely collaborated with the number-theorist S. Erickson, jointly with G. Abrams. In addition in order to deal with these numbers the software *Magma* was used. Concretely with a little bit of effort, Theorem 5.3.6 is proved where a formula for $H_3(n)$ ($n \in \mathbb{N}$) is given in terms of the Narayana numbers $G(n-1)$, $G(n-2) - 1$ and $G(n-3)$; this will be the main result of Section 5.3.

Finally in Section 5.4 we give the main result of the chapter. We explicitly describe the K_0 group for Leavitt path algebras of the form C_n^3 in terms of the integers appearing in the third Haselgrove sequence, together with integers arising from the Narayana's cows sequence. Furthermore we extract enough additional information about the structure of these Grothendieck groups for applying the Kirchberg-Phillips-type result to realize the algebras $L_K(C_n^3)$ as the Leavitt path algebras of graphs having at most four vertices.

Theorem 5.4.1. *Let $n \in \mathbb{N}$. Then*

$$K_0(L_K(C_n^3)) \cong \mathbb{Z}_{d_3(n)} \times \mathbb{Z}_{\frac{d_3'(n)}{d_3(n)}} \times \mathbb{Z}_{\frac{H_3(n)}{d_3'(n)}},$$

where

$$\begin{aligned} d_3(n) &= \gcd\{G(n-1), G(n-2)-1, G(n-3)\} \textit{ and,} \\ d_3'(n) &= \gcd\{G(n-1)G(n-3) - (G(n-2)-1)^2, \\ &\qquad G(n)G(n-3) - G(n-1)(G(n-2)-1), \\ &\qquad G(n-1)^2 - G(n)(G(n-2)-1)\}. \end{aligned}$$

.

In order to finish, we will conclude with a last section named Further Work where we present some of the currently unresolved questions together with future work we could do related to the topics described in this thesis.

Chapter 1

Leavitt path algebras

1.1 Definition and basic properties

For the first two sections of the current chapter we will be dealing with Leavitt path algebras over a commutative ring R with unit. For the remaining sections of this chapter we need to consider the literature about Leavitt path algebras over a field K. We will omit the proofs of the preliminary well-known results concerning Leavitt path algebras described in this chapter.

To begin with we fix all the terminology and notation that we will use in order to work with Leavitt path algebras. After some basic notions on graph theory we give the definition of a Leavitt path algebra together with some examples.

A (directed) *graph* $E = (E^0, E^1, r, s)$ consists of two sets E^0 and E^1 together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called *vertices* and the elements of E^1 *edges*. If a vertex v emits no edges, that is, if $s^{-1}(v)$ is empty, then v is called a *sink*. A vertex v is called a *regular vertex* if $s^{-1}(v)$ is a finite non-empty set, i.e., a vertex $v \in E^0$ is regular if and only if $0 < |s^{-1}(v)| < \infty$. The set of regular vertices is denoted by E^0_{reg}.

If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then we say that the graph is *row-finite*. If E^0 is also finite, then, by the row-finiteness condition, E^1 must necessarily be finite; in this situation we say that E is *finite*. In general in this thesis we will work with arbitrary graphs; so we will use row-finite graphs when mentioned explicitly. So when we consider arbitrary graphs we make no assumptions whatsoever on the size of E^0, E^1 or in the row-finiteness of E.

A path μ in a graph E is a finite sequence of edges $\mu = e_1 \dots e_n$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. In this case, $n = l(\mu)$ is the length of μ; we view the elements of E^0 as paths of length 0. For any $n \in \mathbb{N}$ the set of paths of length n is denoted by E^n. Also, Path(E) stands for the set of all paths, i.e., Path$(E) = \bigcup_{n \in \mathbb{N} \cup \{0\}} E^n$. We denote by Vert$(\mu)$ the set of the vertices of the path μ, that is, the set $\{s(e_1), r(e_1), \dots, r(e_n)\}$.

A path $\mu = e_1 \dots e_n$ is *closed* if $r(e_n) = s(e_1)$, in which case μ is said to be *based at the vertex* $s(e_1)$. The closed path μ based at $s(e_1)$ is *simple* if $s(e_j) \neq s(e_1)$ for every $j > 1$. The closed path μ is called a *cycle* if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$. A cycle of length one is called a *loop*. An *exit* for a path $\mu = e_1 \dots e_n$ is an edge e such that $s(e) = s(e_i)$ for some i and $e \neq e_i$. We say that E satisfies *Condition (L)* if every simple closed path in E has an exit, or, equivalently, every cycle in E has an exit.

Given paths α, β, we say $\alpha \leq \beta$ if $\beta = \alpha\alpha'$, for some path α'.

For each $e \in E^1$, we call e^* a *ghost edge*. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\mu = e_1 \dots e_n$ is a path in E, we write μ^* for the element $e_n^* \dots e_1^*$. We also define $v^* = v$ for all $v \in E^0$.

Let us introduce then our main object of study.

Definition 1.1.1. Given an arbitrary graph E and a commutative ring with unit R, the *Leavitt path algebra with coefficients in* R, denoted $L_R(E)$, is the universal R-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

(1) $s(e)e = e = er(e)$ for all $e \in E^1$.

(2) $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.

(3) (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if $e \neq f$.

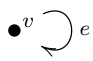
(4) (The "CK-2 relations") For every regular vertex $v \in E^0$,

$$v = \sum_{\{e \in E^1,\ s(e)=v\}} ee^*.$$

An alternative definition for $L_R(E)$ can be given using the extended graph $\widehat{E}$. This graph has the same set of vertices E^0 and same set of edges E^1 together with the so-called ghost edges e^* for each $e \in E^1$, whose directions are opposite to those of the corresponding $e \in E^1$. Thus, $L_R(E)$ can be defined as the usual path algebra $R\widehat{E}$ with coefficients in R subject to the Cuntz-Krieger relations (3) and (4) above.

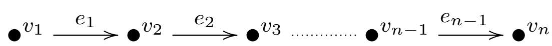
Example 1.1.2. We can realize many well-known algebras as the Leavitt path algebra of some graph. Here we give some well-known examples:

(i) The ring of Laurent polynomials $R[x, x^{-1}]$ is the Leavitt path algebra of the graph R_1 defined by $E^0 = \{v\}$, $E^1 = \{e\}$:



The isomorphism is clear: $v \mapsto 1$, $e \mapsto x$ and $e^* \mapsto x^{-1}$.

(ii) The matrix algebra $M_n(R)$ is the Leavitt path algebra of a line graph with n vertices and $n-1$ edges, i.e., the graph E defined by $E^0 = \{v_1, \ldots, v_n\}$, $E^1 = \{e_1, \ldots, e_{n-1}\}$ and $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, \ldots, n-1$:



Then $M_n(R) \cong L_R(E)$, considering the map $\varphi : L_R(E) \longrightarrow M_n(R)$ given by $v_i \mapsto e(i,i), e_i \mapsto e(i, i+1)$, and $e_i^* \mapsto e(i+1, i)$ (where $e(i,j)$ denotes the standard (i,j)-matrix unit in $M_n(R)$). Let us see that φ is an isomorphism:

– φ is a well-defined homomorphism. The relations defining $L_R(E)$ are satisfied via φ in $M_n(R)$:

(1) $e(i,i)e(j,j) = \delta_{ij}e(i,j)$.

(2) $e(i,i+1) = e(i,i+1)e(i+1,i+1) = e(i,i)e(i,i+1)$. And $e(i+1,i) = e(i+1,i)e(i,i) = e(i+1,i+1)e(i+1,i)$.

(CK1) $e(i+1,i)e(j,j+1) = \delta_{ij}e(i+1,j+1)$.

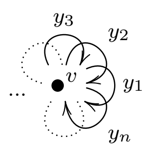
(CK2) $e(i,i) = \sum_{i,j} e(j,j+1)e(j+1,j)$.

– φ is an epimorphism. By definition we know that $e(i,i) = \varphi(v_i)$, $e(i,i+1) = \varphi(e_i)$ and $e(i+1,i) = \varphi(e_i^*)$. Besides, we have that for the standard matrix units in the upper diagonal $e(i,i+2) = \varphi(e_ie_{i+1}), \ldots, e(i,n) = \varphi(e_ie_{i+1}\ldots e_{n-1})$ and for those in the lower diagonal, $e(i+2,i) = \varphi(e_{i+1}^*e_i^*), \ldots, e(n,i) = \varphi(e_{n-1}^*\ldots e_i^*)$.

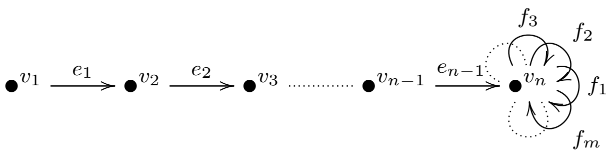
– φ is a monomorphism. The idea is to find a homomorphism $\psi : M_n(R) \longrightarrow L_R(E)$ such that $\psi\varphi = 1_{L_R(E)}$. It is sufficient to define:

$$\psi(e(i,j)) = \begin{cases} e_ie_{i+1}\ldots e_{j+1}, & \text{if } i < j \\ v_i & \text{if } i = j \\ e_{i-1}^*\ldots e_j^*, & \text{if } i > j \end{cases}$$

(iii) Classical Leavitt algebras $L(1,n)$ for $n \geq 2$: consider the graph R_n defined by $E^0 = \{v\}$, $E^1 = \{y_1, \ldots, y_n\}$. Then $L(1,n) \cong L_R(E)$. In fact, R_n is the rose with n petals graph:

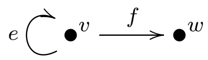


(iv) Taking into account the previous examples we can find that the Leavitt path algebra of the following graph



is $M_n(L(1,m))$, where n denotes the number of vertices in the graph and m denotes the number of loops.

(v) The Toeplitz algebra T is the Leavitt path algebra of the graph E having one loop and one exit:



□

Propositions 3.4 and 4.9 of [61] show this first property for the Leavitt path algebra $L_R(E)$. These results were first proved for a field K in [4, 55].

Proposition 1.1.3. *If E is a graph and R is a commutative ring with unit, then*

$$L_R(E) = span_R\{\alpha\beta^* : \alpha, \beta \in \mathrm{Path}(E) \text{ and } r(\alpha) = r(\beta)\}$$

and $rv \neq 0$ for all $v \in \mathrm{Path}(E)$ and all $r \in R \setminus \{0\}$. (The elements of E^0 are viewed as paths of length 0, so that this set includes elements of the form v with $v \in E^0$).

Moreover, the set of paths $\mathrm{Path}(E)$ in $L_R(E)$ is linearly independent over R. Likewise, the set of ghost paths $\{\mu^* : \mu \in \mathrm{Path}(E)\}$ in $L_R(E)$ is linearly independent over R.

We give now one more basic property of the elements of $L_R(E)$.

Proposition 1.1.4. *Let E be an arbitrary graph. Let $\alpha, \beta, \mu, \nu \in \mathrm{Path}(E)$ with $r(\alpha) = r(\beta)$ and $r(\mu) = r(\nu)$. Then products of monomials in $L_R(E)$ are computed as follows:*

$$(\alpha\beta^*)(\mu\nu^*) = \begin{cases} \alpha\mu'\nu^* & \text{if } \mu = \beta\mu' \text{ for some } \mu' \in \mathrm{Path}(E), \\ \alpha\nu^* & \text{if } \beta = \mu, \\ \alpha\beta'^*\nu^* & \text{if } \beta = \mu\beta' \text{ for some } \beta' \in \mathrm{Path}(E), \\ 0 & \text{otherwise.} \end{cases}$$

We can define an R-linear involution $x \mapsto x^*$ on $L_R(E)$ as follows: if $x = \sum_{i=1}^n r_i\alpha_i\beta_i^*$ then $x^* = \sum_{i=1}^n r_i\beta_i\alpha_i^*$. This operation is R-linear, involutive ($(x^*)^* = x$), and antimultiplicative ($(xy)^* = y^*x^*$).

If E^0 is finite, then $L_R(E)$ is unital with $\sum_{v\in E^0} v = 1_{L_R(E)}$. Otherwise, $L_R(E)$ does not have a unit but is a ring with a set of *local units* $\mathcal{S}$, given by sums of distinct vertices, with the property that for any $x \in L_R(E)$ there exists $t \in \mathcal{S}$ such that $tx = xt = x$.

1.2 The $\mathbb{Z}$-grading and the uniqueness theorems

First we will write here the so-called "reduction theorem" for Leavitt path algebras but referred to the case over a commutative ring with unit. This theorem essentially says that we can "reduce" by convenient multiplications any element of the Leavitt path algebra to a nonzero scalar multiple of a vertex or, in the worst case, to a polynomial on a cycle without exits. This gives us an extremely useful tool in a variety of contexts for Leavitt path algebras; concretely it yields both *graded* and *Cuntz-Krieger uniqueness theorems.*

For a cycle c based at the vertex v in the graph E, we write $c^0 = v$ and $c^{-n} = (c^*)^n$ for all $n \in \mathbb{N}$; for every polynomial $p(x) = \sum_{i=-m}^n r_ix^i \in R[x, x^{-1}]$ we denote by $p(c)$ the element

$$p(c) = \sum_{i=-m}^{n} r_ic^i \in L_R(E).$$

The proof is followed analogously to that given in [3, Theorem 2.2.11].

Theorem 1.2.1. *Let E be an arbitrary graph and R a commutative ring with unit. For any non-zero element $a \in L_R(E)$ there exist $\mu, \nu \in Path(E)$ such that either:*

(i) $0 \neq \mu^*a\nu = rv$*, for some* $r \in R \setminus \{0\}$ *and* $v \in E^0$*, or*

(ii) $0 \neq \mu^*a\nu = p(\lambda)$*, where* λ *is a cycle without exits and* $p(x)$ *is a non-zero polynomial in* $R[x, x^{-1}]$*.*

On the other hand, one of the most important properties of Leavitt path algebras $L_R(E)$ is that it has a natural $\mathbb{Z}$-grading.

Definition 1.2.2. If S is a ring, we say S is $\mathbb{Z}$-*graded* if there is a collection of additive subgroups $\{S_k\}_{k\in\mathbb{Z}}$ of S with the following two properties:

(i) $S = \bigoplus_{k\in\mathbb{Z}} S_k$, and

(ii) $S_j S_k \subseteq S_{j+k}$ for all $j, k \in \mathbb{Z}$.

The subgroup S_k is called the *homogeneus component* of S of degree k.

If $\phi : T \to S$ is ring homomorphism between $\mathbb{Z}$-graded rings, then ϕ is a *graded ring homomorphism* if $\phi(T_k) \subseteq S_k$ for all $k \in \mathbb{Z}$. □

Note that any ring R may be viewed as a $\mathbb{Z}$-ring in the natural way (set $R_0 = R$ and $R_n = 0$ for every $0 \neq n \in \mathbb{Z}$). Then $L_R(E)$ can be decomposed as a direct sum of homogeneous components $L_R(E) = \bigoplus_{n\in\mathbb{Z}} L_R(E)_n$ satisfying $L_R(E)_n L_R(E)_m \subseteq L_R(E)_{n+m}$. Actually,

Proposition 1.2.3. ([4, Lemma 1.7]) *Let E be a graph and R a commutative ring with unit. The Leavitt path algebra $L_R(E)$ is $\mathbb{Z}$-graded, with grading induced by*

$$deg(v_i) = 0 \textit{ for all } v_i \in E^0; deg(e_i) = 1 \textit{ and } deg(e_i^*) = -1 \textit{ for all } e_i \in E^1.$$

That is, $L_R(E) = \bigoplus_{n\in\mathbb{Z}} L_R(E)_n$, where

$$L_R(E)_n = span_R\{pq^* : p, q \in \mathrm{Path}(E), l(p) - l(q) = n\}.$$

Every element $x_n \in L_R(E)_n$ is a homogeneous element of degree n.

We have now all the machinery in hand in order to announce two important results in the theory of Leavitt path algebras which are immediate consequences of Theorem 1.2.1. First we have the *graded uniqueness theorem*, which was given by [61, Theorem 5.3]:

Theorem 1.2.4. *Let E be a graph and let R be a commutative ring with unit. If S is a graded ring and $\phi : L_R(E) \to S$ is a graded ring homomorphism with the property that $\phi(rv) \neq 0$ for all $v \in E^0$ and for all $r \in R \setminus \{0\}$, then ϕ is injective.*

Second we have the *Cuntz-Krieger uniqueness theorem*, originally proved in [61, Theorem 6.5]. Recall that a graph E satisfies Condition (L) if every cycle in E has an exit.

Theorem 1.2.5. *Let E be a graph satisfying Condition (L), and let R be a commutative ring with unit. If S is a ring and $\phi : L_R(E) \to S$ is a ring homomorphism with the property that $\phi(rv) \neq 0$ for all $v \in E^0$ and for all $r \in R \setminus \{0\}$, then ϕ is injective.*

Observe that both uniqueness theorems require some conditions: the first one considers only $\mathbb{Z}$-graded homomorphisms, while the second requires the extra hypothesis on the graph, that is, Condition (L). Our aim will consist of proving more general versions of these uniqueness theorems for Leavitt path algebras (Section 2.5) and for its generalization named as *Kumjian-Pask algebras* (Section 3.4), setting no additional hypotheses on the homomorphism or the graph.

1.3 Purely infinite simplicity

In this section we review the literature about the Leavitt path algebras which are *purely infinite simple*. For this purpose we focus our attention to the case when our commutative ring with unit R is actually a field K; so we consider the Leavitt path algebra $L_K(E)$. They have been explicitly described in [5].

Let S be a ring. A left S-module M is called *infinite* in case $M \cong M \oplus N$ with $N \neq \{0\}$. An idempotent $x \in S$ is called *infinite* in case Sx is infinite. The ring S is called *purely infinite simple* in case S is simple, and each nonzero left ideal of S contains an infinite idempotent. Equivalently, a unital ring S is *purely infinite simple* in case S is not a division ring, and S has the property that for every nonzero element x of S there exist $b, c \in S$ for which $bxc = 1_S$.

The following definitions are extremely important in the theory of Leavitt path algebras and they will play an important role in the sequel.

Definitions 1.3.1. Consider a graph E. Given two vertices $w, v \in E^0$ we say that w *connects to* v if $w = v$ or there is a path $\mu \in \text{Path}(E)$ such that $s(\mu) = w$ and $r(\mu) = v$. If $v \in E^0$ and c is a cycle, we say that v *connects to the cycle* c if there exists $u \in \text{Vert}(c)$ such that v connects to u.

A subset $H \subseteq E^0$ is *hereditary* if whenever $w \in H$ and we have that w connects to v, then it implies that $v \in H$.

We say that H is *saturated* if whenever $s^{-1}(v) \neq \emptyset$ and $\{r(e) : s(e) = v\} \subseteq H$, then $v \in H$; that is, H is saturated if, for any vertex v in E, with the condition that all of the range vertices $r(e)$ for those edges e having $s(e) = v$ are in H, then v must also be in H.

We have all the ingredients in order to announce the main result of the section.

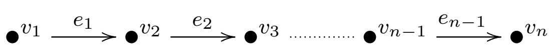
Theorem 1.3.2. ([6, Proposition 4.3]) *Let E be an arbitrary graph. Then the Leavitt path algebra $L_K(E)$ is purely infinite simple if and only if E satisfies the following conditions:*

(i) The only hereditary and saturated subsets of E^0 are $\emptyset$ and E^0.

(ii) E satisfies Condition (L).

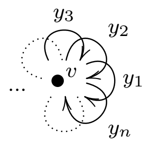
(iii) Every vertex connects to a cycle.

Example 1.3.3. (i) Let E be the graph defined by $E^0 = \{v_1, \ldots, v_n\}$, $E^1 = \{e_1, \ldots, e_{n-1}\}$ and $s(e_i) = v_i$ and $r(e_i) = v_{i+1}$ for $i = 1, \ldots, n-1$:



Then $L_K(E) \cong M_n(R)$. We have that it is not purely infinite simple since no vertex in E^0 connects to a cycle.

(ii) For $n \geq 2$ consider the rose with n petals graph R_n defined by $E^0 = \{v\}$, $E^1 = \{y_1, \dots, y_n\}$:



Then $L_K(E) \cong L(1, n)$, the classical Leavitt algebra. Since $n \geq 2$ we see that all the hypotheses of Theorem 1.3.2 are satisfied, so that $L(1, n)$ is purely infinite simple.

In the context of Leavitt path algebras the purely infinite simple algebras play an especially intriguing role, that we will discover in Section 1.4.

1.4 The Grothendieck group K_0

Inspired by Section 1.3 let us suppose now again for the current section the case when our commutative ring with unit R is actually a field K; so we consider the Leavitt path algebra $L_K(E)$.

For a unital K-algebra A, the set of isomorphism classes of finitely generated projective left A-modules is denoted by $\mathcal{V}(A)$. We denote the elements of $\mathcal{V}(A)$ using brackets: $[A] \in \mathcal{V}(A)$ represents the isomorphism class of the left regular module ${}_AA$. $\mathcal{V}(A)$ is a monoid, with operation $\oplus$, and zero element $[\{0\}]$. The monoid $(\mathcal{V}(A), \oplus)$ is *conical*: this means that the sum of any two nonzero elements of $\mathcal{V}(A)$ is nonzero, i.e., that $\mathcal{V}(A)^* = \mathcal{V}(A) \setminus \{0\}$ is a semigroup under $\oplus$.

By [49, Proposition 0.1] if A has local units, then the well-studied *Grothendieck group* $K_0(A)$ of A is the universal group corresponding to the monoid $\mathcal{V}(A)$. Here universal means that any homomorphism from $\mathcal{V}(A)$ to a group G necessarily factors through $K_0(A)$.

Definition 1.4.1. Let E be a row-finite graph. We define $(M_E, +)$ to be the free abelian *monoid* having generating set $\{a_v \mid v \in E^0\}$ and with relations given by

$$a_v = \sum_{e \in E^1, s(e)=v} a_{r(e)} \text{ for every } v \in E^0_{\text{reg}}.$$

We denote the zero element of M_E by z.

The following property for Leavitt path algebras was established in [14, Theorem 3.5].

Theorem 1.4.2. *Let E be a row-finite graph and K any field. Then*

$$\mathcal{V}(L_K(E)) \cong M_E \textit{ as monoids.}$$

$$\textit{Moreover, } \ [L_K(E)] \leftrightarrow \sum_{v \in E^0} [v] \textit{ under this isomorphism.}$$

It is shown in [13, Corollary 2.2] that if A is a unital purely infinite simple K-algebra, then the semigroup $(\mathcal{V}(A)^*, \oplus)$ is in fact a group, and, moreover, that $\mathcal{V}(A)^* \cong K_0(A)$, the Grothendieck group of A. So when $L_K(E)$ is unital purely infinite simple we have the following isomorphisms of groups:

$$K_0(L_K(E)) \cong \mathcal{V}(L_K(E))^* \cong M_E^*.$$

In particular, in this situation we have $|K_0(L_K(E))| = |M_E^*|$.

Let us present a computational tool which will be quite useful in the discussion. Let $M \in M_n(\mathbb{Z})$ and view M as a linear transformation $M : \mathbb{Z}^n \to \mathbb{Z}^n$ via left multiplication on columns. We have that if P, Q are invertible in $M_n(\mathbb{Z})$ then $\text{Coker}(M) \cong \text{Coker}(PMQ)$. This means that if $N \in M_n(\mathbb{Z})$ is a matrix which is constructed by performing any sequence $\mathbb{Z}$-elementary row and/or column operations starting with M, then $\text{Coker}(M) \cong \text{Coker}(N)$ as abelian groups.

Definition 1.4.3. The *Smith normal form* for any $n \times n$ matrix $M = (m_{i,j})$ with $m_{i,j} \in \mathbb{Z}$ is a diagonal matrix $S = \text{diag}(s_1, s_2, \dots, s_r, 0, \dots, 0)$ where r is the rank of M and $s_i \mid s_{i+1}$ for every $i \in \{1, \dots, r-1\}$; s_i are called the *invariant factors* of M. More concretely, let us define the i^{th} *determinant* *divisor* as

$$\delta_0 := 1,$$
$$\delta_i = \gcd(\text{all } i \times i \text{ minors of } M) \text{ for } i \geq 1.$$

Besides we have that $s_i = \frac{\delta_i}{\delta_{i-1}}$.

For any matrix $M \in M_n(\mathbb{Z})$, the Smith normal form of M exists and is unique. If $D \in M_n(\mathbb{Z})$ is a diagonal matrix with entries $d_1, d_2, \ldots, d_n$ then (viewing D as a linear transformation from $\mathbb{Z}^n$ to $\mathbb{Z}^n$), we clearly have $\mathrm{Coker}(D) \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \mathbb{Z}/d_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}$. Let us interpret $\mathbb{Z}/1\mathbb{Z}$ as the trivial group $\{0\}$ and $\mathbb{Z}/0\mathbb{Z}$ as $\mathbb{Z}$. In the end we have the following.

Proposition 1.4.4. *Let $M \in M_n(\mathbb{Z})$, and let S denote the Smith normal form of M. Suppose the diagonal entries of S are $s_1, s_2, \ldots, s_n$. Then*

$$\mathrm{Coker}(M) \cong \mathbb{Z}/s_1\mathbb{Z} \oplus \mathbb{Z}/s_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/s_n\mathbb{Z}.$$

Suppose now that E is a finite directed graph having n vertices $v_1, v_2, \ldots, v_n$. Let A_E be the usual *incidence matrix* of E, i.e., the matrix $A_E = (a_{i,j})$ where, for each pair $1 \leq i, j \leq n$, the entry $a_{i,j}$ denotes the number of edges e in E for which $s(e) = v_i$ and $r(e) = v_j$. Consider the matrix $I_n - A_E^t$, where I_n is the identity $n \times n$ matrix, and $(\ )^t$ denotes the transpose of a matrix. As mentioned before we view $I_n - A_E^t$ both as a matrix, and as a linear transformation $I_n - A_E^t : \mathbb{Z}^n \to \mathbb{Z}^n$, via left multiplication.

In [2, Section 3] we can find the following situation: in the case $L_K(E)$ is purely infinite simple, so that in particular M_E^* is a group (necessarily isomorphic to $K_0(L_K(E))$), we have that

$$K_0(L_K(E)) \cong M_E^* \cong \mathbb{Z}^n/\mathrm{Im}(I_n - A_E^t) = \mathrm{Coker}(I_n - A_E^t).$$

Under this isomorphism $[v_i] \mapsto \vec{b_i} + \mathrm{Im}(I_n - A_E^t)$, where $\vec{b_i}$ is the element of $\mathbb{Z}^n$ which is 1 in the i^{th} coordinate and 0 elsewhere. In other words, when $L_K(E)$ is purely infinite simple, then $K_0(L_K(E))$ is the cokernel of the linear transformation $I_n - A_E^t : \mathbb{Z}^n \to \mathbb{Z}^n$ induced by matrix multiplication.

Example 1.4.5. Suppose $E = R_m$ the rose with m petals graph ($m \geq 2$). Then $A_E = (m)$, so $I - A_E^t$ is the 1×1 matrix $(1 - m)$, and

$$K_0(L_K(E)) \cong \mathbb{Z}^1/(1-m)\mathbb{Z}^1 \cong \mathbb{Z}/(m-1)\mathbb{Z}.$$

Taking into account Proposition 1.4.4 we have the following result for $L_K(E)$ purely infinite simple.

Proposition 1.4.6. *Suppose E is a finite graph with $|E^0| = n$ and $L_K(E)$ is purely infinite simple. Let S be the Smith normal form of the matrix $I_n - A_E^t$, with diagonal entries $s_1, s_2, \ldots, s_n$. Then*

$$K_0(L_K(E)) \cong \mathbb{Z}/s_1\mathbb{Z} \oplus \mathbb{Z}/s_2\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/s_n\mathbb{Z}.$$

Moreover, if $K_0(L_K(E))$ is finite, then the analysis of the Smith normal form of the matrix $I_n - A_E^t$ yields

$$|K_0(L_K(E))| = |\det(I_n - A_E^t)|.$$

Conversely, $K_0(L_K(E))$ is infinite if and only if $\det(I_n - A_E^t) = 0$.

Finally we present the fundamental result of interest here, and more particularly for Chapters 4 and 5: the *algebraic Kirchberg Phillips theorem*.

Theorem 1.4.7. ([10, Corollary 2.7]) *Suppose E and F are finite graphs for which the Leavitt path algebras $L_K(E)$ and $L_K(F)$ are purely infinite simple. Suppose that there is an isomorphism $\varphi : K_0(L_K(E)) \to K_0(L_K(F))$ for which $\varphi([L_K(E)]) = [L_K(F)]$, and suppose also that the two integers $\det(I_{|E^0|} - A_E^t)$ and $\det(I_{|F^0|} - A_F^t)$ have the same sign (i.e., are either both nonnegative, or both nonpositive). Then $L_K(E) \cong L_K(F)$ as K-algebras.*

Our aim in Chapters 4 and 5 will be to explicitly describe the K_0 group for Leavitt path algebras of the *Cayley graphs* C_n^j ($n \in \mathbb{N}, 0 \leq j \leq n-1$). Furthermore we will extract enough additional information about the structure of these Grothendieck groups to be able to apply the Kirchberg-Phillips-type result to realize the algebras $L_K(C_n^2)$ and $L_K(C_n^3)$ as the Leavitt path algebras of graphs having at most respectively three and four vertices.

Chapter 2

The commutative core of a Leavitt path algebra

2.1 Preliminaries

Let us consider R a commutative ring with unit. Our main goal for this chapter consists of identifying the *commutative core* of the Leavitt path algebra of E with coefficients in R which will result to be a maximal commutative subalgebra of the Leavitt path algebra. As a second step, we will be able to generalize and simplify the result about commutative Leavitt path algebras over fields. Furthermore, our final aim will be to characterize injectivity of representations which gives a generalization of the Cuntz-Krieger uniqueness theorem.

Let us start by defining the analogous relations given for a Leavitt path algebra when we consider a general R-algebra with involution. Afterwards we introduce some more background information.

Definition 2.1.1. Let E be a graph and $\mathcal{A}$ be an R-algebra with involution $*$. A *Cuntz-Krieger E-system in $\mathcal{A}$* is a collection $\Sigma = (S_\mu)_{\mu \in E^0 \cup E^1} \subset \mathcal{A}$ which satisfies the following relations:

(1) For all $v, w \in E^0$, $S_v S_v = S_v$ and $S_v S_w = 0$ if $v \neq w$.

(2) $S_v^* = S_v$ for all $v \in E^0$.

(3) $S_{s(e)} S_e = S_e = S_e S_{r(e)}$ for all $e \in E^1$.

(4) For all $e, f \in E^1$, $S_e^* S_e = S_{r(e)}$ and $S_e^* S_f = 0$ if $e \neq f$.

(5) For every regular vertex $v \in E^0$,

$$S_v = \sum_{\{e \in E^1,\ s(e)=v\}} S_e S_e^*.$$

For convenience we are going to extend the Cuntz-Krieger E-system $\Sigma \subset \mathcal{A}$ by defining all elements $S_\mu \in \mathcal{A}$, $\mu \in \text{Path}(E)$, considering $S_\mu S_\nu = S_{\mu\nu}$ if $r(\mu) = s(\nu)$ and $S_\mu S_\nu = 0$ otherwise. So we shall abuse the notation and write it as an extended system $\Sigma = (S_\mu)_{\mu \in \text{Path}(E)}$.

There exists a new version given for paths of the Cuntz-Krieger relation (5) of Definition 2.1.1.

Proposition 2.1.2. *Let $v \in E^0$ and $k \in \mathbb{N}$. If there are finitely many paths μ with $s(\mu) = v$ and $l(\mu) \leq k$, then*

$$S_v = \sum_{\substack{s(\mu)=v \\ l(\mu)=k}} S_\mu S_\mu^* + \sum_{\substack{s(\mu)=v \\ l(\mu)<k \\ r(\mu) \textit{ is a sink}}} S_\mu S_\mu^*.$$

Proof. The proof follows analogously to [50, Proposition 1.3]. We prove it by induction on k. The statement is clear for $k = 0$. Suppose that it is true for $k \geq 0$ and then:

$$\begin{aligned} S_v &= \sum_{\substack{s(\mu)=v \\ l(\mu)=k \\ r(\mu) \text{ is not a sink}}} S_\mu S_\mu^* + \sum_{\substack{s(\mu)=v \\ l(\mu)\leq k \\ r(\mu) \text{ is a sink}}} S_\mu S_\mu^* \\ &= \sum_{s(\mu)=v} \sum_{\substack{s(e)=r(\mu) \\ l(\mu)=k}} S_\mu S_e S_e^* S_\mu^* + \sum_{\substack{s(\mu)=v \\ l(\mu)\leq k \\ r(\mu) \text{ is a sink}}} S_\mu S_\mu^* \\ &= \sum_{\substack{s(\mu)=v \\ l(\mu)=k+1}} S_\mu S_\mu^* + \sum_{\substack{s(\mu)=v \\ l(\mu)<k+1 \\ r(\mu) \text{ is a sink}}} S_\mu S_\mu^*. \end{aligned}$$

□

Analogously to Proposition 1.1.4, we set here the following proposition for future reference.

Proposition 2.1.3. *Given a Cuntz-Krieger E-system $\Sigma = (S_\mu)_{\mu \in \mathrm{Path}(E)}$ in some R-algebra $\mathcal{A}$ with involution, the collection*

$$G(\Sigma) = \{S_\mu S_\nu^* : \mu, \nu \in \mathrm{Path}(E), r(\mu) = r(\nu)\}$$

satisfies the following:

(i) given paths $\mu, \nu \in \mathrm{Path}(E)$ with $r(\mu) = r(\nu)$, we have $(S_\mu S_\nu^*)^* = S_\nu S_\mu^*$;

(ii) given four paths $\alpha, \beta, \mu, \nu \in \mathrm{Path}(E)$ with $r(\alpha) = r(\beta)$ and $r(\mu) = r(\nu)$ then

$$(S_\alpha S_\beta^*)(S_\mu S_\nu^*) = \begin{cases} S_{\alpha\mu'} S_\nu^* & \text{if } \mu = \beta\mu' \text{ for some } \mu' \in \mathrm{Path}(E), \\ S_\alpha S_{\nu\beta'}^* & \text{if } \beta = \mu\beta' \text{ for some } \beta' \in \mathrm{Path}(E), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1.4. The collection $G(\Sigma)$ given in Proposition 2.1.3 will be called the *standard Cuntz-Krieger generator set associated with* Σ. When we refer to the Leavitt path algebra $L_R(E)$ we will denote it simply by G_E, i.e.,

$$G_E = \{\mu\nu^* : \mu, \nu \in \mathrm{Path}(E), r(\mu) = r(\nu)\}.$$

Note that we have $L_R(E) = \mathrm{span}_R(G_E)$.

Also we introduce a graded uniqueness theorem that follows similarly to the one given in Theorem 1.2.4. We will use this result later.

Theorem 2.1.5. *Let $\Sigma = (S_\mu)_{\mu \in E^0 \cup E^1}$ be a Cuntz-Krieger E-system in a $\mathbb{Z}$-graded R-algebra $\mathcal{A}$ such that for any $\mu \in E^0$, S_μ is homogenous of degree zero and for any $\mu \in E^1$, S_μ is homogenous of degree one. Then the following conditions are equivalent:*

(i) the associated representation $\Phi : L_R(E) \to \mathcal{A}$ is injective;

(ii) $\Phi(rv) \neq 0$ for all $v \in E^0$ and for all $r \in R \setminus \{0\}$.

2.2 The essentially aperiodic representation

In this section we give a particular representation of the Leavitt path algebra, which will be called the "essentially aperiodic representation". Previously we define all the terminology we need for this purpose. These definitions were introduced in [50, Definitions 2.4-2.5].

Definition 2.2.1. Given a graph E, a *trail* in E is either

(i) a finite path $\tau = e_1 \dots e_n$ (possibly of length zero) whose range $r(\tau) = r(e_n)$ is a sink, that is, $s^{-1}(r(\tau)) = \emptyset$; or

(ii) an infinite path, that is, an infinite sequence $\tau = e_1 e_2 \dots$ of edges, such that $r(e_n) = s(e_{n+1})$ for every $n \in \mathbb{N}$.

In some literature about Leavitt path algebras (see for example [6] and [19]), the set of all trails in the graph E is denoted by $E^{\leq\infty}$.

Given a trail τ and some integer $n \geq 0$, we define the *head of length* n to be the finite path:

$$\tau_{(n)} = \begin{cases} s(\tau) & \text{if } \ n = 0, \\ e_1 \dots e_n & \text{if } \ n > 0 \text{ and } \tau \text{ is either infinite, or finite with } l(\tau) > n, \\ \tau & \text{if } \ n > 0 \text{ and } \tau \text{ is finite, with } l(\tau) \leq n. \end{cases}$$

Notice that all heads of τ have the same source, that is, $s(\tau_{(n)}) = s(\tau_{(0)})$, and this special vertex will be denoted simply by $s(\tau)$, and will be referred to as the source of τ. Given a trail τ and a path $\mu \in \mathrm{Path}(E)$, we write $\mu \leq \tau$ if $\mu = \tau_{(n)}$ for some $n \geq 0$. It is evident that if $\mu \leq \tau$, then removing μ from τ still yields a trail. On the other hand, if we have trail τ and a path $\mu \in \mathrm{Path}(E)$ with $r(\mu) = s(\tau)$, the obvious concatenation $\mu\tau$ is again a trail.

Definition 2.2.2. Suppose $\Sigma = (S_\mu)_{\mu \in \mathrm{Path}(E)}$ is a Cuntz-Krieger E-system in an R-algebra $\mathcal{A}$. Consider the subset

$$G^{\Delta}(\Sigma) = \{S_\mu S_\mu^* : \mu \in \mathrm{Path}(E)\}.$$

By Proposition 2.1.3 it is clear that $G^{\Delta}(\Sigma)$ is a commutative set of self-adjoint idempotents in $< \Sigma >$, the R-subalgebra of $\mathcal{A}$ generated by Σ. We will refer to $G^{\Delta}(\Sigma)$ as the *standard diagonal generator set*. The R-subalgebra $< G^{\Delta}(\Sigma) > \subset \mathcal{A}$, which will be denoted by $\Delta(\Sigma)$, is called the *diagonal algebra associated with* Σ. In particular when we refer to the case of the Leavitt path algebra $L_R(E)$, the diagonal generator set will be denoted by G_E^{Δ} and the R-subalgebra generated by it will be denoted by $\Delta(E)$.

Remark 2.2.3. If $\Sigma = (S_\mu)_{\mu \in \mathrm{Path}(E)}$ is a Cuntz-Krieger E-system in an R-algebra $\mathcal{A}$ and $\mathcal{B}$ is another R-algebra such that we have a $*$-homomorphism $\Pi : \mathcal{A} \to \mathcal{B}$, then $\Pi(\Sigma) = (\Pi(S_\mu))_{\mu \in \mathrm{Path}(E)}$ is a Cuntz-Krieger E-system in $\mathcal{B}$ and Π maps $G^{\Delta}(\Sigma)$ onto $G^{\Delta}(\Pi(\Sigma))$ and $\Delta(\Sigma)$ onto $\Delta(\Pi(\Sigma))$. When we specialize to the case of a representation $\Pi : L_R(E) \to \mathcal{A}$, this maps G_E^{Δ} onto $G^{\Delta}(\Sigma)$ and $\Delta(E)$ onto $\Delta(\Sigma)$.

Remark 2.2.4. For $\mu\mu^*, \nu\nu^* \in G_E^{\Delta}$ we say $\mu\mu^* \leq \nu\nu^*$ if and only if $\mu \leq \nu$, that is, $\nu = \mu\mu'$ for some $\mu' \in \mathrm{Path}(E)$. Then for any pair $\mu\mu^*, \nu\nu^* \in G_E^{\Delta}$ we have either $(\mu\mu^*)(\nu\nu^*) = 0$ or $\mu\mu^*$ and $\nu\nu^*$ are comparable (either $\mu\mu^* \leq \nu\nu^*$ or $\nu\nu^* \leq \mu\mu^*$). Let us see that if we consider the diagonal generator set G_E^{Δ} (whose elements are all non-zero self-adjoint idempotents), the finite (resp. countably infinite) maximal totally ordered subsets of G_E^{Δ} are in one-to-one correspondence with the finite (resp. infinite) trails in E. Specifically, every maximal totally ordered subset $\mathcal{P} \subset G_E^{\Delta}$ can be uniquely presented as $\mathcal{P}_\tau = \{\tau_{(n)}\tau_{(n)}^* : n \geq 0\}$ for some trail τ:

First let τ be a finite trail ($\tau = e_1 \dots e_n$ and $r(\tau)$ is a sink). Let us see that $\mathcal{P} = \{\tau_{(0)}\tau_{(0)}^*, \tau_{(1)}\tau_{(1)}^*, \dots, \tau_{(n)}\tau_{(n)}^*\}$ is a maximal totally ordered subset of G_E^{Δ}. Obviously $\mathcal{P}$ is a totally ordered set with order $\leq$. Assume that there exists $\mathcal{P} \subsetneq \mathcal{Q}$ and $\mathcal{Q}$ is a totally ordered subset of G_E^{Δ}; then there exists $\beta\beta^* \in \mathcal{Q} \setminus \mathcal{P}$ and so $\mathcal{P} \cup \{\beta\beta^*\}$ is totally ordered. Then $\beta\beta^* \leq \tau_{(n)}\tau_{(n)}^*$ or $\tau_{(n)}\tau_{(n)}^* \leq \beta\beta^*$. If $\beta\beta^* \leq \tau_{(n)}\tau_{(n)}^*$ then $\beta \leq \tau_{(n)}$ which is a contradiction since $\beta\beta^* \notin \mathcal{P}$; on the other hand if $\tau_{(n)}\tau_{(n)}^* \leq \beta\beta^*$ then $\tau_{(n)} \leq \beta$ and because $r(\tau_{(n)}) = r(\tau)$ is a sink, then $\tau_{(n)} = \beta$ which gives again a contradiction since $\beta\beta^* \notin \mathcal{P}$. Then $\mathcal{P}$ is maximal totally ordered subset of G_E^{Δ}.

Consider now τ an infinite trail given in the form $\tau = e_1e_2e_3\dots$. Let $\mathcal{P} = \{\tau_{(0)}\tau_{(0)}^*, \tau_{(1)}\tau_{(1)}^*, \tau_{(2)}\tau_{(2)}^*, \dots\}$. Let us prove that $\mathcal{P}$ is a maximal totally ordered subset of G_E^Δ. Take into account that $\mathcal{P}$ is totally ordered set with order $\leq$. Suppose there exists a totally ordered subset $\mathcal{Q}$ of G_E^Δ such that $\mathcal{P} \subsetneq \mathcal{Q}$ and consider $\beta\beta^* \in \mathcal{Q}\setminus\mathcal{P}$. Then again $\mathcal{P}\cup\{\beta\beta^*\}$ is totally ordered. If there exists n such that $\beta\beta^* \leq \tau_{(n)}\tau_{(n)}^*$ then $\beta \leq \tau_{(n)}$ which is a contradiction. Therefore for any n, $\tau_{(n)}\tau_{(n)}^* \leq \beta\beta^*$ and then $\tau_{(n)} \leq \beta$ which is impossible since $\beta\beta^* \in G_E^\Delta$. So $\mathcal{P}$ is a maximal totally ordered subset of G_E^Δ.

Conversely let $\mathcal{P}$ be a finite maximal totally ordered subset of G_E^Δ. Then by using Zorn's Lemma we can find a maximal element in $\mathcal{P}$. Assume that $\beta\beta^*$ is a maximal element of $\mathcal{P}$. Then β is a finite path and since $\mathcal{P}$ is maximal totally ordered subset of G_E^Δ then $r(\beta)$ is a sink giving that β is a finite trail.

Finally suppose that $\mathcal{P}$ is a countable infinite subset of G_E^Δ. Then by Zorn's Lemma $\mathcal{P}$ has a minimal element $e_0e_0^*$. Also let $e_1e_1^*$ the minimal element of $\mathcal{P} \setminus \{e_0e_0^*\}$, $e_2e_2^*$ the minimal element of $\mathcal{P} \setminus \{e_0e_0^*, e_1e_1^*\}$ etc. Proceeding in this way we have that $e_0e_1e_2\dots$ is an infinite trail.

Therefore we can say that for any trail there exists a maximal totally ordered subset of G_E^Δ and for any finite or infinite countable maximal totally ordered subset of G_E^Δ there exists a trail. In the end if E is countable, then the maximal totally ordered subsets of G_E^Δ are in one-to-one correspondence with the trails in E.

Definition 2.2.5. Let E be a graph.

(A) An infinite trail $\tau = e_1e_2\dots$ in E is said to be *periodic*, if there exist integers $j, k \geq 1$, such that $e_{n+k} = e_n$ for every $n \geq j$. In this case, it is clear that the path $\rho = e_j \dots e_{j+k-1}$ is closed. If we take j and k such that $j + k$ is the smallest possible value which satisfies the condition $e_{n+k} = e_n$ for every $n \geq j$ and we consider the paths $\alpha = e_1 \dots e_{j-1}$ and $\lambda = e_j \dots e_{j+k-1}$, we will refer to the pair (α, λ) as the *seed of* τ. Of course α may have length zero. In any case, λ is a closed path, which will be called the *period of* τ.

(B) A trail τ is said to be *essentially aperiodic* if:

(i) τ is finite, or

(ii) τ is periodic and its period is a closed path without exits (which means that it has to be a cycle without exits), or

(iii) τ is infinite and not periodic.

The trails of the form (i) and (ii) will be called *discrete*, while the ones of type (iii) will be called *continuous*. In a discrete essentially aperiodic trail τ, we refer to a certain path as the *essential head of* τ, and will denote it by $\tau_{(\text{ess})}$, which is defined accordingly as follows:

(i) if τ is finite, then $\tau_{(\text{ess})} = \tau$;

(ii) if τ is periodic, with seed (α, λ) then $\tau_{(\text{ess})} = \alpha$.

Let us denote by $\mathfrak{T}_E$ the set of all essentially aperiodic trails in E.

Remark 2.2.6. Note that in Definition 2.2.5 (A), the choice of j and k such that $j+k$ is the smallest possible value which satisfies the condition $e_{n+k} = e_n$ for every $n \geq j$, is unique. Assume that there exist $j_1, k_1, j_2, k_2 \geq 1$ such that $j_1 + k_1 = j_2 + k_2$ is the smallest possible value which satisfies the condition $e_{m+k_1} = e_m$ for every $m \geq j_1$ and $e_{n+k_2} = e_n$ for every $n \geq j_2$. Then for every $r \geq 0$, $e_{j_1+r} = e_{j_1+r+k_1} = e_{j_2+r+k_2} = e_{j_2+r}$ (since $j_1 + k_1 = j_2 + k_2$). Suppose that $j_2 > j_1$ (the case $j_1 > j_2$ is similar). So for any $r \geq 0$, $e_{j_1+j_2-j_1+r} = e_{j_2+r} = e_{j_1+r}$ and hence for any $m \geq j_1$, $e_{m+j_2-j_1} = e_m$. By minimality of $k_2 + j_2$ we have that $k_2 = 0$ which is a contradiction. Thus j_1 and k_1 are unique satisfying that $j_1 + k_1$ is the smallest value with this desired property.

Lemma 2.2.7. *For every vertex $v \in E^0$ there exists at least one essentially aperiodic trail τ with $v = s(\tau)$.*

Proof. The proof follows completely [50, Lemma 2.6]. We will include it here for completeness. We will say that a vertex $w \in E^0$ is a *trap* if:

(a) $s^{-1}(w) = \emptyset$, that is, w is a sink, or

(b) w is a vertex visited by a cycle without exits, or

(c) there exist two cycles $\lambda = e_1 \dots e_m$ and $\mu = f_1 \dots f_n$ with $s(\lambda) = r(\lambda) = s(\mu) = r(\mu) = w$ such that $e_1 \neq f_1$.

To prove this result we need to distinguish the following cases:

(i) no path $\alpha \in \mathrm{Path}(E)$ with $s(\alpha) = v$ has $r(\alpha)$ being a trap;

(ii) there exists a path $\alpha \in \mathrm{Path}(E)$ with $s(\alpha) = v$ and $r(\alpha)$ a trap.

In case (i) we construct a sequence $(\alpha_n)_{n=1}^{\infty} \subset \mathrm{Path}(E)$ with $s(\alpha_1) = v$, such that

$$r(\alpha_{n-1}) = s(\alpha_n) \tag{2.1}$$

$$\mathrm{Vert}(\alpha_n) \not\subseteq \mathrm{Vert}(\alpha_1 \dots \alpha_{n-1}) \text{ for all } n \geq 2. \tag{2.2}$$

We start off with an arbitrary length 1 path with $s(\alpha_1) = v$. Imagine that the paths $\alpha_1, \dots, \alpha_N$ have been constructed and the equations (2.1) and (2.2) hold for all $n \leq N$. Let us describe α_{N+1}: suppose $\alpha_1 \dots \alpha_n = e_1 \dots e_k$; since the vertex $r(e_k)$ is not a trap, there exists at least one edge, call it e_{k+1}, such that $s(e_{k+1}) = r(e_k)$. If $r(e_{k+1}) \notin \mathrm{Vert}(\alpha_1 \dots \alpha_n)$, then α_{N+1} is the length 1 path e_{k+1}, and we finish. On the contrary, if $r(e_{k+1}) \in \mathrm{Vert}(\alpha_1 \dots \alpha_n)$ then this means that we have a cycle of the form $\lambda = e_{k+1} e_p e_{p+1} \dots e_k$ for some $p \in \{1, 2, \dots, k+1\}$. Notice that no point in $\mathrm{Vert}(\lambda)$ is a trap by hypothesis, so the cycle λ must have an exit: we denote it by e_{k+2} and then there exists $j \in \{p, p+1, \dots, k, k+1\}$ with $s(e_{k+2}) = s(e_j)$ but $e_{k+2} \neq e_j$. Observe that $r(e_{k+2}) \notin \mathrm{Vert}(\alpha_1 \dots \alpha_N)$ and then we can define the path $\alpha_{N+1} = e_{k+1} e_p e_{p+1} \dots e_{j-1} e_{k+2}$ implying that the set $\mathrm{Vert}(\alpha_{N+1})$ contains the vertex $r(e_{k+2})$ which does not belong to the set $\mathrm{Vert}(\alpha_1 \dots \alpha_N)$. In this manner we have an infinite sequence $(\alpha_n)_{n=1}^{\infty}$ satisfying equations (2.1) and (2.2) so we can form the infinite trail $\tau = \alpha_1 \alpha_2 \dots$ not being periodic.

Assume now the case (ii): there exists a path $\alpha \in \mathrm{Path}(E)$ with $s(\alpha) = v$ with $r(\alpha)$ a trap. There are three possibilities for $r(\alpha)$:

(a) If $r(\alpha)$ is a sink, we simply consider $\tau = \alpha$.

(b) If $r(\alpha)$ sits on a cycle without exits λ which starts and ends at $r(\alpha)$ then take τ to be the periodic trail with seed (α, λ).

(c) Finally suppose we have two cycles $\lambda = e_1 \dots e_m$ and $\mu = f_1 \dots f_n$ such that $s(\lambda) = r(\lambda) = s(\mu) = r(\mu) = r(\alpha)$ and $e_1 \neq f_1$. By replacing λ with λ^n and μ with μ^m we can assume that λ and μ have equal lengths. We construct now the (not periodic) trail $\tau = \alpha\alpha_1\alpha_2 \dots$, where $\alpha_n = \lambda\mu^n$ for every $n \geq 1$. □

We are now in a position to give the so-called "essentially aperiodic representation" for a Leavitt path algebra.

Proposition 2.2.8. *Let E be a graph and R be a commutative ring with unit. If we consider M the R-module generated by a set of generators $\{\xi_\tau^n : n \in \mathbb{Z}, \tau \in \mathfrak{T}_E\}$ then there exist a $\mathbb{Z}$-graded R-algebra $\mathcal{E} \subseteq \mathrm{End}_R(M)$ and a unique Cuntz-Krieger E-system $\Sigma = (S_\alpha)_{\alpha \in Path(E)} \subset \mathcal{E}$ such that the map $S_\alpha : M \to M$ is given by:*

$$S_\alpha(\xi_\tau^n) = \begin{cases} \xi_{\alpha\tau}^{l(\alpha)+n} & \text{if } \ r(\alpha) = s(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

Besides, the associated representation $\Pi_{\mathrm{ap}} : L_R(E) \to \mathcal{E}$ is injective.

Definition 2.2.9. The representation Π_{ap} given in Proposition 2.2.8 is called the *essentially aperiodic representation* of $L_R(E)$.

Proof of Proposition 2.2.8. First let us define for each $i \in \mathbb{Z}$, M_i as the R-module generated by the collection $\{\xi_\tau^i : \tau \in \mathfrak{T}_E\}$. Consider $M = \bigoplus_{i \in \mathbb{Z}} M_i$ as $\mathbb{Z}$-graded R-module. Let $f \in \mathrm{End}_R(M)$, we say that f is *homogeneous of degree k* ($k \in \mathbb{Z}$) if $f(M_i) \subseteq M_{i+k}$ for every i. In this case we set $\deg(f) = k$. Then define $\mathcal{E}_i = \{f \mid f \in \mathrm{End}_R(M), \deg(f) = i\}$ and let $\mathcal{E} = \bigoplus_{i \in \mathbb{Z}} \mathcal{E}_i$ be an R-subalgebra of $\mathrm{End}_R(M)$.

Let us check that $\mathcal{E}$ is $\mathbb{Z}$-graded, that is, for each $i, j \in \mathbb{Z}$, $\mathcal{E}_i\mathcal{E}_j \subseteq \mathcal{E}_{i+j}$. Suppose $f \in \mathcal{E}_i$ and $g \in \mathcal{E}_j$ and consider $f \circ g$. Take some $t \in \mathbb{Z}$, and then $(f \circ g)(M_t) = f(g(M_t))$. Since $\deg(g) = j$, $g(M_t) \subseteq M_{t+j}$. Then taking into account that $\deg(f) = i$, we have $f(g(M_t)) \subseteq f(M_{t+j}) \subseteq M_{t+j+i}$ which means $f \circ g \in \mathcal{E}_{i+j}$.

Now let $\alpha \in \text{Path}(E)$ and $S_\alpha : M \to M$ be an R-homomorphism which is defined on the generators of M as:

$$S_\alpha(\xi_\tau^n) = \begin{cases} \xi_{\alpha\tau}^{l(\alpha)+n} & \text{if } r(\alpha) = s(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

Let us prove that S_α is homogeneous of degree $l(\alpha)$, that is, for any $t \in \mathbb{Z}$ we have $S_\alpha(M_t) \subseteq M_{t+l(\alpha)}$. So consider $\tau \in \mathfrak{T}_E$ and let $\xi_\tau^t \in M_t$. Then

$$S_\alpha(\xi_\tau^t) = \begin{cases} \xi_{\alpha\tau}^{l(\alpha)+t} & \text{if } r(\alpha) = s(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

and therefore $S_\alpha(\xi_\tau^t) \in M_{t+l(\alpha)}$ obtaining the desired statement.

On the other hand, let us define for each $\alpha \in \text{Path}(E)$, $S_\alpha^* = S_{\alpha^*}$, where α^* is the ghost path of α and

$$S_{\alpha^*}(\xi_\tau^n) = \begin{cases} \xi_\beta^{n-l(\alpha)} & \text{if } \alpha \leq \tau \text{ with } \tau = \alpha\beta, \\ 0 & \text{otherwise.} \end{cases}$$

A similar argument shows that S_α^* is homogeneous of degree $-l(\alpha)$.

Let us prove that $\Sigma = (S_\alpha)_{\alpha \in \text{Path}(E)}$ is a Cuntz-Krieger E-system inside $\mathcal{E}$. Consider $\mu, \nu \in \text{Path}(E)$ $n \in \mathbb{Z}$, and $\tau \in \mathfrak{T}_E$ then on the one hand

$$S_\mu S_\nu(\xi_\tau^n) = \begin{cases} S_\mu(\xi_{\nu\tau}^{l(\nu)+n}) & \text{if } r(\nu) = s(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

$$= \begin{cases} \xi_{\mu\nu\tau}^{l(\mu)+l(\nu)+n} & \text{if } r(\mu) = s(\nu\tau) \text{ and } r(\nu) = s(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$S_{\mu\nu}(\xi_\tau^n) = \begin{cases} \xi_{\mu\nu\tau}^{l(\mu)+l(\nu)+n} & \text{if } r(\mu\nu) = s(\tau), \\ 0 & \text{otherwise.} \end{cases}$$

So

$$S_\mu S_\nu = \begin{cases} S_{\mu\nu} & \text{if } r(\mu) = s(\nu), \\ 0 & \text{otherwise.} \end{cases}$$

Similarly to this case also it can be proved that

$$S_\mu S_\nu^* = \begin{cases} S_{\mu\nu^*} & \text{if } \ r(\mu) = r(\nu), \\ 0 & \text{otherwise} \end{cases}$$

and

$$S_\mu^* S_\nu = \begin{cases} S_{\mu^*\nu} & \text{if } \ s(\mu) = s(\nu), \\ 0 & \text{otherwise.} \end{cases}$$

Considering μ, ν also as vertices (paths of length zero), conditions (1) and (3) from Definition 2.1.1 are satisfied. Condition (2) follows from the fact that for any vertex v, $S_v^* = S_{v^*} = S_v$. Again for any edges $e, f \in E^1$, $S_e^* S_e = S_{e^*e} = S_{r(e)}$ and if $e \neq f$, $S_e^* S_f = S_{e^*f} = 0$ having condition (4). It remains to prove equation (5) from Definition 2.1.1: consider v a regular vertex, then for any $n \in \mathbb{Z}$ and $\tau \in \mathfrak{T}_E$,

$$\sum_{\{e \in E^1, \ s(e)=v\}} S_e S_e^*(\xi_\tau^n) = \begin{cases} \xi_\tau^n & \text{if } \ e \leq \tau \text{ for some } e \in s^{-1}(v), \\ 0 & \text{otherwise;} \end{cases}$$

and also

$$S_v(\xi_\tau^n) = \begin{cases} \xi_\tau^n & \text{if } \ v = s(\tau), \\ 0 & \text{otherwise;} \end{cases}$$

which means $(S_v - \sum_{\{e \in E^1, \ s(e)=v\}} S_e S_e^*)(\xi_\tau^n) = 0$ obtaining condition (5).

Let us define now the representation $\Pi_{\text{ap}} : L_R(E) \to \mathcal{E}$. We know that every $x \in L_R(E)$ can be written as $x = \sum_{i=1}^n r_i \alpha_i \beta_i^*$ where $r_i \in R$, $\alpha_i, \beta_i \in \text{Path}(E)$ for $i = 1, \ldots, n$. For every $\alpha\beta^*$ of the previous form let $\Pi_{\text{ap}}(\alpha\beta^*) = S_{\alpha\beta^*}$ and extend the definition R-linearly for every $x \in L_R(E)$. Let us check that Π_{ap} is a well-defined homomorphism: suppose we have $\alpha\beta^* = \alpha'\beta'^* \neq 0$; then $r(\alpha\beta^*) = r(\alpha'\beta'^*)$ and $l(\alpha\beta^*) = l(\alpha'\beta'^*)$ so $S_{\alpha\beta^*} = S_{\alpha'\beta'^*}$ which gives us $\Pi_{\text{ap}}(\alpha\beta^*) = \Pi_{\text{ap}}(\alpha'\beta'^*)$ and it is well-defined. Besides, it is R-linear and assume that we have $\alpha\beta^*$ and $\mu\nu^*$ such that $\alpha\beta^* \cdot \mu\nu^* \neq 0$; by Proposition 2.1.3, we can say that $\alpha\beta^* \cdot \mu\nu^* = \alpha\mu'\nu^*$ or $\alpha\beta^* \cdot \mu\nu^* = \alpha\nu\beta'^*$ for some $\mu', \beta' \in \text{Path}(E)$. Assume that the first option $\alpha\beta^* \cdot \mu\nu^* = \alpha\mu'\nu^*$ holds (the second option is done similarly) and then applying that $\Sigma = (S_\alpha)_{\alpha\in \mathrm{Path}(E)}$ is a Cuntz-Krieger E-system and Proposition 2.1.3, we have:

$$\begin{aligned}\Pi_{\mathrm{ap}}(\alpha\beta^* \cdot \mu\nu^*) &= \Pi_{\mathrm{ap}}(\alpha\mu'\nu^*) = S_{\alpha\mu'\nu^*} = S_{\alpha\mu'}S_\nu^* \\ &= (S_\alpha S_\beta^*)(S_\mu S_\nu^*) = S_{\alpha\beta^*}S_{\mu\nu^*} = \Pi_{\mathrm{ap}}(\alpha\beta^*)\Pi_{\mathrm{ap}}(\mu\nu^*).\end{aligned}$$

To prove that Π_{ap} is a $\mathbb{Z}$-graded homomorphism, consider $n \in \mathbb{Z}$ and $\alpha\beta^* \in L_R(E)_n$, which means $l(\alpha) - l(\beta) = n$. Then $\Pi_{\mathrm{ap}}(\alpha\beta^*) = S_{\alpha\beta^*} = S_\alpha S_\beta^* \in \mathcal{E}_{l(\alpha)}\mathcal{E}_{-l(\beta)} \subseteq \mathcal{E}_{l(\alpha)-l(\beta)} = \mathcal{E}_n$. Hence we obtain,

$$\Pi_{\mathrm{ap}}(L_R(E)_n) \subseteq \mathcal{E}_n.$$

Finally, let us see that Π_{ap} is injective. By Lemma 2.2.7, for every vertex $v \in E^0$ there exists at least one essentially aperiodic trail τ with $v = s(\tau)$. Then $S_v(\xi_\tau^n) = \xi_\tau^n$ for every $n \in \mathbb{Z}$ implying that $\Pi_{\mathrm{ap}}(rv) \neq 0$ for all $r \in R \setminus \{0\}$. Apply Theorem 2.1.5 to obtain the injectivity of the essentially aperiodic representation. □

Remark 2.2.10. Consider M the R-module generated by the set $\{\xi_\tau^n : n \in \mathbb{Z}, \tau \in \mathfrak{T}_E\}$ as in Proposition 2.2.8. Given a path $\alpha \in \mathrm{Path}(E)$, let us define $P_\alpha \in \mathcal{E}$, $P_\alpha : M \to M$ such that: for every $\tau \in \mathfrak{T}_E$ and every $n \in \mathbb{Z}$,

$$P_\alpha(\xi_\tau^n) = \begin{cases} \xi_\tau^n & \text{if } \alpha \leq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Take into account that we have $P_\alpha = S_\alpha S_\alpha^*$.

Given $\tau \in \mathfrak{T}_E$ an essentially aperiodic trail, let us define the projection $Q_\tau \in \mathcal{E}$, $Q_\tau : M \to M$ such that: for every $\tau' \in \mathfrak{T}_E$ and every $n \in \mathbb{Z}$,

$$Q_\tau(\xi_{\tau'}^n) = \begin{cases} \xi_\tau^n & \text{if } \tau' = \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Using this notation we can affirm:

(A) Given a trail $\tau \in \mathfrak{T}_E$, for any $m \in M$ there exists some integer $N \geq 0$ such that for every $n \geq N$,

$$P_{\tau_{(n)}}(m) = Q_\tau(m).$$

In particular if τ is discrete we could take $N = l(\tau_{(\text{ess})})$ since for any $m \in M$ and for every $n \geq l(\tau_{(\text{ess})})$, $P_{\tau_{(n)}}(m) = P_{\tau_{(\text{ess})}}(m) = Q_\tau(m)$. By the injectivity of Π_{ap} the same holds in $L_R(E)$: $\tau_{(n)}\tau^*_{(n)} = \tau_{(\text{ess})}\tau^*_{(\text{ess})}$ for all $n \geq l(\tau_{(\text{ess})})$.

(B) Of course when we restrict the essentially aperiodic representation Π_{ap} to the diagonal $\Delta(E)$ is injective. Suppose $\alpha \in \text{Path}(E)$ is such that $\alpha\alpha^*$ is one of the generators for $\Delta(E)$, then $\Pi_{\text{ap}}(\alpha\alpha^*) = P_\alpha$, so therefore

$$Q_\tau \Pi_{\text{ap}}(\alpha\alpha^*) = \Pi_{\text{ap}}(\alpha\alpha^*)Q_\tau = \begin{cases} Q_\tau & \text{if } \ \alpha \leq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

Then for every $x \in \Delta(E)$ and every $\tau \in \mathfrak{T}_E$ there exists a unique scalar $\varepsilon_\tau(x) \in R$ such that

$$Q_\tau \Pi_{\text{ap}}(x) = \Pi_{\text{ap}}(x)Q_\tau = \varepsilon_\tau(x)Q_\tau.$$

(C) It is straightforward that Q_τ is idempotent and all elements in the collection $(Q_\tau)_{\tau\in\mathfrak{T}_E}$ are mutually orthogonal. Furthermore it satisfies that, for any $m \in M$, there exists a unique minimal finite subset $\{\tau_1, \dots, \tau_n\}$ of $\mathfrak{T}_E$ such that

$$(\sum_{i=1}^{n} Q_{\tau_i})(m) = m.$$

In fact if $m = \sum_{i,j} r_{ij}\xi^{t_j}_{\tau_i}$ with $1 \leq i \leq n, 1 \leq j \leq m$, $r_{ij} \in R$, $\tau_i \in \mathfrak{T}_E$ and $t_j \in \mathbb{Z}$ then $(\sum_{i=1}^{n} Q_{\tau_i})(m) = m$ and for any subset $\{\tau'_1, \dots, \tau'_r\}$ of $\mathfrak{T}_E$ with $(\sum_{i=1}^{r} Q_{\tau'_i})(m) = m$ we have $\{\tau_1, \dots, \tau_n\} \subseteq \{\tau'_1, \dots, \tau'_r\}$.

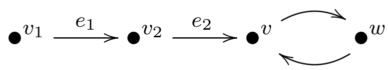
2.3 The commutative core

In this section we describe a commutative subalgebra inside $L_R(E)$, the so-called "commutative core". Previously we need a few definitions and a lemma that will be used in Proposition 2.3.6.

Definition 2.3.1. For any infinite discrete essentially aperiodic trail (Definition 2.2.5(B)(ii)) which is parameterized by the seed (α, λ_α) of the trail (that is, $\alpha \in \mathrm{Path}(E)$ is its essential head and $r(\alpha)$ is visited by the cycle without exits λ_α), the path α will be called a *distinguished* path. In the case $l(\alpha) = 0$, we will call it a *distinguished* vertex.

Remark 2.3.2. Consider a distinguished path α. This means that if $l(\alpha) = 0$ then α is simply a vertex v and the cycle λ_α starts and ends at v. In the case $l(\alpha) \geq 1$, suppose $\alpha = e_1 \dots e_n$, then λ_α is a cycle that starts and ends at $r(\alpha) = r(e_n)$ but does not visit any of the vertices $s(e_1), \dots, s(e_n)$. Of course, for any distinguished path α, $r(\alpha)$ is a distinguished vertex.

Example 2.3.3. For instance, $\alpha = e_1 e_2$ is a distinguished path in the following graph:



Lemma 2.3.4. *A cycle λ has no exits if and only if $\lambda\lambda^* = s(\lambda)$.*

Proof. We proceed similarly to [50, Lemma 3.2]. Suppose $k = l(\lambda)$. Then λ has an exit if and only if there is a path $\mu \neq \lambda$ such that $s(\mu) = s(\lambda)$, and either

(i) $l(\mu) < k$ and $r(\mu)$ is a sink, or

(ii) $l(\mu) = k$.

Since $\mu\mu^* \neq 0$, the result follows from Proposition 2.1.2. □

Definition 2.3.5. An element $x \in L_R(E)$ is said to be *normal* if $xx^* = x^*x$.

Proposition 2.3.6. *Let $\alpha, \beta \in Path(E)$ with $r(\alpha) = r(\beta)$. The generator $\alpha\beta^* \in G_E$ is a normal element in $L_R(E)$ if and only if one of the following holds:*

(a) $\alpha = \beta$;

(b) $\beta \leq \alpha$ and β is a distinguished path, i.e., $\alpha = \beta\lambda_\beta$ and λ_β is a cycle without exits;

(c) *$\alpha \leq \beta$ and α is a distinguished path, i.e., $\beta = \alpha\lambda_\alpha$ and λ_α is a cycle without exits.*

Also, if we denote by G_E^M the set of all such normal generators, then the R-algebra $M_R(E)$ generated by G_E^M, $M_R(E) = < G_E^M > \subseteq L_R(E)$ is commutative.

Definition 2.3.7. The subalgebra $M_R(E)$ given in Proposition 2.3.6 is called the *commutative core* of $L_R(E)$.

Proof of Proposition 2.3.6. The necessary and sufficient condition for normality is proved in a similar way by following the ideas given in [50, Proposition-Definition 3.1]. We will include here the proof for completeness. Consider first $x = \alpha\beta^*$ satisfying one of the conditions (a), (b) or (c). If $\alpha = \beta$ then x is clearly normal. Suppose $\beta \leq \alpha$ (the case $\alpha \leq \beta$ is done similarly). We have $\alpha = \beta\lambda$ and λ is a cycle without exits; by Lemma 2.3.4 $\lambda\lambda^* = s(\lambda) = r(\beta)$ so

$$xx^* = \beta\lambda\beta^*\beta\lambda^*\beta^* = \beta\lambda\lambda^*\beta^* = \beta\beta^* = \beta\alpha^*\alpha\beta^* = x^*x.$$

For the converse implication suppose now that $x = \alpha\beta^*$ is a non-zero normal element of $L_R(E)$. Since $x \neq 0$ then $r(\alpha) = r(\beta)$ and since $xx^* = x^*x$ then $s(\alpha) = s(\beta)$. We have $(xx^*)^2 \neq 0$ but $(xx^*)^2 = x^*xxx^*$ so x^2 is non-zero. But having $x^2 = (\alpha\beta^*)(\alpha\beta^*) \neq 0$ implies that $\beta \leq \alpha$ or $\alpha \leq \beta$ by Proposition 2.1.3. We can assume that $\beta \leq \alpha$. Considering all these facts together necessarily $\alpha = \beta\lambda$ where λ is a closed path. If we suppose that λ has an exit then by Lemma 2.3.4, $\lambda\lambda^* \neq s(\lambda) = r(\beta)$ and finally

$$xx^* = \beta\lambda\lambda^*\beta^* \neq \beta\beta^* = x^*x,$$

which is a contradiction.

In order to prove the commutativity of $M_R(E)$, we are going to check the following items:

(i) Elements of the form (a) commute: this is clear since $\alpha = \beta$ then $\alpha\alpha^* \in G_E^\Delta$.

(ii) Elements of the form (a) commute with elements of the form (b): let $x = \alpha\alpha^*$ of the form (a) and $y = \mu\nu^*$ of the form (b). We know that $\nu \leq \mu$ and $\mu = \nu\lambda_\nu$ where λ_ν is a cycle without exits. Let us see that $xy = yx$. First we show that

$$xy = 0 \Leftrightarrow yx = 0.$$

By Proposition 2.1.3 we have

$$yx = \mu\nu^*\alpha\alpha^* = \begin{cases} \mu\alpha'\alpha^* & \text{if } \alpha = \nu\alpha' \text{ for some } \alpha' \in \text{Path}(E), \\ \mu(\alpha\nu')^* & \text{if } \nu = \alpha\nu' \text{ for some } \nu' \in \text{Path}(E), \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha = \nu\alpha'$ for some $\alpha' \in \text{Path}(E)$, $yx = \nu\lambda_\nu\alpha'\alpha^*$. We know λ_ν is a cycle without exits so then α' is either a subpath of the cycle λ_ν, i.e., $\alpha' \leq \lambda_\nu$, or $\alpha' = \lambda_\nu^t$ for some $t \in \mathbb{N}$. Then $xy = \nu\alpha'\alpha'^*\nu^*\nu\lambda_\nu\nu^* = \nu\alpha'\alpha'^*\lambda_\nu\nu^* \neq 0$.

If $\nu = \alpha\nu'$ then $yx = \mu\nu'^*\alpha^* = \nu\lambda_\nu\nu'^*\alpha^*$. But λ_ν is a cycle without exits so ν' is either $\nu' \leq \lambda_\nu$ or $\nu' = \lambda_\nu^t$ for some t. Then $xy = \alpha\alpha^*\mu\nu^* = \alpha\alpha^*\nu\lambda_\nu\nu'^*\alpha^* = \alpha\alpha^*\alpha\nu'\lambda_\nu\nu'^*\alpha^* = \alpha\nu'\lambda_\nu\nu'^*\alpha^* \neq 0$.

So if $yx \neq 0$ then $xy \neq 0$, which is equivalent to say that if $xy = 0$ we have $yx = 0$. The case $yx = 0$ implies $xy = 0$ can be done similarly.

Now by Proposition 2.1.3 the condition $xy \neq 0$ (which implies $yx \neq 0$) holds if and only if $\alpha = \mu\alpha'$ for some $\alpha' \in \text{Path}(E)$ or $\mu = \alpha\mu'$ for some $\mu' \in \text{Path}(E)$.

Suppose that $\alpha = \mu\alpha'$, then $xy = \alpha\alpha^*\mu\nu^* = \alpha\alpha'^*\mu^*\mu\nu^* = \alpha\alpha'^*\nu^*$ and $\alpha = \nu\lambda_\nu\alpha'$. We have that λ_ν is a cycle without exits, so $\alpha = \nu\lambda_\nu^t\lambda'$ where $t \in \mathbb{N}$ and $\lambda' \leq \lambda_\nu$. Then $xy = \nu\lambda_\nu^t\lambda'(\lambda_\nu^{t-1}\lambda')^*\nu^* = \nu\lambda_\nu^t\lambda'\lambda'^*(\lambda_\nu^{t-1})^*\nu^*$. Now notice that $\lambda'\lambda'^* = s(\lambda_\nu)$: we know that λ_ν is a cycle without exits and $\lambda' \leq \lambda_\nu$ so $\lambda_\nu = \lambda'\lambda''$ for certain $\lambda'' \in \text{Path}(E)$; by Lemma 2.3.4 $s(\lambda_\nu) = \lambda_\nu\lambda_\nu^* = \lambda'\lambda''(\lambda'\lambda'')^* = \lambda'\lambda''\lambda''^*\lambda'^*$ and consequently $s(\lambda_\nu) = \lambda'^*s(\lambda_\nu)\lambda' = \lambda''\lambda''^*$ which in the end gives us $s(\lambda_\nu) = \lambda'(\lambda''\lambda''^*)\lambda'^* = \lambda'\lambda'^*$. So then $xy = \nu\lambda_\nu^t(\lambda_\nu^{t-1})^*\nu^* = \nu\lambda_\nu\nu^*$ by using Lemma 2.3.4 again.

On the other hand, $yx = \mu\nu^*\alpha\alpha^* = \nu\lambda_\nu\nu^*\nu\lambda_\nu^t\lambda'\lambda'^*(\lambda_\nu^t)^*\nu^*$. Again we have that $\lambda_\nu^t\lambda'\lambda'^*(\lambda_\nu^t)^* = s(\lambda_\nu)$ and finally

$$yx = \nu\lambda_\nu\nu^*\nu\nu^* = \nu\lambda_\nu\nu^* = xy.$$

Now assume that $\mu = \alpha\mu'$ for some $\mu' \in \text{Path}(E)$, then by Proposition 2.1.3, $xy = \alpha\mu'\nu^* = \mu\nu^*$. Since $yx \neq 0$, then by Proposition 2.1.3, $\alpha = \nu\alpha''$ for some $\alpha'' \in \text{Path}(E)$ or $\nu = \alpha\nu'$ for some $\nu' \in \text{Path}(E)$. In case $\alpha = \nu\alpha''$, $yx = \mu\alpha''\alpha^*$. Now λ_ν is a cycle without exits so $\alpha'' = \lambda_\nu^t\lambda'$ where $t \in \mathbb{N}$ and $\lambda' \leq \lambda_\nu$; a similar argument to the previous paragraph yields to $xy = yx$. If $\nu = \alpha\nu'$, then by Proposition 2.1.3, $yx = \mu(\alpha\nu')^* = \mu\nu^* = xy$. In the end we have $xy = yx$ as desired.

(iii) Elements of the form (b) commute: consider both $x = \alpha\beta^*$ and $y = \mu\nu^*$ of the form (b), that is, $\beta \leq \alpha$ with $\alpha = \beta\lambda_\beta$ and $\nu \leq \mu$ with $\mu = \nu\lambda_\nu$ where λ_β and λ_ν are cycles without exits. Then arguing similarly to the case (ii) we have that

$$xy = 0 \Leftrightarrow yx = 0.$$

By Proposition 2.1.3 $xy \neq 0$ (which implies $yx \neq 0$) if $\beta \leq \nu$ or $\nu \leq \beta$; but except when $\beta = \nu$, the remaining options are not possible since β and ν are distinguished paths. In this case then $xy = yx$.

Analogously to the previous steps it can be proved that:

(ii)' elements of the form (a) commute with elements of the form (c);

(iii)' elements of the form (b) commute with elements of the form (c) and

(iii)" elements of the form (c) commute.

Finally we get that $M_R(E)$ is commutative. □

The following definition is an important tool in order to prove that $M_R(E)$ is a maximal commutative subalgebra of $L_R(E)$. This will be the next goal of this section.

Definition 2.3.8. Let $\mathcal{A}$ be an R-algebra and $\mathcal{B} \subseteq \mathcal{A}$ an R-subalgebra. A linear map $\mathbb{E} : \mathcal{A} \to \mathcal{B}$ is called an *algebraic conditional expectation of $\mathcal{A}$ onto $\mathcal{B}$* if it satisfies the following conditions:

(i) $\mathbb{E}$ is idempotent and $\mathrm{Im}(\mathbb{E}) = \mathcal{B}$.

(ii) For every $a \in \mathcal{A}$ and $b \in \mathcal{B}$, $\mathbb{E}(ba) = b\mathbb{E}(a)$ and $\mathbb{E}(ab) = \mathbb{E}(a)b$.

For our purposes we need to find an appropriate algebraic conditional expectation of the R-algebra $\mathcal{E}$ defined in Proposition 2.2.8.

Definition 2.3.9. For every $T \in \mathcal{E}$ let us define the map $E_{\text{ap}} : \mathcal{E} \to \mathcal{E}$, $T \mapsto E_{\text{ap}}(T)$ as follows: by Remark 2.2.10 (C), for any $m \in M$ there exists a unique minimal finite subset $\{\tau_1, \ldots, \tau_n\}$ of $\mathfrak{T}_E$ such that $(\sum_{i=1}^n Q_{\tau_i})(m) = m$, we define

$$E_{\text{ap}}(T)(m) := (\sum_{i=1}^{n} Q_{\tau_i} T Q_{\tau_i})(m).$$

Proposition 2.3.10. *Let $\mathcal{Q} = \{Q_\tau\}_{\tau \in \mathfrak{T}_E} \subset \mathcal{E}$. Then the map E_{ap} is an algebraic conditional expectation of $\mathcal{E}$ onto $\mathcal{Q}'$ where*

$$\mathcal{Q}' = \{T \in \mathcal{E} : TQ_\tau = Q_\tau T \text{ for every } \tau \in \mathfrak{T}_E\}.$$

Proof. We need to check the conditions from the definition of algebraic conditional expectation above.

(i) $(E_{\text{ap}})^2 = E_{\text{ap}}$ follows from the fact that the elements in the collection $(Q_\tau)_{\tau \in \mathfrak{T}_E}$ are mutually orthogonal idempotents. Let us see that $\mathrm{Im}(E_{\text{ap}}) = \mathcal{Q}'$:

Consider $T \in \mathrm{Im}(E_{\text{ap}})$, that is, $T = E_{\text{ap}}(T')$ for some $T' \in \mathcal{E}$. For $m \in M$ there exist $\{\tau_1, \ldots, \tau_n\} \subset \mathfrak{T}_E$ such that $(\sum_{i=1}^n Q_{\tau_i})(m) = m$, so we have $T(m) = E_{\text{ap}}(T')(m) = (\sum_{i=1}^n Q_{\tau_i} T' Q_{\tau_i})(m)$. Let us check that $T \in \mathcal{Q}'$: consider Q_τ for some $\tau \in \mathfrak{T}_E$ then

$$(Q_\tau T)(m) = Q_\tau((\sum_{i=1}^{n} Q_{\tau_i} T' Q_{\tau_i})(m)) = (\sum_{i=1}^{n} Q_\tau Q_{\tau_i} T' Q_{\tau_i})(m).$$

Since the elements in the collection $(Q_\tau)_{\tau\in\mathfrak{T}_E}$ are mutually orthogonal idempotents,

$$T((Q_\tau)(m)) = (Q_\tau T' Q_\tau)(Q_\tau(m)) = Q_\tau T' Q_\tau(m).$$

Also since $(\sum_{i=1}^n Q_{\tau_i})(m) = m$, then $Q_\tau(m) = 0$ if $\tau \neq \tau_i$ for each i. In any case since the collection $(Q_\tau)_{\tau\in\mathfrak{T}_E}$ consists of mutually orthogonal idempotents,

$$(TQ_\tau)(m) = (Q_\tau T)(m) = \begin{cases} Q_\tau T' Q_\tau(m) & \text{if } \tau = \tau_i \text{ for some } i = 1, .., n \\ 0 & \text{otherwise.} \end{cases}$$

Conversely consider $T \in \mathcal{Q}'$. For $m \in M$ then again there exist $\{\tau_1, \dots, \tau_n\} \subset \mathfrak{T}_E$ such that $(\sum_{i=1}^n Q_{\tau_i})(m) = m$. So

$$E_{\text{ap}}(T)(m) = (\sum_{i=1}^n Q_{\tau_i} T Q_{\tau_i})(m) = (\sum_{i=1}^n T Q_{\tau_i} Q_{\tau_i})(m) =$$
$$= (\sum_{i=1}^n T Q_{\tau_i})(m) = T(\sum_{i=1}^n Q_{\tau_i})(m) = T(m).$$

Which means that $T \in \text{Im}(E_{\text{ap}})$.

(ii) Let $T' \in \mathcal{Q}'$. For $T \in \mathcal{E}$ and for $m \in M$ there exist $\{\tau_1, \dots, \tau_n\} \subset \mathfrak{T}_E$ such that $(\sum_{i=1}^n Q_{\tau_i})(m) = m$. Then we have

$$E_{\text{ap}}(T'T)(m) = (\sum_{i=1}^n Q_{\tau_i} T' T Q_{\tau_i})(m) = (\sum_{i=1}^n T' Q_{\tau_i} T Q_{\tau_i})(m) =$$
$$= T'(\sum_{i=1}^n Q_{\tau_i} T Q_{\tau_i})(m) = T' E_{\text{ap}}(T)(m).$$

Similarly it can be shown that $E_{\text{ap}}(TT')(m) = E_{\text{ap}}(T)T'(m)$. □

Theorem 2.3.11. *There exists a unique algebraic conditional expectation E_M of $L_R(E)$ onto $M_R(E)$ such that*

$$E_M(x) = \begin{cases} x & \textit{if } x \in G_E^M, \\ 0 & \textit{if } x \in G_E \setminus G_E^M. \end{cases}$$

Furthermore, for the essentially aperiodic representation $\Pi_{\text{ap}} : L_R(E) \to \mathcal{E}$ and the algebraic conditional expectation $E_{\text{ap}} : \mathcal{E} \to \mathcal{E}$ we have

$$\Pi_{\text{ap}} \circ E_M = E_{\text{ap}} \circ \Pi_{\text{ap}}.$$

Proof. First let us prove the following claims:

(a) $E_{\text{ap}}(\Pi_{\text{ap}}(x)) = \Pi_{\text{ap}}(x)$ for every $x \in M_R(E)$.
Consider the collection $\mathcal{P} = \{P_\alpha\}_{\alpha \in \text{Path}(E)} = \{\Pi_{\text{ap}}(\alpha\alpha^*)\}_{\alpha \in \text{Path}(E)}$. By Remark 2.2.10(B), it is clear that $\mathcal{P} \subset \mathcal{Q}'$, where

$$\mathcal{Q}' = \{T \in \mathcal{E} : TQ_\tau = Q_\tau T \text{ for every } \tau \in \mathfrak{T}_E\}.$$

Suppose $T \in \mathcal{E}$ such that T commutes with all P_α, $\alpha \in \text{Path}(E)$. In particular for every $\tau \in \mathfrak{T}_E$ we have that $TP_{\tau_{(n)}} = P_{\tau_{(n)}}T$ for every $n \geq 0$. From Remark 2.2.10(A) we get $TQ_\tau = Q_\tau T$ which implies $\mathcal{P}' \subset \mathcal{Q}'$ where

$$\mathcal{P}' = \{T \in \mathcal{E} : TP_\alpha = P_\alpha T \text{ for every } \alpha \in \text{Path}(E)\}.$$

Therefore we have that

$$E_{\text{ap}}(T) = T \text{ for every } T \in \mathcal{P}'.$$

In particular for all $x \in M_R(E)$, it follows from Proposition 2.3.6 that $\Pi_{\text{ap}}(x) \in \mathcal{P}'$ so we get

$$E_{\text{ap}}(\Pi_{\text{ap}}(x)) = \Pi_{\text{ap}}(x), \text{ for every } x \in M_R(E).$$

(b) $E_{\text{ap}}(\Pi_{\text{ap}}(x)) = 0$ for every $x \in G_E \setminus G_E^M$.
Consider $x = \alpha\beta^* \in G_E \setminus G_E^M$, that is, $r(\alpha) = r(\beta)$. Let $\tau \in \mathfrak{T}_E$ then clearly $Q_\tau S_\alpha S_\beta^* Q_\tau \neq 0$ only if $\beta \leq \tau$ and $\alpha \leq \tau$. In any case then we will have that $\alpha \leq \beta$ or $\beta \leq \alpha$ and since $x \notin G_E^M$, $\alpha \neq \beta$. We can suppose the case $\beta \leq \alpha$; then it follows that $s(\alpha) = s(\beta)$ and necessarily $\alpha = \beta\lambda$ for some closed path λ. Since $\beta \leq \tau$ assume that $\tau = \beta\mu$ for some path μ. The fact that $Q_\tau S_\alpha S_\beta^* Q_\tau \neq 0$ implies that $\beta\lambda\mu = \alpha\mu = \tau = \beta\mu$, and this yields $\lambda\mu = \mu$ which means that τ is periodic with period λ. Since τ is essentially aperiodic then λ has no exits. Then $\mu = r(\lambda)$ and so $\lambda \in E^0$. Thus $\alpha = \beta$ which is impossible. Finally we have that then $E_{\text{ap}}(\Pi_{\text{ap}}(x)) = 0$ for every $x \in G_E \setminus G_E^M$.

So we have that $E_{\rm ap}(\Pi_{\rm ap}(G_E)) \subseteq \Pi_{\rm ap}(G_E)$. Then extending linearly we get that $E_{\rm ap}(\Pi_{\rm ap}(L_R(E))) \subseteq \Pi_{\rm ap}(L_R(E))$. Since $\Pi_{\rm ap}$ is injective by Proposition 2.2.8, there exists a unique linear map $E_M : L_R(E) \to L_R(E)$ such that $\Pi_{\rm ap} \circ E_M = E_{\rm ap} \circ \Pi_{\rm ap}$. Besides, by the injectivity of $\Pi_{\rm ap}$ and by (a) and (b) we have

$$E_M(x) = \begin{cases} x & \text{if } \; x \in G_E^M, \\ 0 & \text{if } \; x \in G_E \setminus G_E^M. \end{cases}$$

It then is immediate that E_M is an algebraic conditional expectation of $L_R(E)$ onto $M_R(E)$. □

We can prove now the main result of this section.

Theorem 2.3.12. *Let E be a graph and R a commutative ring with unit. Consider $M_R(E) \subseteq L_R(E)$. Then*

$$M_R(E) = \{x \in L_R(E) : xd = dx \textit{ for every } d \in \Delta(E)\}.$$

Furthermore, $M_R(E)$ is a maximal commutative $*$-subalgebra of $L_R(E)$.

Proof. For the equality $M_R(E) = \{x \in L_R(E) : xd = dx$ for every $d \in \Delta(E)\}$ we need to prove the double inclusion. First it is clear that

$$M_R(E) \subseteq \{x \in L_R(E) : xd = dx \text{ for every } d \in \Delta(E)\}$$

since $\Delta(E) \subset M_R(E)$ and $M_R(E)$ is commutative by Proposition 2.3.6.

For the other containment consider an element $x \in L_R(E)$ such that $xd = dx$ for every $d \in \Delta(E)$. Then for every path $\alpha \in \text{Path}(E)$ we know that $\alpha\alpha^* \in \Delta(E)$ and then

$$\Pi_{\rm ap}(x)P_\alpha = \Pi_{\rm ap}(x)\Pi_{\rm ap}(\alpha\alpha^*) = \Pi_{\rm ap}(x\alpha\alpha^*) = \Pi_{\rm ap}(\alpha\alpha^* x) = P_\alpha \Pi_{\rm ap}(x).$$

Following the proof of Theorem 2.3.11, we know that $E_{\rm ap}(T) = T$ for every $T \in \mathcal{P}'$ where

$$\mathcal{P}' = \{T \in \mathcal{E} : TP_\alpha = P_\alpha T \text{ for every } \alpha \in \text{Path}(E)\}.$$

In particular it follows that $\Pi_{\text{ap}}(x) = E_{\text{ap}}(\Pi_{\text{ap}}(x))$, using Theorem 2.3.11, $E_{\text{ap}}(\Pi_{\text{ap}}(x)) = \Pi_{\text{ap}}(E_M(x))$ and hence $\Pi_{\text{ap}}(x) = \Pi_{\text{ap}}(E_M(x))$. Since Π_{ap} is injective by Proposition 2.2.8, we have that $x = E_M(x) \in M_R(E)$ and we are done.

In order to prove that $M_R(E)$ is a maximal commutative subalgebra inside $L_R(E)$, consider $\mathcal{C}$ a commutative subalgebra of $L_R(E)$ such that $M_R(E) \subseteq \mathcal{C}$. Since we have $\Delta(E) \subseteq M_R(E) \subseteq \mathcal{C}$ then in particular

$$\mathcal{C} \subseteq \{x \in L_R(E) : xd = dx \text{ for every } d \in \Delta(E)\} = M_R(E).$$

So in the end necessarily $\mathcal{C} = M_R(E)$. □

Finally as a corollary the result concerning the center of a Leavitt path algebra, studied in [17] and [35], is obtained.

Corollary 2.3.13. *The center of the Leavitt path algebra $L_R(E)$, that is*

$$\mathcal{Z}(L_R(E)) = \{x \in L_R(E) : xy = yx \text{ for every } y \in L_R(E)\}$$

satisfies that $\mathcal{Z}(L_R(E)) \subseteq M_R(E)$.

2.4 Commutativity of Leavitt and Cohn path algebras

Once we have found the commutative core of $L_R(E)$, we are going to use it in order to study the commutative Leavitt path algebras from a different point of view. In particular we will extend the result given in [17, Proposition 2.7]. First we need the following proposition.

Proposition 2.4.1. *For every $x \in L_R(E)$ and $\tau \in \mathfrak{T}_E$. If τ is discrete then*

$$\tau_{(\text{ess})}\tau^*_{(\text{ess})}\ x\ \tau_{(\text{ess})}\tau^*_{(\text{ess})} = E_M(x)\ \tau_{(\text{ess})}\tau^*_{(\text{ess})}.$$

Proof. Since τ is discrete then by Remark 2.2.10(A) $P_{\tau_{(\text{ess})}} = Q_\tau$, and then we have:

$$\Pi_{\text{ap}}(\tau_{(\text{ess})}\tau^*_{(\text{ess})}x\tau_{(\text{ess})}\tau^*_{(\text{ess})}) = P_{\tau_{(\text{ess})}}\Pi_{\text{ap}}(x)P_{\tau_{(\text{ess})}} = Q_\tau\Pi_{\text{ap}}(x)Q_\tau.$$

On the other hand by using Theorem 2.3.11 and Proposition 2.3.10,

$$\Pi_{\mathrm{ap}}(E_M(x)\tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})}) = \Pi_{\mathrm{ap}}(E_M(x))Q_\tau = E_{\mathrm{ap}} \circ \Pi_{\mathrm{ap}}(x)Q_\tau = Q_\tau \Pi_{\mathrm{ap}}(x)Q_\tau.$$

So we get $\Pi_{\mathrm{ap}}(\tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})}x\tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})}) = \Pi_{\mathrm{ap}}(E_M(x)\tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})})$. Finally it is sufficient to apply Proposition 2.2.8; since Π_{ap} is injective then the statement follows. □

Remark 2.4.2. For a distinguished path α let $\omega_\alpha = \alpha\lambda_\alpha\alpha^*$, where λ_α is the cycle without exits that starts and ends at $r(\alpha)$. We know by Proposition 2.3.6 that $\omega_\alpha \in G_E^M$. It so happens that the R-algebra $< \omega_\alpha > \subseteq M_R(E)$ generated by ω_α is unital (with unit $\alpha\alpha^*$) and ω_α is invertible in $< \omega_\alpha >$:

$$\omega_\alpha^*\omega_\alpha = \omega_\alpha\omega_\alpha^* = \alpha\alpha^*.$$

We can define then the powers ω_α^n for every $n \in \mathbb{Z}$: if $n < 0$ we let $\omega_\alpha^n := (\omega_\alpha^*)^{-n}$ and $\omega_\alpha^0 := \alpha\alpha^*$. In this case it follows that there exists a unique $*$-isomorphism $\Gamma_\alpha : < \omega_\alpha > \to R[x, x^{-1}]$ by simply defining $\Gamma_\alpha(\alpha\alpha^*) = 1$, $\Gamma_\alpha(\omega_\alpha) = x$ and $\Gamma_\alpha(\omega_\alpha^*) = x^{-1}$.

Notice that we can write G_E^M as a disjoint union

$$G_E^M = G_E^\Delta \cup \{\omega_\alpha^n : \alpha \text{ is a distinguished path, } n \neq 0\}.$$

Remark 2.4.3. Let $\tau \in \mathfrak{T}_E$ be a discrete trail. We can define a corner R-subalgebra inside $L_R(E)$ as follows:

$$M_\tau(E) = \tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})}\ L_R(E)\ \tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})}.$$

By Proposition 2.4.1, $M_\tau(E)$ is contained in $M_R(E)$ and concretely we have:

(i) if τ is finite, then $M_\tau(E) = < \tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})} >$ the R-subalgebra generated by $\tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})}$. In this case we have a natural $*$-isomorphism $< \tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})} > \cong R$, by simply defining $\tau_{(\mathrm{ess})}\tau^*_{(\mathrm{ess})} \mapsto 1$.

(ii) if τ is infinite, then $M_\tau(E) = < \omega_{\tau_{(\mathrm{ess})}} >$. By Remark 2.4.2 we know that there exists a $*$-isomorphism $< \omega_{\tau_{(\mathrm{ess})}} > \cong R[x, x^{-1}]$.

Furthermore from the proof of Proposition 2.3.6, we have that the collection $\{M_\tau(E)\}$, for τ discrete, are pairwise orthogonal: if $\tau_1 \neq \tau_2$ are discrete then

$$M_{\tau_1}(E) \cdot M_{\tau_2}(E) = \{0\}. \tag{2.3}$$

Denote $M_{\text{disc}}(E) \subset M_R(E)$ the R-algebra generated by $\bigcup_{\tau \text{ discrete}} M_\tau(E)$, then taking into account equation (2.3) together with previous (i) and (ii) we have that

$$M_{\text{disc}}(E) \cong \bigoplus_{\tau \text{ finite}} R \oplus \bigoplus_{\tau \text{ infinite}} R[x, x^{-1}],$$

up to $*$-isomorphism.

Let us see now the result about commutativity of Leavitt path algebras.

Corollary 2.4.4. *Let E be a graph and R a commutative ring with unit. The following conditions are equivalent:*

(i) $L_R(E)$ *is commutative; in particular* $L_R(E) = M_R(E)$.

(ii) *For the graph E, the maps r and s coincide and are injective.*

(iii) $E = \bigsqcup_{i \in I} E_i$ *where each subgraph E_i is an isolated vertex or an isolated loop.*

(iv)

$$L_R(E) \cong \bigoplus_{\tau \in \mathfrak{T}^{\text{f}}_{\text{disc}}} R \oplus \bigoplus_{\tau \in \mathfrak{T}^{\text{i}}_{\text{disc}}} R[x, x^{-1}],$$

where $\mathfrak{T}^{\text{f}}_{\text{disc}}$ and $\mathfrak{T}^{\text{i}}_{\text{disc}}$ denote respectively the set of all finite and infinite discrete essentially aperiodic trails of E.

Proof. (i)⇒(ii) Consider an edge $e \in E^1$. Then since $L_R(E) = M_R(E)$, we can write by Proposition 2.3.6, $e = \sum_{i=1}^n r_i \alpha_i \beta_i^*$ where $\alpha_i \beta_i^* \in G_E^M$ and $r_i \in R \setminus \{0\}$ for all $i = 1, \dots, n$. Since $\alpha_i \beta_i^* \in G_E^M$ then $s(\alpha_i) = s(\beta_i)$. But

$$e = s(e)e = s(e)(\sum_{i=1}^n r_i \alpha_i \beta_i^*) = \sum_{i=1}^n r_i s(e) \alpha_i \beta_i^* \Rightarrow s(e) = s(\alpha_i) \text{ for all } i.$$

$$e = er(e) = (\sum_{i=1}^{n} r_i \alpha_i \beta_i^*) r(e) = \sum_{i=1}^{n} r_i \alpha_i \beta_i^* r(e) \Rightarrow r(e) = s(\beta_i) \text{ for all } i.$$

So we get that necessarily $s(e) = r(e)$; then r and s coincide.

Let us prove the injectivity. We know that for every $e \in E^1$, $s(e) = r(e)$. This means that e is a loop. Suppose we have two different loops μ, ν such that $r(\mu) = r(\nu)$; then $\mu\nu \neq \nu\mu$ which is a contradiction since $L_R(E)$ is commutative.

(ii)⇒(iii) It is straightforward.

(iii)⇒(iv) Suppose E is a collection of isolated vertices and isolated loops. Then it is immediate that $L_R(E)$ is commutative; note that the product of any two different isolated vertices is zero, the product of an isolated vertex and an isolated loop is zero too, and the same for the product of two different isolated loops. So by Theorem 2.3.12, $L_R(E) = M_R(E)$.

On the other hand observe that by hypothesis in our graph E, any infinite path is periodic, or in other words, there are no continuous trails; so then $M_R(E) = M_{\text{disc}}(E)$. By Remark 2.4.3, $M_{\text{disc}}(E) \cong (\bigoplus_{\tau \in \mathfrak{T}^{\text{f}}_{\text{disc}}} R) \oplus (\bigoplus_{\tau \in \mathfrak{T}^{\text{i}}_{\text{disc}}} R[x, x^{-1}])$ where $\mathfrak{T}^{\text{f}}_{\text{disc}}$ is the set of all finite discrete essentially aperiodic trails and $\mathfrak{T}^{\text{i}}_{\text{disc}}$ is the set of all infinite discrete essentially aperiodic trails. Finally we get:

$$L_R(E) \cong \bigoplus_{\tau \in \mathfrak{T}^{\text{f}}_{\text{disc}}} R \oplus \bigoplus_{\tau \in \mathfrak{T}^{\text{i}}_{\text{disc}}} R[x, x^{-1}].$$

(iv)⇒(i) It is obvious. □

We now study the commutativity of *Cohn path algebras* taking advantage of all results we have obtained so far.

Definition 2.4.5. Let E be an arbitrary graph and R any commutative ring with unit. The *Cohn path algebra of E with coefficients in R*, denoted by $C_R(E)$, is the free associative R-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$, subject to the relations given in (1), (2), and (3) of Definition 1.1.1.

In other words, $C_R(E)$ is the algebra generated by the same symbols as those which generate $L_R(E)$, but on which we do not impose the (CK-2) relation. It can be proved that for any graph E the Cohn path algebra $C_R(E)$ is isomorphic to the Leavitt path algebra $L_R(F)$ of some graph F, concretely:

Definition 2.4.6. Let E be an arbitrary graph. Let $Y := E^0_{\text{reg}}$ and $Y' = \{v' : v \in Y\}$ be a disjoint copy of Y. For $v \in Y$ and for each edge $e \in r_E^{-1}(v)$, we consider a new symbol e'. We define the graph $F(E)$, as follows:

$$F(E)^0 = E^0 \cup Y' \text{ and } F(E)^1 = E^1 \cup \{e' : r_E(e) \in Y\}.$$

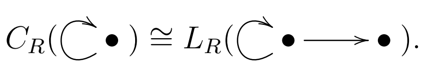
For $e \in E^1$ we define $s_{F(E)}(e) = s_E(e)$ and $r_{F(E)}(e) = r_E(e)$ and define $s_{F(E)}(e') = s_E(e)$ and $r_{F(E)}(e') = r_E(e)'$ for the new symbols e'.

Note that the graph $F(E)$ is built from E by adding a new vertex to E corresponding to each element of E^0_{reg}, and then including new edges to each of these new vertices as appropriate. Observe in particular that each of the new vertices $v' \in Y'$ is a sink in $F(E)$, so that $E^0_{\text{reg}} = F(E)^0_{\text{reg}}$.

As in [3, Theorem 1.5.18] we obtain the desired result about the isomorphism between the Cohn path algebra $C_R(E)$ and the Leavitt path algebra $L_R(F(E))$.

Theorem 2.4.7. ([3, Theorem 1.5.18]) *Let E be an arbitrary graph and R a commutative ring with unit. Then $C_R(E) \cong L_R(F(E))$ where $F(E)$ is the graph given in Definition 2.4.6.*

Example 2.4.8. For $n \geq 1$ consider R_n the rose with n petals graph. If $n = 1$ thus we get that $C_R(R_1)$ is exactly the Toeplitz algebra, which is given by the Leavitt path algebra of the graph having one loop and one exit:



The terminology "Cohn path algebra" owes to the fact that for each $n \geq 1$ and for K a field, the algebra $C_K(R_n)$ is precisely the algebra $U_{1,n}$ described and investigated by Cohn in [33].

In order to analyze the commutative Cohn path algebras we need the following property of the graph $F(E)$.

Lemma 2.4.9. *For every graph E, its corresponding graph $F(E)$ satisfies Condition (L).*

Proof. By way of contradiction suppose we have a graph E such that $F(E)$ has a cycle $c = e_1 \dots e_n$ without exits. Let $v_i = s_{F(E)}(e_i)$ for every $i = 1, \dots, n$. Since c has no exits then each v_i is a regular vertex in $F(E)$. Then $v_1, \dots, v_n$ are regular vertices in E. Consider $v_1 \in E^0$; for the vertex $v_1 \in E^0_{\text{reg}}$ we have the same $e_1 \in E^1$ such that $s_E(e_1) = v_1$. Now $r_E(e_1) = v_2$ and v_2 is regular in E, this means that we have an edge e_1' in $F(E)^1$ with $s_{F(E)}(e_1') = s_E(e_1) = v_1$ which is a contradiction since $e_1' \neq e_1$. □

Corollary 2.4.10. *Let E be a graph and R a commutative ring with unit. The following conditions are equivalent:*

(i) $C_R(E)$ is commutative.

(ii) $E = \bigsqcup_{i \in I} E_i$ where each subgraph E_i is an isolated vertex.

Proof. (i)⇒(ii) We know that $C_R(E) \cong L_R(F(E))$ by Theorem 2.4.7. Since $C_R(E)$ is commutative then $L_R(F(E))$ is commutative too. Using Corollary 2.4.4, this means that $F(E) = \bigsqcup_{i \in I} F_i$ where each subgraph F_i is an isolated vertex or an isolated loop. However by Lemma 2.4.9 it is not possible to have isolated loops inside $F(E)$, so necessarily $F(E) = \bigsqcup_{i \in I} F_i$ where each subgraph F_i is an isolated vertex. This means that then $E = \bigsqcup_{i \in I} E_i$ where each subgraph E_i is an isolated vertex.

(ii)⇒(i) It is straightforward; note that the product of two different vertices in $C_R(E)$ is zero. □

2.5 General uniqueness theorem for Leavitt path algebras

In the final section of this chapter we focus our attention on giving a general version of the uniqueness theorems for Leavitt path algebras mentioned in Section 1.2; concretely we will prove that we only need to check the injectivity of a homomorphism when we restrict to the commutative core $M_R(E)$.

Theorem 2.5.1. *Let E be a graph and R a commutative ring with unit. Consider $\Phi : L_R(E) \to \mathcal{A}$ a ring homomorphism. Then the following conditions are equivalent:*

(i) Φ *is injective;*

(ii) *the restriction of* Φ *to* $M_R(E)$ *is injective;*

(iii) *both these conditions are satisfied:*

(a) $\Phi(rv) \neq 0$*, for all* $v \in E^0$ *and for all* $r \in R \setminus \{0\}$*;*

(b) *for every distinguished path* α *the* $*$ *R-algebra* $< \Phi(\omega_\alpha) >$ *generated by* $\Phi(\omega_\alpha)$ *is* $*$*-isomorphic to* $R[x, x^{-1}]$*; that is,*

$$< \Phi(\omega_\alpha) > \cong R[x, x^{-1}].$$

Proof. Let us prove the equivalence between the statements.

• (i) $\Rightarrow$ (ii) It is obvious.

• (ii) $\Rightarrow$ (iii) First let us check (a). Consider rv, for some $v \in E^0$ and $r \in R \setminus \{0\}$. But $rv = rvv^* \in \Delta(E) \subset M_R(E)$, so it follows immediately.

Now in order to see (b), note that for every distinguished path α and each $i \in \mathbb{Z}$ then $\Phi(\omega_\alpha^i) \neq 0$: this is straightforward since every $\omega_\alpha \in M_R(E)$ by Proposition 2.3.6.

Then we have a natural $*$-isomorphism $< \omega_\alpha > \cong < \Phi(\omega_\alpha) >$. We know on the other hand that $< \omega_\alpha > \cong R[x, x^{-1}]$ is a $*$-isomorphism by Remark 2.4.2. It follows therefore that $< \Phi(\omega_\alpha) > \cong R[x, x^{-1}]$ as desired.

• (iii) $\Rightarrow$ (i) Suppose $0 \neq a \in L_R(E)$ is such that $a \in \text{Ker } \Phi$. It is a well-known fact that Ker Φ is an ideal of $L_R(E)$. By Theorem 1.2.1 there exist $\mu, \nu \in \text{Path}(E)$ satisfying one of the two following possibilities.

Assume $0 \neq \mu^* a \nu = rv$, for some $r \in R \setminus \{0\}$ and $v \in E^0$. But $a \in \text{Ker } \Phi$, so $rv \in \text{Ker } \Phi$ which is a contradiction with hypothesis (a).

So necessarily $0 \neq \mu^* a \nu = p(\lambda)$, where λ is a cycle without exits and $p(x)$ is a non-zero polynomial in $R[x, x^{-1}]$. Since $a \in \text{Ker } \Phi$ we have $p(\lambda) \in \text{Ker } \Phi$, so $\Phi(p(\lambda)) = 0$. Let $v \in \text{Vert}(\lambda)$, then v is a distinguished vertex and $\lambda = \lambda_v$, so by hypothesis (b) we have that $g :< \Phi(\lambda) >\to R[x, x^{-1}]$ is an algebra isomorphism. We know $0 = \Phi(p(\lambda))$, then $g(\Phi(p(\lambda))) = 0$. Assume that $p(\lambda) = r_{-m}\lambda^{-m} + \ldots + r_n\lambda^n$ for certain $m, n \in \mathbb{N}$. Then

$$\begin{aligned} 0 &= g\big(\Phi(p(\lambda))\big) \\ &= g\big(\Phi(r_{-m}v)\Phi(\lambda^*)^m + ... + \Phi(r_n v)\Phi(\lambda)^n\big) \\ &= g\big(\Phi(r_{-m}v)\big)x^{-m} + ... + g\big(\Phi(r_n v)\big)x^n. \end{aligned}$$

This means that $g(\Phi(r_i v)) = 0$ for all $i \in \{-m, \ldots, n\}$ which implies that $\Phi(r_i v) = 0$ for each i giving us a contradiction with (a). □

As a corollary we can have the Cuntz-Krieger uniqueness theorem. If the graph E satisfies Condition (L) then there are no distinguished paths in Path(E), so we get immediately:

Corollary 2.5.2. *Let E be a graph satisfying Condition (L) and let R be a commutative ring with unit. Suppose $\Phi : L_R(E) \to \mathcal{A}$ is a ring homomorphism. Then the following conditions are equivalent:*

(i) *Φ is injective;*

(ii) *the restriction of Φ to $M_R(E)$ is injective;*

(iii) *$\Phi(rv) \neq 0$, for all $v \in E^0$ and for all $r \in R \setminus \{0\}$.*

By using the proof of *(ii)* $\Rightarrow$ *(iii)* in the proof of Theorem 2.5.1 we can recover the graded uniqueness theorem:

Theorem 2.5.3. *Let E be a graph and R a commutative ring with unit. If $\mathcal{A}$ is a $\mathbb{Z}$-graded ring and $\Phi : L_R(E) \to \mathcal{A}$ is a $\mathbb{Z}$-graded ring homomorphism, then the following conditions are equivalent:*

(i) *Φ is injective;*

(ii) *the restriction of Φ to $M_R(E)$ is injective;*

(iii) *$\Phi(rv) \neq 0$, for all $v \in E^0$ and for all $r \in R \setminus \{0\}$.*

Finally by using Lemma 2.4.9 and Theorems 2.4.7, 2.5.1 we obtain a uniqueness theorem for Cohn path algebras as well.

Corollary 2.5.4. *Let E be a graph and R a commutative ring with unit. Consider $\Phi : C_R(E) \to \mathcal{A}$ a ring homomorphism. Then the following conditions are equivalent:*

(i) Φ *is injective;*

(ii) *the restriction of* Φ *to* $M_R(F(E))$ *is injective;*

(iii) $\Phi(rv) \neq 0$, *for all* $v \in F(E)^0$ *and for all* $r \in R \setminus \{0\}$, *where* $F(E)$ *is the graph given in Definition 2.4.6.*

Chapter 3

The cycline subalgebra of a Kumjian-Pask algebra

3.1 Generalization of Leavitt path algebras

In this chapter we study analogs of Leavitt path algebras associated to higher-rank graphs; these algebras are called *Kumjian-Pask algebras*. In particular our aim will be to extend the results given in the previous Chapter 2 to Kumjian-Pask algebras. Here we first give some necessary background which will be used later and we fix some notation too. After some basic notions on higher-rank graphs we give the definition of a Kumjian-Pask algebra.

Let k be a positive integer. We consider the additive semigroup $\mathbb{N}^k$ as a category with one object. We say a countable category $\Lambda = (\Lambda^0, \Lambda, r, s)$ with objects Λ^0, morphisms Λ, range map r and source map s, is a *k-graph* if there exists a functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the *unique factorization property*: if $d(\lambda) = m + n$ for some $m, n \in \mathbb{N}^k$, then there exist unique $\mu, \nu \in \mathbb{N}^k$ such that $r(\nu) = s(\mu)$ and $d(\mu) = m$, $d(\nu) = n$ with $\lambda = \mu\nu$. Since we think of Λ as a generalized graph, we call $\lambda \in \Lambda$ a *path* in Λ and $v \in \Lambda^0$ a *vertex*.

For $n \in \mathbb{N}^k$ define $\Lambda^n = d^{-1}(\{n\})$ and call the elements λ of Λ^n *paths of degree n*; by the factorization property we identify Λ^0 as the paths of degree 0 (or the set of vertices). For any $v \in \Lambda^0$ and $X \subseteq \Lambda$ we denote $vX = \{\lambda \in X \mid r(\lambda) = v\}$ and $Xv = \{\lambda \in X \mid s(\lambda) = v\}$. A k-graph Λ

has *no sources* if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ the set $v\Lambda^n$ is nonempty; Λ is *row-finite* if for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$ the set $v\Lambda^n$ is finite. In this chapter we are concerned only with row-finite k-graphs without sources. The case of not necessarily row-finite higher-rank graphs with sources is undertaken in [28, 32], but their study is significantly more technical.

Example 3.1.1. Let $E = (E^0, E^1, r, s)$ be a directed graph. Then the *path category* Λ_E has object set E^0, and the morphisms in Λ_E from $v \in E^0$ to $w \in E^0$ are finite paths μ with $s(\mu) = v$ and $r(\mu) = w$; composition is defined by concatenation, and the identity morphisms obtained by viewing the vertices as paths of length 0. With the degree functor $d : \Lambda_E \to \mathbb{N}$ as the length function, the path category (Λ_E, d) is a 1-graph. Note that this requires us to use the convention where a path is a sequence of edges $e_1 \dots e_n$ such that $s(e_i) = r(e_{i+1})$, i.e., the opposite as to what we have been doing, and we will be doing in the rest of the chapters of the thesis. This is to follow the normal convention in the theory of Kumjian-Pask algebras.

For $m, n \in \mathbb{N}^k$, $m \leq n$ means $m_i \leq n_i$ for all $1 \leq i \leq k$ and $m \vee n$ denotes the pointwise maximum.

Example 3.1.2. Let $\Omega_k^0 = \mathbb{N}^k$, $\Omega_k = \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q\}$, define $r, s : \Omega_k \to \Omega_k^0$ by $r(p,q) = p$ and $s(p,q) = q$, define composition by $(p,q)(q,r) = (p,r)$, and define $d : \Omega_k \to \mathbb{N}^k$ by $d(p,q) = q - p$. Then $\Omega_k = (\Omega_k, r, s, d)$ is a k-graph.

An *infinite path* in Λ is a degree-preserving functor $x : \Omega_k \to \Lambda$. We denote the set of all infinite paths by Λ^∞. We denote $x(0,0)$ by $r(x)$ and refer to this vertex as the *range* of x.

For $\mu \in \Lambda$ define the *cylinder set*

$$Z(\mu) := \{x \in \Lambda^\infty \mid x(0, d(\mu)) = \mu\}.$$

For $\alpha \in \Lambda$ and $x \in Z(s(\alpha))$, define αx to be the unique infinite path such that for any $n \geq d(\alpha)$, we have that $(\alpha x)(0,n) = \alpha(x(0, n - d(\alpha)))$. We denote

$Z(\alpha)$ by $\alpha\Lambda^\infty$ when α is a vertex. In fact, [43, Proposition 2.8] establishes that the collection of cylinder sets is a basis of compact sets for a Hausdorff topology on Λ^∞.

For $p \in \mathbb{N}^k$ we define a map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ by $\sigma^p(x)(m,n) = x(m+p, n+p)$ for every $x \in \Lambda^\infty$, $(m,n) \in \Omega_k$. Note that by unique factorization $x = x(0,p)\sigma^p(x)$.

We say that a path $x \in \Lambda^\infty$ is *periodic* if there exists $p \neq q \in \mathbb{N}^k$ such that $\sigma^p(x) = \sigma^q(x)$. That is, for all $m, n \in \mathbb{N}^k$, $x(m+p, n+p) = x(m+q, n+q)$. If x is not periodic, we say x is *aperiodic*. A row-finite k-graph with no sources is called *aperiodic* if for every vertex $v \in \Lambda^0$, there exists an aperiodic path in $v\Lambda^\infty$.

If Λ is a k-graph, we let $\Lambda^{\neq 0} := \{\lambda \in \Lambda : d(\lambda) \neq 0\}$. For each $\lambda \in \Lambda^{\neq 0}$ we introduce a *ghost path* λ^* and for $v \in \Lambda^0$ define $v^* := v$. We write $G(\Lambda)$ for the set of ghost paths, or $G(\Lambda^{\neq 0})$ if we wish to exclude vertices. We define d, r and s on $G(\Lambda)$ by $d(\lambda^*) = -d(\lambda)$, $r(\lambda^*) = s(\lambda)$, $s(\lambda^*) = r(\lambda)$; we then define composition on $G(\Lambda)$ by setting $\lambda^*\mu^* = (\mu\lambda)^*$ for $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu^*) = s(\lambda^*)$. The factorization property of Λ induces a similar factorization property on $G(\Lambda)$.

We have now all the tools in hand in order to give the following definition, first given in [16, Definition 3.1].

Definition 3.1.3. Let Λ be a row-finite k-graph without sources and let R be a commutative ring with 1. A *Kumjian-Pask* Λ*-family* (P, S) in an R-algebra A consists of two functions $P : \Lambda^0 \to A$ and $S : \Lambda^{\neq 0} \cup G(\Lambda^{\neq 0}) \to A$ such that:

(KP1) $\{P_v : v \in \Lambda^0\}$ is a family of mutually orthogonal idempotents;

(KP2) for all $\lambda, \mu \in \Lambda^{\neq 0}$ with $r(\mu) = s(\lambda)$, we have

$$S_\lambda S_\mu = S_{\lambda\mu}, S_{\mu^*}S_{\lambda^*} = S_{(\lambda\mu)^*}, P_{r(\lambda)}S_\lambda = S_\lambda = S_\lambda P_{s(\lambda)},$$
$$\text{and } P_{s(\lambda)}S_{\lambda^*} = S_{\lambda^*} = S_{\lambda^*}P_{r(\lambda)};$$

(KP3) for all $\lambda, \mu \in \Lambda^{\neq 0}$ with $d(\lambda) = d(\mu)$, we have $S_{\lambda^*}S_\mu = \delta_{\lambda,\mu}P_{s(\lambda)}$;

(KP4) for all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k \setminus \{0\}$, we have $P_v = \sum_{\lambda \in v\Lambda^n} S_\lambda S_{\lambda^*}$.

In the literature [16, Theorem 3.4] proves that there is an R-algebra $\mathrm{KP}_R(\Lambda)$ generated by a Kumjian-Pask Λ-family (p, s) with the following universal property: whenever (Q, T) is a Kumjian-Pask Λ-family in an R-algebra A, there is a unique R-algebra homomorphism $\pi_{Q,T} : \mathrm{KP}_R(\Lambda) \to A$ such that $\pi_{Q,T}(p_v) = Q_v$, $\pi_{Q,T}(s_\lambda) = T_\lambda$, $\pi_{Q,T}(s_{\mu^*}) = T_{\mu^*}$ for $v \in \Lambda^0$ and $\lambda, \mu \in \Lambda^{\neq 0}$. We call $\mathrm{KP}_R(\Lambda)$ the *Kumjian-Pask algebra* of Λ and the generating family (p, s) the *universal Kumjian-Pask family.*

With the convention that $s_v = p_v$ and $s_{v^*} = p_v$ it follows from the *Kumjian-Pask relations* (KP1)-(KP4) that

$$\mathrm{KP}_R(\Lambda) = \mathrm{span}_R\{s_\mu s_{\nu^*} : \mu, \nu \in \Lambda \text{ with } s(\mu) = s(\nu)\},$$

and $s_\mu s_{\nu^*} \neq 0$ in $\mathrm{KP}_R(\Lambda)$ if $s(\mu) = s(\nu)$. For every nonzero $a \in \mathrm{KP}_R(\Lambda)$ and $n \in \mathbb{N}^k$ there exist $m \geq n$ and a finite subset $F \subseteq \Lambda \times \Lambda^m$ such that $s(\alpha) = s(\beta)$ for all $(\alpha, \beta) \in F$ and

$$a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$$

with $r_{\alpha,\beta} \in R \setminus \{0\}$. In this situation, we say that a is written in *normal form* [16, Lemma 4.2]. We can define an R-linear involution $a \mapsto a^*$ on $\mathrm{KP}_R(\Lambda)$ as follows: if $a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$ then $a^* = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\beta s_{\alpha^*}$.

It is proved in [16, Theorem 3.4] that $\mathrm{KP}_R(\Lambda)$ is graded by $\mathbb{Z}^k$ in such a way that, for each $n \in \mathbb{Z}^k$,

$$\mathrm{KP}_R(\Lambda)_n = \mathrm{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda \text{ and } d(\lambda) - d(\mu) = n\}.$$

[16, Theorem 3.4] also says that $rp_v \neq 0$ for $v \in \Lambda^0$ and $r \in R \setminus \{0\}$.

To conclude this section let us see that the family of Kumjian-Pask algebras is indeed larger than the family of Leavitt path algebras. We need a previous lemma (which tells us how to simplify products $S_{\lambda^*} S_\mu$) and to remind the graded uniqueness theorem for $\mathrm{KP}_R(\Lambda)$ proved in [16].

Lemma 3.1.4. ([16, Lemma 3.3]) *Suppose that (P, S) is a Kumjian-Pask Λ-family, and $\lambda, \mu \in \Lambda$. Then for each $q \geq d(\lambda) \vee d(\mu)$, we have*

$$S_{\lambda^*} S_\mu = \sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} S_\alpha S_{\beta^*}.$$

Theorem 3.1.5. ([16, Theorem 4.1]) *Let Λ be a row-finite k-graph without sources, R a commutative ring with 1, and A a $\mathbb{Z}^k$-graded ring. If $\pi : \mathrm{KP}_R(\Lambda) \to A$ is a $\mathbb{Z}^k$-graded ring homomorphism such that $\pi(rp_v) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$, then π is injective.*

The following example is given in [16, Example 7.1]:

Example 3.1.6. Consider $\Lambda = \mathbb{N}^2$ as a category with a single object v, and let $d : \mathbb{N}^2 \to \mathbb{N}^2$ be the identity map. Then Λ is the unique 2-graph whose skeleton consists of one blue and one red loop at a single vertex. For each $n \in \mathbb{N}^2$ there is a unique path n of degree n, and a Kumjian-Pask family (P, S) in an R-algebra must satisfy

$$P_v^2 = P_v = S_{n^*}S_n = S_nS_{n^*},$$
$$S_mS_n = S_{m+n},\ S_{n^*}S_{m^*} = S_{(m+n)^*},$$
$$P_vS_n = S_n = S_nP_v,\ P_vS_{n^*} = S_{n^*} = S_{n^*}P_v.$$

For $q \geq m \vee n$ in $\mathbb{N}^2$, the sum in Lemma 3.1.4 has exactly one term, and we have $S_{m^*}S_n = S_{q-m}S_{(q-n)^*}$; taking $q = m + n$ gives $S_{m^*}S_n = S_nS_{m^*}$. In particular, $\mathrm{KP}_R(\Lambda)$ is commutative. We will use the graded-uniqueness theorem to show that $\mathrm{KP}_R(\Lambda)$ is isomorphic to the ring $R[x, x^{-1}, y, y^{-1}]$ of Laurent polynomials over R in two commuting indeterminates x and y.

Set $Q_v = 1$, $T_{(i,j)} = x^iy^j$ and $T_{(i,j)^*} = x^{-i}y^{-j}$. Then (Q, T) is a Kumjian-Pask Λ-family in $R[x, x^{-1}, y, y^{-1}]$, and the universal property of $\mathrm{KP}_R(\Lambda)$ gives a homomorphism $\phi : \mathrm{KP}_R(\Lambda) \to R[x, x^{-1}, y, y^{-1}]$ such that $\phi \circ p = Q$ and $\phi \circ s = T$. The groups $A_{(i,j)} := \mathrm{span}\{x^iy^j\}$ for $(i, j) \in \mathbb{Z}^2$ grade $R[x, x^{-1}, y, y^{-1}]$ over $\mathbb{Z}^2$, and ϕ maps $\mathrm{KP}_R(\Lambda)_{(i,j)} = \mathrm{span}\{s_ns_{m^*} : n - m = (i, j)\}$ into $A_{(i,j)}$, so ϕ is graded. Finally, $\phi(rp_v) = r\phi(p_v) = r1 = r \neq 0$ for all $r \in R \setminus \{0\}$, and so Theorem 3.1.5 implies that ϕ is injective. Since the image of ϕ contains a generating set for $R[x, x^{-1}, y, y^{-1}]$, ϕ is an isomorphism.

Let K be a field. We claim that $K[x, x^{-1}, y, y^{-1}]$ cannot be realized as a Leavitt path algebra $L_K(E)$ for any directed graph E. Thus Example 3.1.6 shows that the class of Kumjian-Pask algebras over K is larger than the class of Leavitt path algebras over K. To see the claim, recall from Corollary 2.4.4 that every commutative Leavitt path algebra has the form $(\bigoplus K) \oplus (\bigoplus K[x, x^{-1}])$. Since $K[x, x^{-1}, y, y^{-1}]$ has no zero divisors, if $K[x, x^{-1}, y, y^{-1}]$ had this previous form then it would be isomorphic to either K or $K[x, x^{-1}]$ as rings. But both K and $K[x, x^{-1}]$ are principal ideal domains, whereas $K[x, x^{-1}, y, y^{-1}]$ is not. So $K[x, x^{-1}, y, y^{-1}]$ is not the Leavitt path algebra of any directed graph.

3.2 The diagonal subalgebra of a Kumjian-Pask algebra

In this section we establish some properties of the *diagonal subalgebra* $\mathcal{D}$ of the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$. Define

$$\mathcal{D} := \langle s_\mu s_{\mu^*} : \mu \in \Lambda \rangle;$$

that is, $\mathcal{D}$ is the R-subalgebra of $\mathrm{KP}_R(\Lambda)$ generated by the set $\{s_\mu s_{\mu^*} : \mu \in \Lambda\}$. Observe that for $v \in \Lambda^0$, $s_v s_{v^*} = p_v$.

Lemma 3.2.1. *The diagonal subalgebra $\mathcal{D}$ is commutative.*

Proof. Using the Kumjian-Pask relations it follows that for each n, $\{s_\mu s_{\mu^*} : \mu \in \Lambda^n\}$ is a set of mutually orthogonal idempotents. Now if we consider $\lambda, \mu \in \Lambda$ then by Lemma 3.1.4, for each $q \geq d(\lambda) \vee d(\mu)$ we can write the following

$$s_{\lambda^*} s_\mu = \sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} s_\alpha s_{\beta^*}.$$

Then $(s_\lambda s_{\lambda^*})(s_\mu s_{\mu^*}) = (s_\mu s_{\mu^*})(s_\lambda s_{\lambda^*})$ because:

$$\begin{aligned}
(s_\lambda s_{\lambda^*})(s_\mu s_{\mu^*}) &= s_\lambda (s_{\lambda^*} s_\mu) s_{\mu^*} \\
&= s_\lambda \Big(\sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} s_\alpha s_{\beta^*} \Big) s_{\mu^*} \\
&= \sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} s_\lambda s_\alpha s_{\beta^*} s_{\mu^*} \\
&= \sum_{d(\lambda\alpha)=q,\ \lambda\alpha=\mu\beta} s_{\lambda\alpha} s_{(\mu\beta)^*}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{d(\mu\beta)=q,\ \mu\beta=\lambda\alpha} s_{\mu\beta} s_{(\lambda\alpha)^*} \\
&= \sum_{d(\mu\beta)=q,\ \mu\beta=\lambda\alpha} s_{\mu} s_{\beta} s_{\alpha^*} s_{\lambda^*} \\
&= s_{\mu} \Big(\sum_{d(\mu\beta)=q,\ \mu\beta=\lambda\alpha} s_{\beta} s_{\alpha^*} \Big) s_{\lambda^*} \\
&= s_{\mu} (s_{\mu^*} s_{\lambda}) s_{\lambda^*} \\
&= (s_{\mu} s_{\mu^*})(s_{\lambda} s_{\lambda^*}).
\end{aligned}
$$

So $\mathcal{D}$ is commutative. □

Remark 3.2.2. Observe that

$$s_{\alpha\gamma} s_{(\alpha\gamma)^*} s_{\alpha} s_{\beta^*} s_{\beta\eta} s_{(\beta\eta)^*} = s_{\alpha\gamma} s_{\gamma^*} s_{\eta} s_{(\beta\eta)^*}.$$

In particular if $d(\gamma) = d(\eta)$, then

$$s_{\alpha\gamma} s_{(\alpha\gamma)^*} s_{\alpha} s_{\beta^*} s_{\beta\eta} s_{(\beta\eta)^*} = \begin{cases} s_{\alpha\gamma} s_{(\beta\gamma)^*} & \text{if } \ \gamma = \eta, \\ 0 & \text{otherwise.} \end{cases}$$

We may view elements of the diagonal $\mathcal{D}$ as functions from Λ^∞ to R in the following way. For $\mu \in \Lambda$ let $1_{Z(\mu)}$ denote the characteristic function association to $Z(\mu)$. That is, $1_{Z(\mu)} : \Lambda^\infty \to R$ such that

$$1_{Z(\mu)}(x) = \begin{cases} 1 & \text{if } \ x \in Z(\mu), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$A_{\mathcal{D}} := \operatorname{span}_R\{1_{Z(\mu)} : \ \mu \in \Lambda\}.$$

Then $A_{\mathcal{D}}$ is an R-algebra with addition and scalar multiplication defined pointwise and multiplication of the generators is given by

$$1_{Z(\mu)} \cdot 1_{Z(\nu)} = 1_{Z(\mu) \cap Z(\nu)}.$$

Remark 3.2.3. In fact $A_{\mathcal{D}}$ is the diagonal of the *Steinberg algebra* associated to Λ. (The reader can see Section 5 of [32] for more details).

We need to include here the following result for future reference.

Lemma 3.2.4. ([32, Proposition 5.4]) *The map* $\pi : \mathcal{D} \to A_{\mathcal{D}}$ *such that* $\pi(s_\mu s_{\mu^*}) = 1_{Z(\mu)}$ *is an isomorphism.*

Lemma 3.2.5. *Let* $\mu, \nu \in \Lambda$ *with* $s(\mu) = s(\nu)$ *and* $r \in R$ *such that* $rs_\mu s_{\mu^*} = rs_\nu s_{\nu^*}$. *Then* $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$.

Proof. Let $\pi : \mathcal{D} \to A_{\mathcal{D}}$. Then $\pi(rs_\mu s_{\mu^*}) = \pi(rs_\nu s_{\nu^*})$. Therefore we have $r1_{Z(\mu)} = r1_{Z(\nu)}$, which implies $Z(\mu) = Z(\nu)$. This means $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$ because the map π of Lemma 3.2.4 is injective. □

3.3 The cycline subalgebra

We study a special class of generators for $\mathrm{KP}_R(\Lambda)$ which will be used in the construction of a distinguished subalgebra, called *the cycline subalgebra* of $\mathrm{KP}_R(\Lambda)$. First an element $a \in \mathrm{KP}_R(\Lambda)$ is said to be *normal* if $aa^* = a^*a$.

Proposition 3.3.1. *Let* Λ *be a row-finite* k*-graph with no sources and* R *be a commutative ring with* 1. *Then for* $(\alpha, \beta) \in \Lambda \times \Lambda$ *with* $s(\alpha) = s(\beta)$, *the following conditions are equivalent:*

(i) $s_{\alpha\gamma} s_{(\alpha\gamma)^*} = s_{\beta\gamma} s_{(\beta\gamma)^*}$ *for all* $\gamma \in s(\alpha)\Lambda$;

(ii) $s_\alpha s_{\beta^*}$ *is normal and commutes with* $\mathcal{D}$; *and*

(iii) $\alpha\gamma = \beta\gamma$ *for all* $\gamma \in s(\alpha)\Lambda^\infty$.

Definition 3.3.2. A pair $(\alpha, \beta) \in \Lambda \times \Lambda$ with $s(\alpha) = s(\beta)$ satisfying the equivalent conditions of Proposition 3.3.1 is called a *cycline pair*. Define

$$\mathcal{M} := \langle s_\alpha s_{\beta^*} : (\alpha, \beta) \text{ cycline} \rangle;$$

that is, the R-subalgebra of $\mathrm{KP}_R(\Lambda)$ generated by the cycline pairs. We call $\mathcal{M}$ the *cycline subalgebra* of the Kumjian-Pask algebra $\mathrm{KP}_R(\Lambda)$.

Proof of Proposition 3.3.1. This proof follows exactly as the one given in [25, Proposition 4.1]. We include here for completeness. Let us see the equivalence between the statements.

• (i) ⇒ (ii) Let $\mu \in \Lambda$. We check that $s_\alpha s_{\beta^*}$ commutes with $s_\mu s_{\mu^*}$. Since $s_\mu s_{\mu^*} = \sum_{\eta \in s(\mu)\Lambda^n} s_{\mu\eta} s_{(\mu\eta)^*}$ without loss of generality we can suppose $d(\mu) > d(\alpha) + d(\beta)$. We can find two cases. The first case consists of $\mu = \alpha\gamma$ or $\mu = \beta\gamma$ for some γ. By assumption (i), $s_{\alpha\gamma} s_{(\alpha\gamma)^*} = s_{\beta\gamma} s_{(\beta\gamma)^*}$, so

$$s_\mu s_{\mu^*} s_\alpha s_{\beta^*} = s_{\alpha\gamma} s_{(\alpha\gamma)^*} s_\alpha s_{\beta^*} = s_{\alpha\gamma} s_{(\beta\gamma)^*} = s_\alpha s_{\beta^*} s_{\beta\gamma} s_{(\beta\gamma)^*} = s_\alpha s_{\beta^*} s_\mu s_{\mu^*}.$$

As a second case suppose μ is neither $\alpha\gamma$ nor $\beta\gamma$ for any $\gamma \in \Lambda$. Then $s_\mu s_{\mu^*} s_\alpha s_{\beta^*} = 0 = s_\alpha s_{\beta^*} s_\mu s_{\mu^*}$. So $s_\alpha s_{\beta^*}$ commutes with $\mathcal{D}$. Besides, $(s_\alpha s_{\beta^*})^* = s_\beta s_{\alpha^*}$ which gives that $s_\alpha s_{\beta^*}$ is normal since

$$s_\alpha s_{\beta^*} (s_\alpha s_{\beta^*})^* = s_\alpha s_{\alpha^*} = s_\beta s_{\beta^*} = (s_\alpha s_{\beta^*})^* s_\alpha s_{\beta^*}.$$

• (ii) ⇒ (i) Consider $\gamma \in s(\alpha)\Lambda$. Because of $s_\alpha s_{\beta^*}$ is normal and commutes with $s_{\alpha\gamma} s_{(\alpha\gamma)^*}$ then we have that $s_{\alpha\gamma} s_{(\alpha\gamma)^*} s_\alpha s_{\beta^*} = s_{\alpha\gamma} s_{(\beta\gamma)^*}$ is normal. So it follows

$$s_{\alpha\gamma} s_{(\alpha\gamma)^*} = s_{\alpha\gamma} s_{(\beta\gamma)^*} (s_{\alpha\gamma} s_{(\beta\gamma)^*})^* = (s_{\alpha\gamma} s_{(\beta\gamma)^*})^* s_{\alpha\gamma} s_{(\beta\gamma)^*} = s_{\beta\gamma} s_{(\beta\gamma)^*}.$$

• (i) ⇒ (iii) Let $\gamma \in s(\alpha)\Lambda^\infty$. According to (i), $s_{\alpha(\gamma(0,n))} s_{(\alpha(\gamma(0,n)))^*} = s_{\beta(\gamma(0,n))} s_{(\beta(\gamma(0,n)))^*}$ for all n which implies that $Z(\alpha(\gamma(0,n))) = Z(\beta(\gamma(0,n)))$ for all n. And then we obtain (iii) since

$$\{\alpha\gamma\} = \bigcap_{n \in \mathbb{N}^k} Z(\alpha(\gamma(0,n))) = \bigcap_{n \in \mathbb{N}^k} Z(\beta(\gamma(0,n))) = \{\beta\gamma\}.$$

• (iii) ⇒ (i) Consider $\gamma \in s(\alpha)\Lambda$. Applying (iii), since $\gamma z \in s(\alpha)\Lambda^\infty$ for all $z \in s(\alpha)\Lambda^\infty$ we have $\alpha\gamma z = \beta\gamma z$ for all $z \in s(\gamma)\Lambda^\infty$. Finally $Z(\alpha\gamma) = Z(\beta\gamma)$ and $s_{\alpha\gamma} s_{(\alpha\gamma)^*} = s_{\beta\gamma} s_{(\beta\gamma)^*}$. □

Remark 3.3.3. Observe that (α, α) is cycline and hence $\mathcal{D} \subseteq \mathcal{M}$. Also (α, β) is cycline if and only if (β, α) is. For $(\alpha, \beta), (\mu, \nu) \in \Lambda \times \Lambda$ and $n \in \mathbb{N}^k$, $n \geq d(\alpha) \vee d(\beta)$ we have

$$s_\alpha s_{\beta^*} s_\mu s_{\nu^*} = \sum_{\gamma, \eta \in \Lambda,\ \beta\gamma = \mu\eta,\ d(\beta\gamma) = n} s_{\alpha\gamma} s_{(\nu\eta)^*}. \tag{3.1}$$

If $(\alpha,\beta),(\mu,\nu)$ are cycline and $\lambda\in s(\gamma)\Lambda$, then

$$s_{\nu\eta\lambda}s_{(\nu\eta\lambda)^*}=s_{\mu\eta\lambda}s_{(\mu\eta\lambda)^*}=s_{\beta\gamma\lambda}s_{(\beta\gamma\lambda)^*}=s_{\alpha\gamma\lambda}s_{(\alpha\gamma\lambda)^*}.$$

This means that the pairs $(\alpha\gamma,\nu\eta)$ appearing in the right hand side of (3.1) are cycline too by Proposition 3.3.1 (i).

Lemma 3.3.4. *The cycline subalgebra $\mathcal{M}$ is commutative.*

Proof. Let (α,β) and (μ,ν) be two cycline pairs. It suffices to show that

$$(s_\alpha s_{\beta^*})(s_\mu s_{\nu^*})=(s_\mu s_{\nu^*})(s_\alpha s_{\beta^*}).$$

Now

$$\begin{aligned}
s_\alpha s_{\beta^*}\ s_\mu s_{\nu^*} &= \sum_{\beta\gamma=\mu\eta} s_{\alpha\gamma}s_{(\nu\eta)^*}\\
&= \sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma} s_{\alpha\gamma}\ \left(s_{(\alpha\gamma)^*}s_{\nu\eta}\right)\ s_{(\nu\eta)^*} \text{ by (KP3)}\\
&= \sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma} \left(s_{\alpha\gamma}s_{(\alpha\gamma)^*}\right)\ \left(s_{\nu\eta}s_{(\nu\eta)^*}\right)\\
&= \sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma} \left(s_{\beta\gamma}s_{(\beta\gamma)^*}\right)\ \left(s_{\mu\eta}s_{(\mu\eta)^*}\right) \text{ since } (\alpha,\beta),(\mu,\nu) \text{ cycline}\\
&= \sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma} \left(s_{\mu\eta}s_{(\mu\eta)^*}\right)\ \left(s_{\beta\gamma}s_{(\beta\gamma)^*}\right) \text{ since } \mathcal{D} \text{ is commutative}\\
&= \sum_{\beta\gamma=\mu\eta,\ \nu\eta=\alpha\gamma} s_{\mu\eta}\ \left(s_{(\mu\eta)^*}s_{\beta\gamma}\right)\ s_{(\beta\gamma)^*}\\
&= \sum_{\nu\eta=\alpha\gamma} s_{\mu\eta}s_{(\beta\gamma)^*}\\
&= s_\mu s_{\nu^*}\ s_\alpha s_{\beta^*} \text{ as desired.}
\end{aligned}$$

□

Example 3.3.5. Let E be a 1-graph. Recall that for this chapter we use the convention where a path is a sequence of edges $e_1\dots e_n$ such that $s(e_i)=r(e_{i+1})$. We say that a path $e_1\dots e_n$ has an *entry* if there exists an i and an edge $f\in r(e_i)E^1$ with $f\neq e_i$. Following the ideas given in [25, Example 4.9] let us check that a standard generator $s_\alpha s_{\beta^*}$ is cycline if and only if $\alpha=\beta$, $\alpha=\beta c$ or $\beta=\alpha c$ for some cycle $c\in s(\alpha)Es(\alpha)$ without entry. By Proposition

2.3.6 $s_\alpha s_{\beta^*}$ is normal if and only if $\alpha = \beta$, $\alpha = \beta c$ or $\beta = \alpha c$ for some cycle $c \in s(\alpha)Es(\alpha)$ without entry. It is evident that $s_\alpha s_{\alpha^*}$ is cycline. For $\alpha \neq \beta$ define

$$\gamma = \begin{cases} \beta ccc\dots & \text{if } \alpha = \beta c, \\ \alpha ccc\dots & \text{if } \beta = \alpha c. \end{cases}$$

In either case $\{\gamma\} = Z(\alpha) = Z(\beta) = Z(\alpha\gamma) = Z(\beta\gamma)$ for any $\gamma \in s(\alpha)E$. As a consequence $s_{\alpha\gamma}s_{(\alpha\gamma)^*} = s_{\beta\gamma}s_{(\beta\gamma)^*}$ for all $\gamma \in s(\alpha)E$ and $s_\alpha s_{\beta^*}$ is cycline. Thus the cycline subalgebra $\mathcal{M}$ coincides with *the commutative core* $M_R(E)$ defined in Definition 2.3.7. □

Define $\mathcal{D}'$ such that

$$\mathcal{D}' := \{a \in \mathrm{KP}_R(\Lambda) : ad = da \text{ for every } d \in \mathcal{D}\}. \tag{3.2}$$

Notice that we have the inclusions $\mathcal{D} \subseteq \mathcal{M} \subseteq \mathcal{D}'$, where the second inclusion comes from Proposition 3.3.1 (ii). In fact, $\mathcal{M} = \mathcal{D}$ if Λ is aperiodic (see Lemma 3.4.5). Our main goal in this section is to prove the following.

Theorem 3.3.6. *Let Λ be a row-finite k-graph with no sources, R be a commutative ring with 1, $\mathcal{M}$ be the cycline subalgebra and $\mathcal{D}'$ be defined as in (3.2). Then $\mathcal{M} = \mathcal{D}'$. In particular, $\mathcal{M}$ is a maximal commutative subalgebra of* $\mathrm{KP}_R(\Lambda)$.

To prove Theorem 3.3.6, we establish three lemmas. Recall that $\mathrm{KP}_R(\Lambda)$ is a $\mathbb{Z}^k$-graded algebra such that for $n \in \mathbb{Z}^k$,

$$\mathrm{KP}_R(\Lambda)_n = \mathrm{span}_R\{s_\lambda s_{\mu^*} : \lambda, \mu \in \Lambda \text{ and } d(\lambda) - d(\mu) = n\}.$$

Lemma 3.3.7. *Let $a \in \mathcal{D}'$ and suppose $a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta}s_\alpha s_{\beta^*}$ is in normal form. Then for any $(\alpha,\beta) \in F$, $r_{\alpha,\beta}s_\alpha s_{\beta^*} \in \mathcal{D}'$.*

Proof. First we claim that $a \in \mathcal{D}'$ if and only if $a_n \in \mathcal{D}'$ for all $n \in \mathbb{Z}^k$, where a_n is the homogeneous part of a of degree n. To prove the claim note if for all $n \in \mathbb{Z}^k$, $a_n \in \mathcal{D}'$ then $a \in \mathcal{D}'$. Assume that $a \in \mathcal{D}'$. Then for any

$d \in \mathcal{D}$, $ad = da$. Since elements of $\mathcal{D}$ are homogeneous of degree zero then $a_n d = (ad)_n$. Then

$$a_n d = (ad)_n = (da)_n = da_n,$$

and so $a_n \in \mathcal{D}'$ for every $n \in \mathbb{Z}^k$.

Thus to prove the lemma, we can assume that a is homogeneous of degree n. Since all β have the same degree, then all α have the same degree. For every $(\gamma, \delta) \in F$, $s_\gamma s_{\gamma^*} \in \mathcal{D}$ and $s_\delta s_{\delta^*} \in \mathcal{D}$ so then $s_\gamma s_{\gamma^*} a s_\delta s_{\delta^*} \in \mathcal{D}'$. By (KP3) we have, $s_\gamma s_{\gamma^*} a s_\delta s_{\delta^*} = r_{\gamma,\delta} s_\gamma s_{\delta^*} \in \mathcal{D}'$. □

In the following lemma we adopt some ideas from [62, Lemma 3.2].

Lemma 3.3.8. *Let $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ and $r \in R \setminus \{0\}$ such that $r s_\mu s_{\nu^*} \in \mathcal{D}'$. Then $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$.*

Proof. First since $r s_\mu s_{\nu^*} \in \mathcal{D}'$ we have $r s_\mu s_{\mu^*} s_\mu s_{\nu^*} = r s_\mu s_{\nu^*} s_\mu s_{\mu^*}$, that is, $r s_\mu s_{\nu^*} = r s_\mu s_{\nu^*} s_\mu s_{\mu^*}$. Then multiplying both sides of this equation by s_{μ^*} on the left, we get $r s_{\mu^*} s_\mu s_{\nu^*} = r s_{\mu^*} s_\mu s_{\nu^*} s_\mu s_{\mu^*}$, which means by (KP3) that $r s_{\nu^*} = r s_{\nu^*} s_\mu s_{\mu^*}$. Finally multiplying by s_ν again on the left gives,

$$r s_\nu s_{\nu^*} = r s_\nu s_{\nu^*} s_\mu s_{\mu^*}.$$

Since $r s_\mu s_{\nu^*} \in \mathcal{D}'$, we have $r s_\nu s_{\mu^*} \in \mathcal{D}'$. Using the same argument we get

$$r s_\mu s_{\mu^*} = r s_\mu s_{\mu^*} s_\nu s_{\nu^*}.$$

Now $s_\nu s_{\nu^*}, s_\mu s_{\mu^*} \in \mathcal{D}$ and $\mathcal{D}$ is commutative so

$$r s_\mu s_{\mu^*} = r s_\mu s_{\mu^*} s_\nu s_{\nu^*} = r s_\nu s_{\nu^*} s_\mu s_{\mu^*} = r s_\nu s_{\nu^*}.$$

That is, $r s_\mu s_{\mu^*} = r s_\nu s_{\nu^*}$. Finally by Lemma 3.2.5 we obtain $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$. □

Lemma 3.3.9. *Let $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ and $r \in R \setminus \{0\}$ such that $r s_\mu s_{\nu^*} \in \mathcal{D}'$. Then (μ, ν) is a cycline pair.*

Proof. Let $\gamma \in s(\mu)\Lambda$. We show $s_{\mu\gamma}s_{(\mu\gamma)^*} = s_{\nu\gamma}s_{(\nu\gamma)^*}$. Because $rs_\mu s_{\nu^*} \in \mathcal{D}'$, $rs_\nu s_{\mu^*} \in \mathcal{D}'$. So, on the one hand, we have:

$$\begin{aligned}(rs_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})(s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})^* &= (s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})(rs_\nu s_{\mu^*})s_{\mu\gamma}s_{(\mu\gamma)^*}\\ &= (s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})s_{\mu\gamma}s_{(\mu\gamma)^*}(rs_\nu s_{\mu^*})\\ &= s_{\mu\gamma}s_{(\mu\gamma)^*}(rs_\mu s_{\nu^*})s_{\mu\gamma}s_{(\mu\gamma)^*}s_\nu s_{\mu^*}\\ &= s_{\mu\gamma}s_{(\mu\gamma)^*}s_{\mu\gamma}s_{(\mu\gamma)^*}(rs_\mu s_{\nu^*})s_\nu s_{\mu^*}\\ &= rs_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\mu^*}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})^*(rs_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*}) &= (rs_\nu s_{\mu^*})s_{\mu\gamma}s_{(\mu\gamma)^*}s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*}\\ &= s_{\mu\gamma}s_{(\mu\gamma)^*}(rs_\nu s_{\mu^*})s_\mu s_{\nu^*}\\ &= rs_{\mu\gamma}s_{(\mu\gamma)^*}s_\nu s_{\nu^*}.\end{aligned}$$

But by Lemma 3.3.8, $rs_\mu s_{\nu^*} \in \mathcal{D}'$ implies $s_\mu s_{\mu^*} = s_\nu s_{\nu^*}$. Therefore

$$(rs_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})(s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})^* = (s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})^*(rs_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*}). \tag{3.3}$$

Now:

$$\begin{aligned}rs_{\mu\gamma}s_{(\mu\gamma)^*} &= (rs_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})(s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})^*\\ &= (s_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*})^*(rs_{\mu\gamma}s_{(\mu\gamma)^*}s_\mu s_{\nu^*}) \text{ by (3.3)}\\ &= rs_{\nu\gamma}s_{(\nu\gamma)^*}.\end{aligned}$$

Hence $rs_{\mu\gamma}s_{(\mu\gamma)^*} = rs_{\nu\gamma}s_{(\nu\gamma)^*}$ and by Lemma 3.2.5, we obtain

$$s_{\mu\gamma}s_{(\mu\gamma)^*} = s_{\nu\gamma}s_{(\nu\gamma)^*}.$$

Thus by Proposition 3.3.1 (i), (μ, ν) is a cycline pair. □

We are now ready to prove the main result of this section, which we stated before.

Proof of Theorem 3.3.6. To see that $\mathcal{M} = \mathcal{D}'$, we show $\mathcal{D}' \subseteq \mathcal{M}$. Let $a \in \mathcal{D}'$. Suppose a is written in normal form

$$a = \sum_{(\alpha,\beta)\in F} r_{\alpha,\beta} s_\alpha s_{\beta^*}$$

By Lemma 3.3.7, for each $(\alpha, \beta) \in F$ we have $r_{\alpha,\beta} s_\alpha s_{\beta^*} \in \mathcal{D}'$. Now we apply Lemma 3.3.9 to see that each (α, β) in F is a cycline pair. Thus $a \in \mathcal{M}$.

In order to prove that $\mathcal{M}$ is a maximal commutative subalgebra inside $\mathrm{KP}_R(\Lambda)$, first recall that $\mathcal{M}$ is commutative by Lemma 3.3.4. Now consider $\mathcal{C}$ a commutative subalgebra of $\mathrm{KP}_R(\Lambda)$ such that $\mathcal{M} \subseteq \mathcal{C}$. Since we have $\mathcal{D} \subseteq \mathcal{M} \subseteq \mathcal{C}$ then in particular

$$\mathcal{C} \subseteq \{x \in \mathrm{KP}_R(\Lambda) : xd = dx \text{ for every } d \in \mathcal{D}\} = \mathcal{D}'.$$

Therefore $\mathcal{C} = \mathcal{M}$. □

The following corollary involving the center of a Kumjian-Pask algebra, (studied in [23]) is now immediate.

Corollary 3.3.10. *Let Λ be a row-finite k-graph with no sources and R be a commutative ring with 1. Then the center of the Kumjian-Pask algebra*

$$\mathcal{Z}(\mathrm{KP}_R(\Lambda)) := \{a \in \mathrm{KP}_R(\Lambda) : ax = xa \text{ for every } x \in \mathrm{KP}_R(\Lambda)\} \subseteq \mathcal{M}.$$

3.4 General uniqueness theorem for Kumjian-Pask algebras

In this section we give a new uniqueness theorem for Kumjian-Pask algebras that says a homomorphism on $\mathrm{KP}_R(\Lambda)$ is injective if and only if it is injective on the cycline subalgebra $\mathcal{M}$. First we adapt some of the technical innovations from [25] to our setting.

For any subset $U \subseteq X$ of a topological space X we denote its interior by $\mathrm{Int}(U)$ and its boundary by ∂U. The following definition appears in [25, Definition 5.2]. For a k-graph Λ, define

$$\Sigma := \{(\alpha, \beta) \in \Lambda \times \Lambda \mid s(\alpha) = s(\beta), \alpha \neq \beta\}.$$

Then for any pair $(\alpha, \beta) \in \Sigma$ let

$$F_{\alpha,\beta} := \{x \in \Lambda^\infty \mid x \in Z(\alpha) \cap Z(\beta) \text{ and } \sigma^{d(\alpha)}(x) = \sigma^{d(\beta)}(x)\}.$$

We define the set of *regular paths* in Λ^∞ to be

$$\mathfrak{T}_\Lambda := \Lambda^\infty - \bigcup_{(\alpha,\beta)\in\Sigma} \partial F_{\alpha,\beta}.$$

Remark 3.4.1. We recall here some properties of $F_{\alpha,\beta}$ and $\mathfrak{T}_\Lambda$, which are given in [25, Section 5].

(a) We have $F_{\alpha,\beta} = F_{\beta,\alpha}$.

(b) We have that $F_{\alpha,\beta}$ is closed for all $(\alpha, \beta) \in \Sigma$. Indeed if $x_i \to x$ in Λ^∞ and $n \in \mathbb{N}^k$ then $\sigma^n(x_i) \to \sigma^n(x)$; in particular if $x_i \in F_{\alpha,\beta}$ for all i then $x(0, d(\alpha)) = \alpha$, $x(0, d(\beta)) = \beta$ and $\sigma^{d(\alpha)}(x) = \sigma^{d(\beta)}(x)$, so $x \in F_{\alpha,\beta}$. Therefore, $\partial F_{\alpha,\beta} \subseteq F_{\alpha,\beta}$. Note that $\partial F_{\alpha,\beta}$ is also closed.

(c) The set $\mathfrak{T}_\Lambda$ is dense in Λ^∞ (by the Baire Category Theorem). So for every $v \in \Lambda^0$ there exists an $x \in Z(v) \cap \mathfrak{T}_\Lambda$.

(d) We have $\mathfrak{T}_\Lambda \cap F_{\alpha,\beta} \subseteq \operatorname{Int}(F_{\alpha,\beta})$.

(e) The cylinder set $Z(\mu) \subseteq F_{\alpha,\beta}$ if and only if $Z(\nu\mu) \subseteq F_{\nu\alpha,\nu\beta}$.

(f) If $x \in \mathfrak{T}_\Lambda$ and $\nu \in \Lambda r(x)$, then $\nu x \in \mathfrak{T}_\Lambda$.

(g) If $x \in \mathfrak{T}_\Lambda$ and $n \in \mathbb{N}^k$, then $\sigma^n(x) \in \mathfrak{T}_\Lambda$.

Example 3.4.2. Let E be a 1-graph. An infinite path in E is called *essentially aperiodic* if it is either aperiodic or of the form $\mu ccc\ldots$, where c is a cycle without entry (see Definition 2.2.5). Following [25, Example 5.4], we claim that $\mathfrak{T}_E$ is the set of essentially aperiodic paths of E.

First assume that $x \in E^\infty$ is aperiodic. Then $x \notin F_{\alpha,\beta}$ for any $(\alpha, \beta) \in \Sigma$. Since $F_{\alpha,\beta}$ are closed, $x \notin \partial F_{\alpha,\beta}$ which implies that aperiodic paths are in $\mathfrak{T}_E$. Now imagine x is periodic: there exists some $(\alpha, \beta) \in \Sigma$ such that $x \notin F_{\alpha,\beta}$. We can suppose $d(\beta) > d(\alpha)$ without loss of generality so that $\beta = \alpha c$ for some $c \in s(\alpha)Es(\alpha)$; that is, $x = \alpha ccc\ldots$ and $F_{\alpha,\beta} = \{x\}$. We claim that periodic paths with entries are not in $\mathfrak{T}_E$: if c has an entry then for any $i \geq 1$, $Z(\alpha c^i)$ contains a path of the form $z = \alpha c^i \gamma$ where $\gamma(0, d(c)) \neq c$; we have $\sigma^{d(\alpha)}(z)((i-1)d(c), i\, d(c)) = c \neq \gamma(0, d(c)) = \sigma^{d(\beta)}(z)((i-1)d(c), i\, d(c))$ and then $z \in Z(\alpha c^i) \cap (\Lambda^\infty - F_{\alpha,\beta})$; since i is arbitrary, $x \in \partial F_{\alpha,\beta}$. Finally if c has no entry then $Z(\alpha c) = \{x\} = F_{\alpha,\beta}$ so $F_{\alpha,\beta}$ is clopen (i.e., both an open and closed subset) and $\partial F_{\alpha,\beta} = \emptyset$. Therefore periodic paths without entries are in $\mathfrak{T}_E$.

We previously need the following lemma in order to prove the main theorem of this section.

Lemma 3.4.3. *For $(\alpha, \beta) \in \Sigma$ and $x \in \mathfrak{T}_\Lambda$ we have the following:*

(a) If $x \notin F_{\alpha,\beta}$, then there exists $\mu, \nu \in \Lambda$ such that $x \in Z(\mu) \cap Z(\nu)$ and

$$s_\mu s_{\mu^*} s_\alpha s_{\beta^*} s_\nu s_{\nu^*} = 0.$$

(b) If $x \in F_{\alpha,\beta}$, then there exists $\gamma \in \Lambda$ with $x \in Z(\alpha\gamma) \cap Z(\beta\gamma)$,

$$s_{\alpha\gamma} s_{(\alpha\gamma)^*} s_\alpha s_{\beta^*} s_{\beta\gamma} s_{(\beta\gamma)^*} = s_{\alpha\gamma} s_{(\beta\gamma)^*}, \text{ and } (\alpha\gamma, \beta\gamma) \text{ cycline.}$$

Proof. The proofs is analogous to that of [25, Lemma 5.8] and it is included here for completeness.

(a) First suppose $x \notin Z(\alpha) \cap Z(\beta)$. Then consider $\mu = x(0, d(\alpha))$ and $\nu = x(0, d(\beta))$ in order to have $s_\mu s_{\mu^*} s_\alpha s_{\beta^*} s_\nu s_{\nu^*} = 0$. On the other hand if $x \in Z(\alpha) \cap Z(\beta)$, since $x \notin F_{\alpha,\beta}$ then $\sigma^{d(\alpha)}(x) \neq \sigma^{d(\beta)}(x)$ implying that there exists an $n \in \mathbb{N}^k$ such that $x(d(\alpha), d(\alpha) + n) \neq x(d(\beta), d(\beta) + n)$. Let $\gamma = x(d(\alpha), d(\alpha) + n)$, $\eta = x(d(\beta), d(\beta) + n)$ and $\mu = \alpha\gamma$, $\nu = \beta\eta$. Then $x \in Z(\mu) \cap Z(\nu)$ and $s_\mu s_{\mu^*} s_\alpha s_{\beta^*} s_\nu s_{\nu^*} = 0$.

(b) By Remark 3.4.1(d), $\mathfrak{T}_\Lambda \cap F_{\alpha,\beta} \subseteq \mathrm{Int}(F_{\alpha,\beta})$. So there exists $\epsilon \in \Lambda$ such that $x \in Z(\epsilon) \subseteq F_{\alpha,\beta}$. Let $\alpha' = x(d(\epsilon), d(\epsilon) + d(\alpha))$, $\beta' = x(d(\epsilon), d(\epsilon) + d(\beta))$ and $\mu = \epsilon\alpha'$, $\nu = \epsilon\beta'$. Then, $x \in Z(\mu) \cap Z(\nu)$. Also $x \in Z(\alpha) \cap Z(\beta)$ so by unique factorization we can put $\mu = \alpha\gamma$ and $\nu = \beta\gamma'$ where $\gamma =$

$\sigma^{d(\alpha)}(x)(0,d(\epsilon))$ and $\gamma' = \sigma^{d(\beta)}(x)(0,d(\epsilon))$. But $x \in F_{\alpha,\beta}$ so $\sigma^{d(\alpha)}(x) = \sigma^{d(\beta)}(x)$ and finally $\gamma = \gamma'$. In order to see that $(\alpha\gamma, \beta\gamma)$ is cycline, consider $z \in s(\gamma)\Lambda^\infty$ and let us check that $\alpha\gamma z = \beta\gamma z$. It happens that $\alpha\gamma z = x = \epsilon\alpha' z$ so $\alpha\gamma z \in Z(\epsilon) \subseteq F_{\alpha,\beta}$. Thus $\alpha\gamma z = \beta\gamma z$ as desired. □

Theorem 3.4.4. *Let Λ be a row-finite k-graph with no sources, R be a commutative ring with 1 and $\mathcal{M}$ be the cycline subalgebra of $\mathrm{KP}_R(\Lambda)$. If $\Phi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism, then Φ is injective if and only if $\Phi|_{\mathcal{M}}$ is injective.*

Proof. We show that $\Phi|_{\mathcal{M}}$ injective implies Φ injective. By way of contradiction suppose we have $0 \neq a \in \mathrm{KP}_R(\Lambda)$ such that $a \in \mathrm{Ker}\ \Phi$. Applying [24, Lemma 2.3 (i)] (and writing the sum involved in normal form) we can find $(\delta, \epsilon) \in \Lambda \times \Lambda$ such that

$$0 \neq s_{\delta^*} a s_\epsilon = r_{\delta,\epsilon} p_{s(\delta)} + \sum_{(\alpha,\beta)\in F,\ d(\alpha)\neq d(\beta)} r_{\alpha,\beta} s_\alpha s_{\beta^*},$$

where in particular $r_{\delta,\epsilon} \neq 0$. Let $b := s_{\delta^*} a s_\epsilon \neq 0$. Notice that $b \in \mathrm{Ker}\ \Phi$ because $\mathrm{Ker}\ \Phi$ is an ideal of $\mathrm{KP}_R(\Lambda)$. Now by Remark 3.4.1 (c), fix $x \in Z(s(\delta)) \cap \mathfrak{T}_\Lambda$. For each $(\alpha, \beta) \in F$ we have two possibilities:

- If $x \notin F_{\alpha,\beta}$, then there exists $\mu = \mu_{\alpha,\beta}, \nu = \nu_{\alpha,\beta} \in \Lambda$ as in Lemma 3.4.3(a) such that $x \in Z(\mu) \cap Z(\nu)$ and

$$s_\mu s_{\mu^*} s_\alpha s_{\beta^*} s_\nu s_{\nu^*} = 0.$$

- Otherwise, if $x \in F_{\alpha,\beta}$ then there exists $\gamma = \gamma_{\alpha,\beta} \in \Lambda$ as in Lemma 3.4.3(b) such that $x \in Z(\alpha\gamma) \cap Z(\beta\gamma)$ and

$$s_{\alpha\gamma} s_{(\alpha\gamma)^*} s_\alpha s_{\beta^*} s_{\beta\gamma} s_{(\beta\gamma)^*} = s_{\alpha\gamma} s_{(\beta\gamma)^*} \text{ with } (\alpha\gamma, \beta\gamma) \text{ cycline.}$$

Now let m be the following product:

$$m = \Big(\prod_{\substack{(\alpha,\beta)\in F \\ x\in F_{\alpha,\beta}}} s_{\alpha\gamma} s_{(\alpha\gamma)^*}\Big)\Big(\prod_{\substack{(\alpha,\beta)\in F \\ x\notin F_{\alpha,\beta}}} s_\mu s_{\mu^*}\Big)\, b\, \Big(\prod_{\substack{(\alpha,\beta)\in F \\ x\notin F_{\alpha,\beta}}} s_\nu s_{\nu^*}\Big)\Big(\prod_{\substack{(\alpha,\beta)\in F \\ x\in F_{\alpha,\beta}}} s_{\beta\gamma} s_{(\beta\gamma)^*}\Big).$$

Observe that every $s_{\alpha\gamma}s_{(\alpha\gamma)^*}, s_{\beta\gamma}s_{(\beta\gamma)^*}, s_\mu s_{\mu^*}, s_\nu s_{\nu^*}$ is inside $\mathcal{D}$. We have that $m \in \mathcal{M}$ by construction. Since $b \in \text{Ker } \Phi$, we have $m \in \text{Ker } \Phi$.

Let us see that $m \neq 0$: we show that the zero-graded component of m is non-zero, that is, $m_0 \neq 0$. We have

$$m = (\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ r_{\delta,\epsilon}p_{s(\delta)}\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*})\ +\ m',$$

where

$$m' = (\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ (\sum_{d(\alpha)\neq d(\beta)} r_{\alpha,\beta}s_\alpha s_{\beta^*})\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*}).$$

We claim that

$$m_0 = (\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ r_{\delta,\epsilon}p_{s(\delta)}\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*}) \in \mathcal{D}.$$

To see this, consider $(\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ r_{\alpha,\beta}s_\alpha s_{\beta^*}\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*})$ a term in m'. Since $\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*}, \prod s_\mu s_{\mu^*}, \prod s_\nu s_{\nu^*}, \prod s_{\beta\gamma}s_{(\beta\gamma)^*}$ are each 0-graded,

$$(\prod s_{\alpha\gamma}s_{(\alpha\gamma)^*})(\prod s_\mu s_{\mu^*})\ r_{\alpha,\beta}s_\alpha s_{\beta^*}\ (\prod s_\nu s_{\nu^*})(\prod s_{\beta\gamma}s_{(\beta\gamma)^*})$$

is $d(\alpha) - d(\beta) \neq 0$ graded. Thus m_0 is as claimed.

Recall from Lemma 3.2.4, we have an isomorphism $\pi : \mathcal{D} \to A_{\mathcal{D}}$. To see that $m_0 \neq 0$, it suffices to show $\pi(m_0) \neq 0$. Now

$$\begin{aligned}\pi(m_0) &= r_{\delta,\epsilon}(\prod 1_{Z(\alpha\gamma)})(\prod 1_{Z(\mu)}\ 1_{Z(s(\delta))})\ (\prod 1_{Z(\nu)})(\prod 1_{Z(\beta\gamma)}) \\ &= r_{\delta,\epsilon}1_U\end{aligned}$$

where

$$U = \left(\bigcap Z(\alpha\gamma)\right) \cap \left(\bigcap Z(\mu)\right) \cap Z(s(\delta)) \cap \left(\bigcap Z(\nu)\right) \cap \left(\bigcap Z(\beta\gamma)\right).$$

By construction $x \in U$ and hence $1_U \neq 0$. So $\pi(m_0) \neq 0$ and finally $m \neq 0$ as desired.

This contradicts the assumption that $\Phi|_{\mathcal{M}}$ is injective because we found $0 \neq m \in \mathcal{M}$ and $m \in \text{Ker } \Phi$. Therefore Φ is injective. □

From Theorem 3.4.4 we can recover the usual Cuntz-Krieger uniqueness theorem given in [16, Theorem 4.7]. First we need to consider the following lemma.

Lemma 3.4.5. *If Λ is aperiodic, then (α, β) is a cycline pair if and only if $\alpha = \beta$. In particular, $\mathcal{M} = \mathcal{D}$.*

Proof. This result is proved similarly to [25, Proposition 4.8]. First let us show that if $\alpha \neq \beta$ and there exists an aperiodic path $x \in s(\alpha)\Lambda^\infty$, then (α, β) is not a cycline pair. In this situation, both αx and βx are aperiodic, so $\alpha \neq \beta$ implies $\alpha x \neq \beta x$. Applying Proposition 3.3.1, (α, β) is not cycline. Finally the lemma follows because of the definition of aperiodicity. □

Corollary 3.4.6. *Let Λ be an aperiodic row-finite k-graph without sources and R be a commutative ring with 1. If $\Phi : \mathrm{KP}_R(\Lambda) \to A$ is a ring homomorphism, then Φ is injective if and only if $\Phi(rp_v) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$.*

Proof. We verify that $\Phi(rp_v) \neq 0$ for all $r \in R \setminus \{0\}$ and $v \in \Lambda^0$ implies Φ is injective. We show $\Phi|_{\mathcal{M}}$ is injective, which suffices by Theorem 3.4.4. First, by Lemma 3.4.5, $\mathcal{M} = \mathcal{D}$. By way of contradiction suppose there exist $r \in R \setminus \{0\}$ and $\lambda \in \Lambda$ such that $\Phi(rs_\lambda s_{\lambda^*}) = 0$. But then $\Phi(s_{\lambda^*}(rs_\lambda s_{\lambda^*})s_\lambda) = 0$. Thus $\Phi(rp_{s(\lambda)}) = 0$, which is a contradiction. □

Chapter 4

Leavitt path algebras of generalized Cayley graphs

4.1 Cayley graphs C_n^j

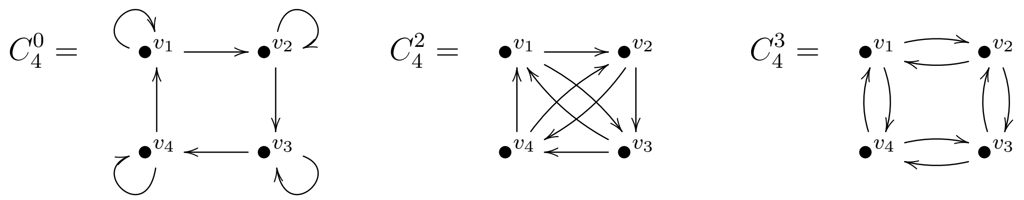
The last part of the thesis is dedicated to the study of the Leavitt path algebras over a field K of some specific graphs: for $n \in \mathbb{N}$ and $0 \leq j \leq n-1$ the generalized *Cayley graphs* of the form C_n^j. Let us start by giving all the background information about the topic: we introduce the Cayley graphs C_n^j and their corresponding *graphs monoids*.

For any finite group G, and any set of generators S of G, the *Cayley graph of* G *with respect to* S is the directed graph $E_{G,S}$ having vertex set $\{v_g \mid g \in G\}$, and in which there is an edge from v_g to v_h in case there exists $s \in S$ with $h = gs$ in G.

Definition 4.1.1. Let n be a positive integer ($n \geq 3$) and let j be an integer for which $0 \leq j \leq n-1$, the graph C_n^j is the Cayley graph $E_{G,S}$ of the cyclic group $G = \mathbb{Z}_n$ with respect to the generating subset $S = \{1, j\}$ of $\mathbb{Z}_n$. Concretely, the *Cayley graph* C_n^j is the directed graph consisting of n vertices $\{v_1, v_2, \ldots, v_n\}$ and $2n$ edges $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ for which $s(e_i) = v_i$, $r(e_i) = v_{i+1}$, $s(f_i) = v_i$, and $r(f_i) = v_{i+j}$, where indices are interpreted mod n.

Intuitively, C_n^j is the graph with n vertices and $2n$ edges, where each vertex v emits two edges, one to its "next" neighboring vertex (which we will draw throughout this article in a clockwise direction), and one to the vertex j places clockwise from v.

Example 4.1.2.



Suppose E is a directed graph with vertices $v_1, v_2, \ldots, v_n$ and incidence matrix $A_E = (a_{i,j})$. We let F_n denote the free abelian monoid on the generators $v_1, v_2, \ldots, v_n$ (so $F_n \cong \bigoplus_{i=1}^n \mathbb{Z}^+$ as monoids). We denote the identity element of this monoid by z. We define a relation $\approx$ on F_n by setting

$$v_i \approx \sum_{j=1}^{n} a_{i,j} v_j$$

for each non-sink v_i, and denote by $\sim_E$ the equivalence relation on F_n generated by $\approx$.

For two $\sim_E$ equivalence classes $[x]$ and $[y]$ we define $[x]+[y] = [x+y]$: this gives a well-defined associative binary operation on the set of $\sim_E$ equivalence classes, such that $[z]$ acts as an identity element for this operation. We denote the resulting *graph monoid* $F_n/\sim_E$ by M_E (see Definition 1.4.1).

Let us focus our attention when the graph E is the Cayley graph C_n^j.

Definition 4.1.3. For any $n \geq 1$ and $0 \leq j \leq n-1$, the graph monoid $M_{C_n^j}$ is the monoid generated by $[v_1], [v_2], \ldots, [v_n]$, subject to the relations

$$[v_i] = [v_{i+1}] + [v_{i+j}]$$

(for all $1 \leq i \leq n$), where subscripts are interpreted mod n.

Let us consider again the Leavitt path algebras over a field K. On the one hand, by Theorem 1.3.2 we have that $L_K(C_n^j)$ is purely infinite simple. This result, together with the discussion given in Section 1.4, immediately yields:

Proposition 4.1.4. ([7, Proposition 1.3]) *For each $n \geq 1$, and for each $0 \leq j \leq n-1$, the K-algebra $L_K(C_n^j)$ is unital purely infinite simple. In particular, $M^*_{C_n^j} = (M_{C_n^j} \setminus \{[z]\}, +)$ is a group.*

Remark 4.1.5. In fact $\sum_{i=1}^n [v_i]$ is the identity element of the group $M^*_{C_n^j}$, because if σ denotes $\sum_{i=1}^n [v_i]$ then by the defining relations in $M^*_{C_n^j}$, we have

$$\sigma = \sum_{i=1}^n [v_i] = \sum_{i=1}^n ([v_{i+1}] + [v_{i+j}]) = \sum_{i=1}^n [v_{i+1}] + \sum_{i=1}^n [v_{i+j}] = \sigma + \sigma.$$

And $\sigma + \sigma = \sigma$ in a group means that σ is the identity element.

In [7, Definition 2.2] the integer $|\det(I_n - A^t_{C_n^k})|$ is denoted by $H_k(n)$, and, for fixed k the sequence

$$H_k(1), H_k(2), H_k(3), ...$$

is referred to as the k^{th} *Haselgrove sequence* since it results that these $H_k(n)$ coincide (up to sign) with the integers investigated by Haselgrove in [42]. Concretely in [7] it was proved the next two results .

Proposition 4.1.6. ([7, Proposition 2.3]) *Let $n \in \mathbb{N}$, and $0 \leq j \leq n-1$.*

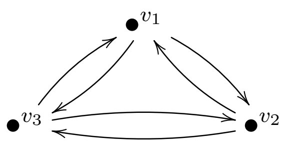
(1) If $H_j(n) > 0$, then $|K_0(L_K(C_n^j))| = H_j(n)$.

(2) If $H_j(n) = 0$, then $K_0(L_K(C_n^j))$ is infinite.

Proposition 4.1.7. ([7, Proposition 2.5]) *For each $n \geq 1$ and $0 \leq j \leq n-1$, $\det(I_n - A^t_{C_n^j}) \leq 0$. In particular, $\det(I_n - A^t_{C_n^j}) = -H_j(n)$ for every such pair n, j.*

Example 4.1.8. As an example let us explicitly describe the graph monoid $M_{C_3^2}$ associated to the Cayley graph C_3^2:

$C_3^2 =$



The generators of this monoid are $[v_1], [v_2]$ and $[v_3]$, subject to the relations

$$[v_1] = [v_2] + [v_3], [v_2] = [v_1] + [v_3], \text{ and } [v_3] = [v_1] + [v_2].$$

Note that $[v_1] + [v_1 + v_2 + v_3] = [v_1 + v_2] + [v_1 + v_3] = [v_3] + [v_2] = [v_1]$, so that $[v_1] = [v_1] + [v_1 + v_2 + v_3]$ in $M_{C_3^2}$. Let us denote $z := v_1 + v_2 + v_3$. A similar computation yields that $[v_2] = [v_2] + [z]$ and $[v_3] = [v_3] + [z]$ in $M_{C_3^2}$. Moreover $[v_1] + [v_1] = [v_1] + [v_2 + v_3] = [z]$ and in similar way we also get $2[v_2] = 2[v_3] = [z]$ in $M_{C_3^2}$. Hence we see that

$$M_{C_3^2} = \{[z], [v_1], [v_2], [v_3], [v_1] + [v_2] + [v_3]\};$$

and $M^*_{C_3^2} = \{[v_1], [v_2], [v_3], [v_1]+[v_2]+[v_3]\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and that $[v_1]+[v_2]+[v_3]$ is the identity element of the group $M^*_{C_3^2}$. Besides, we have that the group $M^*_{C_3^2} \cong K_0(L_K(C_3^2))$ contains $H_2(3)$ elements, that is, $H_2(3) = 4$ elements according to [42].

4.2 A new approach to Leavitt path algebras of the graphs C_n^2

Our main aim in the current section will be to come back to the case C_n^2, originally studied in [7, Section 4], and propose an alternative, more systematic way of computing the Grothendieck group of the Leavitt path algebra $L_K(C_n^2)$ in a totally different point of view from the work done in [7]. Taking into account the ideas presented in the literature from Sections 1.3 and 1.4 and by using the *Smith normal form* of an integer-valued matrix, we will be able to (re)compute the group K_0 of this purely infinite simple Leavitt path algebra. In the end the information provided by the K_0 gives us a natural classification-type answer in the context of Leavitt path algebras which is given by the algebraic Kirchberg-Phillips theorem.

So let us focus our attention to the specific case of the Leavitt path algebras of the form $L_K(C_n^2)$. If $n \in \mathbb{N}$ the generating relations for the group $M^*_{C_n^2}$ are given by

$$[v_i] = [v_{i+1}] + [v_{i+2}]$$

for $1 \leq i \leq n$, where subscripts are interpreted mod n. In this case, a central role is played by the elements of the standard *Fibonacci sequence* F, defined by setting $F(1) = 1$, $F(2) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n \geq 3$. (Define $F(0) = 0$ consistently with the given recursion equation.) Of course, F is the well-known sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

Focusing on the generator $[v_1]$ of $M^*_{C_n^2}$ (thanks to the symmetry of the relations for each $[v_i]$, $1 \leq i \leq n$ in $M^*_{C_n^2}$) notice that

$$\begin{aligned}[v_1] &= [v_2] + [v_3] = ([v_3] + [v_4]) + [v_3] = 2[v_3] + [v_4] \\ &= 2([v_4] + [v_5]) + [v_4] = 3[v_4] + 2[v_5] = \cdots\end{aligned}$$

which inductively gives, for $1 \leq i \leq n$,

$$[v_1] = F(i)[v_i] + F(i-1)[v_{i+1}]$$

in $M^*_{C_n^2}$. Setting $i = n$, and using that $[v_{n+1}] = [v_1]$ by notational convention, we get in particular that $[v_1] = F(n)[v_n] + F(n-1)[v_1]$, so that

$$0 = F(n)[v_n] + (F(n-1) - 1)[v_1] \text{ in } M^*_{C_n^2}.$$

Recall that for integers a, b, $\gcd(a, b)$ denotes the greatest common divisor of a and b. A key role is played by the following integer.

Definition 4.2.1. For any positive integer n we define

$$d_2(n) = \gcd(F(n), F(n-1) - 1).$$

The previous definition comes from [7, Definition 4.1], where $\gcd(F(n), F(n-1) - 1)$ is simply denoted by $d(n)$. In particular, the previous displayed equation yields: if x denotes the well-defined element

$$\frac{F(n)}{d_2(n)}[v_n] + \frac{F(n-1) - 1}{d_2(n)}[v_1]$$

of $M^*_{C_n^2}$, then $d_2(n)x = 0$ in $M^*_{C_n^2}$. In [7, Theorem 4.13], it is shown that

$$K_0(L_K(C_n^2)) \cong \mathbb{Z}_{d_2(n)} \times \mathbb{Z}_{\frac{H_2(n)}{d_2(n)}},$$

for every $n \in \mathbb{N}$ (interpreting $\mathbb{Z}_1$ as the trivial group $\{0\}$). Now our purpose consists of analyzing the structure of the K_0 of the Leavitt path algebras of C_n^2 from a totally different point of view from which was given in [7, Section 4]. We will obtain as a result [7, Theorem 4.13] in a simpler manner. Following the idea given in Proposition 1.4.6 now we compute the Smith normal form of the matrix $I_n - A^t_{C_n^2}$: we are able to show that this computation simply reduces to the case of calculating the Smith normal form of a smaller matrix (of size 2×2), which is very closely related to the Fibonacci sequence.

Definition 4.2.2. The *companion matrix* of the monic polynomial

$$p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-1} x^{n-1} + x^n$$

where $c_i \in \mathbb{Z}$ for $i \in \{0, 1, \ldots, n-1\}$, is the $n \times n$ square matrix defined as

$$\mathrm{M(p)} = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -c_0 \\ 1 & 0 & 0 & \ldots & 0 & -c_1 \\ 0 & 1 & 0 & \ldots & 0 & -c_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -c_{n-2} \\ 0 & 0 & 0 & \ldots & 1 & -c_{n-1} \end{bmatrix}$$

Let us consider the characteristic polynomial associated to the Fibonacci sequence F, that is, $p(x) = x^2 - x - 1$. Then its corresponding companion matrix, which we will denote by M_2, is the 2×2 matrix given as

$$M_2 := \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Lemma 4.2.3. *For $n \geq 1$,*

$$M_2^n = \begin{bmatrix} F(n-1) & F(n) \\ F(n) & F(n+1) \end{bmatrix}$$

where the F's are the Fibonacci numbers.

Proof. Let us prove it by induction. For $n = 1$ we have

$$M_2 = \begin{bmatrix} F(0) & F(1) \\ F(1) & F(2) \end{bmatrix}$$

Suppose it is true that

$$M_2^{n-1} = \begin{bmatrix} F(n-2) & F(n-1) \\ F(n-1) & F(n) \end{bmatrix}$$

So then

$$M_2^{n-1} \cdot M_2 = \begin{bmatrix} F(n-2) & F(n-1) \\ F(n-1) & F(n) \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

which gives

$$M_2^n = \begin{bmatrix} F(n-1) & F(n-2)+F(n-1) \\ F(n) & F(n-1)+F(n) \end{bmatrix} = \begin{bmatrix} F(n-1) & F(n) \\ F(n) & F(n+1) \end{bmatrix}$$

□

Recall that the adjacency matrix for the graph C_n^2 is of the form

$$A_{C_n^2} := \begin{bmatrix} 0 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

Proposition 4.2.4. *The matrices $A_{C_n^2} - I_n$ and $M_2^n - I_3$ have the same Smith normal form.*

Proof. First of all,

$$A_{C_n^2} - I_n = \begin{bmatrix} -1 & 1 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}$$

In particular, the first $n-2$ rows have only nonzero entries of -1 along the diagonal, and 1 in the columns 1 and 2 after the diagonal. More precisely, for $1 \leq i \leq n-2$,

$$a_{ij} = \begin{cases} -1 & \text{if } j = i \\ 1 & \text{if } j = i+1 \text{ or } j = i+2 \\ 0 & \text{otherwise.} \end{cases}$$

Recall that Smith normal form is obtained by elementary row and column operations. The goal is to obtain a diagonal matrix with integers along the diagonal. In the first step, adding column 1 to column 2 and column 3 results in the following matrix:

$$A_{C_n^2} - I_n \sim \begin{bmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 & & 0 & -1 & 1 \\ 1 & 2 & 1 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}$$

Now adding row 1 to row $n-1$ and row n:

$$A_{C_n^2} - I_n \sim \begin{bmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}$$

Finally, multiplying row 1 by -1:

$$A_{C_n^2} - I_n \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & & -1 & 1 & 1 \\ 0 & 1 & 1 & 0 & & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}$$

The resulting matrix now has 1 in the first entry of the diagonal and zeroes along the rest of the first row and first column.

Now we can repeat the process, noting that the only nonzero entries below the diagonal only occur in the last 2 rows. For the i^{th} step, we perform the following operations:

1. Add the i^{th} column to column $i+1$ and column $i+2$.

2. Add the i^{th} row (now containing -1 in the i^{th} entry and zeroes everywhere else) to the bottom two rows in order to make them zero.

3. Multiply the i^{th} row by -1.

For completeness, below is the result of applying the second step of the process:

$$A_{C_n^2} - I_n \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & & -1 & 1 & 1 \\ 0 & 0 & 2 & 1 & & 0 & -1 & 1 \\ 0 & 0 & 3 & 2 & \dots & 0 & 0 & -1 \end{bmatrix}$$

Focusing on what the i^{th} step does to the bottom two rows (at least for $1 \le i \le n-4$):

1. Adds i^{th} column to column $i+1$, while column $i+2$ becomes the i^{th} column (since column $i+2$ is all zeroes in the last three rows).

2. Zeroes out the i^{th} column.

3. Nothing.

We note that the bottom left 2×2 minor of the original matrix $A_{C_n^2} - I_n$ can be written as

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F(1) & F(0) \\ F(2) & F(1) \end{bmatrix}.$$

Applying the first step of the process, the first 2×2 minor at the bottom with nonzero columns is

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} F(0)+F(1) & F(1) \\ F(1)+F(2) & F(2) \end{bmatrix} = \begin{bmatrix} F(2) & F(1) \\ F(3) & F(2) \end{bmatrix}.$$

Proceeding by induction, suppose that i steps of the reduction process results in the first 2×2 minor at the bottom with nonzero columns given by

$$\begin{bmatrix} F(i+1) & F(i) \\ F(i+2) & F(i+1) \end{bmatrix}.$$

Applying the procedure for step $i+1$, the next 2×2 minor will be

$$\begin{bmatrix} F(i)+F(i+1) & F(i+1) \\ F(i+1)+F(i+2) & F(i+2) \end{bmatrix} = \begin{bmatrix} F(i+2) & F(i+1) \\ F(i+3) & F(i+2) \end{bmatrix}.$$

We conclude by induction that for $1 \leq i \leq n$, the result of i steps on the first 2×2 minor at the bottom with nonzero columns is

$$\begin{bmatrix} F(i+1) & F(i) \\ F(i+2) & F(i+1) \end{bmatrix}.$$

Finally, we must apply the procedure to the last two columns. After $n-4$ steps, the matrix is now of the form

$$A_{C_n^2} - I_n \sim \begin{bmatrix} 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & & -1 & 1 & 1 & 0 \\ 0 & & 0 & -1 & 1 & 1 \\ 0 & & F(n-3) & F(n-4) & -1 & 1 \\ 0 & \dots & F(n-2) & F(n-3) & 0 & -1 \end{bmatrix}.$$

Applying the procedure two more times results in the following 2×2 minor in the bottom right corner:

$$\begin{bmatrix} F(n-1)-1 & F(n-2)+1 \\ F(n) & F(n-1)-1 \end{bmatrix}$$

Lastly, we add column $n-1$ to column n:

$$\begin{bmatrix} F(n-1)-1 & F(n) \\ F(n) & F(n+1)-1 \end{bmatrix}$$

and hence

$$A_{C_n^2} - I_n \sim \begin{bmatrix} 1 & \dots & 0 & 0 & 0 \\ \vdots & \ddots & & & \\ 0 & & 1 & 0 & 0 \\ 0 & & 0 & F(n-1)-1 & F(n) \\ 0 & \dots & 0 & F(n) & F(n+1)-1 \end{bmatrix}.$$

Since the bottom corner is equal to $M_2^n - I_2$ by Lemma 4.2.3, we conclude that, except for extra 1 elements in the (bigger) diagonal of $A_{C_n^2} - I_n$ and the different size of the matrices, $A_{C_n^2} - I_n$ and $M_2^n - I_2$ have the same Smith normal form. □

Corollary 4.2.5. *The matrices* $I_n - A^t_{C_n^2}$ *and* $(M_2^n)^t - I_2$ *have the same Smith normal form , that is,*

$$\mathrm{SNF}(I_n - A^t_{C_n^2}) = \mathrm{SNF}\begin{bmatrix} F(n-1)-1 & F(n) \\ F(n) & F(n+1)-1 \end{bmatrix},$$

where F's are the Fibonacci numbers and SNF *denotes the Smith normal form of the matrix.*

At this point recall from Definition 1.4.3 how the Smith normal form is specifically presented for any $n \times n$ matrix $M = (m_{i,j})$ with $m_{i,j} \in \mathbb{Z}$: there exist unimodular matrices U, V so that the product UMV is a diagonal matrix $S = \mathrm{diag}(s_1, s_2, \ldots, s_r, 0, \ldots, 0)$ where r is the rank of M and $s_i \mid s_{i+1}$ for every $i \in \{1, \ldots, r-1\}$; s_i are the invariant factors of M. Let us define the i^{th} determinant divisor as

$$\begin{aligned} &\delta_0 := 1, \\ &\delta_i = \gcd(\text{all } i \times i \text{ minors of } M) \text{ for } i \geq 1. \end{aligned}$$

Then in [57] it was proved that

$$s_i = \frac{\delta_i}{\delta_{i-1}}.$$

Let us focus now our attention to the particular case of the matrix $(M_2^n)^t - I_2 = M_2^n - I_2$.

Definition 4.2.6. Consider $n \in \mathbb{N}$ and

$$(M_2^n)^t - I_2 = \begin{bmatrix} F(n-1)-1 & F(n) \\ F(n) & F(n+1)-1 \end{bmatrix}.$$

For $i = 0, 1, 2$ the corresponding *i-th determinant divisors* are

$$\begin{aligned} &\delta_0(n) := 1, \\ &\delta_1(n) := \gcd\{F(n-1)-1, F(n), F(n+1)-1\}, \\ &\delta_2(n) := |\det((M_2^n)^t - I_2)|. \end{aligned}$$

In conclusion the Smith normal form of the matrix $(M_2^n)^t - I_2$ is given by:

$$\mathrm{SNF}((M_2^n)^t - I_2) = \begin{bmatrix} \delta_1(n) & 0 \\ 0 & \frac{\delta_2(n)}{\delta_1(n)} \end{bmatrix}.$$

As a consequence, Corollary 4.2.5 immediately yields the following result.

Corollary 4.2.7. *Let $n \in \mathbb{N}$. Then*

$$K_0(L_K(C_n^2)) \cong \mathbb{Z}_{\delta_1(n)} \times \mathbb{Z}_{\frac{\delta_2(n)}{\delta_1(n)}},$$

where $\delta_i(n) = \gcd\{$all $i \times i$ minors of the matrix $(M_2^n)^t - I_2\}$ for $i = 1, 2$ (interpreting $\mathbb{Z}_1$ as the trivial group $\{0\}$) where

$$(M_2^n)^t - I_2 = \begin{bmatrix} F(n-1) - 1 & F(n) \\ F(n) & F(n+1) - 1 \end{bmatrix}.$$

Remark 4.2.8. This previous statement about the Smith normal form of the matrix $(M_2^n)^t - I_2$ follows from the fact that the rank of $(M_2^n)^t - I_2$ is 2 for every n. We will see it in Lemma 4.2.10. We include here in order to clarify what now are our main targets.

Corollary 4.2.7 results to be one of the main ingredients in hand to prove the main result about the structure of the $K_0(L_K(C_n^2))$. Then we need to compute explicitly the determinant divisors $\delta_1(n)$, and $\delta_2(n)$. To get it we will use the following well-known property of greatest common divisors:

$$\gcd(a, b) = \gcd(a, b + ka) = \gcd(a + mb, b) \text{ for every } a, b, k, m \in \mathbb{Z}.$$

To establish other results in the sequel as well we will often use that fact, and in our proofs we will note it by ($\stackrel{*}{=}$).

Lemma 4.2.9. *Let $n \in \mathbb{N}$ and $d_2(n) = \gcd(F(n-1)-1, F(n))$. Then $\delta_1(n) = d_2(n)$.*

Proof. Recall from Definition 4.2.6 that $\delta_1(n) = \gcd\{F(n-1)-1, F(n), F(n+1) - 1\}$. So

$$\begin{aligned} \delta_1(n) &= \gcd\{F(n-1) - 1, F(n), F(n+1) - 1\} \\ &= \gcd\{F(n-1) - 1, F(n), F(n) + F(n-1) - 1\} \\ &\stackrel{*}{=} \gcd\{F(n-1) - 1, F(n), F(n-1) - 1\} \\ &= d_2(n). \end{aligned}$$

□

To finish, let us see that the second determinant divisor $\delta_2(n)$ coincides with $H_2(n)$. Note that the proof requires to use some nontrivial equations involving F and H_2.

Lemma 4.2.10. *Let $n \in \mathbb{N}$ and $\delta_2(n) = |\det((M_2^n)^t - I_2)|$. Then $\delta_2(n) = H_2(n) \neq 0$.*

Proof. Recall that

$$(M_2^n)^t - I_2 = \begin{bmatrix} F(n-1)-1 & F(n) \\ F(n) & F(n+1)-1 \end{bmatrix}.$$

Then

$$\begin{aligned}
\delta_2(n) &= |\det((M_2^n)^t - I_2)| \\
&= |(F(n-1)-1)(F(n+1)-1) - F(n)^2| \\
&= |F(n-1)F(n+1) - F(n+1) - F(n-1) + 1 - F(n)^2| \\
&= |-F(n+1) - F(n-1) + 1 + F(n+1)F(n-1) - F(n)^2| \\
&= |-F(n+1) - F(n-1) + 1 + (-1)^n| \text{ by [7, Proposition 4.4]} \\
&= |-H_2(n)| \text{ by [7, Proposition 4.4]} \\
&= H_2(n).
\end{aligned}$$

□

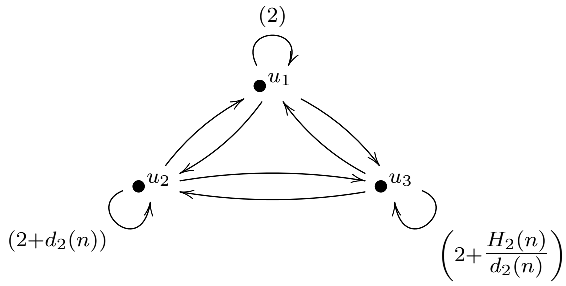
Applying Corollary 4.2.7, Lemma 4.2.9 and Lemma 4.2.10 finally we get the result originally given in [7, Theorem 4.13] .

Corollary 4.2.11. *Let $n \in \mathbb{N}$. Then*

$$K_0(L_K(C_n^2)) \cong \mathbb{Z}_{d_2(n)} \times \mathbb{Z}_{\frac{H_2(n)}{d_2(n)}}.$$

We finish this section with the result given by [7] which explicitly realize the algebras $L_K(C_n^2)$ as the Leavitt path algebras of graphs having three vertices. This is an immediate consequence of Theorem 1.4.7 . Notice that the algebras $\{L_K(C_n^2) \mid n \in \mathbb{N}\}$ are mutually non-isomorphic as their K_0 groups have size $H_2(n)$, and this is an strictly increasing sequence.

Proposition 4.2.12. ([7, Proposition 4.14]) *Let $n \in \mathbb{N}$. The algebra $L_K(C_n^2)$ is isomorphic to the Leavitt path algebra of the graph E_n with three vertices given by*



where the numbers in parentheses indicate the number of loops at a given vertex. Furthermore, when $K_0(L_K(C_n^2))$ is not cyclic, this realization is minimal in the sense that it is not possible to realize $L_K(C_n^2)$ up to isomorphism as the Leavitt path algebra of a graph having less than three vertices.

Chapter 5

Leavitt path algebras of Cayley graphs C_n^3

5.1 The Smith normal form of the matrix $I - A^t_{C_n^3}$

Our next step in the study of the Leavitt path algebras of generalized Cayley graphs consists of investigating the Leavitt path algebras corresponding to the graphs $\{L_K(C_n^3) \mid n \in \mathbb{N}\}$. In order to do that we proceed similarly to the case C_n^2 studied in Section 4.2. Now the generating relations for $M^*_{C_n^3}$ are given by

$$[v_i] = [v_{i+1}] + [v_{i+3}]$$

for $1 \leq i \leq n$, where subscripts are interpreted mod n. We will focus on the generator $[v_1]$ of $M^*_{C_n^3}$; corresponding to any statement established for $[v_1]$ in $M^*_{C_n^3}$, there will be (by the symmetry of the relations) an analogous statement in $M^*_{C_n^3}$ for each $[v_i]$, $1 \leq i \leq n$.

Thanks to Proposition 4.1.6 the sequence $H_3(n)$ will play a key role in this analysis. In [42], Haselgrove establishes that the values of $H_3(n)$ may be defined by setting $H_3(1) = 1, H_3(2) = 3$, $H_3(3) = 1$, $H_3(4) = 3$, $H_3(5) = 11$, $H_3(6) = 9$ and defining the remaining values (for $n \geq 7$) recursively as:

$$H_3(n) = H_3(n-1) - H_3(n-2) + 3H_3(n-3) - H_3(n-4) + H_3(n-5) - H_3(n-6).$$

These integers which appear in the third Haselgrove sequence are also known as the *associated Mersenne numbers* in the Online Encyclopedia of

Integer Sequences [51, Sequence A001351]. The first few terms of the third Haselgrove sequence H_3 are thus:

$$1, 3, 1, 3, 11, 9, 8, 27, 37, 33, 67, 117, 131, 192, 341, ...$$

An important role will be played by the elements of the *Narayana's cows sequence* G, defined by setting $G(1) = 1$, $G(2) = 1$, $G(3) = 1$ and $G(n) = G(n-1) + G(n-3)$ for all $n \geq 4$. (We may also define $G(0) = 0$, $G(-1) = 0$, $G(-2) = 1$ and $G(-3) = 0$ consistently with the given recursion equation). This name is used in the Online Encyclopedia of Integer Sequences [51, Sequence A000930] . The first terms of G are:

$$1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, ...$$

We begin by noticing that

$$\begin{aligned}[v_1] &= [v_2] + [v_4] = ([v_3] + [v_5]) + [v_4] = ([v_4] + [v_6]) + [v_5] + [v_4] \\ &= 2[v_4] + [v_5] + [v_6] = 2([v_5] + [v_7]) + [v_5] + [v_6] \\ &= 3[v_5] + [v_6] + 2[v_7] = \cdots\end{aligned}$$

which inductively gives, for $1 \leq i \leq n$,

$$[v_1] = G(i-1)[v_{i-1}] + G(i-3)[v_i] + G(i-2)[v_{i+1}]$$

in $M^*_{C^3_n}$. Setting $i = n$, and using that $[v_{n+1}] = [v_1]$ by notational convention, we get in particular that $[v_1] = G(n-1)[v_{n-1}] + G(n-3)[v_n] + G(n-2)[v_1]$, so that

$$0 = G(n-1)[v_{n-1}] + G(n-3)[v_n] + (G(n-2) - 1)[v_1] \text{ in } M^*_{C^3_n}. \tag{5.1}$$

Again for integers a, b, $\gcd(a, b)$ denotes the greatest common divisor of a and b. A key role will be played by the following integer; in fact, we will come back to this number in the next section.

Definition 5.1.1. For any positive integer n we define

$$d_3(n) = \gcd\big(G(n-1), G(n-3), G(n-2) - 1\big).$$

In particular Equation 5.1 yields this lemma.

Lemma 5.1.2. *We let x denote the element*

$$\frac{G(n-1)}{d_3(n)}[v_{n-1}] + \frac{G(n-3)}{d_3(n)}[v_n] + \frac{G(n-2)-1}{d_3(n)}[v_1]$$

of $M^*_{C_n^3}$. Then $d_3(n)x = 0$ in $M^*_{C_n^3}$.

We are now in the position to develop our main machinery in order to analyze the structure of the K_0 of the Leavitt path algebras of C_n^3. Following the new approach given in Section 4.2, this way offers a totally different point of view from [7, Section 4] for the Leavitt path algebras of the form $L_K(C_n^2)$. Analogously to the work done in Section 4.2, the computation of the Smith normal form of the matrix $I_n - A^t_{C_n^3}$ is the key tool for determining the K_0 of the Leavitt path algebra C_n^3 (see Proposition 1.4.6). We are able to show that this computation simply reduces to the case of calculating the Smith normal form of a smaller matrix (of size 3×3). In fact, this special 3×3 matrix will very closely related to the Narayana's cows sequence.

Let us consider the characteristic polynomial associated to the Narayana's cows sequence G, that is, $p(x) = x^3 - x^2 - 1$. In this case its corresponding companion matrix, which we will denote by M_3, is the 3×3 matrix given as

$$M_3 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Lemma 5.1.3. *For $n \geq 1$,*

$$M_3^n = \begin{bmatrix} G(n-2) & G(n-1) & G(n) \\ G(n-3) & G(n-2) & G(n-1) \\ G(n-1) & G(n) & G(n+1) \end{bmatrix}$$

where the G's are the Narayana's cows numbers.

Proof. Let us prove it by induction. Extending the $G(n)$ sequence to $G(0) = 0$, $G(-1) = 0$, $G(-2) = 1$, and $G(-3) = 0$ we have

$$M_3 = \begin{bmatrix} G(-1) & G(0) & G(1) \\ G(-2) & G(-1) & G(0) \\ G(0) & G(1) & G(2) \end{bmatrix}$$

Suppose the following holds:

$$M_3^{n-1} = \begin{bmatrix} G(n-3) & G(n-2) & G(n-1) \\ G(n-4) & G(n-3) & G(n-2) \\ G(n-2) & G(n-1) & G(n) \end{bmatrix}$$

So then

$$M_3^{n-1} \cdot M_3 = \begin{bmatrix} G(n-3) & G(n-2) & G(n-1) \\ G(n-4) & G(n-3) & G(n-2) \\ G(n-2) & G(n-1) & G(n) \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

which gives

$$\begin{aligned} M_3^n &= \begin{bmatrix} G(n-2) & G(n-1) & G(n-3)+G(n-1) \\ G(n-3) & G(n-2) & G(n-4)+G(n-2) \\ G(n-1) & G(n) & G(n-2)+G(n) \end{bmatrix} \\ &= \begin{bmatrix} G(n-2) & G(n-1) & G(n) \\ G(n-3) & G(n-2) & G(n-1) \\ G(n-1) & G(n) & G(n+1) \end{bmatrix} \end{aligned}$$

□

Recall that the adjacency matrix for the graph C_n^3 is of the form

$$A_{C_n^3} := \begin{bmatrix} 0 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 1 & 0 & 0 & 0 & & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \end{bmatrix}$$

Proposition 5.1.4. *The matrices $A_{C_n^3} - I_n$ and $M_3^n - I_3$ have the same Smith normal form.*

Proof. First of all,

$$A_{C_n^3} - I_n = \begin{bmatrix} -1 & 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 1 & 0 & 0 & 0 & & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & & 0 & -1 & 1 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}$$

In particular, the first $n-3$ rows have only nonzero entries of -1 along the diagonal, and 1 in the columns 1 and 3 after the diagonal. More precisely, for $1 \leq i \leq n-3$,

$$a_{ij} = \begin{cases} -1 & \text{if } j = i \\ 1 & \text{if } j = i+1 \text{ or } j = i+3 \\ 0 & \text{otherwise.} \end{cases}$$

Our goal is to obtain a diagonal matrix with integers along the diagonal by row and column operations. In the first step, adding column 1 to column 2 and column 4 results in the following matrix:

$$A_{C_n^3} - I_n \sim \begin{bmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 1 & 1 & 0 & 1 & & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & \dots & 0 & 0 & -1 \end{bmatrix}$$

Now adding row 1 to row $n-2$ and row n:

$$A_{C_n^3} - I_n \sim \begin{bmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 1 & 0 & 1 & & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & \dots & 0 & 0 & -1 \end{bmatrix}$$

Finally, multiplying row 1 by -1:

$$A_{C_n^3} - I_n \sim \begin{bmatrix} 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & & 0 & 0 & 0 \\ \vdots & & & & \ddots & & & \vdots \\ 0 & 1 & 0 & 1 & & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 & \dots & 0 & 0 & -1 \end{bmatrix}$$

The resulting matrix now has 1 in the first entry of the diagonal and zeroes along the rest of the first row and first column.

Now we can repeat the process, noting that the only nonzero entries below the diagonal only occur in the last 3 rows. For the i^{th} step, we perform the following operations:

1. Add the i^{th} column to column $i+1$ and column $i+3$.
2. Add the i^{th} row (now containing -1 in the i^{th} entry and zeroes everywhere else) to the bottom three rows in order to make them zero.
3. Multiply the i^{th} row by -1.

Below is the result of applying the second step of the process:

$$A_{C_n^3} - I_n \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & & 0 & 0 & 0 \\ \vdots & & & & & \ddots & & & \vdots \\ 0 & 0 & 1 & 1 & 1 & & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & & 0 & -1 & 1 \\ 0 & 0 & 2 & 1 & 1 & \dots & 0 & 0 & -1 \end{bmatrix}$$

Focusing on what the i^{th} step does to the bottom three rows (at least for $1 \leq i \leq n-6$):

1. Adds i^{th} column to column $i+1$, while column $i+3$ becomes the i^{th} column (since column $i+3$ is all zeroes in the last three rows).
2. Zeroes out the i^{th} column.
3. Nothing.

We note that the bottom left 3×3 minor of the original matrix $A_{C_n^3} - I_n$ can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} G(1) & G(-1) & G(0) \\ G(0) & G(-2) & G(-1) \\ G(2) & G(0) & G(1) \end{bmatrix}.$$

Applying the first step of the process, the first 3×3 minor at the bottom with nonzero columns is

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} G(-1)+G(1) & G(0) & G(1) \\ G(-2)+G(0) & G(-1) & G(0) \\ G(0)+G(2) & G(1) & G(2) \end{bmatrix} = \begin{bmatrix} G(2) & G(0) & G(1) \\ G(1) & G(-1) & G(0) \\ G(3) & G(1) & G(2) \end{bmatrix}.$$

Proceeding by induction, suppose that i steps of the reduction process results in the first 3×3 minor with nonzero columns at the bottom:

$$\begin{bmatrix} G(i+1) & G(i-1) & G(i) \\ G(i) & G(i-2) & G(i-1) \\ G(i+2) & G(i) & G(i+1) \end{bmatrix}.$$

Applying the procedure for step $i+1$, the next 3×3 minor will be

$$\begin{bmatrix} G(i-1)+G(i+1) & G(i) & G(i+1) \\ G(i-2)+G(i) & G(i-1) & G(i) \\ G(i)+G(i+2) & G(i+1) & G(i+2) \end{bmatrix}$$
$$= \begin{bmatrix} G(i+2) & G(i) & G(i+1) \\ G(i+1) & G(i-1) & G(i) \\ G(i+3) & G(i+1) & G(i+2) \end{bmatrix}.$$

We conclude by induction that for $1 \leq i \leq n$, the result of i steps on the first 3×3 minor at the bottom with nonzero columns is

$$\begin{bmatrix} G(i+1) & G(i-1) & G(i) \\ G(i) & G(i-2) & G(i-1) \\ G(i+2) & G(i) & G(i+1) \end{bmatrix}.$$

Finally, we must apply the procedure to the last three columns. After $n-6$ steps, the matrix is now of the form

$$A_{C_n^3} - I_n \sim \begin{bmatrix} 1 & \dots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \ddots & & & & & & \vdots \\ 0 & & -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & & 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & & G(n-5) & G(n-7) & G(n-6) & -1 & 1 & 0 \\ 0 & & G(n-6) & G(n-8) & G(n-7) & 0 & -1 & 1 \\ 0 & \dots & G(n-4) & G(n-6) & G(n-5) & 0 & 0 & -1 \end{bmatrix}.$$

Applying the procedure three more times results in the following 3×3 minor in the bottom right corner:

$$\begin{bmatrix} G(n-2)-1 & G(n-4)+1 & G(n-3) \\ G(n-3) & G(n-5)-1 & G(n-4)+1 \\ G(n-1) & G(n-3) & G(n-2)-1 \end{bmatrix}$$

Lastly, we add column $n-2$ and column $n-1$ to column n, and then add column $n-2$ to column $n-1$:

$$\begin{bmatrix} G(n-2)-1 & G(n-1) & G(n) \\ G(n-3) & G(n-2)-1 & G(n-1) \\ G(n-1) & G(n) & G(n+1)-1 \end{bmatrix}$$

and hence

$$A_{C_n^3} - I_n \sim \begin{bmatrix} 1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & & & & \vdots \\ 0 & & 1 & 0 & 0 & 0 \\ 0 & & 0 & G(n-2)-1 & G(n-1) & G(n) \\ 0 & & 0 & G(n-3) & G(n-2)-1 & G(n-1) \\ 0 & \dots & 0 & G(n-1) & G(n) & G(n+1)-1 \end{bmatrix}.$$

Since the bottom corner is equal to $M_3^n - I_3$ by Lemma 5.1.3, we conclude that $A_{C_n^3} - I_n$ and $M_3^n - I_3$ have the same Smith normal form. □

Corollary 5.1.5. *The matrices $I_n - A^t_{C_n^3}$ and $(M_3^n)^t - I_3$ have the same Smith normal form , that is,*

$$\mathrm{SNF}(I_n - A^t_{C_n^3}) = \mathrm{SNF} \begin{bmatrix} G(n-2)-1 & G(n-3) & G(n-1) \\ G(n-1) & G(n-2)-1 & G(n) \\ G(n) & G(n-1) & G(n+1)-1 \end{bmatrix},$$

where G's are the Narayana's cows numbers and SNF *denotes the Smith normal form of the matrix.*

Proof. The result follows from the fact that $\mathrm{SNF}(I_n - A_{C_n^3}) = \mathrm{SNF}(A_{C_n^3} - I_n)$ and then

$$\mathrm{SNF}((I_n - A_{C_n^3})^t) = \mathrm{SNF}((A_{C_n^3} - I_n)^t) = \mathrm{SNF}((M_3^n - I_3)^t).$$

□

We present here how the Smith normal form is specifically presented for the particular case of the matrix $(M_3^n)^t - I_3$.

Definition 5.1.6. Consider $n \in \mathbb{N}$ and

$$(M_3^n)^t - I_3 = \begin{bmatrix} G(n-2)-1 & G(n-3) & G(n-1) \\ G(n-1) & G(n-2)-1 & G(n) \\ G(n) & G(n-1) & G(n+1)-1 \end{bmatrix}.$$

For $i = 0, 1, 2, 3$ the corresponding *i-th determinant divisors* are

$$\begin{aligned}
\alpha_0(n) &:= 1, \\
\alpha_1(n) &:= \gcd\{G(n-2)-1, G(n-3), G(n-1), G(n), G(n+1)-1\}, \\
\alpha_2(n) &:= \gcd\{(G(n-2)-1)^2 - G(n-1)G(n-3), \\
&\qquad (G(n-2)-1)G(n-1) - G(n)G(n-3), \\
&\qquad (G(n-2)-1)(G(n+1)-1) - G(n)G(n-1), \\
&\qquad (G(n-2)-1)G(n) - G(n-1)^2, \\
&\qquad G(n-1)(G(n+1)-1) - G(n)^2, \\
&\qquad G(n-3)(G(n+1)-1) - G(n-1)^2\}, \text{ and} \\
\alpha_3(n) &:= |\det((M_3^n)^t - I_3)|.
\end{aligned}$$

In conclusion the Smith normal form of the matrix $(M_3^n)^t - I_3$ is given by:

$$\mathrm{SNF}((M_3^n)^t - I_3) = \begin{bmatrix} \alpha_1(n) & 0 & 0 \\ 0 & \frac{\alpha_2(n)}{\alpha_1(n)} & 0 \\ 0 & 0 & \frac{\alpha_3(n)}{\alpha_2(n)} \end{bmatrix}.$$

Remark 5.1.7. This previous statement about the Smith normal form of the matrix $(M_3^n)^t - I_3$ follows from the fact that the rank of $(M_3^n)^t - I_3$ is 3 for every n. We will see it in Corollary 5.3.7. We include it here in order to clarify now which are our main targets.

As a consequence, Corollary 5.1.5 immediately yields the following result.

Corollary 5.1.8. *Let $n \in \mathbb{N}$. Then*

$$K_0(L_K(C_n^3)) \cong \mathbb{Z}_{\alpha_1(n)} \times \mathbb{Z}_{\frac{\alpha_2(n)}{\alpha_1(n)}} \times \mathbb{Z}_{\frac{\alpha_3(n)}{\alpha_2(n)}},$$

where $\alpha_i(n) = \gcd\{$all $i \times i$ minors of the matrix $(M_3^n)^t - I_3\}$ for $i = 1, 2, 3$ (interpreting $\mathbb{Z}_1$ as the trivial group $\{0\}$) where

$$(M_3^n)^t - I_3 = \begin{bmatrix} G(n-2)-1 & G(n-3) & G(n-1) \\ G(n-1) & G(n-2)-1 & G(n) \\ G(n) & G(n-1) & G(n+1)-1 \end{bmatrix}.$$

Corollary 5.1.8 results to be one of the main ingredients in hand to prove the main result about the structure of the $K_0(L_K(C_n^3))$. Then our goal for the remaining sections will be to have a more computationally explicit description of the determinant divisors $\alpha_1(n), \alpha_2(n)$ and $\alpha_3(n)$ than those given in Definition 5.1.6.

5.2 The 1^{st} and the 2^{nd} determinant divisors

To begin with we show that the first determinant divisor $\alpha_1(n)$ coincides with the greatest common divisor $d_3(n)$ given in Definition 5.1.1. To achieve it recall that we will again use the following well-known property of greatest common divisors:

$$\gcd(a,b) = \gcd(a, b+ka) = \gcd(a+mb, b) \text{ for every } a,b,k,m \in \mathbb{Z},$$

which in our proofs we will note it by ($\overset{*}{=}$).

Lemma 5.2.1. *Let* $n \in \mathbb{N}$ *and* $d_3(n) = \gcd(G(n-1), G(n-3), G(n-2)-1)$. *Then* $\alpha_1(n) = d_3(n)$.

Proof. According to Definition 5.1.6,

$$\alpha_1(n) := \gcd\{G(n-2)-1, G(n-3), G(n-1), G(n), G(n+1)-1\}.$$

Then,

$$\begin{aligned}
\alpha_1(n) &= \gcd\{G(n-2)-1, G(n-3), G(n-1), G(n), G(n+1)-1\} \\
&= \gcd\{G(n-2)-1, G(n-3), G(n-1), G(n), G(n)+G(n-2)-1\} \\
&\overset{*}{=} \gcd\{G(n-2)-1, G(n-3), G(n-1), G(n), G(n-2)-1\} \\
&= \gcd\{G(n-2)-1, G(n-3), G(n-1), G(n-1)+G(n-3)\} \\
&\overset{*}{=} \gcd\{G(n-2)-1, G(n-3), G(n-1), G(n-3)\} \\
&= d_3(n).
\end{aligned}$$

□

We focus now our attention in analyzing the second determinant divisor $\alpha_2(n)$ given in Definition 5.1.6. Thanks to property ($\overset{*}{=}$) let us simplify the numbers of terms appearing in its expression.

Definition 5.2.2. Let $n \in \mathbb{N}$. Let us define

$$\begin{aligned} d_3'(n) = \gcd\{ & G(n-1)G(n-3) - (G(n-2)-1)^2, \\ & G(n)G(n-3) - G(n-1)(G(n-2)-1), \\ & G(n-1)^2 - G(n)(G(n-2)-1)\}. \end{aligned}$$

Lemma 5.2.3. *Consider the second determinant divisor $\alpha_2(n)$ from Definition 5.1.6. Then*

$$\alpha_2(n) = d_3'(n).$$

Proof. By definition we have that

$$\begin{aligned} \alpha_2(n) := \gcd\{ & (G(n-2)-1)^2 - G(n-1)G(n-3), \\ & (G(n-2)-1)G(n-1) - G(n)G(n-3), \\ & (G(n-2)-1)(G(n+1)-1) - G(n)G(n-1), \\ & (G(n-2)-1)G(n) - G(n-1)^2, \\ & G(n-1)(G(n+1)-1) - G(n)^2, \\ & G(n-3)(G(n+1)-1) - G(n-1)^2\}. \end{aligned}$$

And then

$$\begin{aligned} & \alpha_2(n) = \\ & = \gcd\{(G(n-2)-1)^2 - G(n-1)G(n-3), \\ & \qquad (G(n-2)-1)G(n-1) - G(n)G(n-3), \\ & \quad (G(n-2)-1)(G(n)+G(n-2)-1) - (G(n-1)+G(n-3))G(n-1), \\ & \qquad (G(n-2)-1)G(n) - G(n-1)^2, G(n-1)(G(n+1)-1) - G(n)^2, \\ & \qquad G(n-3)(G(n+1)-1) - G(n-1)^2\} \\ & \overset{*}{=} \gcd\{(G(n-2)-1)^2 - G(n-1)G(n-3), \\ & \qquad (G(n-2)-1)G(n-1) - G(n)G(n-3), \\ & \qquad (G(n-2)-1)G(n) - G(n-1)^2, \\ & \qquad G(n-1)(G(n+1)-1) - G(n)^2, \\ & \qquad G(n-3)(G(n+1)-1) - G(n-1)^2\} \end{aligned}$$

$$\begin{aligned}
&= \gcd\{(G(n-2)-1)^2 - G(n-1)G(n-3),\\
&\qquad (G(n-2)-1)G(n-1) - G(n)G(n-3),\\
&\qquad (G(n-2)-1)G(n) - G(n-1)^2,\\
&\qquad G(n-1)(G(n)+G(n-2)-1) - G(n)^2,\\
&\qquad G(n-3)(G(n)+G(n-2)-1) - G(n-1)^2\}\\
&= \gcd\{(G(n-2)-1)^2 - G(n-1)G(n-3),\\
&\qquad (G(n-2)-1)G(n-1) - G(n)G(n-3),\\
&\qquad (G(n-2)-1)G(n) - G(n-1)^2,\\
&\qquad G(n-1)(G(n-2)-1) - G(n)G(n-3),\\
&\qquad G(n-3)(G(n)+G(n-2)-1) - G(n-1)^2\}\\
&= \gcd\{(G(n-2)-1)^2 - G(n-1)G(n-3),\\
&\qquad (G(n-2)-1)G(n-1) - G(n)G(n-3),\\
&\qquad (G(n-2)-1)G(n) - G(n-1)^2,\\
&\qquad G(n-3)(G(n)+G(n-2)-1) - G(n-1)^2\}\\
&\overset{*}{=} \gcd\{(G(n-2)-1)^2 - G(n-1)G(n-3),\\
&\qquad (G(n-2)-1)G(n-1) - G(n)G(n-3),\\
&\qquad (G(n-2)-1)G(n) - G(n-1)^2,\\
&\qquad G(n-3)(G(n)+G(n-2)-1) - G(n-1)^2\\
&\qquad + (G(n-2)-1)G(n-1) - G(n)G(n-3)\}\\
&= \gcd\{(G(n-2)-1)^2 - G(n-1)G(n-3),\\
&\qquad (G(n-2)-1)G(n-1) - G(n)G(n-3),\\
&\qquad (G(n-2)-1)G(n) - G(n-1)^2\}\\
&= \gcd\{G(n-1)G(n-3) - (G(n-2)-1)^2,\\
&\qquad G(n)G(n-3) - G(n-1)(G(n-2)-1),\\
&\qquad G(n-1)^2 - G(n)(G(n-2)-1)\}.
\end{aligned}$$

where the last equality comes from the fact that $\gcd(a, b) = \gcd(-a, b)$ for any integers a, b. □

5.3 Relation between H_3 and G. The 3^{rd} determinant divisor

As the following step we will show that the third determinant divisor $\alpha_3(n)$ appearing in Definition 5.1.6 exactly coincides with the third Haselgrove number $H_3(n)$ for every n. To this aim we previously have to establish very interesting relations between the Haselgrove sequence H_3 and the Narayana's cows sequence G. Remember the third Haselgrove sequence H_3 defined by setting $H_3(0) = 0$, $H_3(1) = 1$, $H_3(2) = 3$, $H_3(3) = 1$, $H_3(4) = 3$, $H_3(5) = 11$, $H_3(6) = 9$ and $H_3(n) = H_3(n-1) - H_3(n-2) + 3H_3(n-3) - H_3(n-4) + H_3(n-5) - H_3(n-6)$ for all $n \geq 7$.

In [38, Theorem 3.3] we can find the so-called Binet-Cauchy formula for the Narayana's cows numbers:

Theorem 5.3.1. *If $G(n) = G(n-1) + G(n-3)$ for $n \geq 3$ then*

$$G(n) = \frac{1}{(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)}[\alpha^{n+1}(\gamma-\beta) + \beta^{n+1}(\alpha-\gamma) + \gamma^{n+1}(\beta-\alpha)]$$

where α, β, γ are the solutions of the equation $x^3 - x^2 - 1 = 0$.

Using similar ideas to those appearing in the proof of [38, Theorem 3.3] we can suppose that $H_3(n) = A\alpha^n + B\beta^n + C\gamma^n + D\delta^n + E\epsilon^n + F\eta^n$ where $A, B, C, D, E, F \in \mathbb{C}$. Given the recurrence formula, it results that $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ are the solutions of the equation $x^6 - x^5 + x^4 - 3x^3 + x^2 - x + 1 = 0$.

In fact, it happens that $x^6 - x^5 + x^4 - 3x^3 + x^2 - x + 1 = (x^3 - x^2 - 1)(x^3 + x - 1)$ and α, β, γ are the solutions of the equation $x^3 - x^2 - 1 = 0$ (related to sequence G) and δ, ϵ, η are the solutions of the equation $x^3 + x - 1 = 0$. Since $H_3(0) = 0$, $H_3(1) = 1$, $H_3(2) = 3$, $H_3(3) = 1$, $H_3(4) = 3$, $H_3(5) = 11$ we obtain the following system:

$$\begin{cases} A + B + C + D + E + F = 0, \\ A\alpha + B\beta + C\gamma + D\delta + E\epsilon + F\eta = 1, \\ A\alpha^2 + B\beta^2 + C\gamma^2 + D\delta^2 + E\epsilon^2 + F\eta^2 = 3, \\ A\alpha^3 + B\beta^3 + C\gamma^3 + D\delta^3 + E\epsilon^3 + F\eta^3 = 1, \\ A\alpha^4 + B\beta^4 + C\gamma^4 + D\delta^4 + E\epsilon^4 + F\eta^4 = 3, \\ A\alpha^5 + B\beta^5 + C\gamma^5 + D\delta^5 + E\epsilon^5 + F\eta^5 = 11. \end{cases}$$

According to the software *Mathematica* the solution of the system using the Cramer's rule is:

$$A = 1, B = 1, C = 1, D = -1, E = -1, F = -1.$$

So finally it results $H_3(n) = \alpha^n + \beta^n + \gamma^n - \delta^n - \epsilon^n - \eta^n$ where $\alpha, \beta, \gamma, \delta, \epsilon, \eta$ are the solutions of the equation $x^6 - x^5 + x^4 - 3x^3 + x^2 - x + 1 = 0$. Since α, β, γ are the roots of $x^3 - x^2 - 1$ we have that:

$$\begin{aligned} x^3 - x^2 - 1 &= (x - \alpha)(x - \beta)(x - \gamma) \\ &= x^3 + x^2(-\alpha - \beta - \gamma) + x(\alpha\beta + \alpha\gamma + \beta\gamma) - \alpha\beta\gamma. \end{aligned}$$

Which gives:

$$\begin{cases} \alpha + \beta + \gamma = 1, \\ \alpha\beta + \alpha\gamma + \beta\gamma = 0, \\ \alpha\beta\gamma = 1. \end{cases}$$

If we consider now $(x - \frac{1}{\alpha})(x - \frac{1}{\beta})(x - \frac{1}{\gamma})$, using the previous equations it follows that:

$$\begin{aligned} &\left(x - \frac{1}{\alpha}\right)\left(x - \frac{1}{\beta}\right)\left(x - \frac{1}{\gamma}\right) = \\ &= x^3 + x^2\left(-\frac{1}{\alpha} - \frac{1}{\beta} - \frac{1}{\gamma}\right) + x\left(\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\beta\gamma}\right) - \frac{1}{\alpha\beta\gamma} \\ &= x^3 - x^2\left(\frac{\alpha\beta + \beta\gamma + \beta\gamma}{\alpha\beta\gamma}\right) + x\left(\frac{\alpha + \beta + \gamma}{\alpha\beta\gamma}\right) - \frac{1}{\alpha\beta\gamma} \\ &= x^3 - x^2\frac{0}{1} + x\frac{1}{1} - \frac{1}{1} \\ &= x^3 + x - 1. \end{aligned}$$

So $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$ are the solutions of the equation $x^3 + x - 1 = 0$ which means that they are in fact δ, ϵ and η. According to *Mathematica* it happens that: $\delta = \frac{1}{\alpha}$, $\epsilon = \frac{1}{\beta}$, $\eta = \frac{1}{\gamma}$. Besides since $\alpha\beta\gamma = 1$ then we have that

$$\begin{cases} \delta = \beta\gamma, \\ \epsilon = \alpha\gamma, \\ \eta = \alpha\beta. \end{cases}$$

Corollary 5.3.2. *For every $n \geq 0$,*

$$H_3(n) = \alpha^n + \beta^n + \gamma^n - (\alpha\beta)^n - (\alpha\gamma)^n - (\beta\gamma)^n$$

where α, β, γ are the solutions of the equation $x^3 - x^2 - 1 = 0$.

Corollary 5.3.3. *For every $n \geq 0$,*

$$H_3(n) = (\alpha^n - 1)(\beta^n - 1)(\gamma^n - 1)$$

where α, β, γ are the solutions of the equation $x^3 - x^2 - 1 = 0$.

Proof. It is sufficient to multiply taking into account that $\alpha\beta\gamma = 1$:

$$\begin{aligned}
&(\alpha^n - 1)(\beta^n - 1)(\gamma^n - 1) = \\
&= (\alpha\beta\gamma)^n + \alpha^n + \beta^n + \gamma^n - (\beta\gamma)^n - (\alpha\gamma)^n - (\alpha\beta)^n - 1 \\
&= 1^n + \alpha^n + \beta^n + \gamma^n - (\beta\gamma)^n - (\alpha\gamma)^n - (\alpha\beta)^n - 1 \\
&= \alpha^n + \beta^n + \gamma^n - (\beta\gamma)^n - (\alpha\gamma)^n - (\alpha\beta)^n \\
&= H_3(n) \text{ by Corollary 5.3.2.}
\end{aligned}$$ □

Given Corollary 5.3.2 the idea now is to try to express H_3 in terms of the sequence G. First, according to Theorem 5.3.1, we can express the sequence G as:

$$G(n) = \frac{\alpha}{(\alpha - \beta)(\alpha - \gamma)}\alpha^n + \frac{(-\beta)}{(\alpha - \beta)(\beta - \gamma)}\beta^n + \frac{\gamma}{(\beta - \gamma)(\alpha - \gamma)}\gamma^n;$$

we will use this expression in our next result.

Lemma 5.3.4. *For every $n \geq 2$, $\alpha^n + \beta^n + \gamma^n = G(n) + 3G(n-2)$ where α, β, γ are the solutions of the equation $x^3 - x^2 - 1 = 0$. That is,*

$$G(n) + 3(G(n-2) - 1) = (\alpha^n - 1) + (\beta^n - 1) + (\gamma^n - 1).$$

Proof. To begin with, we are computing the expression $(\alpha - \beta)(\beta - \gamma)(\alpha - \gamma)$ taking into account that $\alpha + \beta + \gamma = 1$ and $\alpha\beta\gamma = 1$:

$$\begin{aligned}
&(\alpha - \beta)(\beta - \gamma)(\alpha - \gamma) = ((\alpha - \beta)(\alpha - \gamma))(\beta - \gamma) = \\
&= (\alpha^2 - \alpha\gamma - \beta\alpha + \beta\gamma)(\beta - \gamma) \\
&= (\alpha^2 - \alpha(\gamma + \beta) + \beta\gamma)(\beta - \gamma) \\
&= (\alpha^2 - \alpha(1 - \alpha) + \frac{1}{\alpha})(\beta - \gamma) \\
&= (2\alpha^2 - \alpha + \frac{1}{\alpha})(\beta - \gamma)
\end{aligned}$$

So then

$$(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma) = (2\alpha^2 - \alpha + \frac{1}{\alpha})(\beta-\gamma).$$

Analogously as before we can also prove that:

$$(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma) = (-2\beta^2 + \beta - \frac{1}{\beta})(\alpha-\gamma).$$

$$(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma) = (2\gamma^2 - \gamma + \frac{1}{\gamma})(\alpha-\beta).$$

Therefore:

$$\begin{aligned}
&\alpha^n + \beta^n + \gamma^n = \\
&= \frac{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\alpha^n + \frac{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\beta^n \\
&\quad + \frac{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\gamma^n
\end{aligned}$$

$$\begin{aligned}
&= \frac{(2\alpha^2 - \alpha + \frac{1}{\alpha})(\beta-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\alpha^n + \frac{(-2\beta^2 + \beta - \frac{1}{\beta})(\alpha-\gamma)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\beta^n \\
&\quad + \frac{(2\gamma^2 - \gamma + \frac{1}{\gamma})(\alpha-\beta)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\gamma^n \\
&= 2\frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n+1} + \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\alpha^{n-2} - \frac{\alpha}{(\alpha-\beta)(\alpha-\gamma)}\alpha^n \\
&\quad + 2\frac{(-\beta)}{(\alpha-\beta)(\beta-\gamma)}\beta^{n+1} + \frac{(-\beta)}{(\alpha-\beta)(\beta-\gamma)}\beta^{n-2} - \frac{(-\beta)}{(\alpha-\beta)(\beta-\gamma)}\beta^n \\
&\quad + 2\frac{\gamma}{(\beta-\gamma)(\alpha-\gamma)}\gamma^{n+1} + \frac{\gamma}{(\beta-\gamma)(\alpha-\gamma)}\gamma^{n-2} - \frac{\gamma}{(\beta-\gamma)(\alpha-\gamma)}\gamma^n \\
&= 2G(n+1) + G(n-2) - G(n) \\
&= 2(G(n) + G(n-2)) + G(n-2) - G(n) \\
&= G(n) + 3G(n-2).
\end{aligned}$$

□

Remember that our main objective is to prove that the third determinant divisor $\alpha_3(n)$ (given in Definition 5.1.6) coincides with the third Haselgrove number $H_3(n)$, so we need to establish a relation for the Haselgrove sequence H_3 in terms of the Narayana's cows numbers $G(n-1), G(n-2)-1$ and $G(n-3)$. Our next step in order to do that consists of proving this result.

Lemma 5.3.5. *For every $n \geq 3$ define*

$$\begin{aligned}\Delta(n) := & -G(n-1)^3 - G(n-3)^2G(n) - G(n)G(n-2)^2 - G(n-2)^3 \\ & + 2G(n-1)^2G(n-2) + 3G(n-1)G(n-2)G(n-3) + 1.\end{aligned}$$

Then $\Delta(n) = 0$ for all $n \geq 3$.

Proof. By induction on $n \geq 3$. For $n = 3$:

$$\begin{aligned}\Delta(3) &= -G(2)^3 - G(0)^2G(3) - G(3)G(1)^2 - G(1)^3 + 2G(2)^2G(1) \\ &\quad + 3G(2)G(1)G(0) + 1 \\ &= -1 - 0 - 1 - 1 + 2 + 0 + 1 \\ &= 0 \text{ as desired.}\end{aligned}$$

Suppose that the equality is satisfied for $n-1$, that is, suppose we have $\Delta(n-1) = 0$. Let us see that $\Delta(n) = 0$. Since $G(i) = G(i-1) + G(i-3)$ for $i = n, n-1$ the following holds:

$$\begin{aligned}\Delta(n) &= -G(n-1)^3 - G(n-3)^2G(n) - G(n)G(n-2)^2 - G(n-2)^3 \\ &\quad + 2G(n-1)^2G(n-2) + 3G(n-1)G(n-2)G(n-3) + 1 \\ &= -G(n-1)^2G(n-1) - G(n-3)^2G(n-1) - G(n-3)^3 \\ &\quad - G(n-1)G(n-2)^2 - G(n-3)G(n-2)^2 - G(n-2)^3 \\ &\quad + 2G(n-1)^2G(n-2) + 3G(n-1)G(n-2)G(n-3) + 1 \\ &= -G(n-1)^2G(n-2) - G(n-1)^2G(n-4) - G(n-3)^2G(n-1) \\ &\quad - G(n-3)^3 - G(n-1)G(n-2)^2 - G(n-3)G(n-2)^2 \\ &\quad - G(n-2)^3 + 2G(n-1)^2G(n-2) + 3G(n-2)^2G(n-3) \\ &\quad + 3G(n-4)G(n-2)G(n-3) + 1 \\ &= -G(n-1)^2G(n-4) - G(n-3)^2G(n-1) - G(n-3)^3 \\ &\quad - G(n-1)G(n-2)^2 - G(n-2)^3 + G(n-1)^2G(n-2) \\ &\quad + 2G(n-2)^2G(n-3) + 3G(n-4)G(n-2)G(n-3) + 1\end{aligned}$$

$$\begin{aligned}
&= -G(n-1)G(n-1)G(n-4) - G(n-3)^2G(n-2)\\
&\quad - G(n-3)^2G(n-4) - G(n-3)^3 - G(n-1)G(n-2)^2\\
&\quad - G(n-2)^3 + G(n-1)G(n-2)^2\\
&\quad + G(n-1)G(n-2)G(n-4) + 2G(n-2)^2G(n-3)\\
&\quad + 3G(n-4)G(n-2)G(n-3) + 1\\
&= -G(n-1)G(n-2)G(n-4) - G(n-1)G(n-4)^2\\
&\quad - G(n-3)^2G(n-2) - G(n-3)^2G(n-4) - G(n-3)^3\\
&\quad - G(n-2)^3 + G(n-1)G(n-2)G(n-4)\\
&\quad + 2G(n-2)^2G(n-3) + 3G(n-4)G(n-2)G(n-3) + 1\\
&= -G(n-2)G(n-4)^2 - G(n-4)^3 - G(n-3)^2G(n-2)\\
&\quad - G(n-3)^2G(n-4) - G(n-3)^3 - G(n-2)^3\\
&\quad + 2G(n-2)^2G(n-3) + 3G(n-4)G(n-2)G(n-3) + 1\\
&= -G(n-2)^3 - G(n-4)^2G(n-2) - G(n-4)^3 - G(n-2)G(n-3)^2\\
&\quad - G(n-4)G(n-3)^2 - G(n-3)^3 + 2G(n-2)^2G(n-3)\\
&\quad + 3G(n-2)G(n-3)G(n-4) + 1\\
&= -G(n-2)^3 - G(n-4)^2G(n-1) - G(n-1)G(n-3)^2 - G(n-3)^2\\
&\quad + 2G(n-2)^2G(n-3) + 3G(n-2)G(n-3)G(n-4) + 1\\
&= \Delta(n-1).
\end{aligned}$$

So $\Delta(n) = \Delta(n-1)$ and $\Delta(n-1) = 0$ by induction hypothesis. Hence $\Delta(n) = 0$. □

In a similar manner that H_2 is related to the Fibonacci sequence F ([7, Proposition 4.4]), we will relate H_3 to the Narayana's cows sequence G.

Theorem 5.3.6. *For every $n \geq 3$,*

$$\begin{aligned}
H_3(n) = &-3G(n-1)(G(n-2)-1)G(n-3) - 2G(n-1)^2(G(n-2)-1)\\
&+ G(n-1)^3 + (G(n-2)-1)^2G(n-1) + (G(n-2)-1)^2G(n-3)\\
&+ (G(n-2)-1)^3 + G(n-3)^2G(n-1) + G(n-3)^3.
\end{aligned}$$

In particular for all $n \geq 3$, $d_3(n)^3$ divides $H_3(n)$ where

$$d_3(n) = \gcd\left(G(n-1), G(n-2)-1, G(n-3)\right).$$

Finally, the desired consequence of Theorem 5.3.6 is the following result: computing the determinant of the matrix $(M_3^n)^t - I_3$ we obtain the third Haselgrove number, as we wanted.

Corollary 5.3.7. *Let $n \geq 1$ and consider*

$$(M_3^n)^t - I_3 = \begin{bmatrix} G(n-2)-1 & G(n-3) & G(n-1) \\ G(n-1) & G(n-2)-1 & G(n) \\ G(n) & G(n-1) & G(n+1)-1 \end{bmatrix},$$

with $\alpha_3(n) = |\det((M_3^n)^t - I_3)|$. Then $\alpha_3(n) = H_3(n) \neq 0$.

Proof. For $n = 1, 2$ it is an immediate computation taking into account that $G(0) = 0$, $G(-1) = 0$ and $G(-2) = 1$. For $n \geq 3$ we use Theorem 5.3.6 in order to get the desired result. The matrix $(M_3^n)^t - I_3$ is indeed:

$$\begin{bmatrix} G(n-2)-1 & G(n-3) & G(n-1) \\ G(n-1) & G(n-2)-1 & G(n-1)+G(n-3) \\ G(n-1)+G(n-3) & G(n-1) & G(n-1)+G(n-3)+G(n-2)-1 \end{bmatrix}.$$

So we have

$$\begin{aligned}
\alpha_3(n) = & |\det((M_3^n)^t - I_3)| \\
=&(G(n-2)-1)^2 G(n-1) + (G(n-2)-1)^2 G(n-3) + (G(n-2)-1)^3 \\
&+ G(n-1)^3 + G(n-1)^2 G(n-3) + G(n-3)^3 + 2G(n-1)G(n-3)^2 \\
&- (G(n-2)-1)G(n-1)^2 - (G(n-2)-1)G(n-1)G(n-3) \\
&- G(n-1)^2 G(n-3) - G(n-1)G(n-3)^2 \\
&- G(n-1)G(n-3)(G(n-2)-1) \\
&- (G(n-2)-1)G(n-1)^2 - G(n-1)G(n-3)(G(n-2)-1) \\
=&-3G(n-1)(G(n-2)-1)G(n-3) - 2G(n-1)^2(G(n-2)-1) \\
&+ G(n-1)^3 + (G(n-2)-1)^2 G(n-1) + (G(n-2)-1)^2 G(n-3) \\
&+ (G(n-2)-1)^3 + G(n-3)^2 G(n-1) + G(n-3)^3 \\
=& H_3(n) \text{ by Theorem 5.3.6.}
\end{aligned}$$

□

Proof of Theorem 5.3.6. By Corollary 5.3.2 , $H_3(n) = \alpha^n + \beta^n + \gamma^n - 3 - ((\alpha\beta)^n + (\alpha\gamma)^n + (\beta\gamma)^n - 3)$ where α, β, γ are the solutions of $x^3 - x^2 - 1 = 0$. To begin with we compute $(\alpha^n + \beta^n + \gamma^n - 3)^3$ taking into account that $\alpha\beta\gamma = 1$:

$$\begin{aligned}
(\alpha^n + \beta^n + \gamma^n - 3)^3 &= \alpha^{3n} + \beta^{3n} + \gamma^{3n} + 3\alpha^n\beta^{2n} + 3\alpha^n\gamma^{2n} + 3\beta^n\alpha^{2n} \\
&\quad + 3\beta^n\gamma^{2n} + 3\gamma^n\alpha^{2n} + 3\gamma^n\beta^{2n} - 18\alpha^n\beta^n - 18\alpha^n\gamma^n - 18\beta^n\gamma^n \\
&\quad - 9\alpha^{2n} - 9\beta^{2n} - 9\gamma^{2n} + 27\alpha^n + 27\beta^n + 27\gamma^n - 21 \\
&= \alpha^{3n} + \beta^{3n} + \gamma^{3n} + 3\alpha^{2n}(\beta^n + \gamma^n) + 3\beta^{2n}(\alpha^n + \gamma^n) + 3\gamma^{2n}(\alpha^n + \beta^n) \\
&\quad - 18(\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n) - 9(\alpha^{2n} + \beta^{2n} + \gamma^{2n}) \\
&\quad + 27(\alpha^n + \beta^n + \gamma^n) - 21 \\
&= \alpha^{3n} + \beta^{3n} + \gamma^{3n} + 3\alpha^{2n}(\beta^n + \gamma^n - 3) + 3\beta^{2n}(\alpha^n + \gamma^n - 3) \\
&\quad + 3\gamma^{2n}(\alpha^n + \beta^n - 3) - 18(\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n) \\
&\quad + 27(\alpha^n + \beta^n + \gamma^n) - 21 \\
&= (-2)(\alpha^{3n} + \beta^{3n} + \gamma^{3n}) + 3(\alpha^{2n} + \beta^{2n} + \gamma^{2n})(\alpha^n + \beta^n + \gamma^n - 3) \\
&\quad - 18(\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n) + 27(\alpha^n + \beta^n + \gamma^n) - 21 \\
&= (-2)(\alpha^{3n} + \beta^{3n} + \gamma^{3n} - 3) + 3(\alpha^{2n} + \beta^{2n} + \gamma^{2n} + 9)(\alpha^n + \beta^n + \gamma^n - 3) \\
&\quad - 18(\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n - 3),
\end{aligned}$$

which implies that:

$$\begin{aligned}
&-(\alpha^n\beta^n + \alpha^n\gamma^n + \beta^n\gamma^n - 3) = \\
&= \frac{(\alpha^n + \beta^n + \gamma^n - 3)^3 + 2(\alpha^{3n} + \beta^{3n} + \gamma^{3n} - 3)}{18} \\
&\quad + \frac{-3(\alpha^{2n} + \beta^{2n} + \gamma^{2n} + 9)(\alpha^n + \beta^n + \gamma^n - 3)}{18}.
\end{aligned}$$

And then we obtain $H_3(n) = \alpha^n + \beta^n + \gamma^n - 3 - ((\alpha\beta)^n + (\alpha\gamma)^n + (\beta\gamma)^n - 3)$

$$\begin{aligned}
&= \frac{(\alpha^n + \beta^n + \gamma^n - 3)^3 + 2(\alpha^{3n} + \beta^{3n} + \gamma^{3n} - 3)}{18} \\
&\quad + \frac{-3(\alpha^{2n} + \beta^{2n} + \gamma^{2n} + 3)(\alpha^n + \beta^n + \gamma^n - 3)}{18}.
\end{aligned}$$

We already know $\alpha^n + \beta^n + \gamma^n - 3 = G(n-1) + 3(G(n-2) - 1) + G(n-3)$ by Lemma 5.3.4, so our following tasks consist of expressing $\alpha^{2n} + \beta^{2n} + \gamma^{2n} + 3$

and $\alpha^{3n}+\beta^{3n}+\gamma^{3n}-3$ appearing in the previous relation for $H_3(n)$, in terms of $G(n-1)$, $G(n-3)$ and $G(n-2)-1$ (or at least $G(n-2)$). The general idea of the proof will be to get the greatest number of summands of the form $\{G(n-1)(G(n-2)-1)G(n-3)$, $G(n-1)^2(G(n-2)-1)$, $G(n-1)^2G(n-3)$, $G(n-1)^3$, $(G(n-2)-1)^2G(n-1)$, $(G(n-2)-1)^2G(n-3)$, $(G(n-2)-1)^3$, $G(n-3)^2G(n-1)$, $G(n-3)^2(G(n-2)-1)$, $G(n-3)^3\}$ and to reduce as far as possible those summands which involve $G(n-2)$ in the given formula:

$H_3(n) = \frac{(\alpha^n+\beta^n+\gamma^n-3)^3+2(\alpha^{3n}+\beta^{3n}+\gamma^{3n}-3)-3(\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3)(\alpha^n+\beta^n+\gamma^n-3)}{18}$.

Step 1:

First let us compute $\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3$ which satisfies: $\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3 = G(2n)+3G(2n-2)+3$ because of Lemma 5.3.4. By [38, item 6 (page 4)]: $G(2n) = G(n+1)^2+G(n-1)^2-G(n-2)^2$. Since $G(n+1) = G(n)+G(n-2)$ (henceforth we will use the definition of G, $G(i) = G(i-1)+G(i-3)$, without explicitly mention) we can rewrite and

$$G(2n) = 2G(n)G(n-2) + G(n)^2 + G(n-1)^2.$$

So then:

$$\begin{aligned}
&\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3 = G(2n)+3G(2n-2)+3\\
&= G(2n)+3G(2(n-1))+3\\
&= 2G(n)G(n-2)+G(n)^2+G(n-1)^2\\
&\quad +3[2G(n-1)G(n-3)+G(n-1)^2+G(n-2)^2]+3\\
&= 2G(n)G(n-2)+G(n)^2+4G(n-1)^2+6G(n-1)G(n-3)\\
&\quad +3G(n-2)^2+3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(-3)(\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3)(\alpha^n+\beta^n+\gamma^n-3) =\\
&= (-3)[\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3][G(n)+3(G(n-2)-1)]
\end{aligned}$$

$$= -3G(n)^2[G(n) + 3(G(n-2) - 1)] - 12G(n-1)^2[G(n) + 3(G(n-2) - 1)] - 18G(n-1)G(n-3)[G(n) + 3(G(n-2) - 1)] - 6G(n)G(n-2)[G(n) + 3(G(n-2) - 1)] - 9G(n-2)^2[G(n) + 3(G(n-2) - 1)] - 9[G(n) + 3(G(n-2) - 1)]$$

$$= -3G(n)^2[G(n) + 3(G(n-2) - 1)] - 12G(n-1)^2[G(n) + 3(G(n-2) - 1)] - 18G(n-1)G(n-3)[G(n) + 3(G(n-2) - 1)] - 6G(n)^2G(n-2) - 18G(n)(G(n-2) - 1)G(n-2) - 9G(n-2)^2G(n) - 27G(n-2)^2(G(n-2) - 1) - 9G(n) - 27(G(n-2) - 1).$$

Step 2:

We have $\alpha^{3n} + \beta^{3n} + \gamma^{3n} - 3 = G(3n) + 3(G(3n-2) - 1)$ by Lemma 5.3.4. In order to proceed we need a formula for $G(3n)$ and $G(3n-2)$ similar to that which appears in [38, item 6 (page 4)] for $G(2n)$. Thanks to [38, item 5 (page 4)] we use

$$G(n+m) = G(n-1)G(m+2) + G(n-2)G(m) + G(n-3)G(m+1)$$

as follows:

$$G(3n) = G(n+2n) =$$
$$= G(n-1)G(2n+2) + G(n-2)G(2n) + G(n-3)G(2n+1)$$
$$= G(n-1)G(2(n+1)) + G(n-2)G(2n) + G(n-3)G((n+1)+n)$$
$$= G(n-1)[2G(n+1)G(n-1) + G(n+1)^2 + G(n)^2] + G(n-2)[2G(n)G(n-2) + G(n)^2 + G(n-1)^2] + G(n-3)[G(n)G(n+2) + G(n-1)G(n) + G(n-2)G(n+1)]$$
$$= 2G(n-1)^2G(n) + 2G(n-1)^2G(n-2) + G(n-1)G(n)^2 + G(n-1)G(n-2)^2 + 2G(n-1)G(n)G(n-2) + G(n-1)G(n)^2 + 2G(n-2)^2G(n) + G(n-2)G(n)^2 + G(n-2)G(n-1)^2 + G(n-3)G(n)^2 + G(n-3)G(n)G(n-2) + G(n-3)G(n)G(n-1) + G(n-3)G(n-1)G(n) + G(n-3)G(n-2)G(n) + G(n-3)G(n-2)^2$$

$$\begin{aligned}&= 2G(n-1)^2G(n) + 2G(n)^2G(n-1) + 3G(n-1)^2G(n-2)\\&\quad + 3G(n-2)^2G(n) + 3G(n-2)G(n)^2\\&\quad + G(n-3)G(n)^2 + 2G(n-3)G(n)G(n-1).\end{aligned}$$

That is,

$$\begin{aligned}G(3n) &= 2G(n-1)^2G(n) + 2G(n)^2G(n-1) + 3G(n-1)^2G(n-2)\\&\quad + 3G(n-2)^2G(n) + 3G(n-2)G(n)^2 + 3G(n-2)G(n)^2\\&\quad + G(n-3)G(n)^2 + 2G(n-3)G(n)G(n-1).\end{aligned}$$

On the other hand,

$$\begin{aligned}&G(3n-2) = G(n+(2n-2)) =\\&= G(n-1)G(2n) + G(n-2)G(2n-2) + G(n-3)G(2n-1)\\&= G(n-1)G(2n) + G(n-2)G(2(n-1)) + G(n-3)G(n+(n-1))\\&= G(n-1)[2G(n)G(n-2) + G(n)^2 + G(n-1)^2]\\&\quad + G(n-2)[2G(n-1)G(n-3) + G(n-1)^2 + G(n-2)^2]\\&\quad + G(n-3)[G(n-1)G(n+1) + G(n-2)G(n-1) + G(n-3)G(n)]\\&= 2G(n)G(n-1)G(n-2) + G(n)^2G(n-1) + G(n-1)^3\\&\quad + 4G(n-2)G(n-1)G(n-3) + G(n-2)G(n-1)^2 + G(n-2)^3\\&\quad + G(n-3)G(n-1)G(n) + G(n-3)^2G(n),\end{aligned}$$

that is,

$$\begin{aligned}G(3n-2) &= 2G(n)G(n-1)G(n-2) + G(n)^2G(n-1) + G(n-1)^3\\&\quad + 4G(n-2)G(n-1)G(n-3) + G(n-2)G(n-1)^2\\&\quad + G(n-2)^3 + G(n-3)G(n-1)G(n) + G(n-3)^2G(n).\end{aligned}$$

Then, taking into account these previous computations, we get:

$$\begin{aligned}&2(\alpha^{3n} + \beta^{3n} + \gamma^{3n} - 3) = 2[G(3n) + 3G(3n-2) - 3] =\\&= 4G(n-1)^2G(n) + 10G(n-1)G(n)^2 + 2G(n-3)G(n)^2 + 6G(n-1)^3\\&\quad + 10G(n-3)G(n)G(n-1) + 6G(n-3)^2G(n) + 6G(n-2)^2G(n)\\&\quad + 2G(n-2)G(n)^2 + 4G(n-3)G(n)G(n-2) + 12G(n-1)^2G(n-2)\end{aligned}$$

$$+ 16G(n)G(n-1)G(n-2) + 24G(n-2)G(n-1)G(n-3)$$
$$+ 6G(n-2)^3 - 6.$$

Step 3:

From the calculations of $2(\alpha^{3n}+\beta^{3n}+\gamma^{3n}-3)$ and $(-3)(\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3)(\alpha^n+\beta^n+\gamma^n-3)$ in terms of $G(n-1)$, $G(n-2)-1$ (or $G(n-2)$) and $G(n-3)$, we continue summing both of them:

$$\begin{aligned}
&2(\alpha^{3n}+\beta^{3n}+\gamma^{3n}-3) - 3(\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3)(\alpha^n+\beta^n+\gamma^n-3) = \\
&= 4G(n-1)^2G(n) + 10G(n-1)G(n)^2 + 2G(n-3)G(n)^2 + 6G(n-1)^3 \\
&\quad + 10G(n-3)G(n)G(n-1) + 6G(n-3)^2G(n) + 6G(n-2)^2G(n) \\
&\quad + 2G(n-2)G(n)^2 + 4G(n-3)G(n)G(n-2) + 12G(n-1)^2G(n-2) \\
&\quad + 16G(n)G(n-1)G(n-2) + 24G(n-2)G(n-1)G(n-3) \\
&\quad + 6G(n-2)^3 - 6 - 3G(n)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 12G(n-1)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 18G(n-1)G(n-3)[G(n) + 3(G(n-2)-1)] - 6G(n)^2G(n-2) \\
&\quad - 18G(n)(G(n-2)-1)G(n-2) - 9G(n-2)^2G(n) \\
&\quad - 27G(n-2)^2(G(n-2)-1) - 9G(n) - 27(G(n-2)-1) \\
&= 4G(n-1)^2G(n) + 10G(n-1)G(n)^2 + 2G(n-3)G(n)^2 + 6G(n-1)^3 \\
&\quad + 10G(n-3)G(n)G(n-1) + 6G(n-3)^2G(n) \\
&\quad - 3G(n)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 12G(n-1)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 18G(n-1)G(n-3)[G(n) + 3(G(n-2)-1)] \\
&\quad + 6G(n-2)^2G(n) + 2G(n-2)G(n)^2 + 4G(n-3)G(n)G(n-2) \\
&\quad + 12G(n-1)^2G(n-2) + 16G(n)G(n-1)G(n-2) \\
&\quad + 24G(n-2)G(n-1)G(n-3) + 6G(n-2)^3 - 6 - 6G(n)^2G(n-2) \\
&\quad - 18G(n)(G(n-2)-1)G(n-2) - 9G(n-2)^2G(n) \\
&\quad - 27G(n-2)^2(G(n-2)-1) - 9G(n) - 27(G(n-2)-1)
\end{aligned}$$

$$
\begin{aligned}
&= 4G(n-1)^2G(n) + 10G(n-1)G(n)^2 + 2G(n-3)G(n)^2 + 6G(n-1)^3 \\
&\quad + 10G(n-3)G(n)G(n-1) + 6G(n-3)^2G(n) \\
&\quad - 3G(n)^2[G(n) + 3(G(n-2)-1)] - 12G(n-1)^2[G(n) \\
&\quad + 3(G(n-2)-1)] - 18G(n-1)G(n-3)[G(n) + 3(G(n-2)-1)] \\
&\quad - 3G(n-2)^2G(n) + \{2G(n-2)G(n)^2 \\
&\quad + 4G(n-3)G(n-2)(G(n-1) + G(n-3)) \\
&\quad + 12G(n-1)^2G(n-2) + 16(G(n-1) + G(n-3))G(n-1)G(n-2) \\
&\quad + 24G(n-2)G(n-1)G(n-3)\} \\
&\quad + 6G(n-2)^3 - 6 - 6G(n)^2G(n-2) - 18G(n)(G(n-2)-1)G(n-2) \\
&\quad - 27G(n-2)^2(G(n-2)-1) - 9G(n) - 27(G(n-2)-1) \\
&= 4G(n-1)^2G(n) + 10G(n-1)G(n)^2 + 2G(n-3)G(n)^2 + 6G(n-1)^3 \\
&\quad + 10G(n-3)G(n)G(n-1) + 6G(n-3)^2G(n) \\
&\quad - 3G(n)^2[G(n) + 3(G(n-2)-1)] - 12G(n-1)^2[G(n) \\
&\quad + 3(G(n-2)-1)] - 18G(n-1)G(n-3)[G(n) + 3(G(n-2)-1)] \\
&\quad - 3G(n-2)^2G(n) + \{6G(n-2)G(n)^2 + 24G(n-1)^2G(n-2) \\
&\quad + 36G(n-2)G(n-1)G(n-3)\} + 6G(n-2)^3 - 6 \\
&\quad - 6G(n)^2G(n-2) - 18G(n)(G(n-2)-1)G(n-2) \\
&\quad - 27G(n-2)^2(G(n-2)-1) - 9G(n) - 27(G(n-2)-1) \\
&= 4G(n-1)^2G(n) + 10G(n-1)G(n)^2 + 2G(n-3)G(n)^2 + 6G(n-1)^3 \\
&\quad + 10G(n-3)G(n)G(n-1) + 6G(n-3)^2G(n) \\
&\quad - 3G(n)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 12G(n-1)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 18G(n-1)G(n-3)[G(n) + 3(G(n-2)-1)] \\
&\quad - 3G(n-2)^2G(n) + 24G(n-1)^2G(n-2) \\
&\quad + 36G(n-2)G(n-1)G(n-3) \\
&\quad + 6G(n-2)^3 - 6 - 18G(n)(G(n-2)-1)G(n-2) \\
&\quad - 27G(n-2)^2(G(n-2)-1) - 9G(n) - 27(G(n-2)-1).
\end{aligned}
$$

And finally with the relations $(G(n-2)-1)^3 = G(n-2)^3 - 3G(n-2)^2 + 3G(n-2) - 1$ and $(G(n-2)-1)^2 = G(n-2)^2 - 2G(n-2) + 1$ in mind we get from the last paragraph:

$$\begin{aligned}
&2(\alpha^{3n} + \beta^{3n} + \gamma^{3n} - 3) - 3(\alpha^{2n} + \beta^{2n} + \gamma^{2n} + 3)(\alpha^n + \beta^n + \gamma^n - 3) = \\
&= 4G(n-1)^2 G(n) + 10G(n-1)G(n)^2 + 2G(n-3)G(n)^2 + 6G(n-1)^3 \\
&\quad + 10G(n-3)G(n)G(n-1) + 6G(n-3)^2 G(n) \\
&\quad - 3G(n)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 12G(n-1)^2[G(n) + 3(G(n-2)-1)] \\
&\quad + 24G(n-1)^2(G(n-2)-1) + 36G(n-1)(G(n-2)-1)G(n-3) \\
&\quad - 21G(n)(G(n-2)-1)^2 - 21(G(n-2)-1)^3 \\
&\quad - 12G(n) - 24G(n)(G(n-2)-1) - 36(G(n-2)-1)^2 \\
&\quad - 36(G(n-2)-1) + 24G(n-1)^2 + 36G(n-1)G(n-3).
\end{aligned}$$

Step 4:

Remember at this point that

$H_3(n) = \frac{(\alpha^n+\beta^n+\gamma^n-3)^3 + 2(\alpha^{3n}+\beta^{3n}+\gamma^{3n}-3) - 3(\alpha^{2n}+\beta^{2n}+\gamma^{2n}+3)(\alpha^n+\beta^n+\gamma^n-3)}{18}$.

Hence,

$$\begin{aligned}
18H_3(n) &= (\alpha^n + \beta^n + \gamma^n - 3)^3 + 2(\alpha^{3n} + \beta^{3n} + \gamma^{3n} - 3) \\
&\qquad - 3(\alpha^{2n} + \beta^{2n} + \gamma^{2n} + 3)(\alpha^n + \beta^n + \gamma^n - 3) = \\
&= (G(n) + 3(G(n-2)-1))^3 + 4G(n-1)^2 G(n) + 10G(n-1)G(n)^2 \\
&\quad + 2G(n-3)G(n)^2 + 6G(n-1)^3 + 10G(n-3)G(n)G(n-1) \\
&\quad + 6G(n-3)^2 G(n) - 3G(n)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 12G(n-1)^2[G(n) + 3(G(n-2)-1)] \\
&\quad - 18G(n-1)G(n-3)[G(n) + 3(G(n-2)-1)] \\
&\quad + 24G(n-1)^2(G(n-2)-1) + 36G(n-1)(G(n-2)-1)G(n-3) \\
&\quad - 21G(n)(G(n-2)-1)^2 - 21(G(n-2)-1)^3 \\
&\quad - 12G(n) - 24G(n)(G(n-2)-1) - 36(G(n-2)-1)^2 \\
&\quad - 36(G(n-2)-1) + 24G(n-1)^2 + 36G(n-1)G(n-3)
\end{aligned}$$

$$
\begin{aligned}
&= G(n)^3 + 9G(n)^2(G(n-2)-1) + 27G(n)(G(n-2)-1)^2 \\
&\quad + 27(G(n-2)-1)^3 + 4G(n-1)^2G(n) + 10G(n-1)G(n)^2 \\
&\quad + 2G(n-3)G(n)^2 + 6G(n-1)^3 + 10G(n-3)G(n)G(n-1) \\
&\quad + 6G(n-3)^2G(n) - 3G(n)^3 - 9G(n)^2(G(n-2)-1) \\
&\quad - 3G(n-1)^2G(n) - 9G(n-1)^2(G(n-2)-1) \\
&\quad - 18G(n-1)G(n-3)G(n) - 54G(n-1)G(n-3)(G(n-2)-1) \\
&\quad - 9G(n-1)^2G(n) - 27G(n-1)^2(G(n-2)-1) \\
&\quad + 24G(n-1)^2(G(n-2)-1) + 36G(n-1)(G(n-2)-1)G(n-3) \\
&\quad - 21G(n)(G(n-2)-1)^2 - 21(G(n-2)-1)^3 - 12G(n) \\
&\quad - 24G(n)(G(n-2)-1) - 36(G(n-2)-1)^2 - 36(G(n-2)-1) \\
&\quad + 24G(n-1)^2 + 36G(n-1)G(n-3)
\end{aligned}
$$

$$
\begin{aligned}
&= -2[G(n-1)+G(n-3)]^3 + 6[G(n-1)+G(n-3)](G(n-2)-1)^2 \\
&\quad + 6(G(n-2)-1)^3 - 8G(n-1)^2[G(n-1)+G(n-3)] \\
&\quad + 10G(n-1)[G(n-1)+G(n-3)]^2 + 2G(n-3)[G(n-1)+G(n-3)]^2 \\
&\quad - 8G(n-1)G(n-3)[G(n-1)+G(n-3)] \\
&\quad + 6G(n-1)^3 + 6G(n-3)^2[G(n-1)+G(n-3)] \\
&\quad - 12G(n-1)^2(G(n-2)-1) - 18G(n-1)G(n-3)(G(n-2)-1) \\
&\quad - 12G(n) - 24G(n)(G(n-2)-1) - 36(G(n-2)-1)^2 \\
&\quad - 36(G(n-2)-1) + 24G(n-1)^2 + 36G(n-1)G(n-3)
\end{aligned}
$$

$$
\begin{aligned}
&= 6G(n-1)^3 + 6G(n-3)^2G(n-1) + 6G(n)(G(n-2)-1)^2 \\
&\quad + 6(G(n-2)-1)^3 + 6G(n-3)^3 - 12G(n-1)^2(G(n-2)-1) \\
&\quad - 18G(n-1)G(n-3)(G(n-2)-1) - 12G(n) \\
&\quad - 24G(n)(G(n-2)-1) - 36(G(n-2)-1)^2 \\
&\quad - 36(G(n-2)-1) + 24G(n-1)^2 + 36G(n-1)G(n-3).
\end{aligned}
$$

Simplifying by 6 and continuing we have:

$$
\begin{aligned}
&3H_3(n) = \\
&= G(n-1)^3 + G(n-3)^2G(n-1) + G(n)(G(n-2)-1)^2 \\
&\quad + (G(n-2)-1)^3 + G(n-3)^3 - 2G(n-1)^2(G(n-2)-1) \\
&\quad - 3G(n-1)G(n-3)(G(n-2)-1) - 2G(n) - 4G(n)(G(n-2)-1) \\
&\quad - 6(G(n-2)-1)^2 - 6(G(n-2)-1) + 4G(n-1)^2 \\
&\quad + 6G(n-1)G(n-3)
\end{aligned}
$$

$$
\begin{aligned}
&= 3G(n-1)^3 + 3G(n-3)^2G(n-1) + 3G(n-1)(G(n-2)-1)^2 \\
&\quad + 3G(n-3)(G(n-2)-1)^2 + 3(G(n-2)-1)^3 + 3G(n-3)^3 \\
&\quad - 6G(n-1)^2(G(n-2)-1) - 9G(n-1)G(n-3)(G(n-2)-1) \\
&\quad + \{-2G(n-1)^3 - 2G(n-3)^2G(n) - 2G(n)(G(n-2)-1)^2 \\
&\quad - 2(G(n-2)-1)^3 + 4G(n-1)^2(G(n-2)-1) \\
&\quad + 6G(n-1)G(n-3)(G(n-2)-1) - 2G(n) \\
&\quad - 4G(n)(G(n-2)-1) - 6(G(n-2)-1)^2 \\
&\quad - 6(G(n-2)-1) + 4G(n-1)^2 + 6G(n-1)G(n-3)\}
\end{aligned}
$$

$$
\begin{aligned}
&= 3G(n-1)^3 + 3G(n-3)^2G(n-1) + 3G(n-1)(G(n-2)-1)^2 \\
&\quad + 3G(n-3)(G(n-2)-1)^2 + 3(G(n-2)-1)^3 + 3G(n-3)^3 \\
&\quad - 6G(n-1)^2(G(n-2)-1) - 9G(n-1)G(n-3)(G(n-2)-1) \\
&\quad + \{-2G(n-1)^3 - 2G(n-3)^2G(n) - 2G(n)G(n-2)^2 \\
&\quad - 2G(n) + +4G(n)G(n-2) - 2G(n-2)^3 \\
&\quad + 6G(n-2)^2 - 6G(n-2) + 2 + 4G(n-1)^2G(n-2) \\
&\quad - 4G(n-1)^2 + 6G(n-1)G(n-3)G(n-2) - 6G(n-1)G(n-3) \\
&\quad - 2G(n) - 4G(n)G(n-2) + 4G(n) - 6G(n-2)^2 - 6 \\
&\quad + 12G(n-2) - 6G(n-2) + 6 \\
&\quad + 4G(n-1)^2 + 6G(n-1)G(n-3)\}
\end{aligned}
$$

$$\begin{aligned}
&= 3G(n-1)^3 + 3G(n-3)^2G(n-1) + 3G(n-1)(G(n-2)-1)^2 \\
&\quad + 3G(n-3)(G(n-2)-1)^2 + 3(G(n-2)-1)^3 + 3G(n-3)^3 \\
&\quad - 6G(n-1)^2(G(n-2)-1) - 9G(n-1)G(n-3)(G(n-2)-1) \\
&\quad + \{2[-G(n-1)^3 - G(n-3)^2G(n) - G(n)G(n-2)^2 \\
&\quad - G(n-2)^3 + 2G(n-1)^2G(n-2) \\
&\quad + 3G(n-1)G(n-2)G(n-3) + 1]\}
\end{aligned}$$

$$\begin{aligned}
&= 3G(n-1)^3 + 3G(n-3)^2G(n-1) + 3G(n-1)(G(n-2)-1)^2 \\
&\quad + 3G(n-3)(G(n-2)-1)^2 + 3(G(n-2)-1)^3 + 3G(n-3)^3 \\
&\quad - 6G(n-1)^2(G(n-2)-1) - 9G(n-1)G(n-3)(G(n-2)-1) \\
&\quad + \{2\Delta(n)\}.
\end{aligned}$$

By Lemma 5.3.5, $\Delta(n) = 0$. So in the end, simplifying by 3, we get the desired formula for $H_3(n)$:

$$\begin{aligned}
H_3(n) = &-3G(n-1)(G(n-2)-1)G(n-3) - 2G(n-1)^2(G(n-2)-1) \\
&+ G(n-1)^3 + (G(n-2)-1)^2G(n-1) + (G(n-2)-1)^2G(n-3) \\
&+ (G(n-2)-1)^3 + G(n-3)^2G(n-1) + G(n-3)^3.
\end{aligned}$$

□

5.4 The K_0 of Leavitt path algebras of the graphs C_n^3

We are now in a position to give the main result of this chapter. Applying Corollary 5.1.8, Lemma 5.2.1, Lemma 5.2.3 and Corollary 5.3.7 finally we get:

Theorem 5.4.1. *Let $n \in \mathbb{N}$. Then*

$$K_0(L_K(C_n^3)) \cong \mathbb{Z}_{d_3(n)} \times \mathbb{Z}_{\frac{d'_3(n)}{d_3(n)}} \times \mathbb{Z}_{\frac{H_3(n)}{d'_3(n)}},$$

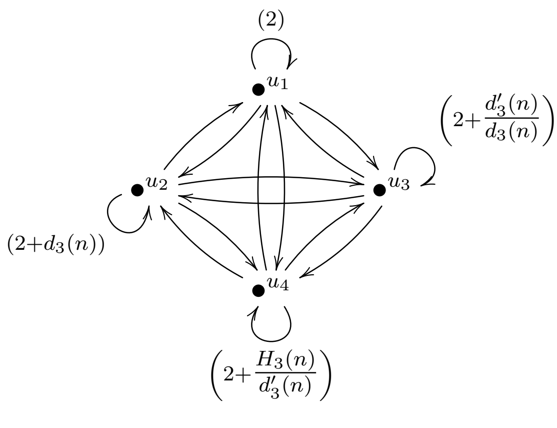
where

$$d_3(n) = \gcd\{G(n-1), G(n-2)-1, G(n-3)\} \text{ and,}$$
$$d_3'(n) = \gcd\{G(n-1)G(n-3) - (G(n-2)-1)^2, G(n)G(n-3) - G(n-1)(G(n-2)-1), G(n-1)^2 - G(n)(G(n-2)-1)\}.$$

Remark 5.4.2. In the Appendix the reader will find a list of values from $n = 1$ to $n = 30$ for the third Haselgrove sequence $H_3(n)$, Narayana's cows numbers $G(n)$, together with the greatest common divisors $d_3(n)$ and $d_3'(n)$.

We are also in position to apply Theorem 1.4.7 to explicitly realize the algebras $L_K(C_n^3)$ as the Leavitt path algebras of graphs having four vertices.

Proposition 5.4.3. *Let $n \in \mathbb{N}$. The algebra $L_K(C_n^3)$ is isomorphic to the Leavitt path algebra of the graph E_n with four vertices given by*



where the numbers in parentheses indicate the number of loops at a given vertex.

Proof. Clearly the graph E_n satisfies the conditions for $L_K(E_n)$ to be unital purely infinite simple. The incidence matrix of E_n is A_{E_n} =

$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2+d_3(n) & 1 & 1 \\ 1 & 1 & 2+\frac{d_3'(n)}{d_3(n)} & 1 \\ 1 & 1 & 1 & 2+\frac{H_3(n)}{d_3'(n)} \end{pmatrix}$, so that

$$I - A_{E_n}^t = -\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+d_3(n) & 1 & 1 \\ 1 & 1 & 1+\frac{d_3'(n)}{d_3(n)} & 1 \\ 1 & 1 & 1 & 1+\frac{H_3(n)}{d_3'(n)} \end{pmatrix}.$$

An easy computation shows that the Smith normal form of $I - A_{E_n}^t$ is just $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d_3(n) & 0 & 0 \\ 0 & 0 & \frac{d_3'(n)}{d_3(n)} & 0 \\ 0 & 0 & 0 & \frac{H_3(n)}{d_3(n)} \end{pmatrix}$, which immediately yields that $K_0(L_K(E_n))$ is isomorphic to $\mathbb{Z}_{d_3(n)} \times \mathbb{Z}_{\frac{d_3'(n)}{d_3(n)}} \times \mathbb{Z}_{\frac{H_3(n)}{d_3'(n)}}$.

Also, it is straightforward to check that $\det(I - A_{E_n}^t) = -H_3(n) < 0$.

Finally, by invoking the relation in $K_0(L_K(E_n))$ at u_1, we have

$$\begin{aligned}&[u_1] + [u_2] + [u_3] + [u_4] = (2[u_1] + [u_2] + [u_3] + [u_4]) + [u_2] + [u_3] + [u_4] \\ &= 2([u_1] + [u_2] + [u_3] + [u_4])\end{aligned}$$

so that $\sigma = [u_1] + [u_2] + [u_3] + [u_4]$ satisfies $\sigma = 2\sigma$ in the group $K_0(L_K(E_n))$, so that $\sigma = [u_1] + [u_2] + [u_3] + [u_4]$ is the identity element of $K_0(L_K(E_n))$.

Thus the purely infinite simple unital Leavitt path algebras $L_K(C_n^3)$ and $L_K(E_n)$ have these properties: $K_0(L_K(C_n^3)) \cong K_0(L_K(E_n))$ (as each is isomorphic to $\mathbb{Z}_{d_3(n)} \times \mathbb{Z}_{\frac{d_3'(n)}{d_3(n)}} \times \mathbb{Z}_{\frac{H_3(n)}{d_3'(n)}}$); this isomorphism takes $[L_K(C_n^3)]$ to $[L_K(E_n)]$ (as each of these is the identity element in their respective K_0 groups); and both $\det(I_n - A_{C_n^3}^t)$ and $\det(I - A_{E_n}^t)$ are negative. Thus the graphs C_n^3 and E_n satisfy the hypotheses of Theorem 1.4.7, and so the desired isomorphism $L_K(C_n^3) \cong L_K(E_n)$ follows. □

Further work

We describe here some of the open questions related to the subjects which have been presented in this thesis.

First we are going to focus on the commutative core $M_R(E)$ described in Section 2.3. Thanks to Theorem 2.3.12 we know that $M_R(E)$ is a maximal commutative subalgebra of $L_R(E)$; in fact, it coincides with the set of elements of the Leavitt path algebra which commutes with every element in the diagonal subalgebra $\Delta(E)$, that is,

$$M_R(E) = \{x \in L_R(E) : xd = dx \text{ for every } d \in \Delta(E)\}.$$

As an important remark we should highlight that $M_R(E)$ might not be the unique maximal commutative subalgebra of $L_R(E)$. What we have proved in Theorem 2.3.12 implies that if we consider $\mathcal{C}$ a commutative subalgebra of $L_R(E)$ such that $M_R(E) \subseteq \mathcal{C}$ then necessarily $\mathcal{C} = M_R(E)$. So $M_R(E)$ is a maximal commutative subalgebra of $L_R(E)$. However,

Question. Could $M_R(E)$ be the unique maximal commutative subalgebra of $L_R(E)$? If not, would there exist any conditions on the graph E such that $M_R(E)$ is the *unique* maximal commutative subalgebra of $L_R(E)$?

On the other hand, by Corollary 2.3.13 the center of the Leavitt path algebra $L_R(E)$, that is

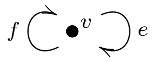
$$\mathcal{Z}(L_R(E)) = \{x \in L_R(E) : xy = yx \text{ for every } y \in L_R(E)\}$$

satisfies that $\mathcal{Z}(L_R(E)) \subseteq M_R(E)$. So then a natural question could be the following.

Question. Can $\mathcal{Z}(L_R(E)) = M_R(E)$? Does there exist an if and only if result about that? Or maybe any sufficient conditions?

And finally related to the commutative core of a Leavitt path algebra consider the following example where we see that not all the normal elements in $L_R(E)$ belong to the commutative core $M_R(E)$; we have that every element in $M_R(E)$ is normal but the other containment may not be true.

Example. Let $E = R_2$ be the 2-petals graph.



Then the element $x = ef^* + fe^*$ is normal since we have

$$x^* = (ef^* + fe^*)^* = fe^* + ef^* = x;$$

hence $xx^* = x^2 = x^*x$. But observe that $x \notin M_R(E)$ because both summands $ef^*, fe^* \notin G_E^M$ (Proposition 2.3.6). □

Question. How big is the difference between the set of all normal elements in the Leavitt path algebra and the set of elements in the core $M_R(E)$?

As a second step we could generalize these questions to higher-rank graphs for the case of the cycline subalgebra $\mathcal{M}$ of a Kumjian-Pask algebra described in Section 3.3. Analogously we obtained in Corollary 3.3.10 that if Λ is a row-finite k-graph with no sources and R is a commutative ring with 1, then the center of the Kumjian-Pask algebra satisfies that:

$$\mathcal{Z}(\mathrm{KP}_R(\Lambda)) := \{a \in \mathrm{KP}_R(\Lambda) : ax = xa \text{ for every } x \in \mathrm{KP}_R(\Lambda)\} \subseteq \mathcal{M}.$$

So as in the case of the commutative core the natural question would be:

Question. Could it happen that $\mathcal{Z}(\mathrm{KP}_R(\Lambda)) = \mathcal{M}$?

To go further, related to the topic of the generalized uniqueness theorem for Leavitt path algebras given in Theorem 2.5.1 and for Kumjian-Pask algebras proved in Theorem 3.4.4, the author would like to remark that both of them have very recently been generalized to Steinberg algebras in [26] by L.O.

Clark, R. Exel and E. Pardo. There the three authors use an analysis of the "interior of the isotropy bundle" to prove a generalized uniqueness theorem, which says that a Steinberg algebra homomorphism is injective if and only if it is injective on the interior of the isotropy group bundle.

With respect to the Leavitt path algebras of Cayley graphs, the unresolved questions are essentially number-theoretic observations. Remember that our main result concerning the Grothendieck group K_0 of the Leavitt path algebras of the form $L_K(C_n^3)$ affirms the following statement.

Theorem 5.4.1. Let $n \in \mathbb{N}$. Then

$$K_0(L_K(C_n^3)) \cong \mathbb{Z}_{d_3(n)} \times \mathbb{Z}_{\frac{d_3'(n)}{d_3(n)}} \times \mathbb{Z}_{\frac{H_3(n)}{d_3'(n)}},$$

where

$$\begin{aligned} d_3(n) &= \gcd\{G(n-1), G(n-2)-1, G(n-3)\} \text{ and,} \\ d_3'(n) &= \gcd\{G(n-1)G(n-3) - (G(n-2)-1)^2, \\ &\qquad G(n)G(n-3) - G(n-1)(G(n-2)-1), \\ &\qquad G(n-1)^2 - G(n)(G(n-2)-1)\}. \end{aligned}$$

.

Question. Which is the relation between the sequences d_3 and d_3'?

The reader could for a moment look at the given tables for the sequences H_3, G, d_3 and d_3' in the Appendix. From the expression of both sequences d_3 and d_3' it immediately happens that $d_3^2(n)$ divides $d_3'(n)$ for every $n \in \mathbb{N}$. Moreover, let us define the sequence b_3 as the following:

$$b_3(n) := \begin{cases} 31 & \text{if } n \equiv 0 \mod 30 \\ 1 & \text{otherwise.} \end{cases}$$

Then it seems apparent that the following is satisfied:

Conjecture. For every $n \in \mathbb{N}$, $d_3'(n) = b_3(n) \cdot d_3^2(n)$.

According to the software *Magma* the previous statement is true for all n from $n = 1$ to $n = 150000$. Why does the number 31 occur? Maybe it is due to the related fact: consider α, β, and γ the solutions of the equation $x^3 - x^2 - 1 = 0$, then

$$-[(\alpha - \beta)(\alpha - \gamma)(\beta - \gamma)]^2 = 31.$$

In other words, -31 is the value of the discriminant of the characteristic polynomial associated the Narayana's cows sequence G. Actually, jointly with G. Abrams and S. Erickson, we are working to obtain that

$$b_3(n) \cdot d_3^2(n) \text{ divides } d_3'(n) \text{ for every } n \in \mathbb{N}.$$

The other direction, i.e., $d_3'(n)$ divides $b_3(n) \cdot d_3^2(n)$, is still an open question.

Bibliography

[1] G. Abrams, Leavitt path algebras: The first decade, *Bull. Math. Sci.* **5**(1) (2015), 59-120.

[2] G. Abrams, P.N. Ánh, A. Louly, E. Pardo, The classification question for Leavitt path algebras, *J. Algebra* **320**(5) (2008), 1983-2026.

[3] G. Abrams, P. Ara, M. Siles Molina, Leavitt path algebras. A primer and handbook, *Lecture Notes Series in Mathematics, Springer Verlag*, to appear.

[4] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, *J. Algebra* **293** (2005), 319-334.

[5] G. Abrams, G. Aranda Pino, Purely infinite simple Leavitt path algebras, *J. Pure App. Alg.* **207**(3) (2006), 553-563.

[6] G. Abrams, G. Aranda Pino, The Leavitt path algebras of arbitrary graphs, *Houston J. Math.* **34** (2008), 423-442.

[7] G. Abrams, G. Aranda Pino, Leavitt path algebras of generalized Cayley graphs, *Mediterranean J. Math.* **13**(1) (2016), 1-27.

[8] G. Abrams, G. Aranda Pino, M. Siles Molina, Locally finite Leavitt path algebras, *Israel J. Math.* **165** (2008), 329-348.

[9] G. Abrams, M. Kanuni, Cohn path algebras have Invariant Basis Number, *Commun. Algebra* **44**(1) (2016), 371-380.

[10] G. Abrams, A. Louly, E. Pardo, C. Smith, Flow invariants in the classification of Leavitt path algebras, *J. Algebra* **333** (2011), 202-231.

[11] G. Abrams, B. Schoonmaker, Leavitt path algebras of Cayley graphs arising from cyclic groups, *Amer. Math. Soc. Contemporary Mathematics Series* **634** (2015), 1-10.

[12] P. Ara, M.A. González Barroso, K. Goodearl, E. Pardo, Fractional skew monoid rings, *J. Algebra* **278** (2004), 104-126.

[13] P. Ara, K. Goodearl, E. Pardo, K_0 of purely infinite simple regular rings, *K-Theory* **26** (2002), 69-100.

[14] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-theory for graph algebras, *Algebr. Represent. Theory* **10** (2007), 157-178.

[15] G. Aranda Pino, On maximal left quotient systems and Leavitt path algebras, *Doctoral dissertation. University of Málaga* (2005).

[16] G. Aranda Pino, J. Clark, A. an Huef, I. Raeburn, Kumjian-Pask algebras of higher rank graphs, *Trans. Amer. Math. Soc.* **365** (2013), 3613-3641.

[17] G. Aranda Pino, K. Crow, The center of a Leavitt path algebra, *Rev. Mat. Iberoam.* **27** (2011), 621-644.

[18] G. Aranda Pino, D. Martín Barquero, C. Martín González, M. Siles Molina, Socle theory for Leavitt path algebras of arbitrary graphs, *Rev. Mat. Iberoam.* **26** (2010), 611-638.

[19] G. Aranda Pino, E. Pardo, M. Siles Molina, Exchange Leavitt path algebras and stable rank, *J. Algebra* **305**(2) (2006), 912-936.

[20] G. Aranda Pino, K.M. Rangaswamy, L. Vaš, *-Regular Leavitt path algebras of arbitrary graphs, *Acta Math. Sin. (Eng. Ser.)* **28**(5) (2012), 957-968.

[21] T. Bates, D. Pask, I. Raeburn, W. Szymański, The C^*-algebras of row-finite graphs, *New York J. Math.* **6** (2000), 307-324.

[22] G.M. Bergman, Coproducts and some universal ring constructions, *Trans. Amer. Math. Soc.* **200** (1974), 33-88.

[23] J.H. Brown, A. an Huef, Centers of algebras associated to higher-rank graphs, *Rev. Mat. Iberoam.* **30** (2014), 1387-1396.

[24] J.H. Brown, A. an Huef, The socle and semisimplicity of a Kumjian-Pask algebra, *Comm. Algebra* **43** (2015), 2703-2723.

[25] J.H. Brown, G. Nagy, S. Reznikoff, A generalized Cuntz-Krieger uniqueness theorem of higher-rank graphs, *J. Funct. Anal.* **266** (2014), 2590-2609.

[26] L.O. Clark, R. Exel, E. Pardo, A generalised uniqueness theorem and the graded ideal structure of Steinberg algebras, arXiv: 1609.02873.

[27] L.O. Clark, C. Farthing, A. Sims, M. Tomforde, A groupoid generalisation of Leavitt path algebras, *Semigroup Forum* **89** (2014), 501-517.

[28] L.O. Clark, C. Flynn, A. an Huef, Kumjian-Pask algebras of locally convex higher-rank graphs, *J. Algebra* **399** (2014), 445-474.

[29] L.O. Clark, C. Gil Canto, A. Nars-Isfahani, The cycline subalgebra of a Kumjian-Pask algebra, *Proc. Amer. Math. Soc.*, to appear, arxiv: 1603.00508.

[30] L.O. Clark, D. Martín Barquero, C. Martín González, M. Siles Molina, Using Steinberg algebras to study decomposability of Leavitt path algebras, arXiv: 1603.01033.

[31] L.O. Clark, D. Martín Barquero, C. Martín González, M. Siles Molina, Using the Steinberg algebra model to determine the center of any Leavitt path algebra, arXiv: 1604.01079.

[32] L.O. Clark, Y.E.P. Pangalela, Kumjian-Pask algebras of finitely-aligned higher-rank graphs, arXiv: 1512.06547.

[33] P.M. Cohn, Some remarks on the invariant basis property, *Topology* **5** (1966), 215-228.

[34] M. G. Corrales García, D. Martín Barquero, C. Martín González, M. Siles Molina, J. F. Solanilla Hernández, Centers of path algebras, Cohn and Leavitt path algebras, *Bull. Malays. Math. Sci. Soc.* (2015), 1-23.

[35] M. G. Corrales García, D. Martín Barquero, C. Martín González, M. Siles Molina, J. F. Solanilla Hernández, Extreme cycles. The center of a Leavitt path algebra, *Publ. Mat.* **60**(1) (2016), 235-263.

[36] J. Cuntz, Simple C^*-algebras generated by isometries, *Commun. Math. Phys.* **57**(2) (1977), 173-185.

[37] J. Cuntz, W. Krieger, A class of C^*-algebras and topological Markov chains, *Invent. Math. Phys.* **56**(3) (1980), 251-268.

[38] C. Flaut, V. Shpakivskyi, On Generalized Fibonacci Quaternions and Fibonacci-Narayana Quaternions, *Advances in Applied Clifford Algebras* **23**(3) (2013), 673-688.

[39] C. Gil Canto, Structure of Leavitt path algebras, *Master thesis. University of Málaga* (2012).

[40] C. Gil Canto, A. Nasr-Isfahani, The maximal commutative subalgebra of a Leavitt path algebra, submitted, arXiv: 1510.03992.

[41] K. R. Goodearl, Leavitt path algebras and direct limits, *Rings, Modules and Representations, Contemporary Mathematics, Amer. Math. Soc., Providence* **480** (2009), 165-187.

[42] C.B. Haselgrove, A note on Fermat's Last Theorem and the Mersenne Numbers, *Eureka: The Archimedians' Journal* **11** (1949), 19-22.

[43] A. Kumjian, D. Pask, Higher rank graph C^*-algebras, *New York J. Math.* **6** (2000), 1-20.

[44] A. Kumjian, D. Pask, I. Raeburn, Cuntz-Krieger algebras of directed graphs, *Pac. J. Math.* **184**(1) (1998), 161-174.

[45] A. Kumjian, D. Pask, I. Raeburn, J. Renault, Graphs, groupoids, and Cuntz-Krieger algebras, *J. Funct. Anal.* **144**(2) (1997), 505-541.

[46] W. G. Leavitt, Modules without invariant basis number, *Proc. Amer. Math. Soc.* **8** (1957), 322-328.

[47] W. G. Leavitt, The module type of homomorphic images, *Duke Math. J.* **32** (1965), 305-311

[48] I. Lehmann, A. Posamentier, The (Fabulous) Fibonacci Numbers, *Prometheus Books* (2008).

[49] P. Menal, J. Moncasi, Lifting units in self-injective rings and an index theory for Rickart C^*-algebras, *Pac. J. Math.* **126**(2) (1987), 295-329.

[50] G. Nagy, S. Reznikoff, Abelian core of graph algebras, *J. London Math. Soc.* **85** (2012), 889-908.

[51] Online Encyclopedia of Integer Sequences, http://oeis.org/.

[52] I. Raeburn, Graph algebras, *CBMS Regional Conference Series in Mathematics, Amer. Math. Soc., Providence* **103** (2005).

[53] K. M. Rangaswamy, The theory of prime ideals of Leavitt path algebras over arbitrary graphs, *J. Algebra* **375** (2013), 73-96.

[54] G. Robertson, T. Steger, Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras, *J. Reine Angew. Math.* **513** (1999), 115-144.

[55] M. Siles Molina, Algebras of quotients of path algebras, *J. Algebra* **319**(12) (2008), 329-348.

[56] H. J. S. Smith, On systems of linear indeterminate equations and congruences, *Phil. Trans. R. Soc. Lond.* **151**(1) (1861), 293-326.

[57] H. J. S. Smith, Arithmetical Notes, *Proc. London Math. Soc.* **S1-4**(1) (1873), 236.

[58] B. Steinberg, A groupoid approach to discrete inverse semigroup algebras, *Adv. Math.* **223** (2010), 689-727.

[59] W. Szymański, General Cuntz-Krieger uniqueness theorem, *Internat. J. Math.* **13** (2002), 549-555.

[60] M. Tomforde, Uniqueness theorems and ideal structure for Leavitt path algebras, *J. Algebra* **318**(1) (2007), 270-299.

[61] M. Tomforde, Leavitt path algebras with coefficients in a commutative ring, *J. Pure Appl. Algebra* **215** (2011), 471-484.

[62] D. Yang, Cycline subalgebras of k-graph C^*-algebras, *Proc. Amer. Math. Soc.* **144**(7) (2016), 2959-2969.

Notation

$K[x, x^{-1}]$	algebra of Laurent polynomials
$\mathbb{E}$	algebraic conditional expectation
$\partial(U)$	boundary of a topological space
C_n^j	Cayley graph
$\mathcal{Z}(S)$	center of a ring
$C_R(E)$	Cohn path algebra
$\mathrm{Coker}(f)$	cokernel
$M_R(E)$	commutative core
R	commutative unital ring
M_p	companion matrix
$\mathbb{C}$	complex numbers
$\mathcal{O}_n$	Cuntz algebra
$C^*(E)$	Cuntz-Krieger algebra
G_E	Cuntz-Krieger generator set
G_E^{Δ}	Cuntz-Krieger diagonal generator set
G_E^{M}	Cuntz-Krieger normal generator set
$\mathcal{M}$	cycline subalgebra
$Z(\mu)$	cylinder set
$\deg$	degree
d	degree map
$\delta_i \;\; \alpha_i$	determinant divisors of a Smith normal form
$\det(M)$	determinant of a matrix
$\Delta(E)$	diagonal subalgebra
$\mathcal{D}$	diagonal (Kumjian-Pask) subalgebra
$E = (E^0, E^1, r, s)$	directed graph
$\oplus$	direct sum

$\sqcup$	disjoint union
$\emptyset$	empty set
$\mathrm{End}_R(M)$	endomorphisms of an R-module
$\widehat{E}$	extended graph
$F(n)$	Fibonacci sequence
K	field
A_g	g-homogeneous component
M_E	graph monoid
$K_0(S)$	Grothendieck group
$H_j(n)$	Haselgrove sequence
$\Lambda = (\Lambda^0, \Lambda, r, s)$	higher rank graph
I_n	identity matrix of size n
$\mathrm{Im}(f)$	image
A_E	incidence matrix
$\mathbb{Z}$	integers
$\mathrm{Int}(U)$	interior of a topological space
$\cap$	intersection
$\cong$	isomorphism
$\mathrm{Ker}(f)$	kernel
δ_{ij}	Kronecker delta
$\mathrm{KP}_R(\Lambda)$	Kumjian-Pask algebra
$L_K(m,n)$ $L(m,n)$	Leavitt algebra
$L_R(E)$ $(L_K(E))$	Leavitt path algebra over R (over K)
$l(\mu)$	length of a path
$\mathbb{M}_n(R)$	matrix ring
$G(n)$	Narayana's cows sequence
$\mathbb{N}$	natural numbers
S	ring
$\mathbb{Z}_n$	ring of n-integers
R_n	rose with n leaves graph
$\mathrm{Path}(E)$	set of all paths
$\mathcal{V}(S)$	set of isomorphism classes of f. g. projective S-modules
E^n	set of paths of length n
E^0_{reg}	set of regular vertices
$\mathrm{Vert}(\mu)$	set of vertices of a path
$\mathrm{SNF}(M)$	Smith normal form of a matrix

α, β, γ	solutions of the equation $x^3 - x^2 - 1 = 0$
$s(e)$ $r(e)$	source and range of an edge
$\subseteq$ $\subset$	subset
$\cup$	union

Index

Appendix

n	$H_3(n)$	$G(n)$	$d_3(n)$	$d'_3(n)$
1	1	1	1	1
2	3	1	1	1
3	1	1	1	1
4	3	2	1	1
5	11	3	1	1
6	9	4	1	1
7	8	6	2	4
8	27	9	3	9
9	37	13	1	1
10	33	19	1	1
11	67	28	1	1
12	117	41	1	1
13	131	60	1	1
14	192	88	4	16
15	341	129	1	1
16	459	189	3	9
17	613	277	1	1
18	999	406	1	1
19	1483	595	1	1
20	2013	872	1	1
21	3032	1278	2	4
22	4623	1873	1	1
23	6533	2745	1	1
24	9477	4023	9	81
25	14311	5896	1	1
26	20829	8641	1	1
27	30007	12664	1	1
28	44544	18560	8	64
29	65657	27201	1	1
30	95139	39865	1	31