Escuela de Ingenierías Industriales
Departamento de Matemática Aplicada
Programa de Doctorado en Ingeniería Mecánica y Eficiencia Energética

TESIS DOCTORAL

Un enfoque lógico a los sistemas de implicaciones y las bases directas

A logic-based approach to deal with implicational systems and direct bases

Dª. Estrella Rodríguez Lorenzo

Directores: Dr. D. Pablo José Cordero Ortega y Dr. D. Ángel Mora Bonilla

Málaga, 2017

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AUTOR: Estrella Rodríguez Lorenzo

http://orcid.org/0000-0001-6115-9080

EDITA: Publicaciones y Divulgación Científica. Universidad de Málaga

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Dr. Pablo José Cordero Ortega, Profesor Titular de Universidad del Departamento de Matemática Aplicada de la Universidad de Málaga

Dr. Ángel Mora Bonilla, Profesor Titular de Universidad del Departamento de Matemática Aplicada de la Universidad de Málaga

HACEN CONSTAR:

Que Dª. Estrella Rodríguez Lorenzo, Licenciada en Matemáticas, ha realizado en el Departamento de Matemática Aplicada de la Universidad de Málaga, bajo nuestra dirección y tutela, el trabajo de investigación correspondiente a su Tesis Doctoral titulada:

Un enfoque lógico a los sistemas de implicaciones y las bases directas

Revisado el presente trabajo, estimamos que puede ser presentado al Tribunal que ha de juzgarlo. Asimismo, informamos que las publicaciones científicas que avalan esta tesis no han sido utilizadas como aval para tesis anteriores.

Y para que conste a efectos de lo establecido en el Real Decreto 99/2011, autorizamos la presentación de este trabajo en la Universidad de Málaga.

Málaga, a 5 de mayo de 2017

Dr. Pablo José Cordero Ortega

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A mis padres,
por la fe ciega que siempre han tenido en mí.

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…También nos educan diciéndonos que es con esfuerzo que se consiguen cosas y que, junto esas cosas, llegará la felicidad. La verdad que yo creo que eso es una gran mentira. Una gran mentira socialmente aceptada, universalmente determinada, pero una mentira al fin. Yo no creo para nada en el esfuerzo como camino para hacer algo. Digo, no creo que haya que esforzarse, sino que hay que dedicarse, que no es lo mismo. La dedicación a algo, la apuesta de todo lo que soy al servicio de un proyecto, no es un esfuerzo. Yo no creo en el esfuerzo, en el sentido de forzarme a hacer lo que no quiero hacer. No creo en los logros que se consiguen desde el esfuerzo. Sí creo en la elección de un camino, sí creo en los rumbos que me fijo.

Jorge Bucay.

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Agradecimientos

Me enfrento en estas líneas a lo que más he temido de esta última etapa de mi doctorado: su fin. Cuesta poder expresar con palabras todo el agradecimiento que he sentido y siento por las personas que han estado ahí, ayudándome, apoyándome y viéndome evolucionar en algo que me ha hecho crecer como investigadora, pero sobre todo como persona.

Gracias a vosotros, mis directores, Pablo José Cordero y Ángel Mora, y a ti, mi otro director, Manuel Enciso, que lo eres en mi corazón aunque no constes en ningún papel. Gracias por todo lo enseñado, sí, pero más aún por todos los momentos compartidos y por hacerme sentir una más de vosotros. Espero que esto, aunque marque el final de una etapa de mi vida, también marque nuestro principio como compañeros en esta ardua tarea que es la investigación y tal vez, por qué no, en la tarea de esculpir la última etapa de formación del ser humano, si es que alguna vez acaba.

Para ser sincera, llegué con un poco de duda, quizá de miedo. Muchos dicen que en los años de doctorado se sufre y sientes en algún momento que quieres tirarlo todo por la borda. Habiendo pasado por eso puedo asegurar que depende mucho de los que te dirigen y orientan. Gracias una vez más por ser como sois; habéis conseguido, probablemente sin daros cuenta, que nunca me haya sentido abatida. Tengo tantas cosas que agradeceros, que es imposible plasmarlo en un papel. Mil veces gracias.

Y... gracias también por haber conseguido que pasase estos últimos meses sin beber ni una mísera gota de café, muy a pesar de Ángel. Son tantas cosas, tantas vivencias...

Un especial agradecimiento a mi otro Manolo por tus lecciones de inglés

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y por tu paciencia infinita para responder a mis eternas preguntas, y a ti, Inma, por ser una de las primeras personas en leerse la tesis aportando un poco más a este trabajo.

Je voudrais remercier aussi à Rokia Missaoui et Karell Bertet. Merci pour m'avoir accouilli dans vos groupes d'investigation, pour traivailler ensemble et obtenir de bons résultats. Vous avez été un example pour moi.

I would also like to thank Kira Adaricheva. Thank you for your efficiency and for enabling the beginning of what I expect it will be a fruitful collaboration.

No puedo olvidaros a vosotros, todos mis compañeros del Departamento de Matemática Aplicada; gracias por ayudarme cuando lo he necesitado, por acogerme como compañera y por siempre saludar con una sonrisa.

Gracias, mamá y papá, por siempre haberme dado la confianza de sentirme capaz. Me cuesta entender cómo el simple hecho de que las personas que más te quieren te digan: "tú eres capaz, mohtrua", haga posible que te lo creas hasta el punto de conseguirlo. Gracias por haberme hecho mirar más allá de lo que mis ojos veían, gracias por crearme aspiraciones, por descubrirme nuevas metas. Poneos la matrícula como padres porque sois unos "monstruos".

Luis Miguel, hermano, gracias muy, muy especiales por acompañarme en la vida cuidando siempre de mí. No sé qué habría hecho esa primera semana en Francia sin tu compañía. Menos mal que te tuve al lado.

También quiero que estéis presentes aquí: Elena, María, Inma, Lourdes, Cristina y David, mi primo. En estos cuatro años siempre estuvisteis a mi lado aunque hubo momentos en los que ni siquiera tuve tiempo para vosotros; os habéis alegrado por mis logros, me habéis acompañado si en algún momento he estado un poco triste y siempre, siempre me habéis apoyado. ¡Sois los mejores! Y Fernando, también aquí, cerca; si bien es verdad que no han sido los cuatro años completos, has estado cuando más apoyo he necesitado. Gracias por los ratitos de traducción simplemente por el hecho de estar conmigo.

Cómo olvidarte, Tomás Caraballo, por haber hecho posible que esté donde estoy, por acordarte de mí cuando salió esta plaza y por ser tan

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rápido en solucionar todas las peticiones del ministerio. Fuiste un profesor magnífico y lo has demostrado con la preocupación que muestras por tus alumnos. Tampoco te olvido, Luis Narváez, por iniciarme en el mundo de la investigación, ni a ti, Juan Núñez, por hacerme alumna interna y enseñarme el apasionante mundo de las matemáticas divulgativas.

Hermano Esteban, usted me enseñó el placer por las matemáticas bien hechas, la pasión por la enseñanza y la pasión por aprender. Siempre estará presenté en mi corazón porque no le olvidaré nunca. Gracias, maestro, allá en el Cielo.

Por último, gracias a todos aquellos a los que no os he citado pero que habéis estado ahí. Sois tantos que es imposible nombraros a todos; sabed que también estáis presentes aquí.

Gracias, muchas gracias.

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Resumen

El tratamiento de la información y el conocimiento es uno de los muchos campos en los que confluyen los métodos matemáticos y computacionales. Una de las áreas donde encontramos de forma clara esta concurrencia es en el Análisis de Conceptos Formales, donde los métodos de almacenamiento, descubrimiento, análisis y manipulación del conocimiento descansan sobre las sólidas bases del Álgebra y de la Lógica.

En el Análisis de Conceptos Formales la información se representa en tablas binarias en las que se relacionan objetos con sus atributos. Dichas tablas, denominadas contextos formales, son el repositorio de datos del que se extrae el conocimiento mediante la utilización de técnicas algebraicas. Este conocimiento se puede representar de diversas formas: retículos de conceptos, operadores de cierre y conjuntos de implicaciones.

Una de las principales ventajas de usar sistemas de implicaciones para representar el conocimiento es que admiten un tratamiento sintáctico por medio de la lógica, segundo pilar matemático en el que se sustenta la tesis. Tradicionalmente se han utilizado los Axiomas de Armstrong [4] como herramienta para razonar con implicaciones aunque las limitaciones inherentes a este sistema axiomático lo inhabilitan como mecanismo de deducción automática. La mejor alternativa de cara al razonamiento automático viene de mano de la Lógica de Simplificación [20]. El conjunto de axiomas y reglas de inferencias de esta lógica lleva directamente a un conjunto de equivalencias que permiten eliminar redundancias en los sistemas de implicaciones, es decir, simplificarlos. Estas mismas equivalencias proporcionan, a su vez, un método de deducción automática.

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En este trabajo nos centramos en las implicaciones como método de representación del conocimiento y mostramos cómo la Lógica de Simplificación se conforma como el núcleo de los nuevos métodos que proponemos.

Cabe destacar que el conocimiento que se puede extraer de un contexto formal queda unívocamente determinado por un retículo de conceptos o, equivalentemente, por un operador de cierre. Sin embargo, el conocimiento representado por éstos puede ser descrito por diferentes sistemas de implicaciones. Esto se debe a que, gracias al tratamiento lógico de estos sistemas, podemos encontrar diversos subconjuntos que caracterizan al conjunto total de las implicaciones que se satisfacen en el contexto.

La extracción de sistemas de implicaciones, y su posterior tratamiento y manipulación, constituyen un tema de actualidad en la comunidad del Análisis de Conceptos Formales. Los conjuntos de implicaciones extraídos pueden contener gran cantidad de información redundante, por lo que el estudio de propiedades que permitan caracterizar conjuntos equivalentes de implicaciones con menor redundancia o sin ella, se erige como uno de los retos más importantes. Sin embargo, como sucede en otras áreas, en algunas ocasiones puede ser interesante almacenar cierta clase de información redundante en función del uso posterior que se le pretenda dar.

Sobresale pues, entre los temas de interés del área, el problema de la búsqueda de representaciones canónicas de sistemas de implicaciones que, satisfaciendo ciertas propiedades, permitan compilar todo el conocimiento extraído del contexto formal. Estas representaciones canónicas para los sistemas de implicaciones suelen recibir el nombre de 'bases'. En esta tesis ponemos nuestra atención en un grupo de bases conocidas como 'bases directas', que son aquellas que permiten calcular el cierre de cualquier conjunto en un único recorrido del sistema de implicaciones.

Los **objetivos generales** de la tesis son dos:

(i) El estudio de las bases directas en Análisis de Conceptos Formales clásico con la finalidad de obtener algoritmos eficientes para calcular dichas bases. Para ello analizamos las definiciones que aparecen en la bibliografía y proponemos una alternativa.

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(ii) Establecer las bases para la extensión de estos resultados al Análisis de Conceptos Triádicos, en particular, introducir una lógica que permita el razonamiento automático sobre implicaciones en esta extensión.

De acuerdo con estos objetivos generales, la tesis se encuentra organizada en dos partes: la primera dedicada a las bases directas en el marco del Análisis de Conceptos Formales clásico y la segunda a introducir un tratamiento lógico para las implicaciones en el Análisis de Conceptos Triádicos. Además, el trabajo comienza con un capítulo de resultados preliminares generales con el fin de hacer, en la medida de lo posible, la tesis autocontenida. Cada una de las partes, a su vez, comienza con un capítulo de preliminares específicos del problema a tratar en esa parte. Por último, la tesis concluye con un capítulo de conclusiones y trabajos futuros, con la bibliografía, y con los habituales índices de términos, figuras y tablas.

Antes de comenzar con la descripción a grandes rasgos de las aportaciones, consideramos necesario resaltar que esta tesis doctoral ha sido, en gran medida, fruto de las estancias en centros de investigación extranjeros realizadas por la doctoranda, y que han supuesto la consolidación de las colaboraciones con la Dra. Kira Adaricheva de la Hofstra University de Nueva York (Estados Unidos), la Dra. Karell Bertet de la Universidad de La Rochelle (Francia) y la Dra. Rokia Missaoui de la Universidad de Quebec en Ottawa (Canadá).

Aportaciones al estudio y tratamiento de bases directas

En la primera parte de la tesis, en el marco clásico del Análisis de Conceptos Formales, estudiamos las bases directas poniendo especial atención en los métodos automáticos para obtenerlas.

En la bibliografía se pueden encontrar diversas definiciones de base. Algunas de ellas coinciden, pero otras no. En el Capítulo 2 presentamos un resumen de las definiciones, propiedades y métodos más importantes de la bibliografía actual del área.

Para abordar el problema de obtención de bases de implicaciones necesitamos caracterizar qué propiedades tiene que satisfacer este conjunto. De

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entre todas las propiedades destacamos en nuestro estudio las siguientes: *minimalidad*, *optimalidad* y la propiedad de ser directa.

La base más citada en la bibliografía del Análisis de Conceptos Formales, la conocida como base de Duquenne-Guigues [29], satisface la primera de las propiedades que garantiza que la base posee el menor número de implicaciones posibles. La propiedad de optimalidad hace referencia a que el tamaño de la base sea el menor posible, considerando que el tamaño de una base es el número total de incidencias de los atributos en el conjunto de implicaciones. Por último, las bases directas tienen como característica que la computación del cierre de cualquier conjunto de atributos se puede obtener en un solo recorrido (una sola iteración) del conjunto de implicaciones.

El estudio de las bases directas tiene una especial relevancia debido a que hay diversos problemas de naturaleza exponencial que requieren del calculo exhaustivo de cierres, de modo que cualquier reducción en el coste de cómputo de estos cierres tiene una enorme repercusión en el problema. Si a la condición de ser directa se le añade que, tanto el número de implicaciones como el número de incidencias de los atributos, sea el menor posible, tendremos una repercusión notable en el coste de los algoritmos que hacen uso masivo de los cierres. De aquí la consideración e importancia de cualquier avance en el tema de las bases directas.

La propiedad de ser directa la podemos encontrar en los albores de este área, en la conocida como base de premisas propias [26], aunque solo estudiada desde el punto de vista teórico y no se desarrolló ningún algoritmo para su cálculo. Se caracteriza por ser una base directa y minimal, es decir, la de menor cardinal entre todas las directas equivalentes. Esta misma noción se ha utilizado en distintas áreas bajo distintos nombres, tal y como se resume en [11]:

- La base implicacional débil [54] usada en la teoría de espacios de conocimiento.

- La base minimal a la izquierda usada en el área de las bases de datos relacionales [36] y más tarde en el marco de las teorías de Horn [31].

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- La base de relación de dependencia [39] estudiada en teoría de retículos y de conjuntos ordenados.

- La base libre de iteración canónica [56] que usa sistemas implicacionales para obtener sistemas de cierre en el área de las bases de datos relacionales.

Base directa-optimal

Es en el marco del Análisis de Conceptos Formales donde estas bases reaparecen en 2004 bajo el nombre de base directa-optimal [12] en la que los autores presentan un método para calcularla. Posteriormente en [9, 11] Bertet et al. proponen métodos mejorados para la obtención de dicha base pero usando sistemas implicacionales unitarios que facilitan la manipulación de las implicaciones a costa de aumentar drásticamente el tamaño del conjunto de implicaciones de partida.

Cabe destacar que estamos abordando un problema no trivial dado que la complejidad en el peor caso de los métodos de cálculo de la base directa-optimal es exponencial respecto al tamaño de la entrada [10]. Precisamente es este reto lo que motiva el estudio de cómo diseñar métodos más eficientes que, aunque no puedan evitar la naturaleza exponencial del problema, sí consigan mejorar los tiempos de cómputo de dichas bases.

En la tesis proponemos nuevos métodos para la obtención de la base directa-optimal más eficientes que los existentes en la bibliografía. Se muestra una comparativa entre todos estos métodos para probar la bondad de nuestra propuesta.

El objetivo ha sido el estudio de cómo la aplicación de las reglas de la Lógica de Simplificación elimina redundancias de los sistemas de implicaciones. Introducimos una nueva regla, denominada *regla de simplificación fuerte*, `[sSimp]`, derivada de la Lógica de Simplificación, que añade nuevas implicaciones de cara a acercarnos a la base directa-optimal sin añadir más redundancia de la necesaria.

El primero de los métodos que proponemos, `doSimp`, tiene tres etapas bien diferenciadas. La primera de ellas usa la equivalencia de simplificación

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(`Si-Eq`) para transformar el sistema de implicaciones inicial en uno simplificado equivalente. La segunda etapa usa la regla `[sSimp]` con el objetivo de añadir aquellas implicaciones necesarias para convertir el conjunto en directo. Tras esta etapa, se obtiene un sistema de implicaciones que es el menor conjunto que contiene al sistema de implicaciones de la entrada y que es cerrado respecto a `[sSimp]`. En la última de las etapas optimizamos el conjunto obtenido anteriormente, consiguiendo la base directa-optimal tras eliminar toda la redundancia posible de las implicaciones sin perder la propiedad de ser directa. Presentamos una comparativa que evidencia que `doSimp` alcanza la base directa-optimal de una manera mucho más eficiente que los algoritmos existentes previamente en la bibliografía.

A pesar de esta mejora, volvemos a hacer un análisis llegando a la conclusión de que los sistemas implicacionales directos obtenidos tras la segunda etapa intermedia adquieren tamaños significativamente grandes. Evidentemente, cuanto mayor sea este tamaño más tiempo necesitará la etapa de optimización para alcanzar la base directa-optimal.

Proponemos entonces un segundo algoritmo, denominado `SLgetdo`, que fusiona las dos últimas etapas de `doSimp` incorporando la regla `[sSimp]` dentro de la función de simplificación. El objetivo es alcanzar la propiedad de ser directa con la menor redundancia posible. El nuevo método desarrollado va generando conjuntos de implicaciones de menor tamaño por lo que la regla `[sSimp]` se aplicará un menor número de veces. Un punto crítico, en este caso, es cerciorarse de que las implicaciones que se añaden con `[sSimp]` después no serán eliminadas con `(Si-Eq)` para evitar bucles infinitos. Probamos que esto no puede suceder.

Se realiza un experimento para comparar los dos métodos propuestos en la tesis. En él, se confirma el mejor comportamiento de `SLgetdo` respecto a `doSimp`, el cual, como se había comentado anteriormente, mejoraba a los ya existentes en la bibliografía.

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D-base

Más recientemente, Adaricheva et al. [3] proponen una nueva definición de base directa, la D-base, que es un subconjunto de la base directa-optimal. Por tanto, ésta suele ser de menor tamaño y sigue permitiendo la computación de los cierres en una sola iteración.

La definición de D-base se basa en el tratamiento separado de las implicaciones según el cardinal de las premisas. Una D-base es un par de dos conjuntos de implicaciones: aquellas con premisa unitaria y las que la tienen no unitaria. La D-base permite calcular los cierres en un solo recorrido del conjunto de implicaciones, siempre que las implicaciones con premisas unitarias sean utilizadas antes que las de premisa no unitaria. Se dice que es una base directa ordenada.

El concepto de recubrimiento minimal es clave en la definición de D-base. A partir de estos recubrimientos se caracterizan los dos conjuntos de implicaciones que forman la D-base. En la bibliografía existente, el único método que la calcula toma como entrada un contexto formal. Cabe destacar que el cálculo de la D-base a partir de un conjunto arbitrario de implicaciones era un problema abierto, resuelto en el presente trabajo. Esto permitirá el uso de las D-bases en las aplicaciones reales. Proponemos dos algoritmos para el cálculo de la D-base comparando este problema con el cálculo de los generadores minimales.

El primer método desarrollado se fundamenta en la relación existente entre los generadores minimales y los recubrimientos. Establecemos, en primer lugar, los fundamentos teóricos de dicha relación. Tras ello, presentamos al primer algoritmo que utiliza, como primera etapa, el algoritmo `MinGen`, presentado en [17] y explicado en el capítulo de preliminares. Esta etapa, que calcula todos los generadores minimales, constituye la parte con complejidad exponencial del algoritmo. A partir de ello resulta sencillo encontrar la D-base en dos pasos: de los generadores minimales obtenemos los recubrimientos para cada atributo y, posteriormente, de la lista de recubrimientos asociada a cada atributo escogemos aquellos que son minimales.

Con la idea en mente de que no todos los generadores minimales con-

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ducen a recubrimientos minimales, la segunda aproximación desarrollada evita el cómputo de todos los generadores minimales. Teniendo en mente las operaciones que realiza `MinGen`, se diseña un nuevo algoritmo inspirado en el citado método que va calculando sólo los generadores minimales que conducen a recubrimientos minimales. Destacamos que el estudio teórico realizado en este tema ha sido un factor clave que nos permitió el diseño del nuevo algoritmo y que, tanto en éste como en el anterior, la Lógica de Simplificación guía todos los pasos.

El nuevo método desarrollado mejora considerablemente al primer método que propusimos tal y como puede comprobarse en la comparativa que aparece más adelante en la presente memoria de tesis. Para ello se han realizado dos experimentos: el primero sobre conjuntos de implicaciones generados de forma aleatoria y el segundo utilizando implicaciones extraídas de las bases de datos del repositorio de la UCI, de la "School of Information and Computer Sciences" de la Universidad de California.

Base dicótoma directa

Como ya hemos comentado, los numerosos beneficios que ofrecen las bases directas tienen, como contraposición, el alto coste de su cálculo. Una vez que hemos propuesto algoritmos más eficientes para las bases directas introducidas por otros autores, proponemos la definición de una nueva base directa con el objetivo último de que pueda ser calculada de manera más eficiente. Tomamos como inspiración el concepto de D-base que divide el conjunto de implicaciones en dos partes en función de su comportamiento de cara al cálculo de cierres. En este caso ponemos nuestra atención en las implicaciones que llamamos quasi-claves. Se trata de una generalización de las claves [16, 38], que juegan un papel relevante en la teoría de bases de datos relacionales.

Llamamos implicación clave a cualquier implicación cuya premisa tenga como cierre todo el conjunto de atributos, y decimos que es propia si todo atributo aparece, o bien en la premisa, o bien en la conclusión. Para poder introducir la noción de implicación quasi-clave como una generaliza-

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ción de implicación clave, nos basamos en las definiciones de los conjuntos Determinado, `Dte`, y Núcleo, `core`, que se introdujeron en [40]. Uno de los resultados que encontramos en ese artículo asegura que los atributos que no aparecen en ninguna conclusión, los del `core`, deben pertenecer a todas las claves. Es importante destacar que cualquier conjunto que contenga a `Dte` es cerrado respecto del conjunto de implicaciones. Una implicación quasi-clave es cualquier implicación cuya premisa cumpla que su cierre contiene a `Dte`. Igualmente, una implicación quasi-clave es propia si todo atributo de `Dte` aparece en la premisa o en la conclusión.

Se analizan las características de dichas implicaciones y su relación con el cálculo del cierres de conjuntos de atributos para discriminar los criterios que hacen realmente más eficiente dicho cálculo. Introducimos así un nuevo tipo de base directa: la base dicótoma directa.

En primer lugar, se define un sistema de implicaciones dicótomo como un par de conjuntos de implicaciones, $\langle \Sigma^*, \Sigma^k \rangle$, tal que las implicaciones del primer conjunto no son quasi-claves y las del segundo son todas quasi-claves propias. Además, los conjuntos de implicaciones deben ser compactos, es decir, no deben tener dos implicaciones diferentes con la misma premisa. Obsérvese que cualquier conjunto de implicaciones puede ser transformado en uno compacto sin más que aplicar (`Co-Eq`) a todos los posibles pares de implicaciones del conjunto. Es más, cualquier sistema de implicaciones puede ser transformado en uno dicótomo equivalente con un procedimiento cuadrático.

Definimos también un operador de iteración $\sigma_{\langle \Sigma^*, \Sigma^k \rangle}$ que, aplicado a un conjunto de atributos, realiza primero una iteración de cierre respecto de Σ^* y, al conjunto resultante, le aplica una iteración respecto de Σ^k. Diremos que un sistema de implicaciones dicótomo $\langle \Sigma^*, \Sigma^k \rangle$ es directo si $\sigma_{\langle \Sigma^*, \Sigma^k \rangle}$ es idempotente, es decir, el cierre de cualquier conjunto de atributos se obtiene con una sola aplicación del operador de iteración.

Como es usual, el término base se conserva para sistemas de implicaciones que satisfacen algún criterio de minimalidad. Siguiendo esta idea, llamamos base dicótoma directa, o DD-base, a aquellos sistemas de implicaciones dicótomos directos cuya primera componente sea simplificada por

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la derecha.

Estudiando la relación existente entre la DD-base y la base directa-optimal llegamos a un teorema de caracterización que asegura que la condición necesaria y suficiente para que un sistema de implicaciones dicótomo sea una DD-base es que su primera componente, Σ^*, esté formada por todas aquellas implicaciones de la base directa-optimal que no son quasi-clave. Este último resultado no es solo una cuestión teórica, sino que proporciona un método de complejidad cuadrática para convertir cualquier base directa-optimal en una DD-base.

Como hemos dicho, el problema que centra esta parte de la tesis consiste en transformar cualquier sistema de implicaciones en uno directo. Cabría considerar la opción de convertirlo en una directa-optimal para luego transformarlo en una DD-base. Sin embargo, los algoritmos que transforman cualquier sistema de implicaciones en una base directa tienen coste exponencial respecto el tamaño del sistema de implicaciones inicial. En este capítulo proponemos un algoritmo alternativo y probamos su mejor comportamiento. El método consiste en transformar la entrada en un sistema de implicaciones dicótomo $\langle\Sigma^*, \Sigma^k\rangle$. Como hemos dicho, esta parte del método tiene coste cuadrático. Después, calculamos la base directa-optimal equivalente a la primera componente Σ^*. De este modo, la reducción de tamaño de la entrada tiene una gran repercusión en el coste de la parte exponencial del método. Finalmente, al resultado se le vuelven a añadir, convenientemente tratadas, las implicaciones quasi-claves del sistema inicial. Específicamente, si Σ_{do} es la base directa optimal que cumple que $\Sigma_{do} \equiv \Sigma^*$ y $\langle\Sigma^*_{do}, \Sigma^k_{do}\rangle$ es la dicotomía de Σ_{do}, la DD-base que buscamos es $\langle\Sigma^*_{do}, \Sigma^k \cup \Sigma^k_{do}\rangle$.

Ésta es la principal ventaja de la DD-base respecto a las alternativas propuestas anteriormente: la base directa-optimal y la D-base. Gracias a este resultado, esta nueva base reduce el tamaño del conjunto de implicaciones que soporta el coste exponencial en el proceso de construcción de la base. Esta reducción se debe a la eliminación de las implicaciones quasi-claves de la tarea exponencial, hecho que tiene una gran repercusión en el coste de la transformación.

En la definición de DD-base se impone cierta restricción de minimalidad

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a la primera componente, pero no a la segunda. Nos encontramos, pues, que existen varias DD-bases equivalentes a un sistema implicacional dado, todas ellas compartiendo la misma primera componente. Es habitual que, cuando usamos el término base, lleve asociado, de por sí, la propiedad de unicidad. Sucede así con la base directa-optimal y con la *D*-base. Esto nos lleva a introducir el concepto de DD-base canónica.

Diremos que una DD-base es canónica si el tamaño de su segunda componente es el menor posible. Esto implica que no puede haber dos implicaciones en la segunda componente que cumplan que la premisa de una esté contenida en la de la otra. De esta forma, la DD-base canónica es la de menor cardinal y menor tamaño de entre todos los sistemas de implicaciones dicótomos equivalentes.

Finalmente, proporcionamos un teorema que caracteriza las DD-bases canónicas de la siguiente forma: consideremos un sistema de implicaciones Σ, la base directa optimal Σ_{do} equivalente a Σ y la base Duquenne-Guigues Σ_{DG} equivalente a Σ. Un par $\langle \Sigma^*, \Sigma^k \rangle$ es la DD-base canónica equivalente a Σ si y solo si Σ^* contiene exactamente las implicaciones no quasi-claves de Σ_{do} y Σ^k contiene exactamente las implicaciones quasi-claves de Σ_{DG}.

Como método para su cálculo, se propone un algoritmo de coste cuadrático que transforma cualquier DD-base en su DD-base canónica equivalente.

A modo de resumen, en este apartado de la tesis se han dado nuevas definiciones de bases directas y algoritmos para calcularlas más eficientes que todos los existentes para la obtención de la base directa-optimal. Probamos que los métodos desarrollados para la obtención de la DD-base y la DD-base canónica constituyen una alternativa real a las técnicas habituales aparecidas en la bibliografía. Para demostrar todas estas afirmaciones se ha llevado a cabo un estudio empírico en el que se observa cómo el tiempo en la computación de la DD-base canónica es menor que el que se requiere para calcular la base directa-optimal.

Marcamos como trabajo futuro, en primer lugar, combinar la aproximación dicótoma con la de *D*-base y, en segundo lugar, la generalización de este paradigma a las implicaciones difusas sobre contextos difusos, donde ya hemos empezado a dar algunos pasos. La siguiente parte de la tesis

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constituye también la base para la extensión de estos resultados al Análisis de Conceptos Triádicos.

Aportaciones al Análisis de Conceptos Triádicos

En la bibliografía aparecen varias extensiones del Análisis de Conceptos Formales y en ellas, como es de esperar, se introducen nuevas definiciones de la noción de implicación como forma de representar conocimiento extraido del contexto extendido. Su objetivo es recoger la mayor potencia expresiva de dichas generalizaciones.

Como segundo bloque de esta tesis doctoral abordamos el diseño de lógicas para razonar sobre implicaciones en una de las extensiones naturales del Análisis de Conceptos Formales que aparece en la bibliografía: el Análisis de Conceptos Triádicos.

Esta denominación surge por considerar, en lugar de un contexto formal clásico, una relación ternaria entre tres conjuntos: los objetos, los atributos y las condiciones. Relaciones similares entre tres o más conjuntos aparecen en aplicaciones reales como los *data cubes* en *data warehouses*, las tablas tridimensionales estadísticas, en los grafos que aparecen en las redes sociales, etc.

El Análisis de Conceptos Triádicos fue desarrollado inicialmente por Lehmann y Wille [33] y, en este marco, la noción de implicación es también una extensión natural del concepto de implicación en el marco clásico. Biedermann [13] propuso la primera definición de implicación para contextos triádicos. Más tarde Ganter y Obiedkov [25] dan nuevas definiciones de implicaciones. Hasta donde sabemos, no existe ningún sistema axiomático para la manipulación de implicaciones en el caso triádico y estudiar lógicas en este contexto surgió como un reto a resolver dada la gran importancia que tiene para el área.

La manera natural de representar un contexto triádico es en una tabla tridimensional [13] donde en cada dimensión se representan los objetos, los atributos y las condiciones. Es fácil darse cuenta de que esa tabla tridimensional estaría formada por todos los contextos diádicos asociados a

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cada condición, es decir, se puede visualizar colocando los contextos diádicos uno tras otro (véase la Figura 6.1) como capas. Esta representación se asemeja a los cubos OLAP en las bases de datos multidimensionales.

Desafortunadamente, este tipo de representación es difícil de manejar. Por este motivo, algunos autores utilizan alguna forma de *aplanamiento*. Una posibilidad es seleccionar uno de los conjuntos y relacionarlo con el producto cartesiano de los otros dos, dando lugar a un contexto diádico. En este caso, habría tres representaciones diferentes, dependiendo de qué conjunto seleccionemos (véase la Figura6.2).

Otra forma alternativa de aplanar el contexto sería la utilizada por Ganter et al. en [25]. Ésta consiste en una tabla en las que cada fila se refiere a cada objeto, cada columna a cada atributo y, en las celdas intersección de objetos y atributos, estarían todas las condiciones que se cumplen para estos objetos y atributos a la vez (véase la Figura 6.3). Esta última alternativa será la que se utilice en este trabajo.

Los conceptos se pueden entender como unidades de pensamiento homogéneas y cerradas [13, 33]. Con homogeneidad nos referimos al hecho de que todos los objetos comparten los mismos atributos bajo las mismas condiciones dentro del concepto. El cierre asegura que el concepto es maximal respecto a la homogeneidad. Por tanto, se pueden entender los conceptos triádicos como cubos maximales en la tabla tridimensional. Sin embargo, esta forma de entenderlos no es muy práctica a la hora de calcularlos. Como alternativa, se proponen los operadores de derivación en [33], una herramienta útil para la caracterización de conceptos triádicos haciendo uso de la propiedad de idempotencia de las conexiones de Galois que forman la composición de dichos operadores de derivación.

Los operadores de derivación nos permiten definir semánticamente el concepto de implicación. Como se ha comentado anteriormente y hasta donde sabemos, la primera definición de implicación triádica se debe a Biedermann [13]. Él considera que una implicación triádica es de la forma $(X \to Y)_{\mathcal{C}}$ y se satisface si "siempre que ocurra X bajo las condiciones de C también ocurre Y bajo las mismas condiciones".

Años más tarde, Ganter et al. [25] extendieron el trabajo de Biedermann

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y definieron tres tipos de implicaciones:

- Las implicaciones atributo×condición (AxCIs). Éstas son implicaciones diádicas en las que los elementos son pares cuyas componentes son atributos y condiciones respectivamente, es decir, tienen la forma $X \to Y$, donde X e Y son subconjuntos del producto cartesiano de atributos y condiciones.

- Las implicaciones de condiciones bajo atributos (ACIs). Son aquellas de la forma $X \xrightarrow{A} Y$, donde X e Y son conjuntos de condiciones y A es un conjunto de atributos. Esta implicación se satisface si "siempre que las condiciones de X se cumplen para cada uno de los atributos de A entonces las condiciones de Y también se cumplen para cada uno de ellos".

- Las implicaciones de atributos bajo condiciones (CAIs). Las implicaciones de atributos bajo condiciones son aquellas de la forma $X \xrightarrow{\mathcal{C}} Y$, donde X e Y son conjuntos de atributos y $\mathcal{C}$ es un conjunto de condiciones. Esta implicación se satisface si "siempre que los atributos de X se cumplen bajo cada una de las condiciones de $\mathcal{C}$ entonces los atributos de Y se cumplen también bajo cada una de ellas".

Como dijo Biedermann en [13], las implicaciones que él define tienen un matiz artificial aunque son adecuadas para la introducción de las implicaciones triádicas. Por otro lado, las AxCIs son exactamente las implicaciones diádicas ya conocidas. Así, centramos nuestro trabajo en las CAIs, aunque todos los resultados se pueden extrapolar a las ACIs ya que ambas tienen la misma estructura.

En este bloque de la tesis, proponemos el primer sistema axiomático para implicaciones de atributos bajo condiciones (CAIs). La lógica desarrollada permitirá diseñar nuevos métodos de razonamiento automático y su uso como herramienta automática en todas las aplicaciones citadas anteriormente. Aunque se ha desarrollado la lógica para implicaciones de atributos bajo condiciones (CAIs), sería directamente traducible a implicaciones de

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condiciones bajo atributos (ACIs) dada la similitud de ambos tipos de implicaciones triádicas, tal y como acabamos de comentar.

Para la definición de la nueva lógica se introducen el lenguaje, la semántica y el sistema axiomático. De hecho, en este bloque se introducen tres sistemas axiomáticos demostrando su corrección y completitud.

Las tres lógicas desarrolladas se denominan:

- CAIL (Conditional Attribute Implication Logic): Lógica de Implicaciones de Atributos bajo Condiciones.

- $\mathscr{B}$ Axiomatic System: Sistema Axiomático $\mathscr{B}$.

- CAISL (Simplification Logic for CAIs): Lógica de Simplificación para Implicaciones de Atributos bajo Condiciones.

Como resultado a destacar se demuestra en la tesis la equivalencia de los tres sistemas axiomáticos desarrollados. En el Capítulo 7 aparecen los detalles del lenguaje y la semántica de estas tres lógicas y sólo mostramos en este resumen las reglas del sistema axiomático de cada una de ellas.

La primera de ellas, CAIL, es una generalización de los ampliamente conocidos Axiomas de Armstrong para la extensión triádica; tiene dos axiomas (Inclusión y No-Restricción) y cuatro reglas de derivación: Aumento, Transitividad, Descomposición Condicional y Composición Condicional. Como es usual en lógica, se define la derivación sintáctica y se muestra cómo se pueden utilizar dichas reglas para razonar con implicaciones triádicas.

Al igual que en los Axiomas de Armstrong en el marco clásico de trabajo consideramos el estudio de reglas derivadas de las reglas primitivas antes citadas, definiendo las siguientes: Descomposición, Pseudotransitividad, Adición, y Acumulación. Estas reglas nos permitirán acortar las cadenas de derivación en los razonamientos.

Por último, extendemos la definición del cierre sintáctico clásico para esta lógica y proponemos un algoritmo para calcularlo. Una vez que estas reglas y el cierre están definidos, se prueba la corrección y completitud del sistema axiomático de CAIL.

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Como alternativa al sistema axiomático de CAIL presentamos el sistema axiomático $\mathscr{B}$ que se inspira en los B-Axiomas de Maier [36]. Tiene, como esquemas de axiomas los mismos que CAIL y, como reglas de inferencia, las de Acumulación, Descomposición, Descomposición condicional y Composición condicional. Demostramos que el sistema axiomático de CAIL y $\mathscr{B}$ son equivalentes, con lo que se prueba su corrección y completitud.

Por último, presentamos la Lógica de Simplificación para implicaciones de atributos bajo condiciones - CAISL. Esta lógica es una generalización al caso triádico de la Lógica de Simplificación del marco clásico. Tiene dos esquemas de axiomas y cuatro reglas de derivación sintáctica: Composición, Descomposición, Simplificación y Composición Condicional. Demostramos en el Capítulo 7 su corrección y completitud probando la equivalencia de CAIL y CAISL.

La característica más interesante de CAISL, frente a las otras dos lógicas, es que las reglas de derivación o inferencia de CAISL inducen reglas de equivalencia (de hecho reciben el mismo nombre) que pueden ser utilizadas directamente para eliminar información redundante de los sistemas de implicaciones, es decir, simplificarlos.

Destacamos el Teorema de la Deducción que, junto con las equivalencias comentadas, permite desarrollar de forma casi directa el primer método de razonamiento automático para CAISL, denominado CAISL-Prover, que decide si una implicación puede ser derivada o inferida del conjunto de implicaciones inicial.

Las aportaciones de esta parte abren la puerta al uso de la lógica en Análisis de Conceptos Triádicos y a una nueva línea de investigación muy prometedora para nuestro grupo de investigación. Hay un gran número de trabajos que pueden plantearse a partir de este punto con la generalización al marco triádico de todos los trabajos sobre eliminación de redundancia, cierres, generadores minimales, bases de implicaciones, etc. Muy especialmente estamos interesados en extender los resultados de la primera parte de la tesis al caso triádico.

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Introduction

The management of information is a wide research area where mathematical and computational methods can be combined to provide solid and efficient results in a proper way. In this work we focus on Formal Concept Analysis (FCA), a robust framework to store information, discover knowledge and manage it efficiently.

In Formal Concept Analysis, the data is represented using tables which relate objects with their attributes. These tables are called formal contexts and collect information which will be used to extract information. There are basically two representations for this knowledge: the concept lattice or the sets of implications. The handling of the latter is more suitable to automatization in terms of logic-based methods. We emphasize that one single concept lattice describes the same knowledge as several equivalent sets of implications. This situation leads to the search for a kind of canonical representation, characterizing those ones that provide a better and more efficient management.

Besides that, implications can be directly mined from a formal context. Frequently, a set of implications obtained in this way has a huge amount of redundancy. Achieving equivalent sets of implications with less redundancy, or without it, is one of the main challenges. On the other hand, it is not always desirable to eliminate all redundancy. It is noteworthy that, for some tasks, storing a certain type of redundancy can improve the efficiency of the algorithms.

There are several incentives to work on the definition of canonical forms for implicational sets, called bases. The importance of the notion of basis

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resides on the fact that the set of all valid implications in a context can be summarized by a smaller subset. In addition, the rest of implications can be syntactically derived from this small set by means of the symbolic computation, a task that cannot be directly tackled over the concept lattice. The logical approach is possible thanks to sound and complete axiomatic systems: Armstrong's Axioms [4] and Simplification Logic [20], among others.

Thus, the search for bases leads to the following questions:

- How to characterize a canonical representation for a given set of implications?
- Which properties does this canonical representation have to fulfill?
- How to design methods to automatically transform an arbitrary subset into its corresponding basis?

Concerning the first question, among the different notions of basis, the Duquenne-Guigues (or stem) basis [29] is the most cited because of its widely acceptation in the FCA area. It is characterized by having the minimum number of implications. Nevertheless, after this first keystone, several authors have worked in depth to provide new definitions of basis, incorporating very interesting properties. As Ryssel et al. stated in [55], "for many years, computing the stem basis has been the default method for extracting a small but complete set of implications from a formal context". Another alternative of Duquenne-Guigues basis was the so-called basis of proper premises [26]. This basis is also a sound and complete set of implications for a context, and improves the Duquenne-Guigues one by adding directness [26, Proposition 22], which provides a new direction to answer the second question: some properties to be fulfilled by the basis are not oriented to its configuration but to its further use. In this line, Rudolph [53] remarks: "one central task when dealing with concept lattices is to represent them in a succinct way while still allowing for their efficient computational usage".

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So, it was in the early days of Formal Concept Analysis when directness appeared although it did not receive that name yet. The benefit offered by the basis of proper premises is evident: to improve the generation of a closure operator by a specific implicational system. Nevertheless, the author presents the characterization of this basis, but not the way to compute it. Because of its benefits, this notion of basis is used in different areas under different names as shown in [11]:

- The *weak-implicational basis.* This basis is used by Rusch [54] to demonstrate that "methods of formal concept analysis can be successfully applied to the theory of knowledge spaces".
- The *left-minimal basis.* This kind of basis appears in the framework of relational databases [36] and later in the area of Horn Theories [31].
- The *dependence relation's basis.* It is based on the dependence relation defined in [39] in lattice and ordered set theory.
- The *canonical iteration free basis.* Wild considers in [56] that it is worth managing implicational bases to work with closure systems. So, he proposes this basis within the theory of relational data bases and Formal Concept Analysis.

In the framework of Formal Concept Analysis the same notion of basis appears again in 2004 under the name of *direct-optimal basis* [12]. In that work, besides giving the definition, the authors also present a method to compute it, opening the door to a discussion to answer the third question. This first method can be enhanced and, this is the reason why Bertet et al. kept on working on its improvement [9, 11] by using unitary implicational systems. The use of unitary implicational systems is due to the fact that their management seems to be easier, even though working with them causes a growth in the size of the implicational sets.

Indeed, the time complexity of the methods for computing the direct-optimal basis from an arbitrary implicational system is exponential in the worst case [10]. This issue motivates the idea of obtaining other methods

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that, without avoiding the intrinsic exponential complexity of the problem, provides a better performance than previous works.

Thus, we can affirm that there exist a wide range of definitions of basis in the literature, providing different properties and minimality measures. The previously mentioned works point out to study not only some minimality guideline regarding the whole set of implications, but also the form of its implications. One of the approaches following this line is the left-minimal direct basis [18]. This basis holds the minimum property of stem basis to have minimal information on the left-hand side as well. The authors propose it because "minimality in the number of implications is a criteria that may be enhanced". By reducing the left-hand side of implications, this approach comes near the optimality, which aims at the lowest number of attributes involved in the implicational set.

From a wider perspective, this work deals with direct bases and their minimality criteria. The main goal is to obtain a single approach covering the three issues: definitions, properties and computational methods. Our starting point was the work developed by Bertet et al. in [12], where the authors proposed an optimal and direct basis explaining that the computation of the closure is more efficient when the implicational system is optimal. This proposed basis is called direct-optimal basis. As the authors said, its importance resides on the fact that "the number of Σ-implications needed to compute the closure can be reduced to 1", where Σ is the implicational system. Obviously, this basis is not minimal in the sense of Duquenne-Guigues basis, but it is minimal among all the direct bases.

Later on, Adaricheva et al. [3] propose another kind of direct basis, the D-basis, being a subset of the direct-optimal basis. Authors consider that "while the D-basis is not direct in this meaning of the term, the closures can still be computed by a single iteration of the basis, provided the basis was put in a specific order prior to computation". The D-basis introduced in [3] exploits the concept of *ordered direct* computation of closures to find the way to shorten the direct-optimal basis without losing its good properties in applications. As the authors mentioned, the D-basis is usually a proper subset of the direct-optimal basis and it preserves the property

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of computation of closures in one iteration, assuming that implications are given in a prescribed order. More theoretical results were obtained in [1], where the authors studied the connection between the *D*-basis and the Duquenne-Guigues basis.

One key point in the definition of D-basis is the separated treatment of implications according to the cardinality of their premises. Thus, D-basis cannot be properly considered a set of implications but a pair of two sets of implications, the one collecting the implications with unitary premises and the other collecting the rest of implications. Thus, the order in Adaricheva's basis is not imposed to the implications but to these two subsets.

Moreover, as an illustration of the incipient development in this area, we remark that the definition of D-basis is not accompanied with a method to transform an arbitrary set of implications into its equivalent D-basis. Based on the fact that the *D*-basis is always a subset of any direct basis, in [3] the authors partially approached the problem of generating the D-basis. They designed a method where the input must be a *direct* basis. In this case, the algorithm of obtaining the *D*-basis is polynomial in the size of the input basis. Nevertheless, the generation of a general method to get the D-basis from an arbitrary implicational set has not been solved.

This open problem is a fundamental issue to promote its use in applications. As a general conclusion, there still are a lot to do in the field of direct bases, with the general objective to balance a compact representation and an efficient method of transformation.

Inspired by the schema of D-basis, we have also analyzed the features of implications and its relation to closure computation looking for a new criterion to discriminate implications belonging to a basis. If this criterium can be evaluated in an efficient way, we can provide a dichotomous representation of the basis which allows an ordered approach to compute closures even in a more efficient way. This new line has been fruitful providing a new kind of basis structured according to the support of the implications. Since our new basis is born with the idea to be used in applications, we also study the development of methods to get a basis from an arbitrary set of implications.

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As we mentioned, our intention is to cover all issues related with direct basis (from its definition to its management), taking into account different properties and optimality criteria. Moreover, we have gone one step beyond, studying implications in a richer framework. Thus, implications have revealed the great power of symbolic representation and automated management of information in classical FCA. In the literature, some authors have proposed several extensions of the FCA and the corresponding notion of implication to improve its expressive power and capture more and more information. Thus, some approaches have incorporated imprecise information [6, 7], negative information [38, 45], etc. In this work, we pay attention to the Triadic Concept Analysis (TCA).

Charles Sanders Peirce, in 1903, developed a system of categories related among them. His three universal categories of *Firstness, Secondness,* and *Thirdness* lead to the notion of triadic concept:

- The first category is a quality of feeling: the Idea of that which is such as it is regardless of anything else.
- The second category is a reaction as an element of the phenomenon: the Idea of that which is such as it is as being Second to some First, regardless of anything else.
- The third category is a representation as an element of the phenomenon: the Idea of that which is such as it is as being a Third, or Medium, between a Second and a First.

Relations among three or more different sets occur in many real-life situations such as data cubes in data warehouses, tridimensional statistical tables, multidimensional social networks and so on.

For these situations where the consideration of one dyadic context is not sufficient because of the complexity of the information, it is desirable to introduce a formalization of a network of contexts or extend formal concept analysis to the preprocessing of multi-relational datasets. The formalization is given by the notion of multicontext [59] and the extension by Relational Concept Analysis [30]. Pierce's pragmatic philosophy together with

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the existence of the situations suggesting an extension of formal concepts with a third component, have motivated the development of TCA. Initially investigated by Lehmann and Wille [33], TCA is a natural extension of Formal Concept Analysis. To the best of our knowledge, Biedermann [13] was the first researcher who investigated the implication issue in triadic contexts. Later on, Ganter and Obiedkov [25] explored other variants of triadic implications.

In the same way as the triadic context, the notion of triadic implication is also a natural extension of the implications in FCA. For this reason, our interest focuses on the study of these triadic implications to develop logics for their automated management. In addition to a formal definition of implications and its language, the introduction of a sound and complete inference system is needed to take the most of implications. Soundness ensures that implications derived by using the axiomatic system hold in the context and completeness guarantees that all implications which are satisfied can be derived from the implicational system. Nowadays, there is not an axiomatic system in triadic concept analysis.

In summary, the **general aims** of this work are the following:

(i) Studying in depth the problem of the direct bases to obtain efficient algorithms that calculate them. We will look for efficient algorithms for the bases already presented in the literature and we will propose a new definition of direct basis, with better properties to be calculated.

(ii) Taking the first steps to extend these results to TCA with the definition of a logic to reason with implications, in this framework, through sound and complete axiomatic systems.

Once we have introduced the framework of our work and our main goals, in the next section we will describe the main contributions of this PhD Thesis.

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Contributions and structure of the work

As previously stated, we study the direct bases as a whole, covering the definitions, the properties fulfilling by them and their automated generation. At the same time, these three dimensions have to be developed over the most outstanding notions appeared in the literature, mainly the direct-optimal basis and the D-basis.

Regarding the direct-optimal basis [12], the background covers the three aspects in a successful way. However, the method to generate them can be improved. Our aim is the development of methods taking advantage of the avoidance of generating extra implications, inherent to the methods based on unitary implicational systems, as well as the benefits offered by the non-unitary implicational systems about the size of the implicational system.

In Chapter 3 we focus on the development of new methods to compute the direct-optimal basis from an arbitrary non-unitary implicational system. This chapter contains the results of the collaboration with Karell Bertet that have been published in [48, 49]. Specifically, in this chapter we traverse the following path:

- A new inference rule is introduced: the *strong Simplification rule*. This rule allows to add new implications avoiding redundant attributes in both, the premises and the conclusions.

- A first algorithm with four separated steps is developed. This algorithm receives as input an arbitrary implicational system, reduces it, simplifies it, adds new rules by applying Strong Simplification rule and, lastly, simplifies it again obtaining the direct-optimal basis.

- A second algorithm where separated steps have been integrated to improve the performance during its execution. In this one, we properly combine the previous step maintaining the implicational system simplified all the time.

- A comparison between the methods is done in order to illustrate the

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benefits of our new methods in practice.

The work with the D-basis [3] has covered an interesting set of results making progress in the issue of the direct bases. The results presented in Chapter 4 are the result of a collaboration with Kira Adaricheva and have been published in [46, 47]. In summary, we develop the following stages:

- A deep study about the relationships between covers, minimal covers, generators and minimal generators. This will be strong point to develop, for first time, a method to compute the *D*-basis from an arbitrary implicational system.
- A first approach to tackle the problem of computing the *D*-basis has been developed. In this approach, the *D*-basis is computed by means of an algorithm generating all the minimal generators and, then, it selects the ones being minimal covers.
- We also develop a second approach that integrates the stages of the previous one in an smart way. This method computes the *D*-basis in a more efficient way that the first one.
- As in the case of the direct-optimal basis, we have shown a comparison between both methods to state their benefits in practice.

Besides studying the notions already appeared in the literature, we provide a new notion of direct basis, covering the three main related issues. The main results presented in Chapter 5, which have been published in [50], are the following:

- The definition of dichotomous implicational system, which will be the pillar of the new basis and is based on the notion of quasi-key.
- The introduction of the notion of direct dichotomous basis (DD-basis) as an alternative to the others, as well as the illustration of its advantages.

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- The development of a well-founded method to compute a dichotomous direct basis.

- The definition of the canonical DD-basis, showing its uniqueness and optimality and providing a quadratic method to compute this basis.

Moreover, we also present contributions related with the extended framework of Triadic Concept Analysis. In particular, we propose a logic, named CAIL, to manage implications in the framework of TCA. Notice that this is the first logic in the literature to manage implications in triadic formal concept analysis. We define its language, semantics and inference system. The language is the set of formulas representing conditional attribute implications defined in [25]. We also prove the soundness and completeness of CAIL.

Moreover, we provide an equivalent system to CAIL to manage this kind of implications, called $\mathscr{B}$, which is inspired in the B-axioms of Maier, and we prove the equivalence of both axiomatic systems.

Finally, in order to get an automated reasoning method, we introduce an alternative equivalent logic, called CAISL. It belongs to the family of Simplification Logics and is inspired in the idea of redundancy removing. To show the advantages of using this logic as methods to manipulate implications, we describe, at the end of the chapter, a method to compute the closure of a set of attributes under a given set of conditions. The proof of the correctness of this algorithm is also provided.

Our contributions to this area, collected in Chapter 7, are the culmination of a collaboration with Rokia Missaoui and have been published in [51, 52]. In summary, in this chapter we enclose the following contributions:

- The extension of some classical notions given in Formal Concept Analysis and the relations among these notions.

- The development of two equivalent sound and complete logics, CAIL and CAISL, which open the door to the automatic management of

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TCA implications. As evidence to confirm it, we have developed an extended version of closure algorithm.

- We introduce two ways to verify the validity of a formula: a first one based on the notion of closure and a second one in a logical style.

Moreover, in Chapter 1, we have collected a set of needed preliminaries related with closure operators, FCA, logic and implicational systems. We have organized the results in two parts: the first one is related with the notions of direct bases, and the second one is related with implications in TCA. At the beginning of each part, we have included a chapter (see Chapters 2 and 6) with the specific preliminaries needed in this part. We end this work with a chapter devoted to conclusions and future works.

For a better understanding and illustration of all the contributions of this PhD Thesis, we have depicted all of them in Figure 1.

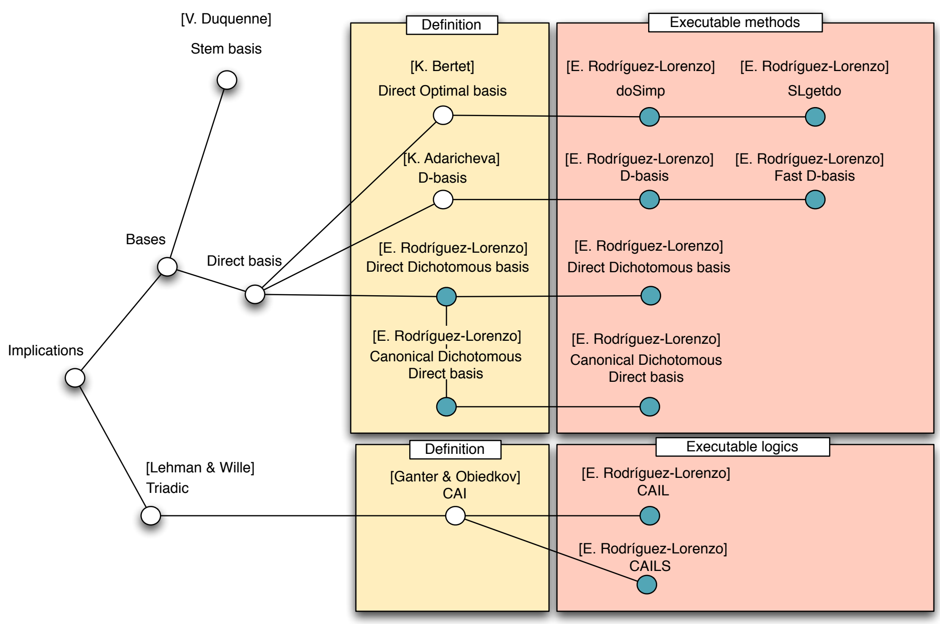

Figure 1: Schema of the state of the art of direct bases and contributions.

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Chapter 1

Preliminaries

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In order to make this work self-contained, in this chapter we introduce the main notions and results about Galois connections and closure operators, which will be used throughout the manuscript. Although the literature concerning these issues is very wide, the most relevant books about this subjects are [28], [14] and [21].

We will also include the main ideas, definitions and properties of Formal Concept Analysis, the main research area of this work. The standard reference book about these topics is [26].

1.1 Closure Operators and Galois Connections

Along this dissertation, we will work on the Boolean algebra $(2^M, \subseteq)$, where M is a finite set. In the following we will refer to it as 2^M.

In this section, the notions of closure operators, closure systems, Moore families and Galois connections are introduced. Moreover, the relationships among them are highlighted.

Definition 1.1.1. *Given a non-empty set M,* a closure operator *on M is a mapping $\varphi : 2^M \to 2^M$ that satisfies the following properties:*

(i) *Extensiveness: $X \subseteq \varphi(X)$ for all $X \in 2^M$.*

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(ii) *Isotonicity:* $X \subseteq Y$ *implies* $\varphi(X) \subseteq \varphi(Y)$ *for all* $X, Y \in 2^M$.

(iii) *Idempotence:* $\varphi(\varphi(X)) = \varphi(X)$ *for all* $X \in 2^M$.

The pair $\langle M, \varphi\rangle$ *is called a* closure system *and a set* $A \subseteq M$ *is called a* closed set *for* φ *if it is a fix-point for* φ*, i.e.* $\varphi(A) = A$.

The idempotence of closure operators leads to the fact that, for all closure system $\langle M, \varphi\rangle$ and all $A \subseteq M$, the set $\varphi(A)$ is closed for φ.

Definition 1.1.2. *Given a non-empty set* M*, a set* $S \subseteq 2^M$ *is said to be a* Moore family[1] *on* 2^M *if it satisfies the following conditions:*

(i) $M \in S$

(ii) *For all* $X, Y \in S$*, we have that* $X \cap Y \in S$.

Thus, a Moore family is closed by intersection and contains the universal set M. It is well known that there is a one-to-one correspondence between closure operators and Moore families.

Proposition 1.1.3. *Let* M *be a non-empty set.*

(i) *Given a closure operator* $\varphi\colon 2^M \to 2^M$*, the family of closed sets of* φ*,*

$$S_\varphi = \{A \subseteq M \text{ such that } \varphi(A) = A\},$$

is a Moore family.

(ii) *Given a Moore family* S *in* 2^M*, the mapping* $\varphi_S\colon 2^M \to 2^M$*, where*

$$\varphi_S(A) = \bigcap\{X \in S \mid A \subseteq X\},$$

is a closure operator.

Moreover, $\varphi = \varphi_{S_\varphi}$ *and* $S = S_{\varphi_S}$.

[1] Moore families are also called closure systems in other research areas.

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The set S_φ is called *Moore family associated with* φ and, φ_S is the *closure operator associated with* S.

There are other notions that are equivalent to closure operators, e.g. Galois connections [43] and implicational systems [26]. In the following, the notion of Galois connection will be introduced and the notion of implicational system will be introduced in Section 1.3.

Definition 1.1.4. *Let G and M be two non-empty sets. A pair $\langle f, g\rangle$ of two mappings $f\colon 2^G \to 2^M$ and $g\colon 2^M \to 2^G$ is said to be a Galois connection between G and M if, for every $A \subseteq G$ and $B \subseteq M$:*

$$A \subseteq g(B) \quad \text{if and only if} \quad B \subseteq f(A)$$

In order to introduce another characterization of Galois connections in the theorem below, we need to recall the following property of functions.

Definition 1.1.5. *Let G and M be non-empty sets. A mapping $f\colon 2^G \to 2^M$ is said to be* antitone *if $A_1 \subseteq A_2$ implies $f(A_2) \subseteq f(A_1)$ for all $A_1, A_2 \subseteq G$.*

Theorem 1.1.6. *Let G and M be two non-empty sets and $f : 2^G \to 2^M$ and $g : 2^M \to 2^G$ two mappings. The following items are equivalent:*

(i) *$\langle f, g\rangle$ is a Galois connection.*

(ii) *f and g are antitone and $f \circ g$ and $g \circ f$ are extensive.*

(iii) *f is antitone and $g(B) = \bigcup\{A \subseteq G \mid B \subseteq f(A)\}$*

(iv) *g is antitone and $f(A) = \bigcup\{B \subseteq M \mid A \subseteq g(B)\}$*

Consequently, in a Galois connection, each mapping directly determines the other one. On the other hand, from the second characterization above it is easy to notice that $g \circ f \circ g = g$ and $f \circ g \circ f = f$. This property induces the following theorem.

Theorem 1.1.7. *If $\langle f, g\rangle$ is a Galois connection between G and M, then $g \circ f$ is a closure operator on G and $f \circ g$ is a closure operator on M.*

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Theorem 1.1.8. *Let $\langle f, g\rangle$ be a Galois connection between G and M. The following conditions hold:*

(i) $f(\bigcup_{i\in I} X_i) = \bigcap_{i\in I} f(X_i)$ *for all family of sets* $\{X_i \mid i \in I\} \subseteq 2^G$.

(ii) $g(\bigcup_{i\in I} Y_i) = \bigcap_{i\in I} g(Y_i)$ *for all family of sets* $\{Y_i \mid i \in I\} \subseteq 2^M$.

Closure operators, Moore families, Galois connections and implicational systems are essentially the same thing. All of these notions and properties are foundations over which Formal Concept Analysis (FCA) is consolidated. In fact, restructuring this theory is the original motivation of developing FCA, which attempts to use the mathematical theory of concepts and concept hierarchies in order to support the rational communication of humans and clarify the connections to Philosophical Logics of human thought [57, 60].

The following section is devoted to the introduction of Formal Concept Analysis and pays particular attention to the notion of implicational system, which plays an important role in this work.

1.2 Formal Concept Analysis

Formal Concept Analysis (FCA) was introduced by Rudolph Wille [57] in the 80's. It is a conceptual environment to structure, analyze, minimize, visualize and reveal hidden knowledge from the data using techniques of data mining about a binary relationship between the elements of two sets: objects and attributes. In the last decades, the FCA techniques have succeeded in diverse research areas such as data mining, social networks analysis, marketing, medical diagnosis, etc.

The starting point is the relationship between a set of objects and its properties.

Definition 1.2.1. *A formal context is a triplet $\mathbb{K} = \langle G, M, I\rangle$ which consists of two non-empty sets G and M and a binary relation I between G and M. The elements of G are called the objects and the elements of M*

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are called the attributes of the context. For $g \in G$ and $m \in M$, we write $\langle g, m\rangle \in I$ or gIm if the object g has the attribute m.

The following example will be often cited in the rest of the work. It is taken from [26] where the authors consider data about 130 countries. For the sake of readability, we only consider a part of this dataset taking into consideration 8 countries.

Example 1.2.2. *The data consists of knowing whether these countries belong or not to* Gr77 *(Group of 77),* NA *(Non-alligned),* LLDC *(Least Developed Countries),* MASC *(Most Seriously Affected Countries),* OPEC *(Organization of Petrol Exporting Countries) and* ACP *(African, Caribbean and Pacific Countries). The triplet $\mathbb{K}_0 = \langle G, M, I\rangle$ is a formal context where the set of objects and the set of attributes are, respectively, the following:*

$$\begin{aligned} G &= \{\text{Afghanistan, Algeria, Benin, Botswana, Cameroon, Gabon, Haiti, Kiribati}\} \\ M &= \{\text{Gr77, NA, LLDC, MASC, OPEC, ACP}\}. \end{aligned}$$

Table 1.1 depicts the binary relation I of the formal context $\mathbb{K}_0$.

I	Gr77	NA	LLDC	MASC	OPEC	ACP
Afghanistan	×	×	×	×		
Algeria	×	×			×	
Benin	×	×	×	×		×
Botswana	×	×	×			×
Cameroon	×	×		×		×
Gabon	×	×			×	×
Haiti	×		×	×		×
Kiribati			×			×

Table 1.1: Membership of countries in supranational groups.

Thus, $\langle$Cameroon, Gr77$\rangle \in I$ means "the object Cameroon has the attribute Group of 77" or "Cameroon belongs to Group of 77". However, $\langle$Cameroon, OPEC$\rangle \notin I$, i.e., "Cameroon does not belong to OPEC".

Knowledge extracted from the formal context needs to be represented. There are two main ways of representation: either by the so-called concept

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lattice or by means of a set of attribute implications. Indeed, both ways are equivalent: they represent the same knowledge and any of them can be built from the other without the need of the formal context.

1.2.1 Formal Concepts

Formal concepts are pairs of subsets of objects and attributes that are somehow related in context. Before introducing them, we start defining the derivation or concept-forming operators.

Definition 1.2.3. *Given a formal context* $\mathbb{K} = \langle G, M, I\rangle$*, two mappings* $(-)^{\uparrow} : 2^G \to 2^M$ *and* $(-)^{\downarrow} : 2^M \to 2^G$*, named* concept-forming operators*, are defined as follows:*

$$A^{\uparrow} = \{m \in M \mid \langle g, m\rangle \in I \text{ for all } g \in A\}$$

$$B^{\downarrow} = \{g \in G \mid \langle g, m\rangle \in I \text{ for all } m \in B\}$$

for any $A \subseteq G$ *and* $B \subseteq M$*.*

The set $A^{\uparrow}$ is the set of all the common attributes shared by all the objects of A, and the set $B^{\downarrow}$ is the set of objects sharing all the attributes of B.

Example 1.2.4. *From the context* $\mathbb{K}_0$ *depicted in Table 1.1, one can obtain:*

$$\begin{array}{rcl@{\qquad}rcl}
\{\text{Algeria}\}^{\uparrow} & = & \{\text{Gr77, NA, OPEC}\} & G^{\uparrow} & = & \varnothing \\
\{\text{Afghanistan}, \text{Algeria}\}^{\uparrow} & = & \{\text{Gr77, NA}\} & \varnothing^{\uparrow} & = & M \\
\{\text{OPEC}\}^{\downarrow} & = & \{\text{Algeria}, \text{Gabon}\} & M^{\downarrow} & = & \varnothing \\
\{\text{OPEC}, \text{ACP}\}^{\downarrow} & = & \{\text{Gabon}\} & \varnothing^{\downarrow} & = & G
\end{array}$$

One of the fundamental results of FCA is the following theorem that relates it to the notions described in the previous section.

Theorem 1.2.5. *Let* $\mathbb{K} = \langle G, M, I\rangle$ *be a formal context. The pair of the concept-forming operators* $\langle (-)^{\uparrow}, (-)^{\downarrow}\rangle$ *is a Galois connection between* G *and* M*.*

Therefore, from the above theorem and Theorems 1.1.6 and 1.1.7, the following corollary is directly obtained.

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Corollary 1.2.6. *In any context $\langle G, M, I\rangle$, the following properties hold:*

(i) $A_1 \subseteq A_2$ *implies* $A_2^{\uparrow} \subseteq A_1^{\uparrow}$ *for all* $A_1, A_2 \subseteq G$.

(ii) $B_1 \subseteq B_2$ *implies* $B_2^{\downarrow} \subseteq B_1^{\downarrow}$ *for all* $B_1, B_2 \subseteq M$.

(iii) $A \subseteq A^{\uparrow\downarrow}$ *and* $B \subseteq B^{\downarrow\uparrow}$ *for all* $A \subseteq G$ *and* $B \subseteq M$.

(iv) $A^{\uparrow} = A^{\uparrow\downarrow\uparrow}$ *and* $B^{\downarrow} = B^{\downarrow\uparrow\downarrow}$ *for all* $A \subseteq G$ *and* $B \subseteq M$.

In addition, both compositions of the two concept-forming operators,

$$(-)^{\uparrow\downarrow}\colon 2^G \to 2^G \quad \text{and} \quad (-)^{\downarrow\uparrow}\colon 2^M \to 2^M,$$

are closure operators.

Moreover, the closed sets of these two mappings, that is, the fixpoints of the closure operators, define the so-called *formal concepts*. As we shall see, formal concept is a key point in FCA which formally describes an *idea* of the model and it allows us to characterize a set of objects by means of the attributes they share and vice versa.

Definition 1.2.7. *Let $\mathbb{K} = \langle G, M, I\rangle$ be a formal context and $A \subseteq G, B \subseteq M$. The pair $\langle A, B\rangle$ is called a* formal concept *if $A^{\uparrow} = B$ and $B^{\downarrow} = A$. The set of objects, A, is said to be the* extent *and the set of attributes, B, the* intent *of the concept $\langle A, B\rangle$.*

In other words, $\langle A, B\rangle$ is a formal concept if A contains all the objects sharing the attributes in B and, analogously, B contains all the attributes sharing the objects in A. The set of all concepts of the context $\mathbb{K}$, denoted by $\mathcal{B}(\mathbb{K})$, constitutes a lattice that is called *concept lattice*, where the order relation is defined as follows:

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \text{ if and only if } A_1 \subseteq A_2 \text{ or, equivalently, } B_2 \subseteq B_1.$$

Example 1.2.8. *There are 26 formal concepts associated with the formal context $\mathbb{K}_0$ introduced in Table 1.1. One concept is, for instance, the pair* $\langle\{\text{Benin, Botswana, Haiti, Kiribati}\}, \{\text{LLDC, ACP}\}\rangle$*, which describes the notion*

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of the least developed countries in a certain region providing its properties and characterizing its countries. Note that the sets {LLDC, ACP} *and* {Benin, Botswana, Haiti, Kiribati} *are closed sets:*

$$\begin{aligned}\{\text{Afghanistan, Benin, Botswana, Haiti}\}^{\uparrow\downarrow} &= \{\text{Afghanistan, Benin, Botswana, Haiti}\}\\ \{\text{LLDC, ACP}\}^{\downarrow\uparrow} &= \{\text{LLCD, ACP}\}\end{aligned}$$

There is an alternative way to define the formal concepts. They can be defined as maximal rectangles in the cross-table of the formal context.

Definition 1.2.9. *A rectangle in* $\mathbb{K} = \langle G, M, I\rangle$ *is a pair* $\langle A, B\rangle$ *such that* $A \times B \subseteq I$. *For rectangles* $\langle A_1, B_1\rangle$ *and* $\langle A_2, B_2\rangle$, *put* $\langle A_1, B_1\rangle \sqsubseteq \langle A_2, B_2\rangle$ *if and only if* $A_1 \subseteq A_2$ *and* $B_1 \subseteq B_2$.

Theorem 1.2.10. *A pair* $\langle A, B\rangle$ *is a formal concept of* $\mathbb{K} = \langle G, M, I\rangle$ *if and only if* $\langle A, B\rangle$ *is a maximal rectangle in* $\mathbb{K}$ *w.r.t.* $\sqsubseteq$.

Example 1.2.11. *In the formal context* $\mathbb{K}_0$ *from Table 1.1, the pair*

$$\langle\{\text{Benin, Botswana}\}, \{\text{Gr77, NA, ACP}\}\rangle$$

is a rectangle in $\mathbb{K}_0$, *but it is not a formal concept because it is not maximal with respect to* $\sqsubseteq$. *The pair*

$$\langle\{\text{Benin, Botswana, Cameroon, Gabon}\}, \{\text{Gr77, NA, ACP}\}\rangle$$

is a maximal rectangle (i.e. formal concept) containing it (see Table 1.2).

As it has been said, FCA is a different view of the notions introduced in the previous section. Thus, on the one hand, Galois connections and Moore families (which can be seen as *concept lattices*) allow a graphical representation. On the other hand, closure operators lead to the notion of implicational systems, which facilitate the reasoning by means of logic. The following section is devoted to this notion.

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I	Gr77	NA	LLDC	MASC	OPEC	ACP
Afghanistan	×	×	×	×		
Algeria	×	×			×	
Benin	×	×	×	×		×
Botswana	×	×	×			×
Cameroon	×	×		×		×
Gabon	×	×			×	×
Haiti	×		×	×		×
Kiribati			×			×

Table 1.2: The concept $\langle\{\text{Benin, Botswana, Cameroon, Gabon}\}, \{\text{Gr77, NA, ACP}\}\rangle$ as a maximal rectangle.

1.3 Implicational Systems

Although later we will give a formal definition of *attribute implication*, we anticipate that it is a pair of attribute sets (*premise* and *conclusion*) and that it is satisfied in a context if every object that has all the attributes of the premise, has also the attributes of the conclusion. The advantage of using attribute implications is two-fold: on the one hand, it allows for a logic-based management of knowledge (with the potential benefit of automated reasoning techniques); on the other hand, attribute implications are the key to more efficient ways of knowledge representation than concept lattices.

We introduce the results related to attribute implications considering the usual components of a logic: language, semantics, axiomatic system or syntactic inference, and its automated reasoning method.

Language

Definition 1.3.1. *Given a finite non-empty set of attributes M, the language is defined as $\mathcal{L}_M = \{A \to B \mid A, B \subseteq M\}$.*

Formulas $A \to B \in \mathcal{L}_M$ are called attribute implications*, or simply,* implications*, and the sets A and B are called the* premise *and the* conclusion *of the implication, respectively.*

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Sets $\Sigma \subseteq \mathcal{L}_M$ *are called* implicational systems *on* M*.*

As usual, in order to distinguish between language and metalanguage, inside implications, the union is denoted by juxtaposition (e.g. XY means $X \cup Y$) and the set difference by the symbol "-" (e.g. X-Y denotes $X \smallsetminus Y$). Moreover, for the sake of readability, inside of the formulae, we omit the brackets and commas (e.g. abc denotes the set $\{a, b, c\}$).

Example 1.3.2. *Given* $M = \{a, b, c, d\}$*, let us consider the sets* $X = \{a, b\}$*,* $Y = \{b, c\}$ *and* $Z = \{c, d\}$*.*

- $X \to Y$ *is written as* $ab \to bc$ *instead of* $\{a, b\} \to \{b, c\}$*.*
- $XY \to Z\text{-}Y$ *denotes* $abc \to d$*.*

Finally, we will use the term *unitary implication* to refer to implications in which the conclusion is a singleton, and the term *unitary implicational system* to refer to sets of unitary implications.

Semantics

Once the language has been defined, we introduce the semantics in order to give a meaning to the formulae of the language.

Definition 1.3.3. *Let* $\mathbb{K} = \langle G, M, I \rangle$ *be a formal context and* $A \to B \in \mathcal{L}_M$*. The context* $\mathbb{K}$ *is said to be a* model *for* $A \to B$ *if* $B \subseteq A^{\downarrow\uparrow}$*. It is denoted by* $\mathbb{K} \models A \to B$*.*

The definition above, and all the results proposed in this section, can be introduced in terms of arbitrary closure operators instead of contexts.

Note that, by Corollary 1.2.6, one has that

$$\mathbb{K} \models A \to B \quad \text{if and only if} \quad A^{\downarrow} \subseteq B^{\downarrow}$$

As usual, the notion of models can be extended to implicational systems: given $\Sigma \subseteq \mathcal{L}_M$, the expression $\mathbb{K} \models \Sigma$ means that $\mathbb{K} \models A \to B$ for all $A \to B \in \Sigma$.

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Example 1.3.4. *Considering the formal context* $\mathbb{K}_0$ *from Table 1.1, one can notice that* $\mathbb{K}_0 \models \text{MASC} \to \text{Gr77}$ *whereas* $\mathbb{K}_0 \not\models \text{MASC} \to \text{LLDC}$.

Definition 1.3.5. *Let* M *be a set of attributes,* $A \to B \in \mathcal{L}_M$ *and* $\Sigma \subseteq \mathcal{L}_M$. *The implication* $A \to B$ *is said to be* semantically derived *from* Σ, *denoted by* $\Sigma \models A \to B$, *if* $\mathbb{K} \models \Sigma$ *implies* $\mathbb{K} \models A \to B$ *for every formal context* $\mathbb{K}$.

On the other hand, two implicational systems $\Sigma_1, \Sigma_2 \subseteq \mathcal{L}_M$ *are* semantically equivalent, *denoted by* $\Sigma_1 \equiv \Sigma_2$, *if the following equivalence holds*

$$\mathbb{K} \models \Sigma_1 \text{ if and only if } \mathbb{K} \models \Sigma_2$$

for every formal context $\mathbb{K}$.

In summary, $\Sigma \models A \to B$ if every model for Σ is a model for $A \to B$ and $\Sigma_1 \equiv \Sigma_2$ if their models are the same, it means, both sets represent the same knowledge.

Example 1.3.6. *Let* $M = \{a, b, c\}$ *be a set of attributes. Observe that* $\{a \to b, a \to c\} \equiv \{a \to bc\}$ *because, for every formal context* $\mathbb{K} = \langle G, M, I\rangle$, $\{b, c\}^{\downarrow} = \{b\}^{\downarrow} \cap \{c\}^{\downarrow}$ *holds. Therefore:*

$$\{a\}^{\downarrow} \subseteq \{b\}^{\downarrow} \text{ and } \{a\}^{\downarrow} \subseteq \{c\}^{\downarrow} \text{ if and only if } \{a\}^{\downarrow} \subseteq \{b, c\}^{\downarrow}$$

This example can be easily extended to ensure that any implicational system is equivalent to a unitary implicational system.

Proposition 1.3.7. *Let* M *be a set of attributes and* $\Sigma \subseteq \mathcal{L}_M$. *The following equivalence holds:*

$$\Sigma \equiv \bigcup_{A \to B \in \Sigma} \{A \to b \mid b \in B\}$$

Armstrong's Axioms

The axiomatic system known as "Armstrong's Axioms" encloses one axiom scheme and two inference rules. Let A, B, C be subsets of M,

Inclusion	`[Inc]`		$\vdash_{\mathcal{A}} AB \to A$
Augmentation	`[Aug]`	$A \to B$	$\vdash_{\mathcal{A}} AC \to BC$
Transitivity	`[Trans]`	$A \to B, B \to C$	$\vdash_{\mathcal{A}} A \to C$

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From the above axiomatic system, the notion of *deduction* ($\vdash$) is defined as usual:

Definition 1.3.8. *Let M be a set of attributes, $\sigma \in \mathcal{L}_M$ and $\Sigma \subseteq \mathcal{L}_M$. We say that σ is* syntactically derived*, or deduced, from Σ, denoted $\Sigma \vdash_{\mathcal{A}} \sigma$, if there exists a sequence $\sigma_1, \ldots, \sigma_n$ such that $\sigma_n = \sigma$ and, for all $1 \leq i \leq n$, we have that $\sigma_i \in \Sigma$, σ_i is an axiom or is obtained by applying the inference rules from Armstrong's Axioms to some formulas in $\{\, \sigma_j \mid 1 \leq j < i \,\}$.*

In this case, the sequence $\{\sigma_i \mid 1 \leq i \leq n\}$ is said to be a proof *for $\Sigma \vdash_{\mathcal{A}} \sigma$.*

This axiomatic system has its origin in [4], where it was used to study the properties of functional dependencies in Codd's relational model [15]. Later on, other works have proposed other equivalent axiomatic systems [5, 22, 31, 44]. The one presented here is the original one.

On the other hand, the axiom scheme is commonly known as "reflexivity". Here, we call it "inclusion" because we reserve the name "reflexivity" for another scheme that fits better with the mathematical idea of reflexivity.

Example 1.3.9. *Let $M = \{a, b, c, d, e\}$ be a set of attributes. The following chain proves that $\{ab \to cd, bc \to e\} \vdash_{\mathcal{A}} ab \to e$:*

$$
\begin{array}{lr}
\sigma_1 = ab \to cd & \textit{by hypothesis} \\
\sigma_2 = bc \to e & \textit{by hypothesis} \\
\sigma_3 = ab \to bcd & \textit{by applying } \texttt{[Augm]} \textit{ to } \sigma_1 \textit{ with } b \\
\sigma_4 = bcd \to de & \textit{by applying } \texttt{[Augm]} \textit{ to } \sigma_2 \textit{ with } d \\
\sigma_5 = ab \to de & \textit{by applying } \texttt{[Trans]} \textit{ to } \sigma_3 \textit{ and } \sigma_4 \\
\sigma_6 = de \to e & \textit{by } \texttt{[Inc]} \\
\sigma_7 = ab \to e & \textit{by applying } \texttt{[Trans]} \textit{ to } \sigma_5 \textit{ and } \sigma_6
\end{array}
$$

The following theorem ensures that the syntactic and semantic derivation coincide. It means that every rule one can deduce with this axiomatic system can be semantically derived (the axiomatic system is sound) and vice versa (the axiomatic system is complete).

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Theorem 1.3.10. *Let M be a finite non-empty set of attributes, $\Sigma \subseteq \mathcal{L}_M$ and $A \to B \in \mathcal{L}_M$. Then $\Sigma \models A \to B$ if and only if $\Sigma \vdash_{\mathcal{A}} A \to B$.*

Simplification Logic

Although Armstrong's axiomatic system is cited in lots of works, it is used in practice just for the theoretical study of implications instead of the development of applications and algorithms. The inherent problem is the fact that the proofs are difficult to automatize. Due to that, in [41] it was presented the Simplification Logic, more appropriated for the automatization of reasoning with implications [42].

Definition 1.3.11. *Simplification Logic considers reflexivity axiom*

`[Ref]` $\vdash_{\mathcal{S}} A \to A$;

and the following inference rules (called fragmentation, composition and simplification, respectively):

`[Frag]` $A \to BC \vdash_{\mathcal{S}} A \to B$;

`[Comp]` $A \to B,\ C \to D \vdash_{\mathcal{S}} AC \to BD$;

`[Simp]` *If* $A \subseteq C$ *and* $A \cap B = \varnothing$, *then* $A \to B,\ C \to D \vdash_{\mathcal{S}} C\text{-}B \to D\text{-}B$.

Theorem 1.3.12 (Cordero et al. [20])**.** *Let M be a non-empty set of attributes. For every $\Sigma \in \mathcal{L}_M$ and every $A \to B \in \mathcal{L}_M, \Sigma \vdash_{\mathcal{S}} A \to B$ if and only if $\Sigma \vdash_{\mathcal{A}} A \to B$.*

Corollary 1.3.13. *Simplification logic is sound and complete.*

Due to the theorem above, from now on, we will omit the subscript and merely write $\vdash$.

The main advantage of Simplification Logic is that inference rules can be considered as equivalence rules. Thus, they have been used as the core of automated methods for removing redundancies, obtaining minimal keys, or computing closures (see [42] for further details and proofs).

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Theorem 1.3.14 (Mora et al. [42]). *The following equivalences hold:*

$$\{A \to B\} \equiv \{A \to B\text{-}A\} \qquad (\texttt{Fr-Eq})$$

$$\{A \to B, A \to C\} \equiv \{A \to BC\} \qquad (\texttt{Co-Eq})$$

If $A \cap B = \varnothing$ and $A \subseteq C$ then

$$\{A \to B, C \to D\} \equiv \{A \to B, C\text{-}B \to D\text{-}B\} \qquad (\texttt{Si-Eq})$$

If $A \cap B = \varnothing$ and $A \subseteq C \cup D$ then

$$\{A \to B, C \to D\} \equiv \{A \to B, C \to D\text{-}B\} \qquad (\texttt{rSi-Eq})$$

These equivalences are called also Fragmentation, Composition and Simplification equivalences respectively. Notice that these equivalences (read from left to right) remove redundant information. Simplification Logic was conceived as a simplification framework, hence its name.

Automated reasoning

For each implicational system, the axiomatic system defines a closure operator in 2^M, which we call *syntactic closure*.

Definition 1.3.15. *Let M be a set of attributes and $\Sigma \subseteq \mathcal{L}_M$. We say that a subset $X \subseteq M$ is closed with respect to Σ if, for each $A \to B \in \Sigma$, we have that $A \subseteq X$ implies $B \subseteq X$.*

The set of all the closed sets with respect to Σ forms a Moore family (see Definition 1.1.2), and, by Proposition 1.1.3, a closure operator in 2^M.

Definition 1.3.16. *Let M be a set of attributes and $\Sigma \subseteq \mathcal{L}_M$. For each $X \subseteq M$, the closure of X with respect to Σ is defined as:*

$$X^+_\Sigma = \bigcap \{C \subseteq M \mid X \subseteq C \text{ and } C \text{ is closed with respect to } \Sigma\}$$

Thus, the closure of X with respect to Σ is the largest subset of M such that $\Sigma \vdash X \to X^+_\Sigma$. When no confusion arises, we will denote the closure of a subset of attributes as X^+ instead of X^+_Σ .

The following theorem is not only essential to compute closures but also to introduce an automated reasoning method.

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Theorem 1.3.17. *Let M be a set of attributes, $\Sigma \subseteq \mathcal{L}_M$ and $A, B \subseteq M$. The following conditions are equivalent:*

$$\Sigma \vdash A \to B \quad \textit{iff} \quad B \subseteq A^+_\Sigma \quad \textit{iff} \quad \{\varnothing \to A\} \cup \Sigma \vdash \varnothing \to B.$$

As a direct consequence of the previous theorem, we have that:

$$X^+_\Sigma = \max\{Y \subseteq M \mid \Sigma \vdash X \to Y\}$$

Based on Theorem 1.3.17, in [42] the method `Cls` is introduced. This method receives as input a set $X \subseteq M$ and a set of implications Σ and renders the pair (X^+_Σ, Σ') where Σ' is the simplified set of implications with respect to $\varnothing \to X^+_\Sigma$ and, thus, $\Sigma' \subseteq \mathcal{L}_{M \smallsetminus X^+_\Sigma}$ (i.e. Σ' is the contraction of Σ to $2^{M \smallsetminus X^+_\Sigma}$). Therefore, this allows to determine whether an implication $X \to Y$ can be deduced from Σ. Moreover, Σ' contains relevant information about the implicational system and the closed set, as it will be shown in the next section.

The `Cls` algorithm computes the closure of a set of attributes X with respect to Σ as follows:

(i) $\varnothing \to X$ is added to Σ. This implication is used as a seed which guides the reasoning to render its closure A^+_Σ. For this reason, it is called the *guide*.

(ii) While it is possible, the procedure compares the guide with the rest of the implications and applies the corresponding equivalence (Theorem 1.3.14) among the following:

Eq. I: If $B \subseteq A$ then $\{\varnothing \to A, B \to C\} \equiv \{\varnothing \to AC\}$

Eq. II: If $C \subseteq A$ then $\{\varnothing \to A, B \to C\} \equiv \{\varnothing \to A\}$

Eq. III: Otherwise, $\{\varnothing \to A, B \to C\} \equiv \{\varnothing \to A, B\text{-}A \to C\text{-}A\}$

(iii) As soon as a fix point is achieved, if the guide is $\varnothing \to A$, then we have that $X^+_\Sigma = A$.

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`Cls` returns the closure of X (the conclusion of the guide) and the set of simplified implications.

If we want to determine whether $\Sigma \vdash X \to Y$ or not, the answer will be affirmative if and only if $Y \subseteq A$. Observe that the procedure can be interrupted before getting the fix point: when $Y \subseteq A$.

Example 1.3.18. *Let Σ be $\{ab \to c, ac \to df, bcd \to ef, f \to c\}$. To compute $\{a, f\}^+_\Sigma$, we consider the implication $\varnothing \to af$ as a guide and apply the equivalences previously described.*

Guide	Σ			
$\varnothing \to af$	$ab \to c$	$ac \to df$	$bcd \to ef$	$f \to c$
$\varnothing \to af$	$\cancel{a}b \to c$	$\cancel{a}c \to d\cancel{f}$	$bcd \to e\cancel{f}$	$\cancel{f} \to c$
$\varnothing \to acf$	$b \to \cancel{c}$	$\cancel{c} \to d$	$b\cancel{c}d \to e$	
$\varnothing \to acdf$			$b\cancel{d} \to e$	
$\varnothing \to acdf$			$b \to e$	

Thus, once the procedure has been applied, it renders $\texttt{Cls}(af, \Sigma) = (acdf, \{b \to e\})$ *where* $\{a, f\}^+_\Sigma = \{a, c, d, f\}$ *and* $\Sigma' = \{b \to e\}$ *has important information that, for instance, will be used for computing minimal generators in the following section.*

The previous method does not only compute the closure of a set of attributes in the guide. It renders also a set of simplified implications that contains relevant information.

1.3.1 Closed Sets and Minimal Generators

In this section we summarize how the inference system of Simplification Logic $\mathbf{SL}_{FD}$ [17, 20] and the output of the algorithm `Cls` can be used to enumerate all closed sets and all minimal generators.

For $X, Y \subseteq M$ satisfying that $X = Y^+_\Sigma$, it is usual to say that Y is a *generator* of the closed set X. Notice that any subset of X containing Y is also a generator of X. As we work with finite sets of attributes, the set of generators of a closed set can be characterized by its minimal ones.

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Definition 1.3.19. *Let M be a finite set of attributes and Σ an implicational system on M. $X \subseteq M$ is said to be a* minimal generator*, or briefly* mingen *if, for all proper subsets $Y \subsetneq X$, one has $Y_\Sigma^+ \subsetneq X_\Sigma^+$.*

A method to compute all minimal generators was presented in [19]. The algorithm called `MinGen` is based on the closure algorithm `Cls` mentioned above and it takes the advantages of the additional information that it provides. That is, `Cls` is used not only to compute closed sets from generators but also a smaller implicational set which guides us in the search of new subsets to be considered as minimal generators. The following definition specifies what the algorithm is assumed to compute.

Definition 1.3.20. *Let M be a set of attributes. For each implicational system Σ over M, the set of* labeled closed sets *(LCS) with respect to Σ is*

$$\{\langle C, mg(C)\rangle \mid C \subseteq M, C_\Sigma^+ = C\}$$

where $mg(C) = \{D \subseteq M \mid D \text{ is a mingen and } D_\Sigma^+ = C\}$.

`MinGen` [19] is outlined in Algorithm 1.1. It is based on the recursive computation of closed sets and generators. Since different parts of the method compute partial information about the set of LCS, some specific operators that allow us to work with this kind of sets are needed. Specifically, given two sets of LCSs Φ and Ψ, the join of Φ and Ψ, $\Phi \sqcup \Psi$, is defined as follows: $\langle A_1, B_1\rangle \in \Phi \sqcup \Psi$ if and only if one of the three following conditions holds:

(i) $\langle A_1, B_1\rangle \in \Phi$ and any $\langle A_2, B_2\rangle \in \Psi$ satisfies $A_1 \neq A_2$.

(ii) $\langle A_1, B_1\rangle \in \Psi$ and any $\langle A_2, B_2\rangle \in \Phi$ satisfies $A_1 \neq A_2$.

(iii) There exist $\langle A_1, B_2\rangle \in \Phi$ and $\langle A_1, B_3\rangle \in \Psi$ such that B_1 is the set of minimal elements of $B_2 \cup B_3$.

Example 1.3.21. *Consider $\Phi = \{\langle abcd, \{ac, ad\}\rangle, \langle ab, \{a\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\}$ and $\Psi = \{\langle abcd, \{c\}\rangle, \langle b, \{b\}\rangle, \{\langle \varnothing, \{\varnothing\}\rangle\}$. Then*

$$\Phi \sqcup \Psi = \{\langle abcd, \{c, ad\}\rangle, \langle ab, \{a\}\rangle, \langle b, \{b\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\}.$$

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Two other operators are required as well. The first one is the trivial operator. It has as input a pair (M, Σ) where M is the set of attributes and Σ is an implicational system, and its output is the following LCS:

$$trv(M, \Sigma) = \{\langle X, \{X\}\rangle \mid X \subseteq M \text{ with } A \not\subseteq X \text{ for all } A \to B \in \Sigma\}$$

Example 1.3.22. *Let $M = \{a, b, c, d\}$ and $\Sigma = \{a \to b, c \to bd, bd \to ac\}$. Then, the trivial operator renders the following:*

$$trv(M, \Sigma) = \{\langle \varnothing, \{\varnothing\}\rangle, \langle b, \{b\}\rangle, \langle d, \{d\}\rangle\}$$

Moreover, we need the operator `Add` given by:

$$\texttt{Add}(\langle C, \{D\}\rangle, \Phi) := \Big\{\big\langle A \cup C, \{X \cup D \mid X \in B\}\big\rangle \mid \langle A, B\rangle \in \Phi\Big\}$$

Example 1.3.23. *Considering the set of attributes from Example 1.3.22, the* LCS $\Phi = \{\langle cd, \{c, d\}\rangle, \langle \varnothing, \varnothing\rangle\}$ *and the pair $\langle ab, \{a\}\rangle$, the* `Add` *operator renders:*

$$\texttt{Add}(\langle ab, \{a\}\rangle, \Phi) = \{\langle abcd, \{ac, ad\}\rangle, \langle ab, \{a\}\rangle\}$$

Now, we have the necessary operators to describe the procedure.

Algorithm 1.1: MinGen

Data: M, Σ
Result: Φ
begin
  Let $\Phi := trv(M, \Sigma)$
  if $\Sigma \neq \varnothing$ **then**
    Let Mnl $:= \{A \subseteq M \mid$ $A \to B \in \Sigma$ for some $B \subseteq M$ and $C \subseteq A$ implies $C = A$ for all $C \to D \in \Sigma\}$
    foreach $A \in$ Mnl **do**
      Let $(A', \Sigma') = \texttt{Cls}(A, \Sigma)$
      Let $\Phi := \Phi \sqcup \texttt{Add}\big(\langle A', \{A\}\rangle, \texttt{MinGen}(M \smallsetminus A', \Sigma')\big)$
  return Φ

The input of `MinGen` is a subset of M and a set of implications Σ and the output is the set of closed sets together with all the minimal generators that generate them, i.e. $\{\langle C, mg(C)\rangle \colon C \text{ is a closed set}\}$ where $mg(C)$ is the set of minimal generators D satisfying $D^+_\Sigma = C$.

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Example 1.3.24. *Let* $M = \{a, b, c, d\}$ *and* $\Sigma = \{a \to b, c \to bd, bd \to ac\}$ *as in Example 1.3.22. Then the* MinGen *algorithm returns the following set:*

$$\texttt{MinGen}(M, \Sigma) = \{\langle abcd, \{c, ad, bd\}\rangle, \langle ab, \{a\}\rangle, \langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\}$$

The trace of the algorithm is the following:

Step 1. $\Phi = trv(M, \Sigma) = \{\langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\}$

$\mathrm{Mnl} = \{a, c, bd\}$

Step 2. $\texttt{Cls}(a, \Sigma) = (ab, \{c \to d, d \to c\})$

$\texttt{MinGen}(\{c, d\}, \{c \to d, d \to c\})$*:*

Step 2.1. $\Phi = trv(\{c, d\}, \{c \to d, d \to c\}) = \{\langle \varnothing, \{\varnothing\}\rangle\}$

$\mathrm{Mnl} = \{c, d\}$

Step 2.2. $\texttt{Cls}(c, \{c \to d, d \to c\}) = (cd, \varnothing)$

$$\begin{aligned} \Phi &= \{\langle \varnothing, \{\varnothing\}\rangle\} \sqcup \texttt{Add}(\langle cd, \{c\}\rangle, \texttt{MinGen}(\varnothing, \varnothing)) \\ &= \{\langle cd, \{c\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \end{aligned}$$

Step 2.3. $\texttt{Cls}(d, \{c \to d, d \to c\}) = (cd, \varnothing)$

$$\begin{aligned} \Phi &= \{\langle cd, \{c\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \sqcup \texttt{Add}(\langle cd, \{d\}\rangle, \texttt{MinGen}(\varnothing, \varnothing)\}) \\ &= \{\langle cd, \{c, d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \end{aligned}$$

Thus, $\texttt{MinGen}(\{c, d\}, \{c \to d, d \to c\}) = \{\langle cd, \{c, d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\}$

$$\begin{aligned} \Phi &= \{\langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \sqcup \\ &\quad \sqcup \texttt{Add}(\langle ab, \{a\}\rangle, \{\langle cd, \{c, d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\}) \\ &= \{\langle abcd, \{ac, ad\}\rangle, \langle ab, \{a\}\rangle, \langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \end{aligned}$$

Step 3. $\texttt{Cls}(c, \Sigma) = (abcd, \varnothing)$

$$\begin{aligned} \Phi &= \{\langle abcd, \{ac, ad\}\rangle, \langle ab, \{a\}\rangle, \langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \sqcup \\ &\quad \sqcup \texttt{Add}(\langle abcd, \{c\}\rangle, \texttt{MinGen}(\varnothing, \varnothing)) \\ &= \{\langle abcd, \{c, ad\}\rangle, \langle ab, \{a\}\rangle, \langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \end{aligned}$$

Step 4. $\texttt{Cls}(bd, \Sigma) = (abcd, \varnothing)$

$$\begin{aligned} \Phi &= \{\langle abcd, \{c, ad\}\rangle, \langle ab, \{a\}\rangle, \langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \sqcup \\ &\quad \sqcup \texttt{Add}(\langle abcd, \{bd\}\rangle, \texttt{MinGen}(\varnothing, \varnothing)) \\ &= \{\langle abcd, \{c, ad, bd\}\rangle, \langle ab, \{a\}\rangle, \langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\} \end{aligned}$$

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Therefore, $\texttt{MinGen}(M, \Sigma)$ *returns*

$$\{\langle abcd, \{c, ad, bd\}\rangle, \langle ab, \{a\}\rangle, \langle b, \{b\}\rangle, \langle d, \{d\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\}$$

Notice that, from the point of view of the implications, some trivial information (e.g. $\langle b, \{b\}\rangle$) is included in the output of this algorithm. In [19], a version of $\texttt{MinGen}$, named $\texttt{MinGen}_0$, is also presented. The difference between this algorithm and the previous one is that here only non-trivial generators are calculated. To do it, we only need to change the function *trv* by the following one:

$$trv_0(M, \Sigma) = \{\langle \varnothing, \{\varnothing\}\rangle\} \text{ for all } \Sigma \subseteq \mathcal{L}_M.$$

Example 1.3.25. *For the same input as in Example 1.3.24, we have:*

$$\texttt{MinGen}_0(M, \Sigma) = \{\langle abcd, \{c, ad, bd\}\rangle, \langle ab, \{a\}\rangle, \langle \varnothing, \{\varnothing\}\rangle\}$$

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Part I

Direct Bases

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Chapter 2

Bases and directness

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Thanks to the existence of a logic to reason with implications, given a context, we do not need to work with the huge set of implications that hold in it. This set is called the *full implicational system*. Instead, we can select a smaller set from which we can derive all the others. The following definition formalizes this idea.

Definition 2.0.1. *Let $\mathbb{K}$ be a context. An implicational system Σ is complete for $\mathbb{K}$ if, for all implication $A \to B$, the following equivalence holds:*

$$\mathbb{K} \models A \to B \quad \textit{if and only if} \quad \Sigma \vdash A \to B.$$

Clearly, any complete implicational system for the same context is contained in the equivalent full implicational system and all the complete implicational systems for a context are equivalent:

$$\text{If } \Sigma_1 \text{ and } \Sigma_2 \text{ are complete for } \mathbb{K} \text{ then } \Sigma_1 \equiv \Sigma_2.$$

In addition, when an implicational system Σ is complete for a context $\mathbb{K}$, it is well-known that the (syntactic) closed sets of attributes with respect to Σ are in bijection with the concepts of $\mathbb{K}$. Specifically, for any attribute set X one has

$$X^{\downarrow\uparrow} = X^+_\Sigma$$

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Given the importance of the notion of complete implicational system, a relevant issue in FCA is to find, among the different equivalent implicational systems, those that have better properties. Before beginning with the contributions, we summarize the state of the art in this topic.

In Section 2.1 below, we introduce the notion of basis as a complete implicational system that satisfies some minimality criteria. Then, in Section 2.2, we summarize the works of Bertet et al. [9, 11, 12] to obtain the direct-optimal basis associated with a formal context whereas Section 2.3 is devoted to present a survey of the work of Adaricheva et al. [3] to obtain the associated D-basis. We summarize these works in great detail since most of the thesis is oriented to propose more efficient alternatives to the algorithms proposed there.

2.1 Bases

Among the different complete implicational systems for a context, a relevant issue is to characterize those ones without redundant information or, satisfying some minimality criteria. The term basis is commonly used to refer to implicational systems where some minimality criteria holds. Among those minimality criteria, the most used are the following:

Definition 2.1.1. *An implicational system Σ is said to be:*

i) minimal *when $\Sigma \smallsetminus \{A \to B\} \not\equiv \Sigma$ for all $A \to B \in \Sigma$;*

ii) minimum *when $|\Sigma| \leq |\Sigma'|$ for all $\Sigma' \equiv \Sigma$;*

iii) optimal *when $\|\Sigma\| \leq \|\Sigma'\|$ for all $\Sigma' \equiv \Sigma$;*

where $|\Sigma|$ is the cardinality of Σ and $\|\Sigma\|$ is its size, i.e.

$$\|\Sigma\| = \sum_{A \to B \in \Sigma} (|A| + |B|).$$

Obviously, any minimum implicational system is minimal. Moreover, Maier showed in [35] that optimal sets of implications are also minimum. So, among these properties, minimality is the weakest one.

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Figure 2.1 shows three equivalent implicational systems. Notice that the optimal one is also minimum (it has three implications), and all of them are also minimal because if one implication is removed, the new implicational system is not equivalent.

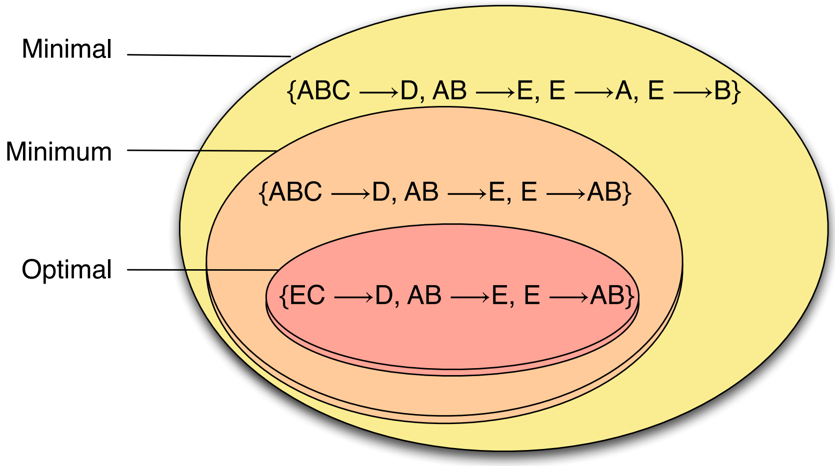

Figure 2.1: Relationships among minimal, minimum and optimal implicational systems.

The term basis is used to refer to a minimal set of implications. Therefore, a minimum basis is then a basis of least cardinality. The following definition formally introduces the notion of basis:

Definition 2.1.2. *An implicational system* Σ *is said to be a* basis *for a formal context* $\mathbb{K}$ *if it is a minimal complete implicational system for* $\mathbb{K}$*.*

Therefore, if Σ is a basis for $\mathbb{K}$, the following conditions are fulfilled:

Correctness: $\mathbb{K} \models \Sigma$.

Completeness: $\mathbb{K} \models A \to B$ if and only if $\Sigma \vdash A \to B$.

Minimality: $\Sigma \smallsetminus \{A \to B\}$ is not complete for $\mathbb{K}$ for all $A \to B \in \Sigma$.

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Since all the complete implicational systems for a context are equivalent, the problem of computing a basis can be stated in two versions, depending on what is the starting point. On the one hand, we could need to compute a basis that is complete for a given context. On the other hand, given an implicational system, we could be interested in finding an equivalent basis. In this thesis, we focus on the latter. Therefore, the following results will be stated with this aim in mind.

Deepening about bases, the most used minimum basis is the so-called Duquenne-Guigues basis or stem basis [29]. This one is defined using the notion of a pseudo-closed set (also named pseudo-intent).

Definition 2.1.3. *Let Σ be an implicational system on M. A set $A \subseteq M$ is* quasi-closed *if, for any $B \subseteq A$, one has $B^+_\Sigma \subseteq A$ or $B^+_\Sigma = A^+_\Sigma$.*

In addition, a quasi-closed set A is pseudo-closed *if $A^+_\Sigma \neq A$ and for any quasi-closed set $B \subsetneq A$ one has $B^+_\Sigma \subsetneq A$.*

An arbitrary set A is quasi-closed if either it is closed or a new Moore family is obtained by adding it to the set of all closed sets. Moreover, every pseudo-closed set is a minimal quasi-closed set in its closure class, defined to be the class containing all quasi-closed with the same closure. Notice that the uniqueness of minimal quasi-closed elements is not ensured: in some closure classes there can be several minimal quasi-closed elements.

The following example illustrates the notions of quasi-closed and pseudo-closed sets.

Example 2.1.4. *Consider the following implicational system:*

$$\begin{aligned}\Sigma = \{\, & \text{OPEC} \to \text{Gr77 NA},\\ & \text{MASC} \to \text{Gr77},\\ & \text{NA} \to \text{Gr77},\\ & \text{MASC OPEC} \to \text{LLDC ACP},\\ & \text{LLDC OPEC} \to \text{MASC ACP}\}\end{aligned}$$

The set Σ is complete for the formal context presented in Table 1.1.

- *The set $A = \{\text{Gr77, OPEC}\}$ is quasi-closed because $\{\text{Gr77}\}^+_\Sigma \subseteq A$ and $\{\text{OPEC}\}^+_\Sigma = A^+_\Sigma$. However, it is not pseudo-closed because $A^+_\Sigma \neq A$ but for the quasi-closed $\{\text{OPEC}\}$, we have that $\{\text{OPEC}\}^+_\Sigma = A^+_\Sigma$.*

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- *The set $B = \{$Gr77, NA, MASC, OPEC$\}$ is pseudo-closed. One can check that it is quasi-closed because, for all subset of B, its closure is a strict subset of B or M (notice that $B^+_\Sigma = M$).*

  Moreover, $B^+_\Sigma \neq B$ and, for any quasi-closed subset, its closure is strictly contained in B. Those quasi-closed sets of B appear in grey colour (see Figure 2.2).

$\{\mathrm{Gr77}\}^+_\Sigma = \{\mathrm{Gr77}\}$	$\{\mathrm{MASC}\}^+_\Sigma = \{\mathrm{MASC}, \mathrm{Gr77}\}$
$\{\mathrm{NA}\}^+_\Sigma = \{\mathrm{NA}, \mathrm{Gr77}\}$	$\{\mathrm{OPEC}\}^+_\Sigma = \{\mathrm{OPEC}, \mathrm{Gr77}, \mathrm{NA}\}$
$\{\mathrm{MASC}, \mathrm{OPEC}\}^+_\Sigma = \mathrm{M}$	$\{\mathrm{NA}, \mathrm{MASC}\}^+_\Sigma = \{\mathrm{MASC}, \mathrm{Gr77}, \mathrm{NA}\}$
$\{\mathrm{Gr77}, \mathrm{NA}\}^+_\Sigma = \{\mathrm{Gr77}, \mathrm{NA}\}$	$\{\mathrm{NA}, \mathrm{OPEC}\}^+_\Sigma = \{\mathrm{OPEC}, \mathrm{Gr77}, \mathrm{NA}\}$
$\{\mathrm{Gr77}, \mathrm{MASC}\}^+_\Sigma = \{\mathrm{Gr77}, \mathrm{MASC}\}$	$\{\mathrm{Gr77}, \mathrm{OPEC}\}^+_\Sigma = \{\mathrm{OPEC}, \mathrm{Gr77}, \mathrm{NA}\}$
$\{\mathrm{Gr77}, \mathrm{MASC}, \mathrm{OPEC}\}^+ = \mathrm{M}$	$\{\mathrm{Gr77}, \mathrm{NA}, \mathrm{OPEC}\}^+_\Sigma = \{\mathrm{OPEC}, \mathrm{Gr77}, \mathrm{NA}\}$
$\{\mathrm{NA}, \mathrm{MASC}, \mathrm{OPEC}\}^+ = \mathrm{M}$	$\{\mathrm{Gr77}, \mathrm{NA}, \mathrm{MASC}\}^+_\Sigma = \{\mathrm{MASC}, \mathrm{Gr77}, \mathrm{NA}\}$

Figure 2.2: Closures of the subsets of $B = \{$Gr77, NA, MASC, OPEC$\}$ with respect to the implicational system from Example 2.1.4.

The taxonomy of closed, quasi-closed and pseudo-closed and their relationships has been thoroughly studied. For a more in-depth study of these notions the reader is referred to [32]. The following proposition is the key for the important role of pseudo-closed sets in the implicational systems.

Proposition 2.1.5 (Ganter et al. [26])**.** *Let Σ be an implicational system. For every pseudo-closed set C with respect to Σ there is $A \to B \in \Sigma$ such that $A \subseteq C$ and $A^+_\Sigma = C^+_\Sigma$.*

Based on the definition of pseudo-closed sets and its properties, the Duquenne-Guigues basis was introduced in [29].

Definition 2.1.6. *Given an implicational system Σ, the* Duquenne-Guigues basis *for Σ is defined as*

$$\Sigma_{DG} = \{A \to A^+_\Sigma\text{-}A \mid A \textit{ is pseudo-closed w.r.t. } \Sigma\}$$

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Example 2.1.7. *The Duquenne-Guigues basis associated with the implicational system introduced in Example 2.1.4, and therefore complete for the formal context presented in Table 1.1, is the following [26]:*

$$\begin{aligned}\Sigma \quad = \{ \quad & \text{OPEC} \to \text{Gr77 NA},\\ & \text{MASC} \to \text{Gr77},\\ & \text{NA} \to \text{Gr77},\\ & \text{Gr77 NA MASC OPEC} \to \text{LLDC ACP},\\ & \text{Gr77 NA LLDC OPEC} \to \text{MASC ACP}\}\end{aligned}$$

Given an implicational system, the existence and the unicity of an equivalent Duquenne-Guigues basis are ensured. Moreover, this basis has minimum cardinality.

Theorem 2.1.8 (Ganter et al. [26])**.** *Let Σ be an implicational system. If Σ_{DG} is its Duquenne-Guigues basis, then $\Sigma \equiv \Sigma_{DG}$.*

In addition, $|\Sigma_{DG}| \leq |\Sigma'|$ for any implicational system Σ' such that $\Sigma' \equiv \Sigma_{DG}$.

Ganter introduced in [24] a well-known algorithm for computing the unique Duquenne-Guigues basis that is complete for a given formal context. This algorithm, known as Next-Closure, is easily adapted for considering an implicational system as the input.

Observe that, since Duquenne-Guigues basis is minimum, it is also minimal, i.e. no implication can be removed because they are *non-redundant*. Some authors call *superfluous* to implications that can be removed and preserve the term *redundant* for a more general notion. There is an extensive literature about these ideas [18–20, 36]. We do not deepen in this line because our objective is different.

As in many other fields of research, some redundant information may be useful for improving the efficiency of methods. The key is to find an equilibrium between the amount of redundant information we handle and the efficiency of the algorithms we use frequently. In the case of implicational systems, there are lots of applications that require the systematic calculation of attribute set closures. For example, there are algorithms for solving problems of exponential nature in which the atomic operation is

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the computation of closures. From now on, we will focus on this idea in this part of the thesis.

2.2 Direct-Optimal bases

In order to formally introduce the notion of directness, in [11], the authors define the operator $\pi_\Sigma : 2^M \to 2^M$, for each implicational system $\Sigma \subseteq \mathcal{L}_M$, as follows:

$$\pi_\Sigma(X) = X \cup \{b \in B \mid A \to B \in \Sigma \text{ for some } A \subseteq X\}. \tag{2.1}$$

The function π_Σ is isotone and extensive, and, therefore, for all $X \in 2^M$, the chain X, $\pi_\Sigma(X)$, $\pi_\Sigma^2(X)$, $\pi_\Sigma^3(X)$, ... reaches a fixpoint. The closure of the set X with respect to Σ, i.e. X_Σ^+, coincides with this fixpoint. For some particular implicational systems, π_Σ is idempotent, i.e. $\pi_\Sigma \circ \pi_\Sigma = \pi_\Sigma$. In these cases, the fixpoint is reached in the first iteration and Σ is called a *direct implicational system* [11].

Definition 2.2.1. *A set $\Sigma \subseteq \mathcal{L}_M$ is said to be* a direct implicational system *if $X_\Sigma^+ = \pi_\Sigma(X)$ for all $X \subseteq M$.*

In [11] Bertet et al. give a characterization of directness that we extend for non-unitary implicational systems. Its proof is straightforward from the definition of closure operator.

Lemma 2.2.2. *A set $\Sigma \subseteq \mathcal{L}_M$ is a direct implicational system if and only if $\pi_\Sigma^2(X) \subseteq \pi_\Sigma(X)$ for all $X \subseteq M$.*

Example 2.2.3. *Consider the following equivalent implicational systems.*

$$\begin{aligned}\Sigma_{DG} = \{\, & \text{OPEC} \to \text{Gr77 NA},\\ & \text{MASC} \to \text{Gr77},\\ & \text{NA} \to \text{Gr77},\\ & \text{Gr77 NA MASC OPEC} \to \text{LLDC ACP},\\ & \text{Gr77 NA LLDC OPEC} \to \text{MASC ACP}\}\end{aligned} \qquad \begin{aligned}\Sigma_{d} = \{\, & \text{OPEC} \to \text{Gr77 NA},\\ & \text{MASC} \to \text{Gr77},\\ & \text{NA} \to \text{Gr77},\\ & \text{MASC OPEC} \to \text{LLDC ACP},\\ & \text{LLDC OPEC} \to \text{MASC ACP}\}\end{aligned}$$

Σ_{DG} is the Duquenne-Guigues basis of Example 2.1.7 and Σ_d is an equivalent direct basis. Notice that Σ_{DG} is not a direct basis:

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$$\pi^2_{\Sigma_{DG}}(\{\text{MASC, OPEC}\}) = \pi_{\Sigma_{DG}}(\{\text{MASC, OPEC, Gr77, NA}\}) = \\ = \{\text{MASC, OPEC, Gr77, NA, ACP, LLDC}\}$$

However, with π_{Σ_d}, the closure is reached in just one iteration:

$$\pi_{\Sigma_d}(\{\text{MASC, OPEC}\}) = \{\text{MASC, OPEC, Gr77, NA, ACP, LLDC}\}$$

It is worth noting that, if an implicational system Σ is direct for a set of attributes M, the complexity of the closure is $O(|\Sigma|)$ instead of $O(|\Sigma|\,|M|)$, which coincides with the complexity in the worst case of classical closure algorithms.

When a huge amount of closures of attribute sets has to be computed, it would be interesting those direct bases which have the smallest size. So, we do not just seek the minimum number of implications but the minimum number of attributes too.

Definition 2.2.4. *A direct implicational system Σ is said to be a* direct-optimal basis *if, for any direct implicational system Σ', one has $\Sigma' \equiv \Sigma$ implies $\|\Sigma\| \leq \|\Sigma'\|$.*

The following theorem ensures the existence and unicity of a direct-optimal basis that is equivalent to any given implicational system.

Theorem 2.2.5 (Bertet et al. [11]). *For any implicational system Σ, there exists a unique direct-optimal basis Σ_{do} such that $\Sigma \equiv \Sigma_{do}$.*

In addition, this unique direct-optimal basis is characterized in the following theorem.

Theorem 2.2.6 (Bertet et al. [12]). *A direct implicational system Σ is direct-optimal if and only if the following properties hold:*

Extensiveness: *If $A \to B \in \Sigma$ then $A \cap B = \varnothing$.*

Isotony: *If $A \to B, C \to D \in \Sigma$ and $C \subsetneq A$ then $B \cap D = \varnothing$.*

Premise: *if $A \to B, A \to B' \in \Sigma$ then $B = B'$.*

Non-empty conclusion: *if $A \to B \in \Sigma$ then $B \neq \varnothing$.*

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Direct-optimal basis combines the directness and optimality properties. On the one hand, directness ensures that the computation of the closure may be done in just one traversal of the implication set. On the other hand, due to its minimal size provided by the optimality, the number of visited implications is reduced to the minimum. Due to these features, it is desirable to design methods to transform an arbitrary set of implications into its equivalent direct-optimal basis. Thus, the problem of building a direct-optimal basis is one of the outstanding problems in FCA.

Given an arbitrary implicational system Σ, the way to proceed to obtain the direct-optimal basis proposed in [12] is the following: first, a direct implicational system Σ_d is built by adding new implications to Σ. Then the direct-optimal basis Σ_{do} is obtained by removing from Σ_d those implications without modifying the directness property up to get a fixpoint.

In the first step, the following rule is exhaustively applied:

$$\textit{Overlap}\ \texttt{[Ovl]}\colon\quad \frac{A \to B, C \to D}{A(C\text{-}B) \to D},\ \text{when } B \cap C \neq \varnothing$$

Then, the direct implicational system generated from an implicational system Σ is defined as the smallest implicational system that contains Σ and is closed for `[Ovl]`.

Definition 2.2.7. *The direct implicational system Σ_d generated from Σ is defined as the smallest implicational system such that:*

(i) $\Sigma \subseteq \Sigma_d$ *and*

(ii) *If* $A \to B, C \to D \in \Sigma_d$ *and* $B \cap C \neq \varnothing$ *then* $A(C\text{-}B) \to D \in \Sigma_d$.

The function which computes this direct implicational system will be called `Bertet-Nebut-Direct` in this work.

The second step of the transformation, the shrinking stage, is inspired by Theorem 2.2.6 and it is mainly based on the following inference rule:

$$\textit{Optimization}\ \texttt{[Opt]}\colon\quad \frac{A \to B, C \to D}{A \to B\text{-}AD},\quad \text{when } C \subsetneq A$$

This rule induces the Optimization Equivalence defined as follows:

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Function Bertet-Nebut-Direct(Σ)

```
input  : An implicational system Σ
output: The direct implicational system Σ_d equivalent to Σ
begin
    Σ_d := Σ
    foreach A → B ∈ Σ_d do
        foreach C → D ∈ Σ_d do
            if B ∩ C ≠ ∅ then add A(C-B) → D to Σ_d
    return Σ_d
```

Proposition 2.2.8. *Let $A, B, C, D \subseteq M$. If $C \subseteq A$, then*

$$\{A \to B, C \to D\} \equiv \{C \to D, A \to B\text{-}AD\} \qquad \texttt{(Opt-Eq)}$$

To compute the unique direct-optimal implicational system equivalent to Σ_d, the sequence (`Co-Eq`)+(`Opt-Eq`)+(`Fr-Eq`) is exhaustively applied. If the application of (`Opt-Eq`) returns a trivial implication $A \to \varnothing$, it is removed from the output as we can see in `Bertet-Nebut-Minimize`.

Function Bertet-Nebut-Minimize(Σ_d)

```
input  : A direct implicational system Σ_d
output: The direct-optimal basis Σ_do equivalent to Σ_d
begin
    Σ_do := ∅
    foreach A → B ∈ Σ_d do
        B' := B
        foreach C → D ∈ Σ_d do
            if C = A then B' := B' ∪ D
            if C ⊊ A then B' := B' ∖ D
        B' := B' ∖ A
        if B' ≠ ∅ then add A → B' to Σ_do
    return Σ_do
```

`Bertet-Nebut-DO` computes the direct-optimal basis Σ_{do} generated from an arbitrary implicational system Σ. Firstly, it computes the set of implications Σ_d using `Bertet-Nebut-Direct` and then minimizes Σ_d using `Bertet-Nebut-Minimize`.

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Function Bertet-Nebut-DO(Σ)

input : An implicational system Σ
output: The direct-optimal basis Σ_{do} equivalent to Σ
begin
  Σ_d := Bertet-Nebut-direct(Σ)
  Σ_{do} := Bertet-Nebut-Minimize(Σ_d)
  return Σ_{do}

The following theorem ensures the correctness of the method above.

Theorem 2.2.9 (Bertet et al. [12]). *Let Σ be an implicational system.*

(i) `Bertet-Nebut-Direct`(Σ) *is an equivalent direct implicational system.*

(ii) `Bertet-Nebut-DO`(Σ) *is the unique direct-optimal basis that is equivalent to Σ.*

Example 2.2.10. *Let Σ be the implicational system considered in Example 2.1.7:*

$$\begin{aligned}\Sigma \quad = \{ \quad & \text{OPEC} \to \text{Gr77 NA},\\ & \text{MASC} \to \text{Gr77},\\ & \text{NA} \to \text{Gr77},\\ & \text{Gr77 NA MASC OPEC} \to \text{LLDC ACP},\\ & \text{Gr77 NA LLDC OPEC} \to \text{MASC ACP}\}\end{aligned}$$

In the first step, `Bertet-Nebut-DO` *builds the following direct implicational system with 31 implications:*

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$$
\begin{array}{lll}
\Sigma_d = \{ & \text{NA} \to \text{Gr77}, & \text{OPEC Gr77 NA LLDC} \to \text{MASC ACP}, \\
& \text{MASC} \to \text{Gr77}, & \text{OPEC Gr77 NA MASC} \to \text{LLDC ACP}, \\
& \text{OPEC} \to \text{Gr77 NA}, & \text{OPEC NA LLDC} \to \text{MASC ACP}, \\
& \text{OPEC} \to \text{Gr77}, & \text{OPEC NA MASC} \to \text{LLDC ACP}, \\
& \text{OPEC MASC} \to \text{LLDC ACP}, & \text{OPEC Gr77 NA LLDC} \to \text{Gr77}, \\
& \text{OPEC MASC} \to \text{Gr77}, & \text{OPEC Gr77 NA MASC} \to \text{MASC ACP}, \\
& \text{OPEC LLDC} \to \text{MASC ACP}, & \text{OPEC NA MASC LLDC} \to \text{MASC ACP}, \\
& \text{OPEC NA LLDC} \to \text{Gr77}, & \text{OPEC NA LLDC} \to \text{LLDC ACP}, \\
& \text{OPEC MASC LLDC} \to \text{LLDC ACP}, & \text{OPEC NA MASC LLDC} \to \text{Gr77}, \\
& \text{OPEC NA MASC} \to \text{Gr77}, & \text{OPEC Gr77 NA MASC} \to \text{Gr77}, \\
& \text{OPEC LLDC} \to \text{Gr77}, & \text{OPEC Gr77 NA MASC LLDC} \to \text{LLDC ACP}, \\
& \text{OPEC MASC LLDC} \to \text{MASC ACP}, & \text{OPEC Gr77 NA MASC LLDC} \to \text{MASC ACP}, \\
& \text{OPEC MASC} \to \text{MASC ACP}, & \text{OPEC NA MASC LLDC} \to \text{LLDC ACP}, \\
& \text{OPEC MASC LLDC} \to \text{Gr77}, & \text{OPEC Gr77 NA LLDC} \to \text{LLDC ACP}, \\
& \text{OPEC LLDC} \to \text{LLDC ACP}, & \text{OPEC Gr77 NA MASC LLDC} \to \text{LLDC ACP}, \\
& \text{OPEC NA MASC} \to \text{MASC ACP}\} &
\end{array}
$$

After this, the function returns, in the second step, the following direct-optimal basis, having 5 implications:

$$
\begin{array}{lll}
\Sigma_{do} = \{ & \text{OPEC} \to \text{NA Gr77}, & \text{LLDC OPEC} \to \text{MASC ACP}, \\
& \text{NA} \to \text{Gr77}, & \text{MASC OPEC} \to \text{ACP LLDC}, \\
& \text{MASC} \to \text{Gr77}\} &
\end{array}
$$

Notice that, in the first step, the function calculates a lot of redundant implications.

Some of this redundancy could have been avoided if we had worked with unitary implications. For this reason, Bertet et al. later introduced a new method that we describe below.

2.2.1 Unitary Direct-Optimal bases

In the same way as in other fields, the use of formulas in a given normal form allows the design of simpler methods with a better performance than those working with arbitrary expressions (e.g. the use of Horn clauses in Logic Programming). Thus, in FCA the usual normal form to improve the methods to get the direct-optimal basis is the unitary implication. Nevertheless, the advantages provided by the limited languages have a counterpart: a significant growth of the input set.

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After Bertet et al. [12] proposed the method that we have just described, Bertet proposed a second method which works with unitary implicational systems [9]. The main advantage in the use of general implicational systems is the minor size of the input implication set while the use of unitary implicational system allows a better performance of the second method.

They provided versions of the above functions for unit implicational systems. First, the method calculates a direct unitary implicational system Σ_d by exhaustively applying the following inference rule:

$$\textit{Pseudo Transitivity}\ \texttt{[PsTran]}:\quad \frac{A \to b, Cb \to d}{AC \to d}, \text{ if } d \neq b \text{ and } d \notin A$$

This procedure is carried out by `Bertet-Unit-Direct`.

Function Bertet-Unit-Direct(Σ)

```
input  : A proper unitary implicational system Σ
output: A direct unitary implicational system Σd equivalent to Σ
begin
    Σd := Σ
    foreach A → b ∈ Σd do
        foreach Cb → d ∈ Σd do
            if b ≠ d and d ∉ A then add AC → d to Σd
    return Σd
```

The second stage is strongly based on a slight derivation of the Armstrong's `[Augm]` Rule:

$$\textit{Unit Augmentation}\ \texttt{[UnAugm]}:\quad \frac{C \to b}{A \to b}, \quad \text{if } C \subsetneq A$$

The above rule is indeed used to narrow the implications. It leads to the following equivalence, which is a particular case of (`Co-Eq`):

$$\text{If } C \subsetneq A \text{ then } \{A \to b, C \to b\} \equiv \{C \to b\} \qquad (\texttt{Na-Eq})$$

Unlike the previous case, we do not have to check whether the conclusion is empty. Working with unitary implicational system, we do not remove attributes from the conclusion so, it will never be the empty set.

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Function Bertet-Unit-Minimize(Σ_d)

```
input  : A direct unitary implicational system Σ_d
output: The direct-optimal unitary basis Σ_do equivalent to Σ
begin
    Σ_do := Σ_d
    foreach A → b ∈ Σ_do do
        foreach C → b ∈ Σ_do do
            if A ⊊ C then remove C → b from Σ_do
    return Σ_do
```

The above functions were used in [9] to build a method which transforms an arbitrary unitary implicational system into an equivalent unitary implicational system with the same properties that the direct-optimal basis for general implicational systems. Since any non-unitary implicational system can be trivially turned into an unitary implicational system, we may encapsulate both functions to provide another method to get the direct-optimal basis from an arbitrary implicational system. Thus, the following function, which solves the problem proposed in this section, incorporates a first step to convert any implicational system into its equivalent unitary implicational system and concludes with the converse switch.

Function Bertet-Unit-DO(Σ)

```
input  : An implicational system Σ
output: The direct-optimal basis Σ_do equivalent to Σ
begin
    Σ_u := {A → b | A → B ∈ Σ and b ∈ B ∖ A}
    Σ_ud := Bertet-Unit-Direct(Σ_u)
    Σ_udo := Bertet-Unit-Minimize(Σ_ud)
    Σ_do := {A → B | B = {b | A → b ∈ Σ_udo} and B ≠ ∅}
    return Σ_do
```

The following theorem ensures the correctness of the `Bertet-Unit-DO` function.

Theorem 2.2.11 (Bertet [9]). *Let Σ be an implicational system. The*

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output of `Bertet-Unit-DO`(Σ) *is the unique direct-optimal implicational system equivalent to* Σ*.*

We illustrate the method in the following example:

Example 2.2.12. *The unitary implicational system equivalent to the implicational system showed in Example 2.1.7 is:*

$$\begin{array}{rll} \Sigma_u = \{ & \text{OPEC} \to \text{Gr77}, & \text{Gr77 NA MASC OPEC} \to \text{LLDC}, \\ & \text{OPEC} \to \text{NA}, & \text{Gr77 NA MASC OPEC} \to \text{ACP}, \\ & \text{NA} \to \text{Gr77}, & \text{Gr77 NA LLDC OPEC} \to \text{MASC}, \\ & \text{MASC} \to \text{Gr77}, & \text{Gr77 NA LLDC OPEC} \to \text{ACP}\} \end{array}$$

First, from this set with 8 unitary implications, the following direct unitary implicational system with 26 implications is generated by `Bertet-Unit-Direct`*:*

$$\begin{array}{rll} \Sigma_{ud} = \{ & \text{OPEC} \to \text{Gr77}, & \text{OPEC} \to \text{NA}, \\ & \text{NA} \to \text{Gr77}, & \text{MASC} \to \text{Gr77}, \\ & \text{OPEC Gr77 NA MASC} \to \text{LLDC}, & \text{OPEC Gr77 NA MASC} \to \text{ACP}, \\ & \text{OPEC Gr77 NA LLDC} \to \text{MASC}, & \text{OPEC Gr77 NA LLDC} \to \text{ACP}, \\ & \text{OPEC NA LLDC} \to \text{ACP}, & \text{OPEC NA MASC LLDC} \to \text{ACP}, \\ & \text{OPEC Gr77 LLDC} \to \text{ACP}, & \text{OPEC NA LLDC} \to \text{MASC}, \\ & \text{OPEC Gr77 LLDC} \to \text{MASC}, & \text{OPEC NA MASC} \to \text{ACP}, \\ & \text{OPEC Gr77 MASC} \to \text{ACP}, & \text{OPEC NA MASC} \to \text{LLDC}, \\ & \text{OPEC Gr77 MASC} \to \text{LLDC}, & \text{OPEC MASC LLDC} \to \text{ACP}, \\ & \text{OPEC NA LLDC} \to \text{Gr77}, & \text{OPEC MASC} \to \text{ACP}, \\ & \text{OPEC MASC} \to \text{LLDC}, & \text{OPEC LLDC} \to \text{ACP}, \\ & \text{OPEC LLDC} \to \text{MASC}, & \text{OPEC LLDC} \to \text{Gr77}, \\ & \text{OPEC NA MASC} \to \text{Gr77}, & \text{OPEC MASC} \to \text{Gr77}\} \end{array}$$

Notice that in this case, the intermediate direct basis is smaller than those presented in Example 2.2.10 for non-unitary implicational systems. Now, `Bertet-Unit-Minimize` *is applied, returning the unitary direct-optimal basis with 8 unitary implications.*

$$\begin{array}{rll} \Sigma_{udo} = \{ & \text{OPEC} \to \text{Gr77}, & \text{OPEC, MASC} \to \text{LLDC}, \\ & \text{OPEC} \to \text{NA}, & \text{OPEC, MASC} \to \text{ACP}, \\ & \text{NA} \to \text{Gr77}, & \text{OPEC, LLDC} \to \text{MASC}, \\ & \text{MASC} \to \text{Gr77}, & \text{OPEC, LLDC} \to \text{ACP}\} \end{array}$$

When it is turned into a non-unitary implicational system, we reach the direct-optimal basis of the Example 2.2.10.

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An analysis of both methods provides a set of interesting conclusions which motivate the design of new methods proposed in this work (see Chapter 4). Although the use of unitary implicational systems causes a significant growth of the set of implications with respect to non-unitary ones (from 5 to 8 in the cardinality and from 19 to 28 in the size, for the case of the implicational system of Example 2.1.7), the method based on unitary implicational system shows a better performance. One reason is that the intermediate direct basis built after the first step is smaller in the unitary implicational system method than in the non-unitary implicational system one (26 vs 31 in cardinality and 95 vs 144 in size respectively). This is a key point in the better performance of the unitary implicational system method because the size of the implicational set has a direct impact on it due to a decreasing number of applications of the rules and equivalences. Thus, the total number of applications is 57 in the case of the non-unitary implicational system method and 36 in the unitary one, i.e. a reduction of 63%. This significant difference is due to the fact that unitary implications have a lower *possibility* to fit the set of conditions imposed in the equivalences, which are based on the operators of inclusion and intersection. Thus, the growth induced by the use of unitary implicational systems provides a greater number of reading of the implication set, but the lower number of further set transformations balances out such initial growth.

Nevertheless, the growth in the use of unitary implicational systems deserves further attention. A right direction to improve even more the efficiency of these methods may be to reduce the cardinality/size of the intermediate direct basis, which strongly influences the cost of the second stage. Thus, our aim is to design a new method which combines the best of these two approaches: to work with implicational systems so that we limit the cardinality and size of the set of implications at any time and to define new rules which reduce the number of applications, avoiding a growth in the first stage that have to be narrowed in the second stage.

In Table 2.1 we summarize the performance of the methods presented in [12] and [9] over Example 2.1.7. This table shows the cardinality and the size of the implicational set at each stage of the methods and the number

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`Bertet-Nebut-DO`

$\lvert\Sigma\rvert$	$\lVert\Sigma\rVert$	$\lvert\Sigma_d\rvert$	$\lVert\Sigma_d\rVert$	Num. Rules Applied	$\lvert\Sigma_{do}\rvert$	$\lVert\Sigma_{do}\rVert$
5	19	31	144	57	5	15

`Bertet-Unit-DO`

$\lvert\Sigma\rvert$	$\lVert\Sigma\rVert$	$\lvert\Sigma_d\rvert$	$\lVert\Sigma_d\rVert$	Num. Rules Applied	$\lvert\Sigma_{do}\rvert$	$\lVert\Sigma_{do}\rVert$
8	28	26	95	36	8	20

Table 2.1: Comparison of `Bertet-Nebut-DO` and `Bertet-Unit-DO`.

of applications of the rules throughout their execution.

2.3 Ordered-direct bases and D-bases

Adaricheva et al. [3] introduce an alternative approach to directness, named ordered-directness. For $\Sigma = \{A_1 \to B_1, \dots, A_n \to B_n\}$ being an indexed implicational set, the notion of ordered-direct implicational system is introduced by replacing the above function π_Σ by a new function

$$\rho_\Sigma : 2^M \to 2^M \quad \text{with} \quad \rho_\Sigma = \pi_{\Sigma_n} \circ \cdots \circ \pi_{\Sigma_2} \circ \pi_{\Sigma_1} \tag{2.2}$$

where $\Sigma_i = \{A_i \to B_i\}$ for each $1 \leq i \leq n$. The operator ρ_Σ will be named *ordered iteration* of Σ.

Both operators, π_Σ and ρ_Σ, have the following common features: they have the same computational cost and they are isotone and extensive. Moreover, for all $X \subseteq M$, we have that $\pi_\Sigma(X) \subseteq \rho_\Sigma(X)$. This fact leads to a faster fixpoint convergence of the chain $X, \rho_\Sigma(X), \rho^2_\Sigma(X), \rho^3_\Sigma(X), \dots$

Following the same scheme used in the definition of the direct implicational system, a set Σ is named an ordered-direct implicational system if ρ_Σ is idempotent.

Definition 2.3.1. *Let $\Sigma = \{A_i \to B_i \mid 1 \leq i \leq n\}$ be an indexed implicational system. Σ is said to be* ordered-direct *if $X^+_\Sigma = \rho_\Sigma(X)$ for all $X \subseteq M$.*

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Note that any *direct* implicational system is *ordered-direct*, but the converse assertion is not always true as the following example illustrates.

Example 2.3.2. *Consider the following implicational system [3]:*

$$\begin{aligned}\Sigma = \{\, & 5 \to 4, && 23 \to 4, && 24 \to 3, && 34 \to 2, \\ & 14 \to 235, && 25 \to 1, && 35 \to 1, && 123 \to 5\}\end{aligned}$$

This set is ordered-direct, but it is not direct because

$$\pi_\Sigma(\{3,5\}) = \{1,3,4,5\} \subsetneq \rho_\Sigma(\{3,5\}) = \{3,5\}^+_\Sigma = \{1,2,3,4,5\}.$$

Adaricheva et al. [3] introduced another definition of basis strongly based on the ρ_Σ operator, called D-basis. It is a specific kind of ordered-direct basis. In that work, they consider a closure operator as the starting point and assume certain property without loss of generality. This property is rewritten in terms of implicational systems as follows:

Remark 2.3.3. *In the rest of this chapter, we will assume that the implicational system Σ satisfies $\varnothing^+_\Sigma = \varnothing$ and the following property:*

$$\textit{for all } x, y \in M, \ \textit{if } \{x\}^+_\Sigma = \{y\}^+_\Sigma \textit{ then } x = y. \tag{2.3}$$

In order to facilitate the readability of the definition of D-basis, we introduce a new operator.

Definition 2.3.4. *Let Σ be an implicational system on M. We define $(-)^*_\Sigma$ as a self-map on 2^M such that $X^*_\Sigma = \bigcup_{x\in X} \{x\}^+_\Sigma$ for all $X \subseteq M$.*

For the sake of readability, for each element $x \in M$, we write x^+_Σ instead of $\{x\}^+_\Sigma$ and x^*_Σ instead of $\{x\}^*_\Sigma$.

The idea underlying this closure operator is to conceive the closure defined as the union of all the "unit closures". Obviously, $X^*_\Sigma \subseteq X^+_\Sigma$ for all $X \subseteq M$.

Example 2.3.5. *Considering the implicational system Σ given in Example 2.1.7, we show how $(-)^*_\Sigma$ works:*

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$$
\begin{aligned}
\{\mathrm{MASC}, \mathrm{OPEC}\}^*_\Sigma &= \mathrm{MASC}^+_\Sigma \cup \mathrm{OPEC}^+_\Sigma = \{\mathrm{MASC}, \mathrm{Gr77}\} \cup \{\mathrm{OPEC}, \mathrm{Gr77}, \mathrm{NA}\} = \\
&= \{\mathrm{MASC}, \mathrm{Gr77}, \mathrm{OPEC}, \mathrm{NA}\} \subseteq \{\mathrm{MASC}, \mathrm{OPEC}\}^+_\Sigma = \mathrm{M}
\end{aligned}
$$

$$
\begin{aligned}
\{\mathrm{NA}, \mathrm{MASC}\}^*_\Sigma &= \mathrm{NA}^+_\Sigma \cup \mathrm{MASC}^+_\Sigma = \{\mathrm{NA}, \mathrm{Gr77}\} \cup \{\mathrm{MASC}, \mathrm{Gr77}\} = \\
&= \{\mathrm{NA}, \mathrm{MASC}, \mathrm{Gr77}\} = \{\mathrm{NA}, \mathrm{MASC}\}^+_\Sigma
\end{aligned}
$$

The following lemma ensures that this new operator is also a closure operator.

Lemma 2.3.6. *For any implicational system Σ, the mapping $(-)^*_\Sigma$ is a closure operator.*

Proof. i) Extensiveness: It is straightforward that $X \subseteq X^*_\Sigma$ because $x \in x^+_\Sigma$ for each $x \in X$.

ii) Isotonicity: If $X \subseteq Y \subseteq M$, then $X^*_\Sigma = \bigcup_{x \in X} x^+_\Sigma \subseteq \bigcup_{x \in Y} x^+_\Sigma = Y^*_\Sigma$.

iii) Idempotency: We only have to prove $(X^*_\Sigma)^*_\Sigma \subseteq X^*_\Sigma$ because the other inclusion is a consequence of extensiveness and isotonicity. Consider $z \in (X^*_\Sigma)^*_\Sigma$. Then, there exist $y \in X^*_\Sigma$ and $x \in X$ such that $z \in y^+_\Sigma$ and $y \in x^+_\Sigma$. Therefore, since $(-)^+_\Sigma$ is a closure operator, one has $z \in y^+_\Sigma \subseteq (x^+_\Sigma)^+_\Sigma = x^+_\Sigma \subseteq X^*_\Sigma$.

□

The notion of *minimal proper covers*[1], which appears in [3], can be redefined in terms of $(-)^+_\Sigma$ and $(-)^*_\Sigma$.

Definition 2.3.7. *Let Σ be an implicational system, $X \subseteq M$ and $y \in M$. The set X is said to be a* proper cover *of y with respect to Σ if $y \in X^+_\Sigma \smallsetminus X^*_\Sigma$. It is denoted by the expression $y \mathrel{\dot{\sim}}_\Sigma X$.*

[1]Although in [3] it was introduced with the name of *minimal cover*, here we name it *minimal proper cover* because in this work we further generalize the notion of *cover* in Chapter 4.

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Example 2.3.8. *Let* $M = \{1,2,3,4,5\}$ *be a set of attributes and* Σ *the implicational system extracted from [11], which is equivalent to the one given in Example 2.3.2.*

$$\begin{aligned}\Sigma = \{\, & 5 \to 4, && 23 \to 4, && 24 \to 3,\\ & 34 \to 2, && 14 \to 235, && 25 \to 13,\\ & 35 \to 12, && 15 \to 23, && 123 \to 5\}\end{aligned}$$

Consider, for instance, $4 \in M$. *We have that:*

$4 \,\dot{\sim}_\Sigma \{2,3\}$ *because* $4 \in \{2,3\}^+_\Sigma \smallsetminus \{2,3\}^*_\Sigma = \{2,3,4\} \smallsetminus \{2,3\} = \{4\}$.

$4 \,\dot{\sim}_\Sigma \{1,2,3\}$ *because* $4 \in \{1,2,3\}^+_\Sigma \smallsetminus \{1,2,3\}^*_\Sigma = M \smallsetminus \{1,2,3\} = \{4,5\}$.

$4 \,\dot{\not\sim}_\Sigma \{1,5\}$ *because* $4 \notin \{1,5\}^+_\Sigma \smallsetminus \{1,5\}^*_\Sigma = M \smallsetminus \{1,4,5\} = \{2,3\}$.

The following table shows the sets of proper covers of each attribute:

Attribute	*Proper Covers*
1	$\{2,5\}, \{3,5\}, \{2,3,5\}, \{2,4,5\}, \{3,4,5\}, \{2,3,4,5\}$
2	$\{1,4\}, \{1,5\}, \{3,4\}, \{3,5\}, \{1,3,4\}, \{1,3,5\}, \{1,4,5\}, \{3,4,5\}, \{1,3,4,5\}$
3	$\{1,4\}, \{1,5\}, \{2,4\}, \{2,5\}, \{1,2,4\}, \{1,2,5\}, \{1,4,5\}, \{2,4,5\}, \{1,2,4,5\}$
4	$\{2,3\}, \{1,2,3\}$
5	$\{1,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}$

Definition 2.3.9. *Let* Σ *be an implicational system,* $X \subseteq M$ *and* $y \in M$. *The set* X *is said to be a* minimal proper cover *of* y *with respect to* Σ, *if the following conditions hold:*

(i) $y \,\dot{\sim}_\Sigma X$.

(ii) $y \,\dot{\sim}_\Sigma Z$ *and* $Z \subseteq X^*_\Sigma$ *imply* $X \subseteq Z$.

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Example 2.3.10. *For the implicational system introduced in Example 2.3.8, the following table shows the minimal proper covers of each attribute.*

Attribute	*Minimal Proper Covers*
1	$\{2,5\},\{3,5\}$
2	$\{1,4\},\{3,4\}$
3	$\{1,4\},\{2,4\}$
4	$\{2,3\}$
5	$\{1,4\},\{1,2,3\}$

The existence of a minimal proper cover is guaranteed by the following lemma. Specifically, it ensures that every proper cover can be reduced to a minimal proper cover under the subset relation combined with the $(-)^*_\Sigma$ operator.

Lemma 2.3.11 (Adaricheva et al. [3]). *Let Σ be an implicational system, $X \subseteq M$ and $y \in M$. If $x \,\dot{\sim}_\Sigma\, X$, then there exists $Y \subseteq X^*_\Sigma$ such that $x \,\dot{\sim}_\Sigma\, Y$ and Y is a minimal proper cover for x.*

These ideas lead to the definition of the implicational system strongly based on the minimal proper covers given in [3], which was named there the D-basis.[2]

Definition 2.3.12 (Adaricheva et al. [3]). *Let Σ be a reduced implicational system. The D-*basis *for Σ is the pair $\langle \Sigma_a, \Sigma_n \rangle$ where*

$$\Sigma_a = \{y \to x \mid y \in M,\ x \in y^+_\Sigma,\ x \neq y\}$$

$$\Sigma_n = \{X \to x \mid X \subseteq M,\ x \in M,\ X \textit{ is a minimal proper cover of } x\}$$

Example 2.3.13. *The D-basis associated with the implicational system given in Example 2.3.8 is the the pair $\langle \Sigma_a, \Sigma_n \rangle$ where $\Sigma_a = \{5 \to 4\}$ and*

$$\begin{aligned} \Sigma_n = \{\, & 23 \to 4,\quad 24 \to 3,\quad 34 \to 2,\quad 14 \to 2, \\ & 14 \to 3,\quad 14 \to 5,\quad 25 \to 1,\quad 35 \to 1,\quad 123 \to 5\} \end{aligned}$$

[2]The name is justified because it is a subset of the so-called dependence relation basis (briefly, D-relation), which is a relevant concept in the study of free lattices [23].

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The subscript “a” is justified because implications in Σ_a are atomic in the sense that both premise and conclusion are singletons, atoms of M. According to its cardinality, both sets are unitary. But we have not used this term to avoid confusion with the unitary implications previously introduced in Section 1.3. Implications in Σ_n have unitary conclusion but n-ary premise.

The following theorem summarizes the properties of the D-bases.

Theorem 2.3.14 (Adaricheva et al. [3]). *Let Σ be an implicational system. If $\langle \Sigma_a, \Sigma_n \rangle$ is its D-basis and Σ_D is the set $\Sigma_a \cup \Sigma_n$ ordered in such a way that implications from Σ_a goes before than implications from Σ_n, then Σ_D is an ordered-direct basis such that $\Sigma_D \equiv \Sigma$.*

In addition, if Σ_{udo} is its unitary direct-optimal basis, $\Sigma_D \subseteq \Sigma_{udo}$.

Notice that the D-basis belongs to the family of the *unit* bases: implications have unitary conclusions. Based on this result, the authors propose there a quadratic method to extract Σ_D from Σ_{udo}.

Now, we justify why it can be done without loss of generality. Following the work of Adaricheva et al. [3], we have done an assumption in Remark 2.3.3. On the one hand, given an implicational system Σ on M, if there exist $x, y \in M$ such that $x \neq y$ but $x^+_\Sigma = y^+_\Sigma$, then we could compute a new implicational system Σ' by replacing each instance of y by x. It is easy to see that $\Sigma \equiv \{x \to y, y \to x\} \cup \Sigma'$. Therefore, by iteratively applying this transformation, any implicational system can be turned into an implicational system that satisfies the property (2.3) described in Remark 2.3.3. In addition, this transformation can be made in polynomial time.

Therefore, given an arbitrary implicational system Σ, it can be transformed into another one, Σ', that satisfies (2.3). Suppose M' is the set of attributes that remains in Σ'. Then, we can compute the D-basis, $\langle \Sigma_a, \Sigma_n \rangle$, that is equivalent to Σ'. Finally, a new ordered implicational system Σ_{od} is obtained as follows:

$$\big\{x \to y, y \to x \mid x \in M', y \in M \smallsetminus M' \text{ such that } x^+_\Sigma = y^+_\Sigma\big\} \cup \Sigma_a \cup \Sigma_n$$

It is easy to prove that Σ_{od} is an ordered-direct basis such that $\Sigma_{od} \equiv \Sigma$.

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On the other hand, given an implicational system Σ on M, if $\varnothing_\Sigma^+ = A$ where A is a non-empty set of attributes, then we could compute a new implicational system Σ' by removing all the attributes in A. It is easy to see that $\Sigma \equiv \{\varnothing \to A\} \cup \Sigma'$. Thus, if $\langle \Sigma_a, \Sigma_n \rangle$ is the equivalent D-basis to Σ', then the one equivalent to Σ is obtained as follows:

$$\left\{\varnothing \to \varnothing_\Sigma^+\right\} \cup \Sigma_a \cup \Sigma_n$$

We have summarized the previous works about the direct-optimal basis and the D-basis. Notice that, up to now, the best way to obtain both bases is to manage unitary implications. However, in this work we propose an alternative way strongly based on non-unitary implications, which improves the efficiency of the computation of these bases.

The two following chapters are devoted to the design of novel algorithms computing the direct-optimal basis as well as the D-basis. They are thoroughly studied in Chapter 3 and Chapter 4, respectively.

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Chapter 3

Computing Direct-Optimal Bases

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The time complexity of the methods for computing the direct-optimal basis from an arbitrary implicational system is exponential in the worst case with respect to the input, as stated in [10]. This issue motivates the idea of obtaining other methods that, without avoiding the intrinsic exponential complexity of the problem, provides a better performance than previous works.

This chapter is focused on the development of new methods to compute the direct-optimal basis from an arbitrary non-unitary implicational system. Our aim is the development of methods that avoid generating extra implications and to take advantage of the benefits offered by the non-unitary implicational systems about the size of the implicational systems.

The methods presented in this chapter are the result of a collaboration with Karell Bertet and have been published in [48, 49].

3.1 Simplified implicational systems

Up to now, the design of logic-based methods to compute the direct-optimal basis from an arbitrary implicational system has shown to be unsuccessful because of the great number of applications of inference rules, which usually produce redundant attributes in both sides of the new implications. These superfluous attributes can be safely removed from the right-hand

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side avoiding, in this way, extra applications of the inference rules in the future. This is the problem that Bertet et al. found in `Bertet-Nebut-DO` and tried to solve in `Bertet-Unit-DO` [9] by using unitary implicational systems. But, using unitary implicational systems has implicit an increasing of the input size that we would like to avoid. Let us show an example to illustrate this situation:

Example 3.1.1. *Consider the implicational system* Σ *from Example 2.1.7. The second and fifth implications are respectively:*

$$\text{MASC} \to \text{Gr77}$$

$$\text{Gr77 NA LLDC OPEC} \to \text{MASC ACP}$$

These implications satisfy the precondition of the rule `[Ovl]` *and, after its application, we obtain:*

$$\text{NA LLDC OPEC MASC} \to \text{MASC ACP}$$

Notice that MASC *attribute is redundant and it will cause new applications of the rule* `[Ovl]` *when the method continues.*

Our goal here is to design a new method admitting arbitrary implicational systems to provide a more compact representation of the new implicational systems. The use of the paradigm of *reduction* to achieve a method working with (not necessarily unitary) implicational systems reduces the extra coupling of implications as well. To this end, we propose to work with reduced implications, which do not have redundant attributes in their right-hand sides. Before describing the new methods, we introduce several definitions that we will use hereafter.

Definition 3.1.2. *An implicational system* Σ *is said to be* reduced *if the following condition holds:*

$$A \to B \in \Sigma \textit{ implies } B \neq \varnothing \textit{ and } A \cap B = \varnothing. \tag{3.1}$$

A reduced implicational system Σ *is said to be* compact *if, in addition, the following condition holds:*

$$A \to B, A \to C \in \Sigma \textit{ implies } B = C. \tag{3.2}$$

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Definition 3.1.3. *An implicational system* Σ *is said to be* left-simplified *if it is compact and the following condition holds:*

$$A \to B, C \to D \in \Sigma \text{ and } A \subsetneq C \text{ imply } C \cap B = \varnothing. \tag{3.3}$$

An implicational system Σ *is said to be* right-simplified *if it is compact and the following condition holds:*

$$A \to B, C \to D \in \Sigma \text{ and } A \subsetneq C \text{ imply } D \cap B = \varnothing. \tag{3.4}$$

An implicational system Σ *is said to be* simplified *if it is left- and right-simplified.*

Note that a set is a reduced implicational system if non-trivial information is included in the conclusions; it is compact if, in addition, there are no two implications with the same premise; and it is a simplified implicational system if the Simplification equivalence (`Si-Eq`) cannot remove redundant information. Clearly, to be simplified implies to be compact, and to be compact implies to be reduced. Notice that an implicational system is simplified if the conclusions in the implications are non-empty and equivalences in Theorem 1.3.14 do not allow to remove redundant attributes.

Now, we introduce a function called `Simplify`, which transforms any implicational system into an equivalent simplified one.

Theorem 3.1.4. *For any implicational system* Σ*,* `Simplify`(Σ) *ends and, if* Σ_s *is the output, then* $\Sigma \equiv \Sigma_s$ *and* Σ_s *is a simplified implicational system.*

Proof. First, we prove $\Sigma \equiv \Sigma_s$. The function, in Line #1, applies (`Fr-Eq`) to the formulas $A \to B \in \Sigma$ such that $B \not\subseteq A$, and discards the rest because they are axioms (i.e. they always hold because they are implications with the empty set as conclusion). The following part of `Simplify` is a loop in which the function exhaustively applies (`Co-Eq`) and (`Si-Eq`) equivalences until a fixpoint is reached. Specifically, each step of the loop copies each implication $A \to B$ from Σ_s to Σ once it has been simplified by comparison with the implications that have been previously added to Σ. Thus, for each $C \to D \in \Sigma$, the function distinguishes the following situations:

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Function Simplify(Σ)

input : An implicational system Σ
output: A simplified implicational system Σ_s equivalent to Σ
begin
#1 $\Sigma := \{A \to B\text{-}A \mid A \to B \in \Sigma, B \not\subseteq A\}$
  repeat
    $\Sigma_s := \Sigma$; $\Sigma := \varnothing$
    foreach $A \to B \in \Sigma_s$ **do**
      $\Gamma := \varnothing$
      foreach $C \to D \in \Sigma$ **do**
#2   **if** $C \subseteq A \subseteq C \cup D$ *or* $A \subseteq C \subseteq A \cup B$ **then**
          $A := A \cap C$; $B := B \cup D$
        else
#3   **if** $A \subsetneq C$ **then**
            if $D \not\subseteq B$ **then** add $C\text{-}B \to D\text{-}B$ to Γ
          else
#4   **if** $C \subsetneq A$ **then** $A := A \smallsetminus D$; $B := B \smallsetminus D$
            add $C \to D$ to Γ
#5   **if** $B = \varnothing$ **then** $\Sigma := \Gamma$ **else** $\Sigma := \Gamma \cup \{A \to B\}$
  until $\Sigma_s = \Sigma$
  return Σ_s

Line #2: It considers two cases:

(i) If $C \subseteq A \subseteq C \cup D$ then, applying (`Si-Eq`) and (`Co-Eq`), and taking into account that there are no common attributes between premises and conclusions in the implications, one has

$$
\begin{aligned}
\{A \to B,\, C \to D\} &\equiv \\
&\equiv \{A\text{-}D \to B\text{-}D, C \to D\} = \{C \to B\text{-}D, C \to D\} \\
&\equiv \{C \to (B\text{-}D)D\} = \{C \to BD\}
\end{aligned}
$$

(ii) If $A \subseteq C \subseteq A \cup B$, by following a similar reasoning, one has $\{A \to B, C \to D\} \equiv \{A \to BD\}$.

In both cases $\{A \to B, C \to D\} \equiv \{E \to BD\}$ where $E = A \cap C$.

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Therefore, the function replaces $A \to B$ by $E \to BD$ and removes $C \to D$ from Σ.

Line #3: When $A \subsetneq C$ but $C \not\subseteq A \cup B$, the function distinguishes two cases again:

(i) If $D \subseteq B$, by (`Si-Eq`) and knowing that an implication with an empty conclusion always holds, one has

$$\{A \to B, C \to D\} \equiv \{A \to B, C\text{-}B \to D\text{-}B\} \equiv \{A \to B\}$$

Then, `Simplify` removes $C \to D$ from Σ.

(ii) Otherwise, using (`Si-Eq`), $C \to D$ is replaced by $C\text{-}B \to D\text{-}B$ in Σ.

Line #4: In the case of $C \subsetneq A$ but $A \not\subseteq C \cup D$, by (`Si-Eq`), one has $\{A \to B, C \to D\} \equiv \{A\text{-}D \to B\text{-}D, C \to D\}$. Therefore, the function replaces $A \to B$ by $A\text{-}D \to B\text{-}D$ and does not modify $C \to D$ in Σ.

The last step in the loop (Line #5) includes the simplified implication $A \to B$ to Σ, only when $B \neq \varnothing$.

Therefore, at the end of each iteration $\Sigma_s \equiv \Sigma$. Finally, the loop ends, because the size of Σ_s strictly decreases in each iteration, and, when a fixpoint is reached, one has that (`Fr-Eq`), (`Co-Eq`) and (`Si-Eq`) does not modify any pair of implications in Σ_s. That is, Σ_s is a simplified implicational system. $\square$

Observe that the complexity of `Simplify`, in the worst case, is $\|\Sigma\| \cdot |\Sigma|^2$.

3.2 `doSimp`: A first Simplification-based method for computing direct-optimal bases

In this section, we introduce Algorithm 3.1, called `doSimp`, having three main stages, each one consisting of the transformation of a previous implicational system into an equivalent one fulfilling some additional property.

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Algorithm 3.1: `doSimp`

input : An implicational system Σ
output: The direct-optimal basis Σ_{do}
begin
  $\Sigma_s :=$ `Simplify`(Σ)
  $\Sigma_{dr} :=$ `Complete`(Σ_s)
  $\Sigma_{do} :=$ `Optimize`(Σ_{dr})
  return Σ_{do}

Notice that the condition being reduced is a natural one in the original specification of the problem, but the inspiration of the method proposed in this section is to maintain intermediate implicational systems reduced at any time by means of the application of inference rules that always produce non-redundant implications. Then, the first stage of the algorithm executes `Simplify` to make the implicational system simplified.

In the second stage of `doSimp`, Σ_s is transformed into an equivalent direct reduced implicational system by exhaustively applying the following rule [1], called *strong Simplification rule*,

$$\texttt{[sSimp]} \quad \frac{A \to B, C \to D}{A(C\text{-}B) \to D\text{-}(AB)}, \text{ if } B \cap C \neq \varnothing \neq D \smallsetminus (A \cup B)$$

In order to prove that Σ_{dr} is a direct reduced implicational system which is equivalent to Σ_s, firstly the following lemma ensures the soundness of the Strong Simplification rule.

Lemma 3.2.1. `[sSimp]` *is a derived inference rule.*

Proof. Assume $B \cap C \neq \varnothing \neq D \smallsetminus (A \cup B)$. The following sequence proves the soundness of `[sSimp]`:

σ_1: $A \to B$ .... by hypothesis.

σ_2: $B \to B \cap C$ .... by `[Inc]`

σ_3: $A \to B \cap C$ .... by σ_1, σ_2 and `[Trans]`

[1] Notice that, if the implicational system is reduced, it is not necessary to put brackets in the premise $A(C\text{-}B)$ because $(AC)\text{-}B = A(C\text{-}B)$.

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σ_4: $C\text{-}B \to C\text{-}B$ .... by `[Ref]`

σ_5: $A(C\text{-}B) \to C$ .... by σ_3, σ_4, `[Comp]`,
and $(B \cap C)(C \smallsetminus B) = C$.

σ_6: $C \to D$ .... by hypothesis

σ_7: $A(C\text{-}B) \to D$ .... by σ_5, σ_6 and `[Trans]`

σ_8: $A(C\text{-}B) \to D\text{-}(AB)$ .... by σ_7, and `[Frag]`

□

Function Complete(Σ_s)

input : A simplified implicational system Σ_s
output: A direct reduced implicational system Σ_{dr} equivalent to Σ_s

```
begin
    Σdr := Σs
    Γ := ∅
    repeat
        Σdr := Σdr ∪ Γ
        Γ := ∅
        foreach A → B ∈ Σdr do
            foreach C → D ∈ Σdr do
                if B ∩ C ≠ ∅ ≠ D ∖ (A ∪ B) then
                    σ := (AC-B → D-(AB))
                    if σ ∉ Σdr then add σ to Γ
    until Γ = ∅
    return Σdr
```

The implicational system Σ_{dr} in `doSimp`, which is the output of `Complete` when it is applied to a simplified implicational system Σ_s, is the smallest set such that $\Sigma_s \subseteq \Sigma_{dr}$ and it is closed with respect to `[sSimp]`. The following theorem characterizes this implicational system.

Theorem 3.2.2. *For any simplified implicational system Σ_s,* `Complete`(Σ_s) *ends and, if Σ_{dr} is the output, then $\Sigma_{dr} \equiv \Sigma_s$ and Σ_{dr} is a direct reduced implicational system.*

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Proof. Since Σ_{dr} is the smallest set such that $\Sigma_s \subseteq \Sigma_{dr}$ and it is closed w.r.t. `[sSimp]`, by Lemma 3.2.1, $\Sigma_{dr} \equiv \Sigma_s$. Moreover, Σ_{dr} is reduced because `[sSimp]` preserves this property. Finally, we prove the directness. By Lemma 2.2.2, it is sufficient to prove that $\pi_{\Sigma_{dr}}(\pi_{\Sigma_{dr}}(X)) \subseteq \pi_{\Sigma_{dr}}(X)$ for all $X \subseteq M$. Consider $Y = \pi_{\Sigma_{dr}}(X)$ and $y \in \pi_{\Sigma_{dr}}(Y)$. We will prove $y \in \pi_{\Sigma_{dr}}(X)$.

First, $y \in \pi_{\Sigma_{dr}}(Y)$ implies $A \to B \in \Sigma_{dr}$ exists with $A \subseteq Y$ and $y \in B$ and, since Σ_{dr} is a reduced implicational system, one has $y \in B \smallsetminus A$. If $A \subseteq X$, then $y \in \pi_{\Sigma_{dr}}(X)$. Otherwise, if $A \not\subseteq X$, there exists a set of implications $\{A_i \to B_i \mid 1 \leq i \leq k\} \subseteq \Sigma_{dr}$ such that $A \smallsetminus X \subseteq \bigcup_{1 \leq i \leq k} B_i$ and $A_i \subseteq X$ for all $1 \leq i \leq k$. Assume without loss of generality that[2]

$$(A \smallsetminus X) \cap B_i \neq \varnothing \quad \text{and} \quad y \notin B_i \smallsetminus X \not\subseteq \bigcup_{\substack{1 \leq j \leq k \\ j \neq i}} B_j \tag{3.5}$$

for all $1 \leq i \leq k$. Consider now:

$$\big(C_1 \to D_1\big) = \big(A_1(A\text{-}B_1) \to B\text{-}(A_1B_1)\big) \text{ and}$$

$$\big(C_i \to D_i\big) = \big(A_i(C_{i-1}\text{-}B_i) \to D_{i-1}\text{-}(C_iD_i)\big) \text{ for all } 1 < i \leq k.$$

By (3.5), $A \cap B_1 \neq \varnothing$, $D_1 \neq \varnothing$, $C_{i-1} \smallsetminus B_i \neq \varnothing$ and $D_i \neq \varnothing$ for all $1 < i \leq k$. Therefore, $\{C_i \to D_i \mid 1 \leq i \leq k\} \subseteq \Sigma_{dr}$ because Σ_{dr} is closed with respect to `[sSimp]`. Finally, it is easy to see that

$$C_k \subseteq \bigcup_{1 \leq i \leq k} A_i \cup \Big(A \smallsetminus \big(\bigcup_{1 \leq i \leq k} B_i\big)\Big) \subseteq \bigcup_{1 \leq i \leq k} A_i \cup \Big(A \smallsetminus (A \smallsetminus X)\Big) \subseteq X$$

and $y \in D_k = B \smallsetminus \bigcup_{1 \leq i \leq k}(A_i \cup B_i)$ and, therefore, $y \in \pi_{\Sigma_{dr}}(X)$. $\square$

The following theorem ensures that `doSimp` transforms an arbitrary implicational system into its equivalent direct-optimal basis.

[2] First, $y \notin B_i \smallsetminus X$ can be assumed because, otherwise, one already has $y \in \pi_{\Sigma_{dr}}(X)$. Second, if there exists $1 \leq i \leq k$ such that $(A \smallsetminus X) \cap B_i = \varnothing$ or $B_i \smallsetminus X \subseteq \bigcup_{1 \leq j \leq k, j \neq i} B_j$, then $\{A_j \to B_j \mid 1 \leq j \leq k, j \neq i\}$ can be considered instead of $\{A_i \to B_i \mid 1 \leq i \leq k\}$.

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Function Optimize(Σ_{dr})

input : A direct reduced implicational system Σ_{dr}
output: The direct-optimal basis Σ_{do} equivalent to Σ_{dr}
begin
  $\Sigma_{do} := \varnothing$
  foreach $A \to B \in \Sigma_{dr}$ **do**
    foreach $C \to D \in \Sigma_{dr}$ **do**
#1   **if** $C \subsetneq A$ **then** $B := B \smallsetminus D$
#2   **if** $C = A$ **then** $B := B \cup D$
#3   **if** $B \neq \varnothing$ **then** add $A \to B$ to Σ_{do}
  return Σ_{do}

Theorem 3.2.3. *For any implicational system Σ, Algorithm 3.1 (*`doSimp`*) ends and, if Σ_{do} is the output, then Σ_{do} is the direct-optimal basis that satisfies $\Sigma_{do} \equiv \Sigma$.*

Proof. From Theorems 3.1.4 and 3.2.2, one has Σ_{dr} is a direct reduced implicational system such that $\Sigma_{dr} \equiv \Sigma$. First, obviously, `Optimize`($\Sigma_{dr}$) ends in $|\Sigma_{dr}|^2$ steps. Second, if Σ_{do} is the output of `Optimize`(Σ_{dr}), it is straightforward that $\pi_{\Sigma_{do}}(X) = \pi_{\Sigma_{dr}}(X)$ for all $X \subseteq M$. Therefore, by Lemma 2.2.2, Σ_{do} is a direct implicational system. Now, by Theorem 2.2.6, Σ_{do} is direct-optimal if and only if the following properties hold:

(i) If $A \to B \in \Sigma_{do}$ then $A \cap B = \varnothing$.

(ii) If $A \to B, C \to D \in \Sigma_{do}$ and $C \subsetneq A$ then $B \cap D = \varnothing$.

(iii) If $A \to B, A \to B' \in \Sigma_{do}$ then $B = B'$.

(iv) If $A \to B \in \Sigma_{do}$ then $B \neq \varnothing$.

Condition (i) is ensured because Σ_{dr} is a reduced implicational system. Conditions (ii), (iii) and (iv) are fulfilled with the transformations done in lines #1, #2 and #3 respectively. Finally, $\Sigma_{do} \equiv \Sigma_{dr}$ is a consequence of (`Co-Eq`) and

$$\{A \to B, C \to D\} \equiv \{A \to B, C \to D\text{-}B\}$$

when $A \subsetneq C$ and $A \cap B = \varnothing$. □

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3.2.1 The performance of `doSimp`

This section begins with an illustrative example that shows some information about the execution of the method and, then, we present the analysis of our experimental evaluation in order to obtain the conclusions on its performance in practice.

Example 3.2.4. *Consider the implicational system from Example 2.1.7:*

$$\begin{aligned}\Sigma = \{\, & \mathrm{OPEC} \to \mathrm{Gr77\ NA},\\ & \mathrm{MASC} \to \mathrm{Gr77},\\ & \mathrm{NA} \to \mathrm{Gr77},\\ & \mathrm{Gr77\ NA\ MASC\ OPEC} \to \mathrm{LLDC\ \ ACP},\\ & \mathrm{Gr77\ NA\ LLDC\ OPEC} \to \mathrm{MASC\ ACP}\}\end{aligned}$$

`Simplify`(Σ) *returns the equivalent simplified implicational system* Σ_s*:*

$$\begin{aligned}\Sigma_s = \{\, & \mathrm{OPEC} \to \mathrm{NA},\\ & \mathrm{NA} \to \mathrm{Gr77},\\ & \mathrm{MASC} \to \mathrm{Gr77},\\ & \mathrm{LLDC\ OPEC} \to \mathrm{MASC},\\ & \mathrm{MASC\ OPEC} \to \mathrm{ACP\ LLDC}\}\end{aligned}$$

And `Complete`(Σ_s) *returns the equivalent direct reduced implicational system* Σ_{dr}*:*

$$\begin{aligned}\Sigma_{dr} = \{\, & \mathrm{OPEC} \to \mathrm{NA},\\ & \mathrm{NA} \to \mathrm{Gr77},\\ & \mathrm{MASC} \to \mathrm{Gr77},\\ & \mathrm{LLDC\ OPEC} \to \mathrm{MASC},\\ & \mathrm{MASC\ OPEC} \to \mathrm{ACP\ LLDC},\\ & \mathrm{OPEC} \to \mathrm{Gr77},\\ & \mathrm{LLDC\ OPEC} \to \mathrm{Gr77},\\ & \mathrm{LLDC\ OPEC} \to \mathrm{ACP}\}\end{aligned}$$

This set of implications is smaller (in size) than those built with the previous methods presented in [12] and [9, 11]. The cardinality of Σ_{dr} *is 8 whereas the previous methods return implicational systems with cardinality 26 (see Example 2.2.12) and 31 (see Example 2.2.10) for unitary implicational systems and non-unitary ones, respectively (see Table 3.1).*

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Algorithm	$\lvert\Sigma\rvert$	$\lVert\Sigma\rVert$	$\lvert\Sigma_d\rvert$	$\lVert\Sigma_d\rVert$	N. Rules App.	$\lvert\Sigma_{do}\rvert$	$\lVert\Sigma_{do}\rVert$
`Bertet-Nebut-DO`	5	19	31	144	57	5	15
`Bertet-Unit-DO`	8	28	26	95	36	8	20
`doSimp`	5	19	8	21	5	5	15

Table 3.1: Comparison of `Bertet-Nebut-DO`, `Bertet-Unit-DO` and `doSimp`.

Moreover, by applying `[sSimp]`*, we achieve an extra reduction in the size of the implicational system,* $\lVert\Sigma_{dr}\rVert = 21$*. We also remark that the size of* Σ_{dr} *is smaller than the size of the direct implicational systems obtained with previous methods for both, unitary and non-unitary implicational systems, which were 95 and 144, respectively.*

In the last stage the direct-optimal basis Σ_{do} *is obtained from* Σ_{dr}*:*

$$\begin{aligned}\Sigma_{do} = \{\, & \text{OPEC} \to \text{NA Gr77},\\ & \text{NA} \to \text{Gr77},\\ & \text{MASC} \to \text{Gr77},\\ & \text{LLDC OPEC} \to \text{MASC ACP},\\ & \text{MASC OPEC} \to \text{ACP LLDC}\}\end{aligned}$$

Regarding the number of rules applied, the total number of rules which have been applied is 5 whereas in the previous methods 57 and 36 rules were necessary for non-unitary implicational systems and unitary ones, respectively.

In summary, the new method improves all the previously published ones (see Table 3.1). The great improvement illustrated in the above example is due to several issues. The key point in our method is to reduce the implications and ensure that the property of being reduced is always preserved at any time dealing with smaller implicational systems than any other method based on unitary implicational systems. It also narrows the input by using (`Si-Eq`) and by adding less implications to compute the intermediate direct implicational system by using `[sSimp]`. As stated previously, this was the main aim of the method presented in this section: to combine the advantages of using both, unitary and non-unitary, implicational systems.

Now, an empirical study to get some practical conclusions on the performance of the algorithm is presented. The methods of Bertet et al. [11,12]

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and our newly proposed method have been implemented in SWI-Prolog. In particular, the input are sets of implications randomly generated, increasing their size until the resources of the computer are exhausted when each algorithm is executed.

		BertetNebutDO		BertetUnitDO		doSimp	
Ex.	$\|\Sigma\|$	Time	Rules	Time	Rules	Time	Rules
1	10	535.67	33 242	36.68	2 778	**0.001**	**15**
2	10	100.86	27 568	55.52	2 781	**0.002**	**47**
3	10	63.65	22 929	6.20	1 471	**0.003**	**54**
4	10	1 741.84	62 901	318.77	5 210	**0.003**	**62**
5	10	428.08	32 658	30.26	2 348	**0.009**	**166**
6	10	412.92	31 280	102.64	4 022	**0.003**	**51**
7	10	4 458.85	74 664	224.05	5 009	**0.017**	**358**
8	10	11 606.55	90 602	1 113.77	7 925	**0.020**	**442**
9	10	755.77	38 970	167.04	4 100	**0.010**	**118**
10	10	281.43	34 805	30.7	2 538	**0.011**	**194**
11	15					**7.920**	**6 484**
12	15					**0.067**	**1 068**
13	15					**0.194**	**1 965**
14	15					**2.191**	**2 838**

Table 3.2: Measures of the better performance of `doSimp` compared with previous methods

Table 3.2 summarizes the results of the execution of the algorithms when they compute the direct-optimal basis. For each method, there are two columns showing the following information: the first one is the execution time in seconds; and the second one stores the number of couples of implications in which a rule is applied. Table 3.2 shows that experiments with 15 implications saturate machine resources for the previous methods. In all parameters, the method proposed in this section obtains much better results.

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3.3 `SLgetdo`: An improved method

In this section, we show a new algorithm, called `SLgetdo`, to compute the direct-optimal basis. In the previous section we designed a method which combines the advantages of Simplification Logic and of the methods proposed by Bertet et al. Here, our aim is to design a new algorithm, which, as we shall see, has a better performance than the previous ones. The main improvement comes from a deeper analysis on the behavior of the inference rules enclosed in the method.

The first difference between `doSimp` and the methods that can be found in the literature is that it begins with a transformation of the input into an equivalent simplified implicational system. It is done by using `Simplify` and its aim is to reduce the size of the input as much as possible, with a polynomial cost. As in the previous method, `SLgetdo` begins also applying `Simplify`.

Once this function is applied, a simplified implicational system is obtained, which, in particular, is a reduced one. Then, instead of applying `[Ovl]`, `doSimp` applies `[sSimp]` in order to maintain the property of being a reduced implicational system. This inference rule is used to achieve the directness property, but with less cost. The main underlying idea is also to reduce the size of the intermediate implicational system as much as possible.

Then, once `[sSimp]` has been exhaustively applied, the method simplifies the implicational system and returns the direct-optimal basis. This last stage is also quadratic. Therefore, the exponential cost of the method lies in the central stage.

This approach presents an intrinsic weak point: the larger output an algorithm creates, the more time and memory space takes [53]. In such approach, the big size of the direct implicational system created in the intermediate steps causes that the next step takes a long time to be executed.

In this section, we avoid the extra-cost of this computation. The key is, as the following theorem shows, that direct-optimal basis is also a simplified implicational system. Therefore, the method can be significantly

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improved if we reach the aim of maintaining the implicational system, in all intermediate stages, being, not only a reduced one, but also a simplified one.

Theorem 3.3.1. *Let Σ be an implicational system.*

(i) *If Σ is direct and Σ' is obtained from Σ by (left-to-right) applying* (`Fr-Eq`)*,* (`Co-Eq`) *or* (`Si-Eq`)*, then Σ' is also direct.*

(ii) *Σ is a direct-optimal basis if and only if Σ is a direct simplified implicational system.*

Proof. Item (i) is straightforward in the cases of (`Fr-Eq`) and (`Co-Eq`). For (`Si-Eq`), we consider $A \to B, C \to D \in \Sigma$ such that $A \cap B = \varnothing$, $A \subseteq C$ and $\Sigma' = \big(\Sigma \smallsetminus \{C \to D\}\big) \cup \{C\text{-}B \to D\text{-}B\}$ and prove $\pi_\Sigma(X) = \pi_{\Sigma'}(X)$ for all $X \subseteq M$. On the one hand, since $\Sigma \equiv \Sigma'$ and Σ is direct, one has $\pi_{\Sigma'}(X) \subseteq X^+_{\Sigma'} = X^+_\Sigma = \pi_\Sigma(X)$. On the other hand, if $x \in \pi_\Sigma(X)$, there exists $U \to V \in \Sigma$ such that $U \subseteq X$ and $x \in V$. Trivially, we only need to study the case of $C \to D$. Assume $C \subseteq X$ and $x \in D$. If $x \in D \smallsetminus B$ then $x \in \pi_{\Sigma'}(X)$ because $C \smallsetminus B \subseteq X$ and $C\text{-}B \to D\text{-}B \in \Sigma'$. Otherwise, $x \in \pi_{\Sigma'}(X)$ because $A \subseteq C \subseteq X$, $x \in B$ and $A \to B \in \Sigma'$.

For item (ii), as a consequence of Theorem 2.2.6, one has Σ a direct-optimal basis if and only if it is a direct right-simplified implicational system (see Definition 3.1.3). In addition, by item (i) and Definitions 2.2.4 and 3.1.3, it is equivalent to Σ being a direct simplified implicational system. □

Algorithm `SLgetdo` embeds `[sSimp]` in `Simplify`. Obviously, this new way of applying both rules, in an integrated way, generates a smaller implicational set, and henceforth, a less number of possible matches between pairs of implications in which the rule `[sSimp]` is applied. As a consequence thereof, the size of the generated implicational system does not grow more than necessary (see Figure 3.1). To successfully achieve a proper behavior of this new approach, it is very important to ensure that the implications added by `[sSimp]` are not removed by (`Si-Eq`) later in order to avoid infinite loops. For this reason, we present a function called `Add-sSimp` that

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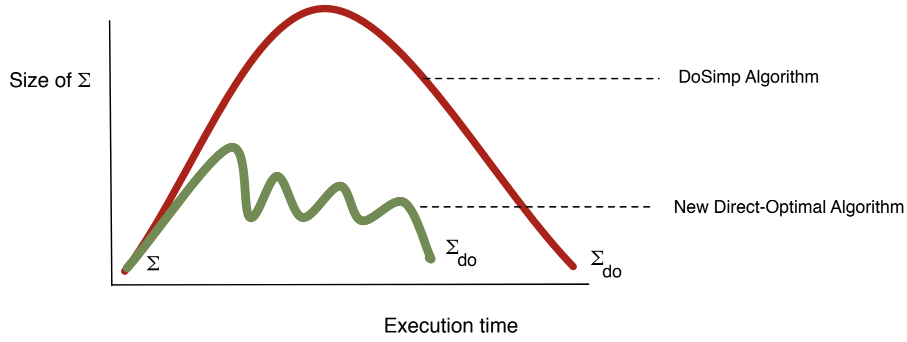

Figure 3.1: Behaviour of `doSimp` vs the new method, `SLgetdo`.

applies `[sSimp]` and checks whether the new implication is simplifiable before including it in the set.

Function Add-sSimp($A \to B, C \to D, \Sigma$)

---

begin
    if $A \nsubseteq C$ *and* $B \cap C \neq \varnothing \neq D \smallsetminus (A \cup B)$ **then**
#1 $E := A \cup (C \smallsetminus B)$; $F := D \smallsetminus (A \cup B)$
        foreach $X \to Y \in \Sigma$ **do**
            if $X \subseteq E$ **then**
#2   **if** $F \subseteq Y$ **then return** $\varnothing$
                else $E := E \smallsetminus Y$; $F := F \smallsetminus Y$
        return $\{E \to F\}$
    else return $\varnothing$

Theorem 3.3.2. *Let Σ be a simplified implicational system. For any pair of implications $A \to B, C \to D \in \Sigma$, if* `Add-sSimp`*($A \to B, C \to D, \Sigma$) returns $\{E \to F\}$ and $\Sigma_s =$* `Simplify`*($\Sigma \cup \{E \to F\}$), the following conditions holds:*

(i) *$F \neq \varnothing$ and $E \cap F = \varnothing$.*

(ii) *For all $X \to Y \in \Sigma$, if $X \subseteq E$ then $Y \cap F = \varnothing$.*

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(iii) *There exists* $X \to Y \in \Sigma_s$ *such that* $X = E$ *and* $F \subseteq Y$.

Proof. Let $\Sigma = \{X_i \to Y_i \mid 1 \leq i \leq n\}$ be a simplified implicational system and $A \to B, C \to D \in \Sigma$. If $\texttt{Add-sSimp}(A \to B, C \to D, \Sigma) = \{E \to F\}$ then $A \not\subseteq C$, $B \cap C \neq \varnothing$ and $D \not\subseteq A \cup B$. Let E_0 and F_0 be the sets defined in Line #1, i.e. $E_0 = A \cup (C \smallsetminus B)$ and $F_0 = D \smallsetminus (A \cup B)$. Moreover, since A and B are disjoint, one has $E_0 = (A \cup C) \smallsetminus B$.

Consider now $i_1, \ldots, i_m \in \{1, \ldots, n\}$ with subscript (in increasing ordering) such that $X_{i_j} \subseteq E_0 \smallsetminus \bigcup_{1 \leq k < j} Y_{i_k}$. Thus, Add-sSimp defines a sequence of implications $\{E_j \to F_j \mid 0 \leq j \leq m\}$ such that $E_m = E$, $F_m = F$ and, for each $0 \leq j \leq m$,

$$E_j = E_0 \smallsetminus \bigcup_{1 \leq k \leq j} Y_{i_k} \quad \text{and} \quad F_j = F_0 \smallsetminus \bigcup_{1 \leq k \leq j} Y_{i_k}.$$

Item (i) follows from Line #2 and the facts that $E_0 \cap F_0 = \varnothing$, $E = E_n \subseteq \cdots \subseteq E_0$ and $F = F_n \subseteq \cdots \subseteq F_0$.

Consider now $X \to Y \in \Sigma$ such that $X \subseteq E$. Then, there exists $j \in \{i, \ldots, m\}$ with $X = X_{i_j} \subseteq E \subseteq E_j$ and $Y = Y_{i_j}$. Therefore, $Y \cap F = Y_{i_j} \cap F_m = \varnothing$, i.e. item (ii) is proved.

Assume now $\Sigma_s = \texttt{Simplify}(\Sigma \cup \{E \to F\})$. From the facts that Σ is a simplified implicational system and items (i) and (ii) hold, we have that Simplify only can apply (Co-Eq) or (Si-Eq) over implications belonging to

$$\{E \to F\} \cup \{C \to D \in \Sigma \mid E \subseteq C\}.$$

Therefore, necessarily, item (iii) holds. □

As a consequence of the previous theorem, given a simplified implicational system, if Add-sSimp adds a new implication and then we simplify it, the new implication remains (probably with a bigger conclusion). Now, we have all the previous necessary results to introduce the new algorithm for computing the direct-optimal basis.

The following theorem ensures that SLgetdo transforms any implicational system to their equivalent direct-optimal basis.

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Algorithm 3.2: `SLgetdo`(Σ)

```
input  : An implicational system Σ.
output: The equivalent direct-optimal basis Σ_do.
begin
    Σ := Simplify(Σ)
    repeat
        Σ_do := Σ;  Σ := ∅
        foreach A → B ∈ Σ_do do
            Γ := ∅
            foreach C → D ∈ Σ do
                if C ⊆ A ⊆ C ∪ D or A ⊆ C ⊆ A ∪ B then
                    A := A ∩ C; B := B ∪ D
                else
                    if A ⊊ C then
                        if D ⊄ B then  add C-B → D-B to Γ
                    else
                        if C ⊊ A then  A := A ∖ D; B := B ∖ D
                        Γ := Γ ∪ {C → D}∪
                              ∪ Add-sSimp(A → B, C → D, Σ)
                              ∪ Add-sSimp(C → D, A → B, Σ)
            if B = ∅ then Σ := Γ else  Σ := Γ ∪ {A → B}
    until Σ_do = Σ
    return Σ_do
```

Theorem 3.3.3. *Algorithm 3.2,* `SLgetdo`*, ends for any implicational system Σ and, if Σ_{do} is the output, then Σ_{do} is the unique direct-optimal basis that is equivalent to Σ.*

Proof. Algorithm 3.2 initializes the implicational system by applying the function `Simplify`, which always ends, as guaranteed by Theorem 3.1.4. The rest of the algorithm is similar to `Simplify` with the only difference that, when it is not possible to apply any equivalence from Theorem 1.3.14, `Add-sSimp` is used in order to include a new implication, which is inferred by `[sSimp]` and remains (probably with a bigger conclusion) in the implicational system. As a consequence of Theorems 2.2.5 and 3.3.2, the loop always ends and the reached fix-point obtained is an equivalent direct sim-

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plified implicational system. Finally, Theorem 3.3.1 ensures that it is the equivalent direct-optimal basis. □

3.3.1 The performance of `SLgetdo`

In this section, first, we show an example which clearly illustrates the different behavior of both methods, `doSimp` and `SLgetdo`, and, then, a complete experiment will be developed.

We have computed the direct-optimal basis corresponding to an implicational system whose cardinality and size are, respectively, 7 and 27. The size and cardinality of its corresponding direct-optimal basis are 78 and 15. This example visualizes how `SLgetdo` achieves a significant better management of resources than `doSimp` (see Figure 3.2). The embedding of `[sSimp]` in the `Simplify` carried out by `SLgetdo` avoids the extra increase of size and cardinality. As the data shows, it always maintains the resources below a maximum value of 17 in cardinality and 95 in size, whereas `doSimp` rises up to 29 in cardinality and 188 in size. This better management of the resources allows `SLgetdo` to deal with greater problems without overtaking the computer capabilities and, at the same time, to produce the final output faster than `doSimp`. Now, we explain in detail such example:

Example 3.3.4. *Let Σ be the following implicational system over the set of attributes $M = \{a, b, c, d, e, f, g, h, i, j\}$:*

$$\Sigma = \{a \to bc, b \to def, aj \to hi, ci \to j, dh \to ae, fg \to be, bei \to gh\}$$

In this example, we compare the algorithms `doSimp` *and* `SLgetdo`*. We have executed both methods over Σ providing the following direct-optimal basis with 15 implications and 78 in size:*

$$\begin{aligned}\Sigma_{do} \;=\; & \{a \to bcdef, b \to def, ai \to ghj, aj \to ghi, bi \to acghj, bh \to ac,\\ & ci \to j, dh \to abcef, fg \to bde, bhj \to gi, dhi \to gj, dhj \to gi,\\ & fgh \to ac, fgi \to achj, fghj \to i\}\end{aligned}$$

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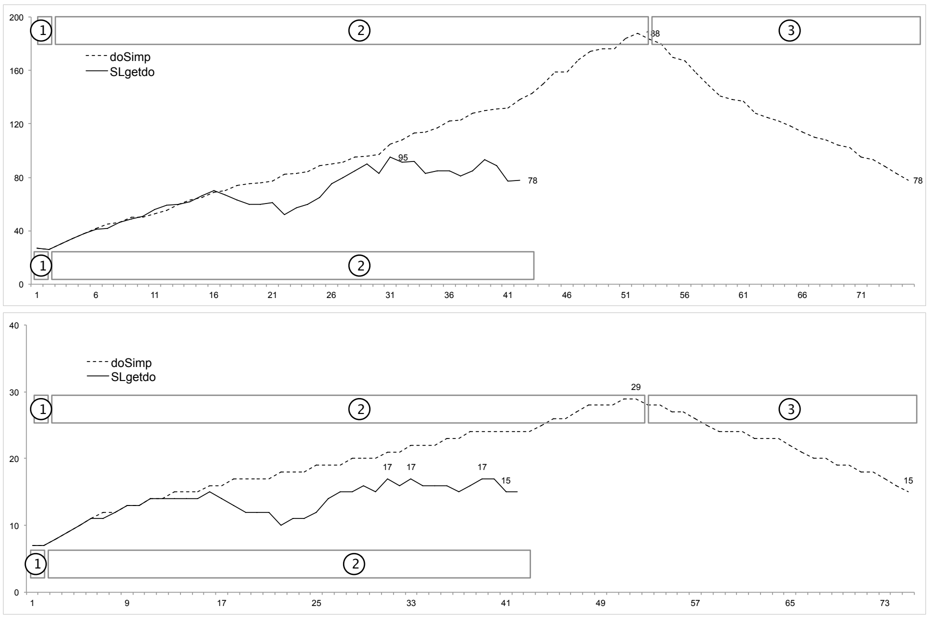

Figure 3.2: Visualization of the behavior of size and cardinality parameters (respectively) in the computation of the direct-optimal basis executing doSimp and SLgetdo methods.

As explained in this section, doSimp is composed of 3 stages whereas SLgetdo has only two stages. In Figure 3.2, we have outlined the corresponding stages of doSimp (at the top of the charts) and SLgetdo (at the botton of the charts). The first stage is the same in both algorithms. Thus, at the end of this first stage, both algorithms produce the same implicational system. In this example, only one attribute has been removed from the original Σ.

The second stage shows a very different behavior of both methods, with a greater increase (almost twice) in size and cardinality of doSimp. In its second stage, this method generates all the necessary implications to ensure directness (remarked in the top of the peak of the two charts). On the contrary, SLgetdo empowers simplification paradigm in its second stage. In this way, it limits the growth of the intermediate implicational system

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size and cardinality, as Figure 3.2 shows. The improved simplification also avoids the execution of a third stage. Such an stage constitutes a critical stage in `doSimp`, needed to clean superfluous information in order to return the direct-optimal basis.

In particular, in the second and third stages, `doSimp` applies 50 times the `[sSimp]` rule and 23 times the (`Si-Eq`) rule. In the second stage, the input implicational system is transformed from a 27-size and 7-cardinality set to a 188-size and 29-cardinality set. The third stage *simplifies* this large intermediate implicational system, returning the direct-optimal basis. Notice that the behavior of this method draws a curve with an obvious inverted V-shape (see Figure 3.2).

On the other hand, `SLgetdo` computes the direct-optimal basis by fusing the last two stages into a single one. It applies 26 times the `[sSimp]` rule and 14 times the (`Si-Eq`) rule in a interweaving way. As Figure 3.2 shows, the curve fluctuates controlling the size and cardinality of the intermediate implicational system and they do not exceed 95 and 17 respectively. The execution of `[sSimp]` implies an increase in the implicational basis size and cardinality, whereas the application of (`Si-Eq`) decreases these measurements. The smart way in which they are combined provides a cautious growth of the implication set during the direct-optimal basis computation.

Now, we focus on the development of an empirical experiment to test the performance of our method, `SLgetdo`. We only compare the `SLgetdo` and `doSimp` methods since, as was shown in Section 3.2.1, `doSimp` has the best performance among all methods existing in the literature to get the direct-optimal basis.

We have designed a test consisting of several randomly generated implication sets. In particular, we have generated 340 sets combining two parameters: the number of implications (varying from 5 to 30) and the number of attributes (varying from 5 to 25). In Table 3.3 we present some data to provide an overall view of the generated test. More specifically, for each value of the number of implications, we show the maximum and minimum sizes, its average size and the average of the density (number of implications divided by the size).

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Implications	Max Size	Min Size	Size (average)	Density (average)
10	65	50	58.9	0.171
15	282	207	244.6	0.062
20	481	107	266.16	0.091
25	588	439	518.15	0.048
30	697	161	405.16	0.092

Table 3.3: Experiment overview.

Function	`doSimp`	`SLgetdo`
Input size correlation	0.31	0.35
Input cardinality correlation	0.18	0.26
Attribute number correlation	0.32	0.32
Output Size correlation	0.45	0.55
Output cardinality correlation	0.46	0.51
Standard deviation	44 973.26	1 643.52
Average	12 388.80	449.09

Table 3.4: Basic statistical information.

Concerning the implementation details, we have also developed a Prolog prototype for the new method: `SLgetdo`. The experiment was run on an iMac computer with a 2.93GHz Intel Core i7 processor, 8GB of RAM, with Mac OS X Yosemite 64 bits.

Implications	`doSimp`	`SLgetdo`
10	1.20	1.40
15	59.75	31.60
20	3 473.40	131.57
25	20 927.95	556.55
30	3 189.56	168.32

Table 3.5: Execution time average (milliseconds).

In a preliminar step, we have studied the (possible) correlation between the execution time and, respectively, some input or output data. Regarding the input parameters, we have analyzed the correlation between the elapsed time and each of the following values: the input size, the input cardinality

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and the total number of attributes. On the other hand, we have also studied the possible correlation between time execution values and the size and the cardinality of the output direct-optimal basis. The correlation values are really low, which denotes that the random generation works quite well, providing different problems for the same parameters and returning a wide variability in the execution times of both algorithms. In Table 3.4 we show the correlation values together with some basic statistical data: standard deviation and time average of the results of each method. This table indicates that the output size of the set of implications fits with the complexity of the problem.

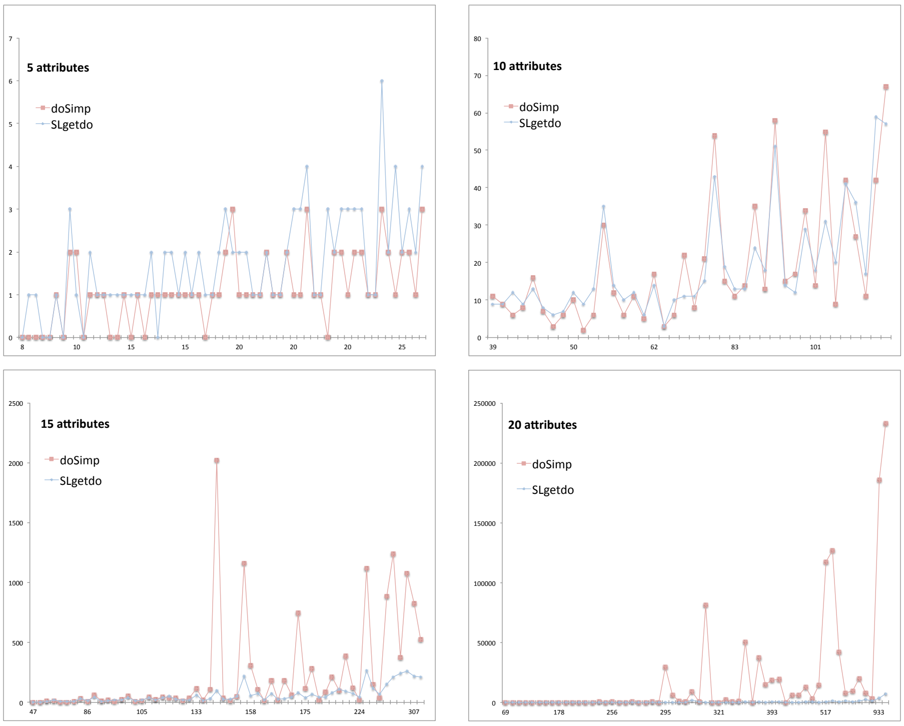

Figure 3.3: Comparison of execution times (milliseconds) of `doSimp` (red color) versus `SLgetdo` (blue color). We have ordered the X axis by the output size and grouped the experiments by the number of attributes.

As a very general conclusion, the experiment establishes a better perfor-

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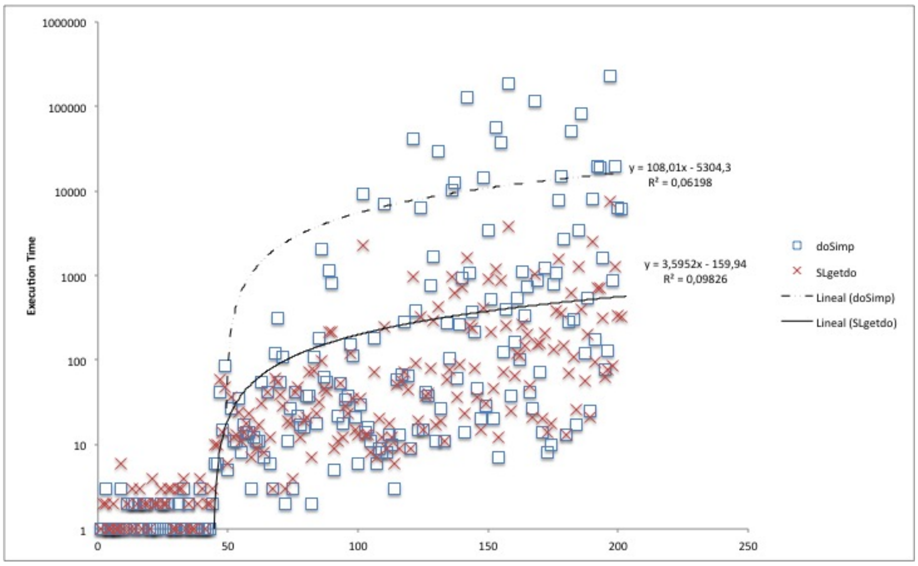

Figure 3.4: Execution times of randomly generated implicational systems. The blue square indicates the execution time with `doSimp` and the red cross with `SLgetdo`.

mance of `SLgetdo`, as the average execution times shows (499.09 ms versus 12 388.8 ms). To get a more detailed view, we have grouped the data by the number of implications of the input (see Table 3.5). We may ensure that the complete integration of the Simplification paradigm in the direct-optimal basis search produces a great benefit. More specifically, apart from the *tiny* problems built just with 5 attributes, in all the other scenes `SLgetdo` beats `doSimp` (see Figure 3.3). This figure shows that the greater the number of attributes and the complexity of the problem (provided by the output size) are, the better the behavior of the new method is. In the experiments with a few number of attributes, our method does not show its advantages because such a few number naturally limits the intermediate implicational system size and cardinality. Consequently, the execution is overloaded by the check in the (possible) use of the (`Si-Eq`), without a significant benefit.

In most cases, `SLgetdo` has a better behavior than `doSimp`, as Figure 3.4 illustrates. Thus, this figure shows how `SLgetdo` remains in the lower part

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of the chart whereas the other method remains in the top. We would like to remark that we have used a vertical axis with logarithmic scale, due to the huge differences between the values returned by both methods. In addition, we have also included the tendency lines corresponding to the two methods, showing that `SLgetdo` has a better tendency.

At the same time, in a significant number of cases, `doSimp` get the solution in a rather longer execution time than `SLgetdo`. Figure 3.5 shows the difference between execution times of `SLgetdo` and `doSimp`. The cases where `doSimp` beats `SLgetdo` (in red color) are grouped in the left part of the horizontal axis, corresponding to problems with less complexity. Moreover, in these cases the execution times are very small (less than 10 milliseconds). Only in 7 experiments these values are greater than this bound, and the greatest one is 89 ms. Regarding the cases where `SLgetdo` wins, they are in the large-scale execution times: hundreds and, even, thousands of milliseconds (the greatest value is 373 841 ms).

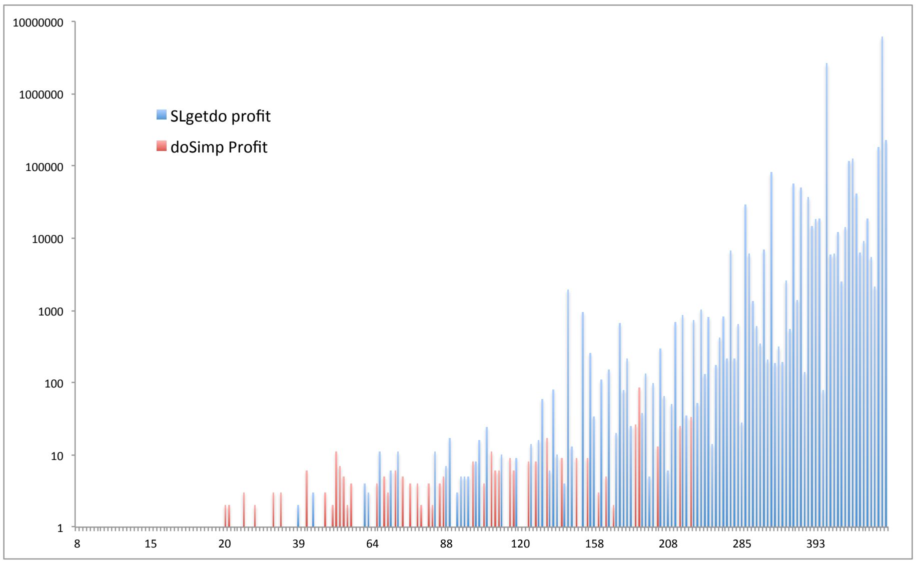

Figure 3.5: Difference of execution time: `SLgetdo` vs `doSimp`. Blue lines represent a better behavior of `SLgetdo` and the red ones the better behavior of `doSimp`.

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To sum up, this experiment has proved the better behavior of the new method proposed. As far as we know, at this time, it is the fastest algorithm to compute the direct-optimal basis.

Again, we show that Simplification Logic is the central tool to obtain new efficient methods to deal with implications.

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Chapter 4

Computing D-bases

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Looking for effective ways to reduce the size of the direct-optimal basis, we focus on the D-basis due to the good balance between the compact representation and the efficiency in obtaining closures. In [2] the authors proposed an algorithm to compute the D-basis taking as input a formal context. However, it remained an open problem to develop an algorithm to generate the D-basis from an arbitrary implicational system. This is the aim of this chapter.

The results presented in this chapter are the culmination of a collaboration with Kira Adaricheva and have been published in [46, 47].

Throughout this chapter we assume that there is no implication with the empty set as premise and there is no pair of attributes whose closures are the same (see Remark 2.3.3) following Adaricheva's approach.

4.1 Covers and Generators: Relationships

In this section, we study the relationships between covers and generators. These links will be used in this chapter as the kernel of the new algorithm to compute the D-basis. The definition of D-basis is strongly based on the notion of a proper minimal cover, and the latter is connected with the notion of a minimal generator.

Recall that the D-basis is defined as a pair of two sets where the first

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one consists of implications where the premises and the conclusions are singletons (atomic implications) and the second one consists of implications where the premise is a proper cover of the conclusion. In order to compute the D-basis in a uniform way, we generalize the notion of proper cover (Definition 2.3.7). This generalization allows us to compute the D-basis without separating the implications where the premise is a singleton from the second ones.

Definition 4.1.1. *Let Σ be an implicational system on M. A set $X \subseteq M$ is said to be a* cover *of an attribute $x \in M$ with respect to Σ if $x \in X^+_\Sigma \smallsetminus X$. It is denoted by $x \sim_\Sigma X$.*

The following proposition illustrates the relationships among proper covers, covers and minimal generators.

Proposition 4.1.2. *Let Σ be an implicational system on M and $C \subseteq M$ be a closed set with respect to Σ. For each $X \subsetneq C$, one has*

(i) *X is a generator of C if and only if $x \sim_\Sigma X$ for all $x \in C \smallsetminus X$.*

(ii) *$x \dot{\sim}_\Sigma X$ implies $x \sim_\Sigma X$ for all $x \in M$.*

Proof. (i) If X is a generator of C, then $C \smallsetminus X = X^+_\Sigma \smallsetminus X$ and, trivially $x \sim_\Sigma X$ for all $x \in C \smallsetminus X$.

Conversely, if $x \sim_\Sigma X$ for all $x \in C \smallsetminus X$, then $C \smallsetminus X \subseteq X^+_\Sigma \smallsetminus X$ and, therefore, $C \subseteq X^+_\Sigma$. Finally, since C is closed w.r.t. Σ and $X \subsetneq C$, one has $C = X^+_\Sigma$, i.e. X is a generator of C.

(ii) It is straightforward because $X \subseteq X^*_\Sigma$ and, then $X^+_\Sigma \smallsetminus X^*_\Sigma \subseteq X^+_\Sigma \smallsetminus X$. □

The second item in the previous proposition ensures that any proper cover is a cover. Notice that the converse result is not true as the following example shows.

Example 4.1.3. *Consider $M = \{a, b, c, d\}$ and $\Sigma = \{a \to b, c \to bd, bd \to ac\}$, we have that $\{a, d\}$ is a cover of b with respect to Σ, $b \sim_\Sigma \{a, d\}$,*

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Attributes	Covers
a	$\{c,d\}$, $\{b,c,d\}$, $\{b,c,e\}$, $\{b,d,e\}$, $\{c,d,e\}$, $\{b,c,d,e\}$
b	$\{a,c\}$, $\{a,e\}$, $\{c,d\}$, $\{a,c,d\}$, $\{a,c,e\}$, $\{a,d,e\}$, $\{c,d,e\}$, $\{a,c,d,e\}$
c	$\{a,b\}$, $\{a,e\}$, $\{a,b,d\}$, $\{a,b,e\}$, $\{a,d,e\}$, $\{b,d,e\}$, $\{a,b,d,e\}$
d	$\{a\}$, $\{a,b\}$, $\{a,c\}$, $\{a,e\}$, $\{a,b,c\}$, $\{a,b,e\}$, $\{a,c,e\}$, $\{b,c,e\}$, $\{a,b,c,e\}$
e	$\{a,b\}$, $\{a,c\}$, $\{c,d\}$, $\{a,b,c\}$, $\{a,b,d\}$, $\{a,c,d\}$, $\{b,c,d\}$, $\{a,b,c,d\}$

Closed Sets	Minimal Generators
$\{a,b,c,d,e\}$	$\{a,b\},\{a,c\},\{a,e\}$, $\{b,c,e\},\{b,d,e\},\{c,d\}$
$\{a,d\}$	$\{a\}$

Table 4.1: Covers, Proper Covers and Minimal Generators.

because $b \in \{a,d\}^+_\Sigma \smallsetminus \{a,d\} = \{b,c\}$. *However, it is not a proper cover because* $b \in a^+_\Sigma \subseteq \{a,d\}^*_\Sigma$.

As a direct consequence of the previous proposition, we have also that, for each closed set $C \subseteq M$ and each $X \subsetneq C$, if X is a minimal generator of C then X is a cover of any $x \in C \smallsetminus X$. The following example illustrates these relationships and gives a counterexample to the converse statement.

Example 4.1.4. *Consider* $M = \{a,b,c,d,e\}$ *and*

$$\Sigma = \{a \to d, bce \to ad, bde \to ac, ade \to bc, cd \to abe, abd \to ce\}.$$

In Table 4.1, the covers of each element of M *and the non-trivial minimal generators of each closed set are shown. Moreover, blue covers are not proper covers whereas the others are.*

Notice that, by Proposition 4.1.2, X *is a cover of an element* x *if and only if there exists a closed set* C *such that* X *is a generator of* C *and* $x \in C \smallsetminus X$. *Thus, there exists a minimal generator* Y *of* C *such that* $Y \subseteq X$. *For example,* $\{c,d\}$ *is a minimal generator of* $\{a,b,c,d,e\}$. *In order to give a counterexample, consider* $\{c,d,e\}$, *which is a cover of* a, *but not a minimal generator because* $\{c,d\} \subseteq \{c,d,e\}$.

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To introduce the notion of minimal cover, the closure operator $(-)^*_\Sigma$ will be used in order to avoid redundancies in the implications of the basis.

Definition 4.1.5. *Let Σ be a set of implications on M. Given $X \subseteq M$ and $x \in M$, X is a* minimal cover *of x if $x \sim_\Sigma X$ and one of the following conditions hold:*

(i) *X is a singleton.*

(ii) *For all $Y \subseteq X^*_\Sigma$, if $x \sim_\Sigma Y$ then $X \subseteq Y$.*

The following example illustrates the definition of minimal covers.

Example 4.1.6. *For the data of Example 4.1.4, the minimal covers are shown in Table 4.2 (cf. Table 4.1).*

Attribute	Minimal Covers
a	$\{c, d\}, \{b, c, e\}, \{b, d, e\}$
b	$\{a, e\}, \{c, d\}$
c	$\{a, b\}, \{a, e\}, \{b, d, e\}$
d	$\{a\}, \{b, c, e\}$
e	$\{a, b\}, \{c, d\}$

Table 4.2: Minimal Covers.

The following proposition relates minimal generators and minimal covers.

Proposition 4.1.7. *Let Σ be a set of implications on M. For any $X \subseteq M$ and $x \in M$, if X is a minimal cover of x, then $X \neq X^+_\Sigma$ and X is a minimal generator of X^+_Σ.*

Proof. Let us assume that X is a minimal cover of x. By Proposition 4.1.2, X is a generator of $X^+_\Sigma \neq X$ and we prove that it is minimal generator. If $|X| = 1$, since $\varnothing^+_\Sigma = \varnothing$ (see Remark 2.3.3) we have X is a minimal generator. Otherwise, if $|X| > 1$, consider any $\varnothing \neq Y \subsetneq X$. Then,

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$Y_\Sigma^+ \subseteq X_\Sigma^+$. Since X is minimal cover of x, we have $x \not\to_\Sigma Y$ (see item (ii) in Definition 4.1.5) i.e. $x \notin Y_\Sigma^+ \smallsetminus Y$. Since $Y \subsetneq X$ and $x \in X_\Sigma^+ \smallsetminus X$, we have $Y_\Sigma^+ \subsetneq X_\Sigma^+$. Therefore, X is a minimal generator of X_Σ^+. □

Therefore, any minimal cover is a non-trivial minimal generator, but the converse result does not hold. For instance, in Example 4.1.6, although $\{a, e\}$ is a minimal generator of $\{a, b, c, d, e\}$, it is not a minimal cover for $d \in \{a, b, c, d, e\}_\Sigma^+ \smallsetminus \{a, e\} = \{b, c, d\}$.

4.2 D-basis by means of Minimal Generators

As stated, the design of an algorithm for computing the D-basis from an arbitrary implicational system was an open problem. In this section, we will introduce the algorithm presented in [46] and based on the strong connection between minimal covers and minimal generators discussed in the previous section.

Basically, the method proposed here has four stages for computing the minimal generators from the implicational system, the covers emanating from minimal generators, the minimal covers from the covers, and finally the D-basis, which is obtained from the minimal covers. This sequence of tasks is illustrated in Figure 4.1.

- In the first stage, the algorithm computes all non-trivial closed sets with their minimal generators by means of MinGen_0, proposed in [17].

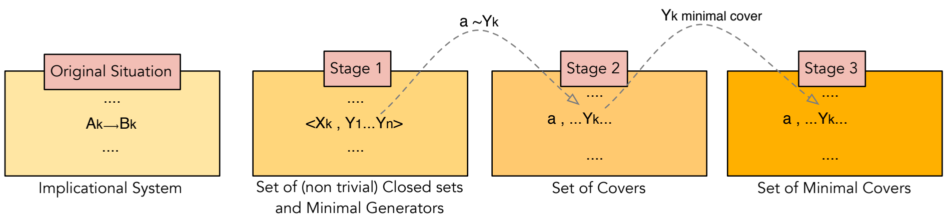

Figure 4.1: Stages of D-basis method described in Algorithm 4.1.

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This algorithm is strongly based on the computation of closures by using Simplification Logic (see Section 1.3.1).

- In the second stage, it associates each minimal generator with all elements for which it is a cover. In this stage, we have calculated the set of covers, which contains all the minimal covers for a given attribute.

- In the third stage, from the set of minimal covers we obtain all minimal covers for each attribute, by means of `MinimalCovers`.

- Finally, the algorithm ends with the fourth stage where the D-basis is computed by applying `OrderedComp`. The D-basis preserves the order proposed in [1].

Algorithm 4.1 outlines this method. As Figure 4.1 shows, the transition from stage 1 to stage 2 needs a way to associate the minimal generators with some of the elements in its closed set. Thus, the relationship between the attribute a and the set of minimal generators whose closure contains a, is denoted as the pair $\langle a, mg_a \rangle$.

Algorithm 4.1: ***D*-basis**

input : An implicational system Σ
output: The D-basis $\langle \Sigma_{\mathrm{a}}, \Sigma_{\mathrm{n}} \rangle$ on M
begin
  MG:=$\texttt{MinGen}_0(M, \Sigma)$
  $\Phi := \varnothing$
  foreach $\langle C, mg(C) \rangle \in$ MG **do**
    foreach $a \in C$ **do** Φ:=`Gather`($\langle a, mg(C) \rangle, \Phi$)
  $\Sigma_D := \varnothing$
  foreach $\langle a, mg_a \rangle \in \Phi$ **do**
    mg_a :=`MinimalCovers`(mg_a)
    foreach $g \in mg_a$ **do** $\Sigma_D := \Sigma_D \cup \{g \to a\}$
  $\langle \Sigma_{\mathrm{a}}, \Sigma_{\mathrm{n}} \rangle$:=`OrderedComp`(Σ_D)
  return $\langle \Sigma_{\mathrm{a}}, \Sigma_{\mathrm{n}} \rangle$

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To carry out this transition, we need a new operator, `Gather`, which builds the set of covers produced in stage 2 as follows:

Let Φ be a set of such pairs of attributes with their covers.

$$\texttt{Gather}(\langle a, mg\rangle, \Phi) = \{\langle a, \{g \in mg | a \notin g\} \cup \{mg_a\}\rangle \mid \langle a, mg_a\rangle \in \Phi\}$$

Function MinimalCovers(L)

input : A set of covers L
output: The set of minimal covers
begin
    foreach $g \in L$ **do**
        if $|g| > 1$ **then**
            foreach $h \in L \setminus \{g\}$ **do**
                if $h \subseteq g$ **then** remove g from L
                else if $h \subseteq g^*_\Sigma$ **then** remove g from L
    return L

We emphasize that `MinimalCovers` removes those covers which are not minimal, that is, when one of them, h, is included in another, g, or h is included in the atomic closure of g. Notice that in this function, although the condition $h \subseteq g$ implies $h \subseteq g^*_\Sigma$ and both of them lead to the same action, it is more efficient to split them off because the cost of the first one is lower. Thus, we previously check the first one and, if it is not fulfilled, then the other one is tested.

The method ends with `OrderedComp` which applies the Composition Rule and, simultaneously, orders the implications in the following sense: the first implications in the D-basis are the atomic ones (those with the left-hand side being a singleton).

In the following, we present the trace of the execution of the proposed method in order to illustrate (stage by stage) how the algorithm works.

A detailed illustrative example

Here, we show how the method computes the D-basis. Let $M = \{1, 2, 3, 4, 5\}$ be a set of attributes and Σ the following set of implications from [11], which

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Function OrderedComp(Σ)

```
input  : A set of unitary implications Σ
output: A pair of implicational systems whose union is equivalent to Σ
begin
    Σa := ∅
    Σn := ∅
    foreach X → y ∈ Σ do
        remove X → y from Σ
        Z := {y}
        foreach V → w ∈ Σ do
            if X = V then Z := Z ∪ {w} and remove V → w from Σ
        if |X| = 1 then Σa := Σa ∪ {X → Z} else Σn := Σn ∪ {X → Z}
    return ⟨Σa, Σn⟩
```

was also used later to illustrate the D-basis definition in [3].

$$\Sigma = \{5 \to 4, 23 \to 4, 24 \to 3, 34 \to 2, 14 \to 235,$$
$$25 \to 134, 35 \to 124, 15 \to 24, 123 \to 45\}$$

In the first step the algorithm $\texttt{MinGen}_0$ returns the following set of pairs of closed sets and their non-trivial minimal generators (see Figure 4.2):

$$\texttt{MinGen}_0(\{1,2,3,4,5\},\Sigma) =$$
$$\{\langle 12345, \{123, 14, 15, 25, 35\}\rangle, \langle 234, \{23, 24, 34\}\rangle, \langle 45, \{5\}\rangle, \langle \varnothing, \varnothing\rangle\}$$

In the second step of the algorithm, the operator `Gather` returns the following set of pairs of atomic attributes with their corresponding covers:

$$\Phi = \{\langle 1, \{25, 35\}\rangle, \langle 2, \{14, 15, 35, 34\}\rangle, \langle 3, \{14, 15, 25, 24\}\rangle,$$
$$\langle 4, \{123, 15, 25, 35, 5, 23\}\rangle, \langle 5, \{123, 14\}\rangle\}$$

Then, for each element of the above list, `MinimalCovers` picks up its minimal covers:

$$\{\langle 1, \{25, 35\}\rangle, \langle 2, \{14, 34\}\rangle, \langle 3, \{14, 24\}\rangle, \langle 4, \{5, 23\}\rangle, \langle 5, \{14, 123\}\rangle\}$$

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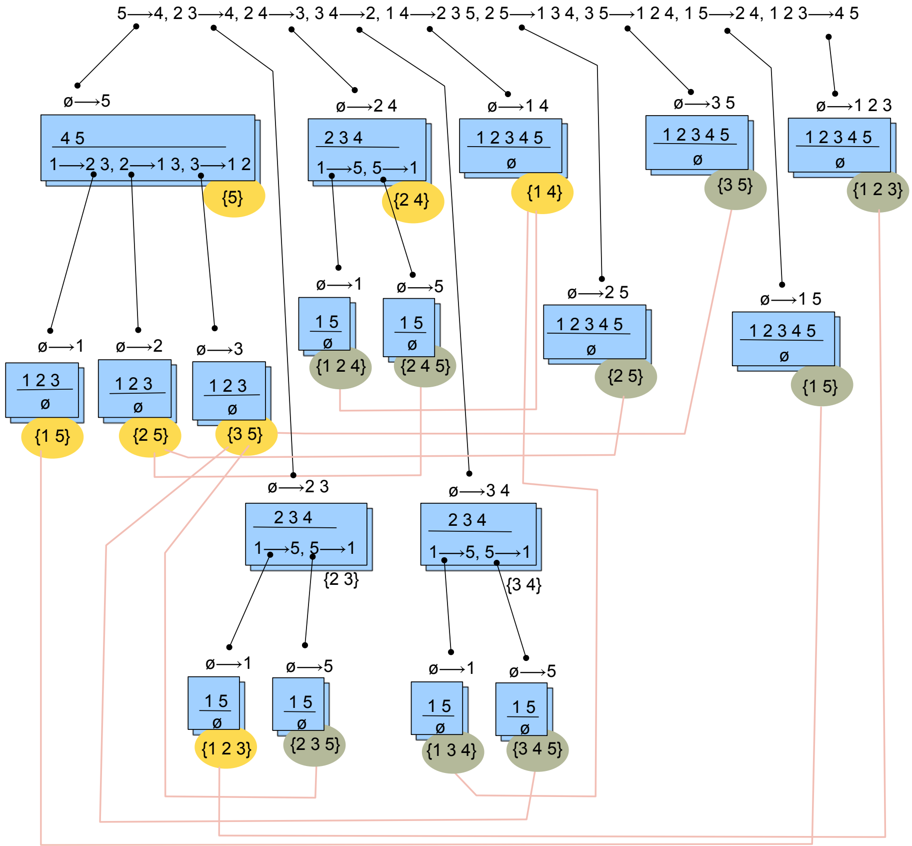

Figure 4.2: Applying $\texttt{MinGen}_0$ to the implicational system from [3].

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Finally, at the last step, the algorithm turns these pairs into implications and applies ordered composition resulting in the D-basis.

$$\langle \Sigma_{\rm a}, \Sigma_{\rm n}\rangle =$$
$$\langle \{5 \to 4\}, \{23 \to 4, 24 \to 3, 34 \to 2, 14 \to 235, 25 \to 1, 35 \to 1, 123 \to 5\}\rangle$$

We emphasize again that, with this algorithm, we have solved an open problem in the literature related to the computation of the D-basis: to directly obtain it from any implicational system. The next challenge is to approach the integration of the computation of the minimal generators and minimal covers, exploiting the theoretical relationships studied between them for searching a more efficient method.

4.3 Fast$D$-basis Algorithm

In this section, we propose an improved algorithm to compute the D-basis from an arbitrary set of implications Σ. This method has two features inspired by Simplification Logic: it traverses the whole set of implications in a uniform way regardless the cardinality of the premises and it does not impose the use of the unitary implicational systems while the new implications are generated.

The algorithm interweaves the selection of the minimal covers in each step rather than in a final step. This idea improves the performance of the method presented in the previous section in two ways: it avoids opening some branches in the search space –by using a minimality test– and it discards some minimal generators during the execution.

This interweaving would imply to check whether a minimal generator does not constitute a minimal cover a lot of times, potentially compromising the efficiency of this approach. In order to avoid this test producing a repetitive computation of closures, we store each implication in a triplet which is defined as follows.

Definition 4.3.1. *Consider a non-empty set M, a subset of $\underline{2}^M \times \underline{2}^M \times \underline{2}^M$, where $\underline{2}^M = 2^M \smallsetminus \{\varnothing\}$, is said to be a* triplet set *on M.*

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In addition, given a triplet set Γ *on* M*, one triplet* $\langle A, B, C\rangle \in \Gamma$ *is said to be* minimal *for* Γ *when* $X \subseteq A$ *implies* $X = A$ *for all* $\langle X, Y, Z\rangle \in \Gamma$.

The main goal in the introduction of the triplets is to compute and collect, in a lazy way, all information needed in the algorithm related to the closures in two separate components: $(-)^+_\Sigma$ and $(-)^*_\Sigma$. Therefore, the set of triplets Γ will be obtained from a given implicational system Σ and all the changes applied to Γ will preserve the spirit of its relation with the closures. Thus, we define:[1]

- $\Gamma_c := \{\, A \to BC \mid \langle A, B, C\rangle \in \Gamma \,\}$ and $X^+_\Gamma := X^+_{\Gamma_c}$ for all $X \subseteq M$.
- $\Gamma_a := \{\, A \to C \mid \langle A, B, C\rangle \in \Gamma \,\}$ and $\pi^*_\Gamma(X) := \pi_{\Gamma_a}(X)$ for all $X \subseteq M$.

Although the properties of the set of triplets, at each stage of the algorithm, will be described step by step, we now advance that the goal is to tend (lazily) to accomplish the following:

- $\Gamma_c \equiv \Sigma$ and, therefore, $X^+_\Gamma = X^+_{\Gamma_c} = X^+_\Sigma$ for all $X \subseteq M$.
- $\pi^*_\Gamma(X) = \pi_{\Gamma_a}(X) = X^*_\Sigma$ for all $X \subseteq M$.

And, in a final stage, for all $\langle A, B, C\rangle \in \Gamma$, if $|A| = 1$, $C = A^+_\Sigma = A^*_\Sigma$, $B = C \smallsetminus A$ and, otherwise, $C = A^*_\Sigma$ and $B = A^+_\Sigma \smallsetminus C$.

Now, we describe the Algorithm `Fast`D`-basis`, which can be divided in the following stages:

- In the first stage of the algorithm, we transform the set Σ of implications into a set Γ of triplets, which constitutes the main input of all further routines of our method (label S1 in the algorithm).
- In the second stage, it computes all the minimal covers for each element of M by means of a recursive function (label S2 in the algorithm).

[1]The subscripts c and a come from *complete* and *atomic* respectively.

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Algorithm 4.2: Fast$D$-basis

Data: A set of implications Σ
Result: The D-basis equivalent to Σ
begin
S1 $\Gamma := \{\langle X, Y \smallsetminus X, X\rangle \mid X \to Y \in \Sigma \text{ with } Y \not\subseteq X\}$
S2 $\Phi :=$ `MinCovers`$(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$
S3 $\Sigma_{\mathrm{a}} := \{A \to B \mid \langle A, B, C\rangle \in \Phi \text{ with } |A| = 1\}$
S4 $\Sigma_{\mathrm{n}} := \{A \to B \mid \langle A, B, C\rangle \in \Phi \text{ with } |A| \neq 1\}$
return $\langle\Sigma_{\mathrm{a}},\Sigma_{\mathrm{n}}\rangle$

- Finally, it turns the set of triplets into a pair of implication sets according to the cardinality of the first component of the triplet: atomic and non-binary implications (labels S3, S4 in the algorithm). As a preliminary remark, we emphasize that this split is due to the induced order for the D-basis definition.

The following example illustrates the transformation of the input into a set of triplets.

Example 4.3.2. *Consider $M = \{1,2,3,4,5,6\}$ and Σ the following implicational system on M:*

$$\Sigma = \{2 \to 1, 3 \to 6, 5 \to 3, 13 \to 24, 14 \to 5, \\ 34 \to 12, 45 \to 13, 125 \to 34, 234 \to 5, 235 \to 1\}$$

In the first stage, the algorithm returns:

$$\Gamma = \{\langle 2,1,2\rangle, \langle 3,6,3\rangle, \langle 5,3,5\rangle, \langle 13,24,13\rangle, \langle 14,5,14\rangle, \langle 34,12,34\rangle, \\ \langle 45,13,45\rangle, \langle 125,34,125\rangle, \langle 234,5,234\rangle, \langle 235,1,235\rangle\}$$

The main routine in the algorithm is `MinCovers`. Firstly, we will outline it at high level and, then, we will go into the details. Basically, the steps of `MinCovers` are the following:

- In the first step (label M1), `Fix` returns a pair with a set of minimal elements and Γ modified in the following way. On the one hand, the

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Function MinCovers($\langle A, B, C\rangle$,Γ)

input : A triplet $\langle A, B, C\rangle$, and a set of triplets Γ
output: A set of triplets $\langle X, Y, Z\rangle$ where X is minimal cover for all $y \in Y$ with respect to Γ

begin
M1 $\quad\langle \mathrm{Mnl}, \Gamma\rangle :=$ `Fix`$(\langle A, B, C\rangle, \Gamma)$
$\quad\Phi := \varnothing$
$\quad$**foreach** $\langle X, Y, Z\rangle \in \Gamma$ *with* $X \in \mathrm{Mnl}$ **do**
M2 $\quad\quad\Psi := \{\langle X, Y, Z\rangle\} \cup$ `MinCovers`$(\langle X, Y, Z\rangle, \Gamma)$
M3 $\quad\quad\Phi :=$ `Join`(Φ, Ψ)
$\quad$**return** Φ

first component of the triplets in Γ will contain A and no component of the triplets in Γ will have elements of B. So, the closure operator is defined in $2^{M \smallsetminus B}$. It is due to the fact that the algorithm builds a tree (see Figure 4.5), so in each branch we work with less and less attributes which guarantees that the algorithm ends. On the other hand, it also returns the minimal elements of this output set.

- For each minimal triplet, `MinCovers` is recursively called (label M2) up to Γ is empty, that is, when the branch of the tree has been explored in depth.

- In the next step, `Join` (label M3) collects in the backtracking all the covers computed in the branches of the tree, returning only the minimal ones.

In the following, we provide the necessary results to prove the soundness and completeness of the method and, in a parallel way, we introduce in more detail the behavior of the rest of the used functions.

We begin with `AddClosure` because of the remarkable role that it plays in the building of the closures. This function is called inside `Fix`, which will be used in the recursive calls to `MinCovers`. The following Proposition 4.3.3 characterizes the set of triplets returned by `AddClosure`.

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Proposition 4.3.3. *Let Γ and Δ be triplet sets. If Δ is the output of* `AddClosure`(A, Γ) *and $\langle A, B, C\rangle$ is minimal for Γ, then $\Gamma_c \equiv \Delta_c$ (therefore $A^+_\Gamma = A^+_\Delta$) and the following conditions hold:*

(i) *If $\langle X, Y, Z\rangle \in \Gamma$ with $A \not\subseteq X$, then $\langle X, Y, Z\rangle \in \Delta$.*

(ii) *If $|A| = 1$, then $\langle A, B', C'\rangle$ is minimal for Δ where*

$$B' = A^+_\Gamma \smallsetminus A \quad \text{and} \quad C' = A^+_\Gamma = \pi^*_\Delta(A) = \pi^*_\Gamma(A).$$

(iii) *If $|A| > 1$, then $\langle A, B', C'\rangle$ is minimal for Δ where*

$$B' = A^+_\Gamma \smallsetminus C' \quad \text{and} \quad C' = \pi^*_\Gamma(A) = \pi^*_\Delta(A).$$

(iv) *If $\langle X, Y, Z\rangle \in \Gamma$ with $A \subsetneq X$ and $Y \not\subseteq A^+_\Gamma$, then $\langle X', Y', Z'\rangle \in \Delta$ where*

$$X' = (X \smallsetminus A^+_\Gamma) \cup A, \quad Y' = Y \smallsetminus A^+_\Gamma \quad \text{and} \quad Z' = Z \cup \pi^*_\Gamma(A).$$

(v) *If $\langle X, Y, Z\rangle$ is minimal for Δ, then either $\langle X, Y, Z\rangle$ is minimal for Γ and $X \neq A$, or it is one of those described in items* (ii)–(iii).

Proof. The equivalence $\Gamma_c \equiv \Delta_c$, and therefore the equality $A^+_\Gamma = A^+_\Delta$, is a consequence of the fact that `AddClosure` systematically applies the following equivalences: (`Co-Eq`) in line Ac1, (`rSi-Eq`) in line Ac2, (`Si-Eq`) in line Ac3 and (`Fr-Eq`) in line Ac5.

Those triplets $\langle X, Y, Z\rangle$ which are minimal for Γ, are only considered in line Ac4. Therefore, these triplets are not modified. Thus, since $A \not\subseteq X$, item (i) holds.

Since $\langle A, B, C\rangle$ is minimal for Γ, `AddClosure` removes from Γ the triplets $\langle X, Y, Z\rangle$ such that $X = A$ and also computes A^+_Γ and $\pi^*_\Gamma(A)$ (in lines Ac1 and Ac2, respectively). Thus, after line Ac4, there is no triplet $\langle X, Y, Z\rangle$ with $X \subseteq A$ in Γ.

On the one hand, if $|A| = 1$, this function appends $\langle A, A^+_\Gamma \smallsetminus A, A^+_\Gamma\rangle$ to the output Δ and the equality $\pi^*_\Delta(A) = A^+_\Gamma$ holds. On the other hand, if

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Function AddClosure(A, Γ)

input : A set of triplets Γ and a set of elements A, associated with a minimal triplet $\langle A, B, C\rangle \in \Gamma$

output: A triplet set fulfilling the conditions of Proposition 4.3.3

begin
  $B := A$; $C := A$
  repeat
    $B_{\text{old}} := B$; $\Gamma_{\text{new}} := \varnothing$
    foreach $\langle X, Y, Z\rangle \in \Gamma$ **do**
Ac1   **if** $A = X$ **then** $B := B \cup Y \cup Z$; $C := C \cup Z$
      else
Ac2   **if** $X \subseteq B$ **then** $B := B \cup Y \cup Z$
Ac3   **if** $A \subsetneq X$ **then**
          if $Y \not\subseteq B$ **then** add $\langle X, Y \smallsetminus B, Z \cup C\rangle$ to Γ_{new}
Ac4   **else** add $\langle X, Y, Z\rangle$ to Γ_{new}
      if $|A| = 1$ **then** $C := B$
    $\Gamma := \Gamma_{\text{new}}$
  until $B_{\text{old}} = B$
Ac5   **if** $|A| = 1$ **then** add $\langle A, B \smallsetminus A, B\rangle$ to Γ **else** add $\langle A, B \smallsetminus C, C\rangle$ to Γ
  return Γ

$|A| > 1$, the triplet $\langle A, A^{+}_{\Gamma} \smallsetminus \pi^{*}_{\Gamma}(A), \pi^{*}_{\Gamma}(A)\rangle$ is added to Δ and $\pi^{*}_{\Delta}(A) = \pi^{*}_{\Gamma}(A)$ (see line Ac5). Therefore, items (ii) and (iii) hold.

Finally, (iv) is a consequence of the fact that `AddClosure` function modifies only triplets $\langle X, Y, Z\rangle$ with $A \subseteq X$ and (v) is straightforward. □

As stated previously, `AddClosure` is used in `Fix`. The goal of `Fix` is to determine only the minimal elements which are necessary to compute the minimal covers needed for the D-basis (see `MinCovers`). In the same way that the previous one, we introduce the theoretical foundations and explain its behavior.

The following results describe the meaning of triplet sets that we use as the data structure and how `Fix` works. By means of them, we characterize the properties of its output, regarding minimality and closure.

Proposition 4.3.4. *Let* Γ *be a triplet set on* M*,* $\langle A, B, C\rangle \in \Gamma$ *and* $\langle \texttt{Mnl}, \Delta\rangle$

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Function Fix($\langle A,B,C\rangle$,Γ)

input : A triplet $\langle A,B,C\rangle$ and a set of triplets Γ
output: A triplet set fulfilling the conditions of Proposition 4.3.4
begin
  $\Gamma_{new} := \varnothing$
  `Minimals` $:= \varnothing$
  foreach $\langle X,Y,Z\rangle \in \Gamma$ **do**
F1 $X := A \cup (X \smallsetminus (B \cup C))$; $Y := Y \smallsetminus (B \cup C)$; $Z := C \cup (Z \smallsetminus B)$
    if $Y \neq \varnothing$ **then** add $\langle X,Y,Z\rangle$ to Γ_{new}
F2   **if** there is no $W \in$ `Minimals` such that $W \subseteq X$ **then**
      remove from `Minimals` all V with $X \subseteq V$
      add X to `Minimals`
  foreach $X \in$ `Minimals` **do** $\Gamma_{new} :=$ `AddClosure`(X,Γ_{new})
  return $\langle$`Minimals`$,\Gamma_{new}\rangle$

be the output of `Fix`$(\langle A,B,C\rangle,\Gamma)$. *Then* $X \in$ `Mnl` *if and only if there exist* $Y,Z \subseteq M$ *such that* $\langle X,Y,Z\rangle$ *is minimal for* Δ.

Proof. Consider the following triplet set (see Line F1)

$$\Theta = \{\langle A \cup (X \smallsetminus \hat{A}), Y \smallsetminus \hat{A}, C \cup (Z \smallsetminus B)\rangle \mid \langle X,Y,Z\rangle \in \Gamma, Y \not\subseteq \hat{A}\} \quad (4.1)$$

where $\hat{A} = B \cup C$. `Mnl` –computed in Line F2– is the family of sets $X \subseteq M$ such that there exist $U,V \subseteq M$ with $\langle X,U,V\rangle$ being minimal for Θ. From Proposition 4.3.3, this minimality for Θ holds if and only if there exist $Y,Z \subseteq M$ such that $\langle X,Y,Z\rangle$ is minimal for Δ. □

Now, we introduce a couple of definitions that are needed for characterizing the triplet sets involved in `Fast`D`-basis`.

Definition 4.3.5. *A triplet set* Γ *is said to be in* normal form *(NF) if, for all* $\langle U,V,W\rangle \in \Gamma$*, the following conditions hold:*

(i) $U \subseteq W \subseteq \pi^*_\Gamma(U)$.

(ii) *If* $|U| > 1$*, then* $V \subseteq U^+_\Gamma \smallsetminus \pi^*_\Gamma(U)$.

(iii) *If* $|U| = 1$*, then* $W = \pi^*_\Gamma(U)$ *and* $V = U^+_\Gamma \smallsetminus U$.

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(iv) *If* $\langle U, V, W\rangle$ *is minimal for* Γ *with* $|U| > 1$*, then* $W = \pi^*_\Gamma(U)$ *and* $V = U^+_\Gamma \smallsetminus \pi^*_\Gamma(U)$*.*

Definition 4.3.6. *Let* Σ *be an implicational system on* M*,* Γ *be a triplet set on* M *and* $A \subseteq M$*. The triplet set* Γ *is said to be* compatible *with* Σ *above* A *if* Γ *is in NF and the following conditions hold:*

(i) *For all* $\langle X, Y, Z\rangle \in \Gamma$*, one has* $A \subsetneq X$ *and* $X \cap A^+_\Sigma = A$*.*

(ii) *For all* $X \subseteq M$ *with* $A \subsetneq X$*, one has* $X^+_\Gamma \smallsetminus A^+_\Sigma = X^+_\Sigma \smallsetminus A^+_\Sigma$ *and* $\pi^*_\Gamma(X) \smallsetminus A^+_\Sigma = X^*_\Sigma \smallsetminus A^+_\Sigma$*.*

The following theorem ensures that the original call to `Fix` in the algorithm preserves the previous definition.

Theorem 4.3.7. *Let* Σ *be a complete implicational system on* M *and* $\Gamma = \{\langle X, Y \smallsetminus X, X\rangle \mid X \to Y \in \Sigma \text{ with } Y \not\subseteq X\}$*. If* $\langle \texttt{Mnl}, \Delta\rangle$ *is the output of* $\texttt{Fix}(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$*, then* Δ *is compatible with* Σ *above* $\varnothing$*.*

Proof. Proposition 4.3.3 ensures that Δ is in NF. From completeness of Σ, by (`Fr-Eq`), one has $X^+_\Sigma = X^+_\Gamma$ for all $X \subseteq M$. Since the first component in the input is $\langle\varnothing,\varnothing,\varnothing\rangle$, line F1 does not modify any triplet in Γ and, by Propositions 4.3.3 and 4.3.4, one has $\Delta_c \equiv \Gamma_c \equiv \Sigma$. Item (i) is consequence of the equality $\varnothing^+_\Sigma = \varnothing$ and (ii) is consequence of $\Delta_c \equiv \Sigma$. $\square$

Example 4.3.8. *Consider the implicational system* Σ *in Example 4.3.2 and the set* Γ *of triples returned in the first step of the algorithm, a call to* `Fix` *returns:*

$$\begin{aligned}\langle Mnl, \Delta\rangle &= \texttt{Fix}(\langle\varnothing,\varnothing,\varnothing\rangle),\Gamma)\\ &= \langle\{2,3,5,14\},\{\langle 2,1,12\rangle,\langle 3,6,36\rangle,\langle 5,36,356\rangle,\langle 13,24,136\rangle,\\ &\quad\ \langle 14,2356,14\rangle,\langle 25,4,12356\rangle,\langle 34,12,346\rangle,\langle 45,1,3456\rangle,\\ &\quad\ \langle 234,5,12346\rangle\}\rangle\end{aligned}$$

The triplet set Δ *produced by* $\texttt{Fix}(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$ *is compatible with* Σ *above* $\varnothing$*, as we will show.*

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On the one hand, we can ensure that Δ is in NF (Definition 4.3.5). The three first conditions are only a matter of contention. The last condition is also fulfilled since the only minimal triplet with cardinality greater than 1 is $\langle 14, 2356, 14\rangle$, and one has:

$$\{1,4\} = \pi^*_\Delta(\{1,4\}) \quad and \quad \{2,3,5,6\} = \{1,4\}^+_\Delta \smallsetminus \pi^*_\Delta(\{1,4\}) = M \smallsetminus \{1,4\}.$$

On the other hand, Δ is compatible with Σ above $\varnothing$ (Definition 4.3.6): since $\varnothing^+_\Sigma = \varnothing$, the first condition is trivial and, as an illustrative example, we show how the second one holds for an specific subset. For instance, for the subset $X = \{2,5\}$:

- $X^+_\Delta \smallsetminus \varnothing^+_\Sigma = X^+_\Sigma \smallsetminus \varnothing^+_\Sigma$ *because* $X^+_\Delta = X^+_\Sigma = M$.
- $\pi^*_\Gamma(X) \smallsetminus \varnothing^+_\Sigma = X^*_\Sigma \smallsetminus \varnothing^+_\Sigma$ *because* $\pi^*_\Delta(\{2,5\}) = X^*_\Sigma = \{1,2,3,5,6\}$.

These two conditions are also fulfilled by the rest of subsets X such that $\varnothing \subsetneq X$, concluding that Δ is compatible with Σ above $\varnothing$.

Taking recursion into account, we need to ensure a similar result in every further call of `Fix`. The following theorem plays this role:

Theorem 4.3.9. *Let Σ be an implicational system on M, $S \subseteq M$, Γ be a triplet set compatible with Σ above S, and $\langle A, B, C\rangle$ be a minimal triplet for Γ. If $\langle \mathtt{Mnl}, \Delta\rangle$ is the output of $\mathtt{Fix}(\langle A, B, C\rangle, \Gamma)$, then Δ is compatible with Σ above A.*

Proof. To check that Δ is in NF is just a matter of computation. Consider the triplet set Θ (4.1) described in the proof of Proposition 4.3.4. Since $S \subsetneq X$ for all $\langle X, Y, Z\rangle \in \Gamma$ and $\langle A, B, C\rangle$ is minimal for Γ, one has $S \subsetneq A$. In addition, $A \subsetneq X$ for all $\langle X, Y, Z\rangle \in \Theta$ and, therefore, it is true for all $\langle X, Y, Z\rangle \in \Delta$. Moreover, if we denote $\hat{A} = B \cup C$, then $\hat{A} = A^+_\Gamma \subseteq A^+_\Sigma \cup S^+_\Sigma = A^+_\Sigma$ because Γ is compatible with Σ above S. Thus, for all $\langle U, V, W\rangle \in \Gamma$, one has $A(U \smallsetminus \hat{A}) \cap A^+_\Sigma = A$.

Assume $U^+_\Gamma \smallsetminus S^+_\Sigma = U^+_\Sigma \smallsetminus S^+_\Sigma$ for all $U \supsetneq S$, and consider a particular U such that $A \subsetneq U$. From Proposition 4.3.3, since $U^+_\Delta = U^+_\Theta$, we prove that $U^+_\Theta \smallsetminus A^+_\Sigma = U^+_\Sigma \smallsetminus A^+_\Sigma$:

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(i) On the one hand, by applying [Simp] and [Frag], every implication $X \to YZ \in \Theta_c$ can be inferred from Γ_c. Therefore, $U_\Theta^+ \subseteq U_\Gamma^+ \subseteq U_\Sigma^+ \cup S_\Sigma^+ \subseteq U_\Sigma^+$ and $U_\Theta^+ \smallsetminus A_\Sigma^+ \subseteq U_\Sigma^+ \smallsetminus A_\Sigma^+$.

(ii) On the other hand, first we prove $U_\Gamma^+ \subseteq U_\Theta^+ \cup B$, i.e. $U \subseteq U_\Theta^+ \cup B$ and $U_\Theta^+ \cup B$ is closed with respect to Γ_c: for all $\langle X, Y, Z\rangle \in \Gamma$, if $X \subseteq U_\Theta^+ \cup B$, then $A \cup (X \smallsetminus \hat{A}) \subseteq U_\Theta^+$, $(Y \smallsetminus \hat{A}) \cup \big(C \cup (Z \smallsetminus B)\big) \subseteq U_\Theta^+$, and $Y \cup Z \subseteq U_\Theta^+ \cup B$.

Finally, $U_\Sigma^+ \subseteq U_\Gamma^+ \cup S_\Sigma^+ \subseteq U_\Theta^+ \cup B \cup S_\Sigma^+ \subseteq U_\Theta^+ \cup A_\Sigma^+$ and, therefore, $U_\Sigma^+ \smallsetminus A_\Sigma^+ \subseteq U_\Theta^+ \smallsetminus A_\Sigma^+$.

Assume now $\pi_\Gamma^*(U) \smallsetminus S_\Sigma^+ = U_\Sigma^* \smallsetminus S_\Sigma^+$, for all $U \supsetneq S$, and consider a particular U such that $A \subsetneq U$. We distinguish two cases:

(i) If $|A| = 1$, then $C = A_\Sigma^+ = A_\Sigma^*$ and $B = A_\Sigma^* \smallsetminus A$, because Γ is in NF. Thus, $\Theta = \{\langle A \cup (X \smallsetminus C), Y \smallsetminus C, C \cup Z\rangle \mid \langle X, Y, Z\rangle \in \Gamma, Y \not\subseteq C\}$. Moreover, since $S \subsetneq A$, one has $S = \varnothing$ and $\pi_\Gamma^*(U) = U_\Sigma^*$ for all $U \neq \varnothing$. Consider now an arbitrary U with $A \subsetneq U$ and we prove $\pi_\Theta^*(U) = U_\Sigma^*$. First, $\pi_\Theta^*(U) \subseteq U_\Sigma^*$ because, for all $\langle X, Y, Z\rangle \in \Gamma$, if $A(X \smallsetminus C) \subseteq U_\Sigma^*$, then $X \smallsetminus C \subseteq U_\Sigma^*$, $X \subseteq U_\Sigma^* \cup A_\Sigma^* = U_\Sigma^*$ and $Z = X_\Sigma^* \subseteq (U_\Sigma^*)_\Sigma^* = U_\Sigma^*$. Conversely, $U_\Sigma^* \subseteq \pi_\Theta^*(U)$ because $U_\Sigma^* = \pi_\Gamma^*(U)$ and $\pi_\Theta^*(U)$ is closed with respect to Γ_a.

(ii) If $|A| > 1$, then $C = \pi_\Gamma^*(A)$, $B = A_\Gamma^+ \smallsetminus \pi_\Gamma^*(A)$ and $\hat{A} = A_\Gamma^+$ because Γ is in NF. Consider U with $S \subsetneq A \subsetneq U$. Then,

- $\pi_\Theta^*(U) \subseteq U_\Sigma^* \cup A_\Sigma^+$ because, for all $\langle X, Y, Z\rangle \in \Gamma$, one has $A \cup (X \smallsetminus \hat{A}) \subseteq U$ implies $C \cup (Z \smallsetminus B) \subseteq C \cup Z \subseteq \pi_\Gamma^*(A) \cup \pi_\Gamma^*(X) \subseteq A_\Sigma^* \cup X_\Sigma^* \cup S_\Sigma^+ = A_\Sigma^* \cup (X_\Sigma^* \smallsetminus B) \cup (X \cap B)_\Sigma^* \cup S_\Sigma^+ \subseteq U_\Sigma^* \cup A_\Sigma^+$.
- In order to prove $U_\Sigma^* \subseteq \pi_\Theta^*(U) \cup A_\Sigma^+$, first, we prove $\pi_\Gamma^*(U) \subseteq \pi_\Theta^*(U) \cup B$. Indeed, for all $\langle X, Y, Z\rangle \in \Gamma$, if $X \subseteq U$, then $A \cup (X \smallsetminus \hat{A}) \subseteq U$ and, by definition of $\pi_\Theta^*(U)$, $Z \smallsetminus B \subseteq C \cup (Z \smallsetminus B) \subseteq \pi_\Theta^*(U)$ i.e. $Z \subseteq \pi_\Theta^*(U) \cup B$. Finally, $U_\Sigma^* \subseteq \pi_\Gamma^*(U) \cup S_\Sigma^+ \subseteq \pi_\Theta^*(U) \cup B \cup S_\Sigma^+ \subseteq \pi_\Theta^*(U) \cup A_\Sigma^+$.

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In summary, $\pi^*_\Theta(U) \smallsetminus A^+_\Sigma = U^*_\Sigma \smallsetminus A^+_\Sigma$, for all $U \supsetneq A$, has been proved in both cases. It is easy to check that this property is preserved in Δ. $\square$

Example 4.3.10. *Consider the set Δ of triplets in Example 4.3.8*

$$\begin{aligned}\Delta \;=\; &\{\langle 2,1,12\rangle, \langle 3,6,36\rangle, \langle 5,36,356\rangle, \langle 13,24,136\rangle, \langle 14,2356,14\rangle,\\ &\langle 25,4,12356\rangle, \langle 34,12,346\rangle, \langle 45,1,3456\rangle, \langle 234,5,12346\rangle\}\end{aligned}$$

and its minimal triplet $\langle 2,1,12\rangle$. The output of $\texttt{Fix}(\langle 2,1,12\rangle,\Gamma)$ *is:*

$$\begin{aligned}\langle Mnl, \Delta'\rangle &:= \texttt{Fix}(\langle 2,1,12\rangle), \Delta)\\ &= \big\langle \{23,24,25\}, \{\langle 23,45,1236\rangle, \langle 24,356,124\rangle, \langle 25,4,12356\rangle\}\big\rangle\end{aligned}$$

On the one hand, Δ' is in NF. By contention, the two first conditions in Definition 4.3.5 are fulfilled. Since all the triplets have a non-binary first component, the third condition is trivially satisfied. Moreover, all the triplets in Δ' are minimal. Following the same scheme as in the previous example, we will just illustrate the procedure to check the last condition with one triplet. Considering the triplet $\langle 23,45,1236\rangle$, one has that: $\{1,2,3,6\} = \pi^*_{\Delta'}(\{2,3\})$ and $\{4,5\} = \{2,3\}^+_{\Delta'} \smallsetminus \pi^*_{\Delta'}(\{2,3\})$.

On the other hand, Δ' fulfills the two conditions left to be compatible with Σ above $\{2\}$ (Definition 4.3.6). Since $\{2\}^+_\Sigma = \{1,2\}$, the first one is trivial because the first component of each triplet never has 1 as an element. Now, note that for each subset X such that $\{2\} \subsetneq X$ we have that $X^+_{\Delta'} = X^+_\Sigma$ and $\pi^*_{\Delta'}(X) = X^*_\Sigma$, except for the subset $\{2,6\}$. Thus, the second condition is trivially satisfied for all the subsets, and then we will only show the second condition for $\{2,6\}$:

$$\begin{aligned}\{2,6\}^+_{\Delta'} \smallsetminus \{2\}^+_\Sigma &= \{2,6\} \smallsetminus \{1,2\} = \{6\}\\ &= \{1,2,6\} \smallsetminus \{1,2\} = \{2,6\}^+_\Sigma \smallsetminus \{2\}^+_\Sigma\\ \pi^*_{\Delta'}(\{2,6\}) \smallsetminus \{2\}^+_\Sigma &= \{2,6\} \smallsetminus \{1,2\} = \{6\}\\ &= \{1,2,6\} \smallsetminus \{1,2\} = \{2,6\}^*_\Sigma \smallsetminus \{2\}^+_\Sigma\end{aligned}$$

To sum up, this example illustrates that the triplet set of the output of $\texttt{Fix}(\langle 2,1,12\rangle,\Gamma)$ *is compatible with Σ above $\{2\}$.*

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The following definition introduces a strong notion of normal form that will be used in further results.

Definition 4.3.11. *A triplet set* Γ *is said to be in* D-normal form *if it is in NF and, for all* $\langle U_1, V_1, W_1\rangle, \langle U_2, V_2, W_2\rangle \in \Gamma$ *with* $|U_2| > 1$*, the following conditions hold:*

(i) *If* $U_1 \subseteq U_2$*, then* $V_2 \subseteq {U_2}^+_\Gamma \smallsetminus {U_1}^+_\Gamma$.

(ii) *If* $U_2 \not\subseteq U_1 \subseteq W_2$*, then* $V_2 \subseteq {U_2}^+_\Gamma \smallsetminus {U_1}^+_\Gamma$.

In the following example we show the idea underlying the previous definition.

Example 4.3.12. *The following set of triplets is in D-normal form*

$$\Gamma_1 = \{\langle 2, 1, 12\rangle, \langle 3, 6, 36\rangle, \langle 13, 245, 136\rangle, \langle 24, 356, 124\rangle, \langle 34, 125, 346\rangle\}$$

Checking that Γ_1 *is in NF is a matter of computation that we have already illustrated in previous examples. Now, we only focus on the item* (i) because no two triplets satisfy the hypothesis of the item (ii).

- For $\langle 2, 1, 12\rangle$ and $\langle 24, 356, 124\rangle$, we have that
$$\{3, 5, 6\} \subseteq \{2, 4\}^+_{\Gamma_1} \smallsetminus \{2\}^+_{\Gamma_1} = M \smallsetminus \{1, 2\}.$$
- For $\langle 3, 6, 36\rangle$ and $\langle 13, 245, 136\rangle$, we have that
$$\{2, 4, 5\} \subseteq \{1, 3\}^+_{\Gamma_1} \smallsetminus \{3\}^+_{\Gamma_1} = M \smallsetminus \{3, 6\}.$$
- For $\langle 3, 6, 36\rangle$ and $\langle 34, 125, 346\rangle$, we have that
$$\{1, 2, 5\} \subseteq \{3, 4\}^+_{\Gamma_1} \smallsetminus \{3\}^+_{\Gamma_1} = M \smallsetminus \{3, 6\}.$$

At this point, we describe how `Join` and `Shorten` gather the set of triplets in the backtracking process of the recursive calls to `MinCovers`, so that they maintain the property of D-normal form and the compatibility with the closure of the associated implicational system.

`Join` receives two triplet sets, Γ_1 and Γ_2, and calls `Shorten`, where the (`Si-Eq`) equivalence is applied for every pair of triplets, one element of the pair is a triplet of Γ_1 and the other of Γ_2.

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Function Join(Γ_1, Γ_2)

input : Two triplet sets in D-normal form
output: A triplet set fulfilling the conditions of Lemma 4.3.13
begin
    $\Gamma_{1\,\text{new}} := \varnothing$; $\Gamma_{2\,\text{new}} := \varnothing$
    foreach $\langle X, Y, Z\rangle \in \Gamma_1$ **do** $\Gamma_{1\,\text{new}} := \Gamma_{1\,\text{new}} \cup$ `Shorten`$(\langle X, Y, Z\rangle, \Gamma_2)$
    foreach $\langle X, Y, Z\rangle \in \Gamma_2$ **do** $\Gamma_{2\,\text{new}} := \Gamma_{2\,\text{new}} \cup$ `Shorten`$(\langle X, Y, Z\rangle, \Gamma_{1\,\text{new}})$
    return $\Gamma_{1\,\text{new}} \cup \Gamma_{2\,\text{new}}$

Function Shorten($\langle A, B, C\rangle$,Γ)

input : A triplet $\langle A, B, C\rangle$ and a set of triplets Γ
output: A triplet if A is minimal cover of some attributes of B or $\varnothing$ otherwise
begin
    if $\mid A \mid \neq 1$ **then**
        foreach $\langle X, Y, Z\rangle \in \Gamma$ **do**
            if $X \subseteq A$ **then** $B := B \smallsetminus Y$
            else if $A \not\subseteq X$ *and* $X \subseteq C$ **then** $B := B \smallsetminus Y$
            if $B = \varnothing$ **then return** $\varnothing$
    return $\{\langle A, B, C\rangle\}$

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Lemma 4.3.13. *Let Γ_1 and Γ_2 be triplet sets. If both Γ_1 and Γ_2 are in D-normal form and Δ is the output of* `Join`(Γ_1, Γ_2)*, the following conditions hold:*

(i) *Δ is in D-normal form.*

(ii) *For all $X \subseteq M$, $X^+_\Delta = X^+_{\Gamma_1\cup\Gamma_2}$ and $\pi^*_\Delta(X) = \pi^*_{\Gamma_1\cup\Gamma_2}(X)$.*

This lemma, whose proof is direct, is illustrated by the following example.

Example 4.3.14. *Consider the set of triplets Γ_1 from Example 4.3.12 and*

$$\Gamma_2 = \{\langle 5, 36, 356\rangle, \langle 15, 24, 1356\rangle, \langle 45, 12, 3456\rangle\}.$$

Both are in D-normal form. The output of `Join` *is the following:*

$$\begin{aligned}\Phi =& \texttt{Join}(\Gamma_1, \Gamma_2)\\ =&\{\langle 2, 1, 12\rangle, \langle 3, 6, 36\rangle, \langle 5, 36, 356\rangle,\\ &\langle 13, 245, 136\rangle, \langle 24, 356, 124\rangle, \langle 34, 125, 346\rangle\}\end{aligned}$$

Notice that $\Phi = \Gamma_1 \cup \{\langle 5, 36, 356\rangle\}$, which is in D-normal form. In order to illustrate that the second condition of Lemma 4.3.13 is also satisfied, we will show that it holds, for instance, for $X = \{2, 6\}$ and $Y = \{4, 5\}$:

- $X^+_\Phi = X^+_{\Gamma_1\cup\Gamma_2} = \{1, 2, 6\}$ *and* $\pi^*_\Phi(X) = \pi^*_{\Gamma_1\cup\Gamma_2}(X) = \{1, 2, 6\}$.
- $Y^+_\Phi = Y^+_{\Gamma_1\cup\Gamma_2} = M$ *and* $\pi^*_\Phi(Y) = \pi^*_{\Gamma_1\cup\Gamma_2}(Y) = \{3, 4, 5, 6\}$.

The following lemma will be used in the proof of the correctness Theorem 4.3.16.

Lemma 4.3.15. *Let Σ be an implicational system on M, $S \subseteq M$, and Γ be a triplet set. If Γ is compatible with Σ above S and $\langle A, B, C\rangle$ is a minimal triplet in Γ, then the output of* `MinCovers`$(\langle A, B, C\rangle, \Gamma)$ *is in D-normal form and is compatible with Σ above A.*

Proof. We will prove it by structural induction:

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- The base case is when the output of $\texttt{Fix}(\langle A, B, C\rangle, \Gamma)$ is $\langle \varnothing, \varnothing\rangle$. In this case $\texttt{MinCovers}(\langle A, B, C\rangle, \Gamma) = \varnothing$, which obviously is in D-normal form and, by Theorem 4.3.9, is compatible with Σ above A.

- Let Γ be a triplet set that is compatible with Σ above S, $\langle A, B, C\rangle$ be a minimal triplet in Γ, and Φ be the output of $\texttt{MinCovers}(\langle A, B, C\rangle, \Gamma)$. Assume $\langle \texttt{Mnl}, \Omega\rangle$ is the output of $\texttt{Fix}(\langle A, B, C\rangle, \Gamma)$ and

$$\Delta_X = \{\langle X, Y, Z\rangle\} \cup \texttt{MinCovers}(\langle X, Y, Z\rangle, \Omega),$$

for all $\langle X, Y, Z\rangle \in \Omega$ with $X \in \texttt{Mnl}$. Assume, by induction hypothesis, that for all $X \in \texttt{Mnl}$, the output of $\texttt{MinCovers}(\langle X, Y, Z\rangle, \Omega)$ is in D-normal form and compatible with Σ above X. On the one hand, Δ_X is in D-normal form and, by Lemma 4.3.13, Φ is also in D-normal form, $U^+_\Phi = U^+_{\cup\Delta_X}$ and $\pi^*_\Phi(U) = \pi^*_{\cup\Delta_X}(U)$ for all $U \subseteq M$. On the other hand, we prove Φ is compatible with Σ above A. Finally, we prove that $U^+_\Phi \smallsetminus A^+_\Sigma = U^+_\Sigma \smallsetminus A^+_\Sigma$ and $\pi^*_\Phi(U) \smallsetminus A^+_\Sigma = U^*_\Sigma \smallsetminus A^+_\Sigma$, for all $U \subseteq M$ with $A \subsetneq U$:

  - If there is no $X \in \texttt{Mnl}$ with $X \subseteq U$, then $X' \not\subseteq U$ for all $\langle X', Y', Z'\rangle \in \Phi$ and, since $A \subseteq U$, one has $X'' \not\subseteq U$ for all $\langle X'', Y'', Z''\rangle \in \Gamma$ (see F1 in the pseudocode). Therefore, $U^+_\Phi = U^+_\Gamma = U$ and $\pi^*_\Phi(U) = \pi^*_\Gamma(U) = U$. Finally, since Γ is compatible with Σ above A, one has $U^+_\Phi \smallsetminus A^+_\Sigma = U^+_\Sigma \smallsetminus A^+_\Sigma$ and $\pi^*_\Phi(U) \smallsetminus A^+_\Sigma = U^*_\Sigma \smallsetminus A^+_\Sigma$.
  - If $U \in \texttt{Mnl}$, there exists $\langle U, V, W\rangle \in \Omega$ which is minimal. By Theorem 4.3.9, Ω is compatible with Σ above A. Then $W = \pi^*_\Omega(U)$, $V = U^+_\Omega \smallsetminus \pi^*_\Omega(U)$, $U \cap A^+_\Sigma = \varnothing$, $U^+_\Omega \smallsetminus A^+_\Sigma = U^+_\Sigma \smallsetminus A^+_\Sigma$ and $\pi^*_\Omega(U) \smallsetminus A^+_\Sigma = U^*_\Sigma \smallsetminus A^+_\Sigma$. Finally, since $\langle U, V, W\rangle$ is also minimal in $\cup\Delta_X$, one has $U^+_\Phi = U^+_{\cup\Delta_X} = U^+_\Omega = V \cup W$ and $\pi^*_\Phi(U) = \pi^*_{\cup\Delta_X}(U) = \pi^*_\Omega(U) = W$. Therefore, $U^+_\Phi \smallsetminus A^+_\Sigma = U^+_\Sigma \smallsetminus A^+_\Sigma$ and $\pi^*_\Phi(U) \smallsetminus A^+_\Sigma = U^*_\Sigma \smallsetminus A^+_\Sigma$.
  - If there exists $X \in \texttt{Mnl}$ with $X \subsetneq U$, then, by induction hypothesis, for all $\langle X, Y, Z\rangle$ minimal in Ω with $X \subsetneq U$, one has

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$U^+_{\Delta_X} \smallsetminus A^+_\Sigma = U^+_\Sigma \smallsetminus A^+_\Sigma$ and $\pi^*_{\Delta_X}(U) \smallsetminus A^+_\Sigma = U^*_\Sigma \smallsetminus A^+_\Sigma$. On the other hand, $U^+_{\Delta_X} = \pi^*_{\Delta_X}(U) = U$ for all $X \in \mathtt{Mnl}$ with $X \not\subseteq U$. Therefore, $U^+_\Phi \smallsetminus A^+_\Sigma = U^+_{\cup\Delta_X} \smallsetminus A^+_\Sigma = U^+_\Sigma \smallsetminus A^+_\Sigma$ and $\pi^*_\Phi(U) \smallsetminus A^+_\Sigma = \pi^*_{\cup\Delta_X}(U) \smallsetminus A^+_\Sigma = U^*_\Sigma \smallsetminus A^+_\Sigma$.

□

The following theorem culminates in the statement that `Fast`D`-basis` Algorithm successfully computes the aggregated D-basis.

Theorem 4.3.16. *Let Σ be a complete implicational system on M. Then,* `Fast`D`-basis` *terminates and returns the unique D-basis equivalent to Σ.*

Proof. As mentioned, this work only refers to finite closure systems and, consequently, to finite implicational systems and to finite triplet sets. Therefore, `MinCovers` terminates because the cardinality of the triplet set strictly decreases in each recursive call to the function. That is, `Fast`D`-basis` always finishes.

Let $\Gamma = \{\, \langle X, Y \smallsetminus X, X\rangle \mid X \to Y \in \Sigma \text{ with } Y \not\subseteq X \,\}$. Then the initial call $\Phi = \mathtt{MinCovers}(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$ returns the unique aggregated D-basis that is equivalent to Σ, and its proof obeys the following schema: First, we prove that Φ is compatible with Σ above $\varnothing$ and it is in D-normal form. Second, we prove that $\langle d, d^*_\Sigma \smallsetminus d, d\rangle \in \Phi$ for all $d \in M$ such that $d^*_\Sigma \smallsetminus d \neq \varnothing$. Third, we prove that, for all $d \in M$ and $D \subseteq M$, if D is a minimal proper cover for d, then there exists $\langle U, V, W\rangle \in \Phi$ such that $U = D$ and $d \in V$. We conclude the proof by showing that U is a minimal proper cover for v for all $\langle U, V, W\rangle \in \Phi$ and $v \in V$. Now, we expand this schema:

As a first step, $\mathtt{MinCovers}(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$ calls to $\mathtt{Fix}(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$. If we denote the output as $\langle \mathtt{Mnl}, \Delta\rangle$, Theorem 4.3.7 ensures that Δ is compatible with Σ above $\varnothing$, and Proposition 4.3.4 ensures that

$$\mathtt{Mnl} = \{\, X \subseteq M \mid \langle X, Y, Z\rangle \text{ is minimal for } \Delta \text{ for some } Y, Z \subseteq M \,\}.$$

By Lemma 4.3.15, for any triplet $\langle A, B, C\rangle$ that is minimal in Δ, the output of $\mathtt{MinCovers}(\langle A, B, C\rangle, \Delta)$ is in D-normal form and is compatible with Σ

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above A. Finally, following the scheme of the last part of the proof in Lemma 4.3.15, one can prove that the output of $\texttt{MinCovers}(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$ is in D-normal form and is compatible with Σ above $\varnothing$.

As a second stage of the schema, since $\Phi = \texttt{MinCovers}(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$ is compatible with Σ above $\varnothing$, for all $d \in M$ such that $d^*_\Sigma \smallsetminus d \neq \varnothing$, one has $\langle d, d^*_\Sigma \smallsetminus d, d\rangle \in \Phi$.

Third, for all $d \in M$, $D \subseteq M$, we prove that if D is a minimal proper cover for d, there exist $\langle X,Y,Z\rangle \in \Delta$ and

$$\langle U,V,W\rangle \in \{\langle X,Y,Z\rangle\} \cup \texttt{MinCovers}(\langle X,Y,Z\rangle,\Delta)$$

such that $X \in \texttt{Mnl}$, $U = D$ and $d \in V$. Indeed, we prove that, for every triplet set Θ that is compatible with Σ above $S \subseteq M$, if D is a minimal proper cover for d, $S \subsetneq D$ and $d \notin S^+_\Sigma$, the following conditions hold:

(i) There exists $\langle A,B,C\rangle \in \Theta$ with $A \subseteq D$ that is minimal for Θ.

(ii) If $\langle A,B,C\rangle$ is minimal for Θ and $A \subseteq D$, there exists $\langle U,V,W\rangle \in \{\langle A,B,C\rangle\} \cup \texttt{MinCovers}(\langle A,B,C\rangle,\Theta)$ with $U = D$ and $d \in V$.

Item (i) is a direct consequence of the compatibility of Θ with Σ above S because $d \in D^+_\Sigma \smallsetminus S^+_\Sigma = D^+_\Theta \smallsetminus S^+_\Sigma$, but $d \notin D$. Item (ii) is proved by structural induction:

- The base case is when the output of $\texttt{Fix}(\langle A,B,C\rangle,\Theta)$ is $\langle\varnothing,\varnothing\rangle$. In this case $\{\langle A,B,C\rangle\} \cup \texttt{MinCovers}(\langle A,B,C\rangle,\Theta) = \{\langle A,B,C\rangle\}$, which is compatible with Σ above S. Therefore, since $d \in D^+_\Sigma \smallsetminus D^*_\Sigma$ and there is no $X \subsetneq D$ with $d \in X^+_\Sigma$, one has $D = A$ and $d \in B$.

- Consider $\langle A,B,C\rangle$ being minimal for Θ with $A \subseteq D$. Consider also $\langle\texttt{Mnl},\Omega\rangle$ being the output of $\texttt{Fix}(\langle A,B,C\rangle,\Theta)$ and

$$\Delta_X = \{\langle X,Y,Z\rangle\} \cup \texttt{MinCovers}(\langle X,Y,Z\rangle,\Omega)$$

for each $\langle X,Y,Z\rangle \in \Omega$ with $X \in \texttt{Mnl}$. Assume, as induction hypothesis, that for all $X \in \texttt{Mnl}$, if $X \subseteq D$, there exists $\langle U,V,W\rangle \in \Delta_X$ such that $U = D$ and $d \in V$. Two cases are distinguished:

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- If $d \in B$, then, since D is minimal proper cover, one has $A = D$ and item (ii) is proved for Θ.
- If $d \notin B$, we prove that there exists $\langle X, Y, Z\rangle \in \Omega$ with $X \in \texttt{Mnl}$ and $X \subseteq D$: since $\langle A, B, C\rangle$ is minimal for Θ and it is in D-normal form, one has $C \subseteq A^*_\Sigma \subseteq A^*_\Sigma \cup S^+_\Sigma \subseteq D^*_\Sigma \cup S^+_\Sigma$ and $d \notin B \cup C = \hat{A} = A^+_\Theta$. Moreover, $d \in D^+_\Theta \smallsetminus A^+_\Theta$ and it is only possible when there exists $\langle X', Y', Z'\rangle \in \Theta$ with $A \neq X' \subseteq D$. Thus, $A \cup (X' \smallsetminus B) \subseteq D$ and there exists $\langle X, Y, Z\rangle \in \Omega$ where $X = A \cup (X' \smallsetminus B)$, $X \in \texttt{Mnl}$ and $X \subseteq D$.

  Now, by induction hypothesis, there exists $\langle U_1, V_1, W_1\rangle \in \Delta_X$ such that $U_1 = D$ and $d \in V_1$, and we prove, by *reductio ad absurdum*, that there exists

  $$\langle U, V, W\rangle \in \{\langle A, B, C\rangle\} \cup \texttt{MinCovers}(\langle A, B, C\rangle, \Theta)$$

  with $U = D$ and $d \in V$. Assume that it is not true, i.e. that `Join` removes d from V_1. Therefore, there exists $\langle U_2, V_2, W_2\rangle \in \texttt{MinCovers}(\langle A, B, C\rangle, \Theta)$ such that $U_1 \not\subseteq U_2$, $U_2 \subseteq W_1$ and $d \in V_2$. Since $\texttt{MinCovers}(\langle A, B, C\rangle, \Theta)$ is compatible with Σ above A, one has $d \in V_2 \subseteq {U_2}^+_\Sigma$ and $U_1 \not\subseteq U_2 \subseteq W_1 \subseteq {U_1}^*_\Sigma \cup A^+_\Sigma$. In addition, since $U_2 \cap A^+_\Sigma = A$, one has $U_2 \subseteq {U_1}^*_\Sigma$, which contradicts the fact that D is minimal proper cover for d.

We conclude the proof by showing that U is a minimal proper cover of v for all $\langle U, V, W\rangle \in \Phi = \texttt{MinCovers}(\langle\varnothing, \varnothing, \varnothing\rangle, \Gamma)$ and $v \in V$. Since Φ is compatible with Σ above $\varnothing$, one has $V \subseteq U^+_\Phi \smallsetminus \pi^*_\Phi(U) = U^+_\Sigma \smallsetminus U^*_\Sigma$ and, therefore, U is a proper cover for all $v \in V$. We have to prove that it is not only a proper cover but also a minimal one. For that, suppose that there exists a triplet $\langle X, Y, Z\rangle \in \Phi$ such that X is a proper cover of v and $X \subseteq U^*_\Sigma$. By *reductio ad absurdum*, suppose that $U \not\subseteq X$. By Lemma 4.3.13, Φ is in D-normal form. Thus, $Y \subseteq X^+_\Phi \smallsetminus U^+_\Phi = X^+_\Sigma \smallsetminus U^+_\Sigma = \varnothing$ yields a contradiction, because there is no triplet in Φ with an empty set as second component. $\square$

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To conclude this section, in Figure 4.5 on page 123, an illustrative example showing the in-depth trace of an execution of `Fast`D`-basis` is introduced. The input is the implicational system Σ from Example 4.3.2. After the call of $\texttt{Fix}(\langle\varnothing,\varnothing,\varnothing\rangle,\Gamma)$, returning the pair $\langle\texttt{Mnl},\Delta\rangle$, the algorithm only opens four branches, corresponding to the minimal triplets in Δ. In each of these branches `MinCovers` is recursively called. For the sake of readability, we detailed the branches associated with the three first minimal triplets in Figures 4.6, 4.7 and 4.8 (on pages 124, 125 and 126, respectively). Finally, the figure shows how `Join` collects the necessary triplets to return the D-basis.

4.4 The performance of `Fast`D`-basis`

In this section, we compare the performance of the two methods to compute the D-basis presented in this chapter: D`-basis` and `Fast`D`-basis`. Both methods have been implemented in R language. To make this comparison, two empirical experiments have been developed: one considering synthetic data and the other one over real data.

The synthetic data for the first experiment has been obtained by using a random generator also implemented in R language. We have generated a collection of 100 implicational systems with sizes of 1500. The experiment was performed on a Mac OS X - Intel Core i5 (3,2 GHz) with 16 GB. We measure, for each implicational set, the execution times in seconds for both methods.

Figure 4.3 illustrates a plot of the cloud of execution times with respect to the input size. It shows a clear improvement achieved with the new method. As expected, the tendencies for both methods are exponential. However, `Fast` D`-basis` Algorithm has a significantly smaller exponential range than the D`-basis` one. For a better illustration of this situation, we have also depicted in Figure 4.4 the execution times with respect to the output size.

For the second experiment, we have used the Mushroom dataset of the UCI repository [34]. We have extracted the implications using the Apriori

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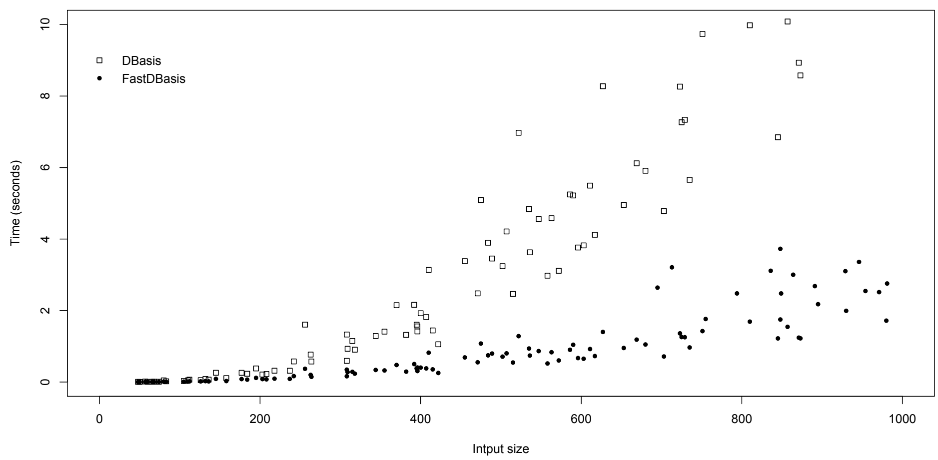

Figure 4.3: Execution times (in seconds) with respect to input sizes of randomly generated implicational systems. The empty squares represent the time of the execution of D-`basis` and the dark dots represent the execution times of `Fast`D-`basis`.

Algorithm by means of the Arules package (R language) considering a support equal or greater than 0.4. This limit is imposed by the Arules package: without any extra-constraint in the support, the R package does not finish. The cardinality of the set of implications is 637. Its corresponding D-basis has 52 implications and its size is 159.

	Execution time	Recursive calls
D-`basis`	6 518.55 s	$9 \cdot 10^6$
`Fast`D-`basis`	1.65 s	1 889

Table 4.3: Comparison over Mushroom dataset

The execution time for `Fast`D-`basis` Algorithm is 1.65 seconds whereas D-`basis` Algorithm needs 6 518.55 seconds. Moreover, the number of recursive calls of the method is 1889 whereas D-`basis` needed 9 millions.

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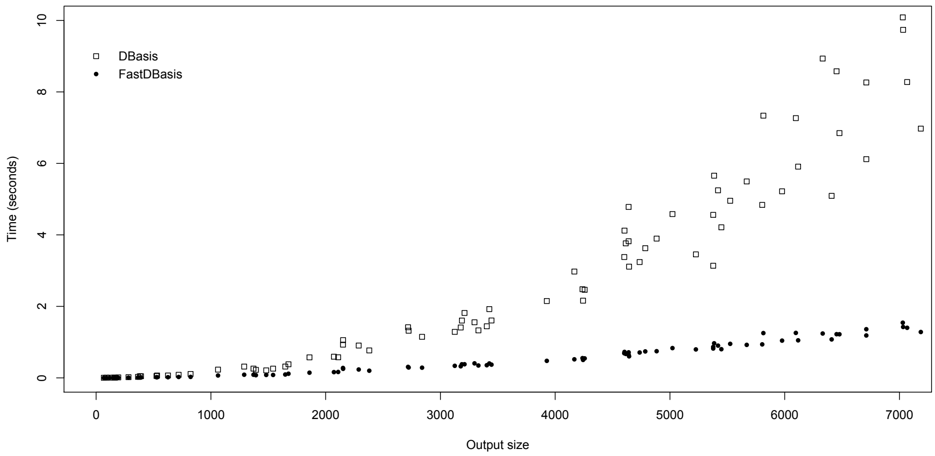

Figure 4.4: Execution times (in seconds) with respect to output sizes. The empty squares represent the time of the execution of D-basis and the dark dots represent the execution times of Fast$D$-basis.

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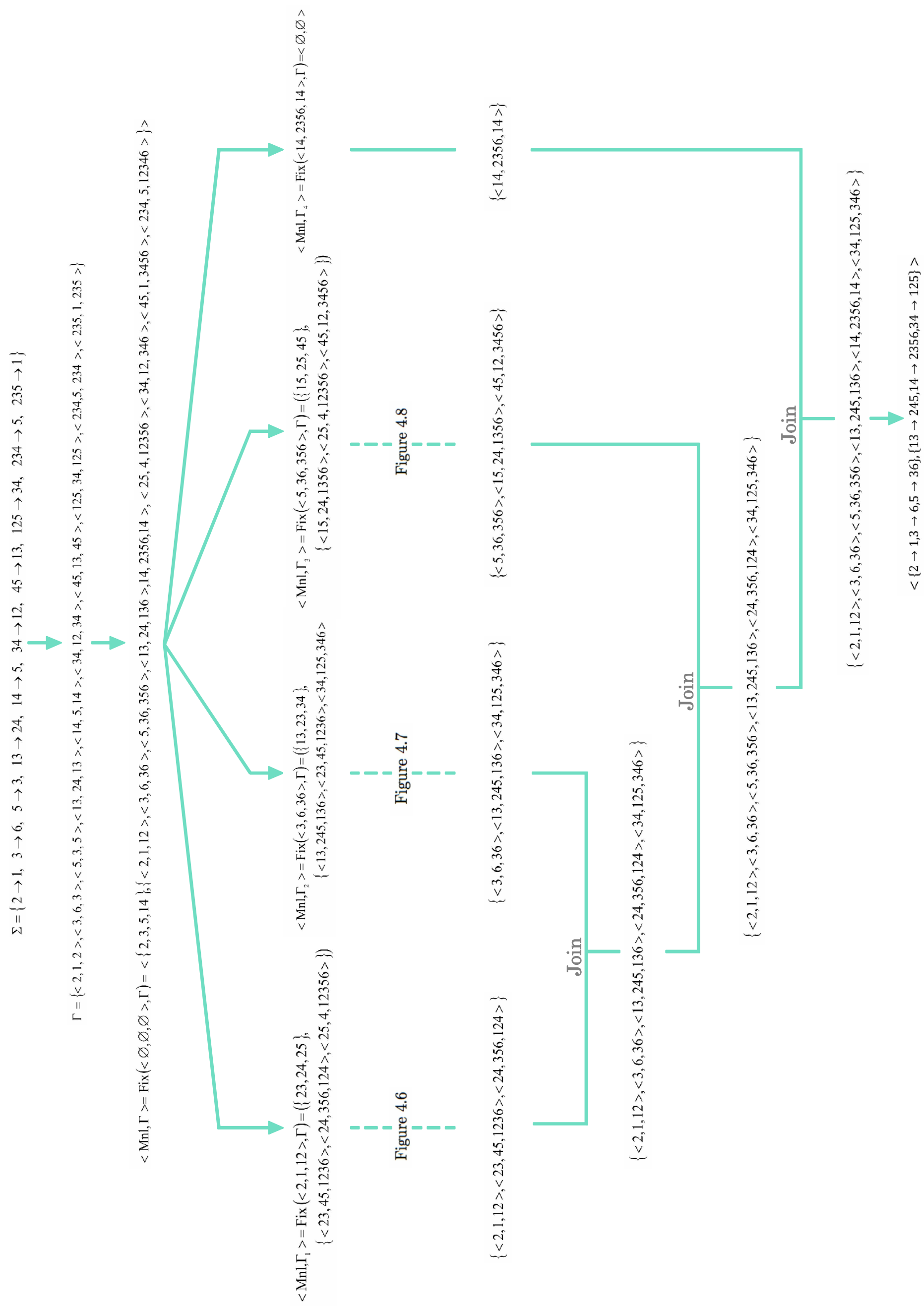

Figure 4.5: Trace of the execution of `Fast`D`-basis` to the implicational system from Example 4.3.2.

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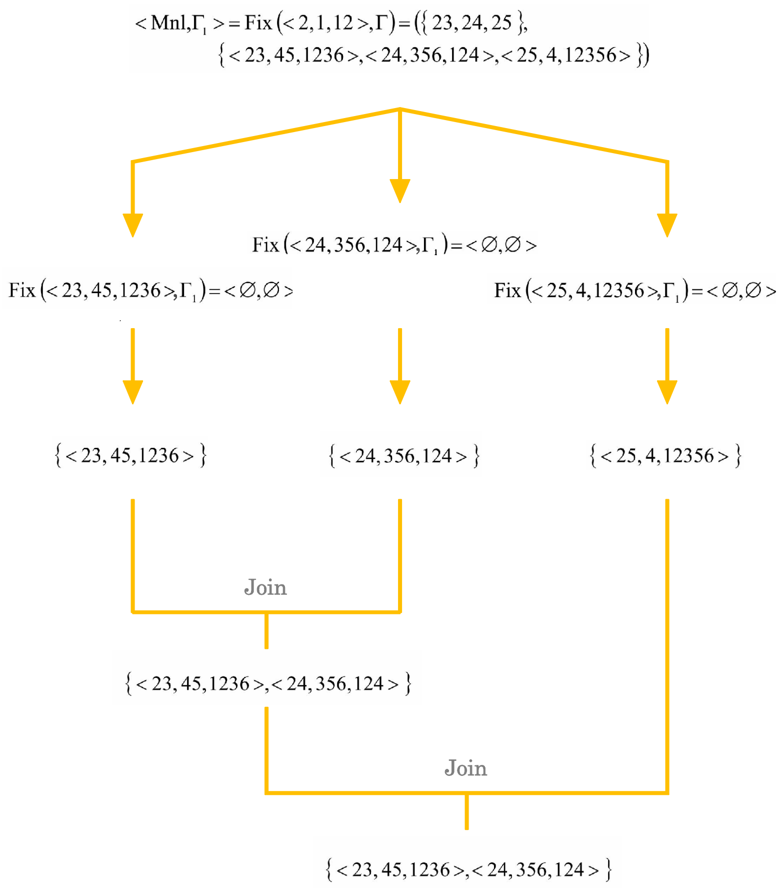

Figure 4.6: Branch of execution of `Fast`D`-basis` applied to the implicational system from Example 4.3.2 associated with the first minimal triplet.

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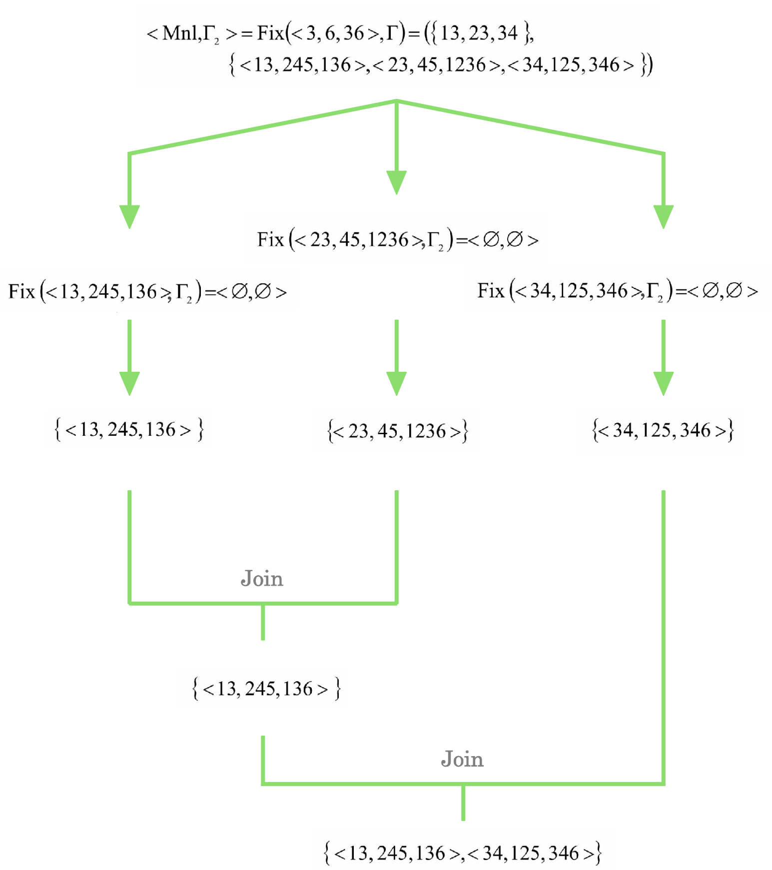

Figure 4.7: Branch of execution of `Fast`D`-basis` applied to the implicational system from Example 4.3.2 associated with the second minimal triplet.

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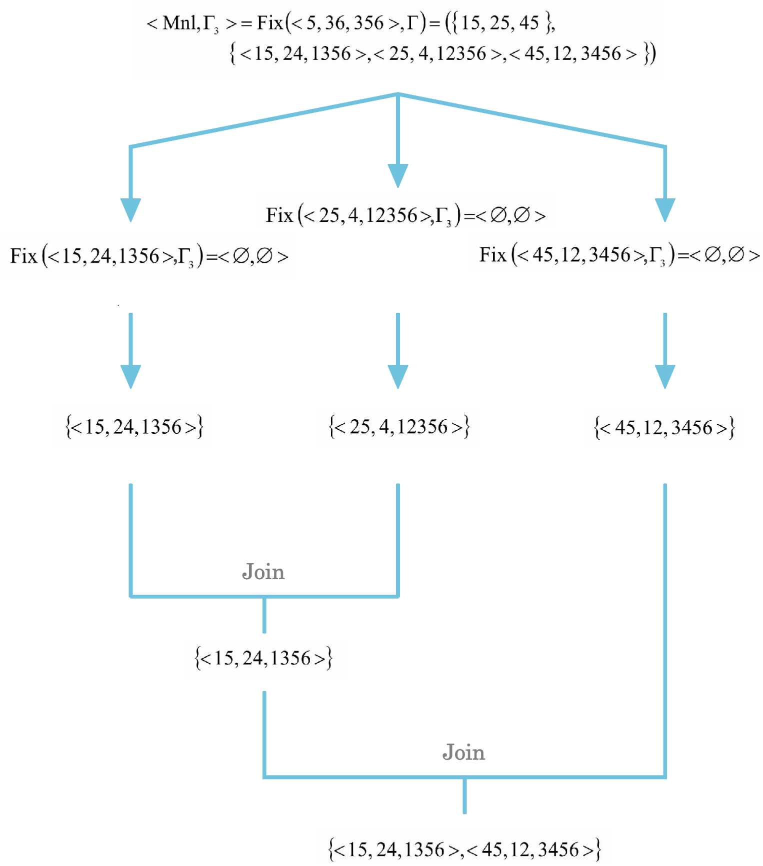

Figure 4.8: Branch of execution of `Fast`D`-basis` applied to the implicational system from Example 4.3.2 associated with the third minimal triplet.

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Chapter 5

Dichotomous Direct Bases

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Despite the benefits of the different definitions of direct bases, their main handicap is the inherent cost of their computation. New algorithms for computing both, the direct-optimal basis and D-basis, have been studied in previous chapters. A hot topic in this line is the definition of new kinds of direct basis whose associated transformation methods have a better performance. In this chapter, in order to deal with this issue, we introduce a new definition of basis together with a method to compute it.

The new basis called dichotomous direct basis, is strongly based on the separated treatment of implications according to their behavior with respect to the closure operator. Thus, we carry out a study of the set of implications, returning a dichotomous partition of the whole set of implications according to their behavior. This dichotomy provides an improvement in the building of the new direct basis by dividing the original problem into two smaller, separated ones: one part will support the hard computation of this construction, whereas the other is carried out almost instantaneously. In addition, the cost of classification must be added, but we accompany the method with a very efficient criterion to classify each implication.

The results presented in this chapter have been published in [50].

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5.1 Dichotomous implicational systems and directness

Before presenting the new kind of direct basis, we characterize the behavior of each implication with respect to the execution of the closure operator and provide an efficient criterion to classify them.

5.1.1 Quasi-key implications

The core of our criterion is the notion of quasi-key, which is based on the concept of key [16, 38].

Definition 5.1.1. *Let Σ be an implicational system on M. A set $A \subseteq M$ is a* key *for Σ if $A^+_\Sigma = M$. An implication $A \to B \in \mathcal{L}_M$ is said to be a* key implication *with respect to Σ if A is a key for Σ. Finally, a key implication $A \to B$ is said to be* proper *if $B = M \smallsetminus A$.*

Notice that the definition of key implication only imposes conditions to the premise, whereas the notion of proper key implications also considers the conclusion. On the other hand, by `[Frag]`, any key implication is inferred, i.e. if $A \to B$ is a key implication with respect to Σ then $\Sigma \vdash A \to B$.

From now on, when no confusion arises about the implicational system Σ, we omit the reference to it and we only say that $A \to B$ is a key implication.

Example 5.1.2. *Consider $M = \{a, b, c, d\}$ and*

$$\Sigma = \{a \to c, bc \to a, d \to c, acd \to b\}.$$

The subset of attributes $\{b, d\}$ is a key for Σ because $\Sigma \vdash bd \to abcd$, as the following sequence proves:

$\sigma_1 : bd \to \varnothing$ .... *by applying* `[Ref]`.

$\sigma_2 : d \to c$ .... *belongs to* Σ.

$\sigma_3 : bd \to c$ .... *by applying* `[Comp]` *to* σ_1 *and* σ_2.

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$\sigma_4 : bc \to a$ .... *belongs to* Σ.

$\sigma_5 : bcd \to ac$............... *by applying* [Comp] *to* σ_3 *and* σ_4.

$\sigma_6 : bd \to a$................. *by applying* [Simp] *to* σ_3 *and* σ_5.

$\sigma_7 : bd \to ac$................ *by applying* [Comp] *to* σ_3 *and* σ_6.

$\sigma_8 : bd \to bd$.............................. *by applying* [Ref].

$\sigma_9 : bd \to abcd$ .... *by applying* [Comp] *to* σ_7 *and* σ_8.

Therefore, for instance, $bd \to abcd$, $bd \to a$ and $bd \to ac$ are key implications with respect to Σ, but only the last one is a proper key implication.

In order to leap to the notion of quasi-key implication as a generalization of the concept of key implication, we will use some definitions and results provided in [40].

Definition 5.1.3. *Given an implicational system Σ on M, the* determinate *for Σ is the set of attributes $\mathtt{Dte}(\Sigma) = \bigcup_{X \to Y \in \Sigma} Y$, and the* core *of Σ is the set $\mathtt{core}(\Sigma) = M \smallsetminus \mathtt{Dte}(\Sigma)$.*

One of the statements given in [40] ensures that the attributes not appearing in any conclusion of the implicational set must belong to all the keys, i.e. if A is a key for Σ then $\mathtt{core}(\Sigma) \subseteq A$. Another interesting property we will often use later is the following.

Proposition 5.1.4. *Let Σ be an implicational system on M. For each $A \subseteq M$, one has $A^+_\Sigma \subseteq A \cup \mathtt{Dte}(\Sigma)$.*

Proof. Consider $a \in A^+_\Sigma$. If $a \in A$, trivially $a \in A \cup \mathtt{Dte}(\Sigma)$. If $a \in A^+_\Sigma \smallsetminus A$, we know that $\Sigma \vdash A \to a$, so there is $X \to Y \in \Sigma$ such that $a \in Y$. Therefore, $a \in \mathtt{Dte}(\Sigma)$. $\square$

The fact that every key with respect to Σ contains $\mathtt{core}(\Sigma)$ leads us to generalize the notion of key as follows:

Definition 5.1.5. *Let Σ be an implicational system on M. An implication $A \to B \in \mathcal{L}_M$ is said to be a* quasi-key implication *with respect to Σ if $B \subseteq \mathtt{Dte}(\Sigma) \subseteq A^+_\Sigma$. In addition, it is said to be* proper quasi-key implication *if $B = \mathtt{Dte}(\Sigma) \smallsetminus A$.*

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Obviously, any key implication is a quasi-key implication. The following assertions straightforwardly follow from Proposition 5.1.4.

Remark 5.1.6. *Given an implicational system Σ, one has:*

(i) $A^+_\Sigma = A \cup \mathtt{Dte}(\Sigma)$ *for any quasi-key implication $A \to B$ w.r.t. Σ.*

(ii) $A^+_\Sigma = A \cup B$ *for any proper quasi-key implication $A \to B$ w.r.t. Σ.*

The following example illustrates these notions.

Example 5.1.7. *Let $\Sigma = \{a \to c, bc \to a, d \to c, acd \to b\}$ be an implicational system on $M = \{a, b, c, d, e\}$. The implication $acde \to b$ is a proper key implication and a proper quasi-key implication for Σ. The implication $bc \to a$ is a proper quasi-key implication, but not a key implication for Σ. Finally, $ad \to c$ is a non-proper quasi-key implication.*

5.1.2 Dichotomous implicational system

Now, we justify that quasi-key implications have a different behavior with respect to the closure computation. It allows us to give them a separated treatment by splitting the implicational system into two well-defined subsets. Thus, we introduce the notion of dichotomous implicational set and a two-fold operator that computes faster the closure of the attribute sets.

Definition 5.1.8. *Let Σ^* and Σ^k be implicational systems on M and consider $\Sigma = \Sigma^* \cup \Sigma^k$. The pair $\langle \Sigma^*, \Sigma^k \rangle$ is said to be a* dichotomous implicational system *if the following conditions hold:*

(i) *Σ is a compact implicational system.*[1]

(ii) *If $A \to B \in \Sigma$ is a quasi-key implication w.r.t. Σ, then $A \to B \in \Sigma^k$.*

(iii) *If $A \to B \in \Sigma^k$ then it is a proper quasi-key implication w.r.t. Σ.*

In addition, we define the operator $\sigma_{\langle \Sigma^*, \Sigma^k \rangle} : 2^M \to 2^M$ as the composition $\sigma_{\langle \Sigma^*, \Sigma^k \rangle} = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}$ where π is defined as in (2.1).

[1] See Definition 3.1.2.

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The underlying idea is that we split implicational systems into two well-defined subsets, Σ^* and Σ^k: implications in Σ^* are not quasi-key implications, whereas every implication in Σ^k is a proper quasi-key implication.

Example 5.1.9. *Let $M = \{a,b,c,d,e,g\}$ be a set of attributes. The pair $\langle\Sigma^*,\Sigma^k\rangle = \langle\{b \to d, d \to c, e \to b\}, \{a \to bcdeg, cg \to bde\}\rangle$ is a dichotomous implicational system. Notice that $cg \to bde$ is a proper quasi-key implication with respect to $\Sigma^* \cup \Sigma^k$ whereas $a \to bcdeg$ is also a proper key implication. We illustrate the application of the new operator $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}$ as follows:*

- $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}(\{b,d\}) = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(\{b,d\}) = \pi_{\Sigma^k}(\{b,c,d\}) = \{b,c,d\}$
- $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}(\{d,g\}) = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(\{d,g\}) = \pi_{\Sigma^k}(\{d,c,g\}) = \{b,c,d,e,g\}$

Hereafter, we abuse the notation and extend the usual definitions for implicational systems to dichotomous implicational systems. Thus, for instance, we will say that two dichotomous implicational systems, $\langle\Sigma_1^*,\Sigma_1^k\rangle$ and $\langle\Sigma_2^*,\Sigma_2^k\rangle$, are equivalent, denoted by $\langle\Sigma_1^*,\Sigma_1^k\rangle \equiv \langle\Sigma_2^*,\Sigma_2^k\rangle$, if the equivalence $(\Sigma_1^* \cup \Sigma_1^k) \equiv (\Sigma_2^* \cup \Sigma_2^k)$ holds.

From the definition of dichotomous implicational system, it is obvious that $\pi_{\Sigma^*}(X) \subseteq X \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k)$ and $\pi_{\Sigma^k}(X) \subseteq X \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k)$, for all $X \subseteq M$. Moreover, this last inclusion is then an equality in those cases in which the attribute set is not a fixpoint with respect to the closure of Σ^k. That is,

$$\pi_{\Sigma^k}(X) = \begin{cases} X \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k) & \text{if } A \subseteq X \text{ for some } A \to B \in \Sigma^k \\ X & \text{otherwise} \end{cases} \tag{5.1}$$

Thus, π_{Σ^k} is idempotent and, therefore, a closure operator.

Regarding the relationship between this operator and those that were introduced in Chapter 2, Equations (2.1) and (2.2), we have that $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}$ is also isotone and extensive, it has the same computational cost than $\pi_{\Sigma^*\cup\Sigma^k}$ and $\rho_{\Sigma^*\cup\Sigma^k}$, and it defines increasing sequences of sets that converge to the closure as well, as the following lemma and theorem ensure.

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Lemma 5.1.10. *Let $\langle \Sigma^*, \Sigma^k \rangle$ be a dichotomous implicational system. For all $n > 0$ we have that $\sigma^n_{\langle \Sigma^*, \Sigma^k \rangle} = \pi_{\Sigma^k} \circ \pi^n_{\Sigma^*}$.*

Proof. The result is straightforwardly obtained by induction from the equality $\pi_{\Sigma^k} \circ \pi_{\Sigma^*} \circ \pi_{\Sigma^k} = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}$, which is proved by distinguishing two separated situations depending on the value of $\pi_{\Sigma^k}(X)$:

In the case of $\pi_{\Sigma^k}(X) = X$, trivially, $\pi_{\Sigma^k} \circ \pi_{\Sigma^*} \circ \pi_{\Sigma^k}(X) = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(X)$.

Otherwise, by (5.1), one has that $\pi_{\Sigma^k}(X) = X \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k)$ and $\pi_{\Sigma^k} \circ \pi_{\Sigma^*} \circ \pi_{\Sigma^k}(X) = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(X \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k)) = X \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k)$. On the other hand, by extensiveness of π_{Σ^*}, we have that $X \subseteq \pi_{\Sigma^*}(X)$ and the isotony of π_{Σ^k} leads to $\pi_{\Sigma^k}(X) \subseteq \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(X)$. Finally, due to $\pi_{\Sigma^k}(X) = X \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k)$, we obtain that $\pi_{\Sigma^k} \circ \pi_{\Sigma^*}(X) = X \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k)$. □

Example 5.1.11. *For the dichotomous implicational system introduced in Example 5.1.9, $\langle \Sigma^*, \Sigma^k \rangle = \langle \{b \to d, d \to c, e \to b\}, \{a \to bcdeg, cg \to bde\} \rangle$, we have that*

$$\begin{aligned}
\sigma^2_{\langle \Sigma^*, \Sigma^k \rangle}(\{b, g\}) =& \pi_{\Sigma^k} \circ \pi_{\Sigma^*} \circ \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(\{b, g\}) \\
=& \pi_{\Sigma^k} \circ \pi_{\Sigma^*} \circ \pi_{\Sigma^k}(\{b, d, g\}) = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(\{b, d, g\}) \\
=& \pi_{\Sigma^k}(\{b, c, d, g\}) = \{b, c, d, e, g\} \\
\pi_{\Sigma^k} \circ \pi^2_{\Sigma^*}(\{b, g\}) =& \pi_{\Sigma^k} \circ \pi_{\Sigma^*} \circ \pi_{\Sigma^*}(\{b, g\}) = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(\{b, d, g\}) \\
=& \pi_{\Sigma^k}(\{b, c, d, g\}) = \{b, c, d, e, g\}
\end{aligned}$$

The following theorem states that the iteration of this operator reaches the closure after finitely many steps.

Theorem 5.1.12. *Let $\langle \Sigma^*, \Sigma^k \rangle$ be a dichotomous implicational system, $\Sigma = \Sigma^* \cup \Sigma^k$ and $X \subseteq M$. The sequence $\sigma^n_{\langle \Sigma^*, \Sigma^k \rangle}(X)$ with $n > 0$ is an increasing sequence and there exists $0 < r \leq \min\{|\Sigma^*|, |\mathtt{Dte}(\Sigma^*)|\}$ such that $\sigma^r_{\langle \Sigma^*, \Sigma^k \rangle}(X) = X^+_\Sigma$.*

Proof. The convergence of $\sigma^n_{\langle \Sigma^*, \Sigma^k \rangle}(X)$ is straightforwardly obtained from the convergence of $\pi^n_\Sigma(X)$ because $\pi_\Sigma(X) \subseteq \sigma_{\langle \Sigma^*, \Sigma^k \rangle}(X)$, and both sequences are increasing and upper-bounded by X^+_Σ, which is the fixpoint for

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$\pi^n_\Sigma(X)$. That is, for all $n > 0$,

$$\pi^n_\Sigma(X) \subseteq \sigma^n_{\langle\Sigma^*,\Sigma^k\rangle}(X) = \pi_{\Sigma^k} \circ \pi^n_{\Sigma^*}(X) \subseteq X^+_\Sigma \subseteq X \cup \mathtt{Dte}(\Sigma)$$

The second part of this proof shows that the fixpoint is achieved after finitely many steps and provides the upper bound.

Consider now $X_0 = X$, $X_n = \sigma^n_{\langle\Sigma^*,\Sigma^k\rangle}(X)$ for each $n > 0$ and r being the smaller integer with $X_r = X_{r+1} = X^+_\Sigma$. It is easy to see that for each $0 \leq n < r$ there exists a different implication $A \to B \in \Sigma^*$ such that $A \subseteq X_n$, $B \not\subseteq X_n$ and $B \subseteq X_{n+1}$. Therefore, $r \leq |\Sigma^*|$. Moreover, $r \leq |\mathtt{Dte}(\Sigma^*)|$ because, in each X_n, at least a new element from $\mathtt{Dte}(\Sigma^*)$ is added. □

To conclude this section, we consider directness in our dichotomous approach by means of the idempotence property, following the same scheme as previous direct approaches (see Chapter 2).

Definition 5.1.13. *A dichotomous implicational system $\langle\Sigma^*,\Sigma^k\rangle$ is said to be direct if $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}$ is idempotent.*

Notice that, since $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}$ is always extensive and isotone, when the dichotomous implicational system is direct, $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}$ is a closure operator. Moreover, if $\Sigma = \Sigma^* \cup \Sigma^k$ then $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}(X) = X^+_\Sigma$ for all $X \subseteq M$.

Theorem 5.1.14. *Let $\langle\Sigma^*,\Sigma^k\rangle$ be a dichotomous implicational system.*

(i) *If Σ^* is a direct implicational system then $\langle\Sigma^*,\Sigma^k\rangle$ is direct.*

(ii) *If $\langle\Sigma^*,\Sigma^k\rangle$ is direct then $\Sigma^* \cup \Sigma^k$ is ordered-direct.*[2]

Proof. Item (i) is a direct consequence of $\sigma^2_{\langle\Sigma^*,\Sigma^k\rangle} = \pi_{\Sigma^k} \circ \pi^2_{\Sigma^*}$, which is ensured by Lemma 5.1.10. Item (ii) is due to $\sigma_{\langle\Sigma^*,\Sigma^k\rangle}(X) \subseteq \rho_{\Sigma^*\cup\Sigma^k}(X) \subseteq X^+_{\Sigma^*\cup\Sigma^k}$ for all $X \subseteq M$. □

The previous theorem establishes sufficient conditions and the following example shows that these conditions are not necessary.

[2]Where any order can be considered, whenever the implications from Σ^* appear before the implications from Σ^k.

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Example 5.1.15. *The dichotomous implicational system*

$$\langle \Sigma^*, \Sigma^k \rangle = \langle \{a \to bc, b \to ac, ce \to d\}, \{ae \to bcd\} \rangle$$

is direct, but Σ^* *is not direct because e.g.* $\pi_{\Sigma^*}(\{b,e\}) = \{a,b,c,e\}$ *and* $\pi^2_{\Sigma^*}(\{b,e\}) = \{a,b,c,d,e\}$.

On the other hand, for $\Sigma^* = \{a \to b, b \to c\}$ *and* $\Sigma^k = \{cd \to abe\}$, *the implicational system* $\Sigma^* \cup \Sigma^k$ *is ordered-direct whereas the dichotomous implicational system* $\langle \Sigma^*, \Sigma^k \rangle$ *is not direct because*

$$\sigma_{\langle \Sigma^*, \Sigma^k \rangle}(\{a,d\}) = \{a,b,d\} \subsetneq \rho_{\Sigma^* \cup \Sigma^k}(\{a,d\}) = \{a,b,c,d,e\} = \{a,d\}^+_{\Sigma^* \cup \Sigma^k}.$$

5.2 Dichotomous direct basis, DD-basis.

As usual, the term "basis" is preserved to implicational systems that satisfy some minimality criteria. In this framework, we call dichotomous direct basis to any dichotomous direct implicational system whose first component is right-simplified. Recall that the dichotomous implicational systems are compact (Definition 5.1.8).

Definition 5.2.1. *A dichotomous implicational system* $\langle \Sigma^*, \Sigma^k \rangle$ *is said to be a* dichotomous direct basis, *briefly* DD-basis, *if the following conditions hold:*

(i) $\sigma_{\langle \Sigma^*, \Sigma^k \rangle}$ *is idempotent (i.e. it is a closure operator).*

(ii) *For all* $A \to B, C \to D \in \Sigma^*$, *if* $A \subsetneq C$ *then* $B \cap D = \varnothing$.

Example 5.2.2. *Let* $M = \{a,b,c,d,e,g\}$ *be a set of attributes and consider the following implicational system on* M*:*

$$\Sigma = \{a \to d, ce \to g, cg \to e, de \to g, bg \to ace, cd \to ab, ab \to ce, aeg \to b\}$$

Among its equivalent implicational systems, the following ones are the direct-optimal basis, the D-basis and a DD-basis, respectively.

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$$\Sigma_{do} = \{a \to d, ab \to ceg, ac \to beg, ae \to bcg, bg \to acde,$$
$$cd \to abeg, ce \to g, cg \to e, de \to g, bce \to ad, bde \to ac\}$$

$$\Sigma_D = \big\langle \{a \to d\}, \{ab \to ceg, ae \to bc, bg \to acde, cd \to abeg,$$
$$ce \to g, cg \to e, de \to g, bce \to ad, bde \to ac\}\big\rangle$$

$$\Sigma_{DD} = \big\langle \{a \to d, ce \to g, cg \to e, de \to g\},$$
$$\{bg \to acde, cd \to abeg, ab \to cdeg, ae \to bcdg\}\big\rangle$$

Notice that, in the previous example, the direct-optimal basis has a greater cardinality than the D-basis and this one also has a greater cardinality than the DD-basis. However, what matters is not the cardinality of the bases but, as we shall see in the next section, the cost of its computation.

In the following, we will provide a result that illustrates the connection between the direct-optimal basis and our DD-basis. Before presenting it, we highlight one result from [26] that will be used in the proof. This result gives a declarative characterization of direct-optimal bases, but it is not suitable to be used in its computation.

Definition 5.2.3. *Let Σ be a set of implications on M. A set $A \subseteq M$ is said to be a proper premise with respect to Σ if $A^{\sharp}_{\Sigma} \neq \varnothing$ where*

$$A^{\sharp}_{\Sigma} = A^{+}_{\Sigma} \smallsetminus \Big(A \cup \bigcup_{x \in A} (A \smallsetminus \{x\})^{+}_{\Sigma}\Big).$$

Proposition 5.2.4 (Ryssel et al. [55])**.** *Let Σ be an implicational system. The unique equivalent direct-optimal basis is*

$$\Sigma_{do} = \{A \to A^{\sharp}_{\Sigma} \mid A \text{ is a proper premise with respect to } \Sigma\}$$

Now, we present the mentioned theorem in whose proof we will need the previous proposition.

Theorem 5.2.5. *Let $\langle \Sigma^*, \Sigma^k \rangle$ be a dichotomous implicational system and Σ_{do} be the unique direct-optimal basis that is equivalent to $\Sigma = \Sigma^* \cup \Sigma^k$. The pair $\langle \Sigma^*, \Sigma^k \rangle$ is a DD-basis if and only if*

$$\Sigma^* = \{A \to B \in \Sigma_{do} \mid A \to B \text{ is not a quasi-key implication w.r.t. } \Sigma_{do}\}$$

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Proof. It is straightforward from the definition, Lemma 5.1.10 and Theorem 2.2.6, that, if Σ^* is the set of non-quasi-key implications from Σ_{do}, then $\langle \Sigma^*, \Sigma^k \rangle$ is a DD-basis.

The proof of the converse result is based on Proposition 5.2.4. Assume that $\langle \Sigma^*, \Sigma^k \rangle$ is a DD-basis and consider $A \to B \in \Sigma^*$. From Definition 5.1.8, we have that $A \to B$ is not a quasi-key implication, $A \cap B = \varnothing$ and $\varnothing \neq B \subseteq \sigma_{\langle \Sigma^*, \Sigma^k \rangle}(A) = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(A) = \pi_{\Sigma^*}(A) = A^+_\Sigma$.

Moreover, if $z \in (A \smallsetminus \{x\})^+_\Sigma$ for some $x \in A$, since $\langle \Sigma^*, \Sigma^k \rangle$ is direct, there exists $A' \to B' \in \Sigma^*$ with $A' \subseteq A \smallsetminus \{x\} \subsetneq A$ such that $z \in B'$. Therefore, by Condition (ii) in the definition of DD-basis, $z \notin B$. Since this reasoning is valid for all $z \in (A \smallsetminus \{x\})^+_\Sigma$, one has $(A \smallsetminus \{x\})^+_\Sigma \cap B = \varnothing$.

So far, we have proved that $\varnothing \neq B \subseteq A^\sharp_\Sigma$. That is, A is a proper premise with respect to Σ. Now, we prove that $A^\sharp_\Sigma \subseteq B$. By definition of $A^\sharp_\Sigma$ and the directness of $\langle \Sigma^*, \Sigma^k \rangle$, for each $x \in A^\sharp_\Sigma$ there exists $A \to B_x \in \Sigma^*$ such that $x \in B_x$. Now, since Σ is a compact implicational system, for all $x, y \in A^\sharp_\Sigma$, we have that $B_x = B_y = B$. Therefore $A^\sharp_\Sigma = B$. □

The previous theorem leads to a set of straightforward corollaries that justify to use the name "basis" for this kind of dichotomous implicational system.

Corollary 5.2.6. *Let $\langle \Sigma_1^*, \Sigma_1^k \rangle$ and $\langle \Sigma_2^*, \Sigma_2^k \rangle$ be two equivalent direct dichotomous implicational systems.*

(i) *If $\langle \Sigma_1^*, \Sigma_1^k \rangle$ is a DD-basis then $|\Sigma_1^*| \leq |\Sigma_2^*|$ and $\|\Sigma_1^*\| \leq \|\Sigma_2^*\|$.*

(ii) *If $\langle \Sigma_1^*, \Sigma_1^k \rangle$ and $\langle \Sigma_2^*, \Sigma_2^k \rangle$ are DD-bases then $\Sigma_1^* = \Sigma_2^*$.*

The following corollary not only ensures the existence of an equivalent DD-basis for any implicational system, but also provides a method to transform any direct-optimal basis into a DD-basis with quadratic cost.

Corollary 5.2.7. *Let Σ be an implicational system and Σ_{do} be its equivalent direct-optimal basis. Consider the following sets:*

$$\begin{aligned}
\Sigma^*_{do} &= \{A \to B \in \Sigma_{do} \mid \mathtt{Dte}(\Sigma_{do}) \not\subseteq \pi_{\Sigma_{do}}(A)\} \\
\Sigma^k_{do} &= \{A \to \mathtt{Dte}(\Sigma_{do})\text{-}A \mid A \to B \in \Sigma_{do} \textit{ with } \mathtt{Dte}(\Sigma_{do}) \subseteq \pi_{\Sigma_{do}}(A)\}
\end{aligned}$$

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Then $\langle \Sigma^*_{do}, \Sigma^k_{do} \rangle$ *is a DD-basis equivalent to* Σ.

Corollary 5.2.7 shows the way to build a DD-basis from an arbitrary direct-optimal basis. However, Corollary 5.2.6 proves that there may be several equivalent DD-bases. The following example illustrates the existence and the non-unicity of the DD-bases.

Example 5.2.8. *Consider* $\Sigma = \Sigma^* \cup \Sigma^k$ *where*

$$\Sigma^* = \{a \to bcd, b \to acd, ce \to f\} \quad \textit{and} \quad \Sigma^k = \{abcde \to fg\}.$$

The dichotomous implicational system $\langle \Sigma^*, \Sigma^k \rangle$ *is a DD-basis. Notice that* Σ^* *itself is not direct. The direct-optimal basis equivalent to* Σ *is*

$$\Sigma_{do} = \{a \to bcd, b \to acd, ce \to f, ae \to fg, be \to fg\}$$

and the DD-basis obtained from Σ_{do} *by using Corollary 5.2.7 is*

$$\big\langle \{a \to bcd, b \to acd, ce \to f\}, \{ae \to bcdfg, be \to acdfg\} \big\rangle,$$

which is equivalent to the initial one.

5.3 Computing the DD-basis

As previously mentioned, the algorithms that transform any implicational system into a direct basis have non-polynomial cost with respect to the size of the original implicational system. The definition of DD-basis and the following theorem allow a reduction in the size of the input and this fact has a huge repercussion on the cost of the transformation.

This is the main advantage of the proposed DD-basis with respect to the alternatives given in previous chapters: direct-optimal basis and D-basis. The new approach reduces the size of the subset of implications withstanding the exponential cost of the basis construction process. This reduction comes from the removal of the quasi-key implications in the exponential task. The following theorem justifies this assertion.

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Theorem 5.3.1. *Let $\langle \Sigma^*, \Sigma^k \rangle$ be a dichotomous implicational system and $\Sigma = \Sigma^* \cup \Sigma^k$. If Σ_{do} is the direct-optimal basis such that $\Sigma_{do} \equiv \Sigma^*$ and*

$$\begin{aligned} \Sigma^*_{do} &= \{A \to B \in \Sigma_{do} \mid \mathtt{Dte}(\Sigma) \not\subseteq \pi_{\Sigma^k} \circ \pi_{\Sigma_{do}}(A)\} \\ \Sigma^k_{do} &= \{A \to \mathtt{Dte}(\Sigma)\text{-}A \mid A \to B \in \Sigma_{do} \text{ with } \mathtt{Dte}(\Sigma) \subseteq \pi_{\Sigma^k} \circ \pi_{\Sigma_{do}}(A)\} \end{aligned}$$

then $\langle \Sigma^*_{do}, \Sigma^k_{do} \cup \Sigma^k \rangle$ is a DD-basis that is equivalent to $\langle \Sigma^*, \Sigma^k \rangle$.[3]

Proof. It is straightforward that $\langle \Sigma^*_{do}, \Sigma^k_{do} \cup \Sigma^k \rangle$ is a dichotomous implicational system. By Theorem 2.2.6, optimality of Σ_{do} implies that Condition (ii) in definition of DD-basis is also ensured. Then, we prove that $\sigma_{\langle \Sigma^*_{do}, \Sigma^k_{do} \cup \Sigma^k \rangle}$ is idempotent. By Lemma 5.1.10, $\sigma^2_{\langle \Sigma^*_{do}, \Sigma^k_{do} \cup \Sigma^k \rangle} = \pi_{\Sigma^k_{do} \cup \Sigma^k} \circ \pi^2_{\Sigma^*_{do}}$. Moreover, since every element in $\Sigma^k_{do} \cup \Sigma^k$ is a quasi-key implication with respect to Σ, one has $\pi_{\Sigma^k_{do} \cup \Sigma^k} = \pi_{\Sigma^k} \circ \pi_{\Sigma^k_{do}}$. On the other hand, by Corollary 5.2.7, $\langle \Sigma^*_{do}, \Sigma^k_{do} \rangle$ is a DD-basis and $\sigma^2_{\langle \Sigma^*_{do}, \Sigma^k_{do} \rangle} = \sigma_{\langle \Sigma^*_{do}, \Sigma^k_{do} \rangle}$. Therefore,

$$\begin{aligned} \sigma^2_{\langle \Sigma^*_{do}, \Sigma^k_{do} \cup \Sigma^k \rangle} &= \pi_{\Sigma^k \cup \Sigma^k_{do}} \circ \pi^2_{\Sigma^*_{do}} = \pi_{\Sigma^k} \circ \pi_{\Sigma^k_{do}} \circ \pi^2_{\Sigma^*_{do}} \\ &= \pi_{\Sigma^k} \circ \sigma^2_{\langle \Sigma^*_{do}, \Sigma^k_{do} \rangle} = \pi_{\Sigma^k} \circ \sigma_{\langle \Sigma^*_{do}, \Sigma^k_{do} \rangle} \\ &= \pi_{\Sigma^k} \circ \pi_{\Sigma^k_{do}} \circ \pi_{\Sigma^*_{do}} = \pi_{\Sigma^k \cup \Sigma^k_{do}} \circ \pi_{\Sigma^*_{do}} = \sigma_{\langle \Sigma^*_{do}, \Sigma^k_{do} \cup \Sigma^k \rangle} \end{aligned}$$

□

Based on the previous theorem, Algorithm 5.1 structures the transformation into three consecutive stages: First, there is an splitting process, with quadratic cost, in which it filters what implications are quasi-keys and reduces Σ^*. Second, Σ^* is transformed into an equivalent direct-optimal basis Σ_{do} by using any of the proposed methods. Finally, in a third stage, the method reorganizes Σ_{do} and Σ^k to obtain a DD-basis with quadratic cost.

The following example illustrates the execution of Algorithm 5.1.

Example 5.3.2. *Let $\Sigma = \{a \to d, ce \to g, cg \to e, de \to g, bg \to ace, cd \to ag, ab \to ce\}$ be an implicational system. In the first stage, we pick up all the*

[3] Notice that Σ_{do} is equivalent to Σ^* but not necessarily equivalent to $\Sigma^* \cup \Sigma^k$, unlike Theorem 5.2.5.

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Algorithm 5.1: DD-basis

input : An implicational system Σ
output: An equivalent DD-basis Σ_{DD}
begin
  $\texttt{Dte}(\Sigma) := \varnothing$
  foreach $A \to B \in \Sigma$ **do** $\texttt{Dte}(\Sigma) = \texttt{Dte}(\Sigma) \cup (B \smallsetminus A)$
                    - - Stage 1: Generation of Σ^* and Σ^k by disjoining of Σ
  $\Sigma^* = \varnothing, \Sigma^k = \varnothing$
  foreach $A \to B \in \Sigma$ **do**
    if $\texttt{Dte}(\Sigma) \subseteq A^+_\Sigma$ **then** add $A \to \texttt{Dte}(\Sigma)\text{-}A$ to Σ^k
    else add $A \to B\text{-}A$ to Σ^*
                    - - Stage 2: Generation of Σ_{do} from Σ^*
  $\Sigma_{do} := DObasis(\Sigma^*)$
                    - - Stage 3: Generation of Σ_{DD} from Σ_{do} and Σ^k
  $\Sigma^* := \varnothing$
  foreach $A \to B \in \Sigma_{do}$ **do**
    if $\texttt{Dte}(\Sigma) \subseteq \pi_{\Sigma^k} \circ \pi_{\Sigma_{do}}(A)$ **then** add $A \to \texttt{Dte}(\Sigma)\text{-}A$ to Σ^k
    else add $A \to B\text{-}A$ to Σ^*
  return $\Sigma_{DD} = \langle \Sigma^*, \Sigma^k \rangle$

quasi-key implications, returning the following dichotomous implicational system:

$$\langle \Sigma^*, \Sigma^k \rangle = \langle \{a \to d, ce \to g, cg \to e, de \to g\}, \\ \{bg \to acde, cd \to aeg, ab \to cdeg\} \rangle$$

In the second stage, we get the direct-optimal basis that is equivalent to Σ^*:

$$\Sigma_{do} = \{a \to d, ae \to g, ce \to g, cg \to e, de \to g\}$$

Finally, we consider Σ_{do} and Σ^k to build the output, a DD-basis:

$$\Sigma_{DD} = \langle \{a \to d, ae \to g, ce \to g, cg \to e, de \to g\}, \\ \{bg \to acde, cd \to aeg, ab \to cdeg\} \rangle$$

In this example, the third stage boils down to the union of both components, since Σ_{do} does not contain quasi-key implications.

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We have shown the advantages of the splitting process of the original implicational system with the goal of obtaining a DD-basis in a efficient way. Going beyond, the next section considers specific criteria to achieve a unique canonical DD-basis.

5.4 Canonical DD-basis

As stated in Corollary 5.2.6, there are different equivalent DD-bases and all of them share the same first component of the dichotomous implicational system. In the following example, three equivalent DD-bases are presented to illustrate this situation:

Example 5.4.1. *The following DD-bases are equivalent:*
$\langle\{a \to bc, b \to c, ae \to d, ce \to d, be \to d\}, \{abcdeg \to h\}\rangle$.
$\langle\{a \to bc, b \to c, ae \to d, ce \to d, be \to d\}, \{aeg \to bcdh, aceg \to bdh\}\rangle$.
$\langle\{a \to bc, b \to c, ae \to d, ce \to d, be \to d\}, \{abeg \to cdh, aceg \to bdh\}\rangle$.

The aim of this section is to define a canonical DD-basis considering minimality in the second component of dichotomous implicational systems.

Definition 5.4.2. *Let $\langle\Sigma^*, \Sigma^k\rangle$ be a DD-basis. It is said to be canonical if the following conditions hold:*

(i) *For each $A \to B \in \Sigma^k$, $\pi_{\Sigma^*}(A) = A$.*

(ii) *For each $A \to B, C \to D \in \Sigma^k$, if $A \subseteq C$, then $A = C$ and $B = D$.*

Example 5.4.3. *The first DD-basis introduced in Example 5.4.1,*

$$\langle\{a \to bc, b \to c, ae \to d, ce \to d, be \to d\}, \{abcdeg \to h\}\rangle,$$

is a canonical DD-basis.

In order to establish the properties of the canonical DD-basis and its unicity, first we introduce several lemmas as basic results to prove the theorem that ensures the existence and the unicity of an equivalent canonical DD-basis for each implicational system (see Theorem 5.4.8 below).

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Lemma 5.4.4. *Let $\langle \Sigma^*, \Sigma^k \rangle$ be a canonical DD-basis and $A \to B \in \Sigma^k$. For any quasi-key implication $C \to D$ with respect to $\Sigma^* \cup \Sigma^k$, if $C \subseteq A$ then $\pi_{\Sigma^*}(C) = A$.*

Proof. If $C \to D$ is a quasi-key implication with respect to $\Sigma^* \cup \Sigma^k$, since $\langle \Sigma^*, \Sigma^k \rangle$ is direct, we have that $\pi_{\Sigma^k} \circ \pi_{\Sigma^*}(C) = C \cup \mathtt{Dte}(\Sigma^* \cup \Sigma^k)$. Therefore, there exists $A' \to B' \in \Sigma^k$ with $A' \subseteq \pi_{\Sigma^*}(C)$. On the other hand, as $C \subseteq A$ and π_{Σ^*} is isotone, we have $\pi_{\Sigma^*}(C) \subseteq \pi_{\Sigma^*}(A) = A$. Finally, by Condition (ii) in the definition of canonical DD-basis, $A = A'$ and $\pi_{\Sigma^*}(C) = A$. □

Lemma 5.4.5. *Let $\langle \Sigma^*, \Sigma^k \rangle$ be a canonical DD-basis, $\Sigma = \Sigma^* \cup \Sigma^k$, and $A \to B$ be a quasi-key implication with respect to Σ. If A is a quasi-closed set with respect to Σ, there exists $C \to D \in \Sigma^k$ such that $C \subseteq A$.*

Proof. First, since A is a quasi-closed set with respect to Σ, for each $E \to F \in \Sigma^*$ with $E \subseteq A$, we have $F \subseteq A$ because $E \to F$ is not a quasi-key implication. Then, $\pi_{\Sigma^*}(A) = A$. Finally, since $A \to B$ is a quasi-key implication with respect to Σ, we have $\pi_{\Sigma^k}(A) = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(A) = A \cup \mathtt{Dte}(\Sigma)$ and, therefore, there exists $C \to D \in \Sigma^k$ such that $C \subseteq A$. □

Lemma 5.4.6. *Let $\langle \Sigma^*, \Sigma^k \rangle$ be a canonical DD-basis and $\Sigma = \Sigma^* \cup \Sigma^k$. For each $A \to B \in \Sigma^k$, one has A is a pseudo-closed set with respect to Σ.*

Proof. First, if $C \subseteq A$, two scenarios are possible. On the one hand, when $\mathtt{Dte}(\Sigma) \not\subseteq C^+_\Sigma$, by Equation (5.1), isotony of π_{Σ^*} and the first condition in definition of canonical DD-basis, $C^+_\Sigma = \pi_{\Sigma^*}(C) \subseteq \pi_{\Sigma^*}(A) = A$. On the other hand, if $\mathtt{Dte}(\Sigma) \subseteq C^+_\Sigma$, by Lemma 5.4.4, $\pi_{\Sigma^*}(C) = A$ and $C^+_\Sigma = \pi_{\Sigma^k} \circ \pi_{\Sigma^*}(C) = \pi_{\Sigma^k}(A) = A \cup B = A^+_\Sigma$. Therefore, A is a quasi-closed set with respect to Σ.

In addition, because of $B \neq \varnothing = A \cap B$ (the implicational system is compact) and $A \to B$ is a proper quasi-key implication, $A^+_\Sigma = A \cup B \neq A$. To conclude the proof it is only needed to see that, if $C \subsetneq A$ is a quasi-closed set, then $C \to C$ is not a quasi-key implication because, in this case, $C^+_\Sigma = \pi_{\Sigma^*}(C) \subsetneq \pi_{\Sigma^*}(A) = A$.

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If $C \subsetneq A$ is a quasi-closed set, then $C \to C$ is not a quasi-key implication with respect to Σ because, in the opposite case, by Lemma 5.4.5, there exists $E \to F \in \Sigma^k$ such that $E \subseteq C \subsetneq A$ in contradiction to Condition (ii) in the definition of the canonical DD-basis. □

The following theorem characterizes those implications belonging to the second component of the canonical DD-basis.

Theorem 5.4.7. *Let $\langle \Sigma^*, \Sigma^k \rangle$ be a canonical DD-basis and $\Sigma = \Sigma^* \cup \Sigma^k$. Then $A \to B \in \Sigma^k$ if and only if $A \to B$ is a proper quasi-key implication with respect to Σ and A is a pseudo-closed set with respect to Σ.*

Proof. The direct implication is ensured by Lemma 5.4.6. Conversely, if $A \to B$ is a proper quasi-key implication with respect to Σ and A is a pseudo-closed set with respect to Σ, by Proposition 2.1.8, there exists $C \to D \in \Sigma$ such that $C \subseteq A$ and $C^+_\Sigma = A^+_\Sigma$. Since $C \to D$ is also a quasi-key implication, $C \to D \in \Sigma^k$ and, by Lemma 5.4.6, C is a pseudo-closed set. Finally, by definition of pseudo-closed sets, due to $C \subseteq A$ and $C^+_\Sigma = A^+_\Sigma$, we have $C = A$ and, therefore, $A \to B \in \Sigma^k$. □

In the following theorem, we state several properties of the canonical DD-basis.

Theorem 5.4.8. *For any implicational system there exists a unique equivalent canonical DD-basis. In addition, if $\langle \Sigma_1^*, \Sigma_1^k \rangle$ is a canonical DD-basis and $\langle \Sigma_2^*, \Sigma_2^k \rangle$ is an equivalent dichotomous implicational system, then $|\Sigma_1^k| \leq |\Sigma_2^k|$, and $\|\Sigma_1^k\| \leq \|\Sigma_2^k\|$.*

Proof. Given an implicational system Σ, if Σ_{do} is its unique equivalent direct-optimal basis and Σ_{DG} is its unique equivalent Duquenne-Guigues basis, by Theorems 5.2.5 and 5.4.7, the pair $\langle \Sigma^*, \Sigma^k \rangle$ where

$$\Sigma^* = \{A \to B \in \Sigma_{do} \mid A \to B \text{ is not a quasi-key implication w.r.t. } \Sigma\}$$
$$\Sigma^k = \{A \to B \in \Sigma_{DG} \mid A \to B \text{ is a quasi-key implication w.r.t. } \Sigma\}$$

is the unique canonical DD-basis that is equivalent to Σ.

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On the other hand, assume that $\langle \Sigma_1^*, \Sigma_1^k \rangle$ is a canonical DD-basis and $\langle \Sigma_2^*, \Sigma_2^k \rangle$ is an equivalent dichotomous implicational system. Let $\Sigma = \Sigma_1^* \cup \Sigma_1^k \equiv \Sigma_2^* \cup \Sigma_2^k$. For each $A_1 \to B_1 \in \Sigma_1^k$, by Theorem 5.4.7 and Proposition 2.1.8, there exists $A_2 \to B_2 \in \Sigma_2^k$ such that $A_2 \subseteq A_1$ and ${A_1}_\Sigma^+ = {A_2}_\Sigma^+$. Since both $A_1 \to B_1$ and $A_2 \to B_2$ are proper quasi-key implications, we have that ${A_1}_\Sigma^+ = A_1 \cup B_1 = {A_2}_\Sigma^+ = A_2 \cup B_2$ (see Remark 5.1.6) and, therefore, $\|A_1 \to B_1\| = \|A_2 \to B_2\|$. Finally, since this reasoning is valid for each implication in Σ_1^k, we conclude that $|\Sigma_1^k| \leq |\Sigma_2^k|$, and $\|\Sigma_1^k\| \leq \|\Sigma_2^k\|$. □

The previous theorem, together with Theorem 5.2.5 and Corollary 5.2.6, ensures that the canonical DD-basis (which exists and is unique) is the direct dichotomous implicational system with the lowest cardinality and the lowest size among all the equivalent dichotomous implicational systems.

Algorithm 5.2: Canonical DD-basis

input : A DD-basis $\langle \Sigma^*, \Sigma^k \rangle$
output: The equivalent canonical DD-basis
begin
  $F := \texttt{Dte}(\Sigma^* \cup \Sigma^k)$
  $\Sigma_c^k := \varnothing$
  foreach $A \to B \in \Sigma^k$ **do**
    $E := \pi_{\Sigma^*}(A)$
    Condition :=true
    foreach $C \to D \in \Sigma_c^k$ **do**
      if $C \subseteq E$ **then** *Condition* :=false and break the loop
      else if $E \subseteq C$ **then** Remove $C \to D$ from Σ_c^k
    if *Condition* **then** Add $E \to F\text{-}E$ to Σ_c^k
  return $\langle \Sigma^*, \Sigma_c^k \rangle$

Notice that, although a canonical DD-basis can be computed via the direct-optimal basis and the Duquenne-Guigues basis, it is an inefficient solution because the algorithms that compute both are exponential. In the previous section, we provide an algorithm for computing DD-bases. We conclude this section with a quadratic algorithm (Algorithm 5.2) that

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transforms any DD-basis into its equivalent canonical DD-basis.

Now, we will show an illustrative example of how to obtain the canonical DD-basis equivalent to a given DD-basis.

Example 5.4.9. *Considering the second DD-basis from Example 5.4.1, the execution of the algorithm would be the following:*

$$\begin{aligned}\langle \Sigma^*, \Sigma^k \rangle = \langle & \{a \to bc, b \to c, ae \to d, ce \to d, be \to d\}, \\ & \{aeg \to bcdh, aceg \to bdh\}\rangle\end{aligned}$$

The algorithm computes $\texttt{Dte}(\Sigma^* \cup \Sigma^k) = \{b, c, d, h\}$ *and traverses* Σ^k*:*

- *For* $aeg \to bcdh \in \Sigma^k$, $\pi_{\Sigma^*}(aeg) = \{a, b, c, d, e, g\}$*. So,* $\Sigma^*_c = \{abcdeg \to h\}$*.*
- *For* $aceg \to bdh \in \Sigma^k$, $\pi_{\Sigma^*}(aceg) = \{a, b, c, d, e, g\}$ *and the loop is broken.*

There are no more implications in Σ^k*, so the algorithm returns the canonical DD-basis:*

$$\Sigma_{DD} = \langle \{a \to bc, b \to c, ae \to d, ce \to d, be \to d\}, \quad \{abcdeg \to h\}\rangle$$

Since the canonical DD-basis is unique, the output of this example is the same basis as the one in Example 5.4.3.

5.5 The performance of DD-basis and Canonical DD-basis

The proposed methods to calculate both, the DD-basis and the canonical DD-basis, allow for the improvement of the efficiency of any method of calculating direct-optimal bases. In this section we support this statement with an empirical study of how they improve the `doSimp` method, which has been presented in Chapter 3. Thus, we compare three methods: do-basis method (`doSimp`), DD-basis method (Algorithm 5.1) and canonical DD-basis method (Algorithm 5.1 + Algorithm 5.2).

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The experiment was performed on a Mac OS X Yosemite 64 bits - Intel Core i7 (2,93 GHz) with 8 GB. By using a random generator for implicational systems, we have obtained a collection of 700 synthetic implicational systems where the number of implications varies from 5 to 35 and the number of attributes varies from 5 to 25. We measure, for each implicational system, the execution times (in seconds) of the methods mentioned above.

First, we study how the three methods behave with respect to the size of the direct-optimal basis. Thus, Figure 5.1 illustrates a plot of the cloud of execution times of the three methods. Due to the huge execution times of do-basis method, we only show the data with execution time lower than 10 seconds. If all the tested implicational systems were plotted, all the relevant points would be clustered in a line that is almost coincident.

The following table completes the study by summarizing the behavior of the cases we have excerpted from in the graphical representation.

Measure	Size	do-basis	DD-basis	canonical DD-b.
Min.	238.0	10.90	0.03	0.03
1st Qu.	426.8	35.12	0.18	0.19
Median	603.0	136.73	0.35	0.38
3rd Qu.	854.5	71.45	3.30	3.34
Max.	1170.0	9 561.85	681.29	681.34
Mean	643.5	1 112.27	33.71	33.73

In addition, we also provide the tendency lines of the execution time for direct-optimal basis and the canonical DD-basis (the tendency lines use the whole set of time values). We have not included the DD-basis tendency line because it is very similar (in this chart) to the canonical one.

A clear difference seems to appear between the do-basis method and the two dichotomous methods (having both of them a similar performance).

In the previous figure, the comparison between the DD-basis and the canonical DD-basis is hardly distinguished. In the following figure we will show such a difference by zooming in the execution times of the two dichotomous methods. Figure 5.2 shows that the execution time of the canonical DD-basis is greater than the one of the DD-basis. The extra time is due to

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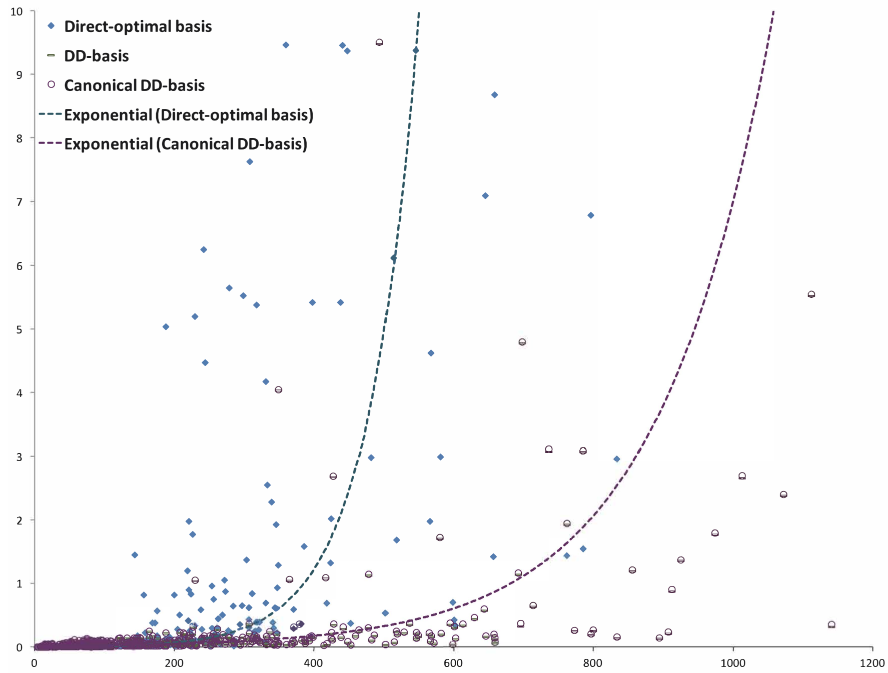

Figure 5.1: Execution times of `doSimp`and both methods of dichotomous direct bases presented in Algorithms 5.1 and 5.2, all of them applied to randomly generated implicational systems.

Algorithm 5.2, which has a quadratic cost, as depicted in Figure 5.3. We remark that the quadratic coefficient of the difference tendency line is very small (nanoseconds).

To sum up, we emphasize that these results are very promising. On the one hand, the theoretical results in the study of DD-basis for direct-optimal basis could be applied to other direct bases directly improving its computation process. On the other hand, the computation of these new kinds of bases are more efficient than previous ones in the literature.

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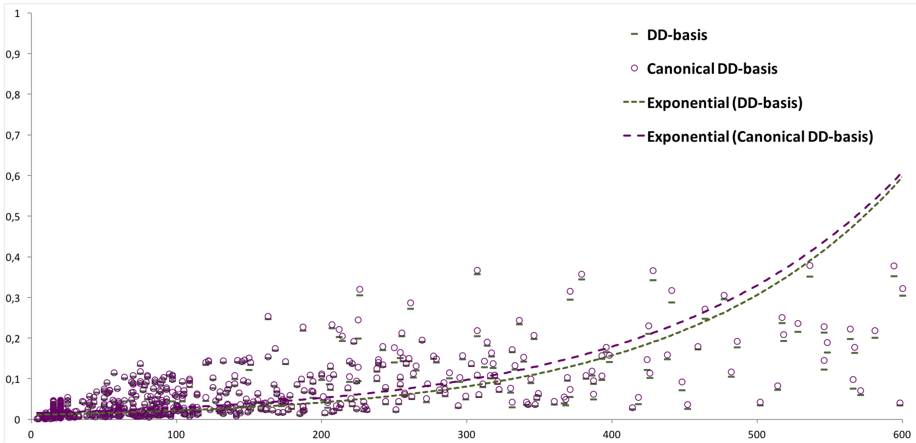

Figure 5.2: Execution times for the computation of a DD-basis (Algorithm 5.1) and its equivalent canonical one (Algorithm 5.2).

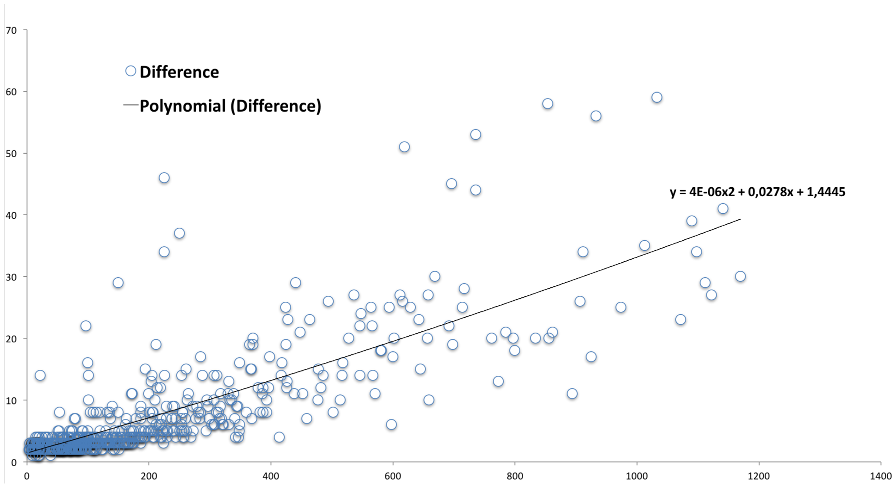

Figure 5.3: Execution time differences between the DD-basis method (Algorithm 5.1) and the canonical one (Algorithm 5.2).

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Part II

Triadic Concept Analysis

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Chapter 6

Triadic concept analysis and implications

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As stated in the introduction, Lehmann and Wille [33,58] introduced *Triadic Concept Analysis (TCA)* as a natural extension of Formal Concept Analysis (FCA). In this chapter, we summarize the basic notions of TCA for a better understanding of the results proposed in the following chapter.

6.1 Triadic context

A triadic formal context is built from a ternary relation and, hence, we need three sets, objects and attributes (as in the previous case) and also a set of conditions. Formally, we have the following:

Definition 6.1.1. *A* triadic context *is a quadruple* $\mathbb{K} = \langle G, M, B, I \rangle$ *consisting of three sets* G *(objects),* M *(attributes) and* B *(conditions) together with a ternary relation* $I \subseteq G \times M \times B$. *A triple* $(g, m, b) \in I$ *means that object* g *possesses attribute* m *under condition* b.

The natural way to represent the triadic context is as a three dimensional cross table [13] where objects, attributes and conditions are represented in each of the dimensions. It is easy to notice that it is formed by all the dyadic contexts associated with each condition, i.e. the triadic context can be visualized by placing the dyadic contexts one after the other (see

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Figure 6.1). Such a representation is similar to the representation of data cubes in the framework of Data Warehousing.

In this section we provide a background on triadic concept analysis by recalling key notions and illustrating them through the following example.

The triadic context presented in Figure 6.3 appears in [37]. It was borrowed from [25] but its meaning was adapted to represent a data cube of three dimensions: Customer, Supplier, and Product.

Example 6.1.2. *It concerns a group G of customers (1 to 5) that purchase from suppliers in M (*P*eter,* N*elson,* R*ick,* K*evin and* S*imon) products found in B (*a*ccessories,* b*ooks,* d*igital music and* e*lectronics). Thus, we consider the triadic context* $\mathbb{K} = \langle G, M, B, I\rangle$*, where* $G = \{1, 2, 3, 4, 5\}$ *is the set of customers,* $M = \{\mathrm{P}, \mathrm{N}, \mathrm{R}, \mathrm{K}, \mathrm{S}\}$ *is the set of suppliers,* $B = \{\mathrm{a}, \mathrm{b}, \mathrm{d}, \mathrm{e}\}$ *is the set of products and the ternary relation I is depicted in the three-dimensional table of Figure 6.1.*

This three-dimensional representation is uncomfortable to handle. For this reason, different authors use some form of flattening. One possibility is to select one of the sets and relate it to the Cartesian product of the other two sets in a binary context. Thus, from a triadic context $\mathbb{K} = \langle G, M, B, I\rangle$ we obtain three flat representations:

- $\mathbb{K}^{(1)} = \langle G, M \times B, I^{(1)}\rangle$ where $(g, (m, b)) \in I^{(1)}$ iff $(g, m, b) \in I$.
- $\mathbb{K}^{(2)} = \langle M, G \times B, I^{(2)}\rangle$ where $(m, (g, b)) \in I^{(2)}$ iff $(g, m, b) \in I$.
- $\mathbb{K}^{(3)} = \langle B, G \times M, I^{(3)}\rangle$ where $(b, (g, m)) \in I^{(3)}$ iff $(g, m, b) \in I$.

Example 6.1.3. *In Figure 6.2 we can view the three flat representations of the triadic context introduced in Example 6.1.2.*

There is another alternative flat representation, which is used by Ganter and Obiedkov in [25]. It consists in a table of the mapping $\mathbb{I}\colon G \times M \to 2^B$ where $\mathbb{I}(g, m) = \{b \in B \mid (g, m, b) \in I\}$. This is the representation that we will use in this work.

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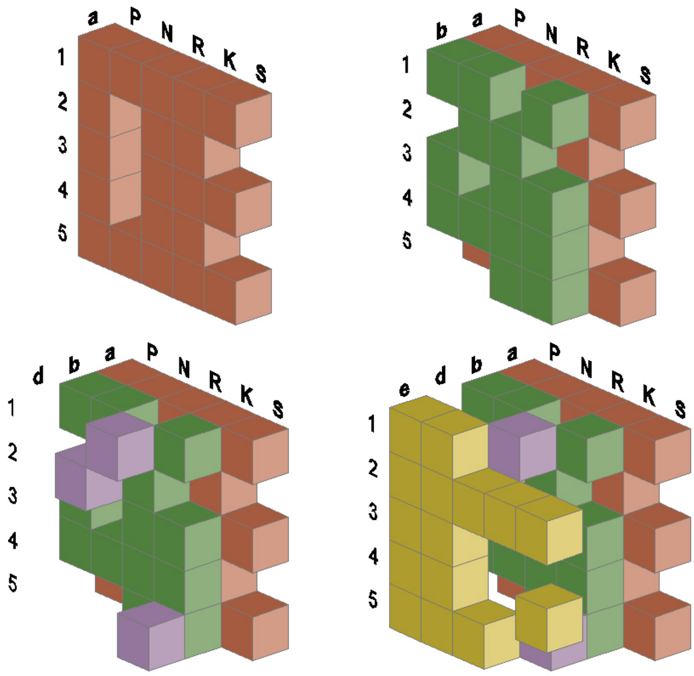

Figure 6.1: The construction of the triadic context from [37] as a three dimensional table.

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$\mathbb{K}^{(1)}$	P				N				R				K				S			
	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e
1	×	×		×	×	×		×	×		×		×	×			×			
2	×			×		×	×	×	×	×		×	×			×				×
3	×	×		×				×	×	×			×	×			×			
4	×	×		×		×		×	×	×			×	×						×
5	×			×	×			×	×	×		×	×	×	×		×			

$\mathbb{K}^{(2)}$	1				2				3				4				5			
	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e
P	×	×		×	×			×	×	×		×	×	×		×	×			×
N	×	×		×		×	×	×				×		×		×	×			×
R	×		×		×	×		×	×	×			×	×			×	×		×
K	×	×			×			×	×	×			×	×			×	×	×	
S	×							×	×							×	×			

$\mathbb{K}^{(3)}$	1					2					3					4					5				
	P	N	R	K	S	P	N	R	K	S	P	N	R	K	S	P	N	R	K	S	P	N	R	K	S
a	×	×	×	×	×	×		×	×		×		×	×	×	×		×	×		×	×	×	×	×
b	×	×		×			×	×			×		×	×		×	×	×	×				×	×	
d			×				×																	×	
e	×	×				×	×	×	×	×	×	×				×	×			×	×	×	×		

Figure 6.2: The three flat representations of the triadic context $\mathbb{K}$ from [37].

Example 6.1.4. *In Figure 6.3, the triadic context introduced in Example 6.1.2 is represented in the style of Ganter and Obiedkov.*

Thus, for instance, the value ad *at the cross of Row* 1 *and Column* R *means that Customer* 1 *orders accesories and digital music from Rick; namely,* $(1, \mathrm{R}, \mathrm{a})$ *and* $(1, \mathrm{R}, \mathrm{d})$ *belongs to* I.

After this brief introduction to triadic contexts, we summarize several important notions needed to introduce later the semantic notion of implication. This definition will lead us to develop a logic to manage these triadic implications.

6.2 Triadic Concepts

In this section we extend the notion of formal concept by defining the triadic concept. Since concepts are seen as units of thought, they tend to be homogeneous and closed [13, 33]. The homogeneity property refers to

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$\mathbb{K}$	P	N	R	K	S
1	abe	abe	ad	ab	a
2	ae	bde	abe	ae	e
3	abe	e	ab	ab	a
4	abe	be	ab	ab	e
5	ae	ae	abe	abd	a

Figure 6.3: Ganter and Obiedkov representation of the triadic context from Example 6.1.2.

the fact that all the objects share all the attributes under all the conditions within the concept. The closure property ensures that the concept is maximal with respect to homogeneity.

Definition 6.2.1. *Given a triadic context* $\mathbb{K} = \langle G, M, B, I \rangle$*, a* cuboid *in* $\mathbb{K}$ *is a triplet* (X_1, X_2, X_3) *such that* $X_1 \times X_2 \times X_3 \subseteq I$*.*

The relation $\sqsubseteq$ *among cuboids is defined as follows:*

$$(X_1, X_2, X_3) \sqsubseteq (Y_1, Y_2, Y_3) \text{ if and only if } X_1 \subseteq Y_1, X_2 \subseteq Y_2 \text{ and } X_3 \subseteq Y_3$$

for any pair of cuboids (X_1, X_2, X_3) *and* (Y_1, Y_2, Y_3)*.*

Due to the property of being closed, a triadic concept (A_1, A_2, A_3) represents a maximal cuboid of ones (or crosses) within a triadic context. Let us introduce this new notion formally.

Definition 6.2.2. *A* triadic concept *(also called* closed tri-set *or* 3-set *for short) of a triadic context* $\mathbb{K}$ *is a cuboid* (A_1, A_2, A_3) *that is maximal w.r.t. the relation* $\sqsubseteq$*. The subsets* A_1*,* A_2 *and* A_3 *are called the* extent*, the* intent *and the* modus *of the triadic concept* (A_1, A_2, A_3)*, respectively.*

Thus, a triadic concept is a cuboid (A_1, A_2, A_3) such that, for any other cuboid (X_1, X_2, X_3), if $(A_1, A_2, A_3) \sqsubseteq (X_1, X_2, X_3)$, then $(A_1, A_2, A_3) = (X_1, X_2, X_3)$.

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As in the dyadic case, all the knowledge extracted from a triadic context $\mathbb{K}$ can be represented by means of the set $\mathfrak{T}(\mathbb{K})$ of all triadic concepts of the triadic context $\mathbb{K}$.

The following example illustrates the difference between a maximal cuboid and another cuboid which is not maximal.

Example 6.2.3. *Consider the triadic context given in Figure 6.3.*

$\mathbb{K}^{(1)}$	P				N				R				K				S			
	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e
1	×	×		×	×	×		×	×		×		×	×			×			
2	×			×		×	×	×	×	×		×	×			×				×
3	×	×		×				×	×	×			×	×			×			
4	×	×		×		×		×	×	×			×	×						×
5	×			×	×			×	×	×		×	×	×	×		×			

In this context, $(135, \mathrm{PN}, \mathrm{e})$ *is a cuboid which is not maximal with respect to* $\sqsubseteq$*. However, the tri-set* $(12345, \mathrm{PN}, \mathrm{e})$ *represents a cuboid which is maximal with respect to* $\sqsubseteq$*.*

$\mathbb{K}^{(1)}$	P				N				R				K				S			
	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e
1	×	×		×	×	×		×	×		×		×	×			×			
2	×			×		×	×	×	×	×		×	×			×				×
3	×	×		×				×	×	×			×	×			×			
4	×	×		×		×		×	×	×			×	×						×
5	×			×	×			×	×	×		×	×	×	×		×			

Example 6.2.4. *From Figure 6.3, we can extract, for example, the triadic concepts (as maximal cuboids) depicted in Figures 6.4, 6.5 and 6.6.*

The tri-set $(135, \mathrm{PN}, \mathrm{e})$ *is not closed since its extent can be augmented without violating the ternary relation to get* $(12345, \mathrm{PN}, \mathrm{e})$*, as shown in Example 6.2.3.*

Once triadic concepts have been introduced, we focus on the derivation operators and the obtention of triadic concepts by means of them.

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$\mathbb{K}^{(1)}$	P				N				R				K				S			
	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e
1	×	×		×	×	×		×	×		×		×	×			×			
2	×			×		×	×	×	×	×		×	×			×				×
3	×	×		×				×	×	×			×	×			×			
4	×	×		×		×		×	×	×			×	×						×
5	×			×	×			×	×	×		×	×	×	×		×			

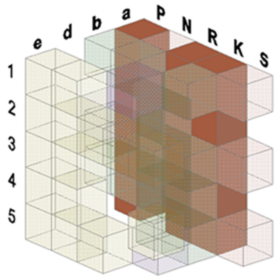

Figure 6.4: The triadic concept $(12345, \mathrm{PRK}, \mathrm{a})$ from the triadic context of Figure 6.3 as a maximal cuboid.

$\mathbb{K}^{(1)}$	P				N				R				K				S			
	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e
1	×	×		×	×	×		×	×		×		×	×			×			
2	×			×		×	×	×	×	×		×	×			×				×
3	×	×		×				×	×	×			×	×			×			
4	×	×		×		×		×	×	×			×	×						×
5	×			×	×			×	×	×		×	×	×	×		×			

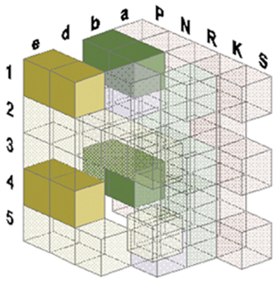

Figure 6.5: The triadic concept $(14, \mathrm{PN}, \mathrm{be})$ from the triadic context of Figure 6.3 as a maximal cuboid.

The derivation operators in triadic concept analysis were defined in [33] as a useful tool for the characterization of triadic concepts. First of all, we will define the (i)-derivation operators and the (j, k)-derivation operators which yield the derivation operators of the dyadic context $\mathbb{K}^{(i)}$ associated with the triadic context (see Figure 6.2).

Definition 6.2.5. *Let $\mathbb{K} = \langle G, M, B, I\rangle$ be a triadic context and $X_1 \subseteq G$, $X_2 \subseteq M$, $X_3 \subseteq B$.*

i) The (i)-derivation operators are defined by:

$$\begin{aligned} X_1' &= \{(m, b) \in M \times B \mid (g, m, b) \in I \text{ for all } g \in X_1\}. \\ X_2' &= \{(g, b) \in G \times B \mid (g, m, b) \in I \text{ for all } m \in X_2\}. \\ X_3' &= \{(g, m) \in G \times M \mid (g, m, b) \in I \text{ for all } b \in X_3\}. \end{aligned}$$

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$\mathbb{K}^{(1)}$	P				N				R				K				S			
	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e	a	b	d	e
1	×	×		×	×	×		×	×		×		×	×			×			
2	×			×		×	×	×	×	×		×	×			×				×
3	×	×		×				×	×	×			×	×			×			
4	×	×		×		×		×	×	×			×	×						×
5	×			×	×			×	×	×		×	×	×	×		×			

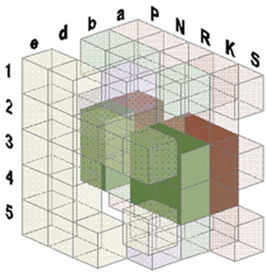

Figure 6.6: The triadic concept (34, PRK, ab) from the triadic context of Figure 6.3 as a maximal cuboid.

ii) the (j,k)-derivation operators are defined by:

$$(X_1, X_2)' = \{b \in B \mid (g,m,b) \in I \text{ for all } (g,m) \in X_1 \times X_2\}.$$
$$(X_1, X_3)' = \{m \in M \mid (g,m,b) \in I \text{ for all } (g,b) \in X_1 \times X_3\}.$$
$$(X_2, X_3)' = \{g \in G \mid (g,m,b) \in I \text{ for all } (m,b) \in X_2 \times X_3\}.$$

Let us see an example to illustrate these derivation operators.

Example 6.2.6. *Consider the formal context from Figure 6.3. Firstly, we will show the behavior of the (i)-derivation operators.*

(1)-derivation operator: $\{2,3\}' = \{(\mathrm{P},\mathrm{a}),(\mathrm{P},\mathrm{e}),(\mathrm{N},\mathrm{e}),(\mathrm{R},\mathrm{a}),(\mathrm{R},\mathrm{b}),(\mathrm{K},\mathrm{a})\}$.

(2)-derivation operator: $\{\mathrm{N},\mathrm{K},\mathrm{S}\}' = \{(1,\mathrm{a}),(2,\mathrm{e}),(5,\mathrm{a})\}$.

(3)-derivation operator: $\{\mathrm{d}\}' = \{(1,\mathrm{R}),(2,\mathrm{N}),(5,\mathrm{K})\}$.

Now, let us see the behavior of the (j,k)-derivation operators.

$(1,2)$-derivation operator: $\big(\{1,2,3,4,5\},\{\mathrm{P},\mathrm{N}\}\big)' = \{\mathrm{e}\}$.

$(2,3)$-derivation operator: $\big(\{\mathrm{P},\mathrm{N}\},\{\mathrm{b},\mathrm{e}\}\big)' = \{1,4\}$.

$(1,3)$-derivation operator: $\big(\{2,5\},\{\mathrm{a}\}\big)' = \{\mathrm{P},\mathrm{R},\mathrm{K}\}$.

The following theorem, presented in [8], will lead to a new definition of triadic concepts in terms of the (j,k)-derivation operators.

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Theorem 6.2.7. *Let $\mathbb{K}$ be a triadic context. A triple (A_1, A_2, A_3) is a triadic concept of $\mathbb{K}$ if and only if $A_1 = (A_2, A_3)', A_2 = (A_1, A_3)'$ and $A_3 = (A_1, A_2)'$.*

Proof. $\Rightarrow$: Consider a triadic context (A_1, A_2, A_3) and $X_1 \subseteq G$ such that $X_1 = (A_2, A_3)'$. From $A_1 \times A_2 \times A_3 \subseteq I$, one has that $A_1 \subseteq X_1$ and $X_1 \times A_2 \times A_3 \subseteq I$. By Definition 6.2.2, since the triadic concept is a maximal cuboid, we have that $X_1 = A_1$. For A_2 and A_3, the proof is analogous.

$\Leftarrow$: Consider $A_1 \subseteq X_1$, $A_2 \subseteq X_2$, and $A_3 \subseteq X_3$. From the chain of inclusions $X_1 \subseteq (X_2, X_3)' \subseteq (X_2, A_3)' \subseteq (A_2, A_3)' = A_1$, we conclude $X_1 \subseteq A_1$ and consequently, $A_1 = X_1$. In a similar way, we obtain $A_2 = X_2$ and $A_3 = X_3$ and, thus, (A_1, A_2, A_3) is a triadic concept. $\square$

As shown in [58], the family of (j, k)-derivation operators, by setting a subset of objects, attributes or conditions (respectively) yields a family of Galois connections and leads us to the generation of triadic concepts.

After defining the derivation operators we can introduce the notion of implication in the triadic framework. The following section surveys the different notions of triadic implications in the literature and formally defines the kind of implications we are interested in.

6.3 Triadic Implications

As far as we know, the first definition of triadic implication is due to Biedermann [13]. He considers that a *triadic implication* has the form $(X \to Y)_{\mathcal{C}}$ and holds if "whenever X occurs under all conditions in $\mathcal{C}$, then Y also occurs under the same conditions". Its definition is formally introduced as follows.

Definition 6.3.1. *Let $\mathbb{K} = \langle G, M, B, I\rangle$ be a triadic context, $X, Y \subseteq M$ and $\mathcal{C} \subseteq B$. The implication $(X \to Y)_{\mathcal{C}}$ holds in the context $\mathbb{K}$ when $(X, \mathcal{C})' \subseteq (Y, \mathcal{C})'$.*

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Later on, Ganter and Obiedkov [25] extended Biedermann's work and defined three types of implications:

Attribute$\times$*condition* implications (AxCIs)

An *attribute*$\times$*condition* implication (AxCI) has the form $X \to Y$, where X and Y are subsets of $M\times B$. Such implications are extracted from the dyadic context $\mathbb{K}^{(1)} = \langle G, M \times B, I^{(1)}\rangle$, being classical dyadic implications (see Definition 1.3.3). For example, the AxCI

$$\text{(R,d)} \to \text{(P,a) (P,b) (P,e) (N,a) (N,b) (N,e) (R,a) (K,a) (K,b) (S,a)}$$

holds in the dyadic context $\mathbb{K}^{(1)}$ from Figure 6.2.

Attributional condition implications (ACIs).

An *attributional condition* implication (ACI) can be considered a proper extension of the classical notion of implication, being an expression of the form $X \xrightarrow{A} Y$, where $X, Y \subseteq B$ and $A \subseteq M$. For instance, the ACI b $\xrightarrow{\text{PN}}$ e holds in $\mathbb{K}$ from Figure 6.3 since whenever *b*ooks are supplied by both *P*eter and *N*elson, then *e*lectronics are also provided by all these two suppliers.

Conditional attribute implications (*CAI*s)

In a dual way, a *conditional attribute* implication CAI takes the form: $X \xrightarrow{\mathcal{C}} Y$, where X and Y are subsets of M, and $\mathcal{C}$ is a subset of B. It means that X *implies* Y *under each condition in* $\mathcal{C}$ and, therefore, *for any subset in* $\mathcal{C}$. Using our context in Figure 6.3, the CAI N $\xrightarrow{\text{ae}} P$ states that whenever *N*elson supplies *a*ccessories and *e*lectronics (or any one of these two products), then *P*eter does so.

As Biedermann says in [13], the implications that he presents "might have seemed a little artificial but they are suitable for an introduction of triadic implications". In addition, the AxCIs are exactly the dyadic implications we know and the other ones have a dual structure. So, we focus on CAIs following [25]. Moreover, if we construct an implication logic for CAIs, analogously we would obtain an implication logic for ACIs.

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First of all, we will give the formal definition of CAIs based on the derivation operators.

Definition 6.3.2. *Let $\mathbb{K} = \langle G, M, B, I\rangle$ be a triadic context, $X, Y \subseteq M$ and $\mathcal{C} \subseteq B$. The implication $X \xrightarrow{\mathcal{C}} Y$ holds in the context $\mathbb{K}$ when $(X, \{c\})' \subseteq (Y, \{c\})'$ for all $c \in \mathcal{C}$.*

So, a CAI holds in a context if it is valid under each single condition. Let us see an example to clarify this notion.

Example 6.3.3. *Let $\mathbb{K} = \langle G, M, B, I\rangle$ be the triadic context depicted in Figure 6.3. The implication $\mathrm{PN} \xrightarrow{\mathrm{abd}} \mathrm{K}$ is satisfied in the triadic context $\mathbb{K}$ because $(\mathrm{PN}, \{c\})' \subseteq (\mathrm{K}, \{c\})'$ for all $c \in \{\mathrm{a}, \mathrm{b}, \mathrm{d}\}$:*

- $(\mathrm{PN}, \{\mathrm{a}\})' = \{1, 5\} \subseteq (\mathrm{K}, \{\mathrm{a}\})' = G$
- $(\mathrm{PN}, \{\mathrm{b}\})' = \{1, 4\} \subseteq (\mathrm{K}, \{\mathrm{b}\})' = \{1, 3, 4, 5\}$
- $(\mathrm{PN}, \{\mathrm{d}\})' = \emptyset \subseteq (\mathrm{K}, \{\mathrm{d}\})' = \{5\}$

Certainly, the previous implication can be linked to Biedermann's definition of triadic implication. The following proposition relates both notions of implications and also shows that Biedermann's approach is weaker than the other one.

Proposition 6.3.4 (Ganter et al. [25])**.** *Let $\mathbb{K} = \langle G, M, B, I\rangle$ be a triadic context, $X, Y \subseteq M$ and $\mathcal{C} \subseteq B$. Then $X \xrightarrow{\mathcal{C}} Y$ holds in $\mathbb{K}$ if and only if $(X \to Y)_{\mathcal{N}}$ also holds in $\mathbb{K}$ for all $\mathcal{N} \subseteq \mathcal{C}$.*

From Theorem 6.3.4 and Definitions 6.3.1 and 6.3.2, the following corollary is straightforward.

Corollary 6.3.5. *Let $\mathbb{K} = \langle G, M, B, I\rangle$ be a triadic context, $X, Y \subseteq M$ and $\mathcal{C} \subseteq B$. Then $X \xrightarrow{\mathcal{C}} Y$ holds in $\mathbb{K}$ if and only if $(X \to Y)_c$ also holds in $\mathbb{K}$ for all $c \in \mathcal{C}$.*

This chapter concludes with an example that illustrates the difference between Biedermann's triadic implications and CAIs.

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Example 6.3.6. *Considering the context* $\mathbb{K}$ *depicted in Figure 6.3,* $\mathrm{N} \xrightarrow{\mathrm{ae}} \mathrm{P}$ *holds since* $(\mathrm{N} \to \mathrm{P})_{\mathcal{C}}$ *is satisfied for any* $\mathcal{C} \subseteq \{\mathrm{a}, \mathrm{e}\}$*. On the other hand, although* $(\mathrm{N} \to \mathrm{P})_{\mathrm{abe}}$ *holds,* $\mathrm{N} \xrightarrow{\mathrm{abe}} \mathrm{P}$ *does not, since* $(\mathrm{N} \to \mathrm{P})_{\mathcal{C}}$ *does not hold for any* $\mathcal{C} \subseteq \{\mathrm{a}, \mathrm{b}, \mathrm{e}\}$*. For instance,* $(\mathrm{N} \to \mathrm{P})_{\mathrm{be}}$ *does not hold.*

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Chapter 7

Conditional Attribute Implications: CAIs

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The notion of triadic implication is a natural extension of the attribute implications in Formal Concept Analysis. Our interest resides on the study of these triadic implications to develop logics to manage them. As stated in the previous chapter, Biedermann [13] was the first who investigated implications in triadic contexts and, later, Ganter and Obiedkov [25] explored other variants of triadic implications. Among their definitions of triadic implications, we focus especially on the so-called *conditional attribute implications*.

In this chapter, we introduce a logic for reasoning with conditional attribute implications. Specifically, we present the three pillars of the logic: the language, the semantics, and a syntactic proof system. In fact, we present three equivalent syntactic proof systems, all of which are sound and complete. Soundness ensures that implications derived by using the axiomatic system hold in the formal context and completeness guarantees that all implications which are satisfied can be derived from the implicational system. As far as we know, no such axiomatic system has been introduced so far in triadic concept analysis.

The results described in this chapter are the culmination of a collaboration with Rokia Missaoui and have been published in [51, 52].

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7.1 CAIL: Conditional Attribute Implication Logic

In this section, we introduce a logic for reasoning on conditional attribute implications in the framework of triadic formal concept analysis. This logic is presented in a classical style by considering its three pillars: the language, the semantics and the inference system.

7.1.1 Language

First, we consider a two-sorted alphabet: we assume the existence of a logical alphabet that consists of a finite set Ω of attribute symbols and a finite set Γ of conditions.

The set of well formed formulas (hereinafter, they will be called *formulas*, *implications* or *CAIs*) is

$$\mathcal{L}_{\Omega,\Gamma} = \{X \xrightarrow{\mathcal{C}} Y \mid X, Y \subseteq \Omega, \mathcal{C} \subseteq \Gamma\}.$$

Following the same scheme than in the previous chapters, we use capital letters $X, Y, Z, W, \dots$, possibly with subscripts, to denote subsets of attributes ($X, Y, Z, W \subseteq \Omega$) and calligraphic capital letters $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2, \dots$ to denote subsets of conditions ($\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2 \subseteq \Gamma$).

For the sake of readability, inside the formulas, we omit the brackets and commas (e.g. *abc* denotes the set $\{a, b, c\}$) and, as usual, the union is denoted by juxtaposition (e.g. XY denotes $X \cup Y$) and the set difference by the symbol "-" (e.g. X-Y denotes $X \smallsetminus Y$).

Example 7.1.1. *Let* $\Omega = \{a, b, c, d\}$ *and* $\Gamma = \{c_1, c_2, c_3\}$ *and consider* $X = \{a, b\}$, $Y = \{b, c\}$, $Z = \{c, d\}$, $\mathcal{C}_1 = \{c_1, c_2\}$ *and* $\mathcal{C}_2 = \{c_2, c_3\}$.

- $X \xrightarrow{\mathcal{C}_1} Y$ *is written as* $ab \xrightarrow{c_1c_2} bc$ *instead of* $\{a, b\} \xrightarrow{\{c_1,c_2\}} \{b, c\}$.
- $XY \xrightarrow{\mathcal{C}_1\text{-}\mathcal{C}_2} Z\text{-}Y$ *denotes* $abc \xrightarrow{c_1} d$.

7.1.2 Semantics

Based on Definition 6.3.2, the semantics is introduced by means of the notions of interpretation and model.

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Definition 7.1.2. *An* interpretation *for the language* $\mathcal{L}_{\Omega,\Gamma}$ *is a triplet* $\langle \mathbb{K}, h_1, h_2 \rangle$ *where* $\mathbb{K}$ *is a triadic context* $\langle G, M, B, I \rangle$*, and* $h_1 : \Omega \to M$ *and* $h_2 : \Gamma \to B$ *are injective mappings.*

The following example illustrates the above definition.

$\mathbb{K}$	**P**eter	**N**elson	**R**ick	**K**evin	**S**imon
1	abe	abe	ad	ab	a
2	ae	bde	abe	ae	e
3	abe	e	ab	ab	a
4	abe	be	ab	ab	e
5	ae	ae	abe	abd	a

Figure 7.1: The triadic context borrowed from [37] presented in Figure 6.3.

Example 7.1.3. *Consider* $\Omega = \{\mathrm{P}, \mathrm{N}, \mathrm{R}, \mathrm{K}, \mathrm{S}\}$ *and* $\Gamma = \{\mathrm{a}, \mathrm{b}, \mathrm{d}, \mathrm{e}\}$*. Consider also the triadic context* $\mathbb{K} = \langle G, M, B, I \rangle$*, about Customers, Suppliers, and Products where*

$$G = \{1, 2, 3, 4, 5\}$$
$$M = \{\text{Peter, Nelson, Rick, Kevin, Simon}\}$$
$$B = \{\textbf{a}\text{ccessories, }\textbf{b}\text{ooks, }\textbf{d}\text{igital music, }\textbf{e}\text{lectronics}\}$$

and I *is the relation depicted in Figure 7.1. The triple* $\langle \mathbb{K}, h_1, h_2 \rangle$ *is an interpretation for* $\mathcal{L}_{\Omega,\Gamma}$ *where* $h_1 : \Omega \to M$ *and* $h_2 : \Gamma \to B$ *are the mappings:*

x	P	N	R	K	S
$h_1(x)$	Peter	Nelson	Rick	Kevin	Simon

x	a	b	d	e
$h_2(x)$	accessories	books	digital music	electronics

Hereafter, for simplicity and without loss of generality we identify interpretations for implications in $\mathcal{L}_{\Omega,\Gamma}$ with triadic contexts $\mathbb{K} = \langle G, M, B, I \rangle$ such that $\Omega \subseteq M$, $\Gamma \subseteq B$, and the mappings h_1 and h_2 are the respective embeddings.

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Definition 7.1.4. *An interpretation* $\mathbb{K}$ *is said to be a* model *for a given formula* $X \xrightarrow{\mathcal{C}} Y \in \mathcal{L}_{\Omega,\Gamma}$ *if the following condition holds:*

$$(X, \{c\})' \subseteq (Y, \{c\})' \quad \textit{for all} \quad c \in \mathcal{C}. \tag{7.1}$$

In addition, the interpretation is model for a set $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ *if it is a model for all* $\sigma \in \Sigma$.

As usual, $\mathbb{K} \models X \xrightarrow{\mathcal{C}} Y$ and $\mathbb{K} \models \Sigma$ denote that $\mathbb{K}$ is a model for $X \xrightarrow{\mathcal{C}} Y$ and $\mathbb{K}$ is model for Σ respectively.

Example 7.1.5. *The interpretation presented in Example 7.1.3 is a model for the formula* $\mathrm{PN} \xrightarrow{\mathrm{abd}} \mathrm{K}$ *because*

- $(\{\mathrm{P}, \mathrm{N}\}, \{\mathrm{a}\})' = \{1, 5\} \subseteq (\{\mathrm{K}\}, \{\mathrm{a}\})' = G$
- $(\{\mathrm{P}, \mathrm{N}\}, \{\mathrm{b}\})' = \{1, 4\} \subseteq (\{\mathrm{K}\}, \{\mathrm{b}\})' = \{1, 3, 4, 5\}$
- $(\{\mathrm{P}, \mathrm{N}\}, \{\mathrm{d}\})' = \varnothing \subseteq (\{\mathrm{K}\}, \{\mathrm{d}\})' = \{5\}$

In addition, $\mathbb{K}$ *is a model for the following set of conditional attribute implications, i.e.,* $\mathbb{K} \models \Sigma$.

$$\begin{array}{lllll}
\Sigma & = \{ & \mathrm{P} \xrightarrow{\mathrm{de}} \mathrm{N}, & \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{P}, & \mathrm{S} \xrightarrow{\mathrm{bde}} \mathrm{P}, \\
 & & \mathrm{P} \xrightarrow{\mathrm{ad}} \mathrm{R}, & \mathrm{R} \xrightarrow{\mathrm{a}} \mathrm{K}, & \mathrm{S} \xrightarrow{\mathrm{ab}} \mathrm{RK}, \\
 & & \mathrm{P} \xrightarrow{\mathrm{d}} \mathrm{S}, & \mathrm{R} \xrightarrow{\mathrm{e}} \mathrm{N}, & \mathrm{PK} \xrightarrow{\mathrm{ade}} \mathrm{R}, \\
 & & \mathrm{P} \xrightarrow{\mathrm{bd}} \mathrm{K}, & \mathrm{K} \xrightarrow{\mathrm{e}} \mathrm{NS}, & \mathrm{NK} \xrightarrow{\mathrm{abde}} \mathrm{P}, \\
 & & \mathrm{N} \xrightarrow{\mathrm{ae}} \mathrm{P}, & \mathrm{K} \xrightarrow{\mathrm{ae}} \mathrm{R}, & \mathrm{RS} \xrightarrow{\mathrm{abde}} \mathrm{K}, \\
 & & \mathrm{N} \xrightarrow{\mathrm{a}} \mathrm{RS}, & \mathrm{S} \xrightarrow{\mathrm{b}} \mathrm{N}\} &
\end{array}$$

Notice that Equation (7.1) implies $(X, \mathcal{C})' \subseteq (Y, \mathcal{C})'$, but they are not equivalent, as the following example shows.

Example 7.1.6. *Considering the context* $\mathbb{K}$ *depicted in Figure 7.1, we have that* $(\{\mathrm{N}\}, \{\mathrm{abe}\})' \subseteq (\{\mathrm{P}\}, \{\mathrm{abe}\})'$ *holds. However,* $\mathbb{K} \not\models \mathrm{N} \xrightarrow{\mathrm{abe}} \mathrm{P}$ *because, for instance,* $(\{\mathrm{N}\}, \{\mathrm{b}\})' \subseteq (\{\mathrm{P}\}, \{\mathrm{b}\})'$ *does not hold.*

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As usual, the notions of interpretation and model leads to the notion of semantic derivation.

Definition 7.1.7. *Consider* $\Sigma, \Sigma_1, \Sigma_2 \subseteq \mathcal{L}_{\Omega,\Gamma}$ *and* $\sigma \in \mathcal{L}_{\Omega,\Gamma}$*.*

- σ *is* semantically derived *from* Σ *if* $\mathbb{K} \models \Sigma$ *implies* $\mathbb{K} \models \sigma$*, for all interpretations* $\mathbb{K}$*. It is denoted by* $\Sigma \models \sigma$*.*

- Σ_1 *and* Σ_2 *are said to be (semantically)* equivalent *if they have the same models, i.e.* $\mathbb{K} \models \Sigma_1$ *if and only if* $\mathbb{K} \models \Sigma_2$ *for any interpretation* $\mathbb{K}$*.*

Finally, the following proposition is trivially obtained from the previous definitions.

Proposition 7.1.8. *For all* $X, Y \subseteq \Omega$ *and* $\mathcal{C} \subseteq \Gamma$*, the following equivalences hold:*

$$\{X \xrightarrow{\mathcal{C}} Y\} \equiv \{X \xrightarrow{\mathcal{C}_1} Y \mid \mathcal{C}_1 \subseteq \mathcal{C}\} \equiv \{X \xrightarrow{c} Y \mid c \in \mathcal{C}\}.$$

7.1.3 Syntactic inference

As we have already stated, one of the main aims of this chapter is to introduce a sound and complete axiomatic system for *CAI*s that represents the counterparts of the well-known Armstrong's axioms [27]. Now, we present this novel axiomatic system and, then, in the following sections, we will prove its soundness and completeness.

Definition 7.1.9. *The CAIL axiomatic system has two axiom schemes:*

$$\textit{Non-constraint } \texttt{[Nc]}: \quad \frac{}{\varnothing \xrightarrow{\varnothing} \Omega}$$

$$\textit{Inclusion } \texttt{[Inc]}: \quad \frac{}{X \xrightarrow{\Gamma} Y}, \textit{ where } Y \subseteq X$$

Together with the following four primitive inference rules:

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Augmentation [Aug]: $\dfrac{X \xrightarrow{\mathcal{C}} Y}{XZ \xrightarrow{\mathcal{C}} YZ}$

Transitivity [Tran]: $\dfrac{X \xrightarrow{\mathcal{C}_1} Y, Y \xrightarrow{\mathcal{C}_2} Z}{X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} Z}$

Conditional Decomposition [ConDec]: $\dfrac{X \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y}{X \xrightarrow{\mathcal{C}_1} Y}$

Conditional Composition [ConCom]: $\dfrac{X \xrightarrow{\mathcal{C}_1} Y, Z \xrightarrow{\mathcal{C}_2} W}{XZ \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y \cap W}$

The notion of *syntactic derivation* is introduced as usual:

Definition 7.1.10. *Consider $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ and $\sigma \in \mathcal{L}_{\Omega,\Gamma}$. The formula σ is said to be* syntactically derived*, or* inferred*, from Σ by using* CAIL*, denoted $\Sigma \vdash_{\mathscr{C}} \sigma$, if there exists a chain of formulas $\sigma_1, \ldots, \sigma_n \in \mathcal{L}_{\Omega,\Gamma}$ such that $\sigma_n = \sigma$ and, for all $1 \leq i \leq n$, one of the following conditions holds:*

(i) *σ_i is an axiom.*

(ii) *$\sigma_i \in \Sigma$.*

(iii) *σ_i is obtained by applying the inference rules in* CAIL *to formulas in $\{\sigma_j \mid 1 \leq j < i\}$.*

In this case, we say that the sequence $\sigma_1, \ldots, \sigma_n$ is a proof *for $\Sigma \vdash_{\mathscr{C}} \sigma$.*

The following example shows how CAIL axiomatic system is used to derive new formulas.

Example 7.1.11. *Consider the set Σ introduced in Example 7.1.5. The following sequence is a proof for $\Sigma \vdash_{\mathscr{C}} \mathrm{PN} \xrightarrow{\mathrm{abd}} \mathrm{K}$.*

$\sigma_1 :\ \mathrm{P} \xrightarrow{\mathrm{bd}} \mathrm{K}$ . . . . *belongs to Σ.*

$\sigma_2 :\ \mathrm{PN} \xrightarrow{\mathrm{bd}} \mathrm{KN}$ . . . . *by applying* [Aug] *to σ_1 with* N.

$\sigma_3 :\ \mathrm{KN} \xrightarrow{\mathrm{bd}} \mathrm{K}$ . . . . *by* [Inc].

$\sigma_4 :\ \mathrm{PN} \xrightarrow{\mathrm{bd}} \mathrm{K}$ . . . . *by applying* [Tran] *to σ_2 and σ_3.*

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σ_5 : N $\xrightarrow{\text{a}}$ RS . . . . *belongs to* Σ.

σ_6 : RS $\xrightarrow{\text{a}}$ R . . . . *by* [Inc].

σ_7 : N $\xrightarrow{\text{a}}$ R . . . . *by applying* [Tran] *to* σ_5 *and* σ_6.

σ_8 : R $\xrightarrow{\text{a}}$ K . . . . *belongs to* Σ.

σ_9 : N $\xrightarrow{\text{a}}$ K . . . . *by applying* [Tran] *to* σ_7 *and* σ_8.

σ_{10} : PN $\xrightarrow{\text{a}}$ PK . . . . *by applying* [Aug] *to* σ_9 *with* P.

σ_{11} : PK $\xrightarrow{\text{a}}$ K . . . . *by* [Inc].

σ_{12} : PN $\xrightarrow{\text{a}}$ K . . . . *by applying* [Tran] *to* σ_{10} *and* σ_{11}.

σ_{13} : PN $\xrightarrow{\text{abd}}$ K . . . . *by applying* [ConCom] *to* σ_4 *and* σ_{12}.

Derived inference rules

Before examining the soundness and completeness of this axiomatic system, we introduce a set of derived inference rules which will be used to shorten the proofs. We name the new derived rules, and provide their schemas and proofs.

Decomposition [Dec]: $\{X \xrightarrow{\mathcal{C}} YZ\} \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y$.

$\sigma_1 : X \xrightarrow{\mathcal{C}} YZ$ . . . . by hypothesis.
$\sigma_2 : YZ \xrightarrow{\Gamma} Y$ . . . . by [Inc].
$\sigma_3 : X \xrightarrow{\mathcal{C}} Y$ . . . . by [Tran] from σ_1 and σ_2.

Pseudotransitivity [PsTr]: $\{X \xrightarrow{\mathcal{C}_1} Y, YZ \xrightarrow{\mathcal{C}_2} W\} \vdash_{\mathscr{C}} XZ \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W$

$\sigma_1 : X \xrightarrow{\mathcal{C}_1} Y$ . . . . by hypothesis.
$\sigma_2 : YZ \xrightarrow{\mathcal{C}_2} W$ . . . . by hypothesis.
$\sigma_3 : XZ \xrightarrow{\mathcal{C}_1} YZ$ . . . . by [Aug] from σ_1 with Z.
$\sigma_4 : XZ \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W$ . . . . by [Tran] from σ_3 and σ_2.

Additivity [Add]: $\{X \xrightarrow{\mathcal{C}_1} Y, X \xrightarrow{\mathcal{C}_2} Z\} \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YZ$

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$\sigma_1 : X \xrightarrow{\mathcal{C}_1} Y$ .... by hypothesis.

$\sigma_2 : X \xrightarrow{\mathcal{C}_2} Z$ .... by hypothesis.

$\sigma_3 : XY \xrightarrow{\mathcal{C}_2} YZ$ .... by `[Aug]` from σ_2 with Y.

$\sigma_4 : X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YZ$ .... by `[PsTr]` from σ_1 and σ_3.

Accumulation `[Acc]`: $\{X \xrightarrow{\mathcal{C}_1} YZ, Z \xrightarrow{\mathcal{C}_2} W\} \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YZW$

$\sigma_1 : X \xrightarrow{\mathcal{C}_1} YZ$ .... by hypothesis.

$\sigma_2 : Z \xrightarrow{\mathcal{C}_2} W$ .... by hypothesis.

$\sigma_3 : X \xrightarrow{\mathcal{C}_1} Z$ .... by `[Dec]` from σ_1.

$\sigma_4 : X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W$ .... by `[Tran]` from σ_3 and σ_2.

$\sigma_5 : X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YZW$ .... by `[Add]` from σ_1 and σ_4.

Hereafter, we can use these derived inference rules within a derivation chain on CAIL.

The following proposition also introduces an initial set of formal theorems (in logic a formal theorem is a formula which can be derived from axioms by using inference rules).

Proposition 7.1.12. *For any $X, Y \subseteq \Omega$, the implication $X \xrightarrow{\varnothing} Y$ is a theorem, that is*

$$\vdash_{\mathscr{C}} X \xrightarrow{\varnothing} Y$$

Proof.

σ_1: $\varnothing \xrightarrow{\varnothing} \Omega$ .... by `[Nc]`.

σ_2: $X \xrightarrow{\varnothing} \Omega$ .... by `[Aug]` from σ_1 with X.

σ_3: $X \xrightarrow{\varnothing} Y$ .... by `[Dec]` from σ_2.

□

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Syntactic closure operator

Now, we extend the definition of classical syntactic closure to triadic concept analysis. For a set of attributes X, its syntactic closure with respect to a set of CAIs Σ under the conditions in $\mathcal{C}$, denoted by $X^+_{\Sigma,\mathcal{C}}$, will be the maximum set of attributes Y satisfying $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y$.

Definition 7.1.13. *Consider $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ and $\mathcal{C} \subseteq \Gamma$. The* syntactic closure operator *with respect to Σ under the conditions in $\mathcal{C}$ is defined as follows:*

$$(-)^+_{\Sigma,\mathcal{C}} \colon 2^\Omega \to 2^\Omega \quad \textit{where} \quad X^+_{\Sigma,\mathcal{C}} = \{m \in \Omega \mid \Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} m\}$$

The following lemma is crucial to prove that this operator is really a closure operator.

Lemma 7.1.14. *Consider $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ and $\mathcal{C} \subseteq \Gamma$. For all $X \subseteq \Omega$ one has*

$$\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} X^+_{\Sigma,\mathcal{C}}$$

Proof. Since Ω is finite, we have that $X^+_{\Sigma,\mathcal{C}}$ is also finite.

Assume $X^+_{\Sigma,\mathcal{C}} = \{m_i \mid 1 \leq i \leq k\}$. For each $1 \leq i \leq k$ there exists a sequence $\sigma_{i1}, \ldots, \sigma_{ij_i}$ that proves $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} m_i$. Then, a proof for $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} X^+_{\Sigma,\mathcal{C}}$ is the sequence $\sigma_{11}, \ldots, \sigma_{1j_1}, \ldots, \sigma_{k1}, \ldots, \sigma_{kj_k}, \sigma_2, \ldots, \sigma_k$ where $\sigma_h = (X \xrightarrow{\mathcal{C}} m_1 \ldots m_h)$ for each $2 \leq h \leq k$, the formula σ_2 is obtained by applying `[Add]` to σ_{1j_1} and σ_{2j_2}, and σ_h is obtained by applying `[Add]` to $\sigma_{(h-1)}$ and σ_{hj_h} for each $2 < h \leq k$. □

Theorem 7.1.15. *Consider $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ and $\mathcal{C} \subseteq \Gamma$. The operator $(-)^+_{\Sigma,\mathcal{C}}$ is a closure operator.*

Proof.

- $(-)^+_{\Sigma,\mathcal{C}}$ is isotone: we assume $X_1 \subseteq X_2 \subseteq \Omega$ and prove the inclusion $(X_1)^+_{\Sigma,\mathcal{C}} \subseteq (X_2)^+_{\Sigma,\mathcal{C}}$. Consider $m \in (X_1)^+_{\Sigma,\mathcal{C}}$. By definition, there exists a sequence $\sigma_1, \ldots, \sigma_n$ that proves $\Sigma \vdash_{\mathscr{C}} X_1 \xrightarrow{\mathcal{C}} m$. Thus, a sequence proving $\Sigma \vdash_{\mathscr{C}} X_2 \xrightarrow{\mathcal{C}} m$ is obtained by adding the following formulas:

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σ_{n+1}: $X_2 \xrightarrow{\mathcal{C}} mX_2$ ...... by applying [Aug] to σ_n with X_2.
σ_{n+2}: $mX_2 \xrightarrow{\Gamma} m$ .... by [Inc].
σ_{n+3}: $mX_2 \xrightarrow{\mathcal{C}} m$ .... by applying [ConDec] to σ_{n+2}.
σ_{n+4}: $X_2 \xrightarrow{\mathcal{C}} m$ .... by applying [Tran] to σ_{n+1} and σ_{n+3}.

Therefore, $m \in (X_2)^+_{\Sigma,\mathcal{C}}$.

- $(-)^+_{\Sigma,\mathcal{C}}$ is inflationary, i.e. $X \subseteq (X)^+_{\Sigma,\mathcal{C}}$: The following sequence proves $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} m$ for all $m \in X$.

  σ_1: $X \xrightarrow{\Gamma} m$ .... by [Inc].
  σ_2: $X \xrightarrow{\mathcal{C}} m$ .... by applying [ConDec] to σ_1.

- $(-)^+_{\Sigma,\mathcal{C}}$ is idempotent: First, since the operator is inflationary, one has $X \subseteq X^+_{\Sigma,\mathcal{C}}$ and, since it is also isotone, $X^+_{\Sigma,\mathcal{C}} \subseteq (X^+_{\Sigma,\mathcal{C}})^+_{\Sigma,\mathcal{C}}$.

  Conversely, we prove $(X^+_{\Sigma,\mathcal{C}})^+_{\Sigma,\mathcal{C}} \subseteq X^+_{\Sigma,\mathcal{C}}$. Consider $m \in (X^+_{\Sigma,\mathcal{C}})^+_{\Sigma,\mathcal{C}}$, that is, $\Sigma \vdash_{\mathscr{C}} X^+_{\Sigma,\mathcal{C}} \xrightarrow{\mathcal{C}} m$. By Lemma 7.1.14, $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} X^+_{\Sigma,\mathcal{C}}$ and, by [Tran], $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} m$. Therefore, $m \in X^+_{\Sigma,\mathcal{C}}$.

□

The following theorem establishes the relationship between the syntactic derivation of a formula and the syntactic closure operator introduced above.

Theorem 7.1.16. *Consider $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$, $\mathcal{C} \subseteq \Gamma$ and $X, Y \subseteq \Omega$. Then*

$$\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y \quad \text{if and only if} \quad Y \subseteq X^+_{\Sigma,\mathcal{C}}.$$

Proof.

- Assume $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y$ and $m \in Y$. The following sequence proves $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} m$, i.e. $m \in X^+_{\Sigma,\mathcal{C}}$:

  σ_1: $Y \xrightarrow{\Gamma} m$ .... by [Inc].
  σ_2: $Y \xrightarrow{\mathcal{C}} m$ .... by applying [ConDec].
  σ_3: $X \xrightarrow{\mathcal{C}} Y$ .... by hypothesis.
  σ_4: $X \xrightarrow{\mathcal{C}} m$ .... by applying [Tran] to σ_3 and σ_2.

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- Conversely, assume $Y \subseteq X^+_{\Sigma,\mathcal{C}}$. By Lemma 7.1.14, there exists a sequence $\sigma_1, \ldots, \sigma_n$ that proves $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} X^+_{\Sigma,\mathcal{C}}$. Thus, a sequence proving $\Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y$ is obtained by adding the following formulas:

  σ_{n+1}: $X^+_{\Sigma,\mathcal{C}} \xrightarrow{\Gamma} Y$ .... by [Inc].

  σ_{n+2}: $X^+_{\Sigma,\mathcal{C}} \xrightarrow{\mathcal{C}} Y$ .... by applying [ConDec] to σ_{n+1}.

  σ_{n+3}: $X \xrightarrow{\mathcal{C}} Y$ .... by applying [Tran] to σ_n and σ_{n+2}.

□

The previous theorem allows the use of the closure operator as a keystone for the definition of an automated reasoning method, as it is usual in dyadic FCA. The syntactic closure becomes the main actor in the automatization of the implication problem. The method presented below is strongly inspired by the classical closure method due to Maier [36] and the following result.

Proposition 7.1.17. *Consider* $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ *and* $\mathcal{C} \subseteq \Gamma$. *Then, for all* $X \subseteq \Omega$,

$$X^+_{\Sigma,\mathcal{C}} = \bigcap_{c\in\mathcal{C}} X^+_{\Sigma,\{c\}}$$

Proof. It is a direct consequence of [ConCom] and [ConDec]. □

The algorithm to compute the operator $X^+_{\Sigma,\mathcal{C}}$ is shown in Function CAIL-Closure. Essentially, this function works as follows: for each atomic condition $c \in C$ (label 1 in the pseudocode) Function CAIL-Closure goes through the set of *CAI*s (label 3) computing the closure of X w.r.t. Σ under c. Finally, (label 4) the intersection of such closures is computed iteratively.

We illustrate this algorithm with an example.

Example 7.1.18. *Let us compute* $\mathrm{PN}^+_{\Sigma,\mathrm{abd}}$ *considering the implicational*

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Function CAIL-Closure(X, C, Σ)

input : A subset of attributes $X \subseteq \Omega$, a subset of conditions $C \subseteq \Gamma$ and a triadic implicational system Σ on $\mathcal{L}_{\Omega,\Gamma}$

output: The closure of X under C for Σ

begin

```
    Closed:=Ω
1   foreach c ∈ C do
        OldAtomicClosed:=∅
        AtomicClosed:=X
        while AtomicClosed ≠ OldAtomicClosed do
2           OldAtomicClosed:=AtomicClosed
3           foreach V -Z-> W ∈ Σ do
                if c ∈ Z and AtomicClosed ⊇ V then
                    AtomicClosed:=AtomicClosed ∪ W
4       Closed:=Closed ∩ AtomicClosed
    return Closed
```

system from Example 7.1.5:

$$
\Sigma \quad = \{ \quad
\begin{array}{lll}
\mathrm{P} \xrightarrow{\mathrm{de}} \mathrm{N}, & \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{P}, & \mathrm{S} \xrightarrow{\mathrm{bde}} \mathrm{P}, \\
\mathrm{P} \xrightarrow{\mathrm{ad}} \mathrm{R}, & \mathrm{R} \xrightarrow{\mathrm{a}} \mathrm{K}, & \mathrm{S} \xrightarrow{\mathrm{ab}} \mathrm{RK}, \\
\mathrm{P} \xrightarrow{\mathrm{d}} \mathrm{S}, & \mathrm{R} \xrightarrow{\mathrm{e}} \mathrm{N}, & \mathrm{PK} \xrightarrow{\mathrm{ade}} \mathrm{R}, \\
\mathrm{P} \xrightarrow{\mathrm{bd}} \mathrm{K}, & \mathrm{K} \xrightarrow{\mathrm{e}} \mathrm{NS}, & \mathrm{NK} \xrightarrow{\mathrm{abde}} \mathrm{P}, \\
\mathrm{N} \xrightarrow{\mathrm{ae}} \mathrm{P}, & \mathrm{K} \xrightarrow{\mathrm{ae}} \mathrm{R}, & \mathrm{RS} \xrightarrow{\mathrm{abde}} \mathrm{K}, \\
\mathrm{N} \xrightarrow{\mathrm{a}} \mathrm{RS}, & \mathrm{S} \xrightarrow{\mathrm{b}} \mathrm{N}\} &
\end{array}
$$

First, the algorithm computes the closure of the set of attributes $X = \mathrm{PN}$ *under each single condition c (label 1) and the partial results are accumulated in the variable* `AtomicClosed`*:*

`Closed=`Ω

Step 1. $c = a$,

Step 1.1 `AtomicClosed=` PNRKS

Step 1.2 `Closed=` $\Omega \cap$ `AtomicClosed=` PNRKS

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Step 2. $c = b$,

Step 2.1 `AtomicClosed`$=$ PNK

Step 2.2 `Closed`$=$ PNRKS $\cap$ `AtomicClosed`$=$ PNK

Step 3. $c = d$,

Step 3.1 `AtomicClosed`$=$ PNRKS

Step 3.2 `Closed`$=$ PNK $\cap$ `AtomicClosed` $=$ PNK

Then, the Function CAIL-Closure returns the intersection of the partial closures for each condition $c \in \{\mathrm{a}, \mathrm{b}, \mathrm{d}\}$*, which coincides with the closure, that is, CAIL-Closure*$(\mathrm{PN}, \mathrm{abc}, \Sigma) = \mathrm{PNK}$.

This algorithm is the most straightforward approach which arises when the computation of the closure is required. In Section 7.2, we will introduce an equivalent logic system that allows us to design more efficient algorithms, avoiding the use of transitivity rule.

7.1.4 Soundness and completeness of CAIL

To begin with, we introduce the following theorem, which plays a central role in the proof of the soundness.

Theorem 7.1.19. *Let* $\mathbb{K} = \langle G, M, B, I\rangle$ *be a triadic context and* $\mathcal{C} \subseteq B$ *a subset of conditions. The pair* $\langle(-, \mathcal{C})', (-, \mathcal{C})'\rangle$ *is a Galois connection between the lattices* $(2^G, \subseteq)$ *and* $(2^M, \subseteq)$*, i.e. for all* $X \subseteq G$ *and* $Y \subseteq M$

$$X \subseteq (Y, \mathcal{C})' \quad \textit{if and only if} \quad Y \subseteq (X, \mathcal{C})'.$$

Proof. By Theorem 1.1.6, it is sufficient to prove that these mappings are antitone and both compositions are inflationary.

(i) Consider $X_1 \subseteq X_2 \subseteq G$. Then

$$\begin{aligned}(X_2, \mathcal{C})' =&\{m \in M \mid (g, m, b) \in I \text{ for all } (g, b) \in X_2 \times \mathcal{C}\}\\ \subseteq&\{m \in M \mid (g, m, b) \in I \text{ for all } (g, b) \in X_1 \times \mathcal{C}\} = (X_1, \mathcal{C})'\end{aligned}$$

Analogously it is proved that $Y_1 \subseteq Y_2 \subseteq M$, implies $(Y_2, \mathcal{C})' \subseteq (Y_1, \mathcal{C})'$.

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(ii) Now, we prove $X \subseteq ((X,\mathcal{C})',\mathcal{C})'$ for all $X \subseteq G$: Consider $g_0 \in X$. By the definition of the derivation operators, $m \in (X,\mathcal{C})'$ if and only if $(g,m,b) \in I$ for all $(g,b) \in X \times \mathcal{C}$ and, then

$$g_0 \in ((X,\mathcal{C})',\mathcal{C})' = \{g \in G \mid (g,m,b) \in I \text{ for all } (m,b) \in (X,\mathcal{C})' \times \mathcal{C}\}.$$

Therefore, $X \subseteq ((X,\mathcal{C})',\mathcal{C})'$.

In a similar way, $Y \subseteq ((Y,\mathcal{C})',\mathcal{C})'$ is proved, for all $Y \subseteq M$.

□

The following corollary is a direct consequence of the previous theorem and Theorem 1.1.8.

Corollary 7.1.20. *Let $\mathbb{K} = \langle G, M, B, I\rangle$ be a triadic context. For any $\mathcal{C} \subseteq B$ and $\{X_i \mid i \in \mathcal{I}\} \subseteq 2^M$, the following equality holds:*

$$\Big(\bigcup_{i\in\mathcal{I}} X_i, \mathcal{C}\Big)' = \bigcap_{i\in\mathcal{I}} (X_i,\mathcal{C})'.$$

This corollary will be often used throughout the proof of the soundness and completeness theorem.

Theorem 7.1.21. *For any $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ and $\sigma \in \mathcal{L}_{\Omega,\Gamma}$, we have that*

$$\Sigma \vdash_{\mathscr{C}} \sigma \quad \textit{if and only if} \quad \Sigma \models \sigma.$$

Proof. First, we prove the *soundness* of the axiomatic system, i.e.

$$\Sigma \vdash_{\mathscr{C}} \sigma \quad \text{implies} \quad \Sigma \models \sigma \tag{7.2}$$

It is sufficient to prove that any interpretation is a model for the axioms and that the primitive inference rules are sound.

- *Non-constraint*: $\mathbb{K} \models \varnothing \xrightarrow{\varnothing} \Omega$ for any interpretation $\mathbb{K}$ because the assertion " $c \in \varnothing$ implies $(\varnothing, \{c\})' \subseteq (\Omega, \{c\})'$ " is trivially true.

- *Inclusion*: Consider $Y \subseteq X \subseteq \Omega$. For each interpretation $\mathbb{K}$, one has $\mathbb{K} \models X \xrightarrow{\Gamma} Y$ because, for all $c \in \Gamma$, by Theorem 7.1.19, the mapping $(-, \{c\})'$ is antitone and $(X, \{c\})' \subseteq (Y, \{c\})'$.

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- *Augmentation*: We have to prove that, for each $X, Y, Z \subseteq \Omega$ and each $\mathcal{C} \subseteq \Gamma$, one has that $\{X \stackrel{\mathcal{C}}{\to} Y\} \models XZ \stackrel{\mathcal{C}}{\to} YZ$ holds. Consider an interpretation such that $\mathbb{K} \models X \stackrel{\mathcal{C}}{\to} Y$, i.e. $(X, \{c\})' \subseteq (Y, \{c\})'$, for all $c \in C$. Then, by Corollary 7.1.20,

$$\begin{aligned}(X \cup Z, \{c\})' =&(X, \{c\})' \cap (Z, \{c\})' \\ \subseteq&(Y, \{c\})' \cap (Z, \{c\})' = (Y \cup Z, \{c\})'\end{aligned}$$

  Therefore, $\mathbb{K} \models XZ \stackrel{\mathcal{C}}{\to} YZ$.

- *Transitivity*: In order to prove $\{X \xrightarrow{\mathcal{C}_1} Y, Y \xrightarrow{\mathcal{C}_2} Z\} \models X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} Z$ for any $X, Y, Z \subseteq \Omega$ and $\mathcal{C}_1, \mathcal{C}_2 \subseteq \Gamma$, consider an interpretation $\mathbb{K}$ such that $\mathbb{K} \models X \xrightarrow{\mathcal{C}_1} Y$ and $\mathbb{K} \models Y \xrightarrow{\mathcal{C}_2} Z$. Then, $(X, \{c\})' \subseteq (Y, \{c\})'$, for all $c \in \mathcal{C}_1$ and $(Y, \{c\})' \subseteq (Z, \{c\})'$, for all $c \in \mathcal{C}_2$. Thus, for all $c \in \mathcal{C}_1 \cap \mathcal{C}_2$, one has $(X, \{c\})' \subseteq (Y, \{c\})' \subseteq (Z, \{c\})'$ and, therefore, $\mathbb{K} \models X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} Z$.

- *Conditional Decomposition*: For any $X, Y \subseteq \Omega$, any $\mathcal{C}, \mathcal{C}_1 \subseteq \Gamma$ and any interpretation $\mathbb{K}$, if $\mathbb{K} \models X \stackrel{\mathcal{C}}{\to} Y$ and $\mathcal{C}_1 \subseteq \mathcal{C}$, then $\mathbb{K} \models X \xrightarrow{\mathcal{C}_1} Y$ is straightforwardly obtained because $(X, \{c\})' \subseteq (Y, \{c\})'$, for all $c \in \mathcal{C}_1 \subseteq \mathcal{C}$.

- *Conditional Composition*: Consider an interpretation $\mathbb{K}$ such that $\mathbb{K} \models X \xrightarrow{\mathcal{C}_1} Y$ and $\mathbb{K} \models Z \xrightarrow{\mathcal{C}_2} W$. Then, $(X, \{c\})' \subseteq (Y, \{c\})'$, for all $c \in \mathcal{C}_1$ and $(Z, \{c\})' \subseteq (W, \{c\})'$, for all $c \in \mathcal{C}_2$. In order to prove that $\mathbb{K} \models XZ \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y{\cap}W$, for $c \in \mathcal{C}_1 \cup \mathcal{C}_2$, three cases are distinguished:

  (i) If $c \in \mathcal{C}_1 \cap \mathcal{C}_2$ then

$$\begin{aligned}(X \cup Z, \{c\})' &= (X, \{c\})' \cap (Z, \{c\})' \\ &\subseteq (Y, \{c\})' \cap (W, \{c\})' = (Y \cup W, \{c\})' \\ &\subseteq (Y \cap W, \{c\})'\end{aligned}$$

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(ii) If $c \in \mathcal{C}_1 \smallsetminus \mathcal{C}_2$ then

$$\begin{aligned}(X \cup Z, \{c\})' &= (X, \{c\})' \cap (Z, \{c\})' \\ &\subseteq (Y, \{c\})' \cap (Z, \{c\})' = (Y \cup Z, \{c\})' \\ &\subseteq (Y, \{c\})' \subseteq (Y \cap W, \{c\})'\end{aligned}$$

(iii) If $c \in \mathcal{C}_2 \smallsetminus \mathcal{C}_1$ then

$$\begin{aligned}(X \cup Z, \{c\})' &= (X, \{c\})' \cap (Z, \{c\})' \\ &\subseteq (X, \{c\})' \cap (W, \{c\})' = (X \cup W, \{c\})' \\ &\subseteq (W, \{c\})' \subseteq (Y \cap W, \{c\})'\end{aligned}$$

Therefore, $\mathbb{K} \models XZ \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y{\cap}W$.

Conversely, we prove now the *completeness* of the axiomatic system, i.e.

$$\Sigma \models \sigma \quad \text{implies} \quad \Sigma \vdash_{\mathscr{C}} \sigma \tag{7.3}$$

Specifically, for a *CAI* system Σ and a formula $X \xrightarrow{\mathcal{C}} Y$, we assume

$$\Sigma \nvdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y \quad \text{and prove} \quad \Sigma \not\models X \xrightarrow{\mathcal{C}} Y.$$

By Proposition 7.1.12, one has $\mathcal{C} \neq \varnothing$. Now we build an interpretation $\mathbb{K}$ that is a model for Σ but not for $X \xrightarrow{\mathcal{C}} Y$. This interpretation is $\mathbb{K} = \langle G, M, B, I\rangle$, where $G = \{1, 2\}$, $M = \Omega$, $Y = \Gamma$ and I is the ternary relation such that:

i. $(1, m, c) \in I$ if and only if one of the following conditions holds:

   a) $c \in \mathcal{C}$ and $m \in X^+_{\Sigma,\{c\}}$, or

   b) $c \notin \mathcal{C}$ and $m \in \varnothing^+_{\Sigma,\{c\}}$.

ii. $(2, m, c) \in I$ for all $m \in M$ and $c \in B$.

As a first step, we show that $\mathbb{K} \models \Sigma$, i.e. $\mathbb{K}$ is a model for every implication in Σ. Given $U \xrightarrow{\mathcal{C}_1} V \in \Sigma$, we prove that $(U, \{c\})' \subseteq (V, \{c\})'$ for all $c \in \mathcal{C}_1$, by distinguishing the following cases:

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(i) When $U = \varnothing$, by conditional decomposition, $V \subseteq \varnothing^+_{\Sigma,\{c\}}$ and $(U, \{c\})' = \{1, 2\}$.

If $c \notin \mathcal{C}$, then $(V, \{c\})' = \{1, 2\}$.

Otherwise, if $c \in \mathcal{C}$, since $(-)^+_{\Sigma,\{c\}}$ is isotone (Proposition 7.1.17), one has $V \subseteq \varnothing^+_{\Sigma,\{c\}} \subseteq X^+_{\Sigma,\{c\}}$ and, therefore, $(V, \{c\})' = \{1, 2\}$.

(ii) In the case of $U \neq \varnothing$ and $c \notin \mathcal{C}$ then $(U, \{c\})' = \{2\} \subseteq (V, \{c\})'$.

(iii) Finally, when $U \neq \varnothing$ and $c \in \mathcal{C}$, if $U \not\subseteq X^+_{\Sigma,\{c\}}$ is straightforward.

On the other hand, if $U \subseteq X^+_{\Sigma,\{c\}}$, by isotonicity and idempotency, one has $U^+_{\Sigma,\{c\}} \subseteq X^+_{\Sigma,\{c\}}$. Since $V \subseteq U^+_{\Sigma,\{c\}}$, one has straightforwardly that $V \subseteq X^+_{\Sigma,\{c\}}$ and, therefore, $(V, \{c\})' = \{1, 2\}$ and $(U, \{c\})' \subseteq (V, \{c\})'$.

We conclude the proof showing that $\mathbb{K}$ is not a model for $X \xrightarrow{\mathcal{C}} Y$ by *reductio ad absurdum*. Assume that $\mathbb{K} \models X \xrightarrow{\mathcal{C}} Y$, i.e. $(X, \{c\})' \subseteq (Y, \{c\})'$ for all $c \in \mathcal{C}$. From item i.a) in the definition of $\mathbb{K}$, by Proposition 7.1.17, one has $Y \subseteq X^+_{\Sigma,\mathcal{C}} = \bigcap_{c \in \mathcal{C}} X^+_{\Sigma,\{c\}}$ which contradicts $\Sigma \not\vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y$ (see Theorem 7.1.16). □

7.1.5 $\mathscr{B}$ axiomatic system

Here, we provide an equivalent system to CAIL inference system, named $\mathscr{B}$, which is strongly inspired in the B-Axioms proposed by Maier [36].

Definition 7.1.22. *The $\mathscr{B}$ axiomatic system has the same axiom schemes:*

`[Nc]` *and* `[Inc]`,

and the following inference rules already introduced:

`[Acc]`, `[Dec]`, `[ConDec]` *and* `[ConCom]`.

The following theorem guarantees that axiomatic systems $\mathscr{B}$ and CAIL are equivalent.

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Theorem 7.1.23. *For any* $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ *and* $X \xrightarrow{\mathcal{C}} Y \in \mathcal{L}_{\Omega,\Gamma}$*, one has*

$$\Sigma \vdash_{\mathscr{B}} X \xrightarrow{\mathcal{C}} Y \quad \textit{if and only if} \quad \Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y$$

Proof. We have shown in Section 7.1.3 that $\mathscr{B}$ rules of inference can be derived from CAIL. So, to prove the equivalence of both systems, we only have to derive `[Aug]` and `[Tran]` from $\mathscr{B}$.

The following derivation chain derives `[Aug]` from $\mathscr{B}$.

σ_1: $X \xrightarrow{\mathcal{C}} Y$ .... by hypothesis.
σ_2: $XZ \xrightarrow{\mathcal{C}} XZ$ .... by `[Inc]`.
σ_3: $XZ \xrightarrow{\mathcal{C}} XYZ$ .... by applying `[Acc]` to σ_2 and σ_1.
σ_4: $XZ \xrightarrow{\mathcal{C}} YZ$ .... by applying `[Dec]` to σ_3.

Finally, we prove that `[Tran]` is also derived from $\mathscr{B}$:

σ_1: $X \xrightarrow{\mathcal{C}_1} Y$ .... by hypothesis.
σ_2: $Y \xrightarrow{\mathcal{C}_2} Z$ .... by hypothesis.
σ_3: $X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YZ$ .... by applying `[Acc]` to σ_1 and σ_2.
σ_4: $X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} Z$ .... by applying `[Dec]` to σ_3.

□

Once the equivalence between both axiomatic systems has been proved, any *CAI* which could be derived from CAIL could be derived from $\mathscr{B}$ axiomatic system too, and vice versa. Thus, we have the following corollary.

Corollary 7.1.24. *The* $\mathscr{B}$ *axiomatic system is sound and complete.*

The following example shows the chain needed to obtain the same *CAI* than in Example 7.1.11 from the new axiomatic system.

Example 7.1.25. *Let us show how to derive the same implication from* Σ *by using* $\mathscr{B}$*, i.e.* $\Sigma \vdash_{\mathscr{B}} \mathrm{PN} \xrightarrow{\mathrm{abd}} \mathrm{K}$.

σ_1**:** $\mathrm{P} \xrightarrow{\mathrm{bd}} \mathrm{K}$ .... *belongs to* Σ.

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σ_2: PN $\xrightarrow{\text{bd}}$ PN . . . . *by* [Inc].

σ_3: PN $\xrightarrow{\text{bd}}$ KPN . . . . *by applying* [Acc] *to* σ_2 *and* σ_1.

σ_4: PN $\xrightarrow{\text{bd}}$ K . . . . *by applying* [Dec] *to* σ_3.

σ_5: N $\xrightarrow{\text{a}}$ RS . . . . *belongs to* Σ.

σ_6: N $\xrightarrow{\text{a}}$ R . . . . *by applying* [Dec] *to* σ_5.

σ_7: R $\xrightarrow{\text{a}}$ K . . . . *belongs to* Σ.

σ_8: N $\xrightarrow{\text{a}}$ RK . . . . *by applying* [Acc] *to* σ_6 *and* σ_7.

σ_9: N $\xrightarrow{\text{a}}$ K . . . . *by applying* [Dec] *to* σ_8.

σ_{10}: PN $\xrightarrow{\text{a}}$ PN . . . . *by* [Inc].

σ_{11}: PN $\xrightarrow{\text{a}}$ K . . . . *by applying* [Acc] *to* σ_{10} *and* σ_9.

σ_{12}: PN $\xrightarrow{\text{abd}}$ K . . . . *by applying* [ConCom] *to* σ_4 *and* σ_{11}.

7.2 CAISL: Simplification Logic for *CAI*s

We now present yet another axiomatic system which is more suitable for automated reasoning. We use the same language and semantics provided in the previous section but giving a novel equivalent axiomatic system based on simplification paradigm [20]. For this axiomatic system, the symbol $\vdash_{\mathscr{S}}$ denotes the syntactic derivation.

Definition 7.2.1. *The* CAISL *axiomatic system has two axiom schemes:*

$$\textit{Non-constraint } \texttt{[Nc]}\text{:}\quad \frac{}{\varnothing \xrightarrow{\varnothing} \Omega}.$$

$$\textit{Reflexivity } \texttt{[Ref]}\text{:}\quad \frac{}{X \xrightarrow{\Gamma} X}.$$

and four inference rules:

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Decomposition `[Dec]`*:* $\dfrac{X \xrightarrow{\mathcal{C}_1\mathcal{C}_2} YZ}{X \xrightarrow{\mathcal{C}_1} Y}.$

Composition `[Com]`*:* $\dfrac{X \xrightarrow{\mathcal{C}_1} Y,\ Z \xrightarrow{\mathcal{C}_2} W}{XZ \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} YW}.$

Conditional Composition `[ConCom]`*:* $\dfrac{X \xrightarrow{\mathcal{C}_1} Y,\ Z \xrightarrow{\mathcal{C}_2} W}{XZ \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y\cap W}.$

Simplification `[Simp]`*:* $\dfrac{X \xrightarrow{\mathcal{C}_1} Y,\ XZ \xrightarrow{\mathcal{C}_2} W}{XZ\text{-}Y \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} W\text{-}Y}$ *if* $X \cap Y = \varnothing$.

The following theorem proves the equivalence between CAIL and CAISL inference systems, which implies the soundness and completeness of CAISL.

Theorem 7.2.2. *For any* $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ *and* $X \xrightarrow{\mathcal{C}} Y \in \mathcal{L}_{\Omega,\Gamma}$*, one has*

$$\Sigma \vdash_{\mathscr{S}} X \xrightarrow{\mathcal{C}} Y \quad \textit{if and only if} \quad \Sigma \vdash_{\mathscr{C}} X \xrightarrow{\mathcal{C}} Y$$

Proof. To prove the equivalence between both logics, we will show that the inference rules of CAISL can be derived from those in CAIL and vice versa.

i) First, we prove that the axiom schemes and the primitive inference rules in CAISL are derived from CAIL.

`[Ref]` Reflexivity:

σ_1: $X \xrightarrow{\Gamma} X$ .... by `[Inc]`.

`[Dec]` Decomposition:

σ_1: $X \xrightarrow{\mathcal{C}_1\mathcal{C}_2} YZ$ ....by hypothesis.

σ_2: $YZ \xrightarrow{\Gamma} Y$ .... by `[Inc]`.

σ_3: $X \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y$ .... by `[Tran]` to σ_1 and σ_2.

σ_4: $X \xrightarrow{\mathcal{C}_1} Y$ .... by `[ConDec]` to σ_3.

`[Comp]` Composition:

σ_1: $X \xrightarrow{\mathcal{C}_1} Y$ .... by hypothesis.

σ_2: $Z \xrightarrow{\mathcal{C}_2} W$ .... by hypothesis.

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σ_3: $XZ \xrightarrow{\mathcal{C}_1} YZ$ .... by `[Aug]` to σ_1.
σ_4: $YZ \xrightarrow{\mathcal{C}_2} YW$ .... by `[Aug]` to σ_2.
σ_5: $XZ \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YW$ .... by `[Tran]` to σ_3 and σ_4.

`[Simp]` Simplification:

σ_1: $X \xrightarrow{\mathcal{C}_1} Y$ .... by hypothesis.
σ_2: $XZ \xrightarrow{\mathcal{C}_2} W$ .... by hypothesis.
σ_3: $XZ\text{-}Y \xrightarrow{\Gamma} X$ .... by `[Inc]`.
σ_4: $W \xrightarrow{\Gamma} W\text{-}Y$ .... by `[Inc]`.
σ_5: $XZ \xrightarrow{\mathcal{C}_2} W\text{-}Y$ .... by `[Tran]` to σ_2 and σ_4.
σ_6: $XZ\text{-}Y \xrightarrow{\mathcal{C}_1} Y$ .... by `[Tran]` to σ_3 and σ_1.
σ_7: $XZ\text{-}Y \xrightarrow{\mathcal{C}_1} XYZ$ .... by `[Aug]` to σ_6.
σ_8: $XYZ \xrightarrow{\mathcal{C}_2} WY$ .... by `[Aug]` to σ_5.
σ_9: $XZ\text{-}Y \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} WY$ .... by `[Tran]` to σ_7 and σ_8.
σ_{10}: $XZ\text{-}Y \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W\text{-}Y$ .... by `[Dec]` to σ_9.

ii) Conversely, we prove that the axiom schemes and the primitive inference rules in CAIL are derived from CAISL.

`[Inc]` Inclusion:

σ_1: $XY \xrightarrow{\Gamma} XY$ .... by `[Ref]`.
σ_2: $XY \xrightarrow{\Gamma} Y$ .... by `[Dec]` to σ_1.

`[Aug]` Augmentation:

σ_1: $X \xrightarrow{\mathcal{C}} Y$ .... by hypothesis.
σ_2: $Z \xrightarrow{\Gamma} Z$ .... by `[Ref]`.
σ_3: $XZ \xrightarrow{\mathcal{C}} YZ$ .... by `[Com]` to σ_1 and σ_2.

`[Tran]` Transitivity:

σ_1: $X \xrightarrow{\mathcal{C}_1} Y$ .... by hypothesis.
σ_2: $Y \xrightarrow{\mathcal{C}_2} Z$ .... by hypothesis.
σ_3: $X \xrightarrow{\mathcal{C}_1} Y\text{-}X$ .... by `[Dec]` to σ_1.
σ_4: $Y \xrightarrow{\mathcal{C}_2} Z\text{-}Y$ .... by `[Dec]` to σ_2.

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σ_5: $X \xrightarrow{\Gamma} X$ .... by [Ref].

σ_6: $X \xrightarrow{\Gamma} \varnothing$ .... by [Dec] to σ_5.

σ_7: $XY \xrightarrow{\mathcal{C}_2} Z\text{-}Y$ .... by [Com] to σ_4 and σ_6.

σ_8: $Y\text{-}X \xrightarrow{\Gamma} Y\text{-}X$ .... by [Ref] .

σ_9: $X \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} Z\text{-}Y$ .... by [Simp] to σ_3 and σ_7.

σ_{10}: $X \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} YZ$ .... by [Com] to σ_1 and σ_9.

σ_{11}: $X \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} Z$ .... by [Dec] to σ_{10}.

□

Since all of the presented axiomatic systems are equivalent, in the sequel we will omit the subscript in the syntactic derivation symbol using simply $\vdash$.

7.2.1 CAISL Equivalences

In this subsection, we introduce several results which constitute the basis of the automated reasoning method that will be introduced in the next section. These results illustrate how we can use CAISL as a framework to syntactically transform and simplify a set of *CAI*s while entirely preserving their semantics. This is the common feature of the family of Simplification Logics.

The notion of equivalence is introduced as usual (see Definition 7.1.7): two sets of *CAI*s, Σ_1 and Σ_2, are equivalent, denoted by $\Sigma_1 \equiv \Sigma_2$, when their models are the same. Since the axiomatic system is sound and complete, the equivalence between sets of *CAI*s can be checked as follows: $\Sigma_1 \equiv \Sigma_2$ if and only if the following two conditions hold:

(i) $\Sigma_1 \vdash \varphi$ for all $\varphi \in \Sigma_2$, and

(ii) $\Sigma_2 \vdash \varphi$ for all $\varphi \in \Sigma_1$.

In the following, we present a set of equivalences that justify de name of the logic, Simplification Logic for *CAI*s, and play a central role in the automated reasoning method.

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Lemma 7.2.3. *The following equivalences hold:*

$$\{X \xrightarrow{\mathcal{C}_1} Y, X \xrightarrow{\mathcal{C}_2} W\} \equiv \{X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YW, X \xrightarrow{\mathcal{C}_1 - \mathcal{C}_2} Y, X \xrightarrow{\mathcal{C}_2 - \mathcal{C}_1} W\} \tag{7.4}$$

$$\{X \xrightarrow{\mathcal{C}_1} Y, XV \xrightarrow{\mathcal{C}_2} W\} \equiv \{X \xrightarrow{\mathcal{C}_1} Y, XV \xrightarrow{\mathcal{C}_2 - \mathcal{C}_1} W, X(V\text{-}Y) \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W\text{-}Y\} \tag{7.5}$$

Proof. For Equivalence (7.4), first, we prove that the formulas

$$X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YW, \ X \xrightarrow{\mathcal{C}_1 - \mathcal{C}_2} Y, \ \text{ and } \ X \xrightarrow{\mathcal{C}_2 - \mathcal{C}_1} W$$

can be inferred from $\{X \xrightarrow{\mathcal{C}_1} Y, X \xrightarrow{\mathcal{C}_2} W\}$:

- $X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YW$ is inferred from $X \xrightarrow{\mathcal{C}_1} Y$ and $X \xrightarrow{\mathcal{C}_2} W$, by `[Com]`.
- $X \xrightarrow{\mathcal{C}_1 - \mathcal{C}_2} Y$ and $X \xrightarrow{\mathcal{C}_2 - \mathcal{C}_1} W$ are obtained by `[Dec]`.

Conversely, by applying `[ConComp]`, $X \xrightarrow{\mathcal{C}_1} Y$ and $X \xrightarrow{\mathcal{C}_2} W$ are inferred from $\{X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} YW, X \xrightarrow{\mathcal{C}_1 - \mathcal{C}_2} Y, X \xrightarrow{\mathcal{C}_2 - \mathcal{C}_1} W\}$.

For Equivalence (7.5), on the one hand, $XV \xrightarrow{\mathcal{C}_2 - \mathcal{C}_1} W$ is inferred from $\{X \xrightarrow{\mathcal{C}_1} Y, XV \xrightarrow{\mathcal{C}_2} W\}$ by applying `[Dec]` and the following sequence proves $\{X \xrightarrow{\mathcal{C}_1} Y, XV \xrightarrow{\mathcal{C}_2} W\} \vdash X(V\text{-}Y) \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W\text{-}Y$.

σ_1: $X \xrightarrow{\mathcal{C}_1} Y$ .... by hypothesis.

σ_2: $X \xrightarrow{\mathcal{C}_1} Y\text{-}X$ .... by applying `[Dec]` to σ_1.

σ_3: $XV \xrightarrow{\mathcal{C}_2} W$ .... by hypothesis.

σ_4: $X(V\text{-}Y) \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W\text{-}(Y\text{-}X)$ by applying `[Simp]` to σ_2 and σ_3 and taking into account that $XV \smallsetminus (Y \smallsetminus X) = X(V \smallsetminus Y)$.

σ_5: $X(V\text{-}Y) \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W\text{-}Y$ .... by applying `[Dec]` to σ_4.

On the other hand, we prove

$$\{X \xrightarrow{\mathcal{C}_1} Y, XV \xrightarrow{\mathcal{C}_2 - \mathcal{C}_1} W, XV\text{-}Y \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} W\text{-}Y\} \vdash XV \xrightarrow{\mathcal{C}_2} W$$

with the following sequence:

σ_1: $XV \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} XV$ .... by `[Ref]`.

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σ_2: $XV \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} XV\text{-}Y$ .... by applying [Dec] to σ_1.

σ_3: $XV\text{-}Y \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} W\text{-}Y$ .... by hypothesis.

σ_4: $XV \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} W\text{-}Y$ .... by applying [Tran] to σ_2 and σ_3.

σ_5: $X \xrightarrow{\mathcal{C}_1} Y$ .... by hypothesis.

σ_6: $XV \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} WY$ .... by applying [Com] to σ_4 and σ_5.

σ_7: $XV \xrightarrow{\mathcal{C}_1\cap\mathcal{C}_2} W$ .... by applying [Dec] to σ_6.

σ_8: $XV \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} W$ .... by hypothesis.

σ_9: $XV \xrightarrow{\mathcal{C}_2} W$ .... by applying [ConCom] to σ_7 and σ_8.

□

The following theorem highlights a common feature of Simplification Logics, which shows that inference rules can be read as equivalences that allow redundancy removal, when they are applied from left to right.

Theorem 7.2.4. *The following equivalences hold:*

$$\{X \xrightarrow{\varnothing} Y\} \equiv \{X \xrightarrow{\mathcal{C}} \varnothing\} \equiv \varnothing \qquad \texttt{(Ax-Eq)}$$

$$\{X \xrightarrow{\mathcal{C}} Y\} \equiv \{X \xrightarrow{\mathcal{C}} Y\text{-}X\} \qquad \texttt{(Dec-Eq)}$$

$$\{X \xrightarrow{\mathcal{C}} Y, X \xrightarrow{\mathcal{C}} W\} \equiv \{X \xrightarrow{\mathcal{C}} YW\} \qquad \texttt{(Com-Eq)}$$

$$\{X \xrightarrow{\mathcal{C}_1} Y, X \xrightarrow{\mathcal{C}_2} Y\} \equiv \{X \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y\} \qquad \texttt{(ConCom-Eq)}$$

If $X \cap Y = \varnothing$, then

$$\{X \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y, XV \xrightarrow{\mathcal{C}_2} W\} \equiv \{X \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y, XV\text{-}Y \xrightarrow{\mathcal{C}_2} W\text{-}Y\} \qquad \texttt{(Simp-Eq)}$$

Proof. (Ax-Eq) is straightforward because both implications are axioms. For the rest of equivalences, the left-to-right inference is directly obtained by applying the homonymous inference rule. Thus, we prove the right-to-left inference:

(Dec-Eq) $X \xrightarrow{\mathcal{C}} XY$ is inferred by [Comp] of $X \xrightarrow{\mathcal{C}} Y\text{-}X$ and $X \xrightarrow{\mathcal{C}} X$ obtained by [Ref]. Then, by applying [Dec], one has $X \xrightarrow{\mathcal{C}} Y$.

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(Com-Eq) $X \xrightarrow{\mathcal{C}} Y$ and $X \xrightarrow{\mathcal{C}} W$ are inferred from $X \xrightarrow{\mathcal{C}} YW$ by applying [Dec].

(ConCom-Eq) $X \xrightarrow{\mathcal{C}_1} Y$ and $X \xrightarrow{\mathcal{C}_2} Y$ are inferred from $X \xrightarrow{\mathcal{C}_1\mathcal{C}_2} Y$ by applying [Dec].

(Simp-Eq) It is a consequence of (Ax-Eq) and (7.5) in Lemma 7.2.3.

□

7.2.2 Automated reasoning in CAISL

Based on the previous equivalences, we will introduce some others where the empty set plays a main role. The Deduction Theorem presented below gives such a role to the empty set. This theorem establishes the necessary and sufficient condition to ensure the derivability of a *CAI* from a set of *CAI*s. Moreover, we present a method that checks whether a *CAI* is derived from a set of *CAI*s. The next theorem is the core of our approach.

Theorem 7.2.5 (Deduction). *For any $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ and $X \xrightarrow{\mathcal{C}} Y \in \mathcal{L}_{\Omega,\Gamma}$, one has*

$$\Sigma \vdash X \xrightarrow{\mathcal{C}} Y \quad \textit{if and only if} \quad \Sigma \cup \{\varnothing \xrightarrow{\mathcal{C}} X\} \vdash \varnothing \xrightarrow{\mathcal{C}} Y$$

Proof. Straightforwardly, we have that

$$\Sigma \vdash X \xrightarrow{\mathcal{C}} Y \quad \text{implies} \quad \Sigma \cup \{\varnothing \xrightarrow{\mathcal{C}} X\} \vdash \varnothing \xrightarrow{\mathcal{C}} Y.$$

Conversely, assuming $\Sigma \cup \{\varnothing \xrightarrow{\mathcal{C}} X\} \vdash \varnothing \xrightarrow{\mathcal{C}} Y$ and, due to soundness and completeness, we prove that $\mathbb{K} \models \Sigma$ implies $\mathbb{K} \models X \xrightarrow{\mathcal{C}} Y$ for each model $\mathbb{K}$.

Consider a model $\mathbb{K} = \langle G, M, B, I\rangle$ for Σ. We will prove $(X, \{c\})' \subseteq (Y, \{c\})'$ for all $c \in \mathcal{C}$ in $\mathbb{K}$. Considering $G_c = (X, \{c\})'$, we build the context $\mathbb{K}_c = \langle G_c, M, B, I_c\rangle$ where $I_c = I \cap (G_c \times M \times B)$.

Since $\mathbb{K} \models \Sigma$, we have $\mathbb{K}_c \models \Sigma \cup \{\varnothing \xrightarrow{\{c\}} X\}$ and therefore, by hypothesis, $\mathbb{K}_c \models \{\varnothing \xrightarrow{\{c\}} Y\}$. That is, $(Y, \{c\})' \supseteq (\varnothing, \{c\})' = G_c = (X, \{c\})'$.

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If we go back to the original triadic context $\mathbb{K}$, $(X, \{c\})'$ remains unchanged whereas $(Y, \{c\})'$ could grow up. Therefore, $(X, \{c\})' \subseteq (Y, \{c\})'$ in $\mathbb{K}$, for all $c \in \mathcal{C}$. $\square$

Theorem 7.2.5 guides the design of the automated prover. To check that the formula $X \xrightarrow{\mathcal{C}} Y$ is inferred from the set Σ, we iteratively apply the family of simplification equivalences, whenever possible, to the set $\Sigma \cup \{\varnothing \xrightarrow{\mathcal{C}} X\}$ searching for $\varnothing \xrightarrow{\mathcal{C}} Y$.

The following proposition revisits Theorem 7.2.4 and Lemma 7.2.3 by instantiating the particular case of having the empty premise.

Proposition 7.2.6. *The following equivalences hold:*

$$\{\varnothing \xrightarrow{\mathcal{C}_1} X, U \xrightarrow{\mathcal{C}_2} V\} \equiv \{\varnothing \xrightarrow{\mathcal{C}_1} X, U\text{-}X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} V\text{-}X, U \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} V\}. \tag{7.6}$$

If $U \subseteq X$ then

$$\{\varnothing \xrightarrow{\mathcal{C}_1} X, U \xrightarrow{\mathcal{C}_2} V\} \equiv \{\varnothing \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} XV, \varnothing \xrightarrow{\mathcal{C}_1\text{-}\mathcal{C}_2} X, U \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} V\}. \tag{7.7}$$

If $V \subseteq X$ then

$$\{\varnothing \xrightarrow{\mathcal{C}_1} X, U \xrightarrow{\mathcal{C}_2} V\} \equiv \{\varnothing \xrightarrow{\mathcal{C}_1} X, U \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} V\}. \tag{7.8}$$

Proof. Equivalence (7.6) is a particular case of Equivalence (7.5). In particular, when $U \subseteq X$, Equivalence (7.7) is obtained from (7.6) by applying (`ConCom-Eq`) and (`Com-Eq`):

$$\begin{aligned}\{\varnothing \xrightarrow{\mathcal{C}_1} X, U \xrightarrow{\mathcal{C}_2} V\} &\equiv \{\varnothing \xrightarrow{\mathcal{C}_1} X, \varnothing \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} V\text{-}X, U \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} V\} \\ &\equiv \{\varnothing \xrightarrow{\mathcal{C}_1\text{-}\mathcal{C}_2} X, \varnothing \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} X, \varnothing \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} V\text{-}X, U \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} V\} \\ &\equiv \{\varnothing \xrightarrow{\mathcal{C}_1\text{-}\mathcal{C}_2} X, \varnothing \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} XV, U \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} V\}\end{aligned}$$

Analogously, Equivalence (7.8) holds as a consequence of Equivalence (7.6) and (`Ax-Eq`): if $V \subseteq X$,

$$\begin{aligned}\{\varnothing \xrightarrow{\mathcal{C}_1} X, U \xrightarrow{\mathcal{C}_2} V\} &\equiv \{\varnothing \xrightarrow{\mathcal{C}_1} X, U\text{-}X \xrightarrow{\mathcal{C}_1 \cap \mathcal{C}_2} \varnothing, U \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} V\} \\ &\equiv \{\varnothing \xrightarrow{\mathcal{C}_1} X, U \xrightarrow{\mathcal{C}_2\text{-}\mathcal{C}_1} V\}\end{aligned}$$

$\square$

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Function CAISL-Prover(Σ,$X \overset{\mathcal{C}}{\to} Y$)

input : A set of implications Σ, and a *CAI* $X \overset{\mathcal{C}}{\to} Y$

output: A boolean answer

```
begin
    Δ_X := X × 𝒞
    Δ_Y := (Y × 𝒞) ∖ (X × 𝒞)
    repeat
        flag:=false
        foreach U --𝒞₁--> V ∈ Σ with 𝒞₁ ∩ 𝒞 ≠ ∅ do
            Δ_C := {c ∈ 𝒞₁ ∩ 𝒞 | U × {c} ⊆ Δ_X}
            if Δ_C ≠ ∅ then ................................Equivalence (7.7)
                Δ_X := Δ_X ∪ (V × Δ_C)
                Δ_Y := Δ_Y ∖ (V × Δ_C)
                Σ := Σ ∖ {U --𝒞₁--> V}
                𝒞₁ := 𝒞₁ ∖ Δ_C
                if 𝒞₁ ≠ ∅ then Σ := Σ ∪ {U --𝒞₁--> V}
                flag:=true
            Δ_C := {c ∈ 𝒞₁ ∩ 𝒞 | V × {c} ⊆ Δ_X}
            if Δ_C ≠ ∅ then ................................Equivalence (7.8)
                if Δ_C = 𝒞₁ then Σ := Σ ∖ {U --𝒞₁--> V}
                else Σ :=(Σ ∖ {U --𝒞₁--> V}) ∪ {U --𝒞₁-Δ_C--> V}
    until (Δ_Y = ∅) or (flag=false)
    return the boolean value (Δ_Y = ∅)
```

The equivalences introduced in Proposition 7.2.6 constitute the core of the function called `CAISL-Prover`, which acts as an automated reasoning method for CAISL. This method works by splitting the original formula into its left and right hand sides (see Theorem 7.2.6) and, by applying Equivalences (7.7) and (7.8), check whether its right hand side can be reduced to the empty set. The derivability is proved if and only if such a reduction is fulfilled, as the following theorem shows.

Theorem 7.2.7. *The algorithm* `CAISL-Prover` *is sound and complete.*

Proof. First, we prove that the algorithm always terminates. Let Δ_i and Σ_i respectively denote the state in which Δ_X and Σ are in the i-th step of the execution of the 'repeat' sentence, for all $0 \leq i$.

Tarski's fixed-point theorem ensures that the algorithm finishes after finitely many steps because the chain $\{\Delta_i \mid 0 \leq i\}$ is strictly increasing in $2^{M\times B}$ that is finite, i.e. $|\Delta_0| < |\Delta_1| < \cdots \leq |M \times B|$. In addition, the sequence Σ_i never increases.

Now, the soundness and completeness will be proved. Let n be the step in which the algorithm finishes. For each $0 \leq i \leq n$, we define:

$$\begin{aligned} X_{i,c} =& \{x \in M \mid (x, c) \in \Delta_i\} \text{ for each } c \in \mathcal{C} \\ \Gamma_i =& \{\varnothing \xrightarrow{c} X_{i,c} \mid c \in \mathcal{C}\} \end{aligned}$$

By (7.7) and (7.8), we can ensure that $\Sigma_i \cup \Gamma_i \equiv \Sigma_j \cup \Gamma_j$ for all $0 \leq i, j \leq n$. Moreover, Theorem 7.2.5 ensures that, for all $\varnothing \xrightarrow{c} X_{n,c} \in \Gamma_n$, one has

$$\Sigma \vdash X \xrightarrow{c} X_{n,c} \tag{7.9}$$

The algorithm returns 'true' when $Y \times \mathcal{C} \subseteq \Delta_n$, which is equivalent to $Y \subseteq X_{n,c}$ for all $c \in \mathcal{C}$. In this case, by Definition 7.1.13 and Theorem 7.1.16, we have $\Sigma \vdash X \xrightarrow{\mathcal{C}} Y$.

On the other hand, we prove that the fact that the algorithm returns 'false' means that $\Sigma \not\vdash X \xrightarrow{\mathcal{C}} Y$. Specifically, we prove that $\Sigma \vdash X \xrightarrow{\mathcal{C}} Y$ implies $Y \subseteq X_{n,c}$ for all $c \in \mathcal{C}$.

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Assume $\Sigma \vdash X \overset{\mathcal{C}}{\to} Y$ and consider any $c \in \mathcal{C}$. By `[Dec]` we have that $\Sigma \vdash X \overset{c}{\to} Y$ and, by `[Com]` and (7.9), we can ensure that $\Sigma \vdash X \overset{c}{\to} X_{n,c}Y$. Now, by Theorem 7.2.5, $\Sigma_0 \cup \Gamma_0 \vdash \varnothing \overset{c}{\to} X_{n,c}Y$ and, since $\Sigma_0 \cup \Gamma_0 \equiv \Sigma_n \cup \Gamma_n$, we have that $\Sigma_n \cup \Gamma_n \vdash \varnothing \overset{c}{\to} X_{n,c}Y$. When the algorithm finishes, `[Dec]` is the unique inference rule that can be applied to $\Sigma_n \cup \Gamma_n$ in order to infer $\varnothing \overset{c}{\to} X_{n,c}Y$. Thus, $X_{n,c}Y \subseteq X_{n,c}$, i.e. $Y \subseteq X_{n,c}$. $\square$

Finally, we conclude this section with an illustrative example of how the `CAISL-Prover` algorithm works.

Step	State
0	$\Delta_X = \{(M,a),(M,b),(M,d),(L,a),(L,b),(L,d)\}$ $\Delta_Y = \{(Q,a),(Q,b),(Q,d)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M},\ \mathrm{M} \xrightarrow{\mathrm{a}} \mathrm{T}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
1	$\Delta_X = \{(M,a),(M,b),(M,d),(L,a),(L,b),(L,d)\}$ $\Delta_Y = \{(Q,a),(Q,b),(Q,d)\}$ $\Sigma = \{\mathbf{Q} \xrightarrow{\cancel{\mathbf{d}}\mathbf{e}} \mathbf{M}, \mathrm{M} \xrightarrow{\mathrm{a}} \mathrm{T}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
2	$\Delta_X = \{(M,a),(M,b),(M,d),(L,a),(L,b),(L,d),(T,a)\}$ $\Delta_Y = \{(Q,a),(Q,b),(Q,d)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{e}} \mathrm{M}, \cancel{\mathbf{M} \xrightarrow{\mathbf{a}} \mathbf{T}}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
3	$\Delta_X = \{(M,a),(M,b),(M,d),(L,a),(L,b),(L,d),(T,a)\}$ $\Delta_Y = \{(Q,a),(Q,b),(Q,d)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{e}} \mathrm{M}, \cancel{\mathbf{Q} \xrightarrow{\mathbf{bd}} \mathbf{L}}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
4	$\Delta_X = \{(M,a),(M,b),(M,d),(L,a),(L,b),(L,d),(T,a),(Q,b),(Q,d)\}$ $\Delta_Y = \{(Q,a)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{e}} \mathrm{M}, \mathbf{ML} \xrightarrow{\cancel{\mathbf{b}}\cancel{\mathbf{d}}\mathbf{e}} \mathbf{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
5	$\Delta_X = \{(M,a),(M,b),(M,d),(L,a),(L,b),(L,d),(T,a),(Q,b),(Q,d),(R,a)\}$ $\Delta_Y = \{(Q,a)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{e}} \mathrm{M}, \mathrm{ML} \xrightarrow{\mathrm{e}} \mathrm{Q}, \mathbf{T} \xrightarrow{\cancel{\mathbf{a}}\mathbf{b}} \mathbf{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
6	$\Delta_X = \{(M,a),(M,b),(M,d),(L,a),(L,b),(L,d),(T,a),(Q,b),(Q,d),(R,a),(Q,a)\}$ $\Delta_Y = \varnothing$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{e}} \mathrm{M}, \mathrm{ML} \xrightarrow{\mathrm{e}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{b}} \mathrm{R}, \mathbf{R} \xrightarrow{\cancel{\mathbf{a}}\mathbf{e}} \mathbf{Q}\}$
Output	Return `TRUE`

Table 7.1: Execution of the derivability of $\mathrm{ML} \xrightarrow{\mathrm{abd}} \mathrm{Q}$

Example 7.2.8. *Consider* $\Omega = \{\mathrm{L, M, P, Q, R, T}\}$, $\Gamma = \{\mathrm{a, b, d, e}\}$ *and the*

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set of CAIs

$$\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathrm{M} \xrightarrow{\mathrm{a}} \mathrm{T}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$$

Table 7.1 shows the trace of `CAISL-Prover` *on the implication* $\mathrm{ML} \xrightarrow{\mathrm{abd}} \mathrm{Q}$ *and the premises* Σ

$$\Sigma \vdash \mathrm{ML} \xrightarrow{\mathrm{abd}} \mathrm{Q}$$

and Table 7.2 shows how `CAISL-Prover` *checks that* $\mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{Q}$ *cannot be inferred*

$$\Sigma \not\vdash \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{Q}$$

7.3 Complete CAI systems

We have introduced two equivalent logics for reasoning about conditional attribute implications in TCA. The first one, CAIL, is closer to the classical Armstrong's axioms, whereas the second one, CAISL, follows the Simplification paradigm. This paradigm leads to introduce a set of equivalences that allows us to remove redundant information in the set of CAIs, i.e. simplifying it. In addition, these equivalences together with the so-called Deduction Theorem provide an efficient automated reasoning method whose advantages will be explored in the near future.

A different issue is to find a set of CAIs that characterizes all the CAIs that are true in a given triadic context. The following definition captures this idea.

Definition 7.3.1. *Consider* $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ *and let* $\mathbb{K}$ *be an interpretation for* $\mathcal{L}_{\Omega,\Gamma}$*. The set* Σ *is said to be a* complete CAI system *for* $\mathbb{K}$ *if, for all* $X \xrightarrow{\mathcal{C}} Y \in \mathcal{L}_{\Omega,\Gamma}$*, the following equivalence holds:*

$$\mathbb{K} \models X \xrightarrow{\mathcal{C}} Y \quad \textit{if and only if} \quad \Sigma \vdash X \xrightarrow{\mathcal{C}} Y$$

As a consequence of Theorems 7.1.16, 7.1.19 and 7.1.21, we have the following theorem that relates syntactic and semantic closures.

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Step	State
0	$\Delta_X = \{(T,a),(T,b)\}$ $\Delta_Y = \{(Q,a),(Q,b)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathrm{M} \xrightarrow{\mathrm{a}} \mathrm{T}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
Loop 1	
1	$\Delta_X = \{(T,a),(T,b)\}$ $\Delta_Y = \{(Q,a),(Q,b)\}$ $\Sigma = \{\mathbf{Q} \xrightarrow{\mathbf{de}} \mathbf{M}, \mathrm{M} \xrightarrow{\mathrm{a}} \mathrm{T}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
2	$\Delta_X = \{(T,a),(T,b)\}$ $\Delta_Y = \{(Q,a),(Q,b)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \cancel{\mathbf{M} \xrightarrow{\mathbf{a}} \mathbf{T}}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
3	$\Delta_X = \{(T,a),(T,b)\}$ $\Delta_Y = \{(Q,a),(Q,b)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathbf{Q} \xrightarrow{\mathbf{bd}} \mathbf{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
4	$\Delta_X = \{(T,a),(T,b)\}$ $\Delta_Y = \{(Q,a),(Q,b)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathbf{ML} \xrightarrow{\mathbf{bde}} \mathbf{Q}, \mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{RL}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
5	$\Delta_X = \{(T,a),(T,b),(R,a),(R,b),(L,a),(L,b)\}$ $\Delta_Y = \{(Q,a),(Q,b)\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \cancel{\mathbf{T} \xrightarrow{\mathbf{ab}} \mathbf{RL}}, \mathrm{R} \xrightarrow{\mathrm{ae}} \mathrm{Q}\}$
6	$\Delta_X = \{(T,a),(T,b),(R,a),(R,b),(L,a),(L,b),(Q,a)\}$ $\Delta_Y = \{(Q,b))\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathbf{R} \xrightarrow{\cancel{\mathbf{a}}\mathbf{e}} \mathbf{Q}\}$
Loop 2	
7	$\Delta_X = \{(T,a),(T,b),(R,a),(R,b),(L,a),(L,b),(Q,a)\}$ $\Delta_Y = \{(Q,b))\}$ $\Sigma = \{\mathbf{Q} \xrightarrow{\mathbf{de}} \mathbf{M}, \mathrm{Q} \xrightarrow{\mathrm{bd}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{R} \xrightarrow{\mathrm{e}} \mathrm{Q}\}$
8	$\Delta_X = \{(T,a),(T,b),(R,a),(R,b),(L,a),(L,b),(Q,a)\}$ $\Delta_Y = \{(Q,b))\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathbf{Q} \xrightarrow{\cancel{\mathbf{b}}\mathbf{d}} \mathbf{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathrm{R} \xrightarrow{\mathrm{e}} \mathrm{Q}\}$
9	$\Delta_X = \{(T,a),(T,b),(R,a),(R,b),(L,a),(L,b),(Q,a)\}$ $\Delta_Y = \{(Q,b))\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathrm{Q} \xrightarrow{\mathrm{d}} \mathrm{L}, \mathbf{ML} \xrightarrow{\mathbf{bde}} \mathbf{Q}, \mathrm{R} \xrightarrow{\mathrm{e}} \mathrm{Q}\}$
10	$\Delta_X = \{(T,a),(T,b),(R,a),(R,b),(L,a),(L,b),(Q,a)\}$ $\Delta_Y = \{(Q,b))\}$ $\Sigma = \{\mathrm{Q} \xrightarrow{\mathrm{de}} \mathrm{M}, \mathrm{Q} \xrightarrow{\mathrm{d}} \mathrm{L}, \mathrm{ML} \xrightarrow{\mathrm{bde}} \mathrm{Q}, \mathbf{R} \xrightarrow{\mathbf{e}} \mathbf{Q}\}$
Output	Return `FALSE`

Table 7.2: Execution of the non derivability of $\mathrm{T} \xrightarrow{\mathrm{ab}} \mathrm{Q}$

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Theorem 7.3.2. *Consider* $\Sigma \subseteq \mathcal{L}_{\Omega,\Gamma}$ *and let* $\mathbb{K} = \langle G, M, B, I\rangle$ *be an interpretation for* $\mathcal{L}_{\Omega,\Gamma}$ *where* $\Omega = M$ *and* $\Gamma = B$. *If* Σ *is a complete CAI system for* $\mathbb{K}$, *then, for all* $X \subseteq M$ *and* $\mathcal{C} \subseteq B$, *one has*

$$X^+_{\Sigma,\mathcal{C}} = \bigcap_{c\in\mathcal{C}}((X,\{c\})',\{c\})' \subseteq ((X,\mathcal{C})',\mathcal{C})'$$

Obviously, given a triadic context, there is a big number of complete CAI systems, which are equivalent among them. There are several interesting problems related with this notion that we leave as a further problem to be studied:

(i) Designing an efficient algorithm for computing a complete CAI system from a given triadic context.

(ii) Studying the suitable notion of base aiming at minimality properties among the equivalent CAI systems.

(iii) Studying the directness property in this more general framework.

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Chapter 8

Conclusions and future works

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In this thesis, we have focused on Formal Concept Analysis, specifically the goal has been to establish a formal framework for implications in this area. Nowadays, the representation and management of this kind of knowledge is probably one of most active topics in FCA.

Providing a well founded logic-algebraic theory promotes the development of efficient and automated methods which can be used in practical applications. Moving from theory to practice and vice versa is a major challenge to achieve that Formal Concept Analysis becomes a fruitful tool for the representation, management and analysis of knowledge in real situations.

As shown, the set of implications retrieved from a formal context has a lot of redundancy and the search for a canonical form from an initial set is one of the main challenges. In this direction, the research presented in this work moves towards the study of the foundations of the most interesting canonical form, the basis of implications.

Duquenne-Guigues basis is the most cited basis but it is not suitable for all applications. Thus, other kinds of bases have been introduced fulfilling different properties depending on their further use. One of the main aims of this work is to design algorithms to compute direct bases, which are very helpful for applications because they lead to a fast computation of closures and ease automated management.

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We have focussed on directness because it is strongly related to the closure problem. In Formal Concept Analysis, the computation of the closure of a set of attributes is an important topic. For this reason, Bertet et al. propose a type of basis called direct-optimal basis [9, 11, 12], which combines directness and a condensed shape: it lets you compute closures of attribute sets in just one traversal and it fulfills a minimality criteria too.

In Chapter 3, we have proposed two new methods to compute the direct-optimal basis from an arbitrary non-unitary implicational system. To this aim, we have introduced a new inference rule, the *strong Simplification rule*, to add new implications avoiding redundancy. This rule is derived from the $\mathbf{SL}_{FD}$ axiomatic system as proved in Lemma 3.2.1.

The first method, `doSimp`, is divided in three separated stages while the second one, `SLgetdo`, integrates these stages to improve its performance. The improvement of this last method is due to the fact that it maintains the intermediate implicational systems simplified all the time. Theorems 3.2.3 and 3.3.3 ensure that both of them are sound and complete.

Finally, we carry out two empirical studies. One of them is a comparison among `doSimp` and the previous direct-optimal basis methods in the literature to show the efficiency of our method (see Table 3.2). The second study compares the two methods proposed in this chapter and illustrate the benefits of `SLgetdo` with respect to `doSimp` in practice (see Table 3.5).

With the same goal of computing efficiently closures and with the directness property in mind, Adaricheva et al. proposed the D-basis [3]. This basis is subsumed in the direct-optimal basis, so it has smaller size. Obviously, a basis which computes the closure in just one traversal and which has less size than the direct-optimal basis will allow to improve the applications which demand the execution of a huge number of closures.

In Chapter 4, we present a theoretical study regarding the relationships between covers and generators. As minimal covers is the main concept to compute the D-basis and the well-founded connection between minimal generators and minimal covers presented in Proposition 4.1.7 has allowed to develop new transformation algorithms.

Despite considerable achievements in understanding the connection be-

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tween direct-optimal basis and D-basis, no algorithm was previously developed to directly produce the D-basis from an arbitrary set of implications. Thus, it was an open problem and it makes our methods the first in the literature.

Specifically, we propose two methods. The first one, Algorithm 4.1 is a three-stage method based on the direct relationship between minimal generators and minimal covers. The second method is completely based on a theoretical study to interleave the computation of the minimal covers whereas just the necessary minimal generators are computed. As this second method is not a direct consequence of the above mentioned relationship, Theorem 4.3.16 proves its soundness and completeness.

Finally, an empirical study between both methods is done to prove the benefits of the second approach (see Figure 4.3).

The generation of direct bases is a problem that has an exponential complexity. This fact led us to think about other alternative definitions of direct bases, with the intention to avoid the exponential complexity throughout the execution of all the stages of the transformation method. That's how it appears the notion of dichotomous direct basis with the aim of reducing the cost in its computation and maintaining directness property.

In Chapter 5, we introduce a new kind of direct basis: dichotomous direct basis or DD-basis for short. First of all, we carry out a theoretical study about the behavior of some kinds of implications with respect to the closure operator: (proper) key implications and (proper) quasi-key implications. In this way, we divide the set of implications in a pair and introduce a two-fold closure operator.

The definition of DD-basis requires the idempotence of the closure operator in order to make it direct. Once this new notion is introduced, we present a method to compute it. This method consists of three stages where the only one having exponential complexity is the second one. In this specific stage we compute the direct-optimal basis but the input has smaller cardinality because previously the input set has been split.

Being used to dealing with unique basis, we present the notion of canonical DD-basis. The existence and the uniqueness of this basis in ensured in

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Theorem 5.4.8. To compute it, we only need to add to the previous algorithm a stage with cuadratic cost to reach the canonical DD-basis equivalent.

Finally, we compare the computation of the direct-optimal basis with the computation of the canonical DD-basis. The conclusion showed in Figure 5.1 is that the new method is more efficient than the one computing the direct-optimal basis.

In the second part of this work, the one devoted to Triadic Concept Analysis, we have firstly summarized some results about the main notions of this framework: triadic context, triadic concept and triadic implications.

The major contribution to this area is located in Chapter 7. Here, we develop the first axiomatic system to manage conditional attribute implications, CAIs. We present CAIL as an extension of Armstrong's Axioms and prove its soundness and completeness (Theorem 7.1.21). Since this logic is not suitable for automated reasoning, we also extend Simplification Logic calling it CAISL. The axiomatic system associated with this logic is proved to be equivalent to the one of CAIL in Theorem 7.2.2. As a consequence, CAISL is also sound and complete.

Finally, we design an automated prover which is able to ensure whether a CAI could be derived from a set of CAIs or not. This automated prover is directly related to closure computation as said in Theorem 7.1.16.

Future works

To conclude, we outline some tasks we have in mind to continue the work presented in this PhD Thesis.

Following the order of the contributions, the first topic we are considering to go beyond is the issue of the direct-optimal basis. We have proposed new methods, more efficient than the previous ones in the literature, but we keep studying the way to improve our algorithms. With the aim of applying reductions in the set of attributes, we plan a collaboration with Dr. Bertet who has already begun working on reductions. Because of the connection points of this idea and our simplification paradigm, we consider

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to integrate both works to provide more efficient methods, which could be applied not only to direct-optimal bases but also to bases in general.

Regarding the D-basis, we intend to work with Dr. Adaricheva in the application of `Fast`D`-basis` in real environments as described in [2]. These applications are related to optimization problems in data base theory, artificial intelligence and game theory. In particular, one of its applications is devoted to the analysis of gene expression data related to a particular phenotypic variable with data provided by the University of Hawaii Cancer Center.

The previous work will motivate the study of further improvements of the performance of `Fast`D`-basis` to get even a more efficient method to get the D-basis.

In addition, the creation of an algorithm to mining the D-basis from a formal context can be interesting to compare it with the only method available by now [2]. That method was solved using the hypergraph dualization algorithm, which can be considered as an indirect technique. The results obtained in this work lead us to think about promising results for this problem, approached with a direct technique.

The progress we have achieved with the definition and the methods proposed for dichotomous direct bases makes us believe that it is possible the integration of any direct basis in the new formalism of dichotomous bases. Particularly, instead of computing the direct-optimal basis as a second step, we can substitute it by the D-basis, which is smaller. In this case, to maintain the directness property the closure operator has to be changed by the composition of ρ_{Σ^*} and π_{Σ^k}.

Finally as a medium and long term work, the generalization of all our work to Triadic Concept Analysis is a challenge. We intend to define a triadic basis based on some criteria of minimality and, later on, extend this definition to direct bases. In this way, we plan to carry out a theoretical study about what properties are needed to compute the closure in one traversal. In addition, it would be a interesting goal to achieve an axiomatic system for Biedermann's implications and introduce the notion of basis for this kind of implications.

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Moreover, this generalization can be also made in the fuzzy framework. The progress obtained for members of our group in fuzzy logic makes us consider this task as feasible. To conclude, we emphasize we have the tool to achieve this challenge, the Simplification Logic proposed for fuzzy implications [7].

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