F-transforms for the definition of contextual fuzzy partitions*

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Abstract. Fuzzy partitions are defined in many different ways but usually, by taking into account aspects of the whole universe. In this paper, we present a method to define fuzzy partitions for elements in the universe holding certain fuzzy attribute. Specifically, we show how to define those fuzzy partitions by means of F-transforms.

1 Introduction

The notion of Fuzzy partition is, in the most cases, the core of the first step of every fuzzy systems, i.e., the fuzzification procedure [1,6]. Although the formal definition of fuzzy partition differs for many authors (see [2,4,5,10]), in general the idea is to divide the universe in a set of classes (usually disjoint and linked to linguistic labels). There exist in the literature many different methods to define fuzzy partitions; the most known are those given by experts, by uniform partitions [9] and by techniques based on clustering or statistics [10,3].

Independently how partitions are defined, they must depend on the context. It is not the same to do a partition according to the attribute *"height"* if the universe is *"the set of players of the NBA"* or *"students of the fourth degree in a public school"*. In this paper we present a technique to define fuzzy partitions according to a fuzzy context (e.g., hotness during the morning) based on Fuzzy transforms [8,7].

The structure of the paper is the following. In Section 2 we recall the basics of the Fuzzy transforms and subsequently, in Section 3 we show how to build conditional partitions.

2 Preliminaries

The theory of F-transforms pivots on the notion of fuzzy partition, which is defined as follows.

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Definition 1. *A fuzzy partition Δ of a universe $\mathcal{U}$ is a set of fuzzy sets $\Delta_1, \ldots, \Delta_n$ on $\mathcal{U}$ fulfilling the covering property, i.e. for all $x \in [0,1]$ there exists $k \in \{1, \ldots, n\}$ such that $\Delta_k(x) > 0$. The membership functions $\Delta_k(x)$, $k = 1, \ldots, n$ are called the* basic functions *of Δ.*

The basic function forming fuzzy partitions can be defined in multiple ways. For the sake of simplicity and a better applicational environment, it is common to define basic function in terms of parameters. In this approach we consider trapezoidal fuzzy sets on the universe of real numbers, which are defined from four parameters $a, b, c, d \in \mathbb{R}$ as follows

$$trp(a,b,c,d)(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x \leq b \\ 1 & \text{if } b < x \leq c \\ \frac{d-x}{d-c} & \text{if } c < x \leq d \\ 0 & \text{if } d < x \end{cases}$$

If $b = c$, the fuzzy set $trp(a,b,c,d)$ is called triangular. In some case we do an abuse of notation by considering trapezoidal membership functions with parameters a, b and c, d equal to $-\infty$ and ∞, respectively. In such a cases, the basic functions refer to the followings:

$$trp(-\infty, -\infty, c, d)(x) = \begin{cases} 1 & \text{if } x \leq c \\ \frac{d-x}{d-c} & \text{if } c < x \leq d \\ 0 & \text{if } d < x \end{cases}$$

and

$$trp(a, b, \infty, \infty)(x) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a < x \leq b \\ 1 & \text{if } b < x \end{cases}$$

In general there are many ways to define fuzzy partitions in an universe $\mathcal{U}$, but they can be clasified into two families: fuzzy partition based on linguistic labels given by an expert and fuzzy partitions defined by clustering or statistic parameters (i.e., learning). In Example 1 we present a fuzzy partition given by an expert to represent a division of a day.

Example 1. Human beings usually divides the 24 hours of the day in five temporal intervals, namely: *Early morning, Morning, Afternoon, Evening*, and *Night.* Those temporal intervals are fuzzy and can be described as trapezoidal fuzzy sets as follows:

- Early morning is identified with: $trp(00,\ 02,\ 05,\ 07)$
- Morning is identified with: $trp(05,\ 07,\ 11,\ 13)$
- Afternoon is identified with: $trp(11,\ 13,\ 14,\ 15)$
- Evening is identified with: $trp(14,\ 15,\ 18,\ 20)$
- Night is identified with: $trp(18,\ 20,\ 23,\ 01)$

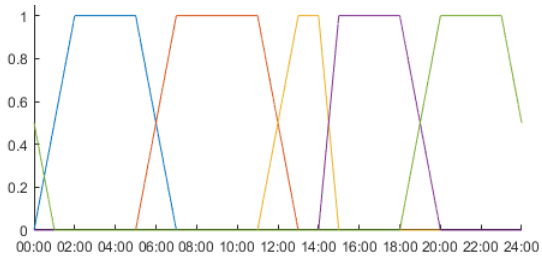


Fig. 1. Fuzzy Partition given by an expert for the division of the day in *"Early morning"*, *"Morning"*, *"Afternoon"*, *"Evening"* and *"Night"*.

where time is given in notation $24h$. The graphics of the previous partition is given in the Figure 1. □

Below we recall the definition and some basic properties of Fuzzy transforms [7]. Hereafter we consider a subset $\mathbf{T} = \{(x_i, y_i)\}_{i\in\mathbb{I}}$ of $\mathcal{U} \times [0,1]$ without functional structure. The discrete F-transform for non functional data is defined at following.

Definition 2. *Let* $\mathbf{T} = \{(x_i, y_i)\}_{i\in\mathbb{I}} \subseteq \mathcal{U} \times [0,1]$ *and let* $\Delta = \{\Delta_1, \dots, \Delta_n\}$ *be a fuzzy partition of* $\mathcal{U}$*. We say that the* n*-tuple* $\mathbf{F}_\Delta[\mathbf{T}] = [F_1, \dots, F_n] \in [0,1]^n$ *is the direct F-transform of* $\mathbf{T}$ *w.r.t.* Δ *if*

$$F_k = \frac{\sum_{i\in\mathbb{I}} y_i \Delta_k(x_i)}{\sum_{i\in\mathbb{I}} \Delta_k(x_i)} \tag{1}$$

The definition above extends the original one [8] by identifying a function $f\colon \mathcal{U} \to [0,1]$ with the subset $\mathbf{T}_f = \{(x, f(x) \mid x \in \mathcal{U}\} \subseteq \mathcal{U}\times[0,1]$, i.e.,$\mathbf{F}_\Delta[\mathbf{T_f}] = \mathbf{F}_\Delta[f]$. As in the original definition [8], the components of the direct F-transform coincide with the *least squares weighted* by the basic functions of Δ.

Proposition 1. *Let* $\mathbf{T} = \{(x_i, y_i)\}_{i\in\mathbb{I}} \subseteq \mathcal{U} \times [0,1]$ *and let* $\Delta = \{\Delta_1, \dots, \Delta_n\}$ *be a fuzzy partition of* $\mathcal{U}$*. Then, the* k*th component of the F-transform is the minimum of the following function:*

$$\phi(z) = \sum_{i\in\mathbb{I}} (y_i - z)^2 \Delta_k(x_i) \tag{2}$$

As in the original approach, the inverse F-transform is a function defined from the direct F-transform.

Definition 3. *Let* $\mathbf{T} = \{(x_i, y_i)\}_{i\in\mathbb{I}} \subseteq \mathcal{U}\times[0,1]$ *and let* $\mathbf{F}_\Delta[\mathbf{T}] = [F_1, \dots, F_n] \in [0,1]^n$ *be the* direct F-transform *of* $\mathbf{T}$ *w.r.t.* Δ*. Then the function*

$$\mathbf{T}^F_\Delta(x) = \frac{\sum_{k=1}^n F_k \Delta_k(x)}{\sum_{k=1}^n \Delta_k(x)} \tag{3}$$

is called the inverse F-transform *of* $\mathbf{T}$*. w.r.t.* Δ*.*

Some remarks about the previous definition. Firstly, the inverse F-transform $\mathbf{T}^F_{\Delta}(x)$ is a function independently whether the set $\mathbf{T}$ has the structure of a function or not. Secondly, note that the domain of the inverse F-transform is $\mathcal{U}$, so it is defined even for those $x \in \mathcal{U}$ such that there is not $(x, y) \in \mathbf{T}$. Thirdly, it extends the original definition of F-transforms [8]. Finally, as we show below, the inverse F-transform is closely related to the function obtained by assigning to each $x \in \mathcal{U}$ the mean among all the y_i such that $(x, y_i) \in \mathbf{T}$. For the sake of a better understanding, let us be more formal about such a function. Let us define the sets

$$D_x = \{y \in [0, 1] \mid (x, y) \in \mathbf{T}\} \qquad \text{for any } x \in \mathcal{U} \tag{4}$$

and

$$D_{\mathbf{T}} = \{x \in \mathcal{U} \mid \text{ there exists } (x, y) \in \mathbf{T}\} \tag{5}$$

On $D_{\mathbf{T}}$ we define the function $m_{\mathbf{T}}(x)$ that assigns to each $x \in D_{\mathbf{T}}$ the value

$$m_{\mathbf{T}}(x) = \frac{\sum_{y_i \in D_x} y_i}{|D_x|} \tag{6}$$

where $|D_x|$ denotes the cardinality of the set D_x. Note that $m_{\mathbf{T}}$ is a function that assigns to each x the mean of all the y_i such that $(x, y_i) \in \mathbf{T}$.

Theorem 1. *Let $\mathbf{T} = \{(x_i, y_i)\}_{i \in \mathbb{I}} \subseteq \mathcal{U} \times [0, 1]$. Then, there exists a fuzzy partition Δ such that $\mathbf{T}^F_{\Delta}(x) = m_{\mathbf{T}}(x)$; where $m_{\mathbf{T}}$ is the function defined by equation* (6)*.*

From the theorem above, we can assert somehow that the inverse F-transform of a data set $\mathbf{T}$ approximates the function mean $m_{\mathbf{T}}$.

3 Defining conditional partitions

Let us assume now that we have a universe $\mathcal{U}$ with two quantitative attributes X and Y. The goal of this section is to define a fuzzy partition on $\mathcal{U}$ according to the variable Y conditioned to the information we already know about the variable X. We assume a fuzzy partition Δ of $\mathcal{U}$ according to the values of the attribute X. Note that each coordinate F_k of the direct F-transform of $\mathbf{T}_X = \{(x_i, Y(x_i))\}_{i \in \mathbb{I}} \subseteq \mathcal{U} \times [0, 1]$ w.r.t. Δ, can be considered as a local mean of the attribute Y for those data with attribute X in Δ_k. So, if the attribute of X is in Δ_k, the attribute Y is medium if its value is close to F_k. Therefore, if the conditioned information of X is given by one class of Δ, the respective coordinate of the direct F-transform can be considered a good referential parameter for a fuzzy partition of Y. In order to determine other parameters to represent when the attribute Y is high or low, we are going to proceed at following:

1. Compute the direct F-transform of $\mathbf{T}_Y = \{(x_i, Y(x_i))\}_{i \in \mathbb{I}}$ according to a partition Δ_X values of the attribute X;

2. Compute the inverse F-transform $(\mathbf{T}_Y)^F_\Delta(x)$ of $\mathbf{T}_Y = \{(x_i, Y(x_i))\}_{i\in\mathbb{I}}$;
3. Define two data sets:

$$\mathbf{T}^{\uparrow}_Y = \{(x_i, Y(x_i)) \mid Y(x_i) \geq (\mathbf{T}_Y)^F_\Delta(x_i)\}_{i\in\mathbb{I}}$$
$$\mathbf{T}^{\downarrow}_Y = \{(x_i, Y(x_i)) \mid Y(x_i) \leq (\mathbf{T}_Y)^F_\Delta(x_i)\}_{i\in\mathbb{I}}$$

4. Compute the direct F-transforms $F_{\Delta_X}[\mathbf{T}^{\uparrow}_Y]$ and $F_{\Delta_X}[\mathbf{T}^{\uparrow}_Y]$ of $\mathbf{T}^{\uparrow}_Y$ and $\mathbf{T}^{\downarrow}_Y$ respectively, w.r.t. the fuzzy partition Δ_X;

For the sake of the presentation, let us denote as $F^{\uparrow}_k$, $F^{\downarrow}_k$ and F^*_k the parameter F_k in $F_{\Delta_X}[\mathbf{T}^{\uparrow}_Y]$, in $F_{\Delta_X}[\mathbf{T}^{\downarrow}_Y]$ and in $F_{\Delta_X}[\mathbf{T}_Y]$, respectively. Moreover, note that the parameter $F^{\uparrow}_k$ (resp. $F^{\downarrow}_k$) represents the mean of the attribute Y for those elements in $\mathcal{U}$ belonging to $(\Delta_X)_k$ and with attribute Y greater (resp. lesser) than the mean. Therefore, the parameters $F^{\uparrow}_k$ and $F^{\downarrow}_k$ can be considered as a kind of index of dispersion and they have a similar meaning than expectiles; i.e., it is expected that the half of the data belongs to the interval $[F^{\uparrow}_k, F^{\downarrow}_k]$. That interpretation of $F^{\uparrow}_k$ and $F^{\downarrow}_k$ motivates the last step of the method:

5 For each $(\Delta_X)_k$, compute the partition of three triangular classes given by the triples $[0, F^{\downarrow}_k, F^*_k]$, $[F^{\downarrow}_k, F^*_k, F^{\uparrow}_k]$ and $[F^*_k, F^{\uparrow}_k, 1]$.

In such a respect, we say that if an object in the universe is in $(\Delta_X)_k$, the attribute Y is low if it is in $[0, F^{\downarrow}_k, F^*_k]$, medium if it is in $[F^{\downarrow}_k, F^*_k, F^{\uparrow}_k]$ and high if it is in $[F^*_k, F^{\uparrow}_k, 1]$.

Application to generation of electricity power.

In order to show the potential application of this kind of conditional partition, we consider the generation of electricity in the Spanish part of the Iberian Peninsula in January 2018[1]. The direct F-transform is computed according to the fuzzy partition given in Example 1 and the results are:

$$\mathbf{F}_\Delta[\mathbf{T}] = [F_1, F_2, F_3, F_4, F_5] = [23891, 28802, 30826, 29877, 29391](MWh)$$

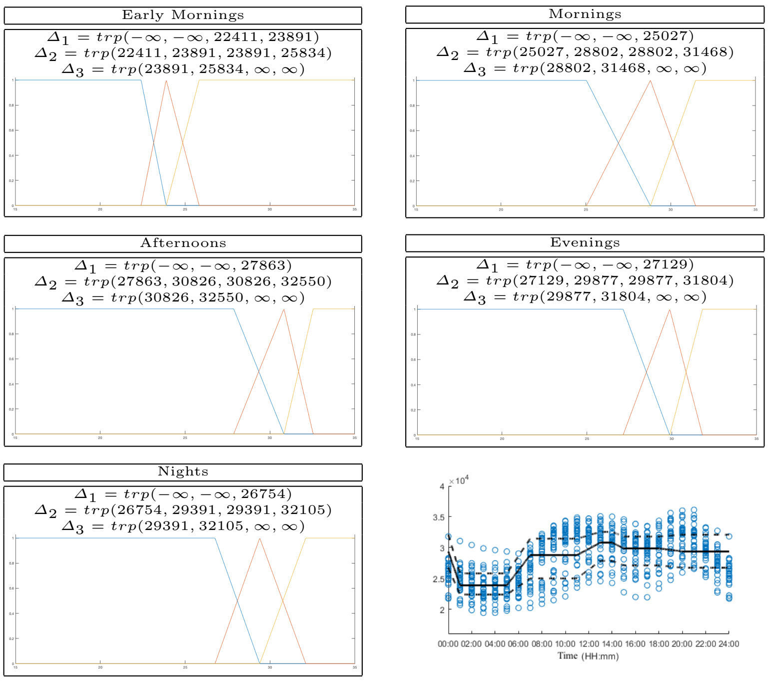
As an example, the interpretation of the value $F_2 = 28802$ is that the mean of the power generation in the mornings during January 2018 was 28802 MWh. The corresponding direct F-transforms $\mathbf{F}_\Delta[\mathbf{T}^{\uparrow}]$ and $\mathbf{F}_\Delta[\mathbf{T}^{\downarrow}]$ are

$$\mathbf{F}_\Delta[\mathbf{T}^{\uparrow}] = [F^{\uparrow}_1, F^{\uparrow}_2, F^{\uparrow}_3, F^{\uparrow}_4, F^{\uparrow}_5] = [25834, 31468, 32550, 31804, 32105]$$
$$\mathbf{F}_\Delta[\mathbf{T}^{\downarrow}] = [F^{\downarrow}_1, F^{\downarrow}_2, F^{\downarrow}_3, F^{\downarrow}_4, F^{\downarrow}_5] = [22411, 25027, 27863, 27129, 26754]$$

Below (down-right) we show scatter plot of the data with the three inverses F-transform of $\mathbf{T}$, $\mathbf{T}^{\uparrow}$ and $\mathbf{T}^{\downarrow}$. The five partitions are also represented

[1] The data is available in `www.minetad.gob.es`

graphically below with the expression of the basic function in the notation of trapezoidal fuzzy numbers. Note the interpretation of *Middle electrical generation* is different in each period of the day. In particular, as a remark, what is a middle electrical generation during mornings, is a low electrical generation during noons.



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