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Kian Sierra McGettigan

CLASSICAL OPERATORS ON WEIGHTED BERGMAN AND MIXED NORM SPACES

ACADEMIC DISSERTATION

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Kian Sierra McGettigan
Classical operators on weighted Bergman and mixed norm spaces
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ABSTRACT

This thesis introduces new results concerning classical operators of the type $T_\mu : X \to Y$ or $T : X \to L^q_\mu(\mathbb{D})$, where μ is a Borel measure over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and X, Y are spaces of functions over the unit disk. The space X in most cases will be one of the following; a weighted Bergman spaces A^p_ω, a tent space $T^p_q(h,\omega)$ or a weighted mixed norm space $A^{p,q}_\omega$, where the weight ω is a radial weight that satisfies the doubling condition $\int_r^1 \omega(s)ds \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)ds$, among other possible conditions. Our goal in this thesis will be to characterize properties of the operators such as boundedness, compactness or belonging to a certain Schatten class, in terms of (geometric) conditions over the measure μ.

RESUMEN

Esta tesis contiene resultados originales sobre operadores clásicos de tipo $T_\mu : X \to Y$ o bien $T : X \to L^q_\mu(\mathbb{D})$, donde μ es una medida de Borel definida sobre el disco unidad $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ y X, Y son espacios de funciones, normalmente analíticas, definidas sobre el disco unidad. El espacio X, suele ser alguno de los siguientes espacios de funciones; un espacio de Bergman con pesos A^p_ω, un espacio de tipo tienda $T^p_q(h,\omega)$ o un espacio de norma mixta $A^{p,q}_\omega$, donde el peso ω es radial y satisface la propiedad doblante $\int_r^1 \omega(s)ds \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)ds$. Nuestro objetivo en esta tesis es caracterizar propiedades de estos operadores, tales como la acotación, compacidad o pertenencia a clases de Schatten, en términos de condiciones (geométricas) sobre la medida μ.

MSC 2010: 32A36, 47G10, 42B25, 47B35, 30H20, 46E15, 47B38.
Keywords: Bergman space, Mixed norm space, Tent space, weight, Carleson measure, reproducing kernel, Bergman projection, area operator, Toeplitz operatos, atomic decomposition.

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Joensuu, April 25, 2018

Kian Sierra McGettigan

LIST OF PUBLICATIONS

This thesis consists of the present review of the author's work in the field of operator theory and the following selection of the author's publications:

I J. Á. Peláez, J. Rättyä and K. Sierra, "Embedding Bergman spaces into tent spaces," *Math. Z.* **281** (2015), no. 3-4, 1215–1237.

II J. Á. Peláez, J. Rättyä and K. Sierra, "Berezin transform and Toeplitz operators on weighted Bergman spaces induced by regular weights" *J. Geom. Anal.* DOI: 10.1007/s12220-017-9837-9 (2017), 1-32.

III J. Á. Peláez, J. Rättyä and K. Sierra, "Atomic decomposition and Carleson measures for weighted mixed norm spaces" Submitted preprint. http://arxiv.org/abs/1709.07239

Throughout the overview, these papers will be referred to by Roman numerals.

AUTHOR'S CONTRIBUTION

The publications selected in this dissertation are original research papers on operator theory.

Paper **I** is a continuation of research done by the other two authors. All authors have made an equal contribution.

Paper **II** is a continuation of research done in Joensuu. All authors have made an equal contribution.

In Paper **III**, all authors have made an equal contribution.

TABLE OF CONTENTS

1 Introduction

The main aim of this thesis is to study the boundedness of certain operators of the type $T_\mu : X \to Y$ with a symbol μ or $T : X \to L^q_\mu(\mathbb{D})$. Here μ is a positive Borel measure over the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, X is a space of analytic functions over the unit disk $\mathbb{D}$ and Y is a certain space of functions.

Our main goal is to characterize the boundedness and compactness of the operators T_μ in terms of (geometric) conditions over the measure μ. We will work with a variety of weighted function spaces such as weighted Bergman spaces A^p_ω, tent spaces $T^p_q(h, \omega)$, mixed norm spaces $A^{p,q}_\omega$. Most of our results are given for weights in the class $\widehat{\mathcal{D}}$, these are radial weights $\omega(z) = \omega(|z|)$, that satisfy

$$\int_r^1 \omega(s)ds \lesssim \int_{\frac{1+r}{2}}^1 \omega(s)ds.$$

Basic properties of these weights can be found in [51, 54], additional conditions on the weights might be required in each case. The standard weights $\omega(z) = (1 - |z|)^\alpha$ for $\alpha > -1$ all belong to the class $\widehat{\mathcal{D}}$. In particular for every weight $\omega \in \widehat{\mathcal{D}}$, there exists $\alpha(\omega) > -1$ such that $H^p \subset A^p_\omega \subset A^p_\alpha$. Some weights in the class $\widehat{\mathcal{D}}$ satisfy the more restrictive embedding

$$H^p \subset A^p_\omega \subset \cap_{\alpha > -1} A^p_\alpha,$$

which gives us an idea of why in certain problems an approach more common of Hardy spaces is effective.

In order to characterize the boundedness of the operator T_μ we will work in understanding the functions of the given spaces, equivalent norms in these spaces or characterizations of their duals, among other techniques.

In order to study the functions in the given space we will find a suitable family of functions $\{f_n\}$ belonging to the space X, and we will see that all functions of the form $f = \sum c_n f_n$ belong to X, where the sequence $\{c_n\}$ belongs to a given sequence space S, and satisfies the inequality $\|f\|_X \lesssim \|\{c_n\}\|_S$. These inequalities will allow us to discretize the problem at hand. When possible we will also prove that every function in the space X can be written as $f = \sum c_n f_n$.

Another way of identifying the functions in the space X is to find a projection that is bounded onto X. If the projection $P : Z \to X$ is bounded, we can see every $f \in X$ as $f = P(g)$, where $g \in Z$, and Z is a function space of which we already have some information.

Many of the conditions that characterize the boundedness of the given operator T_μ, will consist in proving that μ is a certain Carleson measure. We say μ is a $q-$Carleson measure for the space X if $Id : X \to L^q_\mu(\mathbb{D})$ is bounded. Some of the classical results on Carleson measures were given by Carleson, Duren, Luecking, Jevtic [11, 24, 35, 41] for the spaces H^p, A^p, $A^{p,q}$. Peláez and Rättyä characterized Carleson measure for A^p_ω when $\omega \in \widehat{\mathcal{D}}$ in [55].

The remainder of this survey is organized as follows. In section 2 we give some notation on the subject and recall certain properties on weights and function spaces.

Section 3 contains results on atomic decomposition of standard Bergman spaces A^p_α and mixed norm spaces $A^{p,q}$. In section 4 we discuss some of the previous results on operators, such as Carleson measures, area operators and projections. Finally section 5 summarizes papers **I-III**.

2 Notation

2.1 BASIC NOTATION

We use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$, which depends on certain parameters that will be specified if necessary such that $a \le Cb$. This constant may change from line to line, and we define $a \gtrsim b$ in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$ we will write $a \asymp b$.

We define the euclidean unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and its boundary $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. We denote with $D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$ the euclidean disk of center $a \in \mathbb{C}$ and radius $r \in (0,\infty)$. The pseudohyperbolic distance is $\varrho(a,z) = \left|\frac{z-a}{1-\overline{a}z}\right|$, and $\Delta(a,r) = \{z \in \mathbb{C} : \varrho(z,a) < r\}$ is the pseudohyperbolic disk with center $a \in \mathbb{D}$ and radius $0 < r < 1$. The hyperbolic distance is defined as $d(a,z) = \frac{1}{2}\log\left(\frac{1+|\varrho(a,z)|}{1-|\varrho(a,z)|}\right)$, and the hyperbolic disk is $\Lambda(a,t) = \{z \in \mathbb{D} : d(a,z) < t\}$ for all $a \in \mathbb{D}$ and $t > 0$.

A sequence $Z = \{z_k\}_{k=0}^{\infty} \subset \mathbb{D}$ is called separated if it is separated in the pseudohyperbolic metric, it is an ε-net for $\varepsilon \in (0,1)$ if $\mathbb{D} = \bigcup_{k=0}^{\infty} \Delta(z_k, \varepsilon)$, and finally it is a δ-lattice if it is a 5δ-net and separated with constant $\delta/5$.

Given two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ we say that a linear operator $T : X \to Y$ is bounded if

$$\|T\|_{(X,Y)} = \sup_{x \in X : \|x\|_X = 1} \{\|T(x)\|_Y\} < \infty.$$

Given $\zeta \in \mathbb{T}$ we define the non-tangential region with vertex at ζ as follows

$$\Gamma(\zeta) = \left\{\xi \in \mathbb{D} : |\theta - \arg(\xi)| < \frac{1}{2}(1 - |\xi|)\right\}, \quad \zeta = e^{i\theta} \in \mathbb{T}.$$

These sets can be generalized to non-tangential regions with vertex at z in the punctured unit disk $\mathbb{D} \setminus \{0\}$,

$$\Gamma(z) = \left\{\xi \in \mathbb{D} : |\theta - \arg(\xi)| < \frac{1}{2}\left(1 - \frac{|\xi|}{r}\right)\right\}, \quad z = re^{i\theta} \in \overline{\mathbb{D}} \setminus \{0\}.$$

The associated tents are defined by $T(\zeta) = \{z \in \mathbb{D} : \zeta \in \Gamma(z)\}$ for all $\zeta \in \mathbb{D} \setminus \{0\}$. When we are working with a radial weight ω we set $\omega(T(0)) = \lim_{r \to 0^+} \omega(T(r))$ to deal with the origin.

The Carleson square $S(I)$ based on an interval $I \subset \mathbb{T}$ is the set $S(I) = \{re^{it} \in \mathbb{D} : e^{it} \in I, 1 - |I| \le r < 1\}$, where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{T}$. We associate to each $a \in \mathbb{D} \setminus \{0\}$ the interval $I_a = \{e^{i\theta} : |\arg(ae^{-i\theta})| \le \frac{1-|a|}{2}\}$, and denote $S(a) = S(I_a)$. For the case $a = 0$ we set $I_0 = \mathbb{T}$, hence $S(0) = \mathbb{D}$.

The polar rectangle associated with an arc $I \subset \mathbb{T}$ is

$$R(I) = \left\{z \in \mathbb{D} : \frac{z}{|z|} \in I,\ 1 - \frac{|I|}{2\pi} \le |z| < 1 - \frac{|I|}{4\pi}\right\}.$$

Write $z_I = (1 - |I|/2\pi)\xi$, where $\xi \in \mathbb{T}$ is the midpoint of I.

Let Υ denote the family of all dyadic arcs of $\mathbb{T}$. Every arc $I \in \Upsilon$ is of the form

$$I_{n,k} = \left\{ e^{i\theta} : \frac{2\pi k}{2^n} \leq \theta < \frac{2\pi(k+1)}{2^n} \right\},$$

where $k = 0, 1, 2, \dots, 2^n - 1$ and $n \in \mathbb{N} \cup \{0\}$.

The family $\{R(I) : I \in \Upsilon\}$ consists of pairwise disjoint rectangles whose union covers $\mathbb{D}$. For $I_j \in \Upsilon \setminus \{I_{0,0}\}$, we will write $z_j = z_{I_j}$. For convenience, we associate the arc $I_{0,0}$ with the point $1/2$.

2.2 WEIGHTS

An integrable function $\omega : \mathbb{D} \to [0, \infty)$ is called a weight. We say it is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$, and we write $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s) ds$. Most of the thesis will focus on radial weights that satisfy the doubling condition $\widehat{\omega}(r) \lesssim \widehat{\omega}(\frac{1+r}{2})$. This class of weights is denoted by $\widehat{\mathcal{D}}$. Given a radial weight ω, we define its associated weight by

$$\omega^\star(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} s \, ds, \quad z \in \mathbb{D} \setminus \{0\}.$$

The following lemma will show some of the characterizations of the weights in this class.

Lemma 2.2.1. *[51, Lemma 2.1] Let ω be a radial weight. Then the following assertions are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) There exists $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that

$$\widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t} \right)^\beta \widehat{\omega}(t), \quad 0 \leq r \leq t < 1;$$

(iii) There exists $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that

$$\int_0^t \left(\frac{1-t}{1-s} \right)^\gamma \omega(s) \, ds \leq C \widehat{\omega}(t), \quad 0 \leq t < 1;$$

(iv) $\omega^\star(z) \asymp \widehat{\omega}(z)(1 - |z|)$, $|z| \to 1^-$;

(v) There exists $\lambda = \lambda(\omega) \geq 0$ such that

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1 - \overline{\zeta} z|^{\lambda+1}} dA(z) \asymp \frac{\widehat{\omega}(\zeta)}{(1 - |\zeta|)^\lambda}, \quad \zeta \in \mathbb{D};$$

(vi) There exists $C = C(\omega) > 0$ such that $\omega_x = \int_0^1 r^x \omega(r) \, dr$, $x \geq 0$ satisfies $\omega_n \leq C\omega_{2n}$ for $n \in \mathbb{N}$.

We say a radial weight satisfies the reverse doubling condition if it satisfies one of the following equivalent conditions.

Lemma 2.2.2. *[59] Let ω be a radial weight. Then the following statements are equivalent:*

(i) There exists $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right) \quad \text{for all } 0 \leq r < 1; \tag{2.1}$$

(ii) There exists $C = C(\omega) > 0$ and $\alpha = \alpha(\omega) > 0$ such that

$$\widehat{\omega}(t) \leq C\left(\frac{1-t}{1-r}\right)^{\alpha} \widehat{\omega}(r), \quad 0 \leq r \leq t < 1;$$

(iii) There exists $C = C(\omega) > 0$ such that

$$\int_r^1 \frac{\widehat{\omega}(s)}{1-s}\, ds \leq C\widehat{\omega}(r).$$

The class of weights that satisfies both the doubling condition and the reverse doubling condition is denoted with $\mathcal{D}$.

We say that a radial weight ω is regular if $\omega \in \widehat{\mathcal{D}}$ and $\frac{\widehat{\omega}(r)}{1-r} \asymp \omega(r)$ for $0 \leq r < 1$, and we denote this class of weights with $\mathcal{R}$. The class of weights $\mathcal{R}$ is contained in $\mathcal{D}$. The standard weights $\omega(r) = (1-r)^{\alpha}$ with $\alpha > -1$ belong to $\mathcal{R}$. We define the class of rapidly increasing weights denoted with $\mathcal{I}$ as those radial weights which are continuous and $\lim_{r \to 1^-} \frac{\widehat{\omega}(r)}{\omega(r)(1-r)} = \infty$. Some examples of weights in the class $\mathcal{I}$ are

$$v_\alpha(r) = \frac{1}{(1-r)\left(\log \frac{e}{1-r}\right)^{\alpha}}, \quad 1 < \alpha < \infty.$$

Both these classes $\mathcal{R}$ and $\mathcal{I}$ are subclasses of the class of doubling weights $\widehat{\mathcal{D}}$ by [51].

We denote $\omega(E) = \int_E \omega(\zeta)dA(\zeta)$ where E is a measurable subset of $\mathbb{D}$.

Lemma 2.2.3. *[54, Lemma 1.6]*

(i) if ω is a radial weight, then

$$\omega^{\star}(z) \asymp \omega(T(z)), \quad |z| \geq \frac{1}{2}.$$

(ii) if $\omega \in \widehat{\mathcal{D}}$, then

$$\omega(T(z)) \asymp \omega(S(z)), \quad z \in \mathbb{D}.$$

The proof in [54, Lemma 1.6] is restricted to the class $\mathcal{R} \cup \mathcal{I}$ but also works for the class $\widehat{\mathcal{D}}$, as appears in [51, Pag. 55, Eq. 26].

2.3 FUNCTION SPACES AND SOME OF THEIR PROPERTIES

In this section we will define the function spaces that appear throughout this thesis.

2.3.1 Hardy spaces

If $0 < r < 1$ and $f \in \mathcal{H}(\mathbb{D})$, we set

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{it})|^p dt \right)^{\frac{1}{p}}, \quad 0 < p < \infty, \tag{2.2}$$

$$M_\infty(r,f) = \max_{0 \le t < 2\pi} |f(re^{it})|. \tag{2.3}$$

Since $f \in \mathcal{H}(\mathbb{D})$, the maximum modulus principle and the subharmonicity of $|f|^p$ tells us that the integral means M_p and M_∞ are non-decreasing functions of r.

For $0 < p \le \infty$ we define the Hardy space H^p as the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(r,f) < \infty.$$

Given a function $f : \mathbb{D} \to \mathbb{C}$ we define its non-tangential maximal function as

$$N(f)(e^{i\theta}) = \sup_{z \in \Gamma(e^{i\theta})} |f(z)|, \quad e^{i\theta} \in \mathbb{T}. \tag{2.4}$$

If $f \in H^p$, then $N(f) < \infty$ almost everywhere on $\mathbb{T}$. In particular for $f \in \mathcal{H}(\mathbb{D})$ we have the equivalent norm

$$\|f\|_{H^p} \asymp \|N(f)\|_{L^p(\mathbb{T})}, \tag{2.5}$$

where $L^p(\mathbb{T})$ refers to the standard Lebesgue space on $\mathbb{T}$.

We also recall the Hardy-Stein-Spencer identity, which can be found in [30]

$$\begin{aligned} \|f\|_{H^p}^p &= \frac{1}{2} \int_{\mathbb{D}} \triangle \left(|f|^p \right) \log \frac{1}{|z|} dA(z) + |f(0)|^p \\ &= \frac{p^2}{2} \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \log \frac{1}{|z|} dA(z) + |f(0)|^p \\ &\asymp \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \,(1 - |z|)\, dA(z) + |f(0)|^p, \end{aligned} \tag{2.6}$$

where $dA(z) = \frac{dxdy}{\pi}$ is the normalized Lebesgue measure on $\mathbb{D}$.

2.3.2 Weighted Bergman spaces

The Bergman spaces were introduced by Bergman [9] and Dzrbashian [22], they worked with the weight $\omega \equiv 1$. Given a weight ω and $0 < p < \infty$, we define the weighted Bergman space A^p_ω as the space of functions $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|_{A^p_\omega} = \left(\int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) \right)^{\frac{1}{p}} < \infty. \tag{2.7}$$

In particular the Bergman spaces induced by the standard weights $\omega(z) = (1 - |z|)^\alpha$ are denoted by A^p_α.

It is clear that for every radial weight ω we have the inclusion $H^p \subset A^p_\omega$. In addition if $\omega \in \widehat{\mathcal{D}}$ we have the inclusion $A^p_\omega \subset A^p_\alpha$ for all $\alpha > \beta(\omega)$, where $\beta(\omega)$ is given by condition (ii) of Lemma E.

If ω is a radial weight we can see that the dilated functions $f_r(z) = f(rz)$ approximate the function f in A^p_ω, that is, $\lim_{r\to 1^-} \|f - f_r\|_{A^p_\omega} = 0$. This implies that polynomials are dense in A^p_ω whenever the weight is radial.

For a function in $f \in \mathcal{H}(\mathbb{D})$ we define its non-tangential maximal function as

$$N(f)(\zeta) = \sup_{z\in\Gamma(\zeta)} |f(z)|, \quad \zeta \in \mathbb{D}\setminus\{0\}. \tag{2.8}$$

By applying (2.5) to the dilated functions f_r we obtain the following equivalent norm for A^p_ω whenever ω is a radial weight

$$\|f\|_{A^p_\omega} \asymp \|N(f)\|_{L^p_\omega}. \tag{2.9}$$

The proof can be found in [54, Lemma 4.4], along with additional information on the constants of equivalence. Using (2.6) on the dilated functions f_r together with Fubini's theorem we obtain the following equivalent norm in A^p_ω whenever ω is radial

$$\|f\|^p_{A^p_\omega} = p^2 \int_{\mathbb{D}} |f(z)|^{p-2}|f'(z)|^2\omega^\star(z)dA(z) + \omega(\mathbb{D})|f(0)|^p. \tag{2.10}$$

This equivalent norm comes in very handy in the case $p = 2$, since we get an equivalent norm in terms of the derivatives exclusively, the proof of this equivalence can be found in [54, Lemma 4.2]. Another relevant norm in terms of the nth-derivative is the one we can extrapolate from the Littlewood-Paley identity

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} \left(\int_{\Gamma(u)} |f^{(n)}(\zeta)|^2 \left(1 - \left|\frac{\zeta}{u}\right|\right)^{2n-2}\right)^{\frac{p}{2}} \omega(u)dA(u) + \sum_{j=0}^{n-1} |f^{(j)}(0)|^p, \tag{2.11}$$

the proof of this equivalence can be found in [54, Lemma 4.2].

2.3.3 Weighted mixed norm spaces

For $0 < p \leq \infty$, $0 < q < \infty$ and a radial weight ω, the weighted mixed norm space $A^{p,q}_\omega$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^q_{A^{p,q}_\omega} = \int_0^1 M^q_p(r,f)\omega(r)\,dr < \infty.$$

These spaces generalize the weighted Bergman space A^p_ω, since $A^p_\omega = A^{p,q}_\omega$ when $q = p$ and we have the obvious inclusions $A^p_\omega \subset A^{p,q}_\omega$ when $q < p$. The mixed norm spaces were introduced by Benedek and Panzone in [8].

For $0 < p \leq \infty$, $0 < q < \infty$ and a radial weight ω, the space $L^{p,q}_\omega$ consists of measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^q_{L^{p,q}_\omega} = \int_0^1 M^q_p(r,f)\omega(r)\,dr < \infty.$$

2.3.4 Tent spaces

The tent spaces were introduced by Coifman, Meyer and Stein in [17] and later work appeared involving these spaces, such as the work done by Cohn and Verbitsky in [16]. In their context, they work on the upper half-space $\mathbb{R}^{n+1}_{+} = \{(x,t) \in \mathbb{R}^{n+1} : t > 0\}$. We will first introduce the tent spaces on the unit disk, as described by Cohn in [15] and we will follow with the generalization of these spaces that appears in [55]. We will divide the traditional tent spaces in to three groups as follows.

For $0 < p < \infty$, the tent spaces T^p_∞ consist of those (equivalence classes of) measurable functions $u : \mathbb{D} \to \mathbb{C}$ such that

$$\|u\|_{T^p_\infty} = \left(\int_{\mathbb{T}} \left(N(u)(e^{i\theta}) \right)^p d\theta \right)^{\frac{1}{p}} < \infty.$$

Given $0 < p, q < \infty$, the tent spaces T^p_q consist of those (equivalence classes of) measurable functions $u : \mathbb{D} \to \mathbb{C}$ such that

$$\|u\|_{T^p_q} = \left(\int_{\mathbb{T}} \left(\int_{\Gamma(e^{i\theta})} |u(z)|^q \frac{dA(z)}{1-|z|} \right)^{\frac{p}{q}} d\theta \right)^{\frac{1}{q}} < \infty.$$

For $0 < q < \infty$, the tent spaces T^∞_q consist of those (equivalence classes of) measurable functions $u : \mathbb{D} \to \mathbb{C}$ such that

$$\|u\|_{T^\infty_q} = \sup_{I \subset \mathbb{T}} \left(\frac{1}{|I|} \int_{T(I)} |u(z)|^q \frac{dA(z)}{1-|z|} \right)^{\frac{1}{q}} < \infty.$$

For $0 < q < \infty$, a positive Borel measure ν on $\mathbb{D}$ finite on compact sets, and a function $f : \mathbb{D} \to \mathbb{C}$, we denote $A^q_{q,\nu}(f)(\zeta) = \int_{\Gamma(\zeta)} |f(z)|^q \, d\nu(z)$ and $A_{\infty,\nu}(f)(\zeta) = \nu\text{-ess}\sup_{z\in\Gamma(\zeta)} |f(z)|$, for all $\zeta \in \mathbb{D}$. For $0 < p < \infty$, $0 < q \le \infty$ and $\omega \in \widehat{\mathcal{D}}$, the tent space $T^p_q(\nu,\omega)$ consists of the (ν-equivalence classes of) ν-measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{T^p_q(\nu,\omega)} = \|A_{q,\nu}(f)\|_{L^p_\omega} < \infty.$$

For $0 < q < \infty$ we define

$$C^q_{q,\nu}(f)(\zeta) = \sup_{a \in \Gamma(\zeta)} \frac{1}{\omega(T(a))} \int_{T(a)} |f(z)|^q \omega(T(z)) \, d\nu(z), \quad \zeta \in \mathbb{D} \setminus \{0\}.$$

A quasi-norm in the tent space $T^\infty_q(\nu,\omega)$ is defined by $\|f\|_{T^\infty_q(\nu,\omega)} = \|C_{q,\nu}(f)\|_{L^\infty}$.

2.3.5 Schatten classes

Let H be a separable Hilbert space. For any non-negative integer n, the n:th singular value of a bounded operator $T : H \to H$ is defined by

$$\lambda_n(T) = \inf \{ \|T - R\| : \text{rank}(R) \le n \},$$

where $\|\cdot\|$ denotes the operator norm. It is clear that

$$\|T\| = \lambda_0(T) \ge \lambda_1(T) \ge \lambda_2(T) \ge \cdots \ge 0.$$

For $0 < p < \infty$, the Schatten p-class $\mathcal{S}_p(H)$ consists of those compact operators $T : H \to H$ whose sequence of singular values $\{\lambda_n\}_{n=0}^{\infty}$ belongs to the space ℓ^p of p-summable sequences. For $1 \leq p < \infty$, the Schatten p-class $\mathcal{S}_p(H)$ is a Banach space with respect to the norm $|T|_p = \|\{\lambda_n\}_{n=0}^{\infty}\|_{\ell^p}$. Therefore all finite rank operators belong to every $\mathcal{S}_p(H)$, and the membership of an operator in $\mathcal{S}_p(H)$ measures in some sense the size of the operator. We refer to [23] and [69, Chapter 1] for more information about $\mathcal{S}_p(H)$.

2.4 RADEMACHER FUNCTIONS AND KHINCHINE'S INEQUALITY

The Rademacher functions are defined as $\varphi_n(t) = \operatorname{sgn}(\sin(2^n \pi t))$, $0 \leq t \leq 1$, these functions form an orthonormal system over the interval $[0,1]$. In particular the first function is

$$\varphi_1(t) = \begin{cases} 1, & 0 < t < \frac{1}{2}; \\ -1, & \frac{1}{2} < t < 1; \\ 0, & t = 0, \frac{1}{2}, 1. \end{cases}$$

Given a sequence $\{a_n\} \in \ell^2$, we define the function $\Phi(t) = \sum_n a_n \varphi_n(t), \quad 0 \leq t \leq 1$, which is well defined almost everywhere. These functions satisfy the following estimate for all $0 < p < \infty$.

$$\left(\int_0^1 |\Phi(t)|^p \, dt \right)^{\frac{1}{p}} \asymp \left(\sum_n |a_n|^2 \right)^{\frac{1}{2}}. \tag{2.12}$$

This estimate is known as Khinchine's inequality and was proven in [37]. For more information on the topic of Rademacher functions and Khinchine's inequality the reader may check [25, Appendix 1] or [73, Chapter 5.8].

3 Atomic decomposition

3.1 ATOMIC DECOMPOSITION IN STANDARD BERGMAN SPACES

The idea behind an atomic decomposition of a Banach space X is to obtain a representation of every function in the space of the form $f = \sum_n c_n g_n$, where the functions $g_n \in X$ are a fixed sequence which are called atoms. These atoms satisfy certain properties and $\{c_n\}$ is a sequence which belongs to a certain sequence space ℓ. We will also have the norm estimate $\|f\|_X \asymp \|\{c_n\}\|_\ell$. Obtaining an atomic decomposition for a certain Banach space X helps the study of certain properties of operators defined on X, such as boundedness and compactness, among others. In the case of a standard Bergman space the following theorem is known.

Theorem 3.1.1. *[18] Suppose $p > 0$, $\alpha > -1$ and*

$$b > \max\left(1, \frac{1}{p}\right) + \frac{\alpha+1}{p}.$$

Then there exists a constant $\sigma > 0$ such that for any r-lattice $\{a_k\}$ in the Bergman metric, where $0 < r < \sigma$, the function space A^p_α consists exactly of functions of the form

$$f(z) = \sum_{k=1}^{\infty} c_k \frac{(1-|a_k|^2)^{(pb-2-\alpha)/p}}{(1-z\overline{a_k})^b}, \tag{3.1}$$

where $\{c_k\} \in \ell^p$, the series in (3.1) converges in norm A^p_α, and $\|f\|_{A^p_\alpha}$ is comparable to

$$\inf\left\{\|c\|_{\ell^p} : c = \{c_k\} \text{ satisfies } (3.1)\right\}.$$

The proof of this result was given by Coifman and Rochberg in [18] by extending the proof of certain results given by Amar [6].

As an extension of Theorem 3.1.1, Ricci and Taibleson [63] obtained an atomic decomposition for the mixed norm spaces $A^{p,q}$, but first in order to state the result we shall introduce the space of doubled indexed sequences. For $0 < p, q < \infty$ we define the set $\ell^{p,q}$ of doubled indexed sequences $\lambda_{j,k}$ such that $\|\lambda\|_{\ell^{p,q}} = \left(\sum_j \left(\sum_k |\lambda_{j,k}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} < \infty$, and in a similar fashion we define the sets $\ell^{\infty,q}$, $\ell^{p,\infty}$ and $\ell^{\infty,\infty}$ by using $\sup |\cdot|$. We have the following inclusions between these spaces

$$\begin{aligned} \ell^{p,q} &\subset \ell^{r,q}, \quad 0 < p \le r \le \infty, \\ \ell^{p,q} &\subset \ell^{p,s}, \quad 0 < q \le s \le \infty. \end{aligned} \tag{3.2}$$

We will work with these spaces of sequences with one of the indexes finite. The following theorem from Nakamura [46, Theorem 1] describes the duals of these spaces.

Theorem 3.1.2. *[46, Theorem 1] For any $0 < p \leq \infty$, we define $p' = \frac{p}{p-1}$ if $1 < p < \infty$, $p' = 1$ if $p = \infty$ and $p' = \infty$ if $0 < p \leq 1$. Then*

$$\sup \left\{ \left| \sum_{i,j} c_{i,j} b_{i,j} \right| : \|c\|_{p,q} = 1 \right\} = \|b\|_{p',q'} .$$

The next theorem was originally proven in [63, Theorem 1.5] in the context of $\mathbb{R}^2_+$. Given a sequence $\{z_k\}$ we re-index it in the following way for each $j \in \mathbb{N} \cup \{0\}$ we set $\{z_{i,j}\}$ as those $\{z_k\}$ such that $2^{-j} \leq 1 - |z_{i,j}| < 2^{-j+1}$, $i = 0, \dots, 2^j - 1$.

Theorem 3.1.3. *Define the operator S on double indexed sequences by*

$$S(\{a_{i,j}\})(z) = \sum_{n} a_{i,j} \frac{(1 - |z_{i,j}|^2)^{M - \frac{1}{p} - \frac{1}{q}}}{(1 - \overline{z_{i,j}} z)^M},$$

where $M > \max\left\{1, \frac{1}{q}\right\} + \frac{1}{p}$. Then S is bounded from $\ell^{p,q}$ to $A^{p,q}$ whenever $\{z_{i,j}\}$ is separated.

Some results on atomic decomposition in weighted Bergman spaces A^p_ω were given by Constantin [19] when the weight ω belongs to the class of Békollé-Bonami weights.

4 Operators on spaces of analytic functions

4.1 CARLESON MEASURES

Given a space X of analytic functions in $\mathbb{D}$ and a positive Borel measure μ on $\mathbb{D}$, we say μ is a q-Carleson measure for X if the identity operator $Id : X \to L^q_\mu(\mathbb{D})$ is bounded.

Before we dive into Carleson measures for weighted Bergman spaces with $\omega \in \widehat{\mathcal{D}}$, we will state the results for Hardy and standard Bergman spaces. Recall that there exists $\alpha = \alpha(\omega, p)$ such that $H^p \subset A^p_\omega \subset A^p_\alpha$. These inclusions tell us that the conditions that characterize Carleson measures for Hardy spaces will be necessary conditions for μ to be a Carleson measure for A^p_ω. In an analogous manner the conditions that characterize Carleson measures for standard Bergman spaces A^p_α will be sufficient conditions (for α big enough) for μ to be a Carleson measure for A^p_ω.

4.1.1 Carleson measures for Hardy spaces

The characterization of q-Carleson measures in Hardy spaces is divided into two main cases. The case $q = p$ was proven by Carleson in [11], and later on was extended by Duren [24] to the case $p \leq q$ by using test functions, a covering Lemma and the maximal operator given by $M(\phi)(z) = \sup_{I: I_z \subset I} \frac{1}{|I|} \int_I |\phi(\zeta)| d\zeta$.

Theorem 4.1.1. *[24] Let $0 < p \leq q < \infty$, and μ a positive Borel measure on $\mathbb{D}$. Then μ is a q-Carleson measure for H^p if and only if*

$$\sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|^{\frac{q}{p}}} < \infty.$$

For the case where $0 < q < p < \infty$ Luecking [40] uses Khinchine's inequality (2.12), subharmonicity, and the theory of tent spaces T^p_q related to Hardy spaces to obtain the following result.

Theorem 4.1.2. *[40] Let $0 < q < p < \infty$, and μ a positive Borel measure on $\mathbb{D}$. Then μ is a q-Carleson measure for H^p if and only if the function*

$$A_\mu(e^{i\theta}) = \int_{\Gamma(e^{i\theta})} \frac{d\mu(\zeta)}{1 - |\zeta|^2}, \quad e^{i\theta} \in \mathbb{T},$$

belongs to $L^{\frac{p}{p-q}}(\mathbb{T})$.

4.1.2 Carleson measures for standard Bergman spaces

The first results that characterize Carleson measures on standard Bergman spaces were given by Hastings [32] offering a description for $1 \leq p \leq q < \infty$, and Oleinik and Pavlov [47] giving a characterization for $0 < p \leq q < \infty$.

Theorem 4.1.3. *[47] Let $0 < p \leq q < \infty$, and μ a positive Borel measure on $\mathbb{D}$. Then the following assertions are equivalent:*

(*i*) *μ is a q-Carleson measure for A^p_α;*

(*ii*)
$$\sup_{a \in \mathbb{D}} \frac{\mu\left(S(a)\right)}{\left(1 - |a|^2\right)^{(2+\alpha)\frac{q}{p}}} < \infty;$$

(*iii*) *For every $0 < r < 1$*
$$\sup_{a \in \mathbb{D}} \frac{\mu\left(\Delta(a,r)\right)}{\left(1 - |a|^2\right)^{(2+\alpha)\frac{q}{p}}} < \infty.$$

The condition $\sup_{a \in \mathbb{D}} \frac{\mu(\Delta(a,r))}{(1-|a|^2)^{(2+\alpha)\frac{q}{p}}} < \infty$ is a necessary condition for μ to be a q-Carleson measure for A^p_α for any $0 < p, q < \infty$. To prove this it suffices to estimate the $L^q_\mu(\mathbb{D})$ norm of the normalized reproducing kernels. However it is not a sufficient condition when $0 < q < p < \infty$. In this case, a characterization of Carleson measures for standard Bergman spaces was proven by Luecking [41], using Rademacher functions and Khinchine's inequality (2.12), together with the atomic decomposition of standard Bergman spaces given by Theorem 3.1.1 and the properties of $\delta-$lattices.

Theorem 4.1.4. *[41] Let $0 < q < p < \infty$, and μ a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(*i*) *μ is a q-Carleson measure for A^p_α;*

(*ii*) *For every $0 < r < 1$ the function $\widehat{\mu_r}(z) = \frac{\mu(\Delta(z,r))}{(1-|z|)^{2+\alpha}}$ belongs to $L_\alpha^{\frac{p}{p-q}}(\mathbb{D})$;*

(*iii*) *The function $\widetilde{\mu}(z) = \frac{\mu(S(z))}{(1-|z|)^{2+\alpha}}$ belongs to $L_\alpha^{\frac{p}{p-q}}(\mathbb{D})$.*

Since these theorems work for the standard weights $\omega(z) = (1 - |z|^2)^\alpha$, and $\omega(S(z)) \asymp \omega(\Delta(z,r)) \asymp (1 - |z|^2)^{2+\alpha}$, it is reasonable to try to find a characterization of Carleson measure for a more general ω, by replacing $(1 - |z|)^{2+\alpha}$ with either $\omega(\Delta(z,r))$ or $\omega(S(z))$ in the previous theorems.

4.1.3 Carleson measures for weighted Bergman spaces

In this section we discuss Carleson measures on weighted Bergman spaces A^p_ω with $\omega \in \widehat{\mathcal{D}}$. In order to study the Carleson measures in A^p_ω we need to define the following Hörmander-type weighted maximal function. Given a positive Borel measure μ on $\mathbb{D}$ and $\alpha > 0$, we define the weighted maximal function

$$M_{\omega,\alpha}(\mu)(z) = \sup_{z \in S(a)} \frac{\mu(S(a))}{\left(\omega\left(S(a)\right)\right)^\alpha}, \quad z \in \mathbb{D}. \tag{4.1}$$

In the case $\alpha = 1$, we omit it from the notation and write $M_{\omega,1}(\mu) = M_\omega(\mu)$. If ϕ is a non-negative function we write $M_{\omega,\alpha}(\phi)$ as $M_{\omega,\alpha}$ acting over the measure $\phi \omega dA$.

Theorem 4.1.5. *[51] Let* $0 < p \leq q < \infty$ *and* $0 < \gamma < \infty$ *such that* $p\gamma > 1$. *Let* $\omega \in \widehat{\mathcal{D}}$ *and* μ *be a positive Borel measure on* $\mathbb{D}$. *Then* $[M_\omega((\cdot)^{\frac{1}{\gamma}})]^\gamma : L^p_\omega \to L^q_\mu(\mathbb{D})$ *is bounded if and only if* $M_{\omega,q/p}(\mu) \in L^\infty$. *Moreover,*

$$\|[M_\omega((\cdot)^{\frac{1}{\gamma}})]^\gamma\|^q_{\left(L^p_\omega, L^q(\mu)\right)} \asymp \|M_{\omega,q/p}(\mu)\|_{L^\infty}.$$

The boundedness of this operator plays a big role in characterizing the Carleson measures for the weighted Bergman space A^p_ω.

Theorem 4.1.6. *[55] Let* $0 < p, q < \infty$, $\omega \in \widehat{\mathcal{D}}$ *and* μ *be a positive Borel measure on* $\mathbb{D}$.

(a) If $p > q$, *the following conditions are equivalent:*

(i) μ *is a q-Carleson measure for* A^p_ω;

(ii) The function

$$B_\mu(z) = \int_{\Gamma(z)} \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D} \setminus \{0\},$$

belongs to $L_\omega^{\frac{p}{p-q}}(\mathbb{D})$;

(iii) $M_\omega(\mu) \in L_\omega^{\frac{p}{p-q}}(\mathbb{D})$.

(b) μ *is a p-Carleson measure for* A^p_ω *if and only if* $M_\omega(\mu) \in L^\infty(\mathbb{D})$.

(c) If $q > p$, *the following conditions are equivalent:*

(i) μ *is a q-Carleson measure for* A^p_ω;

(ii) $M_{\omega,q/p}(\mu) \in L^\infty(\mathbb{D})$;

(iii) $z \mapsto \dfrac{\mu\left(\Delta(z,r)\right)}{\left(\omega\left(S(z)\right)\right)^{\frac{q}{p}}}$ *belongs to* $L^\infty(\mathbb{D})$ *for any fixed* $r \in (0,1)$.

There are many results about Carleson measures on A^p_ω for other classes of weights. We will state some of them such as the characterization of Carleson measures on weighted Bergman spaces induced by rapidly decreasing weights $\mathcal{W}$ defined in [48], which include the family of weights

$$\omega_\alpha(r) = \exp\left(\frac{-1}{(1-r)^\alpha}\right), \quad \alpha > 0. \tag{4.2}$$

Theorem 4.1.7. *[48, Theorem 1] Let* $\omega \in \mathcal{W}$ *and let* μ *be a finite positive Borel measure on* $\mathbb{D}$.

(i) Let $0 < p \leq q < \infty$. *Then* μ *is a q-Carleson measure for* A^p_ω *if and only if for each sufficiently small* $\delta > 0$ *we have*

$$\sup_{a\in\mathbb{D}} \frac{1}{\tau(a)^{\frac{2q}{p}}} \int_{D(a,\delta\tau(a))} \omega(z)^{\frac{-q}{p}} d\mu(z) < \infty.$$

(ii) Let $0 < q < p < \infty$. Then μ is a q-Carleson measure for A^p_ω if and only if for each sufficiently small $\delta > 0$ the function

$$z \to \frac{1}{\tau(z)^{\frac{2q}{p}}} \int_{D(z,\delta\tau(z))} \omega(\zeta)^{\frac{-q}{p}} d\mu(\zeta) \in L^{\frac{p}{p-q}}(\mathbb{D}).$$

Here $\tau : \mathbb{D} \to (0,1)$ is a radial function which depends on ω and has the property that it decreases to 0 as $|z| \to 1^-$. For more information on the class $\mathcal{W}$ and the function τ we refer to [48].

There is literature on the characterization of Carleson measures for A^p_ω when ω is a non-radial weight, like the results given by O.Constantin [21] for the Békolle weight class. A weight ω is said to belong to the Békolle class $B_{p_0}(\eta)$, where $p_0 > 1$ and $\eta > -1$ if there exists $k > 0$ such that

$$\int_{S(a)} \omega(z) dA_\eta(z) \left(\int_{S(a)} \omega(z)^{-\frac{p_0'}{p_0}} dA_\eta(z) \right)^{\frac{p_0}{p_0'}} \leq k \left(\int_{S(a)} dA_\eta(z) \right)^{p_0}, \quad a \in \mathbb{D}, \tag{4.3}$$

where $dA_\eta(z) = (\eta+1)(1-|z|)^\eta dA(z)$.

Theorem 4.1.8. *[21, Theorem 3.1] Let ω be a weight such that $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to $B_{p_0}(\eta)$ for some $p_0 > 0$, $\eta > -1$. Consider a positive finite Borel measure μ on $\mathbb{D}$ and assume $q \geq p > 0$, $n \in \mathbb{N}$. Then there exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{D}} |f^{(n)}(z)|^q d\mu(z) \right)^{\frac{1}{q}} \leq C \|f\|_{A^p_\omega} \tag{4.4}$$

if and only if μ satisfies

$$\mu(D(a, \alpha(1-|a|))) \leq C'(1-|a|^2)^{nq} \left(\int_{D(a,\alpha(1-|a|))} \omega(z) dA(z) \right)^{\frac{q}{p}} \tag{4.5}$$

for some constant $C' > 0$ independent of a, and for some $\alpha \in (0,1)$.

Theorem 4.1.9. *[21, Theorem 3.2] Let ω be a weight such that $\frac{\omega(z)}{(1-|z|)^\eta}$ belongs to $B_{p_0}(\eta)$ for some $p_0 > 0$, $\eta > -1$. Consider a positive finite Borel measure μ on $\mathbb{D}$ and assume $p > q > 0$, $n \in \mathbb{N}$. Then there exists a constant $C > 0$ such that*

$$\left(\int_{\mathbb{D}} |f^{(n)}(z)|^q d\mu(z) \right)^{\frac{1}{q}} \leq C \|f\|_{A^p_\omega} \tag{4.6}$$

if and only if the function

$$a \to \frac{\mu(D(a, \alpha(1-|a|)))}{(1-|a|^2)^{nq} \int_{D(a,\alpha(1-|a|))} \omega(z) dA(z)} \tag{4.7}$$

belongs to $L_\omega^{\frac{p}{p-q}}$, for some $\alpha \in (0,1)$.

4.1.4 Carleson measures for mixed norm spaces

Jevtić gives a characterization of Carleson measures for weighted mixed norm spaces in the half-plane $\mathbb{R}^2_+$. We reindex the the family $\{R(I) : I \in \Upsilon\}$ as $\{R_{i,j}\}$, $j \in \mathbb{N} \cup \{0\}$, $i = 0, \ldots, 2^j - 1$.

Theorem 4.1.10. *[35, Theorem 3.3] Let μ be a positive Borel measure on $\mathbb{D}$, $0 < s < p < \infty$ and $0 < t < q < \infty$. Then there exists $C > 0$ such that*

$$\left(\sum_j \left(\sum_i \int_{R_{i,j}} |f(z)|^s d\mu(z) \right)^{\frac{t}{s}} \right)^{\frac{1}{t}} \leq C \|f\|_{A^{p,q}}, \tag{4.8}$$

if and only if

$$\sum_j \left(\sum_i \mu(R_{i,j})^{\frac{1}{s}} 2^{ju\left(\frac{1}{p}+\frac{1}{q}\right)} \right)^{\frac{v}{u}} < \infty, \tag{4.9}$$

where $\frac{1}{u} = \frac{1}{s} - \frac{1}{p}$ and $\frac{1}{v} = \frac{1}{t} - \frac{1}{q}$.

Note that in the case where $s = t$ and $p = q$, Jevtić gives a characterization of the s-Carleson measures for A^p. Luecking extended the result given by Jevtić by eliminating the restriction over the coefficients and by adding a continuous characterization.

Theorem 4.1.11. *[41, Theorem 2] Let $0 < p, q, s < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) There exists $C > 0$ such that $\|f^{(n)}\|_{L^s_\mu} \leq C\|f\|_{A^{p,q}}$ for all $f \in A^{p,q}$;

(ii) The sequence $\left\{ \mu(R_{i,j}) 2^{sj(n+\frac{1}{p}+\frac{1}{q})} \right\}$ belongs to $\ell^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}$;

(iii) For all $0 < r < 1$ the function k_r belongs to $L^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}$, where

$$k_r(z) = \begin{cases} \frac{\mu(\Delta(z,r))}{(1-|z|)^{2+sn}}, & s < \min\{p,q\}; \\ \frac{\mu(\Delta(z,r))}{(1-|z|)^{1+\frac{s}{p}+sn}}, & p \leq s < q; \\ \frac{\mu(\Delta(z,r))}{(1-|z|)^{1+\frac{s}{q}+sn}}, & q \leq s < p; \\ \frac{\mu(\Delta(z,r))}{(1-|z|)^{\frac{s}{p}+\frac{s}{q}+sn}}, & \max\{p,q\} \leq s. \end{cases}$$

In the proof of this theorem, Luecking uses Theorem 3.1.3 given by Ricci and Taibleson [63], Rademacher functions and Khinchine's inequality (2.12), and an equivalent norm for the functions in $A^{p,q}$.

4.2 AREA OPERATORS

4.2.1 Area operators in Hardy spaces

As Ahern and Bruna showed in [1], a function $f \in H^p$ if and only if for a given $\alpha > 0$

$$\int_{\mathbb{T}}\left(\int_{\Gamma(e^{i\theta})}(1-r)^{2\alpha}\,|D^{\alpha}(z)|^2\,\frac{dA(z)}{(1-|z|)^2}\right)^{\frac{p}{2}}d\theta<\infty, \tag{4.10}$$

where $D^{\alpha}f$ is the radial fractional derivative of order α. In the paper [15] Cohn extracted the following area operator

$$\mathcal{A}_{\mu}(f)(e^{i\theta})=\int_{\Gamma(e^{i\theta})}|f(z)|\frac{d\mu(z)}{1-|z|}, \tag{4.11}$$

and characterized its boundedness from H^p to $L^p(\mathbb{T})$ in terms of the measure μ.

Theorem 4.2.1. *[15, Theorem 1] Let $0<p<\infty$ and μ a positive Borel measure on $\mathbb{D}$. Then $\mathcal{A}_{\mu}$ is bounded from H^p to $L^p(\mathbb{T})$ if and only if μ is a p-Carleson measure for H^p.*

In a later work Gong, Lou, and Wu characterized in [31] the boundedness of $\mathcal{A}_{\mu}$ from H^p to $L^q(\mathbb{T})$ with the following results.

Theorem 4.2.2. *[31, Theorem 3.1] Let $0<p\leq q<\infty$ and μ a non-negative measure on $\mathbb{D}$. Then $\mathcal{A}_{\mu}$ is bounded from H^p to $L^q(\mathbb{T})$ if and only if μ is a $\left(1+\frac{1}{p}-\frac{1}{q}\right)$-Carleson measure for H^1.*

In order to prove this result, they use the equivalent norm given in (2.5), the test functions $f_a(z)=\frac{(1-|a|)^m}{(1-\bar{a}z)^{m+\frac{1}{p}}}$, the boundedness of the Riesz projection from $L^q(\mathbb{T})$ to H^q when $q>1$ and Calderon-Zygmund decompositions among other techniques.

Theorem 4.2.3. *[31, Theorem 3.2] Let $1\leq q<p\leq\infty$ and μ a non-negative measure on $\mathbb{D}$. Then $\mathcal{A}_{\mu}$ is bounded from H^p to $L^q(\mathbb{T})$ if and only if $\widehat{\mu}(\zeta)=\int_{\Gamma(\zeta)}\frac{d\mu(z)}{1-|z|}$ for $\zeta\in\mathbb{T}$ belongs to $L^{\frac{pq}{p-q}}(\mathbb{T})$.*

For this result they relied again on the equivalent norm (2.5), the characterization of Carleson measures in Hardy spaces described in Theorem 4.1.2 and some estimates involving the non-tangential maximal operator N.

4.2.2 Area operators in standard Bergman spaces

Given a non-negative Borel measure μ on $\mathbb{D}$ and $\tau>0$, the area operator $\mathcal{A}^{\tau}_{\mu}$ on the Bergman space A^p_{α} is defined as

$$\mathcal{A}^{\tau}_{\mu}(\zeta)=\int_{\Gamma_{\tau}(\zeta)}|f(z)|\frac{d\mu(z)}{1-|z|},\quad \zeta\in\mathbb{T},$$

where the non-tangential regions $\Gamma_{\tau}(\zeta)$ as defined by Wu in [65] are

$$\Gamma_{\tau}(\zeta)=\{z\in\mathbb{D}:|z-\zeta|<(1+\tau)(1-|z|)\},\quad \zeta\in\mathbb{T}.$$

He characterized the boundedness of the area operator $\mathcal{A}^{\tau}_{\mu}$ in terms of the growth of the α-density function $\rho_{\alpha}(\mu)(z,t)=\frac{\mu(\Lambda(z,t))}{|\Lambda(z,t)|_{\alpha}}$, where $|\Lambda(z,t)|_{\alpha}=\int_{\Lambda(z,t)|}(1-|w|^2)^{\alpha}dA(w)$.

Theorem 4.2.4. *[65, Theorem 1] Suppose $\alpha > -1$, $0 < p \leq q < \infty$ and $0 < p \leq 1$. For a non-negative Borel measure μ on $\mathbb{D}$ the following statements are equivalent:*

(i) *$\mathcal{A}^1_\mu$ is bounded from A^p_α to $L^q(\mathbb{T})$;*

(ii) *There exists $\tau > 0$ such that $\mathcal{A}^\tau_\mu$ is bounded from A^p_α to $L^q(\mathbb{T})$;*

(iii) *There exists $\delta > 0$ such that the α-density function $\rho_\alpha(\mu)$ satisfies*

$$\rho_\alpha(\mu)(z,\delta) \lesssim (1-|z|)^{(\alpha+2)(\frac{1}{p}-1)+1-\frac{1}{q}}, \quad z \in \mathbb{D};$$

(iv) *For any fixed $t > 0$ the α-density function $\rho_\alpha(\mu)$ satisfies*

$$\rho_\alpha(\mu)(z,t) \lesssim (1-|z|)^{(\alpha+2)(\frac{1}{p}-1)+1-\frac{1}{q}}, \quad z \in \mathbb{D}.$$

In this proof, Wu groups the conditions (i) with (ii) and (iii) with (iv). In one direction he used the test functions $f_a(z) = \frac{(1-|a|^m)}{(1-\bar{a}z)^{m+\frac{\alpha+2}{p}}}$, which are uniformly bounded in the A^p_α norm and essentially constant in $\Delta(a,r)$, and the boundedness of $\mathcal{A}^\tau_\mu$ from A^p_α to $L^q(\mathbb{T})$, to obtain the growth estimate on $\rho_\alpha(\mu)$. For the reverse implication he uses a suitable δ-lattice on $\mathbb{D}$, the subharmonicity of the functions in A^p_α, together with some geometric arguments and the conditions on p and q.

Theorem 4.2.5. *[65, Theorem 2] Suppose $\alpha > -1$ and $1 < p \leq q < \infty$. For a non-negative Borel measure μ on $\mathbb{D}$ the following statements are equivalent:*

(i) *$\mathcal{A}^1_\mu$ is bounded from A^p_α to $L^q(\mathbb{T})$;*

(ii) *There exists $\tau > 0$ such that $\mathcal{A}^\tau_\mu$ is bounded from A^p_α to $L^q(\mathbb{T})$;*

(iii) *There exists $\delta > 0$ such that the α-density function $\rho_\alpha(\mu)(z,\delta)$ satisfies that the measure*

$$\rho_\alpha(\mu)(z,\delta)^{\frac{p}{p-1}} dA_\alpha(z)$$

is a $\frac{p(q-1)}{q(p-1)}$-Carleson measure for H^1;

(iv) *For any fixed $t > 0$ the α-density function $\rho_\alpha(\mu)(z,t)$ satisfies that the measure*

$$\rho_\alpha(\mu)(z,t)^{\frac{p}{p-1}} dA_\alpha(z)$$

is a $\frac{p(q-1)}{q(p-1)}$-Carleson measure for H^1.

Here, in order to prove this theorem, Wu uses the duality of $L^q(\mathbb{T})$ and ℓ^q when $q > 1$, a characterization of Carleson measures for Hardy spaces, together with Rademacher functions and Khinchine's inequality (2.12), δ-lattices and their properties and the test functions given by an atomic decomposition like the one given in (3.1), among other tools.

Theorem 4.2.6. *[65, Theorem 3] Suppose $\alpha > -1$ and $1 \leq q < p < \infty$. For a non-negative Borel measure μ on $\mathbb{D}$ the following are equivalent:*

(i) $\mathcal{A}_\mu^1$ *is bounded from* A_α^p *to* $L^q(\mathbb{T})$;

(ii) *There exists* $\tau > 0$ *such that* $\mathcal{A}_\mu^\tau$ *is bounded from* A_α^p *to* $L^q(\mathbb{T})$;

(iii) *There exists* $\delta, \tau > 0$ *such that the* α*-density function* $\rho_\alpha(\mu)(z,\delta)$ *satisfies the property*

$$\int_{\Gamma_\tau(\zeta)} (\rho_\alpha(\mu)(z,\delta))^{\frac{p}{p-1}} \frac{dA_\alpha(z)}{1-|z|} \in L^{\frac{q(p-1)}{p-q}}(\mathbb{T});$$

(iv) *For any fixed* $t, \tau > 0$ *the* α*-density function* $\rho_\alpha(\mu)(z,t)$ *satisfies the property*

$$\int_{\Gamma_\tau(\zeta)} (\rho_\alpha(\mu)(z,t))^{\frac{p}{p-1}} \frac{dA_\alpha(z)}{1-|z|} \in L^{\frac{q(p-1)}{p-q}}(\mathbb{T}).$$

To prove this Theorem Wu uses the duality of L^q, Khinchine's inequality (2.12), the properties of δ-lattices and the Poisson integral.

Theorem 4.2.7. *[65, Theorem 4] Suppose* $\alpha > -1$, $q < p < \infty$, $0 < q < 1$, μ *a non-negative Borel measure on* $\mathbb{D}$. *Then* $\mathcal{A}_\mu$ *is bounded from* A_α^p *to* $L^q(\mathbb{T})$ *if and only if for any fixed* $\tau > 0$, δ*-lattice* $\{z_j\}$ *in* $\mathbb{D}$, *any* $\{a_j\} \in \ell^p$, *the* α*-density function* $\rho_\alpha(\mu)$ *satisfies*

$$\int_{\mathbb{T}} \left(\sum_{j: z_j \in \Gamma_\tau(\zeta)} |a_j| \frac{\rho_\alpha(\mu)(z_j,\delta)}{(1-|z_j|)^{(\alpha+2)(\frac{1}{p}-1)+1}} \right)^q |d\zeta| \lesssim \|\{a_j\}\|_{\ell^p}^q.$$

In order to prove the Theorem Wu uses the test functions $F_t(z) = \sum_j a_j r_j(t) \frac{(1-|z_j|^m)}{(1-\overline{z_j}z)^{m+\frac{\alpha+2}{p}}}$, which satisfy the estimate $\|F_t\|_{A_\alpha^p} \lesssim \|\{a_j\}\|_{\ell^p}$, together with a generalization of Khinchine's inequality given by Kalton [36] and the properties of δ-lattices.

4.3 PROJECTIONS AND REPRODUCING KERNELS

4.3.1 Projections on standard Bergman spaces

For each $a \in \mathbb{D}$, we define the point evaluation functionals

$$\begin{aligned} L_a : A_\alpha^p &\to \mathbb{C}, \\ f &\mapsto f(a), \end{aligned}$$

which is bounded for each $a \in \mathbb{D}$. Focusing on the case $p = 2$, A_α^2 is a Hilbert space with the scalar product

$$\langle f, g \rangle_{A_\alpha^2} = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z).$$

Since L_a is a bounded and linear functional, due to Riesz representation theorem [64, Theorem 4.12, pag.80], there exists $K_a^\alpha \in A_\alpha^2$ such that

$$f(a) = L_a(f) = \langle f, K_a^\alpha \rangle_{A_\alpha^2} = \int_{\mathbb{D}} f(z)\overline{K_a^\alpha(z)} dA_\alpha(z), \quad f \in A_\alpha^2, \quad a \in \mathbb{D}. \tag{4.12}$$

The family of analytic functions $\{K_a^\alpha\}_{a \in \mathbb{D}}$ is called the reproducing kernel of A_α^2. The reproducing kernel K_a^α has an explicit formula, which can be obtained as follows.

Theorem 4.3.1. *[69, Theorem 4.19] Let $\{e_n\}_{n=0}^{\infty}$ an ortonormal basis of A^2_α, then*

$$K_z^\alpha(\xi) = \sum_{n=0}^{\infty} e_n(\xi)\overline{e_n(z)}, \quad z, \xi \in \mathbb{D}. \tag{4.13}$$

Next, we take a a concrete basis of A^2_α. For each $n \in \mathbb{N} \cup \{0\}$, the canonical basis is given by

$$e_n(z) = \frac{z^n}{\left(2(\alpha+1)\int_0^1 s^{2n+1}(1-s^2)^\alpha\, ds\right)^{\frac{1}{2}}}, \quad z \in \mathbb{D}.$$

So, by Theorem 4.3.1,

$$K_z^\alpha(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n \overline{z}^n}{2(\alpha+1)\int_0^1 r^{2n+1}(1-r^2)^\alpha dr}, \quad z, \xi \in \mathbb{D}. \tag{4.14}$$

Then by using some properties of the Gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt, \quad x > 0,$$

we can rewrite the reproducing kernel K_z^α in the following way.

Corollary 4.3.2. *[69, Corollary 4.20] For $\alpha > -1$ the reproducing kernel of A^2_α is given by*

$$K_z^\alpha(\xi) = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+2)}{\Gamma(\alpha+2)\Gamma(n+1)}(\xi\overline{z})^n = \frac{1}{(1-\overline{z}\xi)^{\alpha+2}}, \quad z, \xi \in \mathbb{D}.$$

This together with (4.12) allows us to write the following reproducing formula.

Theorem 4.3.3. *[69, Proposition 4.23] Let $\alpha > -1$, then*

$$f(a) = \int_{\mathbb{D}} \frac{f(z)}{(1-\overline{z}a)^{2+\alpha}} dA_\alpha(z), \quad f \in A^1_\alpha, a \in \mathbb{D}.$$

Since A^2_α is a closed subspace of $L^2_\alpha(\mathbb{D})$, there exists an orthogonal projection from $L^2_\alpha(\mathbb{D})$ to A^2_α, which we shall denote by P_α. Since P_α is an orthogonal projection it satisfies $P^2 = P$, and it is self-adjoint

$$\langle P(f), g\rangle_{L^2_\alpha(\mathbb{D})} = \langle f, P(g)\rangle_{L^2_\alpha(\mathbb{D})}.$$

If $\phi \in L^2_\alpha(\mathbb{D})$ then $P_\alpha(\phi) \in A^2_\alpha$, and the following formula holds

$$\begin{aligned} P_\alpha(\phi)(z) &= \langle P_\alpha(\phi), K_z^\alpha\rangle_{L^2_\alpha(\mathbb{D})} = \langle \phi, P_\alpha(K_z^\alpha)\rangle_{L^2_\alpha(\mathbb{D})} \\ &= \langle \phi, K_z^\alpha\rangle_{L^2_\alpha(\mathbb{D})} = \int_{\mathbb{D}} \phi(\xi)\overline{K_z^\alpha(\xi)} dA_\alpha(\xi) \\ &= \int_{\mathbb{D}} \frac{\phi(\xi)}{(1-\overline{\xi}z)^{\alpha+2}} dA_\alpha(\xi). \end{aligned}$$

If $\phi \in L^p_\alpha(\mathbb{D})$ for $p \geq 2$ the previous formula is also true since $L^p_\alpha(\mathbb{D}) \subset L^2_\alpha(\mathbb{D})$. The projection P_α also makes sense for $\phi \in L^1_\alpha(\mathbb{D})$ since for $z \in \mathbb{D}$ we have

$$\left|\int_{\mathbb{D}} \frac{\phi(\xi)}{(1-\bar{\xi}z)^{\alpha+2}} dA_\alpha(\xi)\right| \leq \frac{1}{(1-|z|)^{2+\alpha}} \int_{\mathbb{D}} |\phi(\xi)|\, dA_\alpha(\xi) = \frac{\|\phi\|_{L^1_\alpha}}{(1-|z|)^{2+\alpha}}, \quad z \in \mathbb{D}.$$

Next we are going to characterize the boundedness of the projection P_γ on the corresponding $L^p_\alpha(\mathbb{D})$ spaces, alongside with analogous result for the maximal projection $P^+_\gamma(\phi)(z) = \int_{\mathbb{D}} \frac{|\phi(\xi)|}{|1-\bar{\xi}z|^{2+\gamma}} dA_\gamma(\xi)$, which is a sublinear operator. Forelli and Rudin proved in [28] the case $\alpha = 0$.

Theorem 4.3.4. *[69, Theorem 4.24] Let $\gamma, \alpha > -1$ and $1 \leq p < \infty$. Then the following statements are equivalent:*

(a) $P_\gamma : L^p_\alpha(\mathbb{D}) \to A^p_\alpha$ *is bounded;*

(b) $P^+_\gamma : L^p_\alpha(\mathbb{D}) \to L^p_\alpha(\mathbb{D})$ *is bounded;*

(c) $p(\gamma+1) > \alpha + 1$.

Now that we have studied the boundedness of the Bergman projection, we will use this to identify $\left(A^p_\alpha\right)^*$, the dual of A^p_α, with $A^{p'}_\alpha$, for $\alpha > -1$ and $1 < p < \infty$.

Theorem 4.3.5. *[69, Theorem 4.25] Let $1 < p < \infty$, and $\alpha > -1$. Then the following statements are true:*

(i) Every $g \in A^{p'}_\alpha$ defines a functional $T_g \in \left(A^p_\alpha\right)^*$ as follows

$$T_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z),$$

with $\|T_g\| \leq \|g\|_{A^{p'}_\alpha}$.

(ii) For every $T \in \left(A^p_\alpha\right)^*$, there exists $g \in A^{p'}_\alpha$ such that

$$T(f) = T_g(f) = \int_{\mathbb{D}} f(z)\overline{g(z)} dA_\alpha(z), \tag{4.15}$$

with $\|g\|_{A^{p'}_\alpha} \leq C\|T\|$, where $C = C(p)$ is a constant.

For more information on the boundedness of Bergman projection P_α the reader may check [26,57,69].

4.3.2 Toeplitz operator and Berezin transform in standard Bergman spaces

Some of the early results with respect to Toeplitz operators were introduced by McDonald and Sundberb [44] and Coburn [14]. Given $\beta > -1$ and a positive Borel measure μ on $\mathbb{D}$, define the Toeplitz operator T^β_μ as follows:

$$T^\beta_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^{2+\beta}}\, d\mu(w) = \int_{\mathbb{D}} f(w) K^\beta_z(w)\, d\mu(w), \quad z \in \mathbb{D}.$$

Pau and Zhao [49] characterized the boundedness of the Toeplitz operator in $\mathbb{C}^n$ from $A^{p_1}_{\alpha_1}$ to $A^{p_2}_{\alpha_2}$. The following result is mostly due to Luecking ([39] and [41]) for the case $\alpha = 0$.

Theorem 4.3.6. *[49, Theorem B] Let $\alpha > -1$, $0 < q < p < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) μ is a $q-$Carleson measure for A^p_α;

(ii) The function

$$\widehat{\mu}_r(z) = \frac{\mu(\Delta(z,r))}{(1-|z|^2)^{(2+\alpha)}}, \quad z \in \mathbb{D},$$

is in $L^{p/(p-q)}_\alpha(\mathbb{D})$ for any (some) fixed $r \in (0,1)$;

(iii) For any r-lattice $\{a_k\}$ and $D_k = \Delta(a_k, r)$, the sequence

$$\{\mu_k\} = \left\{ \frac{\mu(D_k)}{(1-|a_k|^2)^{(2+\alpha)\frac{q}{p}}} \right\}$$

belongs to $\ell^{p/(p-q)}$ for any (some) fixed $r \in (0,1)$;

(iv) For any $s > 0$, the Berezin-type transform $B_{s,\alpha}(\mu)$ belongs to $L^{\frac{p}{p-q}}_\alpha(\mathbb{D})$.

Furthermore, with $\lambda = q/p$, one has

$$\|\widehat{\mu}_r\|_{L^{\frac{p}{p-q}}_\alpha(\mathbb{D})} \asymp \|\{\mu_k\}\|_{\ell^{p/(p-q)}} \asymp \|B_{s,\alpha}(\mu)\|_{L^{\frac{p}{p-q}}_\alpha(\mathbb{D})} \asymp \|Id\|^q_{(A^p_\alpha, L^q_\mu(\mathbb{D}))}.$$

Here, for a positive measure μ, the Berezin-type transform $B_{s,\alpha}(\mu)$ is

$$B_{s,\alpha}(\mu)(z) = \int_{\mathbb{D}} \frac{(1-|z|^2)^s}{|1-\overline{w}z|^{2+s+\alpha}}\, d\mu(w), \quad z \in \mathbb{D}.$$

For more information on the Berezin transform we refer to the book [33]. For additional information with respect to the Toeplitz operator, see [69].

4.3.3 Projection on weighted Bergman spaces

From now on, we assume that the norm convergence in the Bergman space A^2_ω implies the uniform convergence on compact subsets, then the point evaluations L_z (at the point $z \in \mathbb{D}$) are bounded linear functionals on A^2_ω. Therefore, there are reproducing kernels $B^\omega_z \in A^2_\omega$ with $\|L_z\| = \|B^\omega_z\|_{A^2_\omega}$ such that

$$L_z f = f(z) = \langle f, B^\omega_z \rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta)\, \overline{B^\omega_z(\zeta)}\, \omega(\zeta)\, dA(\zeta), \quad f \in A^2_\omega. \tag{4.16}$$

In a similar fashion as in (4.13) for any orthonormal basis of A^2_ω

$$B^\omega_z(\zeta) = \sum_{n=0}^{\infty} e_n(\zeta)\overline{e_n(z)}, \quad z, \zeta \in \mathbb{D}.$$

When ω is radial, we can use the standard orthonormal basis $\{z^j/\sqrt{2\omega_{2j+1}}\}$, $j \in \mathbb{N} \cup \{0\}$, of A^2_ω to obtain the following formula for the Bergman reproducing kernels

$$B^\omega_z(\zeta) = \sum_{n=0}^{\infty} \frac{(\zeta\overline{z})^n}{2\omega_{2n+1}}, \quad z, \zeta \in \mathbb{D}. \tag{4.17}$$

Even if ω is a radial weight, the reproducing kernels B_z^ω, $z \in \mathbb{D}$, do not necessarily have the good properties that the standard reproducing kernels B_z^α have. The fact that it is not always possible to obtain a closed formula for the reproducing kernel B_z^ω makes it harder to know relevant information about the kernel such as; its behaviour in pseudohyperbolic bounded regions, the existence of zeros or a norm estimate. One of the main issues with these reproducing kernels B_z^ω is the existence of zeros, which can appear even in radial weights with apparently good properties as Zeytuncu [67] and Perälä [62] proved.

Theorem 4.3.7. *[67, Theorem 1.5] There exists a radial weight ω on $\mathbb{D}$, comparable to 1, such that the reproducing kernel B_a^ω has zeros.*

Since A_ω^2 is a closed subspace of $L_\omega^2(\mathbb{D})$, we may consider the orthogonal Bergman projection P_ω from L_ω^2 to A_ω^2. This projection is usually called the Bergman projection and it is given by the following formula

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\,\omega(\zeta)dA(\zeta), \quad z \in \mathbb{D}. \tag{4.18}$$

The maximal Bergman projection is the following sublinear operator

$$P_\omega^+(f)(z) = \int_{\mathbb{D}} |f(\zeta)B_z^\omega(\zeta)|\,\omega(\zeta)dA(\zeta), \quad z \in \mathbb{D}.$$

Theorem 4.3.8. *[58, Theorem 12] Let $1 < p < \infty$ and $\omega \in \mathcal{R}$. Then the following statements are true:*

(i) $P_\omega^+ : L_\omega^p(\mathbb{D}) \to L_\omega^p(\mathbb{D})$ *is bounded. In particular,* $P_\omega : L_\omega^p(\mathbb{D}) \to A_\omega^p$ *is bounded.*

(ii) $P_\omega : L^\infty(\mathbb{D}) \to \mathcal{B}$ *is bounded.*

The first part of this theorem is a direct consequence of [57, Theorem 3]. The $L_\omega^p(\mathbb{D})$ norm estimates of the Bergman reproducing kernels were obtained using estimates on the moments ω_n by Peláez and Rättyä.

Theorem 4.3.9. *[57, Theorem 1] Let $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and $N \in \mathbb{N} \cup \{0\}$. Then the following assertions hold:*

(i) $$M_p^p\left(r, (B_a^\omega)^{(N)}\right) \asymp \int_0^{|a|r} \frac{dt}{\widehat{\omega}(t)^p(1-t)^{p(N+1)}}, \quad r, |a| \to 1^-.$$

(ii) If $v \in \widehat{\mathcal{D}}$, then

$$\| (B_a^\omega)^{(N)} \|_{A_v^p}^p \asymp \int_0^{|a|} \frac{\widehat{v}(t)}{\widehat{\omega}(t)^p(1-t)^{p(N+1)}}\, dt, \quad |a| \to 1^-.$$

When P_ω is bounded on L_ω^p we can use it to obtain the dual of A_ω^p, hence under the same conditions as in Theorem 4.3.8 Peláez and Rättyä obtained the following result.

Corollary 4.3.10. *[57, Corollary 2] Let $1 < p < \infty$ and $\omega \in \mathcal{R}$. Then the following equivalences hold under the A_ω^2 pairing: $(A_\omega^p)^\star \simeq A_\omega^{p'}$ and $(A_\omega^1)^\star \simeq \mathcal{B}$.*

5 Summary of papers I-III

5.1 SUMMARY OF PAPER I

In this paper we characterize the embedding $A^p_\omega \subset T^q_s(\nu,\omega)$ in terms of Carleson measures for A^p_ω. This result can be interpreted as an additional characterization of Carleson measures for the space A^p_ω. We add the technical condition $\nu(\{0\}) = 0$ because the tents $\Gamma(z)$ are not defined for $z = 0$, this condition does not carry any real restriction.

Theorem 5.1.1. *Let $0 < p, q, s < \infty$ such that $1 + \frac{s}{p} - \frac{s}{q} > 0$, $\omega \in \widehat{\mathcal{D}}$ and let ν be positive Borel measure on $\mathbb{D}$, finite on compact sets, such that $\nu(\{0\}) = 0$. Write $\nu_\omega(\zeta) = \omega(T(\zeta))\,d\nu(\zeta)$ for all $\zeta \in \mathbb{D} \setminus \{0\}$. Then the following assertions hold:*

(i) $I_d : A^p_\omega \to T^q_s(\nu,\omega)$ is bounded if and only if ν_ω is a $(p + s - \frac{ps}{q})$-Carleson measure for A^p_ω. Moreover,

$$\|I_d\|^s_{A^p_\omega \to T^q_s(\nu,\omega)} \asymp \|I_d\|^{p+s-\frac{ps}{q}}_{A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\nu_\omega}}.$$

(ii) $I_d : A^p_\omega \to T^q_s(\nu,\omega)$ is compact if and only if $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\nu_\omega}(\mathbb{D})$ is compact.

We can generalize this theorem by extracting the following area operator and studying its boundedness. For $0 < s < \infty$, the generalized area operator induced by positive measures μ and v on $\mathbb{D}$ is defined by

$$G^v_{\mu,s}(f)(z) = \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))}\right)^{\frac{1}{s}}, \quad z \in \mathbb{D} \setminus \{0\}.$$

Minkowski's inequality shows that $G^v_{\mu,s}$ is sublinear if $s \geq 1$. This is not the case for $0 < s < 1$, but instead we have $\left(G^v_{\mu,s}(f+g)\right)^s \leq \left(G^v_{\mu,s}(f)\right)^s + \left(G^v_{\mu,s}(g)\right)^s$. Write μ^ω_v for the positive measure such that

$$d\mu^\omega_v(z) = \frac{\omega(T(z))}{v(T(z))}\,d\mu(z)$$

for μ-almost every $z \in \mathbb{D}$. Fubini's theorem shows that

$$\begin{aligned}
\|G^v_{\mu,s}(f)\|^s_{L^s_\omega(\mathbb{D})} &= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))}\right) \omega(z)\,dA(z) \\
&= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\right) \omega(z)\,dA(z) \\
&= \int_{\mathbb{D}\setminus\{0\}} |f(\zeta)|^s \left(\frac{1}{\omega(T(\zeta))}\int_{T(\zeta)} \omega(z)\,dA(z)\right) d\mu^\omega_v(\zeta) \\
&= \int_{\mathbb{D}\setminus\{0\}} |f(\zeta)|^s\,d\mu^\omega_v(\zeta) = \|f\|^s_{L^s_{\mu^\omega_v}(\mathbb{D})} - |f(0)|^s \mu^\omega_v(\{0\}),
\end{aligned} \tag{5.1}$$

and hence $G^v_{\mu,s}: A^p_\omega \to L^s_\omega(\mathbb{D})$ is bounded if and only if μ^ω_v is an s-Carleson measure for A^p_ω. In order to study this operator the equivalent norm given by (2.9) will play an important role.

Theorem 5.1.2. *Let $0 < p, q, s < \infty$ such that $s > q - p$, $\omega \in \widehat{\mathcal{D}}$ and let μ, v be positive Borel measures on $\mathbb{D}$ such that $\mu(\{z \in \mathbb{D} : v(T(z)) = 0\}) = 0 = \mu(\{0\})$. Then the following assertions hold:*

(i) μ^ω_v *is a q-Carleson measure for* A^p_ω *if and only if* $G^v_{\mu,s}: A^p_\omega \to L^{\frac{ps}{p+s-q}}_\omega(\mathbb{D})$ *is bounded. Moreover,*

$$\|G^v_{\mu,s}\|^s_{A^p_\omega \to L^{\frac{ps}{p+s-q}}_\omega(\mathbb{D})} \asymp \|I_d\|^q_{A^p_\omega \to L^q_{\mu^\omega_v}(\mathbb{D})}.$$

(ii) $I_d: A^p_\omega \to L^q_{\mu^\omega_v}(\mathbb{D})$ *is compact if and only if* $G^v_{\mu,s}: A^p_\omega \to L^{\frac{ps}{p+s-q}}_\omega(\mathbb{D})$ *is compact.*

In order to prove the previous theorems we need to estimate the norm of the identity operator $Id: A^p_\omega \to L^q_\mu(\mathbb{D})$. The equivalence between the conditions of the following theorem was proven by Peláez and Rättyä in [55, Theorem 1].

Theorem 5.1.3. *Let $0 < p, q < \infty$, $\omega \in \widehat{\mathcal{D}}$ and let μ be a positive Borel measure on $\mathbb{D}$. Further, let $dh(z) = dA(z)/(1-|z|^2)^2$ denote the hyperbolic measure.*

(i) If $p \le q$, then μ is a q-Carleson measure for A^p_ω if and only if $\sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{q}{p}}} < \infty$. Moreover,

$$\|I_d\|^q_{A^p_\omega \to L^q_\mu(\mathbb{D})} \asymp \sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{q}{p}}}.$$

(ii) If $p \le q$, then $I_d: A^p_\omega \to L^q_\mu(\mathbb{D})$ is compact if and only if

$$\lim_{|a|\to 1^-} \frac{\mu(S(a))}{(\omega(S(a)))^{q/p}} = 0. \tag{5.2}$$

(iii) If $q < p$, then the following conditions are equivalent:

(a) $I_d: A^p_\omega \to L^q_\mu(\mathbb{D})$ *is compact;*

(b) $I_d: A^p_\omega \to L^q_\mu(\mathbb{D})$ *is bounded;*

(c) The function

$$B^\omega_\mu(z) = \int_{\Gamma(z)} \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D}\setminus\{0\},$$

belongs to $L^{\frac{p}{p-q}}_\omega(\mathbb{D})$*;*

(d) For each fixed $r \in (0,1)$, the function

$$\Phi^\omega_\mu(z) = \Phi^\omega_{\mu,r}(z) = \int_{\Gamma(z)} \frac{\mu(\Delta(\zeta,r))}{\omega(T(\zeta))}\, dh(\zeta), \quad z \in \mathbb{D}\setminus\{0\},$$

belongs to $L^{\frac{p}{p-q}}_\omega(\mathbb{D})$*;*

(e) For each sufficiently large $\lambda = \lambda(\omega) > 1$, the function

$$\Psi_\mu^\omega(z) = \Psi_{\mu,\lambda}^\omega(z) = \int_{\mathbb{D}} \left(\frac{1-|\zeta|}{|1-\bar{z}\zeta|} \right)^\lambda \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D},$$

belongs to $L_\omega^{\frac{p}{p-q}}(\mathbb{D})$.

Moreover,

$$\begin{aligned} \|I_d\|^q_{A^p_\omega \to L^q_\mu(\mathbb{D})} &\asymp \|M_\omega(\mu)\|_{L_\omega^{\frac{p}{p-q}}(\mathbb{D})} \asymp \|B_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}(\mathbb{D})} + \mu(\{0\}) \\ &\asymp \|\Psi_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}(\mathbb{D})} \asymp \|\Phi_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}(\mathbb{D})} + \mu(\{0\}). \end{aligned} \tag{5.3}$$

To prove the compactness of these operators, we use the following lemma, the proof of which can be obtained by adapting the proof in [25, Lemma 1 p. 21]

Lemma 5.1.4. *Let ν be a positive Borel measure on $\mathbb{D}$ and $0 < p < \infty$. If $\{\varphi_n\}_{n=0}^\infty \subset L^p_\nu(\mathbb{D})$ and $\varphi \in L^p_\nu(\mathbb{D})$ satisfy $\lim_{n\to\infty} \|\varphi_n\|_{L^p_\nu(\mathbb{D})} = \|\varphi\|_{L^p_\nu(\mathbb{D})}$ and $\lim_{n\to\infty} \varphi_n(z) = \varphi(z)$ ν-a.e. on $\mathbb{D}$, then $\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^p_\nu(\mathbb{D})} = 0$.*

We can use Theorem 5.1.2 to study the boundedness of other operators such as the following integral operators, also called Volterra type operators. Given $g \in \mathcal{H}(\mathbb{D})$ we define the integral operator

$$T_g(f)(z) = \int_0^z g'(\zeta) f(\zeta)\, d\zeta, \quad z \in \mathbb{D},$$

acting on $\mathcal{H}(\mathbb{D})$. Some results of this operator in the context of Hardy spaces are due to Aleman and Cima [2] and Aleman and Siskakis [3], while in the context of Bergman spaces we find the results given by Aleman and Siskakis in [4]. This type of integral operators have been extensively studied during recent decades and have interesting connections with other areas of mathematical analysis, see [51, 54] and the references therein.

Theorem 5.1.5. *Let $0 < p, q < \infty$ such that $q > \frac{2p}{2+p}$ and $\omega \in \widehat{\mathcal{D}}$. Let $g \in \mathcal{H}(\mathbb{D})$ and denote $d\mu_g(z) = |g'(z)|^2 \omega^\star(z)\, dA(z)$. Then $T_g : A^p_\omega \to A^q_\omega$ is bounded (resp. compact) if and only if $I_d : A^p_\omega \to L_{\mu_g}^{p+2-\frac{2p}{q}}(\mathbb{D})$ is bounded (resp. compact).*

This result is a direct consequence of Theorem 5.1.2 with $\nu = \omega dA$, $\mu = \mu_g$, and $s = 2$ together with the equivalent norm given in (2.10).

5.2 SUMMARY OF PAPER II

In this paper we work with weights ω which belong to either $\mathcal{R}$ or $\widehat{\mathcal{D}}$. The Bergman projection P_ω is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta) \overline{B_z^\omega(\zeta)} \omega(\zeta)\, dA(\zeta), \quad z \in \mathbb{D},$$

where B_z^ω are the reproducing kernels stated in (4.17). Recently, those regular weights ω and ν for which $P_\omega : L^p_\nu(\mathbb{D}) \to L^p_\nu(\mathbb{D})$ is bounded were characterized in terms of Bekollé-Bonami type conditions [57]. In this paper we consider operators which are natural extensions of the projection P_ω. For a positive Borel measure μ on $\mathbb{D}$, the Toeplitz operator associated with μ is

$$\mathcal{T}_\mu(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B_z^\omega(\zeta)}\, d\mu(\zeta).$$

If $d\mu = \Phi\omega dA$ for a non-negative function Φ, then write $\mathcal{T}_\mu = \mathcal{T}_\Phi$ so that $\mathcal{T}_\Phi(f) = P_\omega(f\Phi)$. The operator $\mathcal{T}_\Phi$ has been extensively studied since the seventies [14,44,68]. Luecking was probably the one who introduced Toeplitz operators $\mathcal{T}_\mu$ with measures as symbols in [39], where he provides, among other things, a description of Schatten class Toeplitz operators $\mathcal{T}_\mu : A^2_\alpha \to A^2_\alpha$ in terms of an ℓ^p-condition involving a hyperbolic lattice of $\mathbb{D}$.

Before presenting our main results we will need some additional results on the reproducing kernels B_z^ω, apart from those given in Theorem 4.3.9. First we will estimate the norm in the Bloch space and H^∞.

Lemma 5.2.1. *Let $\omega \in \widehat{\mathcal{D}}$. Then*

$$\|B_z^\omega\|_{\mathcal{B}} \asymp \frac{1}{\omega(S(z))} \asymp \|B_z^\omega\|_{H^\infty}, \quad z \in \mathbb{D}.$$

Here the Bloch norm is given by $\|f\|_{\mathcal{B}} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|)|f'(z)|$. Additionally to prove the main result of this section we will need the following pointwise estimates of the reproducing Bergman kernels B_a^ω in certain sets induced by a point $a \in \mathbb{D}$.

Lemma 5.2.2. *Let $\omega \in \widehat{\mathcal{D}}$. Then there exists $r = r(\omega) \in (0,1)$ such that $|B_a^\omega(z)| \asymp B_a^\omega(a)$ for all $a \in \mathbb{D}$ and $z \in \Delta(a,r)$.*

In this first result we used the definition of B_a^ω in terms of a basis and the norm estimates $\|B_a^\omega\|_{A^2_\omega}$ from Theorem 4.3.9.

Lemma 5.2.3. *Let $\omega \in \widehat{\mathcal{D}}$. Then there exists constants $c = c(\omega) > 0$ and $\delta = \delta(\omega) \in (0,1]$ such that*

$$|B_a^\omega(z)| \geq \frac{c}{\omega(S(a))}, \quad z \in S(a_\delta), \quad a \in \mathbb{D}\setminus\{0\}. \tag{5.4}$$

To prove this result we use estimates of the derivative of B_a^ω, the fact that $B_a^\omega(a_\delta) = B^\omega_{\sqrt{|aa_\delta|}}(\sqrt{|aa_\delta|})$ and Lemma E.

These pointwise estimates will allow us to avoid some of the issues of weighted reproducing Bergman kernels B_a^ω such as the possible existence of zeros in $\mathbb{D}$.

Theorem 5.2.4. *Let $1 < p \leq q < \infty$, $\omega \in \mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is bounded;*

(ii) $\dfrac{\widetilde{\mathcal{T}_\mu}(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \in L^\infty(\mathbb{D})$;

(iii) μ *is a* $\frac{s(p+q')}{pq'}$*-Carleson measure for* A^s_ω *for some (equivalently for all)* $0 < s < \infty$;

(iv) $\frac{\mu(S(\cdot))}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}}} \in L^\infty(\mathbb{D})$.

Moreover,

$$\|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \asymp \left\| \frac{\widetilde{\mathcal{T}}_\mu(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \right\|_{L^\infty(\mathbb{D})} \asymp \|Id\|^{\frac{s(p+q')}{pq'}}_{A^s_\omega \to L_\mu^{\frac{s(p+q')}{pq'}}(\mathbb{D})} \asymp \left\| \frac{\mu(S(\cdot))}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}}} \right\|_{L^\infty(\mathbb{D})}.$$

In order to prove this result we use the norm estimates given in Theorem 4.3.9, the pointwise estimate Lemma 3 and the duality of the space A^q_ω given in Corollary 4.3.10.

Theorem 5.2.5. *Let $1 < p \leq q < \infty$, $\omega \in \mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is compact;*

(ii) $\lim_{|z|\to 1^-} \frac{\widetilde{\mathcal{T}}_\mu(z)}{\omega(S(z))^{\frac{1}{p}+\frac{1}{q'}-1}} = 0$;

(iii) $Id : A^s_\omega \to L_\mu^{\frac{s(p+q')}{pq'}}(\mathbb{D})$ *is compact for some (equivalently for all)* $0 < s < \infty$;

(iv) $\lim_{|I|\to 0} \frac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p}+\frac{1}{q'}}} = 0$.

To prove this result we proceed in a similar fashion as in Theorem 11, together with common techniques usually employed to study the compactness of concrete operators, and the weak convergence of $\frac{B^\omega_z}{\|B^\omega_z\|_{A^p_\omega}} \to 0$ in A^p_ω as $|z| \to 1^-$.

Proposition 5.2.6. *Let $1 < p < \infty$, $\omega \in \mathcal{R}$ and $\{z_j\}_{j=1}^\infty \subset \mathbb{D} \setminus \{0\}$ be a separated sequence. Then*

$$F = \sum_{j=1}^\infty c_j \frac{B^\omega_{z_j}}{\|B^\omega_{z_j}\|_{A^p_\omega}} \in A^p_\omega \tag{5.5}$$

with $\|F\|_{A^p_\omega} \lesssim \|\{c_j\}_{j=1}^\infty\|_{\ell^p}$ for all $\{c_j\}_{j=1}^\infty \in \ell^p$.

For us to prove the reverse case when $1 < q < p < \infty$, we will need the test functions F given by the result above, which is obtained using Theorem 4.3.10, the reproducing formula for $B^\omega_{z_j}$ and the subharmonicity of these functions.

Theorem 5.2.7. *Let $1 < q < p < \infty$, $0 < r < 1$, $\omega \in \mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is compact;*

(ii) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is bounded;*

(iii) $\widehat{\mu}_r(\cdot) = \frac{\mu(\Delta(\cdot,r))}{\omega(\Delta(\cdot,r))} \in L_\omega^{\frac{pq}{p-q}}(\mathbb{D})$;

(iv) μ *is a* $\left(p+1-\frac{p}{q}\right)$*-Carleson measure for* A^p_ω;

(v) $Id : A^p_\omega \to L^{p+1-\frac{p}{q}}_\mu(\mathbb{D})$ *is compact;*

(vi) $\widetilde{\mathcal{T}_\mu} \in L^{\frac{pq}{p-q}}_\omega(\mathbb{D})$.

Moreover,

$$\|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \asymp \|\widehat{\mu}_r\|_{L^{\frac{qp}{p-q}}_\omega(\mathbb{D})} \asymp \|Id\|^{p+1-\frac{p}{q}}_{A^p_\omega \to L^{p+1-\frac{p}{q}}_\mu} \asymp \|\widetilde{\mathcal{T}_\mu}\|_{L^{\frac{qp}{p-q}}_\omega(\mathbb{D})}.$$

The tools used to prove this result, are the characterization of Carleson measures for A^p_ω given by Theorem 4.1.6, the test functions given by Proposition 3, Rademacher functions and Khinchine's inequality and the boundedness of the sublinear operator P^+_ω given by Theorem 4.3.8.

Theorem 5.2.8. *Let $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$;

(ii) $\sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{\omega^\star(z_j)}\right)^p < \infty$;

(iii) $\frac{\mu(\Delta(\cdot,r))}{\omega^\star(\cdot)}$ *belongs to* $L^p\left(\frac{dA}{(1-|\cdot|)^2}\right)$ *for some* $0 < r < 1$.

Moreover, if $\omega \in \mathcal{R}$ such that $\frac{(\omega^\star(z))^p}{(1-|z|)^2}$ is also a regular weight, then $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$ if and only if $\widetilde{\mathcal{T}_\mu} \in L^p_{\omega/\omega^\star}(\mathbb{D})$.

This final result is an extension of [56, Theorem 1], with the additional characterization in terms of the Berezin transform when $\omega \in \mathcal{R}$. The condition $\frac{(\omega^\star(z))^p}{(1-|z|)^2} \in \mathcal{R}$ is not a restriction when $p \geq 1$, and equates to the condition $p > \frac{1}{2+\alpha}$ from [69, Corollary 7.17] when working with the standard Bergman weights.

5.3 SUMMARY OF PAPER III

In this article we give an atomic decomposition of the weighted mixed norm spaces $A^{p,q}_\omega$, with ω in the doubling class $\mathcal{D}$. As we saw in Proposition 3, all the functions given by a series as in (5.5) belong to the space A^p_ω, but in order to get an atomic decomposition for A^p_ω we need the reverse implication, that every function in A^p_ω could be expressed as in (5.5). This first theorem proves an analogue to Proposition 3, for the more general spaces $A^{p,q}_\omega$ with different atoms.

Theorem 5.3.1. *Let $0 < p \leq \infty$, $0 < q < \infty$, $\omega \in \mathcal{D}$, and $\{z_k\}_{k=0}^\infty$ a separated sequence in $\mathbb{D}$. Let $\beta = \beta(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ be those of Lemma E(ii) and (iii). If*

$$M > 1 + \frac{1}{p} + \frac{\beta+\gamma}{q} \tag{5.6}$$

and $\lambda = \{\lambda_{j,l}\} \in \ell^{p,q}$, then

$$F(z) = \sum_{j,l} \lambda_{j,l} \frac{(1-|z_{j,l}|)^{M-\frac{1}{p}} \widehat{\omega}(z_{j,l})^{-\frac{1}{q}}}{\left(1-\overline{z_{j,l}}z\right)^M} \in \mathcal{H}(\mathbb{D}) \tag{5.7}$$

and there exists a constant $C = C(M, \omega, p, q) > 0$ *such that*

$$\|F\|_{A^{p,q}_\omega} \leq C \|\lambda\|_{\ell^{p,q}} . \tag{5.8}$$

To prove this result we use the properties of the weights in $\mathcal{D}$ given in Lemma E and Lemma 8.7, together with the results of Muckenhoupt in [45]. The next theorem proves the reverse of the previous result, that is it is proved that every function in $A^{p,q}_\omega$ can be expressed as a series of the form (5.7). In order to prove this result we shall introduce the appropriate dyadic polar rectangles induced by $K \in \mathbb{N} \setminus \{1\}$, $K > 1$.

For each $K \in \mathbb{N} \setminus \{1\}$, $j \in \mathbb{N} \cup \{0\}$ and $l = 0, 1, \ldots, K^{j+3} - 1$, the dyadic polar rectangle is defined as

$$Q_{j,l} = \left\{ z \in \mathbb{D} : r_j \leq |z| < r_{j+1}, \arg z \in \left[2\pi \frac{l}{K^{j+3}}, 2\pi \frac{l+1}{K^{j+3}} \right) \right\},$$

where $r_j = r_j(K) = 1 - K^{-j}$ as before, and its center is denoted by $\zeta_{j,l}$. For each $M \in \mathbb{N}$ and $k = 1, \ldots, M^2$, the rectangle $Q^k_{j,l}$ is defined as the result of dividing $Q_{j,l}$ into M^2 pairwise disjoint rectangles of equal Euclidean area, and the centres of these squares are denoted by $\zeta^k_{j,l}$. It is worth noticing that the cubes $Q_{j,l}$ and $Q^k_{j,l}$ as well as their centres $\zeta^k_{j,l}$ depend on the value of K.

Theorem 5.3.2. *Let* $0 < p \leq \infty$, $0 < q < \infty$, $\omega \in \mathcal{D}$ *and* $K \in \mathbb{N} \setminus \{1\}$ *such that* (8.1) *holds. Then, for each* $f \in A^{p,q}_\omega$ *there exists* $\lambda(f) = \{\lambda(f)^k_{j,l}\} \in \ell^{p,q}$ *and* $M = M(p, q, \omega) > 0$ *such that*

$$f(z) = \sum_{j,l,k} \lambda(f)^k_{j,l} \frac{(1 - |\zeta^k_{j,l}|^2)^{M - \frac{1}{p}} \widehat{\omega}(r_j)^{-\frac{1}{q}}}{(1 - \overline{\zeta^k_{j,l}} z)^M}, \quad z \in \mathbb{D}, \tag{5.9}$$

and

$$\left\| \{\lambda(f)^k_{j,l}\} \right\|_{\ell^{p,q}} \lesssim \|f\|_{A^{p,q}_\omega}. \tag{5.10}$$

To prove Theorem 18 some definitions and lemmas are needed. For $f \in \mathcal{H}(\mathbb{D})$, define $f_{j,l} = \sup_{z \in Q_{j,l}} |f(z)|$ and write $\lambda(f) = \{\lambda(f)_{j,l}\}$, where $\lambda(f)_{j,l} = K^{-\frac{j}{p}} \widehat{\omega}(r_j)^{\frac{1}{q}} f_{j,l}$ for all $j \in \mathbb{N} \cup \{0\}$ and $l = 0, 1, \ldots, K^{j+3} - 1$.

Lemma 5.3.3. *Let* $0 < p, q < \infty$, $\omega \in \mathcal{D}$ *and* $K \in \mathbb{N} \setminus \{1\}$ *such that* (8.1) *holds. Then* $\|f\|_{A^{p,q}_\omega} \asymp \|\lambda(f)\|_{\ell^{p,q}}$ *for all* $f \in \mathcal{H}(\mathbb{D})$.

This equivalent norm plays a key role in the proof of Theorem 18 as it gives us a discretization of the $A^{p,q}_\omega$-norm.

Lemma 5.3.4. *Let* $0 < p \leq \infty$, $0 < q < \infty$ *and* $\omega \in \widehat{\mathcal{D}}$, *and let* $\beta = \beta(\omega) > 0$ *be that of Lemma E(ii). Then* $A^{p,q}_\omega \subset A^1_\eta$ *for all* $\eta > \frac{\beta}{q} + \frac{1}{p} - 1$.

In order to demonstrate Theorem 18, this lemma allows us to represent each $f \in A^{p,q}_\omega$ as $P_\eta(f)$ which help us to estimate $|f - S_\eta(f)| = |P_\eta(f) - S_\eta(f)|$, where

$$S_\eta(f)(z) = (\eta + 1) \sum_{j,l,k} f(\zeta^k_{j,l}) \frac{(1 - |\zeta^k_{j,l}|^2)^\eta}{\left(1 - \overline{\zeta^k_{j,l}} z\right)^{\eta+2}} \left| Q^k_{j,l} \right|.$$

By dividing the dyadic squares $Q_{j,l}$ in sufficiently small pieces, we can obtain $\|Id - S_\eta\|_{A^{p,q}_\omega \to A^{p,q}_\omega} \leq \frac{1}{2}$. Furthermore, we define the sequence $f_1 = S_\eta(f)$ and $f_n = S_\eta\left(f - \sum_{m=1}^{n-1} f_m\right)$ for $n \in \mathbb{N} \setminus \{1\}$ and from the estimate

$$\left\| f - \sum_{m=1}^{n} f_m \right\|_{A^{p,q}_\omega} \leq \frac{1}{2^n} \|f\|_{A^{p,q}_\omega}, \tag{5.11}$$

we reach the equality $f = \sum_{n=1}^{\infty} f_n$.

Finally we will characterize the Carleson measures for the spaces $A^{p,q}_\omega$, together with the boundedness of the differentiation operator $D^{(n)} : A^{p,q}_\omega \to L^s_\mu(\mathbb{D})$ defined by $D^{(n)}(f) = f^{(n)}$. We introduce the function

$$T_{r,u,v}(z) = \frac{\mu(\Delta(z,r))}{(1-|z|)^u \widehat{\omega}(z)^v}, \quad z \in \mathbb{D},$$

where $0 < r < 1$ and $0 < u, v < \infty$.

Theorem 5.3.5. *Let $0 < p, q, s < \infty$, $n \in \mathbb{N} \cup \{0\}$, $0 < r < 1$, μ a positive Borel measure on $\mathbb{D}$, $\omega \in \mathcal{D}$ and let $K = K(\omega) \in \mathbb{N} \setminus \{1\}$ such that* (8.1) *holds. Then the following statements are equivalent:*

(i) $D^{(n)} : A^{p,q}_\omega \to L^s_\mu(\mathbb{D})$ *is bounded;*

(ii) The sequence $\left\{ \mu(Q_{j,l}) K^{sj\left(n+\frac{1}{p}\right)} \widehat{\omega}(r_j)^{-\frac{s}{q}} \right\}_{j,l}$ *belongs to* $\ell^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}$;

(iii) The function $T_{r,u,v}$ *belongs to* $L_\omega^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}(\mathbb{D})$, *where*

(a) $u = sn + 1$ *and* $v = 1$ *if* $s < \min\{p, q\}$;

(b) $u = sn + \frac{1}{p}$ *and* $v = 1$ *if* $p \leq s < q$;

(c) $u = sn + 1$ *and* $v = \frac{s}{q}$ *if* $q \leq s < p$;

(d) $u = sn + \frac{1}{p}$ *and* $v = \frac{s}{q}$ *if* $s \geq \max\{p, q\}$.

Moreover,

$$\|D^{(n)}\|^s_{A^{p,q}_\omega \to L^s_\mu} \asymp \left\| \left\{ \mu(Q_{j,l}) 2^{sj\left(n+\frac{1}{p}\right)} \widehat{\omega}(r_j)^{-\frac{s}{q}} \right\}_{j,l} \right\|_{\ell^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}} \asymp \|T_{r,u,v}\|_{L_\omega^{\left(\frac{p}{s}\right)', \left(\frac{q}{s}\right)'}(\mathbb{D})}.$$

In order to prove this theorem we use Theorem 5.3.1, Rademacher functions and Khinchine's inequality (2.12) and the equivalent norm given by Theorem 5.3.3.

6 Paper I

6.1 INTRODUCTION AND MAIN RESULTS

The theory of tent spaces introduced by Coifman, Meyer and Stein [17], and further studied by Cohn and Verbitsky [16] among others, shows the importance of maximal and square area functions and other objects from harmonic analysis [27] in the study of Hardy spaces in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ [25,30]. The recent studies [54,55] show that tent spaces have natural analogues for Bergman spaces, and they may play a role in the theory of weighted Bergman spaces similar to that of the original tent spaces in the Hardy space case. The tent space $T_s^q(\nu,\omega)$ consists of ν-equivalence classes of ν-measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{T_s^q(\nu,\omega)}^q = \|A_{s,\nu}(f)\|_{L^q_\omega}^q = \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |f(z)|^s \, d\nu(z) \right)^{\frac{q}{s}} \omega(\zeta)\, dA(\zeta) < \infty, \quad 0 < q, s < \infty.$$

Here ν is assumed to be a positive Borel measure on $\mathbb{D}$, finite on compact sets, and $\omega \in \widehat{\mathcal{D}}$, that is, ω is radial and $\widehat{\omega}(r) = \int_r^1 \omega(s)\, ds$ has the doubling property $\sup_{0\le r<1} \widehat{\omega}(r)/\widehat{\omega}(\frac{1+r}{2}) < \infty$. Moreover,

$$\Gamma(z) = \left\{ \xi \in \mathbb{D} : |\theta - \arg(\xi)| < \frac{1}{2}\left(1 - \frac{|\xi|}{r}\right) \right\}, \quad z = re^{i\theta} \in \overline{\mathbb{D}} \setminus \{0\},$$

are non-tangential approach regions with vertexes inside the disc, and the related tents are defined by $T(\zeta) = \{z \in \mathbb{D} : \zeta \in \Gamma(z)\}$ for all $\zeta \in \mathbb{D} \setminus \{0\}$. We also set $\omega(T(0)) = \lim_{r\to 0^+} \omega(T(r))$ to deal with the origin.

The purpose of this paper is three fold. First, we are interested in the question of when the weighted Bergman space A^p_ω, consisting of analytic functions in the unit disc $\mathbb{D}$ such that

$$\|f\|_{A^p_\omega}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty, \quad 0 < p < \infty,$$

is continuously or compactly embedded into the tent space $T_s^q(\nu,\omega)$. Analogous problems for Hardy and Hardy-Sobolev spaces have been considered in [12,15,31]. It turns out that the containment $A^p_\omega \subset T_s^q(\nu,\omega)$ is naturally described in terms of Carleson measures for A^p_ω. For $0 < p, q < \infty$, a positive Borel measure μ on $\mathbb{D}$ is a q-Carleson measure for A^p_ω if there exists a constant $C > 0$ such that $\|f\|_{L^q_\mu} \le C\|f\|_{A^p_\omega}$ for all $f \in A^p_\omega$.

Theorem 1. *Let $0 < p, q, s < \infty$ such that $1 + \frac{s}{p} - \frac{s}{q} > 0$, $\omega \in \widehat{\mathcal{D}}$ and ν a positive Borel measure on $\mathbb{D}$, finite on compact sets, such that $\nu(\{0\}) = 0$. Write $\nu_\omega(\zeta) = \omega(T(\zeta))\, d\nu(\zeta)$ for all $\zeta \in \mathbb{D}$. Then the following assertions hold:*

(i) $I_d : A^p_\omega \to T_s^q(\nu,\omega)$ is bounded if and only if ν_ω is a $(p + s - \frac{ps}{q})$-Carleson measure for A^p_ω. Moreover,

$$\|I_d\|^s_{A^p_\omega \to T_s^q(\nu,\omega)} \asymp \|I_d\|^{p+s-\frac{ps}{q}}_{A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\nu_\omega}}.$$

(ii) $I_d : A^p_\omega \to T^q_s(\nu,\omega)$ *is compact if and only if* $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\nu_\omega}$ *is compact.*

The requirement $\nu(\{0\}) = 0$, which does not carry any real restriction, is a technical hypotheses caused by the geometry of the tents $\Gamma(z)$.

Theorem 1 can be interpret as a characterization of Carleson measures. This is the second aim of our study and becomes more apparent when an operator is extracted from Theorem 1. For $0 < s < \infty$, the generalized area operator induced by positive measures μ and v on $\mathbb{D}$ is defined by

$$G^v_{\mu,s}(f)(z) = \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))} \right)^{\frac{1}{s}}, \quad z \in \mathbb{D} \setminus \{0\}.$$

Minkowski's inequality shows that $G^v_{\mu,s}$ is sublinear if $s \geq 1$. This not the case for $0 < s < 1$, but instead we have $\left(G^v_{\mu,s}(f+g)\right)^s \leq \left(G^v_{\mu,s}(f)\right)^s + \left(G^v_{\mu,s}(g)\right)^s$. Anyway, we say that $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded if there exists $C > 0$ such that $\|G^v_{\mu,s}(f)\|_{L^q_\omega} \leq C\|f\|_{A^p_\omega}$ for all $f \in A^p_\omega$. Write μ^ω_v for the positive measure such that

$$d\mu^\omega_v(z) = \frac{\omega(T(z))}{v(T(z))}\, d\mu(z)$$

for μ-almost every $z \in \mathbb{D}$. Fubini's theorem shows that

$$\begin{aligned}
\|G^v_{\mu,s}(f)\|^s_{L^s_\omega} &= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))} \right) \omega(z)\, dA(z) \\
&= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))} \right) \omega(z)\, dA(z) \\
&= \int_{\mathbb{D}\setminus\{0\}} |f(\zeta)|^s \left(\frac{1}{\omega(T(\zeta))} \int_{T(\zeta)} \omega(z)\, dA(z) \right) d\mu^\omega_v(\zeta) \\
&= \int_{\mathbb{D}\setminus\{0\}} |f(\zeta)|^s\, d\mu^\omega_v(\zeta) = \|f\|^s_{L^s_{\mu^\omega_v}} - |f(0)|^s \mu^\omega_v(\{0\}),
\end{aligned} \tag{6.1}$$

and hence $G^v_{\mu,s} : A^p_\omega \to L^s_\omega$ is bounded if and only if μ^ω_v is an s-Carleson measure for A^p_ω. For any $s > 0$, we say that $G^v_{\mu,s} : A^p_\omega \to L^q_\mu$ is compact if for every bounded sequence $\{f_n\}$ in A^p_ω there exists a subsequence $\{f_{n_k}\}$ such that $G^v_{\mu,s}(f_{n_k})$ converges in $L^q(\mu)$.

The next theorem gives a characterization of Carleson measures for Bergman spaces by using the generalized area operator $G^v_{\mu,s}$. Theorem 1 is a special case of this result, see also Theorem 4 in Section 6.3.

Theorem 2. *Let* $0 < p, q, s < \infty$ *such that* $s > q - p$, $\omega \in \widehat{\mathcal{D}}$ *and let* μ, v *be positive Borel measures on* $\mathbb{D}$ *such that* $\mu\left(\{z \in \mathbb{D} : v(T(z)) = 0\}\right) = 0 = \mu(\{0\})$. *Then the following assertions hold:*

(i) μ^ω_v *is a q-Carleson measure for* A^p_ω *if and only if* $G^v_{\mu,s} : A^p_\omega \to L^{\frac{ps}{p+s-q}}_\omega$ *is bounded. Moreover,*

$$\|G^v_{\mu,s}\|^s_{A^p_\omega \to L^{\frac{ps}{p+s-q}}_\omega} \asymp \|I_d\|^q_{A^p_\omega \to L^q_{\mu^\omega_v}}.$$

(ii) $I_d : A^p_\omega \to L^q_{\mu^\omega_v}$ *is compact if and only if* $G^v_{\mu,s} : A^p_\omega \to L^{\frac{ps}{p+s-q}}_\omega$ *is compact.*

The third motivation of this study comes from the equivalent A^p_ω-norm involving square functions, given by

$$\|f\|^p_{A^p_\omega} \asymp \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |D^\alpha f(z)|^2 \left(1 - \left|\frac{z}{\zeta}\right|\right)^{2\alpha-2} dA(z) \right)^{\frac{p}{2}} \omega(\zeta)\, dA(\zeta) + \sum_{j=0}^{[\alpha]-1} |f^{(j)}(0)|^p.$$

Here $0 < \alpha, p < \infty$, ω is a radial weight, $D^\alpha f$ denotes the fractional derivative of order α and $[\alpha]$ is the integer such that $[\alpha] \le \alpha < [\alpha] + 1$ [54, Theorem 4.2]. This comparability shows that the operator

$$F_\alpha(f)(\zeta) = \left(\int_{\Gamma(\zeta)} |D^\alpha f(z)|^2 \left(1 - \left|\frac{z}{\zeta}\right|\right)^{2\alpha-2} dA(z) \right)^{\frac{1}{2}}, \quad \zeta \in \mathbb{D} \setminus \{0\},$$

is bounded from A^p_ω to L^p_ω for each $\alpha > 0$. This is no longer true when $\alpha = 0$, and therefore the definition of $G^v_{\mu,s}$ is also motivated by the study of this limit case. This was the starting point in the study by Cohn on the area operator $G_\mu(f)(z) = \int_{\Gamma(z)} |f(\zeta)| \frac{d\mu(\zeta)}{1-|\zeta|}$, defined for z on the boundary $\mathbb{T}$ of $\mathbb{D}$, acting from the Hardy space H^p to $L^p(\mathbb{T})$ [15, Theorem 1]. The approach by Cohn relies on ideas by John and Nirenberg [30, Theorem 2.1] and Calderon-Zygmund decompositions. More recently, similar ideas together the classical factorization of Hardy spaces $H^p = H^{p_1} \cdot H^{p_2}$, $p^{-1} = p_1^{-1} + p_2^{-1}$, were used in [31] to study the case $G_\mu : H^p \to L^q(\mathbb{T})$. We do not employ these techniques in the proof of Theorem 2 but instead we use a description of the boundedness of a weighted maximal function of Hörmander-type. To give the precise statement we need to introduce some notation. The Carleson square $S(I)$ based on an interval $I \subset \mathbb{T}$ is the set $S(I) = \{re^{it} \in \mathbb{D} : e^{it} \in I, 1 - |I| \le r < 1\}$, where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{T}$. We associate to each $a \in \mathbb{D} \setminus \{0\}$ the interval $I_a = \{e^{i\theta} : |\arg(ae^{-i\theta})| \le \frac{1-|a|}{2}\}$, and denote $S(a) = S(I_a)$. For a positive Borel measure μ on $\mathbb{D}$ and $\alpha > 0$, define the weighted maximal function

$$M_{\omega,\alpha}(\mu)(z) = \sup_{z \in S(a)} \frac{\mu(S(a))}{(\omega(S(a)))^\alpha}, \quad z \in \mathbb{D}.$$

In the case $\alpha = 1$ simply write $M_\omega(\mu)$, and if μ is of the form $\varphi\omega\, dA$, then $M_{\omega,\alpha}(\mu)$ is the weighted maximal function $M_{\omega,\alpha}(\varphi)$ of φ. The following result is [55, Theorem 3].

Theorem A. *Let* $0 < p \le q < \infty$ *and* $0 < \gamma < \infty$ *such that* $p\gamma > 1$. *Let* $\omega \in \widehat{\mathcal{D}}$ *and* μ *be a positive Borel measure on* $\mathbb{D}$. *Then* $[M_\omega((\cdot)^{\frac{1}{\gamma}})]^\gamma : L^p_\omega \to L^q(\mu)$ *is bounded if and only if* $M_{\omega,q/p}(\mu) \in L^\infty$. *Moreover,*

$$\|[M_\omega((\cdot)^{\frac{1}{\gamma}})]^\gamma\|^q_{(L^p_\omega, L^q(\mu))} \asymp \|M_{\omega,q/p}(\mu)\|_{L^\infty}.$$

Another maximal operator we will face is defined by $N(f)(z) = \sup_{\zeta \in \Gamma(z)} |f(\zeta)|$. It is known that $N : A^p_\omega \to L^p_\omega$ is bounded for each radial weight ω and $\|N(f)\|_{L^p_\omega} \asymp \|f\|_{A^p_\omega}$ by [54, Lemma 4.4].

Theorems 1 and 2 would perhaps not be of much significance if we did not understand the q-Carleson measures for A^p_ω sufficiently well. In fact, another important ingredient in the proof of our main results is the following refinement of the characterizations of Carleson measures for Bergman spaces given in [55, Theorem 1]. The estimates for the norm of the identity operator $I_d : A^p_\omega \to L^q_\mu$ are of special importance for us.

Theorem 3. *Let $0 < p, q < \infty$, $\omega \in \widehat{\mathcal{D}}$ and let μ be a positive Borel measure on $\mathbb{D}$. Further, let $dh(z) = dA(z)/(1-|z|^2)^2$ denote the hyperbolic measure.*

(i) If $p \le q$, then μ is a q-Carleson measure for A^p_ω if and only if $\sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{q}{p}}} < \infty$.

Moreover,

$$\|I_d\|^q_{A^p_\omega \to L^q_\mu} \asymp \sup_{a\in\mathbb{D}} \frac{\mu(S(a))}{\omega(S(a))^{\frac{q}{p}}}.$$

(ii) If $p \le q$, then $I_d : A^p_\omega \to L^q_\mu$ is compact if and only if

$$\lim_{|a|\to 1^-} \frac{\mu(S(a))}{(\omega(S(a)))^{q/p}} = 0. \tag{6.2}$$

(iii) If $q < p$, then the following conditions are equivalent:

(a) $I_d : A^p_\omega \to L^q_\mu$ is compact;

(b) $I_d : A^p_\omega \to L^q_\mu$ is bounded;

(c) The function

$$B^\omega_\mu(z) = \int_{\Gamma(z)} \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D}\setminus\{0\},$$

belongs to $L^{\frac{p}{p-q}}_\omega$;

(d) For each fixed $r \in (0,1)$, the function

$$\Phi^\omega_\mu(z) = \Phi^\omega_{\mu,r}(z) = \int_{\Gamma(z)} \frac{\mu(\Delta(\zeta,r))}{\omega(T(\zeta))}\, dh(\zeta), \quad z \in \mathbb{D}\setminus\{0\},$$

belongs to $L^{\frac{p}{p-q}}_\omega$;

(e) For each sufficiently large $\lambda = \lambda(\omega) > 1$, the function

$$\Psi^\omega_\mu(z) = \Psi^\omega_{\mu,\lambda}(z) = \int_{\mathbb{D}} \left(\frac{1-|\zeta|}{|1-\bar{z}\zeta|}\right)^\lambda \frac{d\mu(\zeta)}{\omega(T(\zeta))}, \quad z \in \mathbb{D},$$

belongs to $L^{\frac{p}{p-q}}_\omega$.

Moreover,

$$\|I_d\|^q_{A^p_\omega\to L^q_\mu} \asymp \|M_\omega(\mu)\|_{L^{\frac{p}{p-q}}_\omega} \asymp \|B^\omega_\mu\|_{L^{\frac{p}{p-q}}_\omega} + \mu(\{0\}) \asymp \|\Psi^\omega_\mu\|_{L^{\frac{p}{p-q}}_\omega} \asymp \|\Phi^\omega_\mu\|_{L^{\frac{p}{p-q}}_\omega} + \mu(\{0\}).$$

An analogue of the above result for Hardy spaces is essentially known. It can be obtained by using [40, Section 7] and [52]. Going further, the implications of the techniques used in this paper can be employed to extend the known results on the area operator on Hardy spaces [15,31] as well as to the study of the integral operator $T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)\,d\zeta$ acting on Hardy and Bergman spaces. These results are briefly discussed in Section 6.5.

6.2 CARLESON MEASURES

In this section we prove Theorem 3. For this aim we need some preliminary results and definitions. The following lemma provides useful characterizations of weights in $\widehat{\mathcal{D}}$. For a proof, see [51,58].

Lemma A. *Let ω be a radial weight. Then the following conditions are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) *There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that*

$$\widehat{\omega}(r) \le C\left(\frac{1-r}{1-t}\right)^{\beta}\widehat{\omega}(t), \quad 0 \le r \le t < 1; \tag{6.3}$$

(iii) *There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that*

$$\int_0^t \left(\frac{1-t}{1-s}\right)^{\gamma}\omega(s)\,ds \le C\widehat{\omega}(t), \quad 0 \le t < 1; \tag{6.4}$$

(iv) *There exists $\lambda = \lambda(\omega) \ge 0$ such that*

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{\zeta}z|^{\lambda+1}}\,dA(z) \asymp \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^{\lambda}}, \quad \zeta \in \mathbb{D};$$

(v) *The associated weight*

$$\omega^{\star}(z) = \int_{|z|}^{1} \omega(s)\log\frac{s}{|z|}s\,ds, \quad z \in \mathbb{D}\setminus\{0\}.$$

satisfies

$$\omega(S(z)) \asymp \omega(T(z)) \asymp \omega^{\star}(z), \quad |z| \to 1^{-}.$$

If $\omega \in \widehat{\mathcal{D}}$, then Lemma A shows that for each $a \in \mathbb{D}$ and $\gamma = \gamma(\omega) > 0$ large enough, the function

$$F_{a,p}(z) = \left(\frac{1-|a|^2}{1-\overline{a}z}\right)^{\frac{\gamma+1}{p}}, \quad z \in \mathbb{D},$$

belongs to A^p_ω and satisfies $\|F_{a,p}\|^p_{A^p_\omega} \asymp \omega(S(a))$ and $|F_{a,p}(z)| \asymp 1$ for all $z \in S(a)$. This family of test functions will be frequently used in the sequel.

Apart from the tent spaces $T^q_s(\nu,\omega)$, $0 < q, s < \infty$, defined in the introduction, we will need to consider the case $q = \infty$. For $0 < s < \infty$, define

$$C^s_{s,\nu}(f)(\zeta) = \sup_{a\in\Gamma(\zeta)} \frac{1}{\omega(T(a))}\int_{T(a)} |f(z)|^s\omega(T(z))\,d\nu(z).$$

A quasi-norm in the tent space $T^\infty_s(\nu,\omega)$ is defined by $\|f\|_{T^\infty_s(\nu,\omega)} = \|C_{s,\nu}(f)\|_{L^\infty}$.

The pseudohyperbolic distance from z to w is defined by $\varrho(z,w) = \left|\frac{1-w}{1-\overline{z}w}\right|$, and the pseudohyperbolic disc of center $a \in \mathbb{D}$ and radius $r \in (0,1)$ is denoted by

$\Delta(a,r) = \{z : \varrho(a,z) < r\}$. The Euclidean discs are denoted by $D(a,r) = \{z \in \mathbb{C} : |a-z| < r\}$. Recall that $Z = \{z_k\}_{k=0}^{\infty} \subset \mathbb{D}$ is called a separated sequence if it is separated in the pseudohyperbolic metric, it is an ε-net if $\mathbb{D} = \bigcup_{k=0}^{\infty} \Delta(z_k, \varepsilon)$, and finally it is a δ-lattice if it is a 5δ-net and separated with constant $\gamma = \delta/5$. If we have a discrete measure $\nu = \sum_k \delta_{z_k}$, where $\{z_k\}$ is a separated sequence, then we write $T_q^p(\nu,\omega) = T_q^p(\{z_k\},\omega)$.

Recall that $I_d : A^p_\omega \to L^q(\mu)$ is compact if it maps bounded sets of A^p_ω to relatively compact (precompact) sets of $L^q(\mu)$. Equivalently, $I_d : A^p_\omega \to L^q(\mu)$ is compact if and only if for every bounded sequence $\{f_n\}$ in A^p_ω there exists a subsequence that converges in $L^q(\mu)$.

Proof of Theorem 3. (i). There are several ways to bound the operator norm of $I_d : A^p_\omega \to L^q_\mu$ from above by the claimed supremum. See [55, Theorem 9] or [51, Theorem 3.3], and also [55, (11)] for the particular case $q = p$. The lower bound is obtained by using test functions, for details, see either [55, Lemma 8] or [51, Theorem 3.3].

(ii). This case can be done by following the proof of [54, Theorem 2.1(ii)], with Lemma A in hand.

(iii). We first show that

$$\|I_d\|^q_{A^p_\omega \to L^q_\mu} \le \|B^\omega_\mu\|_{L_\omega^{\frac{p}{p-q}}} + \mu(\{0\}) \asymp \|\Psi^\omega_\mu\|_{L_\omega^{\frac{p}{p-q}}} \asymp \|\Phi^\omega_\mu\|_{L_\omega^{\frac{p}{p-q}}} + \mu(\{0\}) \lesssim \|M_\omega(\mu)\|_{L_\omega^{\frac{p}{p-q}}} \lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu}$$

Fubini's theorem, Hölder's inequality and [54, Lemma 4.4] yield

$$\begin{aligned}
\|f\|^q_{L^q_\mu} &= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^q \frac{d\mu(\zeta)}{\omega(T(\zeta))} \right) \omega(z)\, dA(z) + \mu(\{0\})|f(0)|^q \\
&\le \int_{\mathbb{D}} N(f)^q(z) \int_{\Gamma(z)} \frac{d\mu(\zeta)}{\omega(T(\zeta))} \omega(z)\, dA(z) + \mu(\{0\})|f(0)|^q \\
&\lesssim \|N(f)\|^q_{L^p_\omega} \|B^\omega_\mu\|_{L_\omega^{\frac{p}{p-q}}} + \mu(\{0\})\|f\|^q_{A^p_\omega} \asymp \|f\|^q_{A^p_\omega} \left(\|B^\omega_\mu\|_{L_\omega^{\frac{p}{p-q}}} + \mu(\{0\} \right),
\end{aligned}$$

and hence $\|I_d\|^q_{A^p_\omega \to L^q_\mu} \lesssim \|B^\omega_\mu\|_{L_\omega^{\frac{p}{p-q}}} + \mu(\{0\})$. Moreover, $\|B^\omega_\mu\|_{L_\omega^{\frac{p}{p-q}}} + \mu(\{0\}) \asymp \|\Psi^\omega_\mu\|_{L_\omega^{\frac{p}{p-q}}}$ by [55, Lemma 4].

Now write $d\nu(z) = \frac{d\mu(z)}{\omega(T(z))}$ and $h_\lambda(\zeta,z) = \left(\frac{1-|\zeta|}{|1-\bar{\zeta}z|}\right)^\lambda$ for short. Then Fubini's theorem, [55, Lemma 4] and the fact $\omega(T(\zeta)) \asymp \omega(T(u))$ and $h_\lambda(\zeta,z) \asymp h_\lambda(u,z)$ for

$\zeta \in \Delta(u,r)$, $r \in (0,1)$, yield

$$
\begin{aligned}
\|\Psi_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}}^{\frac{p}{p-q}} &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{h_\lambda(\zeta,z)}{|\Delta(\zeta,r)|} |\Delta(\zeta,r)| \, d\nu(\zeta) \right)^{\frac{p}{p-q}} \omega(z) \, dA(z) \\
&\asymp \int_{\mathbb{D}} \left(\int_{\mathbb{D}} h_\lambda(u,z) \nu(\Delta(u,r)) \, dh(u) \right)^{\frac{p}{p-q}} \omega(z) \, dA(z) \\
&\asymp \int_{\mathbb{D}} \left(\int_{\Gamma(z)} \nu(\Delta(u,r)) \, dh(u) \right)^{\frac{p}{p-q}} \omega(z) \, dA(z) + \mu(\{0\}) \qquad (6.5) \\
&\asymp \int_{\mathbb{D}} \left(\int_{\Gamma(z)} \frac{\mu(\Delta(u,r))}{\omega(T(u))} \, dh(u) \right)^{\frac{p}{p-q}} \omega(z) \, dA(z) + \mu(\{0\}) \\
&= \|\Phi_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}}^{\frac{p}{p-q}} + \mu(\{0\}) = \|g\|_{T_1^{\frac{p}{p-q}}(h,\omega)}^{\frac{p}{p-q}} + \mu(\{0\}),
\end{aligned}
$$

where $g(u) = \frac{\mu(\Delta(u,r))}{\omega(T(u))}$. Now [55, Lemma 7] implies

$$
\|g\|_{T_1^{\frac{p}{p-q}}(h,\omega)} \asymp \|C_{1,h}(g)\|_{L_\omega^{\frac{p}{p-q}}}. \tag{6.6}
$$

Fubini's theorem gives

$$
\begin{aligned}
C_{1,h}(g)(\zeta) &= \sup_{a \in \Gamma(\zeta)} \frac{1}{\omega(T(a))} \int_{T(a)} \mu(\Delta(z,r)) \, dh(z) \\
&= \sup_{\zeta \in T(a)} \frac{1}{\omega(T(a))} \int_{\mathbb{D}} \left(\int_{T(a) \cap \Delta(u,r)} dh(z) \right) d\mu(u).
\end{aligned}
$$

The points $u \in \mathbb{D}$ for which $T(a) \cap \Delta(u,r) \neq \emptyset$ are contained in some tent $T(a')$, where $\arg a' = \arg a$ and $1 - |a'| \asymp 1 - |a|$, for all $a \in \mathbb{D} \setminus D(0,\rho)$, where $\rho = \rho(r) \in (0,1)$. Therefore

$$
\begin{aligned}
\frac{1}{\omega(T(a))} \int_{\mathbb{D}} \left(\int_{T(a) \cap \Delta(u,r)} dh(z) \right) d\mu(u) &= \frac{1}{\omega(T(a))} \int_{T(a')} \left(\int_{T(a) \cap \Delta(u,r)} dh(z) \right) d\mu(u) \\
&\lesssim \frac{\mu(T(a'))}{\omega(T(a))} \asymp \frac{\mu(T(a'))}{\omega(T(a'))}, \quad a \in \mathbb{D} \setminus D(0,\rho),
\end{aligned}
$$

and it follows that

$$
C_{1,h}(g)(\zeta) \asymp \sup_{\zeta \in T(a)} \frac{\mu(T(a))}{\omega(T(a))} \lesssim M_\omega(\mu)(\zeta), \quad \zeta \in \mathbb{D} \setminus \{0\}. \tag{6.7}
$$

By combining (6.5), (6.6) and (6.7), we deduce

$$
\begin{aligned}
\|\Psi_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}} &\asymp \|\Phi_\mu^\omega\|_{L_\omega^{\frac{p}{p-q}}} + \mu(\{0\}) = \|g\|_{T_1^{\frac{p}{p-q}}(h,\omega)} + \mu(\{0\}) \\
&\asymp \|C_{1,h}(g)\|_{L_\omega^{\frac{p}{p-q}}} + \mu(\{0\}) \lesssim \|M_\omega(\mu)\|_{L_\omega^{\frac{p}{p-q}}}.
\end{aligned}
$$

It remains to prove $\|M_\omega(\mu)\|_{L_\omega^{\frac{p}{p-q}}} \lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu}$. To do this, we will show that $M_\omega(\mu)$ is pointwise equivalent to the sum of two dyadic maximal functions.

Let $I_{n,k+l} = \{e^{i\theta} : \frac{\pi(k+l)}{2^{n+2}} \leq \theta < \frac{\pi(k+l+1)}{2^{n+2}}\}$, and denote $\Upsilon_l = \{I_{n,k+l} : n \in \mathbb{N} \cup \{0\}, k = 0, 1, \dots, 2^{n+2}-1\}$ and $l \in \{0, \frac{1}{2}\}$. Define the dyadic maximal functions

$$\widetilde{M}^d_{\omega,l}(\mu)(z) = \max\left\{\sup_{z\in T(I), I\in\Upsilon_l} \frac{\mu(T(I))}{\omega(T(I))}, \frac{\mu(\mathbb{D})}{\omega(\mathbb{D})}\right\}, \quad z \in \mathbb{D}, \quad l \in \left\{0, \frac{1}{2}\right\},$$

and set

$$\widetilde{M}^d_{\omega}(\mu)(z) = \widetilde{M}^d_{\omega,0}(\mu)(z) + \widetilde{M}^d_{\omega,\frac{1}{2}}(\mu)(z), \quad z \in \mathbb{D}.$$

If $\omega \in \widehat{\mathcal{D}}$, then $\widetilde{M}^d_{\omega}(\mu)(z) \lesssim M_\omega(\mu)(z)$ for all $z \in \mathbb{D}$ because $\sup_{I\subset\mathbb{T}} \frac{\omega(S(I))}{\omega(T(I))} < \infty$ by Lemma A. For the converse inequality, given $I \subset \mathbb{T}$ such that $z \in S(I)$ there exist intervals $I_{n,k}, I_{n,k+1}$ (if $k = 2^{n+2}$, take $I_{n,k} = I_{n,0}$) such that $|I_{n+1,0}| < |I| \leq |I_{n,0}|$, $I_{n,k} \cap I \neq \emptyset$ and $I_{n,k-1} \cap I = \emptyset$. We may assume that $n \geq 3$, for otherwise the inequality we are searching for is immediate. Then $I \subset I_{n,k} \cup I_{n,k+1}$, and there exists $I_{n-3,m} \in \Upsilon_0 \cup \Upsilon_{\frac{1}{2}}$ such that $\bigcup_{i=k-2}^{k+3} I_{n,i} \subset I_{n-3,m}$ and $S(I) \subset T(I_{n-3,m})$. Therefore Lemma A yields

$$\frac{\mu(S(I))}{\omega(S(I))} \leq \frac{\mu(T(I_{n-3,m}))}{\omega(T(I))} \leq \frac{\mu(T(I_{n-3,m}))}{\omega(T(I_{n+1,0}))} \asymp \frac{\mu(T(I_{n-3,m}))}{\omega(T(I_{n-3,m}))} \leq \widetilde{M}^d_{\omega}(\mu)(z), \quad z \in S(I).$$

It follows that $M_\omega(\mu)(z) \lesssim \widetilde{M}^d_{\omega}(\mu)(z)$, for all $z \in \mathbb{D}$, and hence

$$M_\omega(\mu)(z) \asymp \widetilde{M}^d_{\omega}(\mu)(z), \quad z \in \mathbb{D}. \tag{6.8}$$

To estimate the norm of $\widetilde{M}^d_{\omega}(\mu)$ upwards, let choose $\{z_k\}$ be a separated sequence and define

$$S_\lambda(f)(z) = \sum_k f(z_k)\left(\frac{1-|z_k|}{1-\bar{z}_k z}\right)^\lambda, \quad z \in \mathbb{D}.$$

By [55, Lemma 6] there exists $\lambda = \lambda(\omega) > 1$ such that $S_\lambda : T^p_2(\{z_k\}, \omega) \to A^p_\omega$ is bounded. By denoting $\{b_k\} = \{f(z_k)\}$, this implies

$$\int_{\mathbb{D}} \left|\sum_k b_k h_\lambda(z_k, z)\right|^q d\mu(z) = \|S_\lambda(f)\|^q_{L^q(\mu)} \lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu} \|S_\lambda\|^q_{T^p_2(\{z_k\},\omega)\to A^p_\omega} \|\{b_k\}\|^q_{T^p_2(\{z_k\},\omega)}.$$

By replacing b_k by $r_k(t)b_k$, where r_k denotes the kth Rademacher function, using the fact that $|h_\lambda(z_k, z)| \gtrsim \chi_{T(z_k)}(z)$ for $z \in T(z_k)$, and applying Khinchine's inequality, we deduce

$$\int_{\mathbb{D}} \left(\sum_k |b_k|^2 \chi_{T(z_k)}(z)\right)^{\frac{q}{2}} d\mu(z) \lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu} \|S_\lambda\|^q_{T^p_2(\{z_k\},\omega)\to A^p_\omega} \|\{b_k\}\|^q_{T^p_2(\{z_k\},\omega)}. \tag{6.9}$$

Let $l \in \{0, \frac{1}{2}\}$ be fixed. For each $k \in \mathbb{Z}$, let $\mathcal{E}_k$ denote the collection of maximal dyadic tents $T \in \{T(I) : I \in \Upsilon_l\} \cup \{\mathbb{D}\}$ with respect to inclusion such that $\mu(T) > 2^k\omega(T)$, and let $E_k = \cup_{T\in\mathcal{E}_k} T$. Then $2^k < \widetilde{M}^d_{\omega,l}(\mu)(z) \leq 2^{k+1}$ for $z \in E_k \setminus E_{k+1}$. Let now $\{b_T\}$ be a sequence indexed by $T \in \mathcal{E} = \cup_k \mathcal{E}_k$. Assume for a moment that μ has compact support. Then $\{b_T\}$ is a finite sequence. For $T \in \mathcal{E}$, let $G(T) = T - \cup\{T' \in$

$\mathcal{E} : T' \subsetneq T\}$, and hence $G(T) = T - \cup\{T' \in \mathcal{E}_{k+1} : T' \subsetneq T\}$ for $T \in \mathcal{E}_k$. If $T_1, T_2 \in \mathcal{E}$ are different (either one is strictly included in the other or they are disjoint), the sets $G(T_1)$ and $G(T_2)$ are disjoint, and hence

$$\left(\sum_{T\in\mathcal{E}} |b_T|^2 \chi_T(z)\right)^{\frac{q}{2}} \geq \left(\sum_{T\in\mathcal{E}} |b_T|^2 \chi_{G(T)}(z)\right)^{\frac{q}{2}} = \sum_{T\in\mathcal{E}} |b_T|^q \chi_{G(T)}(z). \tag{6.10}$$

Index the tents in $\mathcal{E}$ according to which $\mathcal{E}_k$ with maximal index they belong to, by writing $\mathcal{E}_k \setminus \cup_{m<k}\mathcal{E}_m = \{T_j^k : j \in \mathbb{N}\}$. Further, denote $b_{k,j} = b_{T_j^k}$ and let $z_{k,j}$ denote the vertex of T_j^k i.e. $T_j^k = T(z_{k,j})$ (with the convenience that the vertex of $\mathbb{D}$ is the origin). The estimates (8.14) and (8.26) yield

$$\begin{aligned}\sum_{k,j} b_{k,j}^q \left(\mu(T_j^k) - \sum_{T_i^{k+1}\subset T_j^k} \mu(T_i^{k+1})\right) &\lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu} \|\{b_{k,j}\}\|^q_{T_2^p(\{z_{k,j}\},\omega)} \\ &= \|I_d\|^q_{A^p_\omega \to L^q_\mu} \left(\int_{\mathbb{D}} \left(\sum_{k,j} |b_{k,j}|^2 \chi_{T_j^k}(z)\right)^{\frac{p}{2}} \omega(z)\, dA(z)\right)^{\frac{q}{p}}.\end{aligned} \tag{6.11}$$

Write $r = \frac{p}{q}$ for short, and choose $b_{k,j}^q = 2^{k(r'-1)}$ for each k and j. Then, by using the inequality $2^k < \widetilde{M}^d_{\omega,l}(\mu)(z) \leq 2^{k+1}$ for $z \in E_k \setminus E_{k+1}$, the left hand side of (8.16) can be estimated as

$$\begin{aligned}&\sum_{k,j} b_{k,j}^q \left(\mu(T_j^k) - \sum_{T_i^{k+1}\subset T_j^k} \mu(T_i^{k+1})\right) = \sum_k 2^{k(r'-1)} \mu(E_k) - \sum_k 2^{k(r'-1)} \sum_j \mu(T_j^k \cap E_{k+1}) \\ &= \sum_k (2^{k(r'-1)} - 2^{(k-1)(r'-1)}) \mu(E_k) = \left(1 - \frac{1}{2^{r'-1}}\right) \sum_k 2^{kr'} 2^{-k} \sum_j \frac{\mu(T_j^k)}{\omega(T_j^k)} \omega(T_j^k) \\ &\gtrsim \sum_k 2^{kr'} \omega(E_k) \gtrsim \sum_k \int_{E_k\setminus E_{k+1}} \left(\widetilde{M}^d_{\omega,l}(\mu)(z)\right)^{r'} \omega(z)\, dA(z) = \|\widetilde{M}^d_{\omega,l}(\mu)\|^{r'}_{L^{r'}_\omega},\end{aligned}$$

while the integral on the right hand side of (8.16) with the notation $\eta = 2^{(r'-1)\frac{2}{q}}$ becomes

$$\begin{aligned}&\int_{\mathbb{D}} \left(\sum_{k,j} |b_{k,j}|^2 \chi_{T_j^k}(z)\right)^{\frac{p}{2}} \omega(z)\, dA(z) = \int_{\mathbb{D}} \left(\sum_{k,j} \eta^k \chi_{T_j^k}(z)\right)^{\frac{p}{2}} \omega(z)\, dA(z) \\ &= \int_{\mathbb{D}} \left(\sum_k \eta^k \chi_{E_k}(z)\right)^{\frac{p}{2}} \omega(z)\, dA(z) = \int_{\mathbb{D}} \left(\frac{\eta}{\eta-1} \sum_k \left(\eta^k - \eta^{k-1}\right) \chi_{E_k}(z)\right)^{\frac{p}{2}} \omega(z)\, dA(z) \\ &\asymp \int_{\mathbb{D}} \left(\sum_k \eta^k \left(\chi_{E_k}(z) - \chi_{E_{k+1}}(z)\right)\right)^{\frac{p}{2}} \omega(z)\, dA(z) = \int_{\mathbb{D}} \left(\sum_k \eta^k \chi_{E_k\setminus E_{k+1}}(z)\right)^{\frac{p}{2}} \omega(z)\, dA(z) \\ &= \int_{\mathbb{D}} \sum_k \eta^{\frac{kp}{2}} \chi_{E_k\setminus E_{k+1}}(z) \omega(z)\, dA(z) = \int_{\mathbb{D}} \sum_k 2^{r'k} \chi_{E_k\setminus E_{k+1}}(z) \omega(z)\, dA(z) \\ &\asymp \sum_k \int_{E_k\setminus E_{k+1}} \left(\widetilde{M}^d_{\omega,l}(\mu)(z)\right)^{r'} \omega(z)\, dA(z) = \|\widetilde{M}^d_{\omega,l}(\mu)\|^{r'}_{L^{r'}_\omega}.\end{aligned}$$

Consequently, $\|\widetilde{M}^d_{\omega,l}(\mu)\|^{r'}_{L^{r'}_\omega} \lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu} \|\widetilde{M}^d_{\omega,l}(\mu)\|^{\frac{r'}{r}}_{L^{r'}_\omega}$, and thus $\|\widetilde{M}^d_{\omega,l}(\mu)\|_{L^{\frac{p}{p-q}}_\omega} \lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu}$ because $r' = \left(\frac{p}{q}\right)' = \frac{p}{p-q}$. Since this is valid for $l \in \{0, \frac{1}{2}\}$, using Minkowski's inequality and (6.8) we get

$$\|M_\omega(\mu)\|_{L^{\frac{p}{p-q}}_\omega} \asymp \|\widetilde{M}^d_\omega(\mu)\|_{L^{\frac{p}{p-q}}_\omega} \leq \|\widetilde{M}^d_{\omega,0}(\mu)\|_{L^{\frac{p}{p-q}}_\omega} + \|\widetilde{M}^d_{\omega,\frac{1}{2}}(\mu)\|_{L^{\frac{p}{p-q}}_\omega} \lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu}$$

for μ with compact support. If μ is positive, then the above estimate, applied to the compactly supported $\mu_r = \chi_{D(0,r)}\mu$, and the standard limiting argument with monotone convergence theorem gives $\|M_\omega(\mu)\|_{L^{\frac{p}{p-q}}_\omega} \lesssim \|I_d\|^q_{A^p_\omega \to L^q_\mu}$. Hence the claimed operator norm estimates are valid and, in particular, (b)-(e) are equivalent.

To complete the proof of (iii), it suffices to show that $I_d : A^p_\omega \to L^q_\mu$ is compact if (e) is satisfied. By the hypothesis (e), (6.5) and the dominated convergence theorem,

$$\begin{aligned}
0 &= \lim_{R\to 1^-} \int_{\mathbb{D}} \left(\int_{\{R<|z|<1\}} \left(\frac{1-|z|}{|1-\overline{\zeta}z|} \right)^\lambda \frac{d\mu(z)}{\omega(T(z))} \right)^{\frac{p}{p-q}} \omega(\zeta)\, dA(\zeta) \\
&\gtrsim \lim_{R\to 1^-} \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)\setminus\overline{D(0,R)}} \frac{d\mu(z)}{\omega(T(z))} \right)^{\frac{p}{p-q}} \omega(\zeta)\, dA(\zeta).
\end{aligned} \tag{6.12}$$

Let $\{f_n\}$ be a bounded sequence in A^p_ω. Then $\{f_n\}$ is locally bounded and thus constitutes a normal family. Hence we may extract a subsequence $\{f_{n_k}\}$ that converges uniformly on compact subsets of $\mathbb{D}$ to $f \in A^p_\omega$. Write $g_k = f_{n_k} - f$. For $\varepsilon > 0$, by (6.12), there exists $R_0 = R_0 \in (0,1)$ such that

$$\int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)\setminus\overline{D(0,R_0)}} \frac{d\mu(z)}{\omega(T(z))} \right)^{\frac{p}{p-q}} \omega(\zeta)\, dA(\zeta) < \varepsilon^{\frac{p}{p-q}}.$$

By the uniform convergence, we may choose $k_0 \in \mathbb{N}$ such that $|g_k(z)| < \varepsilon^{1/q}$ for all $k \geq k_0$ and $z \in \overline{D(0,R_0)}$. Then Fubini's theorem, Hölder's inequality and [54, Lemma 4.4] yield

$$\begin{aligned}
\|g_k\|^q_{L^q_\mu} &= \int_{\overline{D(0,R_0)}} |g_k(\zeta)|^q\, d\mu(\zeta) + \int_{\mathbb{D}\setminus\overline{D(0,R_0)}} |g_k(\zeta)|^q\, d\mu(\zeta) \\
&\leq \varepsilon\mu(\mathbb{D}) + \int_{\mathbb{D}} \left(\int_{\Gamma(z)\setminus\overline{D(0,R_0)}} |g_k(\zeta)|^q \frac{d\mu(\zeta)}{\omega(T(\zeta))} \right) \omega(z)\, dA(z) \\
&\leq \varepsilon\mu(\mathbb{D}) + \int_{\mathbb{D}} N(g_k)^q(z) \left(\int_{\Gamma(z)\setminus\overline{D(0,R_0)}} \frac{d\mu(\zeta)}{\omega(T(\zeta))} \right) \omega(z)\, dA(z) \\
&\leq \varepsilon\mu(\mathbb{D}) + \|N(g_k)\|^q_{L^p_\omega} \varepsilon \asymp \varepsilon\mu(\mathbb{D}) + \|g_k\|^q_{A^p_\omega} \varepsilon \lesssim \varepsilon,
\end{aligned}$$

and thus $I_d : A^p_\omega \to L^q_\mu$ is compact. □

6.3 BOUNDED OPERATOR

Theorem 2 is equivalent to the following result.

Theorem 4. *Let $0 < p, q, s < \infty$ such that $1 + \frac{s}{p} - \frac{s}{q} > 0$, $\omega \in \widehat{\mathcal{D}}$ and let μ, v be positive Borel measures on $\mathbb{D}$ such that $\mu\left(\{z \in \mathbb{D} : v(T(z)) = 0\}\right) = 0 = \mu(\{0\})$. Then the following assertions hold:*

(i) $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ *is bounded if and only if μ^ω_v is a $\left(p + s - \frac{ps}{q}\right)$-Carleson measure for A^p_ω. Moreover,*

$$\|G^v_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega} \asymp \|I_d\|^{p+s-\frac{ps}{q}}_{A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}}.$$

(ii) $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ *is compact if and only if $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}$ is compact.*

Theorem 4(i) will be proved in two parts. We first deal with the case $q \geq p$.

Theorem 5. *Let $0 < p \leq q < \infty$, $0 < s < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let μ, v be positive Borel measures on $\mathbb{D}$ such that $\mu\left(\{z \in \mathbb{D} : v(T(z)) = 0\}\right) = 0 = \mu(\{0\})$. Then $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded if and only if μ^ω_v is a $\left(p + s - \frac{ps}{q}\right)$-Carleson measure for A^p_ω. Moreover,*

$$\|G^v_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega} \asymp \|I_d\|^{p+s-\frac{ps}{q}}_{A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}} \asymp \sup_{a \in \mathbb{D}} \frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}}.$$

Proof. Let first $q > s$. Assume that $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded. Let $a \in \mathbb{D}$ and choose $\gamma = \gamma(p, q, s)$ sufficiently large so that $\|F_{a,p}\|^p_{A^p_\omega} \asymp \omega(S(a))$ and $\|F_{a,(\frac{q}{s})'}\|^{(\frac{q}{s})'}_{A^{(\frac{q}{s})'}_\omega} \asymp \omega(S(a))$. Then Fubini's theorem and Hölder's inequality yield

$$\begin{aligned}
\mu^\omega_v(S(a)) &\asymp \int_{S(a)} |F_{a,p}(z)|^s \, d\mu^\omega_v(z) \\
&\asymp \int_{S(a)} |F_{a,p}(z)|^s \left(\frac{1}{v(T(z))} \int_{T(z)} |F_{a,(\frac{q}{s})'}(\zeta)| \omega(\zeta) \, dA(\zeta) \right) d\mu(z) \\
&\lesssim \int_{\mathbb{D}} |F_{a,(\frac{q}{s})'}(\zeta)| \left(\int_{\Gamma(\zeta)} |F_{a,p}(z)|^s \frac{d\mu(z)}{v(T(z))} \right) \omega(\zeta) \, dA(\zeta) \\
&\leq \|F_{a,(\frac{q}{s})'}\|_{A^{(\frac{q}{s})'}_\omega} \|G^v_{\mu,s}(F_{a,p})\|^s_{L^q_\omega} \leq \|F_{a,(\frac{q}{s})'}\|_{A^{(\frac{q}{s})'}_\omega} \|G^v_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega} \|F_{a,p}\|^s_{A^p_\omega} \\
&\asymp \|G^v_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega} \omega(S(a))^{\frac{1}{(\frac{q}{s})'}} \omega(S(a))^{\frac{s}{p}} \asymp \|G^v_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega} \omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}.
\end{aligned}$$

Hence $\sup_{a \in \mathbb{D}} \frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}} \lesssim \|G^v_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega}$ and μ^ω_v is a $\left(p + s - \frac{ps}{q}\right)$-Carleson measure for A^p_ω by Theorem 3.

Conversely, let $q > s$ and μ^ω_v be a $\left(p + s - \frac{ps}{q}\right)$-Carleson measure for A^p_ω. Write $t = t(p, q, s) = p + s - \frac{ps}{q} > s$ for short. Since $\frac{t}{p} = (\frac{t}{s})'/(\frac{q}{s})'$, [55, Theorem 3] shows that $M_\omega : L^{(\frac{q}{s})'}_\omega \to L^{(\frac{t}{s})'}_{\mu^\omega_v}$ is bounded with $\|M_\omega\|^{(\frac{t}{s})'}_{L^{(\frac{q}{s})'}_\omega \to L^{(\frac{t}{s})'}_{\mu^\omega_v}} \asymp \sup_{a \in \mathbb{D}} \frac{\mu^\omega_v(S(a))}{\omega(S(a))^{t/p}}$. This together with Theorem 3, Fubini's theorem and Hölder's inequality give

$$
\begin{aligned}
\|G^v_{\mu,s}(f)\|^s_{L^q_\omega} &= \sup_{\|h\|_{L^{(\frac{q}{s})'}_\omega}\le 1} \int_{\mathbb{D}} |h(z)| \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\right)\omega(z)\,dA(z)\\
&= \sup_{\|h\|_{L^{(\frac{q}{s})'}_\omega}\le 1} \int_{\mathbb{D}} |f(\zeta)|^s \left(\frac{1}{\omega(T(\zeta))}\int_{T(\zeta)}|h(z)|\omega(z)\,dA(z)\right) d\mu^\omega_v(\zeta)\\
&\lesssim \sup_{\|h\|_{L^{(\frac{q}{s})'}_\omega}\le 1} \int_{\mathbb{D}} |f(\zeta)|^s M_\omega(h)(\zeta)\,d\mu^\omega_v(\zeta) \le \sup_{\|h\|_{L^{(\frac{q}{s})'}_\omega}\le 1} \|f\|^s_{L^t_{\mu^\omega_v}} \|M_\omega(h)\|_{L^{(\frac{t}{s})'}_{\mu^\omega_v}}\\
&\le \sup_{\|h\|_{L^{(\frac{q}{s})'}_\omega}\le 1} \|I_d\|^s_{A^p_\omega\to L^t_{\mu^\omega_v}} \|f\|^s_{A^p_\omega} \|M_\omega\|_{L^{(\frac{q}{s})'}_\omega \to L^{(\frac{t}{s})'}_{\mu^\omega_v}} \|h\|_{L^{(\frac{q}{s})'}_\omega}\\
&\asymp \left(\sup_{a\in\mathbb{D}} \frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}}\right)^{\frac{s}{t}+\frac{1}{(\frac{t}{s})'}} \|f\|^s_{A^p_\omega},
\end{aligned}
$$

and hence $G^v_{\mu,s}: A^p_\omega \to L^q_\omega$ is bounded and $\|G^v_{\mu,s}(f)\|^s_{L^q_\omega} \lesssim \sup_{a\in\mathbb{D}} \frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}}$.

Since the assertion is valid for $q = s$ by (6.1), it remains to consider the case $q < s$. Let first μ^ω_v be a $\left(p + s - \frac{ps}{q}\right)$-Carleson measure for A^p_ω, and let $0 < x < s$. Hölder's inequality and Fubini's theorem yield

$$
\begin{aligned}
\|G^v_{\mu,s}(f)\|^q_{L^q_\omega} &= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^{x+s-x} \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\right)^{\frac{q}{s}} \omega(z)\,dA(z)\\
&\le \int_{\mathbb{D}} N(f)(z)^{\frac{qx}{s}} \left(\int_{\Gamma(z)} |f(\zeta)|^{s-x} \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\right)^{\frac{q}{s}} \omega(z)\,dA(z)\\
&\le \left(\int_{\mathbb{D}} N(f)(z)^{\frac{qx}{s-q}}\omega(z)\,dA(z)\right)^{1-\frac{q}{s}} \left(\int_{\mathbb{D}}\int_{\Gamma(z)} |f(\zeta)|^{s-x} \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\omega(z)\,dA(z)\right)^{\frac{q}{s}}\\
&\lesssim \|N(f)\|^{\frac{qx}{s}}_{L^{\frac{qx}{s-q}}_\omega} \left(\int_{\mathbb{D}} |f(\zeta)|^{s-x}\,d\mu^\omega_v(\zeta)\right)^{\frac{q}{s}} \asymp \|N(f)\|^{\frac{qx}{s}}_{L^{\frac{qx}{s-q}}_\omega} \|f\|^{\frac{q(s-x)}{s}}_{L^{(s-x)}_{\mu^\omega_v}}.
\end{aligned}
$$

Take $x = \frac{p(s-q)}{q} < s$ so that $s - x = s + p - \frac{ps}{q}$. Then the estimates above together with [54, Lemma 4.4] and Theorem 3 give

$$
\|G^v_{\mu,s}(f)\|^q_{L^q_\omega} \lesssim \|f\|^{\frac{qx}{s}}_{A^p_\omega} \|I_d\|^{\frac{q(s-x)}{s}}_{A^p_\omega \to L^{s+p-\frac{ps}{q}}_{\mu^\omega_v}} \|f\|^{\frac{q(s-x)}{s}}_{A^p_\omega} \asymp \left(\sup_{a\in\mathbb{D}} \frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}}\right)^{\frac{q}{s}} \|f\|^q_{A^p_\omega},
$$

and hence $G^v_{\mu,s}: A^p_\omega \to L^q_\omega$ is bounded with $\|G^v_\mu\|^s_{A^p_\omega\to L^q_\omega} \lesssim \sup_{a\in\mathbb{D}} \frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}}$.

Conversely, let $q < s$ and $G^v_{\mu,s}: A^p_\omega \to L^q_\omega$ be bounded. Choose $\alpha > \beta > 1$ such that $\frac{\beta}{\alpha} = \frac{q}{s}$. Fubini's theorem and Hölder's inequality yield

$$
\begin{aligned}
\mu_v^{\omega}(S(a)) &= \int_{\mathbb{D}} \chi_{S(a)}(z)\, d\mu_v^{\omega}(z) \asymp \int_{\mathbb{D}} \chi_{S(a)}(z) |F_{a,p}(z)|^s \frac{1}{v(T(z))} \int_{T(z)} \omega(\zeta)\, dA(\zeta)\, d\mu(z) \\
&= \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} \chi_{S(a)}(z) |F_{a,p}(z)|^s \frac{d\mu(z)}{v(T(z))} \right)^{\frac{1}{\alpha}+\frac{1}{\alpha'}} \omega(\zeta)\, dA(\zeta) \\
&\leq \left(\int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |F_{a,p}(z)|^s \frac{d\mu(z)}{v(T(z))} \right)^{\frac{\beta}{\alpha}} \omega(\zeta)\, dA(\zeta) \right)^{\frac{1}{\beta}} \\
&\quad \cdot \left(\int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} \chi_{S(a)}(z) \frac{d\mu(z)}{v(T(z))} \right)^{\frac{\beta'}{\alpha'}} \omega(\zeta)\, dA(\zeta) \right)^{\frac{1}{\beta'}} \\
&= \|G_{\mu,s}^{v}(F_{a,p})\|_{L_\omega^q}^{\frac{q}{\beta}} \|G_{\mu,1}^{v}(\chi_{S(a)})\|_{L_\omega^{\frac{\beta'}{\alpha'}}}^{\frac{1}{\alpha'}}.
\end{aligned}
\tag{6.13}
$$

Now $\frac{\beta'}{\alpha'} > 1$ because $\beta < \alpha$, and hence $\|G_{\mu,1}^{v}(\chi_{S(a)})\|_{L_\omega^{\frac{\beta'}{\alpha'}}}$ can be estimated by duality arguments. Namely, since $\left(\frac{\beta'}{\alpha'}\right)' = \frac{\beta(\alpha-1)}{\alpha-\beta}$, Fubini's theorem, Hölder's inequality and [55, Theorem 3] give

$$
\begin{aligned}
\|G^v_{\mu,1}(\chi_{S(a)})\|_{L^{\frac{\beta'}{\alpha'}}_\omega} &= \sup_{\|h\|_{L^{\frac{\beta(\alpha-1)}{\alpha-\beta}}_\omega}\le 1} \int_{\mathbb{D}} |h(\zeta)| G^v_{\mu,1}(\chi_{S(a)})(\zeta)\omega(\zeta)\,dA(\zeta) \\
&= \sup_{\|h\|_{L^{\frac{\beta(\alpha-1)}{\alpha-\beta}}_\omega}\le 1} \int_{\mathbb{D}} |h(\zeta)| \left(\int_{\Gamma(\zeta)} \chi_{S(a)}(z)\frac{d\mu(z)}{v(T(z))}\right)\omega(\zeta)\,dA(\zeta) \\
&= \sup_{\|h\|_{L^{\frac{\beta(\alpha-1)}{\alpha-\beta}}_\omega}\le 1} \int_{\mathbb{D}} \chi_{S(a)}(z) \left(\frac{1}{v(T(z))}\int_{T(z)} |h(\zeta)|\omega(\zeta)\,dA(\zeta)\right) d\mu(z) \\
&= \sup_{\|h\|_{L^{\frac{\beta(\alpha-1)}{\alpha-\beta}}_\omega}\le 1} \int_{\mathbb{D}} \chi_{S(a)}(z) \left(\frac{1}{\omega(T(z))}\int_{T(z)} |h(\zeta)|\omega(\zeta)\,dA(\zeta)\right) d\mu^\omega_v(z) \\
&\lesssim \sup_{\|h\|_{L^{\frac{\beta(\alpha-1)}{\alpha-\beta}}_\omega}\le 1} \int_{\mathbb{D}} \chi_{S(a)}(z) M_\omega(h)(z)\,d\mu^\omega_v(z) \\
&\le \sup_{\|h\|_{L^{\frac{\beta(\alpha-1)}{\alpha-\beta}}_\omega}\le 1} \left(\int_{\mathbb{D}} \chi_{S(a)}(z)\,d\mu^\omega_v(z)\right)^{\frac{\alpha'}{\beta'}} \\
&\quad\cdot \left(\int_{\mathbb{D}} M_\omega(h)(z)^{\frac{\beta(\alpha-1)}{\alpha-\beta}} \chi_{S(a)}(z)\,d\mu^\omega_v(z)\right)^{1-\frac{\alpha'}{\beta'}} \\
&\lesssim \mu^\omega_v(S(a))^{\frac{\alpha'}{\beta'}} \left(\sup_{z\in\mathbb{D}} \frac{\mu^\omega_v(S(z)\cap S(a))}{\omega(S(z))}\right)^{1-\frac{\alpha'}{\beta'}}, \quad a\in\mathbb{D}.
\end{aligned}
\tag{6.14}
$$

By combining this with (6.13) and using the norm estimate $\|F_{a,p}\|^p_{A^p_\omega} \asymp \omega(S(a))$, we deduce

$$
\begin{aligned}
\mu^\omega_v(S(a)) &\lesssim \|G^v_{\mu,s}(F_{a,p})\|^{\frac{q}{\beta}}_{L^q_\omega} \|G^v_{\mu,1}(\chi_{S(a)})\|^{\frac{1}{\alpha'}}_{L^{\frac{\beta'}{\alpha'}}_\omega} \\
&\lesssim \|G^v_{\mu,s}\|^{\frac{q}{\beta}}_{A^p_\omega\to L^q_\omega} \omega(S(a))^{\frac{q}{p\beta}} \mu^\omega_v(S(a))^{\frac{1}{\beta'}} \left(\sup_{z\in\mathbb{D}} \frac{\mu^\omega_v(S(z)\cap S(a))}{\omega(S(z))}\right)^{\frac{1}{\alpha'}-\frac{1}{\beta'}},
\end{aligned}
$$

which yields

$$
\mu^\omega_v(S(a))^{\frac{1}{\beta}} \omega(S(a))^{-\frac{q}{p\beta}} \lesssim \|G^v_{\mu,s}\|^{\frac{q}{\beta}}_{A^p_\omega\to L^q_\omega} \left(\sup_{z\in\mathbb{D}} \frac{\mu^\omega_v(S(z)\cap S(a))}{\omega(S(z))}\right)^{\frac{1}{\alpha'}-\frac{1}{\beta'}}, \quad a\in\mathbb{D}.
\tag{6.15}
$$

Define $d\mu_r(z) = \chi_{D(0,r)}(z)d\mu(z)$ for $0 < r < 1$. Then

$$\begin{aligned}\left(G^v_{\mu_r,s}(f)(z)\right)^s &= \int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu_r(\zeta)}{v(T(\zeta))} = \int_{\Gamma(z)\cap D(0,r)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))} \\ &\leq \int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))} = \left(G^v_{\mu,s}(f)(z)\right)^s, \quad z \in \mathbb{D}\setminus\{0\},\end{aligned} \tag{6.16}$$

and hence $\|G^v_{\mu_r,s}\|_{A^p_\omega\to L^q_\omega} \leq \|G^v_{\mu,s}\|_{A^p_\omega\to L^q_\omega}$ for all $0 < r < 1$.

If $q = p$, then (6.15) applied to μ_r implies

$$(\mu^\omega_v)_r(S(a))^{\frac{1}{\beta}}\omega(S(a))^{-\frac{1}{\beta}} \lesssim \|G^v_{\mu_r,s}\|^{\frac{p}{\beta}}_{A^p_\omega\to L^p_\omega}\left(\sup_{z\in\mathbb{D}}\frac{(\mu^\omega_v)_r(S(z))}{\omega(S(z))}\right)^{\frac{1}{\alpha'}-\frac{1}{\beta'}}, \quad a \in \mathbb{D},$$

and hence

$$\left(\sup_{z\in\mathbb{D}}\frac{(\mu^\omega_v)_r(S(z))}{\omega(S(z))}\right)^{\frac{1}{\beta}} \lesssim \|G^v_{\mu,s}\|^{\frac{p}{\beta}}_{A^p_\omega\to L^p_\omega}\left(\sup_{z\in\mathbb{D}}\frac{(\mu^\omega_v)_r(S(z))}{\omega(S(z))}\right)^{\frac{1}{\alpha'}-\frac{1}{\beta'}}.$$

Consequently,

$$\sup_{z\in\mathbb{D},r\in(0,1)}\frac{(\mu^\omega_v)_r(S(z))}{\omega(S(z))} \lesssim \|G^v_{\mu,s}\|^s_{A^p_\omega\to L^p_\omega}.$$

So Fatou's lemma and Theorem 3 show that μ^ω_v is a p-Carleson measure for A^p_ω with

$$\|I_d\|^p_{A^p_\omega\to L^p_{\mu^\omega_v}} \asymp \sup_{a\in\mathbb{D}}\frac{\mu^\omega_v(S(a))}{\omega(S(a))} \lesssim \|G^v_{\mu,s}\|^s_{A^p_\omega\to L^p_\omega}.$$

If $q > p$, then applying (6.15) to μ_r and bearing in mind that ω is radial

$$\begin{aligned}\frac{(\mu^\omega_v)_r(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}} &\lesssim \|G^v_{\mu_r,s}\|^q_{A^p_\omega\to L^q_\omega}\omega(S(a))^{\frac{q}{p}-1-\frac{s}{p}+\frac{s}{q}}\left(\sup_{z\in\mathbb{D}}\frac{(\mu^\omega_v)_r(S(z)\cap S(a))}{\omega(S(z))}\right)^{\frac{\beta}{\alpha'}-\frac{\beta}{\beta'}} \\ &= \|G^v_{\mu_r,s}\|^q_{A^p_\omega\to L^q_\omega}\omega(S(a))^{\frac{-(q-p)(s-q)}{pq}}\left(\sup_{z:S(z)\subset S(a)}\frac{(\mu^\omega_v)_r(S(z)\cap S(a))}{\omega(S(z))}\right)^{1-\frac{q}{s}} \\ &= \|G^v_{\mu_r,s}\|^q_{A^p_\omega\to L^q_\omega}\left(\sup_{z:S(z)\subset S(a)}\frac{(\mu^\omega_v)_r(S(z)\cap S(a))}{\omega(S(a))^{s\frac{q-p}{pq}}\omega(S(z))}\right)^{1-\frac{q}{s}} \\ &\leq \|G^v_{\mu_r,s}\|^q_{A^p_\omega\to L^q_\omega}\left(\sup_{z:S(z)\subset S(a)}\frac{(\mu^\omega_v)_r(S(z))}{\omega(S(z))^{1+\frac{s}{p}-\frac{s}{q}}}\right)^{1-\frac{q}{s}}, \quad a \in \mathbb{D}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sup_{a\in\mathbb{D}}\left(\frac{(\mu^\omega_v)_r(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}}\right) &\lesssim \|G^v_{\mu,s}\|^q_{A^p_\omega\to L^q_\omega}\left(\sup_{a\in\mathbb{D}}\sup_{z:S(z)\subset S(a)}\frac{(\mu^\omega_v)_r(S(z))}{\omega(S(z))^{1+\frac{s}{p}-\frac{s}{q}}}\right)^{1-\frac{q}{s}} \\ &= \|G^v_{\mu,s}\|^q_{A^p_\omega\to L^q_\omega}\left(\sup_{a\in\mathbb{D}}\frac{(\mu^\omega_v)_r(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}}\right)^{1-\frac{q}{s}},\end{aligned}$$

and thus

$$\sup_{a\in\mathbb{D},\, r\in(0,1)} \frac{(\mu_v^\omega)_r(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}} \lesssim \|G^v_{\mu,s}\|^s_{A^p_\omega\to L^q_\omega}.$$

Fatou's lemma and Theorem 3 show that μ_v^ω is a $\left(p+s-\frac{ps}{q}\right)$-Carleson measure for A^p_ω with the corresponding inequality of norms. □

Theorem 6. *Let $0<q<p<\infty$ and $0<s<\infty$ such that $1+\frac{s}{p}-\frac{s}{q}>0$, $\omega\in\widehat{\mathcal{D}}$ and let μ, v be positive Borel measures on $\mathbb{D}$ such that $\mu\left(\{z\in\mathbb{D} : v(T(z))=0\}\right)=0=\mu(\{0\})$. Then $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded if and only if μ_v^ω is a $p\left(1+\frac{s}{p}-\frac{s}{q}\right)$-Carleson measure for A^p_ω. Moreover,*

$$\|G^v_{\mu,s}\|^s_{A^p_\omega\to L^q_\omega} \asymp \|I_d\|^{p+s-\frac{ps}{q}}_{A^p_\omega\to L^{p+s-\frac{ps}{q}}_{\mu_v^\omega}} \asymp \|B^v_\mu\|_{L_\omega^{\frac{qp}{s(p-q)}}}.$$

Proof. The equivalence $\|I_d\|^{p+s-\frac{ps}{q}}_{A^p_\omega\to L^{p+s-\frac{ps}{q}}_{\mu_v^\omega}} \asymp \|B^v_\mu\|_{L_\omega^{\frac{qp}{s(p-q)}}}$ follows from Theorem 3.

If $B^v_\mu \in L_\omega^{\frac{qp}{s(p-q)}}$, then Hölder's inequality and [54, Lemma 4.4] give

$$\begin{aligned}
\|G^v_{\mu,s}(f)\|^q_{L^q_\omega} &= \int_{\mathbb{D}} \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))}\right)^{\frac{q}{s}} \omega(z)\, dA(z)\\
&\le \int_{\mathbb{D}} N(f)^q(z) \left(\int_{\Gamma(z)} \frac{d\mu(\zeta)}{v(T(\zeta))}\right)^{\frac{q}{s}} \omega(z)\, dA(z)\\
&\le \|N(f)\|^q_{L^p_\omega} \|B^v_\mu\|^{\frac{q}{s}}_{L_\omega^{\frac{qp}{s(p-q)}}} \asymp \|f\|^q_{A^p_\omega} \|B^v_\mu\|^{\frac{q}{s}}_{L_\omega^{\frac{qp}{s(p-q)}}},
\end{aligned}$$

and hence $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded and $\|G^v_{\mu,s}\|^s_{A^p_\omega\to L^q_\omega} \lesssim \|B^v_\mu\|_{L_\omega^{\frac{qp}{p-q}}}$.

Assume now that $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded, and let first $q>s$ and write $t=t(p,q,s)=(p+s-\frac{ps}{q})=s+p(1-\frac{s}{q})$. It suffices to show that μ_v^ω is a t-Carleson measure for A^p_ω. Fubini's theorem, Hölder's inequality and [54, Lemma 4.4] yield

$$\begin{aligned}
\|f\|^t_{L^t_{\mu_v^\omega}} &= \int_{\mathbb{D}} |f(z)|^t \left(\int_{T(z)} \omega(\zeta)\, dA(\zeta)\right) \frac{d\mu(z)}{v(T(z))}\\
&= \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |f(z)|^{s+p(1-\frac{s}{q})} \frac{d\mu(z)}{v(T(z))}\right) \omega(\zeta)\, dA(\zeta)\\
&\le \int_{\mathbb{D}} N(f)^{p(1-\frac{s}{q})}(\zeta) \left(\int_{\Gamma(\zeta)} |f(z)|^s \frac{d\mu(z)}{v(T(z))}\right) \omega(\zeta)\, dA(\zeta)\\
&\lesssim \|N(f)\|^{p(1-\frac{s}{q})}_{L^p_\omega} \|G^v_{\mu,s}(f)\|^s_{L^q_\omega} \asymp \|f\|^{p(1-\frac{s}{q})}_{A^p_\omega} \|G^v_{\mu,s}(f)\|^s_{L^q_\omega}\\
&\le \|G^v_\mu\|^s_{A^p_\omega\to L^q_\omega} \|f\|^t_{A^p_\omega}
\end{aligned}$$

and hence $I_d : A^p_\omega \to L^t_{\mu_v^\omega}$ is bounded with $\|I_d\|^t_{A^p_\omega\to L^t_{\mu_v^\omega}} \lesssim \|G^v_\mu\|^s_{A^p_\omega\to L^q_\omega}$.

If $q=s$, then $t=s$ and the result follows from (6.1).

If $q<s$, then $t=p+s-\frac{ps}{q}<s$ and $\frac{q}{t}>1$ because $\min\{p,s\}>q$. Hence Hölder's inequality gives

$$
\begin{aligned}
\|f\|^t_{L^t_{\mu^\omega_v}} &= \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |f(z)|^t \frac{d\mu(z)}{v(T(z))} \right) \omega(\zeta)\, dA(\zeta) \\
&\lesssim \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |f(z)|^s \frac{d\mu(z)}{v(T(z))} \right)^{\frac{t}{s}} \left(\int_{\Gamma(\zeta)} \frac{d\mu(z)}{v(T(z))} \right)^{1-\frac{t}{s}} \omega(\zeta)\, dA(\zeta) \\
&\le \left(\int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |f(z)|^s \frac{d\mu(z)}{v(T(z))} \right)^{\frac{q}{s}} \omega(\zeta)\, dA(\zeta) \right)^{\frac{t}{q}} \\
&\cdot \left(\int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} \frac{d\mu(z)}{v(T(z))} \right)^{\frac{s-t}{s}\frac{q}{q-t}} \omega(\zeta)\, dA(\zeta) \right)^{\frac{q-t}{q}} \\
&= \|G^v_{\mu,s}(f)\|^t_{L^q_\omega} \|B^v_\mu\|^{1-\frac{t}{s}}_{L^{\frac{qp}{s(p-q)}}_\omega} \\
&\le \|G^v_{\mu,s}\|^t_{A^p_\omega \to L^q_\omega} \|B^v_{\mu,s}\|^{1-\frac{t}{s}}_{L^{\frac{qp}{s(p-q)}}_\omega} \|f\|^t_{A^p_\omega}.
\end{aligned}
$$

This applied to μ_r yields

$$
\left(\frac{\|f\|_{L^t_{(\mu^\omega_v)_r}}}{\|f\|_{A^p_\omega}} \right)^t \lesssim \|G^v_{\mu_r,s}\|^t_{A^p_\omega \to L^q_\omega} \|B^v_{\mu_r}\|^{1-\frac{t}{s}}_{L^{\frac{qp}{s(p-q)}}_\omega}, \quad f \in A^p_\omega, \quad f \not\equiv 0,
$$

and hence

$$
\|I_d\|^t_{A^p_\omega \to L^t_{(\mu^\omega_v)_r}} \le \|G^v_{\mu_r,s}\|^t_{A^p_\omega \to L^q_\omega} \|B^v_{\mu_r}\|^{1-\frac{t}{s}}_{L^{\frac{qp}{s(p-q)}}_\omega}.
$$

Since $\|I_d\|^t_{A^p_\omega \to L^t_{(\mu^\omega_v)_r}} \asymp \|B^v_{\mu_r}\|_{L^{\frac{qp}{s(p-q)}}_\omega}$ by Theorem 3, we deduce

$$
\|B^v_{\mu_r}\|_{L^{\frac{qp}{s(p-q)}}_\omega} \lesssim \|G^v_{\mu_r,s}\|^t_{A^p_\omega \to L^q_\omega} \|B^v_{\mu_r}\|^{1-\frac{t}{s}}_{L^{\frac{qp}{s(p-q)}}_\omega},
$$

and hence

$$
\|B^v_{\mu_r}\|_{L^{\frac{qp}{s(p-q)}}_\omega} \lesssim \|G^v_{\mu_r,s}\|^s_{A^p_\omega \to L^q_\omega} \le \|G^v_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega},
$$

which together with Fatou's lemma gives

$$
\|B^v_\mu\|_{L^{\frac{qp}{s(p-q)}}_\omega} \le \liminf_{r\to 1} \|B^v_{\mu_r}\|_{L^{\frac{qp}{s(p-q)}}_\omega} \lesssim \|G^v_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega}.
$$

This finishes the proof. □

6.4 COMPACT OPERATOR

In this section we prove Theorem 4(ii).

Lemma 1. *Let ν be a finite positive Borel measure on $\mathbb{D}$ and $0 < p < \infty$. If $\{\varphi_n\}_{n=0}^{\infty} \subset L^p_\nu$ and $\varphi \in L^p_\nu$ satisfy $\lim_{n\to\infty} \|\varphi_n\|_{L^p_\nu} = \|\varphi\|_{L^p_\nu}$ and $\lim_{n\to\infty} \varphi_n(z) = \varphi(z)$ ν-a.e. on $\mathbb{D}$, then $\lim_{n\to\infty} \|\varphi_n - \varphi\|_{L^p_\nu} = 0$.*

Proof. See the proof of [25, Lemma 1 p. 21]. $\square$

Theorem 7. *Let $0 < p \le q < \infty$, $0 < s < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let μ, v be positive Borel measures on $\mathbb{D}$ such that $\mu(\{z \in \mathbb{D} : v(T(z)) = 0\}) = 0 = \mu(\{0\})$. Then $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is compact if and only if $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}$ is compact.*

Proof. Let first $q > s$. Assume that $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is compact. For each $a \in \mathbb{D}$, let $f_{a,p}(z) = (\omega(S(a)))^{-\frac{1}{p}} F_{a,p}(z) = (\omega(S(a)))^{-\frac{1}{p}} \left(\frac{1-|a|^2}{1-\bar{a}z}\right)^{\frac{\gamma+1}{p}}$. By Lemma A we may choose $\gamma = \gamma(p,q,s,\omega)$ sufficiently large such that $\sup_{a\in\mathbb{D}} \|f_{a,p}\|^p_{A^p_\omega} \asymp 1$, $\sup_{a\in\mathbb{D}} \|f_{a,(\frac{q}{s})'}\|^{(\frac{q}{s})'}_{A^{(\frac{q}{s})'}_\omega} \asymp 1$ and $f_{a,p}$ converges uniformly to zero on compact subsets of $\mathbb{D}$ as $|a| \to 1^-$. A standard argument shows that

$$\lim_{|a|\to 1^-} \|G^v_{\mu,s}(f_{a,p})\|_{L^q_\omega} = 0. \tag{6.17}$$

Fubini's theorem and Hölder's inequality yield

$$\begin{aligned}
\frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}} &\asymp \frac{1}{\omega(S(a))^{1-\frac{s}{q}}} \int_{S(a)} |f_{a,p}(z)|^s \, d\mu^\omega_v(z) \\
&\asymp \int_{S(a)} |f_{a,p}(z)|^s \left(\frac{1}{v(T(z))} \int_{T(z)} |f_{a,(\frac{q}{s})'}(\zeta)| \omega(\zeta) \, dA(\zeta) \right) d\mu(z) \\
&\lesssim \int_{\mathbb{D}} |f_{a,(\frac{q}{s})'}(\zeta)| \left(\int_{\Gamma(\zeta)} |f_{a,p}(z)|^s \frac{d\mu(z)}{v(T(z))} \right) \omega(\zeta) \, dA(\zeta) \\
&\le \|f_{a,(\frac{q}{s})'}\|_{A^{(\frac{q}{s})'}_\omega} \|G^v_{\mu,s}(f_{a,p})\|^s_{L^q_\omega} \lesssim \|G^v_{\mu,s}(f_{a,p})\|^s_{L^q_\omega}.
\end{aligned}$$

This together with (6.17) gives $\lim_{|a|\to 1^-} \frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}} = 0$ and hence $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}$ is compact by Theorem 3.

Conversely, let $q > s$ and assume that $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}$ is compact. Write $t = t(p,q,s) = p + s - \frac{ps}{q} > s$ for short. Let $\{f_n\}$ be a bounded sequence in A^p_ω. Then, we may extract a subsequence $\{f_{n_k}\}$ that converges in $L^t_{\mu^\omega_v}$ and uniformly on compact subsets to some $f \in A^p_\omega$. Write $g_{n_k} = f_{n_k} - f$. By Fubini's theorem, [55, Theorem 3] and Hölder's inequality,

$$\|G^v_{\mu,s}(g_{n_k})\|^s_{L^q_\omega} = \sup_{\|h\|_{L^{(\frac{q}{s})'}_\omega} \le 1} \int_{\mathbb{D}} |h(z)| \left(\int_{\Gamma(z)} |g_{n_k}(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))} \right) \omega(z) \, dA(z) \lesssim \|g_{n_k}\|^s_{L^t_{\mu^\omega_v}}$$

and hence

$$\lim_{k\to\infty} \|G^v_{\mu,s}(g_{n_k})\|^s_{L^q_\omega} = 0.$$

If $s \geq 1$, two applications of Minkowski's inequality gives

$$\left| \|G^v_{\mu,s}(f_{n_k})\|_{L^q_\omega} - \|G^v_{\mu,s}(f)\|_{L^q_\omega} \right| \leq \|G^v_{\mu,s}(g_{n_k})\|_{L^q_\omega} \to 0, \quad k \to \infty.$$

Moreover, since $\{f_{n_k}\}$ converges uniformly on compact subsets of $\mathbb{D}$ to f, then $\varphi_k(z) = \left(\int_{\Gamma(z)} |f_{n_k}(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\right)^{1/s}$ converges to $\varphi(z) = \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\right)^{1/s}$ for each $z \in \mathbb{D}$. Therefore Lemma 1 yields

$$\lim_{k\to\infty} \|G^v_{\mu,s}(f_{n_k}) - G^v_{\mu,s}(f)\|_{L^q_\omega} = \lim_{k\to\infty} \|\varphi_k - \varphi\|_{L^q_\omega} = 0,$$

and thus $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is compact.

If $0 < s \leq 1$,

$$\int_{\Gamma(z)} |f_{n_k}(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))} \leq \int_{\Gamma(z)} |g_{n_k}(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))} + \int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))},$$

which together with Minkowski's inequality yields

$$\left| \|G^v_{\mu,s}(f_{n_k})\|^s_{L^q_\omega} - \|G^v_{\mu,s}(f)\|^s_{L^q_\omega} \right| \leq \|G^v_{\mu,s}(g_{n_k})\|^s_{L^q_\omega}.$$

Now, by arguing as in the previous case we see that $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is compact. In view of (6.1), a similar reasoning also applies in the case $q = s$.

It remains to consider the case $q < s$. Assume first that $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}$ is compact. Let $\{f_n\}$ be a bounded sequence in A^p_ω. Then we may extract a subsequence $\{f_{n_k}\}$ that converges on $L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}$ and uniformly on compact subsets of $\mathbb{D}$ to some $f \in A^p_\omega$. Write $g_{n_k} = f_{n_k} - f$, and let $0 < x < s$. Hölder's inequality and Fubini's theorem yield

$$\begin{aligned} \|G^v_{\mu,s}(g_{n_k})\|^q_{L^q_\omega} &\leq \int_{\mathbb{D}} N(g_{n_k})(z)^{\frac{qx}{s}} \left(\int_{\Gamma(z)} |g_{n_k}(\zeta)|^{s-x} \frac{d\mu^\omega_v(\zeta)}{\omega(T(\zeta))}\right)^{\frac{q}{s}} \omega(z)\, dA(z) \\ &\lesssim \|N(g_{n_k})\|^{\frac{qx}{s}}_{L^{\frac{qx}{s-q}}_\omega} \|g_{n_k}\|^{\frac{q(s-x)}{s}}_{L^{(s-x)}_{\mu^\omega_v}}. \end{aligned}$$

Take $x = \frac{p(s-q)}{q} < s$ so that $s - x = s + p - \frac{ps}{q}$. Then the estimates above together with [54, Lemma 4.4] give

$$\|G^v_{\mu,s}(g_{n_k})\|^q_{L^q_\omega} \lesssim \|g_{n_k}\|^{\frac{qx}{s}}_{A^p_\omega} \|g_{n_k}\|^{\frac{q(s-x)}{s}}_{L^{s+p-\frac{ps}{q}}_{\mu^\omega_v}} \lesssim \|g_{n_k}\|^{\frac{q(s-x)}{s}}_{L^{s+p-\frac{ps}{q}}_{\mu^\omega_v}},$$

and hence

$$\lim_{k\to\infty} \|G^v_{\mu,s}(g_{n_k})\|_{L^q_\omega} = 0.$$

Now, by using Lemma 1 and arguing as in the case $q > s$, we conclude that $\lim_{k\to\infty} \|G^v_{\mu,s}(f_{n_k}) - G^v_{\mu,s}(f)\|_{L^q_\omega} = 0$, that is, $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is compact.

Conversely, let $q < s$ and assume that $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded. Choose $\alpha > \beta > 1$ such that $\frac{\beta}{\alpha} = \frac{q}{s}$. By arguing as in (6.13) and (6.14), we get

$$\frac{\mu^\omega_v(S(a))}{\omega\left(S(a)\right)^{s/p}} \lesssim \|G^v_{\mu,s}(f_{a,p})\|^{\frac{q}{\beta}}_{L^q_\omega} \|G^v_{\mu,1}(f^s_{a,p}(z)\chi_{S(a)})\|^{\frac{1}{\alpha'}}_{L^{\frac{\beta'}{\alpha'}}_\omega}$$

and

$$\|G^v_{\mu,1}(f^s_{a,p}(z)\chi_{S(a)})\|_{L^{\frac{\beta'}{\alpha'}}_\omega} \lesssim \frac{\mu^\omega_v(S(a))^{\frac{\alpha'}{\beta'}}}{\omega\left(S(a)\right)^{s/p}} \left(\sup_{z\in\mathbb{D}} \frac{\mu^\omega_v(S(z)\cap S(a))}{\omega(S(z))}\right)^{1-\frac{\alpha'}{\beta'}}, \quad a \in \mathbb{D},$$

respectively. These estimates yield

$$\frac{\mu^\omega_v(S(a))}{\omega\left(S(a)\right)^{s/p}} \lesssim \|G^v_{\mu,s}(f_{a,p})\|^{\frac{q}{\beta}}_{L^q_\omega} \frac{\mu^\omega_v(S(a))^{\frac{1}{\beta'}}}{\omega\left(S(a)\right)^{\frac{s}{p\alpha'}}} \left(\sup_{z\in\mathbb{D}} \frac{\mu^\omega_v(S(z)\cap S(a))}{\omega(S(z))}\right)^{\frac{1}{\alpha'}-\frac{1}{\beta'}}, \quad a \in \mathbb{D},$$

and thus

$$\frac{\mu^\omega_v(S(a))}{\omega(S(a))^{\frac{q}{p}}} \lesssim \|G^v_{\mu,s}(f_{a,p})\|^q_{L^q_\omega} \left(\sup_{z\in\mathbb{D}} \frac{\mu^\omega_v(S(z)\cap S(a))}{\omega(S(z))}\right)^{\frac{\beta}{\alpha'}-\frac{\beta}{\beta'}}, \quad a \in \mathbb{D}. \tag{6.18}$$

If $q = p$, we may use $\lim_{|a|\to 1^-} \|G^v_{\mu,s}(f_{a,p})\|_{L^p_\omega} = 0$ and Theorem 5 to deduce that the right-hand side of (6.18) tends to zero as a approaches the boundary. Therefore $I_d : A^p_\omega \to L^p_{\mu^\omega_v}$ is compact by Theorem 3(ii).

If $q > p$, by using (6.18) and arguing as in the corresponding part of the proof of Theorem 5, we get

$$\frac{\mu^\omega_v(S(a))}{\omega(S(a))^{1+\frac{s}{p}-\frac{s}{q}}} \leq \|G^v_{\mu,s}(f_{a,p})\|^q_{L^q_\omega} \left(\sup_{b\in\mathbb{D}} \frac{\mu^\omega_v(S(b))}{\omega(S(b))^{1+\frac{s}{p}-\frac{s}{q}}}\right)^{1-\frac{q}{s}}, \quad a \in \mathbb{D}.$$

from which arguments similar to those applied in the previous paragraph show that $I_d : A^p_\omega \to L^{p+s-\frac{ps}{q}}_{\mu^\omega_v}$ is compact. This finishes the proof. □

Theorem 8. *Let $0 < q < p < \infty$ and $0 < s < \infty$ such that $1 + \frac{s}{p} - \frac{s}{q} > 0$, $\omega \in \widehat{\mathcal{D}}$ and let μ, v be positive Borel measures on $\mathbb{D}$ such that $\mu\left(\{z \in \mathbb{D} : v(T(z)) = 0\}\right) = 0 = \mu(\{0\})$. Then the following conditions are equivalent:*

(i) $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ *is compact;*

(ii) $G^v_{\mu,s} : A^p_\omega \to L^q_\omega$ *is bounded;*

(iii) $I_d : A^p_\omega \to L^{p\left(1+\frac{s}{p}-\frac{s}{q}\right)}_{\mu^\omega_v}$ *is compact;*

(iv) $I_d : A^p_\omega \to L^{p\left(1+\frac{s}{p}-\frac{s}{q}\right)}_{\mu^\omega_v}$ *is bounded.*

Proof. The conditions (ii)–(iv) are equivalent by Theorems 4(i) and 3(iii). To complete the proof, it suffices to show that $G^v_{\mu,s}: A^p_\omega \to L^q_\omega$ is compact if $I_d: A^p_\omega \to L^{p\left(1+\frac{s}{p}-\frac{s}{q}\right)}_{\mu^\omega_v}$ is bounded. To see this, let $\{f_n\}$ be a bounded sequence in A^p_ω, let $\{f_{n_k}\}$ be a subsequence that converges uniformly on compact subsets of $\mathbb{D}$ to $f \in A^p_\omega$. Write $g_k = f_{n_k} - f$ as before. By using Theorem 3 and the last part of the proof of Theorem 3(iii), we deduce

$$\lim_{R\to 1^-} \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)\setminus\overline{D(0,R)}} \frac{d\mu(z)}{\omega(T(z))} \right)^{\frac{pq}{s(p-q)}} \omega(\zeta)\, dA(\zeta) = 0.$$

Therefore, for a fixed $\varepsilon > 0$, there exists $R_0 \in (0,1)$ such that

$$\int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)\setminus\overline{D(0,R_0)}} \frac{d\mu(z)}{\omega(T(z))} \right)^{\frac{pq}{s(p-q)}} \omega(\zeta)\, dA(\zeta) < \varepsilon^{\frac{p}{p-q}}.$$

Choose $k_0 \in \mathbb{N}$ such that $|g_k(z)| < \varepsilon^{1/q}$ for all $k \geq k_0$ and $z \in \overline{D(0,R_0)}$. Then Hölder's inequality and [54, Lemma 4.4] give

$$\begin{aligned}
\|G^v_{\mu,s}(g_k)\|^q_{L^q_\omega} &\lesssim \int_{\mathbb{D}} \left(\int_{\Gamma(z)\cap\overline{D(0,R_0)}} |g_k(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))} \right)^{\frac{q}{s}} \omega(z)\, dA(z) \\
&\quad + \int_{\mathbb{D}} \left(\int_{\Gamma(z)\setminus\overline{D(0,R_0)}} |g_k(\zeta)|^s \frac{d\mu(\zeta)}{v(T(\zeta))} \right)^{\frac{q}{s}} \omega(z)\, dA(z) \\
&\lesssim \varepsilon + \int_{\mathbb{D}} N(g_k)^q(z) \left(\int_{\Gamma(z)\setminus\overline{D(0,R_0)}} \frac{d\mu(\zeta)}{v(T(\zeta))} \right)^{\frac{q}{s}} \omega(z)\, dA(z) \\
&\leq \varepsilon + \|N(g_k)\|^q_{L^p_\omega} \left(\int_{\mathbb{D}} \left(\int_{\Gamma(z)\setminus\overline{D(0,R_0)}} \frac{d\mu(\zeta)}{v(T(\zeta))} \right)^{\frac{pq}{(p-q)s}} \omega(z)\, dA(z) \right)^{\frac{p-q}{p}} \lesssim \varepsilon,
\end{aligned}$$

and consequently, $\lim_{k\to\infty} \|G^v_{\mu,s}(g_k)\|^q_{L^q_\omega} = 0$. Finally, by using Lemma 1 and arguing as in the proof of Theorem 7, we deduce $\lim_{k\to\infty} \|G^v_{\mu,s}(f_k) - G^v_{\mu,s}(f)\|^q_{L^q_\omega} = 0$, that is, $G^v_{\mu,s}: A^p_\omega \to L^q_\omega$ is compact. □

6.5 APPLICATIONS AND FURTHER COMMENTS

6.5.1 Area operators in Hardy spaces

For $0 < s < \infty$, define

$$G_{\mu,s}(f)(z) = \left(\int_{\Gamma(z)} |f(\zeta)|^s \frac{d\mu(\zeta)}{1-|\zeta|} \right)^{\frac{1}{s}}, \quad z \in \mathbb{T}.$$

The method of proof of Theorem 4, combined with the results in [40, Section 7] and [52], can be used to obtain the following result. The details of the proof does not reveal anything new, and are therefore omitted. Here $L^q(\mathbb{T})$ refers to the classical L^q-space on $\mathbb{T}$.

Theorem 9. *Let $0 < p, q < \infty$ such that $1 + \frac{s}{p} - \frac{s}{q} > 0$, and let μ be a positive Borel measure on $\mathbb{D}$ such that $\mu(\{0\}) = 0$. Then $G_{\mu,s} : H^p \to L^q(\mathbb{T})$ is bounded (resp. compact) if and only if $I_d : H^p \to L_\mu^{p+s-\frac{ps}{q}}$ is bounded (resp. compact). Moreover,*

$$\|G_{\mu,s}\|^s_{H^p \to L^q(\mathbb{T})} \asymp \|Id\|^{p+s-\frac{ps}{q}}_{H^p \to L_\mu^{p+s-\frac{ps}{q}}} \asymp \sup_{a \in \mathbb{D}} \frac{\mu(S(a))}{(1-|a|)^{1+\frac{s}{p}-\frac{s}{q}}}, \quad p \leq q,$$

and

$$\|G_{\mu,s}\|^s_{H^p \to L^q(\mathbb{T})} \asymp \|Id\|^{p+s-\frac{ps}{q}}_{H^p \to L_\mu^{p+s-\frac{ps}{q}}} \asymp \|B_\mu\|_{L^{\frac{qp}{s(p-q)}}}, \quad q < p,$$

where

$$B_\mu(\zeta) = \int_{\Gamma(\zeta)} \frac{d\mu(z)}{1-|z|}, \quad \zeta \in \mathbb{T}.$$

In particular, this result proves the conjecture in [31, p. 365] in the case $1 + \frac{1}{p} - \frac{1}{q} > 0$.

6.5.2 Integral operator T_g on Bergman and Hardy spaces

Each $g \in \mathcal{H}(\mathbb{D})$ induces the integral operator

$$T_g(f)(z) = \int_0^z g'(\zeta) f(\zeta)\, d\zeta, \quad z \in \mathbb{D},$$

acting on $\mathcal{H}(\mathbb{D})$. This type of integral operators have been extensively studied during the last decades and have interesting connections with other areas of mathematical analysis, see [51, 54] and the references therein. In particular, the symbols g for which T_g is bounded or compact from A^p_ω to A^q_ω can be described in terms of the following spaces of analytic functions when $q \geq p$.

We say that $g \in \mathcal{H}(\mathbb{D})$ belongs to $\mathcal{C}^{q,p}(\omega^\star)$, $0 < p, q < \infty$, if the measure $|g'(z)|^2 \omega^\star(z)\, dA(z)$ is a q-Carleson measure for A^p_ω. Moreover, $g \in \mathcal{C}^{q,p}_0(\omega^\star)$ if the identity operator $I_d : A^p_\omega \to L^q(|g'|^2\omega^\star dA)$ is compact. If $q \geq p$ and $\omega \in \widehat{\mathcal{D}}$, then Theorem 3 shows that these spaces only depend on the quotient $\frac{q}{p}$. Consequently, for $q \geq p$ and $\omega \in \widehat{\mathcal{D}}$, we simply write $\mathcal{C}^{q/p}(\omega^\star)$ instead of $\mathcal{C}^{q,p}(\omega^\star)$. Thus, if $\alpha \geq 1$ and $\omega \in \widehat{\mathcal{D}}$, then $\mathcal{C}^\alpha(\omega^\star)$ consists of those $g \in \mathcal{H}(\mathbb{D})$ such that

$$\|g\|^2_{\mathcal{C}^\alpha(\omega^\star)} = |g(0)|^2 + \sup_{I \subset \mathbb{T}} \frac{\int_{S(I)} |g'(z)|^2 \omega^\star(z)\, dA(z)}{(\omega(S(I)))^\alpha} < \infty. \quad (6.19)$$

An analogue of this identity is valid for the little space $\mathcal{C}^\alpha_0(\omega^\star)$. We refer to [54, Chapter 5] for further information about these spaces.

Theorem 10. *Let $0 < p, q < \infty$ such that $q > \frac{2p}{2+p}$ and $\omega \in \widehat{\mathcal{D}}$. Let $g \in \mathcal{H}(\mathbb{D})$ and denote $d\mu_g(z) = |g'(z)|^2 \omega^\star(z)\, dA(z)$. Then $T_g : A^p_\omega \to A^q_\omega$ is bounded (resp. compact) if and only if $I_d : A^p_\omega \to L^{p+2-\frac{2p}{q}}_{\mu_g}$ is bounded (resp. compact).*

Proof. By [54, Theorem 4.2], $T_g : A^p_\omega \to L^q_\omega$ is bounded (resp. compact) if and only if $G^\omega_{\mu_g,2} : A^p_\omega \to L^q_\omega$ is bounded (resp. compact), and $\|T_g\|_{A^p_\omega \to L^q_\omega} \asymp \|G^\omega_{\mu_g,2}\|_{A^p_\omega \to L^q_\omega}$. Theorem 4 implies

$$\|G^\omega_{\mu_g,2}\|^2_{A^p_\omega \to L^q_\omega} \asymp \|I_d\|^{p+2-\frac{2p}{q}}_{A^p_\omega \to L^{p+2-\frac{2p}{q}}_{\mu_g}},$$

and this finishes the proof. □

It is worth mentioning that Theorem 3 yields

$$\|T_g\|^2_{A^p_\omega \to L^q_\omega} \asymp \sup_{a \in \mathbb{D}} \frac{\mu_g(S(a))}{\omega(S(a))^{2(\frac{1}{p}-\frac{1}{q})+1}}, \quad q \geq p$$

Thus $T_g : A^p_\omega \to L^q_\omega$ is bounded if and only if $g \in \mathcal{C}^{2(\frac{1}{p}-\frac{1}{q})+1}(\omega^\star)$. If $p > q$, Theorem 3 also gives

$$\|T_g\|^2_{A^p_\omega \to L^q_\omega} \asymp \int_{\mathbb{D}} \left(\int_{\Gamma(\zeta)} |g'(z)|^2 \, dA(z) \right)^{\frac{qp}{2(p-q)}} \omega(\zeta) \, dA(\zeta),$$

and thus $T_g : A^p_\omega \to L^q_\omega$ is bounded if and only if $g \in A^{\frac{qp}{p-q}}_\omega$ by [54, Theorem 4.2].

Consequently, whenever $q > \frac{2p}{2+p}$ Theorem 10 improves [54, Theorem 4.1] because the hypothesis on ω are stronger in the original result. In particular, if $q < p$, the weight ω is assumed to be continuous and strictly positive with the local regularity

$$\omega(t) \asymp \omega(r), \quad 1 - t \asymp 1 - r.$$

This hypothesis allows one to use the strong factorization $A^p_\omega = A^{p_1}_\omega \cdot A^{p_2}_\omega$, $p^{-1} = p_1^{-1} + p_2^{-1}$, [54, Theorem 3.1], which is a principal ingredient in the proof of [54, Theorem 4.1].

However, the defect of Theorem 10 is the extra hypothesis $q > \frac{2p}{2+p}$ which is a restriction only in the case $p > q$. This condition is inherited from Theorem 4 and appears there because Carleson measures are finite measures. This is not true in general for μ_g when $g \in A^{\frac{qp}{p-q}}_\omega$ and $q < \frac{2p}{2+p}$. The case of compact operators can be analyzed in the same way.

If $q > \frac{2p}{2+p}$ one may characterize bounded and compact operators $T_g : H^p \to H^q$ by using Section 6.5.1. In order to avoid unnecessary repetition, we omit the details.

7 Paper II

7.1 INTRODUCTION AND MAIN RESULTS

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. For $0 < p < \infty$ and a nonnegative integrable function ω on $\mathbb{D}$, the weighted Bergman space A^p_ω consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^p_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dA(z) < \infty,$$

where $dA(z) = \frac{dx\,dy}{\pi}$ is the normalized Lebesgue area measure on $\mathbb{D}$. As usual, A^p_α denotes the weighted Bergman space induced by the standard radial weight $(1-|z|^2)^\alpha$.

A radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\, ds$ satisfies the doubling condition $\widehat{\omega}(r) \le C\widehat{\omega}(\frac{1+r}{2})$. Further, a radial weight $\omega \in \widehat{\mathcal{D}}$ is regular, denoted by $\omega \in \mathcal{R}$, if $\omega(r)$ behaves as its integral average over $(r, 1)$, that is,

$$\omega(r) \asymp \frac{\int_r^1 \omega(s)\, ds}{1-r}, \quad 0 \le r < 1.$$

Every standard weight as well as those given in [4, (4.4)–(4.6)] are regular. It is easy to see that for each radial weight ω, the norm convergence in A^2_ω implies the uniform convergence on compact subsets of $\mathbb{D}$, and hence the Hilbert space A^2_ω is a closed subspace of L^2_ω and the orthogonal Bergman projection P_ω from L^2_ω to A^2_ω is given by

$$P_\omega(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\omega(\zeta)\, dA(\zeta),$$

where B^ω_z are the reproducing kernels of A^2_ω. Recently, those regular weights ω and ν for which $P_\omega : L^p_\nu \to L^p_\nu$ is bounded were characterized in terms of Bekollé-Bonami type conditions [57]. In this paper we consider operators which are natural extensions of the orthogonal projection P_ω. For a positive Borel measure μ on $\mathbb{D}$, the Toeplitz operator associated with μ is

$$\mathcal{T}_\mu(f)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\, d\mu(\zeta).$$

If $d\mu = \Phi\omega dA$ for a non-negative function Φ, then write $\mathcal{T}_\mu = \mathcal{T}_\Phi$ so that $\mathcal{T}_\Phi(f) = P_\omega(f\Phi)$. The operator $\mathcal{T}_\Phi$ has been extensively studied since the seventies [14,44,68]. Luecking was probably the one who introduced Toeplitz operators $\mathcal{T}_\mu$ with measures as symbols in [39], where he provides, among other things, a description of Schatten class Toeplitz operators $\mathcal{T}_\mu : A^2_\alpha \to A^2_\alpha$ in terms of an ℓ^p-condition involving a hyperbolic lattice of $\mathbb{D}$. More recently, Zhu [71] gave an alternative characterization in terms of $L^p\left(\frac{dA}{(1-|\cdot|)^2}\right)$-integrability of the Berezin transform of $\mathcal{T}_\mu$ in the widest possible range of the paremeters p and α. We refer to [69, Chapter 7] for the theory of Toeplitz operators $\mathcal{T}_\mu$ acting on A^2_α and to [13, 49] for descriptions in terms of Carleson measures and the Berezin transform of bounded and compact Toeplitz operators $\mathcal{T}_\mu : A^p_\alpha \to A^q_\alpha$, $1 < p, q < \infty$. The Berezin transform of a bounded linear operator $T : A^2_\omega \to A^2_\omega$ is

$$\widetilde{T}(z) = \langle T(b^\omega_z), b^\omega_z \rangle_{A^2_\omega}, \tag{7.1}$$

where $b^\omega_z = \frac{B^\omega_z}{\|B^\omega_z\|_{A^2_\omega}}$ are the normalized reproducing kernels of A^2_ω. Given $0 < p, q < \infty$ and a positive Borel measure μ on $\mathbb{D}$, we say that μ is a q-Carleson measure for A^p_ω if the identity operator $Id : A^p_\omega \to L^q_\mu$ is bounded. A description of q-Carleson measures for A^p_ω induced by doubling weights was recently given in [55], see also [60].

One of the main purposes of this study is to characterize, in terms of Carleson measures and the Berezin transform $\widetilde{\mathcal{T}_\mu}$, those positive Borel measures μ such that the Toeplitz operator $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$, where $1 < p, q < \infty$ and $\omega \in \mathcal{R}$, is bounded or compact. We also describe Schatten class Toeplitz operators $\mathcal{T}_\mu : A^2_\omega \to A^2_\omega$ in terms of their Berezin transforms and show how this result can be used to study Schatten class composition operators induced by symbols of bounded valence.

A simple fact that is repeatedly used in the study of Toeplitz operators on standard Bergman spaces A^p_α is the closed formula $(1 - \overline{z}\zeta)^{-(2+\alpha)}$ of the Bergman reproducing kernel of A^2_α. This shows that the kernels never vanish, and allows one to easily establish useful pointwise and norm estimates. However, the situation in the case of A^2_ω with $\omega \in \mathcal{R}$ is more complicated because of the lack of such an explicit expression for B^ω_z. In fact a little perturbation in the weight, that does not change the space itself, might introduce zeros to the kernel functions [67]. This difference causes severe difficulties in the study related to Toeplitz operators on A^p_ω, and forces us to circumvent several obstacles in a different manner. We will shortly indicate the main tools used in the proofs after each result is stated.

We need a bit more of notation to state our first result. For each $1 < p < \infty$ we write p' for its conjugate exponent, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. The Carleson square $S(I)$ based on an interval I on the boundary $\mathbb{T}$ of $\mathbb{D}$ is the set $S(I) = \{re^{it} \in \mathbb{D} : e^{it} \in I, 1 - |I| \le r < 1\}$, where $|E|$ denotes the Lebesgue measure of $E \subset \mathbb{T}$. We associate to each $a \in \mathbb{D} \setminus \{0\}$ the interval $I_a = \{e^{i\theta} : |\arg(ae^{-i\theta})| \le \frac{1-|a|}{2}\}$, and denote $S(a) = S(I_a)$.

Theorem 11. *Let $1 < p \le q < \infty$, $\omega \in \mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is bounded;*

(ii) $\dfrac{\widetilde{\mathcal{T}_\mu}(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \in L^\infty$;

(iii) μ *is a* $\frac{s(p+q')}{pq'}$*-Carleson measure for* A^s_ω *for some (equivalently for all)* $0 < s < \infty$;

(iv) $\dfrac{\mu(S(\cdot))}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}}} \in L^\infty$.

Moreover,

$$\|\mathcal{T}_\mu\|_{A^p_\omega\to A^q_\omega} \asymp \left\| \frac{\widetilde{\mathcal{T}}_\mu(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \right\|_{L^\infty} \asymp \|Id\|^{\frac{s(p+q')}{pq'}}_{A^s_\omega\to L_\mu^{\frac{s(p+q')}{pq'}}} \asymp \left\| \frac{\mu(S(\cdot))}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}}} \right\|_{L^\infty}.$$

The equivalence between (ii) and (iv) shows that the Berezin transform $\widetilde{\mathcal{T}}_\mu$ behaves asymptotically as the average $\mu(S(\cdot))/\omega(S(\cdot))$. By using Fubini's theorem and the reproducing formula

$$L_z(f) = f(z) = \langle f, B^\omega_z\rangle_{A^2_\omega} = \int_{\mathbb{D}} f(\zeta)\,\overline{B^\omega_z(\zeta)}\,\omega(\zeta)\,dA(\zeta), \quad f\in A^1_\omega, \tag{7.2}$$

we deduce

$$\langle \mathcal{T}_\mu(f), g\rangle_{A^2_\omega} = \langle f, g\rangle_{L^2_\mu} \tag{7.3}$$

for each compactly supported positive Borel measure μ and all $f, g\in A^2_\omega$. This identity shows that Carleson measures and Toeplitz operators are intimately connected, and thus the use of Carleson measures in the proof of Theorem 11 does not come as a surprise. Another key tools in the proof are the L^p-estimates of the kernels B^ω_z, obtained in [57, Theorem 1], and a pointwise estimate for B^ω_z in a sufficiently small Carleson square contained in $S(z)$, given in Lemma 3 below. We also prove a counterpart of Theorem 11 for compact Toeplitz operators. This result is stated as Theorem 15 and its proof relies, among other things, on the duality relation $(A^p_\omega)^\star \simeq A^{p'}_\omega$ under the pairing $\langle\cdot,\cdot\rangle_{A^2_\omega}$, valid for all $\omega\in\mathcal{R}$ [57, Corollary 7].

To describe the positive Borel measures such that $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ is bounded on the range $1<q<p<\infty$, we write $\varrho(a,z) = |\varphi_a(z)| = \left|\frac{a-z}{1-\bar{a}z}\right|$, for the pseudohyperbolic distance between z and a, and $\Delta(a,r) = \{z : \varrho(a,z) < r\}$ for the pseudohyperbolic disc of center $a\in\mathbb{D}$ and radius $r\in(0,1)$.

Theorem 12. *Let $1<q<p<\infty$, $0<r<1$, $\omega\in\mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega\to A^q_\omega$ *is compact;*

(ii) $\mathcal{T}_\mu : A^p_\omega\to A^q_\omega$ *is bounded;*

(iii) $\widehat{\mu}_r(\cdot) = \frac{\mu(\Delta(\cdot,r))}{\omega(\Delta(\cdot,r))} \in L^{\frac{pq}{p-q}}_\omega$;

(iv) μ *is a* $\left(p+1-\frac{p}{q}\right)$*-Carleson measure for* A^p_ω;

(v) $Id : A^p_\omega \to L^{p+1-\frac{p}{q}}_\mu$ *is compact;*

(vi) $\widetilde{\mathcal{T}}_\mu \in L^{\frac{pq}{p-q}}_\omega$.

Moreover,

$$\|\mathcal{T}_\mu\|_{A^p_\omega\to A^q_\omega} \asymp \|\widehat{\mu}_r\|_{L^{\frac{qp}{p-q}}_\omega} \asymp \|Id\|^{p+1-\frac{p}{q}}_{A^p_\omega\to L_\mu^{p+1-\frac{p}{q}}} \asymp \|\widetilde{\mathcal{T}}_\mu\|_{L^{\frac{qp}{p-q}}_\omega}.$$

Apart from standard techniques, such as a duality relation for Bergman spaces and the use of Rademacher functions along with Khinchine's inequality, the boundedness of the maximal Bergman projection

$$P_\omega^+(f)(z) = \int_{\mathbb{D}} |f(\zeta)||B_z^\omega(\zeta)|\,\omega(\zeta)dA(\zeta)$$

on L^p_ω for $p \in (1,\infty)$ and $\omega \in \mathcal{R}$ [57, Theorem 5] plays a crucial role in the proof of Theorem 12. Another important fact employed is that, even if the kernels may vanish, by Lemma 4 for each $\omega \in \widehat{\mathcal{D}}$ they obey the relation $|B_a^\omega| \asymp B_a^\omega(a)$ on sufficiently small pseudohyperbolic discs centered at a. This is used when (iii) is considered, but (iii) involves pseudohyperbolic discs of all sizes, and therefore a suitably chosen covering of $\mathbb{D}$ will be used to deal with this technical obstacle.

As for the statements of our results on Schatten classes, some notation are in order. The polar rectangle associated with an arc $I \subset \mathbb{T}$ is

$$R(I) = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I,\ 1 - \frac{|I|}{2\pi} \le |z| < 1 - \frac{|I|}{4\pi} \right\}.$$

Write $z_I = (1 - |I|/2\pi)\xi$, where $\xi \in \mathbb{T}$ is the midpoint of I. Let Υ denote the family of all dyadic arcs of $\mathbb{T}$. Every arc $I \in \Upsilon$ is of the form

$$I_{n,k} = \left\{ e^{i\theta} : \frac{2\pi k}{2^n} \le \theta < \frac{2\pi(k+1)}{2^n} \right\},$$

where $k = 0, 1, 2, \dots, 2^n - 1$ and $n = \mathbb{N} \cup \{0\}$. The family $\{R(I) :\ I \in \Upsilon\}$ consists of pairwise disjoint rectangles whose union covers $\mathbb{D}$. For $I_j \in \Upsilon \setminus \{I_{0,0}\}$, we will write $z_j = z_{I_j}$. For convenience, we associate the arc $I_{0,0}$ with the point $1/2$. Given a radial weight ω, we write

$$\omega^\star(z) = \int_{|z|}^1 \omega(s) \log \frac{s}{|z|} s\,ds, \quad z \in \mathbb{D} \setminus \{0\}.$$

Theorem 13. *Let $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu$ *belongs to the Schatten p-class $\mathcal{S}_p(A^2_\omega)$;*

(ii) $\sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{\omega^\star(z_j)} \right)^p < \infty$;

(iii) $\frac{\mu(\Delta(\cdot,r))}{\omega^\star(\cdot)}$ *belongs to* $L^p\left(\frac{dA}{(1-|\cdot|)^2}\right)$ *for some (equivalently for all)* $0 < r < 1$.

Moreover,

$$|\mathcal{T}_\mu|_p^p \asymp \sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{\omega^\star(z_j)} \right)^p \asymp \int_{\mathbb{D}} \left(\frac{\mu(\Delta(z,r))}{\omega^\star(z)} \right)^p \frac{dA(z)}{(1-|z|)^2}.$$

If $\omega \in \mathcal{R}$ such that $\frac{(\omega^\star(\cdot))^p}{(1-|\cdot|)^2}$ is also a regular weight, then $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$ if and only if $\widetilde{\mathcal{T}}_\mu \in L^p_{\omega/\omega^\star}$, and $|\mathcal{T}_\mu|_p \asymp \|\widetilde{\mathcal{T}}_\mu\|_{L^p_{\omega/\omega^\star}}$.

The equivalence of the first three statements were proved in [56, Theorem 1], and hence the novelty of Theorem 13 stems from the last part involving the Berezin transform. The hypothesis $\frac{\omega^\star(\cdot)^p}{(1-|\cdot|)^2} \in \mathcal{R}$ is not a restriction for $p \geq 1$, and for $\omega(z) = (1-|z|^2)^\alpha$ it reduces to the inequality $p(2+\alpha) > 1$. Therefore Theorem 13 is an extension of [69, Theorem 7.18], see also [71]. Since each standard weight is regular, the cut-off condition $\frac{\omega^\star(\cdot)^p}{(1-|\cdot|)^2} \in \mathcal{R}$ is in a sense the best possible.

The proof of the last statement of Theorem 13 for $p \geq 1$ follows by standard techniques once the pointwise kernel estimate given in Lemma 4 is available. However, the proof for $0 < p < 1$ is more involved because the reproducing kernels of A^2_ω with $\omega \in \mathcal{R}$ do not necessarily remain essentially constant in hyperbolically bounded regions, a property which the standard kernels $(1-\overline{z}\zeta)^{2+\alpha}$ trivially admit and is used in the proof of [69, Theorem 7.18] concerning the weighted Bergman spaces A^p_α. This obstacle is circumvent by using subharmonicity and estimates for the A^p_ν-norm of B^ω_z for doubling weights $\omega, \nu \in \widehat{\mathcal{D}}$, obtained in [57, Theorem 1].

Theorem 13 can be applied to study Schatten class composition operators when the inducing symbol φ is of finite valence. To state the result, some more notation and motivation are in order. For an analytic self-map φ of $\mathbb{D}$, let $\zeta \in \varphi^{-1}(z)$ denote the set of the points $\{\zeta_n\}$ in $\mathbb{D}$, organized by increasing moduli and repeated according to their multiplicities, such that $\varphi(\zeta_n) = z$ for all n. For a radial weight ω and φ as above, the generalized Nevanlinna counting function is

$$N_{\varphi,\omega^\star}(z) = \sum_{\zeta \in \varphi^{-1}(z),} \omega^\star(\zeta), \quad z \in \mathbb{D} \setminus \{\varphi(0)\}.$$

In [56, Theorem 3] it was shown that, for each $\omega \in \widehat{\mathcal{D}}$, the composition operator C_φ belongs to the Schatten p-class $\mathcal{S}_p(A^2_\omega)$ if and only if $N_{\varphi,\omega^\star} \in L^p\left(\frac{dA}{(1-|\cdot|)^2}\right)$. This condition might be difficult to test in praxis because of the counting function $N_{\varphi,\omega^\star}$. Therefore it is natural to look for more workable descriptions. As for this, we observe that by using [57, Theorem 1] one can show that the Berezin transform of $C_\varphi C^\star_\varphi$ behaves asymptotically as $\frac{\omega^\star(\cdot)}{\omega^\star(\varphi(\cdot))}$, and moreover, the condition $\frac{\omega^\star(z)}{\omega^\star(\varphi(z))} \to 0$, $|z| \to 1^-$, characterizes compact operators $C_\varphi : A^2_\omega \to A^2_\omega$ when $\omega \in \mathcal{R}$ by [56, Theorem 20 and Lemma 23]. Therefore one may ask how close is the condition

$$\int_{\mathbb{D}} \left(\frac{\omega^\star(z)}{\omega^\star(\varphi(z))}\right)^{\frac{p}{2}} \frac{\omega(z)}{\omega^\star(z)}\, dA(z) < \infty \tag{7.4}$$

to describe Schatten class composition operators? The next result shows that this is a description in the case $p > 2$ under the hypothesis of φ being of bounded valence.

Theorem 14. *Let $2 < p < \infty$ and $\omega \in \mathcal{R}$, and let φ be a bounded valent analytic self-map of $\mathbb{D}$. Then $C_\varphi \in \mathcal{S}_p(A^2_\omega)$ if and only if* (7.4) *holds.*

Theorem 14 is an extension of [70, Theorem 1.1] to the setting of regular weights. If $\omega(z) = (1-|z|^2)^\alpha$, then the statement in Theorem 14 is not valid for $p(\alpha+2) \leq 2$ because in this case the condition (7.4) fails for all analytic self-maps φ. More generally, by using [54, p. 10 (ii)] one can show that if $\omega \in \mathcal{R}$ and p is small enough, then (7.4) fails for each φ. Moreover, [66, Theorem 3] shows that the statement in Theorem 14 does not remain valid for $\omega \equiv 1$ without the additional hypothesis regarding the valence of φ.

It is easy to see that each regular weight ω satisfies $\omega(r) \asymp \omega(t)$ whenever $1-r \asymp 1-t$. This asymptotic relation shows that $\omega \in \mathcal{R}$ must be essentially constant in each hyperbolically bounded region, and hence, in particular, ω may not have zeros. This apparently severe requirement does not cause too much loss of generality in our study. This because in the next section we will show that if $\omega \in \widehat{\mathcal{D}}$ satisfies the reverse doubling property $\widehat{\omega}(r) \geq C\widehat{\omega}\left(1-\frac{1-r}{K}\right)$ for some $K > 1$ and $C > 1$, a condition that is satisfied for each $\omega \in \mathcal{R}$, then there exists a differentiable strictly positive weight $W \in \mathcal{R}$ such that $\|\cdot\|_{A^p_\omega}$ and $\|\cdot\|_{A^p_W}$ are comparable. In Section 7.2 we also discuss the kernel estimates and other auxiliary results. Section 8.3 is devoted to the study of bounded and compact Toeplitz operators. Schatten class Toeplitz and composition operators are discussed in Sections 7.4 and 7.5, respectively.

7.2 POINTWISE AND NORM ESTIMATES OF BERGMAN REPRODUCING KERNELS

We begin with considering the classes of weights appearing in this study and their basic properties. Then we will prove several pointwise and norm estimates for the reproducing kernels, and finally an auxiliary result on weak convergence of normalized kernels is established.

The first auxiliary lemma contains several characterizations of doubling weights and will be repeatedly used throughout the rest of the paper. For a proof, see [51, Lemma 2.1]. All along we will assume $\widehat{\omega}(r) > 0$ for all $0 \leq r < 1$ without mentioning it, for otherwise $A^p_\omega = \mathcal{H}(\mathbb{D})$.

Lemma B. *Let ω be a radial weight. Then the following conditions are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that

$$\widehat{\omega}(r) \leq C\left(\frac{1-r}{1-t}\right)^{\beta}\widehat{\omega}(t), \quad 0 \leq r \leq t < 1;$$

(iii) There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that

$$\int_0^t \left(\frac{1-t}{1-s}\right)^{\gamma} \omega(s)\,ds \leq C\widehat{\omega}(t), \quad 0 \leq t < 1;$$

(iv) The asymptotic equality

$$\int_0^1 s^x \omega(s)\,ds \asymp \widehat{\omega}\left(1-\frac{1}{x}\right), \quad x \in [1,\infty),$$

is valid;

(v) $\omega^\star(z) \asymp \widehat{\omega}(z)(1-|z|)$, $|z| \to 1^-$;

(vi) There exists $\lambda = \lambda(\omega) \geq 0$ such that

$$\int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{\zeta}z|^{\lambda+1}}\,dA(z) \asymp \frac{\widehat{\omega}(\zeta)}{(1-|\zeta|)^{\lambda}}, \quad \zeta \in \mathbb{D};$$

(vii) *There exists* $C = C(\omega) > 0$ *such that the moments* $\omega_n = \int_0^1 r^n\omega(r)\,dr$ *satisfy the condition* $\omega_n \leq C\omega_{2n}$.

We next briefly discuss radial weights having a kind of reversed doubling property, and then show how this is related to the pointwise condition that defines the class $\mathcal{R}$ of regular weights. More precisely, we show that if $\omega \in \widehat{\mathcal{D}}$ satisfies the reverse doubling condition appearing in part (i) of Lemma 8.7 below, then one can find a strictly positive n times differentiable weight which belongs to $\mathcal{R}$ and induces the same Bergman space as ω. The next lemma can be find in [59].

Lemma C. *Let* ω *be a radial weight. For each* $K > 1$, *let* $\rho_n = \rho_n(\omega, K)$ *be the sequence defined by* $\widehat{\omega}(\rho_n) = \widehat{\omega}(0)K^{-n}$, *and for each* $\beta \in \mathbb{R}$, *write* $\omega_{[\beta]}(z) = \omega(z)(1-|z|)^\beta$. *Then the following statements are equivalent:*

(i) *There exist* $K = K(\omega) > 1$ *and* $C = C(\omega) > 1$ *such that* $\widehat{\omega}(r) \geq C\widehat{\omega}\left(1 - \frac{1-r}{K}\right)$ *for all* $0 \leq r < 1$;

(ii) *There exist* $C = C(\omega) > 0$ *and* $\beta = \beta(\omega) > 0$ *such that*

$$\widehat{\omega}(t) \leq C\left(\frac{1-t}{1-r}\right)^\beta \widehat{\omega}(r), \quad 0 \leq r \leq t < 1;$$

(iii) *For some (equivalently for each)* $\beta \in (0,\infty)$, *there exists* $C = C(\beta,\omega) \in (0,1)$ *such that*

$$\frac{\int_r^1 \widehat{\omega}(t)\beta(1-t)^{\beta-1}\,dt}{(1-r)^\beta} \leq C\widehat{\omega}(r), \quad 0 < r < 1.$$

By Lemma 8.7 and [54, Lemma 1.1] each $\omega \in \mathcal{R}$ satisfies the reverse doubling condition.The next result shows that if $\omega \in \widehat{\mathcal{D}}$ satisfies the reverse doubling condition, then there exists a continuous and locally smooth weight W that induces the same Bergman space as ω.

Proposition 1. *Let* $0 < p < \infty$ *and* $\omega \in \widehat{\mathcal{D}}$, *and write* $W(r) = W_\omega(r) = \widehat{\omega}(r)/(1-r)$ *for all* $0 \leq r < 1$. *Then* $\|f\|_{A^p_W} \asymp \|f\|_{A^p_\omega}$ *for all* $f \in \mathcal{H}(\mathbb{D})$ *if and only if* ω *satisfies the reverse doubling condition appearing in part (i) of Lemma 8.7.*

Proof. Since ω belongs to $\widehat{\mathcal{D}}$ by the hypothesis, so does W. Therefore $\|f\|_{A^p_W} \asymp \|f\|_{A^p_\omega}$ for all $f \in \mathcal{H}(\mathbb{D})$ by [55, Theorem 1] if $W(S(a)) \asymp \omega(S(a))$ for all $a \in \mathbb{D}\setminus\{0\}$. Since ω and W are radial, this is the case if

$$\widehat{W}(r) = \widehat{\omega}(r)\int_r^1 \frac{\widehat{\omega}(t)}{\widehat{\omega}(r)}\frac{1}{1-t}\,dt \asymp \widehat{\omega}(r), \quad 0 \leq r < 1.$$

If now $\omega \in \widehat{\mathcal{D}}$ satisfies the reverse doubling condition, then Lemma E(ii) and Lemma 8.7(ii) applied to the middle term above imply the asymptotic equality we are after.

Conversely, assume that $\omega \in \widehat{\mathcal{D}}$ and $\|f\|_{A^p_W} \asymp \|f\|_{A^p_\omega}$ for all $f \in \mathcal{H}(\mathbb{D})$. Write $f_a(z) = (1-\overline{a}z)^{-\frac{\lambda+1}{p}}$ for all $a \in \mathbb{D}$. By Lemma E(vi) there exists $\lambda = \lambda(\omega) \geq 0$ such that

$$\frac{\widehat{\omega}(a)}{(1-|a|)^{\lambda}} \asymp \int_{\mathbb{D}} \frac{\omega(z)}{|1-\overline{a}z|^{\lambda+1}}\, dA(z) = \|f_a\|^p_{A^p_\omega} \asymp \|f_a\|^p_{A^p_W} \asymp \int_0^1 \frac{\widehat{\omega}(r)}{(1-|a|r)^{\lambda}(1-r)}\, dr$$
$$\geq \int_{|a|}^1 \frac{\widehat{\omega}(r)}{(1-|a|r)^{\lambda}(1-r)}\, dr \gtrsim \frac{\widehat{\widehat{\omega}}(r)}{(1-|a|)^{\lambda+1}},$$

and thus ω satisfies the Lemma 8.7(iii) with $\beta = 1$. □

Consider now $\omega \in \widehat{\mathcal{D}}$ satisfying the reverse doubling condition. Then $A^p_\omega = A^p_{W_\omega}$ and $W_\omega \in \mathcal{R}$ by the first part of the proof of Proposition 1. The weight W_ω is continuous and strictly positive. Further, the differentiable weight $\widehat{W_\omega}(r)/(1-r)$ belongs to $\mathcal{R}$ and induces the same Bergman space as ω. Therefore, by repeating the process, for a given $\omega \in \widehat{\mathcal{D}}$ satisfying the reverse doubling condition, we can always find a strictly positive n times differentiable weight that induces the same Bergman space as the original weight ω. Therefore assuming $\omega \in \mathcal{R}$ instead of the two doubling conditions is not a severe restriction in our study.

The true advantage of the class $\mathcal{R}$ is the local smoothness of its weights. It is clear that if $\omega \in \mathcal{R}$, then for each $s \in [0,1)$ there exists a constant $C = C(s,\omega) > 1$ such that

$$C^{-1}\omega(t) \leq \omega(r) \leq C\omega(t), \quad 0 \leq r \leq t \leq r + s(1-r) < 1. \tag{7.5}$$

Therefore, for $\omega \in \mathcal{R}$ and $r \in (0,1)$,

$$\omega\left(S(z)\right) \asymp \widehat{\omega}(z)(1-|z|) \asymp \omega(z)(1-|z|)^2 \asymp \omega\left(\Delta(z,r)\right), \quad z \in \mathbb{D}, \tag{7.6}$$

where the constants of comparison depend on ω and also on r in the last case. This observation finishes our discussion on basic properties of different classes of weights.

We next turn to kernel estimates. In order to prove our main results, and in particular to deal with the Berezin transform of a Toeplitz operator, we will need asymptotic estimates for the norm of the Bergman reproducing kernel in several spaces of analytic functions in $\mathbb{D}$. The next result follows by [57, Theorem 1] (see also [54, Lemma 6.2]), Lemma E and (7.6).

Theorem B. *Let $\omega, \nu \in \widehat{\mathcal{D}}$, $0 < p < \infty$ and $n \in \mathbb{N} \cup \{0\}$. Then*

$$\|(B^\omega_z)^{(n)}\|^p_{A^p_\nu} \asymp \int_0^{|z|} \frac{\widehat{\nu}(t)}{\widehat{\omega}(t)^p(1-t)^{p(n+1)}}\, dt, \quad |z| \to 1^-. \tag{7.7}$$

In particular, if $1 < p < \infty$, $\omega \in \mathcal{R}$ and $r \in (0,1)$, then

$$\|B^\omega_z\|^p_{A^p_\omega} \asymp \frac{1}{\omega(S(z))^{p-1}} \asymp \frac{1}{\omega(\Delta(z,r))^{p-1}}, \quad z \in \mathbb{D}. \tag{7.8}$$

As usual, we write H^∞ for the space of bounded analytic functions in $\mathbb{D}$, and $\mathcal{B}$ stands for the Bloch functions, that is, the space of $f \in \mathcal{H}(\mathbb{D})$ such that $\|f\|_{\mathcal{B}} = \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|) + |f(0)| < \infty$.

Lemma 2. *Let $\omega \in \widehat{\mathcal{D}}$. Then*

$$\|B^\omega_z\|_{\mathcal{B}} \asymp \frac{1}{\omega(S(z))} \asymp \|B^\omega_z\|_{H^\infty}, \quad z \in \mathbb{D}.$$

Proof. Since

$$B_z^\omega(\zeta)=\sum_{n=0}^\infty \frac{(\zeta\overline{z})^n}{2\omega_n},\quad (B_z^\omega)'(\zeta)=\sum_{n=1}^\infty \frac{n\zeta^{n-1}\overline{z}^n}{2\omega_n},\quad z,\zeta\in\mathbb{D},$$

the estimate [57, (20)], with $p=1$, $N=2$ and $r=|z|^2$, together with Lemma E yields

$$\begin{split}\left|(B_z^\omega)'(z)\right| &\asymp \sum_{n=1}^\infty \frac{n|z|^{2(n-1)}}{\omega_n}\asymp\int_0^{|z|^2}\frac{1}{\widehat{\omega}(t)(1-t)^3}\,dt\\ &\asymp\frac{1}{\widehat{\omega}(z^2)(1-|z|^2)^2}\asymp\frac{1}{\omega(S(z))(1-|z|)},\quad |z|\to 1^-,\end{split}\tag{7.9}$$

and hence

$$\frac{1}{\omega(S(z))}\lesssim\|B_z^\omega\|_{\mathcal{B}},\quad |z|\to 1^-.$$

Since $\|B_z^\omega\|_{\mathcal{B}}\le 2\,\|B_z^\omega\|_{H^\infty}$, it remains to establish the desired upper estimate for the H^∞-norm. To see this, observe first that

$$|B_z^\omega(\zeta)|\le\sum_{n=0}^\infty\frac{|z|^n}{2\omega_n},\quad z,\zeta\in\mathbb{D}.$$

Then, by using again the estimate [57, (20)], but now with $p=1$, $N=1$ and $r=|z|$, it follows that

$$\|B_z^\omega\|_{H^\infty}\le\sum_{n=0}^\infty\frac{|z|^n}{2\omega_n}\asymp\int_0^{|z|}\frac{dt}{\widehat{\omega}(t)(1-t)^2}\asymp\frac{1}{\omega(S(z))},\quad |z|\to 1^-.$$

This finishes the proof. □

We next establish two local pointwise estimates for the Bergman reproducing kernels. To do this, for each $\delta\in(0,1]$ and $a\in\mathbb{D}\setminus\{0\}$, write $a_\delta=(1-\delta(1-|a|))e^{i\arg a}$. Then $a_1=a$, $|a_\delta|>|a|$ for all $\delta\in(0,1)$, and $\lim_{\delta\to 0^+}a_\delta=a/|a|$.

Lemma 3. *Let $\omega\in\widehat{\mathcal{D}}$. Then there exist constants $c=c(\omega)>0$ and $\delta=\delta(\omega)\in(0,1]$ such that*

$$|B_a^\omega(z)|\ge\frac{c}{\omega(S(a))},\quad z\in S(a_\delta),\quad a\in\mathbb{D}\setminus\{0\}.\tag{7.10}$$

Proof. By Theorem B there exists a constant $C_1=C_1(\omega)>0$ such that $\|B_a^\omega\|^2_{A^2_\omega}\ge C_1/\omega(S(a))$ for all $a\in\mathbb{D}\setminus\{0\}$, and hence

$$\begin{split}|B_a^\omega(z)|&\ge|B_a^\omega(a_\delta)|-|B_a^\omega(a_\delta)-B_a^\omega(z)|=|B^\omega_{\sqrt{|aa_\delta|}}(\sqrt{|aa_\delta|})|-|B_a^\omega(a_\delta)-B_a^\omega(z)|\\ &=\left\|B^\omega_{\sqrt{|aa_\delta|}}\right\|^2_{A^2_\omega}-|B_a^\omega(a_\delta)-B_a^\omega(z)|\ge\frac{C_1}{\omega(S(\sqrt{|aa_\delta|})}-|B_a^\omega(a_\delta)-B_a^\omega(z)|\\ &\ge\frac{C_1}{\omega(S(a))}-|B_a^\omega(a_\delta)-B_a^\omega(z)|,\quad z\in\mathbb{D}.\end{split}\tag{7.11}$$

Moreover, by (7.9) and Lemma E,

$$
\begin{aligned}
|B_a^\omega(a_\delta)-B_a^\omega(z)| &\le \sup_{\zeta\in[a_\delta,z]}|(B_a^\omega)'(\zeta)||z-a_\delta| \le 2\delta(1-|a|)\sup_{\zeta\in[a_\delta,z]}|(B_a^\omega)'(\zeta)|\\
&\le \delta(1-|a|)\sum_{n=1}^{\infty}\frac{n|a|^n}{\omega_{2n+1}} \le \frac{\delta C_2}{\omega(S(a))}.
\end{aligned}
$$

By combining this with (7.11), and choosing $\delta = C_1/2C_2$ we deduce the assertion for $c = C_1/2$. □

Lemma 4. *Let $\omega \in \widehat{\mathcal{D}}$. Then there exists $r = r(\omega) \in (0,1)$ such that $|B_a^\omega(z)| \asymp B_a^\omega(a)$ for all $a \in \mathbb{D}$ and $z \in \Delta(a,r)$.*

Proof. The proof is similar to that of [54, Lemma 6.4]. First, use the Cauchy-Schwarz inequality, Theorem B and Lemma E to obtain

$$
\begin{aligned}
|B_a^\omega(z)| &\le \sum_n \frac{|az|^n}{2\omega_{2n+1}} \le \left(\sum_n \frac{|z|^{2n}}{2\omega_{2n+1}}\right)^{\frac12}\left(\sum_n \frac{|a|^{2n}}{2\omega_{2n+1}}\right)^{\frac12} = |B_a^\omega(a)|^{\frac12}|B_z^\omega(z)|^{\frac12}\\
&\asymp \frac{|B_a^\omega(a)|^{\frac12}}{\sqrt{\widehat{\omega}(z)(1-|z|)}} \asymp \frac{|B_a^\omega(a)|^{\frac12}}{\sqrt{\widehat{\omega}(a)(1-|a|)}} \asymp |B_a^\omega(a)|, \quad z\in\Delta(a,r),
\end{aligned}
\tag{7.12}
$$

for all $a \in \mathbb{D}$. This gives the claimed upper bound. To obtain the same lower bound, let $r \in (0,1)$ and note first that

$$
\begin{aligned}
|B_a^\omega(z)| &\ge |B_a^\omega(a)| - \max_{\zeta\in[a,z]}|(B_a^\omega)'(\zeta)||z-a|\\
&\ge |B_a^\omega(a)| - \max_{\zeta\in[a,z]}|(B_a^\omega)'(\zeta)|rC(1-|a|),
\end{aligned}
$$

where $C = C(r) > 0$ is a constant for which $\sup_{0<r<r_0} C(r) < \infty$ for each $r_0 \in (0,1)$. Now the Cauchy integral formula and a reasoning similar to that in (7.12) yield

$$
\max_{\zeta\in[a,z]}|(B_a^\omega)'(\zeta)| \lesssim \frac{|B_a^\omega(a)|}{1-|a|}, \quad a\in\mathbb{D},
$$

and the desired lower bound follows by choosing r sufficiently small. □

The last aim of this section is to show that for each $\omega \in \mathcal{R}$, the normalized reproducing kernels $b_{p,z}^\omega = B_z^\omega/\|B_z^\omega\|_{A_\omega^p}$ converge weakly to zero in A_ω^p, as $|z| \to 1^-$. To do this, the following growth estimate is used.

Lemma 5. *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$. Then*

$$
|f(z)| = \mathrm{o}\left(\frac{1}{(\widehat{\omega}(z)(1-|z|))^{\frac1p}}\right), \quad |z| \to 1^-,
$$

for all $f \in A_\omega^p$.

Proof. Let $f \in A^p_\omega$ and $\varepsilon > 0$. Then there exists $r \in (0,1)$ such that

$$\varepsilon > \int_r^1 M_p^p(s,f) s\omega(s)\,ds \geq M_p^p(r,f) r\widehat{\omega}(r),$$

which together with the well-known estimate

$$M_\infty(r,f) \lesssim \frac{M_p\left(\frac{1+r}{2},f\right)}{(1-r)^{\frac{1}{p}}}, \quad 0 < r < 1,$$

and the hypothesis $\omega \in \widehat{\mathcal{D}}$ yields the assertion. $\square$

The proof of the weak convergence we are after relies on the following known duality relation [57, Corollary 7].

Theorem C. *Let $1 < p < \infty$ and $\omega \in \mathcal{R}$. Then $(A^p_\omega)^\star \simeq A^{p'}_\omega$, with equivalence of norms, under the pairing*

$$\langle f,g\rangle_{A^2_\omega} = \int_{\mathbb{D}} f(z)\overline{g(z)}\omega(z)\,dA(z). \tag{7.13}$$

With these preparations we can prove the last result of the section.

Lemma 6. *Let $1 < p < \infty$ and $\omega \in \mathcal{R}$. Then $b^\omega_{p,z} \to 0$ weakly in A^p_ω, as $|z| \to 1^-$.*

Proof. Let $1 < p < \infty$ and $\omega \in \mathcal{R}$. By Theorem C it suffices to show that

$$\left|\left\langle b^\omega_{p,z}, g\right\rangle_{A^2_\omega}\right| = \frac{|g(z)|}{\|B^\omega_z\|_{A^p_\omega}} \to 0, \quad |z| \to 1^-,$$

for all $g \in A^{p'}_\omega$. But since $\|B^\omega_z\|^p_{A^p_\omega} \asymp (\widehat{\omega}(z)(1-|z|))^{1-p}$ by Theorem B, and $1-p = -p/p'$, the assertion follows by Lemma 5. $\square$

7.3 BOUNDED AND COMPACT TOEPLITZ OPERATORS

The main objective of this section is to prove Theorems 11 and 12, stated in the introduction, and establish a characterization analogous to Theorem 11 for compact operators $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$, given as Theorem 15 below. We begin with the following technical result.

Lemma 7. *Let μ be a finite positive Borel measure on $\mathbb{D}$. Then* (7.3) *is satisfied for all $f(z) = \sum_{n=0}^\infty \widehat{f}(n)z^n$ and $g(z) = \sum_{n=0}^\infty \widehat{g}(n)z^n$ such that $f \in H^\infty$ and $\sum_{n=0}^\infty |\widehat{g}(n)| < \infty$.*

Proof. Fubini's theorem and the dominated convergence theorem yield

$$\begin{aligned}
\langle \mathcal{T}_\mu(f), g\rangle &= \lim_{s\to 1^-} \int_{|u|<s} \left(\int_{\mathbb{D}} f(\zeta) B^\omega_\zeta(u)\,d\mu(\zeta)\right) \overline{g(u)}\omega(u)\,dA(u) \\
&= \lim_{s\to 1^-} \int_{\mathbb{D}} f(\zeta) \left(\int_{|u|<s} \overline{g(u)} B^\omega_\zeta(u)\omega(u)\,dA(u)\right) d\mu(\zeta) \\
&= \lim_{s\to 1^-} \int_{\mathbb{D}} f(\zeta) \left(\sum_{n=0}^\infty \frac{\overline{\widehat{g}(n)\zeta^n}\int_0^s x^{2n+1}\omega(x)\,dx}{\omega_{2n+1}}\right) d\mu(\zeta) = \int_{\mathbb{D}} f(\zeta)\overline{g(\zeta)}\,d\mu(\zeta),
\end{aligned}$$

and the assertion is proved. $\square$

Recall that $b_z^\omega = B_z^\omega / \|B_z^\omega\|_{A^2_\omega}$ for all $z \in \mathbb{D}$. If μ is a finite positive Borel measure on $\mathbb{D}$ and $\omega \in \widehat{\mathcal{D}}$, then by using the definition (7.1) of Berezin transform, Lemma 7 and Theorem B, we deduce

$$\widetilde{\mathcal{T}}_\mu(z) = \langle \mathcal{T}_\mu(b_z^\omega), b_z^\omega \rangle_{A^2_\omega} = \frac{\|B_z^\omega\|^2_{L^2_\mu}}{\|B_z^\omega\|^2_{A^2_\omega}} \asymp \omega\left(S(z)\right) \|B_z^\omega\|^2_{L^2_\mu}, \quad z \in \mathbb{D}. \tag{7.14}$$

We now embark on the proofs by considering the cases $p \leq q$ and $p > q$ separately.

7.3.1 Case $1 < p \leq q < \infty$

We first consider bounded Toeplitz operators.

Proof of Theorem 11. Since $\frac{p+q'}{pq'} \geq 1$ by the hypothesis $q \geq p$, the equivalence (iii)⇔(iv) and the estimate

$$\|Id\|^{\frac{s(p+q')}{pq'}}_{A^s_\omega \to L_\mu^{\frac{s(p+q')}{pq'}}} \asymp \sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p}+\frac{1}{q'}}}$$

follow by [55, Theorem 1], see also [60, Theorem 3] and [54, Theorem 2.1].

If $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ is bounded, then Hölder's inequality and Theorem B yield

$$\begin{aligned}
\left|\widetilde{\mathcal{T}}_\mu(z)\right| &= \left|\langle \mathcal{T}_\mu(b_z^\omega), b_z^\omega \rangle_{A^2_\omega}\right| \leq \|\mathcal{T}_\mu(b_z^\omega)\|_{A^q_\omega} \|b_z^\omega\|_{A^{q'}_\omega} \leq \|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \|b_z^\omega\|_{A^p_\omega} \|b_z^\omega\|_{A^{q'}_\omega} \\
&= \|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \frac{\|B_z^\omega\|_{A^p_\omega} \|B_z^\omega\|_{A^{q'}_\omega}}{\|B_z^\omega\|^2_{A^2_\omega}} \asymp \|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \frac{\omega(S(z))}{\omega(S(z))^{1-\frac{1}{p}} \omega(S(z))^{1-\frac{1}{q'}}} \\
&\lesssim \|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \frac{1}{\omega(S(z))^{1-\frac{1}{p}-\frac{1}{q'}}}, \quad z \in \mathbb{D},
\end{aligned}$$

and hence $\left\| \frac{\widetilde{\mathcal{T}}_\mu(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \right\|_{L^\infty} \lesssim \|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega}$.

Assume next $\frac{\widetilde{\mathcal{T}}_\mu(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \in L^\infty$, and let $\delta = \delta(\omega)$ and $c = c(\omega)$ be those of Lemma 3. Then Theorem B and (7.14) give

$$\begin{aligned}
\frac{\widetilde{\mathcal{T}}_\mu(z)}{\omega(S(z))} &\asymp \|B_z^\omega\|^2_{A^2_\omega} \widetilde{\mathcal{T}}_\mu(z) = \|B_z^\omega\|^2_{L^2_\mu} \\
&\geq \int_{S(z_\delta)} |B_z^\omega(\zeta)|^2 \, d\mu(\zeta) \geq c^2 \frac{\mu(S(z_\delta))}{\omega(S(z))^2}, \quad z \in \mathbb{D} \setminus \{0\},
\end{aligned}$$

and hence $\mu(S(z_\delta)) \lesssim \widetilde{\mathcal{T}}_\mu(z)\omega(S(z))$ for all $z \in \mathbb{D} \setminus \{0\}$. It follows from Lemma E that

$$\sup_I \frac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p}+\frac{1}{q'}}} \lesssim \left\| \frac{\widetilde{\mathcal{T}}_\mu(\cdot)}{\omega(S(\cdot))^{\frac{1}{p}+\frac{1}{q'}-1}} \right\|_{L^\infty},$$

and hence (ii)⇒(iv).

If now μ is a $\frac{s(p+q')}{pq'}$-Carleson measure for A^s_ω, that is, μ is a 1-Carleson measure for $A_\omega^{\frac{pq'}{p+q'}}$ by [55, Theorem 1], then Lemma 7, [60, Theorem 3] and Hölder's inequality yield

$$
\begin{aligned}
\left|\langle \mathcal{T}_\mu(f), g\rangle_{A^2_\omega}\right| &\le \int_{\mathbb{D}} |f(z)g(z)|\,d\mu(z) \lesssim \|Id\|_{A_\omega^{\frac{pq'}{p+q'}} \to L^1_\mu} \left(\int_{\mathbb{D}} |f(z)g(z)|^{\frac{pq'}{p+q'}}\omega(z)\,dA(z)\right)^{\frac{p+q'}{pq'}} \\
&\lesssim \|Id\|_{A_\omega^{\frac{pq'}{p+q'}} \to L^1_\mu} \|f\|_{A^p_\omega}\|g\|_{A^{q'}_\omega}
\end{aligned}
$$

for all polynomials f and g. Since polynomials are dense in both A^p_ω and $A^{q'}_\omega$, and $(A^q_\omega)^\star \simeq A^{q'}_\omega$ by Theorem C, it follows that $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ is bounded and $\|\mathcal{T}_\mu\|_{A^p_\omega \to A^q_\omega} \lesssim \|Id\|_{A_\omega^{\frac{pq'}{p+q'}} \to L^1_\mu}$. This is the right upper bound for $s = \frac{pq'}{p+q'}$, and the general case follows by an application of [60, Theorem 3]. □

Now we turn to compact Toeplitz operators.

Proposition 2. *Let $1 < p \le q < \infty$ and $\omega \in \mathcal{R}$. If $T : A^p_\omega \to A^q_\omega$ is a compact linear operator, then*

$$
\lim_{|z|\to 1^-} \frac{\widetilde{T}(z)}{\omega(S(z))^{\frac{1}{p}+\frac{1}{q'}-1}} = 0.
$$

Proof. Since $b^\omega_{p,z} \to 0$ weakly in A^p_ω, as $|z| \to 1^-$, by Lemma 6, and $T : A^p_\omega \to A^q_\omega$ is compact, and in particular completely continuous, by the hypothesis, we deduce

$$
\left\|T\left(b^\omega_{p,z}\right)\right\|_{A^q_\omega} \to 0, \quad |z| \to 1^-.
$$

By Hölder's inequality this implies

$$
\left|\left\langle T\left(b^\omega_{p,z}\right), b^\omega_{q',z}\right\rangle_{A^2_\omega}\right| \to 0, \quad |z| \to 1^-.
$$

Moreover, by Theorem B,

$$
\begin{aligned}
\|B^\omega_z\|_{A^p_\omega}\|B^\omega_z\|_{A^{q'}_\omega} &\asymp \frac{1}{\widehat{\omega}(z)^{1-\frac{1}{p}}(1-|z|)^{1-\frac{1}{p}}} \frac{1}{\widehat{\omega}(z)^{1-\frac{1}{q'}}(1-|z|)^{1-\frac{1}{q'}}} \\
&\asymp \|B^\omega_z\|^2_{A^2_\omega} \frac{1}{\widehat{\omega}(z)^{1-\frac{1}{p}-\frac{1}{q'}}(1-|z|)^{1-\frac{1}{p}-\frac{1}{q'}}} \asymp \|B^\omega_z\|^2_{A^2_\omega} \frac{1}{\omega(S(z))^{1-\frac{1}{p}-\frac{1}{q'}}},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\left|\widetilde{T}(z)\right| \omega(S(z))^{1-\frac{1}{p}-\frac{1}{q'}} &= \left|\langle T(b^\omega_z), b^\omega_z\rangle_{A^2_\omega}\right| \omega(S(z))^{1-\frac{1}{p}-\frac{1}{q'}} \\
&\asymp \left|\left\langle T\left(b^\omega_{p,z}\right), b^\omega_{q',z}\right\rangle_{A^2_\omega}\right| \to 0, \quad |z| \to 1^-,
\end{aligned}
$$

and the assertion is proved. □

The following result is the analogue of Theorem 11 for compact Toeplitz operators.

Theorem 15. *Let $1 < p \leq q < \infty$, $\omega \in \mathcal{R}$ and μ be a positive Borel measure on $\mathbb{D}$. Then the following statements are equivalent:*

(i) $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ *is compact;*

(ii) $\lim_{|z|\to 1^-} \dfrac{\widetilde{\mathcal{T}_\mu}(z)}{\omega(S(z))^{\frac{1}{p}+\frac{1}{q'}-1}} = 0;$

(iii) $Id : A^s_\omega \to L_\mu^{\frac{s(p+q')}{pq'}}$ *is compact for some (equivalently for all)* $0 < s < \infty$*;*

(iv) $\lim_{|I|\to 0} \dfrac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p}+\frac{1}{q'}}} = 0.$

Proof. The equivalence (iii)$\Leftrightarrow$(iv) follows from [60, Theorem 3], see also [54, Theorem 2.1]. If $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ is compact, then $\lim_{|z|\to 1^-} \frac{\widetilde{\mathcal{T}_\mu}(z)}{\omega(S(z))^{\frac{1}{p}+\frac{1}{q'}-1}} = 0$ by Proposition 2. Assume next that (ii) is satisfied, and let $\delta = \delta(\omega) \in (0,1)$ be that of Lemma 3. By the proof of Theorem 11, there exists a constant $C = C(\omega) > 0$ such that $\mu(S(z_\delta)) \leq C\widetilde{\mathcal{T}_\mu}(z)\omega(S(z))$ for all $z \in \mathbb{D} \setminus \{0\}$. By applying Lemma E, and letting $|z| \to 1^-$, it follows by the assumption (ii) that $\lim_{|z|\to 1^-} \frac{\mu(S(z))}{\omega(S(z))^{\frac{p+q'}{pq'}}} = 0$, and thus $Id : A^s_\omega \to L_\mu^{\frac{s(p+q')}{pq'}}$ is compact by [60, Theorem 3].

Assume now that $Id : A^s_\omega \to L_\mu^{\frac{s(p+q')}{pq'}}$ is compact for some (equivalently for all) $0 < s < \infty$. Then, by [60, Theorem 3], $Id : A^p_\omega \to L_\mu^{\frac{p+q'}{q'}}$ and $Id : A^{q'}_\omega \to L_\mu^{\frac{p+q'}{p}}$ are compact. Let $\{f_n\}$ be a bounded sequence in A^p_ω. Then the proof of [54, Theorem 2.1] shows that there exists a subsequence $\{f_{n_k}\}$ and $f \in A^p_\omega$ such that $\lim_{k\to\infty} \|f_{n_k} - f\|_{L_\mu^{\frac{p+q'}{q'}}} = 0$. Write $\mu_r = \chi_{D(0,r)}\mu$ for $0 < r < 1$. Then Theorem 11 yields

$$\begin{aligned} \|\mathcal{T}_\mu(f_{n_k} - f)\|_{A^q_\omega} &\leq \|\mathcal{T}_{\mu_r}(f_{n_k} - f)\|_{A^q_\omega} + \|(\mathcal{T}_\mu - \mathcal{T}_{\mu_r})(f_{n_k} - f)\|_{A^q_\omega} \\ &\lesssim \|\mathcal{T}_{\mu_r}(f_{n_k} - f)\|_{A^q_\omega} + \|\mathcal{T}_\mu - \mathcal{T}_{\mu_r}\|_{A^p_\omega \to A^q_\omega}, \end{aligned}$$

where

$$\|\mathcal{T}_\mu - \mathcal{T}_{\mu_r}\|_{A^p_\omega \to A^q_\omega} \lesssim \sup_I \frac{\mu(S(I) \setminus D(0,r))}{\omega(S(I))^{\frac{1}{p}+\frac{1}{q'}}} \lesssim \sup_{|I|\leq 1-r} \frac{\mu(S(I))}{\omega(S(I))^{\frac{1}{p}+\frac{1}{q'}}} \to 0, \quad r \to 1^-,$$

by Theorem 11 and [60, Theorem 3], because $Id : A^1_\omega \to L_\mu^{\frac{p+q'}{pq'}}$ is compact by the hypothesis. Moreover, (7.3), Theorem 11 and Hölder's inequality yield

$$\begin{aligned}\left|\langle \mathcal{T}_{\mu_r}(f_{n_k}-f),g\rangle_{A^2_\omega}\right| &\le \int_{\mathbb{D}} |(f_{n_k}-f)(z)g(z)|\,d\mu_r(z) \le \|f_{n_k}-f\|_{L_{\mu_r}^{\frac{p+q'}{q'}}}\|g\|_{L_{\mu_r}^{\frac{p+q'}{p}}}\\ &\le \|f_{n_k}-f\|_{L_{\mu_r}^{\frac{p+q'}{q'}}}\|Id\|_{A^{q'}_\omega\to L_{\mu_r}^{\frac{p+q'}{p}}}\|g\|_{A^{q'}_\omega}\\ &\le \|f_{n_k}-f\|_{L_{\mu}^{\frac{p+q'}{q'}}}\|Id\|_{A^{q'}_\omega\to L_{\mu}^{\frac{p+q'}{p}}}\|g\|_{A^{q'}_\omega}.\end{aligned}$$

Since $(A^q_\omega)^\star \simeq A^{q'}_\omega$ by Theorem C, we obtain

$$\begin{aligned}\|\mathcal{T}_{\mu_r}(f_{n_k}-f)\|_{A^q_\omega} &\asymp \sup_{\left\{g:\|g\|_{A^{q'}_\omega}\le 1\right\}}\left|\langle \mathcal{T}_{\mu_r}(f_{n_k}-f),g\rangle_{A^2_\omega}\right|\\ &\le \|Id\|_{A^{q'}_\omega\to L_{\mu}^{\frac{p+q'}{p}}}\|f_{n_k}-f\|_{L_{\mu}^{\frac{p+q'}{q'}}}\to 0,\quad k\to\infty.\end{aligned}$$

Thus $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ is compact, and the proof is complete. □

7.3.2 Case $1 < q < p < \infty$

We begin with constructing appropriate test functions to be used in the proof of Theorem 12. To do this, some notation is needed. The Euclidean discs are denoted by $D(a,r) = \{z \in \mathbb{C} : |a-z| < r\}$. A sequence $Z = \{z_k\}_{k=0}^\infty \subset \mathbb{D}$ is called separated if it is separated in the pseudohyperbolic metric, it is an ε-net for $\varepsilon \in (0,1)$ if $\mathbb{D} = \bigcup_{k=0}^\infty \Delta(z_k,\varepsilon)$, and finally it is a δ-lattice if it is a 5δ-net and separated with constant $\delta/5$.

Proposition 3. *Let $1 < p < \infty$, $\omega \in \mathcal{R}$ and $\{z_j\}_{j=1}^\infty \subset \mathbb{D}\setminus\{0\}$ be a separated sequence. Then $F = \sum_{j=1}^\infty c_j b^\omega_{p,z_j} \in A^p_\omega$ with $\|F\|_{A^p_\omega} \lesssim \|\{c_j\}_{j=1}^\infty\|_{\ell^p}$ for all $\{c_j\}_{j=1}^\infty \in \ell^p$.*

Proof. Let $\{c_j\}_{j=1}^\infty \in \ell^p$, $0 < r < 1$ and $z \in \overline{D(0,\rho)}$ with $0 < \rho < 1$. Then Hölder's inequality and Theorem B yield

$$\left|\sum_{j=1}^\infty c_j b^\omega_{p,z_j}(z)\right| \lesssim \|\{c_j\}_{j=1}^\infty\|_{\ell^p}\left(\sum_{j=1}^\infty \omega(\Delta(z_j,r))|B^\omega_{z_j}(z)|^{p'}\right)^{1/p'} \le C(\omega,\rho)\|\{c_j\}_{j=1}^\infty\|_{\ell^p}\omega(\mathbb{D}),$$

and hence $F \in \mathcal{H}(\mathbb{D})$. Moreover, by Hölder's inequality, Theorem B, (7.6), the subharmonicity of $|g|^{p'}$ and (7.5),

$$\begin{aligned}\left|\langle g,F\rangle_{A^2_\omega}\right| &= \left|\sum_{j=1}^\infty \overline{c_j}\frac{g(z_j)}{\|B^\omega_z\|_{A^p_\omega}}\right| \lesssim \|\{c_j\}_{j=1}^\infty\|_{\ell^p}\left(\sum_{j=1}^\infty \omega(\Delta(z_j,r))|g(z_j)|^{p'}\right)^{1/p'}\\ &\lesssim \|\{c_j\}_{j=1}^\infty\|_{\ell^p}\left(\sum_{j=1}^\infty \omega(z_j)\int_{\Delta(z_j,r)}|g(z)|^{p'}\,dA(z)\right)^{1/p'}\\ &\asymp \|\{c_j\}_{j=1}^\infty\|_{\ell^p}\left(\sum_{j=1}^\infty \int_{\Delta(z_j,r)}|g(z)|^{p'}\omega(z)\,dA(z)\right)^{1/p'} \lesssim \|\{c_j\}_{j=1}^\infty\|_{\ell^p}\|g\|_{A^{p'}_\omega},\end{aligned}$$

where in the last step the fact that each $z \in \mathbb{D}$ belongs to at most N of the discs $\Delta(z_j, r)$ is also used. Therefore F defines a bounded linear functional on $A^{p'}_\omega$ with norm bounded by a constant times $\|\{c_j\}_{j=1}^\infty\|_{\ell^p}$. Since $(A^{p'}_\omega)^\star \simeq A^p_\omega$ by Theorem C, this implies $F \in A^p_\omega$ with $\|F\|_{A^p_\omega} \lesssim \|\{c_j\}_{j=1}^\infty\|_{\ell^p}$. $\square$

Proof of Theorem 12. Write $x = x(p,q) = p + 1 - \frac{p}{q}$ for short. Assume first (ii). Take $\{a_j\}_{j=1}^\infty \subset \mathbb{D} \setminus \{0\}$ a separated sequence. Then Proposition 3 gives

$$\left\| \mathcal{T}_\mu \left(\sum_{j=1}^\infty c_j b^\omega_{p,a_j} \right) \right\|^q_{A^q_\omega} \lesssim \|\mathcal{T}_\mu\|^q_{A^p_\omega \to A^q_\omega} \|\{c_j\}_{j=1}^\infty\|^q_{\ell^p}.$$

By replacing c_k by $r_k(t)c_k$, where r_k denotes the kth Rademacher function, and applying Khinchine's inequality, we deduce

$$\begin{aligned} \|\mathcal{T}_\mu\|^q_{A^p_\omega \to A^q_\omega} \|\{c_j\}_{j=1}^\infty\|^q_{\ell^p} &\gtrsim \int_{\mathbb{D}} \left(\sum_{j=1}^\infty |c_j|^2 |\mathcal{T}_\mu(b^\omega_{p,a_j})(z)|^2 \right)^{q/2} \omega(z)\, dA(z) \\ &\gtrsim \sum_{j=1}^\infty |c_j|^q \int_{\Delta(a_j,s)} |\mathcal{T}_\mu(b^\omega_{p,a_j})(z)|^q \omega(z)\, dA(z), \quad 0 < s < 1, \end{aligned} \tag{7.15}$$

where in the last step the fact that each $z \in \mathbb{D}$ belongs to at most $N = N(s)$ of the discs $\Delta(a_j, s)$ is also used. By using the subharmonicity of $|\mathcal{T}_\mu(b^\omega_{p,a_j})|^q$ together with (7.5) and (7.6), and then applying Lemma 4 and Theorem B, we obtain

$$\begin{aligned} \int_{\Delta(a_j,s)} |\mathcal{T}_\mu(b^\omega_{p,a_j})(z)|^q \omega(z)\, dA(z) &\gtrsim \omega\left(\Delta(a_j,s)\right) |\mathcal{T}_\mu(b^\omega_{p,a_j})(a_j)|^q \\ &= \frac{\omega\left(\Delta(a_j,s)\right)}{\|B^\omega_{a_j}\|^q_{A^p_\omega}} \left(\int_{\mathbb{D}} |B^\omega_{a_j}(\zeta)|^2\, d\mu(\zeta) \right)^q \\ &\geq \frac{\omega\left(\Delta(a_j,s)\right)}{\|B^\omega_{a_j}\|^q_{A^p_\omega}} \left(\int_{\Delta(a_j,s)} |B^\omega_{a_j}(\zeta)|^2\, d\mu(\zeta) \right)^q \\ &\gtrsim \frac{\omega\left(\Delta(a_j,s)\right)}{\|B^\omega_{a_j}\|^q_{A^p_\omega}} \mu\left(\Delta(a_j,s)\right)^q |B^\omega_{a_j}(a_j)|^{2q} \\ &\asymp \left(\frac{\mu\left(\Delta(a_j,s)\right)}{\omega\left(\Delta(a_j,s)\right)^{1+\frac{1}{p}-\frac{1}{q}}} \right)^q, \quad 0 < s \leq r(\omega), \end{aligned}$$

where $r(\omega)$ is that of Lemma 4. This together with (7.15) yields

$$\sum_{j=1}^\infty |c_j|^q \left(\frac{\mu\left(\Delta(a_j,s)\right)}{\omega\left(\Delta(a_j,s)\right)^{1+\frac{1}{p}-\frac{1}{q}}} \right)^q \lesssim \|\mathcal{T}_\mu\|^q_{A^p_\omega \to A^q_\omega} \|\{c_j\}_{j=1}^\infty\|^q_{\ell^p}, \quad 0 < s \leq r(\omega). \tag{7.16}$$

Let now $s \in (r(\omega), 1)$ and $Z = \{z_j\}_{j=1}^\infty \subset \mathbb{D} \setminus \{0\}$ a δ-lattice with $5\delta \leq r(\omega)$. For each z_j choose $N = N(s, r(\omega))$ points $z_{k,j}$ of the δ-lattice Z such that $\Delta(z_j, s) \subset$

$\cup_{k=1}^{N}\Delta(z_{k,j},r(\omega))$. Then, by (7.5), (7.6) and (7.16),

$$\sum_{j=1}^{\infty}|c_j|^q\left(\frac{\mu\left(\Delta(z_j,s)\right)}{\omega\left(\Delta(z_j,s)\right)^{1+\frac{1}{p}-\frac{1}{q}}}\right)^q \lesssim \sum_{j=1}^{\infty}\sum_{k=1}^{N}|c_j|^q\left(\frac{\mu\left(\Delta(z_{k,j},r(\omega))\right)}{\omega\left(\Delta(z_{k,j},r(\omega))\right)^{1+\frac{1}{p}-\frac{1}{q}}}\right)^q$$
$$= \sum_{k=1}^{N}\sum_{j=1}^{\infty}|c_j|^q\left(\frac{\mu\left(\Delta(z_{k,j},r(\omega))\right)}{\omega\left(\Delta(z_{k,j},r(\omega))\right)^{1+\frac{1}{p}-\frac{1}{q}}}\right)^q$$
$$\lesssim \|\mathcal{T}_\mu\|^q_{A^p_\omega\to A^q_\omega}\|\{c_j\}_{j=1}^{\infty}\|^q_{\ell^p}.$$

Therefore (7.16) holds for each $0<s<1$ and any δ-lattice $\{z_j\}_{j=1}^{\infty}\subset\mathbb{D}\setminus\{0\}$ with $5\delta\le r(\omega)$. The classical duality relation $\left(\ell^{p/q}\right)^\star\simeq\ell^{\frac{p}{p-q}}$ now implies

$$\sum_{j=1}^{\infty}\left(\frac{\mu\left(\Delta(z_j,s)\right)}{\omega\left(\Delta(z_j,s)\right)}\right)^{\frac{qp}{p-q}}\omega\left(\Delta(z_j,s)\right)=\sum_{j=1}^{\infty}\left(\frac{\mu\left(\Delta(z_j,s)\right)}{\omega\left(\Delta(z_j,s)\right)^{1+\frac{1}{p}-\frac{1}{q}}}\right)^{\frac{qp}{p-q}}\lesssim\|\mathcal{T}_\mu\|^{\frac{pq}{p-q}}_{A^p_\omega\to A^q_\omega}.$$

Let $0<r<1$, and choose $s=s(r,\delta)\in(0,1)$ such that $\Delta(z,r)\subset\Delta(z_j,s)$ for all $z\in\Delta(z_j,5\delta)$ and $j\in\mathbb{N}$. Then (7.5) and (7.6) imply

$$\begin{aligned}\|\widehat{\mu}_r\|^{\frac{qp}{p-q}}_{L^{\frac{qp}{p-q}}_\omega}&\le\sum_{j=1}^{\infty}\int_{\Delta(z_j,5\delta)}\widehat{\mu}_r(z)^{\frac{qp}{p-q}}\omega(z)\,dA(z)\\&\asymp\sum_{j=1}^{\infty}\frac{\omega(z_j)}{(\omega(z_j)(1-|z_j|)^2)^{\frac{qp}{p-q}}}\int_{\Delta(z_j,5\delta)}\mu(\Delta(z,r))^{\frac{qp}{p-q}}\,dA(z)\\&\asymp\sum_{j=1}^{\infty}\frac{\mu(\Delta(z_j,s))^{\frac{qp}{p-q}}}{(\omega(z_j)(1-|z_j|)^2)^{\frac{qp}{p-q}-1}}\asymp\sum_{j=1}^{\infty}\left(\frac{\mu\left(\Delta(z_j,s)\right)}{\omega\left(\Delta(z_j,s)\right)^{1+\frac{1}{p}-\frac{1}{q}}}\right)^{\frac{qp}{p-q}}.\end{aligned}\tag{7.17}$$

Thus (iii) is satisfied and $\|\widehat{\mu}_r\|_{L^{\frac{qp}{p-q}}_\omega}\lesssim\|\mathcal{T}_\mu\|_{A^p_\omega\to A^q_\omega}$ for each fixed $0<r<1$.

Assume next (iii). By using the subharmonicity of $|f|^x$ together with (7.5) and (7.6), and then Fubini's theorem and Hölder's inequality we deduce

$$\begin{aligned}\int_{\mathbb{D}}|f(z)|^x\,d\mu(z)&\lesssim\int_{\mathbb{D}}\frac{\int_{\Delta(z,r)}|f(\zeta)|^x\omega(\zeta)\,dA(\zeta)}{\omega\left(\Delta(z,r)\right)}\,d\mu(z)\\&\asymp\int_{\mathbb{D}}\frac{\mu\left(\Delta(\zeta,r)\right)}{\omega\left(\Delta(\zeta,r)\right)}|f(\zeta)|^x\omega(\zeta)\,dA(\zeta)\le\|f\|^x_{A^p_\omega}\|\widehat{\mu}_r\|_{L^{\frac{qp}{p-q}}_\omega}.\end{aligned}$$

Therefore μ is a $\left(p+1-\frac{p}{q}\right)$-Carleson measure for A^p_ω, that is, (iv) is satisfied, and $\|Id\|^x_{A^p_\omega\to L^q_\mu}\lesssim\|\widehat{\mu}_r\|_{L^{\frac{qp}{p-q}}_\omega}$. In fact, it follows from [21, Theorem 3.2] and [54, Lemma 1.4] that $Id:A^p_\omega\to L^x_\mu$ is bounded if and only if (iii) is satisfied.

The equivalence (iv)$\Leftrightarrow$(v) follows from [60, Theorem 3].

Let us now prove (iv)⇒(ii). Since μ is an x-Carleson measure for A^p_ω by the hypothesis (iv), Lemma 7, Hölder's inequality and [60, Theorem 3] together with the equality $\frac{x}{p} = \frac{x'}{q'}$ give

$$\begin{aligned}\left|\langle \mathcal{T}_\mu(f), g\rangle_{A^2_\omega}\right| &\le \int_{\mathbb{D}} |f(z)g(z)|\,d\mu(z) \le \|f\|_{L^x_\mu}\|g\|_{L^{x'}_\mu}\\ &\le \|Id\|_{A^p_\omega\to L^x_\mu}\|Id\|_{A^{q'}_\omega\to L^{x'}_\mu}\|f\|_{A^p_\omega}\|g\|_{A^{q'}_\omega}\\ &\lesssim \|Id\|^x_{A^p_\omega\to L^x_\mu}\|f\|_{A^p_\omega}\|g\|_{A^{q'}_\omega}\end{aligned} \tag{7.18}$$

for polynomials f and g. Since polynomials are dense in both $A^{q'}_\omega$ and A^p_ω, and $(A^q_\omega)^\star \simeq A^{q'}_\omega$ by Theorem C, it follows that $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ is bounded and $\|\mathcal{T}_\mu\|_{A^p_\omega\to A^q_\omega} \lesssim \|Id\|^x_{A^p_\omega\to L^x_\mu}$.

The implication (ii)⇒(i) follows by a general argument. Namely, for $1<p<\infty$, A^p_ω is isomorphic to ℓ^p by [43, Corollary 2.6] and Lemma B. Moreover, each bounded linear operator $L : \ell^p \to \ell^q$, $0<q<p<\infty$, is compact by [38, Theorem I. 2.7, p. 31]. Thus $\mathcal{T}_\mu : A^p_\omega \to A^q_\omega$ is compact.

It remains to prove (iii)⇔(vi) and the equivalence of norms $\|\widehat{\mu}_r\|_{L^{\frac{qp}{p-q}}_\omega} \asymp \|\widetilde{\mathcal{T}}_\mu\|_{L^{\frac{qp}{p-q}}_\omega}$ for each fixed $r\in(0,1)$. Assume $\widetilde{\mathcal{T}}_\mu \in L^{\frac{pq}{p-q}}_\omega$, and let first $r\in(0,r(\omega)]$, where $r(\omega)$ is that of Lemma 4. Then Lemma 7 and Theorem B give

$$\widetilde{\mathcal{T}}_\mu(z) = \int_{\mathbb{D}} |b^\omega_z(\zeta)|^2\,d\mu(\zeta) \ge \int_{\Delta(z,r)} |b^\omega_z(\zeta)|^2\,d\mu(\zeta) \asymp |b^\omega_z(z)|^2\mu(\Delta(z,r)) \asymp \widehat{\mu}_r(z). \tag{7.19}$$

Hence $\widehat{\mu}_r \in L^{\frac{pq}{p-q}}_\omega$ and $\|\widehat{\mu}_r\|_{L^{\frac{qp}{p-q}}_\omega} \lesssim \|\widetilde{\mathcal{T}}_\mu\|_{L^{\frac{qp}{p-q}}_\omega}$. Let now $r\in(r(\omega),1)$, and let $\{z_j\}$ be a δ-lattice. Further, let $s=s(r,\delta)$ be that of (7.17), and choose $r'=r'(r(\omega))$ such that $\Delta(z,r')\subset\Delta(w,r(\omega))$ for all $z\in\Delta(w,r')$ and $w\in\mathbb{D}$. Furthermore, choose $z^n_j\in\Delta(z_j,s)$, $n=1,\dots,N$, such that $\Delta(z_j,s)\subset\cup_{n=1}^N\Delta(z^n_j,r')$ for all j and $\inf_j\min_{n\ne m}\varrho(z^n_j,z^m_j)>0$. Then (7.17), Lemma E, Lemma 4, Theorem B and (7.14) yield

$$\begin{aligned}\|\widehat{\mu}_r\|^{\frac{qp}{p-q}}_{L^{\frac{qp}{p-q}}_\omega} &\lesssim \sum_{j=1}^\infty \frac{\mu(\Delta(z_j,s))^{\frac{qp}{p-q}}}{\omega(\Delta(z_j,s))^{\frac{qp}{p-q}-1}} \lesssim \sum_{j=1}^\infty\sum_{n=1}^N \omega(\Delta(z_j,s))\left(\frac{\mu(\Delta(z^n_j,r'))}{\omega(\Delta(z_j,s))}\right)^{\frac{qp}{p-q}}\\ &\asymp \sum_{j=1}^\infty\sum_{n=1}^N \omega(\Delta(z^n_j,r'))\widehat{\mu}_{r'}(z^n_j)^{\frac{qp}{p-q}}\\ &\asymp \sum_{j=1}^\infty\sum_{n=1}^N \int_{\Delta(z^n_j,r')}\left(\int_{\Delta(z^n_j,r')}|b^\omega_z(\zeta)|^2\,d\mu(\zeta)\right)^{\frac{qp}{p-q}}\omega(z)\,dA(z)\\ &\le \sum_{j=1}^\infty\sum_{n=1}^N \int_{\Delta(z^n_j,r')}\left(\widetilde{\mathcal{T}}_\mu(z)\right)^{\frac{qp}{p-q}}\omega(z)\,dA(z) \asymp \|\widetilde{\mathcal{T}}_\mu\|^{\frac{qp}{p-q}}_{L^{\frac{qp}{p-q}}_\omega}.\end{aligned}$$

Now assume (iii), and let h be a positive subharmonic function in $\mathbb{D}$. Then (7.5), (7.6) and Fubini's theorem yield

$$\int_{\mathbb{D}} h(z)\,d\mu(z) \lesssim \int_{\mathbb{D}} \left(\frac{1}{(1-|z|)^2} \int_{\Delta(z,r)} h(\zeta)\,dA(\zeta) \right) d\mu(z)$$
$$\asymp \int_{\mathbb{D}} \left(\int_{\Delta(z,r)} h(\zeta) \frac{\omega(\zeta)}{\omega(\Delta(\zeta,r))}\,dA(\zeta) \right) d\mu(z) = \int_{\mathbb{D}} h(\zeta)\widehat{\mu}_r(\zeta)\omega(\zeta)\,dA(\zeta).$$

This together with (7.14), Theorem B, and Lemma 2 yield

$$\begin{aligned}\widetilde{\mathcal{T}}_\mu(z) &= \int_{\mathbb{D}} |b_z^\omega(\zeta)|^2\,d\mu(\zeta) \lesssim \int_{\mathbb{D}} |b_z^\omega(\zeta)|^2 \widehat{\mu}_r(\zeta)\omega(\zeta)\,dA(\zeta)\\ &= \int_{\mathbb{D}} |B_z^\omega(\zeta)| \frac{|B_z^\omega(\zeta)|}{\|B_z^\omega\|_{A^2_\omega}^2} \widehat{\mu}_r(\zeta)\omega(\zeta)\,dA(\zeta)\\ &\le \frac{\|B_z^\omega\|_{H^\infty}}{\|B_z^\omega\|_{A^2_\omega}^2} \int_{\mathbb{D}} |B_z^\omega(\zeta)|\widehat{\mu}_r(\zeta)\omega(\zeta)\,dA(\zeta) \asymp P_\omega^+(\widehat{\mu}_r)(z).\end{aligned}$$

But $P_\omega^+ : L_\omega^{\frac{pq}{p-q}} \to L_\omega^{\frac{pq}{p-q}}$ is bounded by [57, Theorem 5] because $\frac{pq}{p-q} > 1$ and $\omega \in \mathcal{R}$, and hence

$$\|\widetilde{\mathcal{T}}_\mu\|_{L_\omega^{\frac{pq}{p-q}}} \lesssim \|P_\omega^+(\widehat{\mu}_r)\|_{L_\omega^{\frac{pq}{p-q}}} \lesssim \|\widehat{\mu}_r\|_{L_\omega^{\frac{pq}{p-q}}} < \infty.$$

This finishes the proof. □

7.4 SCHATTEN CLASS TOEPLITZ OPERATORS

The purpose of this section is to prove Theorem 13, or more precisely, the last part of it, and then show that it can not be extended to the whole class $\widehat{\mathcal{D}}$ of doubling weights. We begin with some necessary notation and definitions, and preliminary results which are well-known in the setting of standard weights [69].

Let H be a separable Hilbert space. For any non-negative integer n, the n:th singular value of a bounded operator $T : H \to H$ is defined by

$$\lambda_n(T) = \inf\{\|T - R\| : \operatorname{rank}(R) \le n\},$$

where $\|\cdot\|$ denotes the operator norm. It is clear that

$$\|T\| = \lambda_0(T) \ge \lambda_1(T) \ge \lambda_2(T) \ge \cdots \ge 0.$$

For $0 < p < \infty$, the Schatten p-class $\mathcal{S}_p(H)$ consists of those compact operators $T : H \to H$ whose sequence of singular values $\{\lambda_n\}_{n=0}^\infty$ belongs to the space ℓ^p of p-summable sequences. For $1 \le p < \infty$, the Schatten p-class $\mathcal{S}_p(H)$ is a Banach space with respect to the norm $|T|_p = \|\{\lambda_n\}_{n=0}^\infty\|_{\ell^p}$. Therefore all finite rank operators belong to every $\mathcal{S}_p(H)$, and the membership of an operator in $\mathcal{S}_p(H)$ measures in some sense the size of the operator. We refer to [23] and [69, Chapter 1] for more information about $\mathcal{S}_p(H)$.

The first auxiliary result is well known and its proof is straightforward, so the details are omitted.

Lemma D. *Let H be a separable Hilbert space and $T : H \to H$ a bounded linear operator such that $\sum_n |\langle T(e_n), e_n\rangle_H| < \infty$ for every orthonormal basis $\{e_n\}$. Then $T : H \to H$ is compact.*

The next result characterizes positive operators in the trace class $\mathcal{S}_1(A^2_\omega)$ in terms of their Berezin transforms.

Theorem 16. *Let $\omega \in \widehat{\mathcal{D}}$ and $T : A^2_\omega \to A^2_\omega$ a positive operator. Then $T \in \mathcal{S}_1(A^2_\omega)$ if and only if $\widetilde{T} \in L^1_{\omega/\omega^\star}$. Moreover, the trace of T satisfies*

$$tr(T) = \int_{\mathbb{D}} \widetilde{T}(z) \|B^\omega_z\|^2_{A^2_\omega} \omega(z)\, dA(z) \asymp \int_{\mathbb{D}} \widetilde{T}(z) \frac{\omega(z)}{\omega^\star(z)}\, dA(z).$$

Proof. The proof is similar to that of [69, Theorem 6.4], and is included for the convenience of the reader. Fix an orthonormal basis $\{e_n\}_{n=1}^\infty$ for A^2_ω. Since T is positive, [69, Theorem 1.23] and Lemma D show that $T \in \mathcal{S}_1(A^2_\omega)$ if and only if $\sum_{n=1}^\infty \langle T(e_n), e_n\rangle_{A^2_\omega} < \infty$, and further, $tr(T) = \sum_{n=1}^\infty \langle T(e_n), e_n\rangle_{A^2_\omega}$. Let $S = \sqrt{T}$. By the reproducing formula (7.2) and Parseval's identity, Theorem B and Lemma E, we have

$$\begin{aligned}
tr(T) &= \sum_{n=1}^\infty \langle T(e_n), e_n\rangle_{A^2_\omega} = \sum_{n=1}^\infty \|S(e_n)\|^2_{A^2_\omega} \\
&= \sum_{n=1}^\infty \int_{\mathbb{D}} |S(e_n)(z)|^2 \omega(z)\, dA(z) = \int_{\mathbb{D}} \left(\sum_{n=1}^\infty |S(e_n)(z)|^2 \right) \omega(z)\, dA(z) \\
&= \int_{\mathbb{D}} \left(\sum_{n=1}^\infty \left| \langle S(e_n), B^\omega_z\rangle_{A^2_\omega} \right|^2 \right) \omega(z)\, dA(z) \\
&= \int_{\mathbb{D}} \left(\sum_{n=1}^\infty \left| \langle e_n, S(B^\omega_z)\rangle_{A^2_\omega} \right|^2 \right) \omega(z)\, dA(z) \\
&= \int_{\mathbb{D}} \|S(B^\omega_z)\|^2_{A^2_\omega} \omega(z)\, dA(z) = \int_{\mathbb{D}} \langle T(B^\omega_z), B^\omega_z\rangle_{A^2_\omega} \omega(z)\, dA(z) \\
&= \int_{\mathbb{D}} \widetilde{T}(z) \|B^\omega_z\|^2_{A^2_\omega} \omega(z)\, dA(z) \asymp \int_{\mathbb{D}} \widetilde{T}(z) \frac{\omega(z)}{\omega^\star(z)}\, dA(z),
\end{aligned}$$

and the assertion is proved. □

By combining Theorem 16 with [69, Proposition 1.31] we obtain the following result.

Lemma 8. *Let $\omega \in \widehat{\mathcal{D}}$ and $T : A^2_\omega \to A^2_\omega$ a positive operator.*

(i) If $1 \le p < \infty$ and $T \in \mathcal{S}_p(A^2_\omega)$, then $\widetilde{T} \in L^p_{\omega/\omega^\star}$ with $\|\widetilde{T}\|_{L^p_{\omega/\omega^\star}} \lesssim |T|^p_p$.

(ii) If $0 < p \le 1$ and $\widetilde{T} \in L^p_{\omega/\omega^\star}$, then $T \in \mathcal{S}_p(A^2_\omega)$ with $|T|^p_p \lesssim \|\widetilde{T}\|_{L^p_{\omega/\omega^\star}}$.

Recall that

$$\mathcal{T}_\Phi(f)(z) = P_\omega(f\Phi)(z) = \int_{\mathbb{D}} f(\zeta)\overline{B^\omega_z(\zeta)}\,\Phi(\zeta)\omega(\zeta)dA(\zeta), \quad f \in A^2_\omega,$$

for each non-negative function Φ on $\mathbb{D}$. We next establish a sufficient condition for $\mathcal{T}_\Phi$ to belong to $\mathcal{S}_p(A^2_\omega)$ for $1 \le p < \infty$.

Proposition 4. *Let $1 \le p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and $\Phi \in L^p_{\omega/\omega^\star}$ positive. Then $\mathcal{T}_\Phi \in \mathcal{S}_p(A^2_\omega)$ with $|\mathcal{T}_\Phi|_p \lesssim \|\Phi\|_{L^p_{\omega/\omega^\star}}$.*

Proof. We will follow the proof of [69, Proposition 7.11]. Assume first that Φ has compact support in $\mathbb{D}$. Then $\mathcal{T}_\Phi$ is a positive compact operator with canonical decomposition

$$\mathcal{T}_\Phi(f) = \sum_{n=1}^{\infty} \lambda_n \langle f, e_n \rangle_{A^2_\omega} e_n,$$

where $\{\lambda_n\}$ is the sequence of eigenvalues of $\mathcal{T}_\Phi$, and $\{e_n\}$ is an orthonormal set of A^2_ω. Therefore

$$\lambda_n = \langle \mathcal{T}_\Phi(e_n), e_n \rangle_{A^2_\omega} = \int_{\mathbb{D}} |e_n(z)|^2 \Phi(z)\omega(z)\, dA(z), \quad n \in \mathbb{N},$$

by (7.3). Since $p \geq 1$, the Hölder's inequality yields

$$\lambda_n^p \leq \int_{\mathbb{D}} |e_n(z)|^2 \Phi(z)^p \omega(z)\, dA(z),$$

and hence

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n^p &\leq \int_{\mathbb{D}} \sum_{n=1}^{\infty} |e_n(z)|^2 \Phi(z)^p \omega(z), dA(z) \\ &\leq \int_{\mathbb{D}} B_z^\omega(z) \Phi(z)^p \omega(z)\, dA(z) \asymp \int_{\mathbb{D}} \Phi(z)^p \frac{\omega(z)}{\omega^\star(z)}\, dA(z) \end{aligned} \tag{7.20}$$

by Theorem B. Thus $\mathcal{T}_\Phi \in \mathcal{S}_p(A^2_\omega)$.

To prove the general case, assume $\Phi \in L^p_{\omega/\omega^\star}$. Then Hölder's inequality and Lemma E yield

$$\begin{aligned} \lim_{|a|\to 1^-} \frac{\int_{S(a)} \Phi(z)\omega(z)\, dA(z)}{\omega(S(a))} &\leq \lim_{|a|\to 1^-} \left(\frac{\int_{S(a)} \Phi(z)^p \omega(z)\, dA(z)}{\omega(S(a))} \right)^{\frac{1}{p}} \\ &\lesssim \lim_{|a|\to 1^-} \left(\int_{S(a)} \Phi(z)^p \frac{\omega(z)}{\omega^\star(z)}\, dA(z) \right)^{\frac{1}{p}} = 0, \end{aligned}$$

and hence $\mathcal{T}_\Phi : A^2_\omega \to A^2_\omega$ is compact by Theorem 15.

Now write $\Phi_r = \Phi \chi_{D(0,r)}$, where $\chi_{D(0,r)}$ is the characteristic function of $D(0,r)$. Arguing as in (7.20) it follows that $\{\mathcal{T}_{\Phi_r}\}_{r\in(0,1)}$ is Cauchy in the Banach space $\big(\mathcal{S}_p(A^2_\omega), |\cdot|_p\big)$. Hence there exists $T \in \mathcal{S}_p(A^2_\omega)$ such that $\lim_{r\to 1^-} |\mathcal{T}_{\Phi_r} - T|_p = 0$. On the other hand, if f is a polynomial and $z \in \mathbb{D}$, then Lemma 7 and Hölder's inequality yield

$$\begin{aligned} |(\mathcal{T}_{\Phi_r} - \mathcal{T}_\Phi)(f)(z)| &= |\langle (\mathcal{T}_{\Phi_r} - \mathcal{T}_\Phi)(f), B_z^\omega \rangle_{A^2_\omega}| \\ &= \left| \int_{r<|\zeta|<1} f(\zeta) B_\zeta^\omega(z) \Phi(\zeta)\omega(\zeta)\, dA(\zeta) \right| \\ &\leq C\|f\|_{H^\infty} \int_{r<|\zeta|<1} \Phi(\zeta)\omega(\zeta)\, dA(\zeta) \\ &\leq C\|f\|_{H^\infty} \left(\int_{r<|\zeta|<1} \Phi(\zeta)^p \frac{\omega(\zeta)}{\omega^\star(\zeta)}\, dA(\zeta) \right)^{\frac{1}{p}} \\ &\quad \cdot \left(\int_{\mathbb{D}} \omega^\star(\zeta)^{p'-1} \omega(\zeta)\, dA(\zeta) \right)^{\frac{1}{p'}} \to 0, \quad r \to 1^-, \end{aligned}$$

where $C = C(z)$ is a constant. Thus $\mathcal{T}_{\Phi_r}(f) \to \mathcal{T}_\Phi(f)$ pointwise for any polynomial f. Since $\mathcal{T}_{\Phi_r}$ and $\mathcal{T}_\Phi$ are bounded on A^2_ω, and polynomials are dense in A^2_ω, we deduce that $\mathcal{T}_{\Phi_r}(f) \to \mathcal{T}_\Phi(f)$ pointwise for all $f \in A^2_\omega$. Therefore $\mathcal{T}_\Phi = T \in \mathcal{S}_p(A^2_\omega)$. □

We will need one more auxiliary result in the proof of Theorem 13.

Proposition 5. *Let $\omega \in \mathcal{R}$, $0 < r < 1$ and μ be a finite positive Borel measure on $\mathbb{D}$ such that $\mathcal{T}_{\widehat{\mu}_r} : A^2_\omega \to A^2_\omega$ is bounded. Then $\mathcal{T}_\mu : A^2_\omega \to A^2_\omega$ is bounded, and there exists $C = C(\omega, r) > 0$ such that $\langle \mathcal{T}_\mu(f), f\rangle_{A^2_\omega} \le C\langle \mathcal{T}_{\widehat{\mu}_r}(f), f\rangle_{A^2_\omega}$ for all $f \in A^2_\omega$.*

Proof. Note first that $\mathcal{T}_\mu : A^2_\omega \to A^2_\omega$ is bounded by Theorem 11 and [55, Theorem 1], see also [21, Theorem 3.1 and Theorem 4.1]. Let f be a polynomial. Then

$$|f(\zeta)|^2 \lesssim \frac{1}{(1-|\zeta|)^2}\int_{\Delta(\zeta,r)} |f(z)|^2\,dA(z) \asymp \int_{\Delta(\zeta,r)} \frac{|f(z)|^2}{(1-|z|)^2}\,dA(z), \quad \zeta \in \mathbb{D},$$

and hence Fubini's theorem, Lemma 7, Lemma E and (7.6) yield

$$\begin{aligned}
\langle \mathcal{T}_\mu(f), f\rangle_{A^2_\omega} &= \int_{\mathbb{D}} |f(\zeta)|^2\,d\mu(\zeta) \lesssim \int_{\mathbb{D}} \left(\int_{\Delta(\zeta,r)} \frac{|f(z)|^2}{(1-|z|)^2}\,dA(z)\right) d\mu(\zeta) \\
&= \int_{\mathbb{D}} \frac{|f(z)|^2}{(1-|z|)^2\omega(z)} \left(\int_{\Delta(z,r)} d\mu(\zeta)\right) \omega(z)\,dA(z) \\
&\asymp \int_{\mathbb{D}} \frac{|f(z)|^2}{\omega(\Delta(z,r))} \left(\int_{\Delta(z,r)} d\mu(\zeta)\right) \omega(z)\,dA(z) = \langle \mathcal{T}_{\widehat{\mu}_r}(f), f\rangle_{A^2_\omega}.
\end{aligned}$$

Since $\mathcal{T}_{\widehat{\mu}_r} : A^2_\omega \to A^2_\omega$ and $\mathcal{T}_\mu : A^2_\omega \to A^2_\omega$ are bounded, and polynomials are dense in A^2_ω, it follows that

$$\langle \mathcal{T}_\mu(f), f\rangle_{A^2_\omega} \lesssim \langle \mathcal{T}_{\widehat{\mu}_r}(f), f\rangle_{A^2_\omega}, \quad f \in A^2_\omega,$$

and the proof is complete. □

Proof of Theorem 13. The conditions (i)–(iii) are equivalent by [56, Theorem 1], so it suffices to prove the last claim which concerns the Berezin transform.

The assertion is valid for $p = 1$ and $\omega \in \widehat{\mathcal{D}}$ by Theorem 16. For $1 < p < \infty$, Lemma 8 shows that $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$ implies $\widetilde{\mathcal{T}}_\mu \in L^p_{\omega/\omega^\star}$ with $\|\widetilde{\mathcal{T}}_\mu\|_{L^p_{\omega/\omega^\star}} \lesssim |\mathcal{T}_\mu|_p$. To see the converse implication, let $r \in (0, r(\omega))$, where $r(\omega)$ is that of Lemma 4. If $\widetilde{\mathcal{T}}_\mu \in L^p_{\omega/\omega^\star}$, then $\widehat{\mu}_r \in L^p_{\omega/\omega^\star}$ with $\|\widehat{\mu}_r\|_{L^p_{\omega/\omega^\star}} \lesssim \|\widetilde{\mathcal{T}}_\mu\|_{L^p_{\omega/\omega^\star}}$ by (7.19). Therefore $\mathcal{T}_{\widehat{\mu}_r} \in \mathcal{S}_p(A^2_\omega)$ by Proposition 4, which in turn implies $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$ with $|\mathcal{T}_\mu|_p \lesssim \|\widetilde{\mathcal{T}}_\mu\|_{L^p_{\omega/\omega^\star}}$ by Proposition 5 and [69, Theorem 1.27].

Let now $0 < p < 1$. If $\widetilde{\mathcal{T}}_\mu \in L^p_{\omega/\omega^\star}$, then $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$ with $|\mathcal{T}_\mu|_p \lesssim \|\widetilde{\mathcal{T}}_\mu\|_{L^p_{\omega/\omega^\star}}$ by Lemma 8. Conversely, assume that $\mathcal{T}_\mu \in \mathcal{S}_p(A^2_\omega)$. Then (7.14) yields

$$\begin{aligned}
(\widetilde{\mathcal{T}}_\mu(z))^p &\asymp (\omega^\star(z))^p \|B^\omega_z\|^{2p}_{L^2_\mu} = (\omega^\star(z))^p \left(\sum_{R_j \in \Upsilon} \int_{R_j} |B^\omega_z(\zeta)|^2\,d\mu(\zeta)\right)^p \\
&\le (\omega^\star(z))^p \sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{\omega^\star(z_j)}\right)^p |B^\omega_z(\widetilde{z}_{j,z})|^{2p} (\omega^\star(z_j))^p, \quad |z| \ge \frac{1}{2},
\end{aligned}$$

where $\widetilde{z}_{j,z} \in \overline{R}_j$ such that $\sup_{\zeta \in R_j} |B_z^\omega(\zeta)| = |B_z^\omega(\widetilde{z}_{j,z})|$. Consequently,

$$
\begin{aligned}
\|\widetilde{\mathcal{T}}_\mu\|^p_{L^p_{\omega/\omega^\star}} &\le \sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{\omega^\star(z_j)} \right)^p (\omega^\star(z_j))^p \int_{\mathbb{D}} |B_z^\omega(\widetilde{z}_{j,z})|^{2p} (\omega^\star(z))^{p-1} \omega(z)\, dA(z) \\
&\asymp \sum_{R_j \in \Upsilon} \left(\frac{\mu(R_j)}{\omega^\star(z_j)} \right)^p (\omega^\star(z_j))^p \int_{\mathbb{D}} |B_z^\omega(\widetilde{z}_{j,z})|^{2p} \frac{\omega^\star(z))^p}{(1-|z|)^2}\, dA(z)
\end{aligned}
\tag{7.21}
$$

because $\omega \in \mathcal{R}$. Now, fix $0 < r < 1$ and $\delta = \delta(r) \in (0,1)$ such that $\Delta(z,r) \subset \Delta(z_j,\delta)$ for all $z \in R_j$. Then, by the subharmonicity of $|B_z^\omega|^{2p}$ and Fubini's theorem,

$$
\begin{aligned}
&\int_{\mathbb{D}} |B_z^\omega(\widetilde{z}_{j,z})|^{2p} \frac{\omega^\star(z)^p}{(1-|z|)^2}\, dA(z) \\
&\lesssim \int_{\mathbb{D}} \frac{1}{(1-|\widetilde{z}_{j,z}|)^2} \left(\int_{\Delta(\widetilde{z}_{j,z},r)} |B_z^\omega(\zeta)|^{2p}\, dA(\zeta) \right) \frac{\omega^\star(z)^p}{(1-|z|)^2}\, dA(z) \\
&\lesssim \frac{1}{(1-|z_j|)^2} \int_{\mathbb{D}} \left(\int_{\Delta(z_j,\delta)} |B_\zeta^\omega(z)|^{2p}\, dA(\zeta) \right) \frac{\omega^\star(z)^p}{(1-|z|)^2}\, dA(z) \\
&= \frac{1}{(1-|z_j|)^2} \int_{\Delta(z_j,\delta)} \left(\int_{\mathbb{D}} |B_\zeta^\omega(z)|^{2p} \frac{\omega^\star(z)^p}{(1-|z|)^2}\, dA(z) \right) dA(\zeta).
\end{aligned}
$$

An application of Theorem B together with Lemma E and the hypothesis that $\omega^\star(\cdot)/(1-|\cdot|)^2$ is a regular weight show that the inner integral above is dominated by a constant times

$$
\int_0^{|\zeta|} \frac{\int_t^1 \frac{\omega^\star(s)^p}{(1-s)^2}\, ds}{\widehat{\omega}(t)^{2p}(1-t)^{2p}}\, dt \asymp \int_0^{|\zeta|} \frac{1}{\widehat{\omega}(t)^p(1-t)^{p+1}}\, dt \asymp \frac{1}{\widehat{\omega}(\zeta)^p(1-|\zeta|)^p},
$$

and hence

$$
\int_{\mathbb{D}} |B_z^\omega(\widetilde{z}_{j,z})|^{2p} \frac{\omega^\star(z))^p}{(1-|z|)^2}\, dA(z) \lesssim \frac{1}{\widehat{\omega}(z_j)^p(1-|z_j|)^p} \asymp \frac{1}{\omega^\star(z_j)^p}, \quad |z_j| \to 1^-,
$$

by Lemma E. This combined with (7.21) and the equivalence (i)$\Leftrightarrow$(iii), proved in [56, Theorem 1], gives the assertion. $\square$

In view of Theorems 13 and 16 it is natural to ask whether or not the condition $\widetilde{\mathcal{T}}_\mu \in L^p_{\omega/\omega^\star}$ characterizes the Schatten class Toeplitz operators for the whole class $\widehat{\mathcal{D}}$ of doubling weights. The next result answers this question in negative.

Proposition 6. *For each $1 < p < \infty$ there exist $\omega \in \widehat{\mathcal{D}}$ and a positive Borel measure μ on $\mathbb{D}$ such that $\widetilde{\mathcal{T}}_\mu \in L^p_{\omega/\omega^\star}$ but $\mathcal{T}_\mu \notin \mathcal{S}_p(A^2_\omega)$.*

Proof. Let $\omega(z) = \left[(1-|z|) \left(\log \frac{e}{1-|z|} \right)^\alpha \right]^{-1}$, where $\alpha > -1$, and let $d\mu(z) = v(z)\, dA(z)$, where $v(z) = (1-|z|)^{-1+\frac{1}{p}} \left(\log \frac{e}{1-|z|} \right)^{-\alpha+1-\beta}$ and $0 < \beta < \frac{1}{p}$. Then (7.14), Theorem B and Lemma E yield

$$
\widetilde{\mathcal{T}}_\mu(z) \asymp \omega^\star(z) \int_{\mathbb{D}} |B_z^\omega(\zeta)|^2 v(\zeta)\, dA(\zeta) \asymp \frac{\widehat{v}(z)}{\widehat{\omega}(z)} \asymp (1-|z|)^{\frac{1}{p}} \left(\log \frac{e}{1-|z|} \right)^{-\beta}, \quad |z| \ge \frac{1}{2}.
$$

Therefore

$$
\begin{aligned}
&\int_{\mathbb{D}\setminus D(0,\frac{1}{2})} \left(\widetilde{\mathcal{T}}_\mu(z)\right)^p \frac{\omega(z)}{\omega^\star(z)}\,dA(z)\\
&\asymp \int_{\mathbb{D}\setminus D(0,\frac{1}{2})} \left((1-|z|)^{\frac{1}{p}}\left(\log\frac{e}{1-|z|}\right)^{-\beta}\right)^p \frac{dA(z)}{(1-|z|)^2\log\frac{e}{1-|z|}}\\
&= \int_{\mathbb{D}\setminus D(0,\frac{1}{2})} \frac{dA(z)}{(1-|z|)\left(\log\frac{e}{1-|z|}\right)^{\beta p+1}} < \infty,
\end{aligned}
$$

and thus $\widetilde{\mathcal{T}}_\mu \in L^p_{\omega/\omega^\star}$. However, for each $r\in(0,1)$,

$$
\ddot{\mu}_r(z) = \frac{\mu(\Delta(z,r))}{\omega^\star(z)} \asymp \frac{(1-|z|)^2 v(z)}{\omega^\star(z)} \asymp (1-|z|)^{\frac{1}{p}}\left(\log\frac{e}{1-|z|}\right)^{-\beta}, \quad |z|\geq\frac{1}{2},
$$

and hence

$$
\begin{aligned}
\|\ddot{\mu}_r\|^p_{L^p\left(\frac{dA(z)}{(1-|z|)^2}\right)} &\gtrsim \int_{\mathbb{D}\setminus D(0,\frac{1}{2})} \left((1-|z|)^{\frac{1}{p}}\left(\log\frac{e}{1-|z|}\right)^{-\beta}\right)^p \frac{dA(z)}{(1-|z|)^2}\\
&= \int_{\mathbb{D}\setminus D(0,\frac{1}{2})} \frac{dA(z)}{(1-|z|)\left(\log\frac{e}{1-|z|}\right)^{p\beta}} = \infty.
\end{aligned}
$$

Consequently, $\mathcal{T}_\mu \notin \mathcal{S}_p(A^2_\omega)$ by Theorem 13. □

The asymptotic relation $\omega(z)/\omega^\star(z) \asymp (1-|z|)^{-2}$, valid for each $\omega\in\mathcal{R}$ and $z\in\mathbb{D}$ uniformly bounded away from the origin, has been repeatedly used in this paper. This relation fails for $\omega\in\widehat{\mathcal{D}}\setminus\mathcal{R}$ and, for example, the doubling weight $\omega(z)=\left[(1-|z|)\left(\log\frac{e}{1-|z|}\right)^{\alpha}\right]^{-1}$, where $\alpha>-1$, satisfies $\omega(z)(1-|z|)^2/\omega^\star(z) \asymp \left(\log\frac{e}{1-|z|}\right)^{-1} \to 0$, as $|z|\to 1^-$. The last result of this section shows that this innocent looking difference is significant concerning the conditions $\widetilde{\mathcal{T}}_\mu \in L^1\left(\frac{dA}{(1-|\cdot|)^2}\right)$ and $\widetilde{\mathcal{T}}_\mu\in L^1_{\omega/\omega^\star}$. Therefore one may not replace $L^1_{\omega/\omega^\star}$ by $L^1\left(\frac{dA}{(1-|\cdot|)^2}\right)$ in the statement of Theorem 16.

Proposition 7. *There exists $\omega\in\widehat{\mathcal{D}}$ and a positive Borel measure μ on $\mathbb{D}$ such that $\mathcal{T}_\mu\in\mathcal{S}_1(A^2_\omega)$ and $\widetilde{\mathcal{T}}_\mu\notin L^1\left(\frac{dA}{(1-|\cdot|)^2}\right)$.*

Proof. Choose $\omega(z)=\left[(1-|z|)\left(\log\frac{e}{1-|z|}\right)^{\alpha}\right]^{-1}$, where $\alpha>2$, and $d\mu(z)=u(z)\,dA(z)$, where $u(z)=\left(\log\frac{e}{1-|z|}\right)^{-\beta-\alpha}$ and $0<\beta<\min\{1,\alpha-2\}$. Then, by Lemma E,

$$
\|B^\omega_z\|^2_{L^2_\mu} = \sum_{n=0}^\infty \frac{|z|^{2n}}{\left[(v_\alpha)_{2n+1}\right]^2} u_n \asymp \sum_{n=1}^\infty \frac{|z|^{2n}}{(n+1)}(\log n)^{\alpha-\beta-2} \asymp \left(\log\frac{e}{1-|z|}\right)^{\alpha-\beta-1},
$$

and hence

$$\widetilde{\mathcal{T}}_\mu(z) \asymp \omega^\star(z)\|B_z^\omega\|^2_{L^2_\mu} \asymp (1-|z|)\left(\log\frac{e}{1-|z|}\right)^{-\beta}, \quad |z| \geq \frac{1}{2},$$

by (7.14). It follows that $\widetilde{\mathcal{T}}_\mu \notin L^1\left(\frac{dA}{(1-|\cdot|)^2}\right)$. However,

$$\int_{\mathbb{D}\setminus D(0,\frac{1}{2})} \widetilde{\mathcal{T}}_\mu(z)\frac{\omega(z)}{\omega^\star(z)}\,dA(z) \asymp \int_{\mathbb{D}} \frac{dA(z)}{(1-|z|)\left(\log\frac{e}{1-|z|}\right)^{\beta+1}} < \infty,$$

and hence $\mathcal{T}_\mu \in \mathcal{S}_1(A^2_\omega)$ by Theorem 16. □

7.5 SCHATTEN CLASS COMPOSITION OPERATORS

The main purpose of this section is to prove Theorem 14. The following result of its own interest plays a role in the proof.

Proposition 8. *Let $0<p<\infty$ and $\omega \in \widehat{\mathcal{D}}$, and let φ be an analytic self-map of $\mathbb{D}$. Then the condition* (7.4) *is sufficient if $0<p\leq 2$ and necessary if $2\leq p<\infty$ for C_φ to belong to $\mathcal{S}_p(A^2_\omega)$.*

Proof. First observe that

$$\langle f, C_\varphi^\star(b_z^\omega)\rangle_{A^2_\omega} = \langle C_\varphi(f), b_z^\omega\rangle_{A^2_\omega} = \|B_z^\omega\|^{-1}_{A^2_\omega}\langle C_\varphi(f), B_z^\omega\rangle_{A^2_\omega} = \|B_z^\omega\|^{-1}_{A^2_\omega} f(\varphi(z)), \tag{7.22}$$

and hence $C_\varphi^\star(b_z^\omega) = \|B_z^\omega\|^{-1}_{A^2_\omega} B^\omega_{\varphi(z)}$. Consequently,

$$\|C_\varphi^\star(b_z^\omega)\|^2_{A^2_\omega} = \frac{\|B^\omega_{\varphi(z)}\|^2_{A^2_\omega}}{\|B_z^\omega\|^2_{A^2_\omega}} \asymp \frac{\omega(S(z))}{\omega((S(\varphi(z))))}, \quad z\in\mathbb{D}, \tag{7.23}$$

by Theorem B. This and Lemma E yield

$$\int_{\mathbb{D}}\left(\frac{\omega^\star(z)}{\omega^\star(\varphi(z))}\right)^{\frac{p}{2}}\frac{\omega(z)}{\omega^\star(z)}\,dA(z) \asymp \int_{\mathbb{D}} \|C_\varphi^\star(b_z^\omega)\|^p_{A^2_\omega}\frac{\omega(z)}{\omega^\star(z)}\,dA(z) = \int_{\mathbb{D}} \|\widetilde{T}(z)\|^{\frac{p}{2}}_{A^2_\omega}\frac{\omega(z)}{\omega^\star(z)}\,dA(z),$$

where $T = C_\varphi C_\varphi^\star$. The assertion follows from [69, Theorem 1.26] and Lemma 8.

An alternative way to establish the assertions is to follow the reasoning in [42, p. 1143]. □

Proof of Theorem 14. Since $C_\varphi^\star$ can be formally computed as

$$\begin{aligned} C_\varphi^\star(f)(z) &= \langle C_\varphi^\star f, B_z^\omega\rangle_{A^2_\omega} = \langle f, C_\varphi(B_z^\omega)\rangle_{A^2_\omega} = \langle f, B_z^\omega(\varphi)\rangle_{A^2_\omega} \\ &= \int_{\mathbb{D}} f(\zeta)B_z^\omega(\varphi(\zeta))\omega(\zeta)\,dA(\zeta), \end{aligned}$$

it follows that

$$C_\varphi^\star C_\varphi(f)(z) = \int_{\mathbb{D}} f(\varphi(\zeta))B_z^\omega(\varphi(\zeta))\omega(\zeta)\,dA(\zeta).$$

Let μ be the pull-back measure defined by $\mu(E) = \omega\left(\varphi^{-1}(E)\right)$. Then

$$C_\varphi^\star C_\varphi(f)(z) = \int_{\mathbb{D}} f(u) B_z^\omega(u)\, d\mu(u) = \mathcal{T}_\mu(f)(z),$$

and hence $C_\varphi \in \mathcal{S}_p(A^2_\omega)$ if and only if $\mathcal{T}_\mu \in \mathcal{S}_{p/2}(A^2_\omega)$ by [69, Theorem 1.26]. Therefore, by Theorems 13 and 8, it suffices to show that (7.4) implies $\widetilde{\mathcal{T}_\mu} \in L^{\frac{p}{2}}_{\omega/\omega^\star}$. To see this, we use Theorem B to write

$$\widetilde{\mathcal{T}_\mu}(z) = \langle \mathcal{T}_\mu(b_z^\omega), b_z^\omega \rangle_{A^2_\omega} = \int_{\mathbb{D}} \frac{|B_z^\omega(\zeta)|^2}{\|B_z^\omega\|^2_{A^2_\omega}}\, d\mu(\zeta) \asymp \omega(S(z)) \int_{\mathbb{D}} |B_z^\omega(\varphi(\zeta))|^2 \omega(\zeta)\, dA(\zeta).$$

We will now argue as in [70, p. 180]. Note first that [54, Theorem 4.2] gives

$$\widetilde{\mathcal{T}_\mu}(z) \asymp \omega(S(z)) |B_z^\omega(\varphi(0))|^2 + \omega(S(z)) \int_{\mathbb{D}} |(B_z^\omega)'(\varphi(\zeta))|^2 |\varphi'(\zeta)|^2 \omega^\star(\zeta)\, dA(\zeta).$$

Hence it suffices to show that

$$\Phi(z) = \omega(S(z)) \int_{\mathbb{D}} |(B_z^\omega)'(\varphi(\zeta))|^2 |\varphi'(\zeta)|^2 \omega^\star(\zeta)\, dA(\zeta)$$

belongs to $L^{p/2}_{\omega/\omega^\star}$. To do this we will use Shur's test with two measures [69, Theorem 3.8]. Let

$$\psi(\zeta) = \frac{\omega^\star(\zeta)}{\omega^\star(\varphi(\zeta))}, \quad d\nu(\zeta) = \frac{\omega(\varphi(\zeta))}{\omega^\star(\varphi(\zeta))} |\varphi'(\zeta)|^2\, dA(\zeta)$$

and

$$H(z,\zeta) = \frac{|(B_z^\omega)'(\varphi(\zeta))|^2 \omega(S(z)) \omega^\star(\varphi(\zeta))^2}{\omega(\varphi(\zeta))},$$

so that the operator

$$T(f) = \int_{\mathbb{D}} H(z,\zeta) f(\zeta)\, d\nu(\zeta)$$

satisfies $T(\psi) = \Phi$. Since φ is of bounded valence, we obtain

$$\begin{aligned}
\int_{\mathbb{D}} H(z,\zeta)\, d\nu(\zeta) &= \omega(S(z)) \int_{\mathbb{D}} |(B_z^\omega)'(\varphi(\zeta))|^2 \omega^\star(\varphi(\zeta)) |\varphi'(\zeta)|^2\, dA(\zeta) \\
&\asymp \omega(S(z)) \int_{\mathbb{D}} |(B_z^\omega)'(\xi)|^2 \omega^\star(\xi)\, dA(\xi) \asymp 1
\end{aligned}$$

by Theorem B. Moreover, by Theorem B,

$$\begin{aligned}
\int_{\mathbb{D}} H(z,\zeta) \frac{\omega(z)}{\omega^\star(z)}\, dA(z) &\asymp \frac{\omega^\star(\varphi(\zeta))^2}{\omega(\varphi(\zeta))} \int_{\mathbb{D}} |(B_z^\omega)'(\varphi(\zeta))|^2 \omega(z)\, dA(z) \\
&= \frac{\omega^\star(\varphi(\zeta))^2}{\omega(\varphi(\zeta))} \int_{\mathbb{D}} |(B_{\varphi(\zeta)}^\omega)'(z)|^2 \omega(z)\, dA(z) \lesssim 1,
\end{aligned}$$

because $\omega \in \mathbb{R}$. Now $\psi \in L^{p/2}_{\omega/\omega^\star}$ by the assumption (7.4), and $\nu \lesssim \omega/\omega^\star$ by the Schwarz-Pick lemma and the assumption $\omega \in \mathbb{R}$, so $\psi \in L^{p/2}_\nu$. Therefore we may apply Schur's test (with both test functions equal to 1) to deduce that T is a bounded operator from $L^{p/2}_{\nu}$ into $L^{p/2}_{\omega/\omega^\star}$, and thus, in particular, $T(\psi) = \Phi \in L^{p/2}_{\omega/\omega^\star}$. Therefore $\widetilde{\mathcal{T}}_\mu \in L^{p/2}_{\omega/\omega^\star}$ as desired. □

The following result is parallel to Proposition 8. By the Schwarz-Pick lemma, (7.4) implies (7.24) for all $0 < p < \infty$ and $\omega \in \mathbb{R}$, and therefore the case $0 < p < 2$ is of particular interest.

Proposition 9. *Let $0 < p < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let φ be an analytic self-map of $\mathbb{D}$. Then the condition*

$$\int_{\mathbb{D}} \left(\frac{\omega^\star(z)}{\omega^\star(\varphi(z))} \right)^{\frac{p}{2}} \frac{|\varphi'(z)|^p (1-|z|^2)^{p-2}}{(1-|\varphi(z)|^2)^p} \, dA(z) < \infty \tag{7.24}$$

is sufficient if $0 < p \le 2$ and necessary if $2 \le p < \infty$ for C_φ to belong to $\mathcal{S}_p(A^2_\omega)$.

Proof. Let first $p \ge 2$. The Schwarz-Pick lemma, a change of variable and a standard inequality yield

$$\begin{aligned}
&\int_{\mathbb{D}} \left(\frac{\omega^\star(z)}{\omega^\star(\varphi(z))} \right)^{\frac{p}{2}} \frac{|\varphi'(z)|^p (1-|z|^2)^{p-2}}{(1-|\varphi(z)|^2)^p} \, dA(z) \\
&\le \int_{\mathbb{D}} \left(\frac{\omega^\star(z)}{\omega^\star(\varphi(z))} \right)^{\frac{p}{2}} \frac{|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \, dA(z) \\
&= \int_{\mathbb{D}} \frac{N_{\varphi,(\omega^\star)^{p/2}}(\zeta)}{\omega^\star(\zeta)^{\frac{p}{2}}} \frac{dA(\zeta)}{(1-|\zeta|^2)^2} \le \int_{\mathbb{D}} \left(\frac{N_{\varphi,\omega^\star}(\zeta)}{\omega^\star(\zeta)} \right)^{\frac{p}{2}} \frac{dA(\zeta)}{(1-|\zeta|)^2},
\end{aligned}$$

and hence the assertion follows by [56, Theorem 3]. A similar reasoning shows the case $p < 2$. □

8 Paper III

8.1 INTRODUCTION AND MAIN RESULTS

Let $\mathcal{H}(\mathbb{D})$ denote the space of all analytic functions in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$. Further, let $\mathbb{T}$ stand for the boundary of $\mathbb{D}$ and $D(a,r) = \{z : |z-a| < r\}$ for the Euclidean disc of center $a \in \mathbb{C}$ and radius $r > 0$. For $0 < r < 1$ and $f \in \mathcal{H}(\mathbb{D})$, set

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^p\,dt\right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \sup_{|z|=r} |f(z)|.$$

An integrable function $\omega : \mathbb{D} \to [0,\infty)$ is called a weight. It is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. For a radial weight ω, write $\widehat{\omega}(z) = \int_{|z|}^1 \omega(s)\,ds$ for all $z \in \mathbb{D}$.

For $0 < p \le \infty$, $0 < q < \infty$ and a radial weight ω, the weighted mixed norm space $A^{p,q}_\omega$ consists of $f \in \mathcal{H}(\mathbb{D})$ such that

$$\|f\|^q_{A^{p,q}_\omega} = \int_0^1 M_p^q(r,f)\omega(r)\,dr < \infty.$$

If $q = p$, then $A^{p,q}_\omega$ coincides with the Bergman space A^p_ω induced by the weight ω. As usual, A^p_α denotes the weighted Bergman space induced by the standard radial weight $(1-|z|^2)^\alpha$. Weighted mixed norm spaces arise naturally in operator and function theory, for example, in the study of the boundedness, compactness and Schatten classes of the generalized Hilbert operator $H_g(f)(z) = \int_0^1 f(t)g'(tz)\,dt$ acting on Bergman spaces [53,61].

A weight ω belongs to the class $\widehat{\mathcal{D}}$ if there exists a constant $C = C(\omega) \ge 1$ such that $\widehat{\omega}(r) \le C\widehat{\omega}(\frac{1+r}{2})$ for all $0 \le r < 1$. Moreover, if there exist $K = K(\omega) > 1$ and $C = C(\omega) > 1$ such that

$$\widehat{\omega}(r) \ge C\widehat{\omega}\left(1 - \frac{1-r}{K}\right), \quad 0 \le r < 1, \tag{8.1}$$

then we write $\omega \in \check{\mathcal{D}}$. Weights ω belonging to $\mathcal{D} = \widehat{\mathcal{D}} \cap \check{\mathcal{D}}$ are called doubling. The classes of weights $\widehat{\mathcal{D}}$ and $\mathcal{D}$ emerge from fundamental questions in operator theory: recently the first two authors showed that the weighted Bergman projection P_ω, induced by a radial weight ω, is bounded from L^∞ to the Bloch space $\mathcal{B} = \{f \in \mathcal{H}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|) < \infty\}$ if and only if $\omega \in \widehat{\mathcal{D}}$, and further, it is bounded and onto if and only if $\omega \in \mathcal{D}$ [59].

The primary aim of this study is to establish a representation theorem, commonly known as an atomic decomposition, for functions in $A^{p,q}_\omega$ in the sense of Coifman and Rochberg [18]. This last-mentioned celebrated result concerning classical weighted Bergman spaces has been extended to the vector-valued Bergman spaces [20], the

Bergman spaces induced by exponential weights [7], the classical Dirichlet spaces [29], the Fock spaces [72] and the classical mixed norm spaces on the upper half plane [63]. In concrete means we will prove that each function in the mixed norm space $A^{p,q}_\omega$ with $\omega \in \mathcal{D}$ can be written as an adequate sum of normalized translates and dilates of powers of the Cauchy kernel in such a way that the coefficients belong to the doubled indexed complex-valued sequence space $\ell^{p,q}$. For $0 < p, q \leq \infty$, the space $\ell^{p,q}$ consists sequences $\lambda = \{\lambda_{j,l}\}_{j,l}$ such that

$$\|\lambda\|_{\ell^{p,q}} = \left\| \left\{ \left\| \left\{ \lambda_{j,l} \right\}_l \right\|_{\ell^p} \right\}_j \right\|_{\ell^q} < \infty,$$

where $\|\{a_n\}_n\|_{\ell^\infty} = \sup_n |a_n|$ and $\|\{a_n\}_n\|^s_{\ell^s} = \sum_n |a_n|^s$ for all $0 < s < \infty$.

In order to state our main results we need to introduce some notation and recall that the class $\widehat{\mathcal{D}}$ can be described by the equivalent conditions given in the following lemma [51, Lemma 2.1].

Lemma E. *Let ω be a radial weight. Then the following conditions are equivalent:*

(i) $\omega \in \widehat{\mathcal{D}}$;

(ii) There exist $C = C(\omega) > 0$ and $\beta = \beta(\omega) > 0$ such that

$$\widehat{\omega}(r) \leq C \left(\frac{1-r}{1-t} \right)^\beta \widehat{\omega}(t), \quad 0 \leq r \leq t < 1;$$

(iii) There exist $C = C(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ such that

$$\int_0^t \left(\frac{1-t}{1-s} \right)^\gamma \omega(s)\, ds \leq C\widehat{\omega}(t), \quad 0 \leq t < 1.$$

We write $\varrho(a,z) = |\varphi_a(z)| = \left| \frac{a-z}{1-\bar{a}z} \right|$ for the pseudohyperbolic distance between z and a, and $\Delta(a,r) = \{z : \varrho(a,z) < r\}$ for the pseudohyperbolic disc of center $a \in \mathbb{D}$ and radius $r \in (0,1)$. A sequence $\{z_k\}_{k=0}^\infty$ in $\mathbb{D}$ is called separated if $\inf_{k\neq j} \varrho(z_k, z_j) > 0$. Now for each $K > 1$, a sequence $\{z_k\}$ in $\mathbb{D}$, is re-indexed in the following way depending on K: For each $j \in \mathbb{N} \cup \{0\}$, let $\{z_{j,l}\}_l$ denote the points of the sequence $\{z_k\}$ in the annulus $A_j = A_j(K) = \{z : r_j \leq |z| < r_{j+1}\}$, where $r_j = r_j(K) = 1 - K^{-j}$. The following result contains a half of the aforementioned atomic decomposition for functions in $A^{p,q}_\omega$.

Theorem 17. *Let $0 < p \leq \infty$, $0 < q < \infty$, $1 < K < \infty$, $\omega \in \mathcal{D}$, and $\{z_k\}_{k=0}^\infty$ a separated sequence in $\mathbb{D}$. Let $\beta = \beta(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ be those of Lemma E(ii) and (iii). If*

$$M > 1 + \frac{1}{p} + \frac{\beta + \gamma}{q} \tag{8.2}$$

and $\lambda = \{\lambda_{j,l}\} \in \ell^{p,q}$, then the function F defined by

$$F(z) = \sum_{j,l} \lambda_{j,l} \frac{(1-|z_{j,l}|)^{M-\frac{1}{p}} \widehat{\omega}(z_{j,l})^{-\frac{1}{q}}}{\left(1 - \overline{z_{j,l}} z\right)^M}$$

belongs to $\mathcal{H}(\mathbb{D})$, and there exists a constant $C = C(K, M, \omega, p, q) > 0$ such that

$$\|F\|_{A^{p,q}_\omega} \leq C \|\lambda\|_{\ell^{p,q}}. \tag{8.3}$$

One important tool in the proof of Theorem 5.3.1, to be given in Section 7.2, is the description due to Muckenhoupt [45] of the weights U and V such that the Hardy operators $\int_0^x f(t)\,dt$ and $\int_x^\infty f(t)\,dt$ are bounded from the Lebesgue space $L^s(U^s,(0,\infty))$ to $L^s(V^s,(0,\infty))$.

To complete the atomic decomposition we are after, for each $K\in\mathbb{N}\setminus\{1\}$, $j\in\mathbb{N}\cup\{0\}$ and $l=0,1,\dots,K^{j+3}-1$, define the dyadic polar rectangle as

$$Q_{j,l}=\left\{z\in\mathbb{D}: r_j\le|z|<r_{j+1}, \arg z\in\left[2\pi\frac{l}{K^{j+3}},2\pi\frac{l+1}{K^{j+3}}\right)\right\}, \tag{8.4}$$

where $r_j=r_j(K)=1-K^{-j}$ as before, and denote its center by $\zeta_{j,l}$. For each $M\in\mathbb{N}$ and $k=1,\dots,M^2$, the rectangle $Q_{j,l}^k$ is defined as the result of dividing $Q_{j,l}$ into M^2 pairwise disjoint rectangles of equal Euclidean area, and the centers of these squares are denoted by $\zeta_{j,l}^k$, respectively. Write $\lambda=\{\lambda_{j,l,k}\}\in\ell^{p,q}$ if

$$\|\lambda\|_{\ell^{p,q}}=\left(\sum_{j=0}^{\infty}\left(\sum_{l=0}^{K^{j+3}-1}\sum_{k=1}^{M^2}|\lambda_{j,l,k}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}}<\infty.$$

The representation part of our result reads as follows and will be proven in Section 8.3.

Theorem 18. *Let $0<p\le\infty$, $0<q<\infty$, $K\in\mathbb{N}\setminus\{1\}$ and $\omega\in\mathcal{D}$ such that (8.1) holds. Then there exists $M=M(p,q,\omega)>0$ such that $A^{p,q}_\omega$ consists of functions of the form*

$$f(z)=\sum_{j,l,k}\lambda(f)_{j,l}^k\frac{(1-|\zeta_{j,l}^k|^2)^{M-\frac{1}{p}}\widehat{\omega}(r_j)^{-\frac{1}{q}}}{(1-\overline{\zeta_{j,l}^k}z)^M},\quad z\in\mathbb{D}, \tag{8.5}$$

where $\lambda(f)=\{\lambda(f)_{j,l}^k\}\in\ell^{p,q}$ and

$$\left\|\{\lambda(f)_{j,l}^k\}\right\|_{\ell^{p,q}}\asymp\|f\|_{A^{p,q}_\omega}. \tag{8.6}$$

Atomic decompositions, and even partial results of the same fashion, for functions in spaces of analytic functions are very useful in operator theory. In particular, they can be used to describe dual spaces [63] or to study basic questions such as the boundedness, the compactness or the Schatten class membership of concrete operators [5,7,21,29,48,54,55,69,72]. In this study we will use Theorem 5.3.1 to describe those positive Borel measures μ on $\mathbb{D}$ such that the differentiation operator defined by $D^{(n)}(f)=f^{(n)}$ for $n\in\mathbb{N}\cup\{0\}$ is bounded from $A^{p,q}_\omega$ to the Lebesgue space L^s_μ. The special case $n=0$ gives a description of the s-Carleson measures for $A^{p,q}_\omega$. Carleson measures have attracted a lot of attention during the last decades because of their numerous applications in the operator theory and elsewhere, and descriptions of these measures have been obtained for many spaces of analytic functions such as the Hardy spaces [10, 11, 24, 25, 40], the classical Bergman spaces [41, 69], the Bergman spaces induced by Bekollé-Bonami, rapidly decreasing or doubling weights [21,48,54,55], the Fock spaces [72] and the classical mixed norm spaces [41], to name a few instances.

To state our result on the differentiation operator, write

$$T_{r,u,v}(z) = \frac{\mu(\Delta(z,r))}{(1-|z|)^u \widehat{\omega}(z)^v}, \quad z \in \mathbb{D},$$

for a positive Borel measure μ on $\mathbb{D}$, $0 < r < 1$ and $0 < u, v < \infty$.

Theorem 19. *Let $0 < p, q, s < \infty$, $n \in \mathbb{N} \cup \{0\}$, $\omega \in \mathcal{D}$, $0 < r < 1$, μ a positive Borel measure on $\mathbb{D}$, and let $K = K(\omega) \in \mathbb{N} \setminus \{1\}$ such that* (8.1) *holds. Then the following statements are equivalent:*

(i) $D^{(n)} : A^{p,q}_\omega \to L^s_\mu$ *is bounded;*

(ii) $\left\{ \mu(Q_{j,l}) K^{sj\left(n+\frac{1}{p}\right)} \widehat{\omega}(r_j)^{-\frac{s}{q}} \right\}_{j,l} \in \ell^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}$;

(iii) $T_{r,u,v} \in L_\omega^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}$*, where*

(a) $u = sn + 1$ *and* $v = 1$ *if* $s < \min\{p, q\}$;

(b) $u = sn + \frac{1}{p}$ *and* $v = 1$ *if* $p \le s < q$;

(c) $u = sn + 1$ *and* $v = \frac{s}{q}$ *if* $q \le s < p$;

(d) $u = s(n + \frac{1}{p})$ *and* $v = \frac{s}{q}$ *if* $s \ge \max\{p, q\}$.

Moreover,

$$\|D^{(n)}\|^s_{A^{p,q}_\omega \to L^s_\mu} \asymp \left\| \left\{ \mu(Q_{j,l}) K^{sj\left(n+\frac{1}{p}\right)} \widehat{\omega}(r_j)^{-\frac{s}{q}} \right\}_{j,l} \right\|_{\ell^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}} \asymp \|T_{r,u,v}\|_{L_\omega^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}}.$$

Theorem 19 will be proven in Section 8.4.

8.2 PROOF OF THEOREM 17

Throughout the proof and in several other occasions in this work we will use the fact that a radial weight ω belongs to $\check{\mathcal{D}}$ if and only if there exist $C = C(\omega) > 0$ and $\alpha = \alpha(\omega) > 0$ such that

$$\widehat{\omega}(t) \le C \left(\frac{1-t}{1-r} \right)^\alpha \widehat{\omega}(r), \quad 0 \le r \le t < 1. \tag{8.7}$$

This equivalence can be proved by following the ideas used in the proof of [51, Lemma 2.1].

To see that the function F defined in Theorem 5.3.1 is analytic, observe first that $\#\{z_k \in A_j\} \lesssim K^j$ for all $j \in \mathbb{N} \cup \{0\}$ since $\{z_k\}$ is a separated sequence by the hypothesis. This together with Lemma E(ii) and the hypothesis (8.2) yields

$$\begin{aligned} |F(z)| &\le \sum_{j,l} |\lambda_{j,l}| \frac{(1-|z_{j,l}|)^{M-\frac{1}{p}} \widehat{\omega}(z_{j,l})^{-\frac{1}{q}}}{|1-\overline{z_{j,l}} z|^M} \lesssim \frac{\|\lambda\|_{\ell^\infty}}{(1-r)^M} \sum_{j=0}^\infty K^{-j\left(M-\frac{1}{p}-1\right)} \widehat{\omega}(r_j)^{-\frac{1}{q}} \\ &\lesssim \frac{\|\lambda\|_{\ell^\infty}}{\widehat{\omega}(0)^{\frac{1}{q}} (1-r)^M} \sum_{j=0}^\infty K^{-j\left(M-\frac{1}{p}-1-\frac{\beta}{q}\right)} \asymp \frac{\|\lambda\|_{\ell^\infty}}{(1-r)^M}, \quad |z| < r < 1, \end{aligned}$$

and hence $F \in \mathcal{H}(\mathbb{D})$.

From now on we write $c_j^p = \sum_l |\lambda_{j,l}|^p$. By following the idea used in the half plane case [63, (1.5) Theorem], we will split the proof into six cases according to the values of p and q:

Case 1.1: $0 < p \leq 1$ and $q \leq p$;

Case 1.2: $0 < p \leq 1$ and $p < q$;

Case 2.1: $1 < p < \infty$ and $q \leq p$;

Case 2.2: $1 < p < \infty$ and $p < q$;

Case 3.1: $p = \infty$ and $0 < q \leq 1$;

Case 3.2: $p = \infty$ and $1 < q < \infty$.

Assume first $0 < p \leq 1$. Then

$$|F(z)|^p \leq \sum_{j,l} |\lambda_{j,l}|^p \frac{(1-|z_{j,l}|)^{pM-1}\widehat{\omega}(z_{j,l})^{-\frac{p}{q}}}{\left|1-\overline{z_{j,l}}z\right|^{pM}}, \quad z \in \mathbb{D}.$$

By using $pM > 1$, which follows from the hypothesis (8.2), and Lemma E(ii) we obtain

$$M_p^p(r,F) \lesssim \sum_{j,l} |\lambda_{j,l}|^p \frac{(1-|z_{j,l}|)^{pM-1}\widehat{\omega}(z_{j,l})^{-\frac{p}{q}}}{\left(1-|z_{j,l}|r\right)^{pM-1}} \asymp \sum_{j=0}^{\infty} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}}. \tag{8.8}$$

Case 1.1: $0 < q \leq p \leq 1$. Observe first that $M > \frac{\gamma}{q} + \frac{1}{p}$ by the hypothesis (8.2), and hence $qM - \frac{q}{p} > \gamma$. By using (8.8) and Lemma E(iii) we deduce

$$\begin{aligned}
\|F\|_{A^{p,q}_\omega}^q &\lesssim \int_0^1 \left(\sum_{j=0}^{\infty} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \omega(r)\,dr \\
&\lesssim \sum_{j=0}^{\infty} c_j^q \frac{(1-r_j)^{qM-\frac{q}{p}}}{\widehat{\omega}(r_j)} \int_0^1 \frac{\omega(r)}{(1-r_jr)^{qM-\frac{q}{p}}}\,dr \\
&\asymp \sum_{j=0}^{\infty} c_j^q + \sum_j c_j^q \frac{(1-r_j)^{qM-\frac{q}{p}}}{\widehat{\omega}(r_j)} \int_0^{r_j} \frac{\omega(r)}{(1-r_jr)^{qM-\frac{q}{p}}}\,dr \asymp \sum_{j=0}^{\infty} c_j^q,
\end{aligned}$$

and thus (8.3) is satisfied.

Case 1.2: $0 < p \leq 1$ and $p < q$. By (8.8) and standard estimates

$$
\begin{aligned}
\|F\|^q_{A^{p,q}_\omega} &\lesssim \int_0^1 \left(\sum_{j=0}^{\infty} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \omega(r)\,dr \\
&= \sum_{k=0}^{\infty} \int_{r_k}^{r_{k+1}} \left(\sum_{j=0}^{\infty} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \omega(r)\,dr \\
&\asymp \sum_{k=0}^{\infty} \int_{r_k}^{r_{k+1}} \left(\sum_{j=0}^{\infty} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr_k)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \omega(r)\,dr \\
&\leq \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr_k)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \widehat{\omega}(r_k) \\
&\lesssim \sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\infty} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr_k)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \widehat{\omega}(r_k) \\
&\quad + \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr_k)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \widehat{\omega}(r_k) \\
&\leq \sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\infty} c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr_k)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \widehat{\omega}(r_k) \\
&\quad + \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \frac{c_j^p}{\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \widehat{\omega}(r_k) = S_1(F) + S_2(F).
\end{aligned}
$$

To prove the estimate $S_l(F) \lesssim \|\lambda\|^q_{\ell^{p,q}}$ for $l = 1, 2$ we will use the characterization, obtained by Muckenhoupt [45], of the weights U and V such that the Hardy operators $\int_0^x f(t)\,dt$ and $\int_x^\infty f(t)\,dt$ are bounded from $L^s(U^s, (0,\infty))$ to $L^s(V^s, (0,\infty))$, where $s = \frac{q}{p} > 1$. To do this, consider first the step functions

$$
\begin{aligned}
U(x) &= \frac{\widehat{\omega}(r_k)^{\frac{p}{q}}}{(1-r_k)^{pM-1}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\}; \\
f(x) &= c_k^p \frac{(1-r_k)^{pM-1}}{\widehat{\omega}(r_k)^{\frac{p}{q}}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\}; \\
V(x) &= \frac{\widehat{\omega}(r_k)^{\frac{p}{q}}}{(1-r_k)^{pM-1}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\}.
\end{aligned}
$$

With this notation

$$
\|\lambda\|^q_{\ell^{p,q}} = \sum_{k=0}^{\infty} c_k^q = \int_0^\infty (V(x) f(x))^{\frac{q}{p}}\,dx
$$

and

$$\int_0^\infty \left(U(x)\int_x^\infty f(y)\,dy\right)^{\frac{q}{p}} dx \geq \sum_{k=0}^\infty \left(\sum_{j=k+1}^\infty c_j^p \frac{(1-r_j)^{pM-1}}{(1-r_jr_k)^{pM-1}\,\widehat{\omega}(r_j)^{\frac{p}{q}}}\right)^{\frac{q}{p}} \widehat{\omega}(r_k) = S_1(F).$$

Therefore the estimate $S_1(F) \lesssim \|\lambda\|_{\ell^{p,q}}^q$ follows by [45, Theorem 2] once it is shown that

$$\sup_{x\geq 0}\left(\int_0^x U(y)^{\frac{q}{p}}\,dy\right)^{\frac{p}{q}}\left(\int_x^\infty V(y)^{-\left(\frac{q}{p}\right)'}\,dy\right)^{\frac{1}{\left(\frac{q}{p}\right)'}} < \infty. \tag{8.9}$$

To see this, let $x \geq 0$, and take $N = N(x) \in \mathbb{N}\cup\{0\}$ such that $N \leq x < N+1$. Then Lemma E(ii) and the inequality $M > \frac{1}{p}+\frac{\beta}{q}$, which follows by the hypothesis (8.2), imply

$$\begin{aligned}
\left(\int_0^x U(y)^{\frac{q}{p}}\,dy\right)^{\frac{p}{q}} &\leq \left(\sum_{k=0}^N \int_k^{k+1} U(y)^{\frac{q}{p}}\,dy\right)^{\frac{p}{q}} = \left(\sum_{k=0}^N U(k)^{\frac{q}{p}}\right)^{\frac{p}{q}} \\
&= \left(\sum_{k=0}^N \frac{\widehat{\omega}(r_k)}{(1-r_k)^{qM-\frac{q}{p}}}\right)^{\frac{p}{q}} \\
&\lesssim \left(\frac{\widehat{\omega}(r_N)}{(1-r_N)^\beta}\sum_{k=0}^N \frac{1}{(1-r_k)^{qM-\frac{q}{p}-\beta}}\right)^{\frac{p}{q}} \\
&\asymp \left(\frac{\widehat{\omega}(r_N)}{(1-r_N)^{qM-\frac{q}{p}}}\right)^{\frac{p}{q}} = \frac{\widehat{\omega}(r_N)^{\frac{p}{q}}}{(1-r_N)^{pM-1}}.
\end{aligned} \tag{8.10}$$

Another application of Lemma E(ii) and $M > \frac{1}{p}+\frac{\beta}{q}$ give

$$\begin{aligned}
\left(\int_x^\infty V(y)^{-\left(\frac{q}{p}\right)'}\,dy\right)^{\frac{1}{\left(\frac{q}{p}\right)'}} &\leq \left(\sum_{k=N}^\infty V(k)^{-\left(\frac{q}{q-p}\right)}\right)^{\frac{q-p}{q}} = \left(\sum_{k=N}^\infty \left(\frac{\widehat{\omega}(r_k)^{\frac{p}{q}}}{(1-r_k)^{pM-1}}\right)^{-\frac{q}{q-p}}\right)^{\frac{q-p}{q}} \\
&\lesssim \left(\frac{(1-r_N)^{\frac{p\beta}{q-p}}}{\widehat{\omega}(r_N)^{\frac{p}{q-p}}}\sum_{k=N}^\infty \frac{1}{(1-r_k)^{\frac{p}{q-p}\left(\beta-q\left(M-\frac{1}{p}\right)\right)}}\right)^{\frac{q-p}{q}} \asymp \frac{(1-r_N)^{pM-1}}{\widehat{\omega}(r_N)^{\frac{p}{q}}},
\end{aligned}$$

which together with (8.10) gives (8.9).

We next establish $S_2(F) \lesssim \|\lambda\|_{\ell^{p,q}}^q$. Define

$$\begin{aligned}
U(x) &= \widehat{\omega}(r_k)^{\frac{p}{q}}, \quad x \in [k,k+1), \quad k \in \mathbb{N}\cup\{0\}; \\
f(x) &= \frac{c_k^p}{\widehat{\omega}(r_k)^{\frac{p}{q}}}, \quad x \in [k,k+1), \quad k \in \mathbb{N}\cup\{0\}; \\
V(x) &= \widehat{\omega}(r_k)^{\frac{p}{q}}, \quad x \in [k,k+1), \quad k \in \mathbb{N}\cup\{0\}.
\end{aligned}$$

Then

$$\|\lambda\|_{\ell^{p,q}}^q = \sum_{k=0}^{\infty} c_k^q = \int_0^\infty (V(x)f(x))^{\frac{q}{p}} \, dx$$

and

$$\int_0^\infty \left(U(x) \int_0^x f(y)\,dy \right)^{\frac{q}{p}} dx \geq \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k-1} \frac{c_j^p}{\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \widehat{\omega}(r_k),$$

and thus

$$S_2(F) \lesssim \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k-1} \frac{c_j^p}{\widehat{\omega}(r_j)^{\frac{p}{q}}} \right)^{\frac{q}{p}} \widehat{\omega}(r_k) + \sum_{k=0}^{\infty} c_k^q \leq \int_0^\infty \left(U(x) \int_0^x f(y)\,dy \right)^{\frac{q}{p}} dx + \|\lambda\|_{\ell^{p,q}}^q.$$

Therefore the estimate $S_2(F) \lesssim \|\lambda\|_{\ell^{p,q}}^q$ we are after, follows by [45, Theorem 1] if

$$\sup_{x\geq 0} \left(\int_x^\infty U(y)^{\frac{q}{p}}\,dy \right)^{\frac{p}{q}} \left(\int_0^x V(y)^{-\left(\frac{q}{p}\right)'}\,dy \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} < \infty. \tag{8.11}$$

To prove this, let $x \geq 0$ and choose $N = N(x) \in \mathbb{N} \cup \{0\}$ such that $N \leq x < N+1$. Then (8.7) yields

$$\left(\int_x^\infty U(y)^{\frac{q}{p}} dy \right)^{\frac{p}{q}} \leq \left(\sum_{k=N}^{\infty} U(k)^{\frac{q}{p}} \right)^{\frac{p}{q}} = \left(\sum_{k=N}^{\infty} \widehat{\omega}(r_k) \right)^{\frac{p}{q}} \lesssim \left(\sum_{k=N}^{\infty} \left(\frac{1-r_k}{1-r_N} \right)^{\alpha} \widehat{\omega}(r_N) \right)^{\frac{p}{q}} \asymp \widehat{\omega}(r_N)^{\frac{p}{q}}$$

and

$$\begin{aligned}\left(\int_0^x V(y)^{-\left(\frac{q}{p}\right)'} dy \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} &\leq \left(\sum_{k=0}^{N} V(k)^{-\left(\frac{q}{p}\right)'} \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} = \left(\sum_{k=0}^{N} \frac{1}{\widehat{\omega}(r_k)^{\frac{p}{q-p}}} \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} \\ &\lesssim \left(\sum_{k=0}^{N} \left(\frac{1-r_N}{1-r_k} \right)^{\frac{p\alpha}{q-p}} \frac{1}{\widehat{\omega}(r_N)^{\frac{p}{q-p}}} \right)^{\frac{q-p}{q}} \asymp \frac{1}{\widehat{\omega}(r_N)^{\frac{p}{q}}},\end{aligned}$$

from which (8.11) follows. Case 1.2 is now proved.

Assume next $1 < p < \infty$. Before dealing with Cases 2.1 and 2.2, we will estimate $M_p(r,F)$. We claim that there exist $\eta, \theta \in (0,1)$ such that

$$\begin{aligned} &M(1-\theta)p' > 1, \\ &M(\eta-\theta) + \frac{1-\eta}{p} > 0, \\ &p'(1-\eta)(M - \frac{1}{p} - \frac{\beta}{q}) > 1, \\ &pM\theta > 1, \\ &q\eta(M - \frac{1}{p}) > \gamma, \\ &M(\eta-\theta) + \frac{1-\eta}{p} < \frac{\alpha\eta}{q}, \end{aligned} \tag{8.12}$$

where α is that of (8.7) and M, β, γ are those in the statement of the theorem. We postpone the proof of this fact for a moment and estimate $M_p(r, F)$ first. By Hölder's inequality,

$$|F(z)|^p \leq \sum_{j,l} |\lambda_{j,l}|^p \frac{(1-|z_{j,l}|)^{(pM-1)\eta}}{\left|1-\overline{z_{j,l}}z\right|^{pM\theta} \widehat{\omega}(z_{j,l})^{\frac{p\eta}{q}}} \left(\sum_{j,l} \frac{(1-|z_{j,l}|)^{p'\left(M-\frac{1}{p}\right)(1-\eta)}}{\left|1-\overline{z_{j,l}}z\right|^{p'M(1-\theta)} \widehat{\omega}(z_{j,l})^{\frac{p'(1-\eta)}{q}}} \right)^{\frac{p}{p'}} \tag{8.13}$$

for all $z \in \mathbb{D}$. Since $\{z_k\}$ is separated by the hypothesis, there exists $\delta = \delta(K) > 0$ such that $\Delta(z_{j,l}, \delta) \subset \{r_{j-1} \leq |w| < r_{j+2}\}$ for all l with the convenience that $r_{-1} = r_0 = 0$, and $\Delta(z_{j,l_1}, \delta) \cap \Delta(z_{j,l_2}, \delta) = \emptyset$ if $l_1 \neq l_2$. Therefore by using the subharmonicity and the first case in (8.12) we obtain

$$\begin{aligned} \sum_l \frac{1}{\left|1-\overline{z_{j,l}}z\right|^{p'M(1-\theta)}} &\lesssim K^{2j} \sum_l \int_{\Delta(z_{j,l},\delta)} \frac{dA(w)}{|1-\overline{w}z|^{p'M(1-\theta)}} \\ &\leq K^{2j} \int_{D(0,r_{j+2}) \setminus D(0,r_{j-1})} \frac{dA(w)}{|1-\overline{w}z|^{p'M(1-\theta)}} \\ &\lesssim \frac{K^j}{(1-r_{j+2}|z|)^{p'M(1-\theta)-1}} \asymp \frac{K^j}{(1-r_j|z|)^{p'M(1-\theta)-1}}, \end{aligned} \tag{8.14}$$

and hence

$$\begin{aligned} \sum_{j,l} \frac{(1-|z_{j,l}|)^{p'\left(M-\frac{1}{p}\right)(1-\eta)}}{\left|1-\overline{z_{j,l}}z\right|^{p'M(1-\theta)} \widehat{\omega}(z_{j,l})^{\frac{p'(1-\eta)}{q}}} &\lesssim \sum_j \frac{(1-r_j)^{p'\left(M-\frac{1}{p}\right)(1-\eta)-1}}{(1-r_j|z|)^{p'M(1-\theta)-1} \widehat{\omega}(r_j)^{\frac{p'(1-\eta)}{q}}} \\ &\asymp \sum_{r_j \leq |z|} \frac{(1-r_j)^{p'\left(M-\frac{1}{p}\right)(1-\eta)-1}}{(1-r_j|z|)^{p'M(1-\theta)-1} \widehat{\omega}(r_j)^{\frac{p'(1-\eta)}{q}}} \\ &\quad + \sum_{r_j > |z|} \frac{(1-r_j)^{p'\left(M-\frac{1}{p}\right)(1-\eta)-1}}{(1-r_j|z|)^{p'M(1-\theta)-1} \widehat{\omega}(r_j)^{\frac{p'(1-\eta)}{q}}} \\ &\leq \frac{1}{\widehat{\omega}(z)^{\frac{p'(1-\eta)}{q}}} \sum_{r_j \leq |z|} (1-r_j)^{p'\left(M-\frac{1}{p}\right)(1-\eta)-p'M(1-\theta)} \\ &\quad + \frac{1}{(1-|z|)^{p'M(1-\theta)-1}} \sum_{r_j > |z|} \frac{(1-r_j)^{p'\left(M-\frac{1}{p}\right)(1-\eta)-1}}{\widehat{\omega}(r_j)^{\frac{p'(1-\eta)}{q}}} \\ &= S_3(F) + S_4(F). \end{aligned} \tag{8.15}$$

Since $M(\eta-\theta) + \frac{1-\eta}{p} > 0$ by the second case in (8.12),

$$S_3(F) \leq \frac{1}{\widehat{\omega}(z)^{\frac{p'(1-\eta)}{q}}} \sum_{r_j \leq |z|} \frac{1}{(1-r_j)^{p'\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \asymp \frac{1}{\widehat{\omega}(z)^{\frac{p'(1-\eta)}{q}} (1-|z|)^{p'\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}. \tag{8.16}$$

Now, by using Lemma E(ii) and the third case of (8.12), we deduce

$$\begin{aligned} S_4(F) &\lesssim \frac{(1-|z|)^{\frac{\beta p'(1-\eta)}{q}-(p'M(1-\theta)-1)}}{\widehat{\omega}(z)^{\frac{p'(1-\eta)}{q}}} \sum_{r_j>|z|} (1-r_j)^{p'\left(M-\frac{1}{p}\right)(1-\eta)-1-\frac{\beta p'(1-\eta)}{q}} \\ &\asymp \frac{1}{\widehat{\omega}(z)^{\frac{p'(1-\eta)}{q}} (1-|z|)^{p'\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}. \end{aligned} \tag{8.17}$$

Consequently, by combining the estimates (8.13)–(8.17) we deduce

$$|F(z)|^p \lesssim \frac{\widehat{\omega}(z)^{-\frac{p(1-\eta)}{q}}}{(1-|z|)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j,l} |\lambda_{j,l}|^p \frac{(1-|z_{j,l}|)^{(pM-1)\eta}}{\left|1-\overline{z_{j,l}}z\right|^{pM\theta} \widehat{\omega}(z_{j,l})^{\frac{p\eta}{q}}}, \quad z \in \mathbb{D},$$

and hence the fourth case of (8.12) gives

$$\begin{aligned} M_p^p(r,F) &\lesssim \frac{\widehat{\omega}(r)^{-\frac{p(1-\eta)}{q}}}{(1-r)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j,l} |\lambda_{j,l}|^p \frac{(1-r_j)^{(pM-1)\eta}}{\widehat{\omega}(r_j)^{\frac{p\eta}{q}}} \int_0^{2\pi} \frac{dt}{\left|1-\overline{z_{j,l}}re^{it}\right|^{pM\theta}} \\ &\lesssim \frac{\widehat{\omega}(r)^{-\frac{p(1-\eta)}{q}}}{(1-r)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_j c_j^p \frac{(1-r_j)^{(pM-1)\eta}}{(1-r_jr)^{pM\theta-1} \widehat{\omega}(r_j)^{\frac{p\eta}{q}}}. \end{aligned} \tag{8.18}$$

By using this estimate we will deal with Cases 2.1 and 2.2.

Case 2.1: $p>1$ and $0<q\le p$. By (8.18),

$$\begin{aligned} \|F\|_{A^{p,q}_\omega}^q &\lesssim \int_0^1 \frac{\widehat{\omega}(r)^{-(1-\eta)}}{(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_j c_j^q \frac{(1-r_j)^{q\left(M-\frac{1}{p}\right)\eta}\widehat{\omega}(r_j)^{-\eta}}{(1-r_jr)^{qM\theta-\frac{q}{p}}}\omega(r)\,dr \\ &= \sum_j c_j^q (1-r_j)^{q\left(M-\frac{1}{p}\right)\eta}\widehat{\omega}(r_j)^{-\eta} \int_0^1 \frac{\widehat{\omega}(r)^{-(1-\eta)}}{(1-r_jr)^{qM\theta-\frac{q}{p}}(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}\omega(r)\,dr. \end{aligned} \tag{8.19}$$

Lemma E(iii) together with the fifth case of (8.12) yields

$$\begin{aligned} \int_0^{r_j} \frac{\widehat{\omega}(r)^{-(1-\eta)}\omega(r)}{(1-r_jr)^{qM\theta-\frac{q}{p}}(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}\,dr &\le \int_0^{r_j} \frac{\widehat{\omega}(r)^{-(1-\eta)}\omega(r)}{(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)+qM\theta-\frac{q}{p}}}\,dr \\ &\le \frac{1}{\widehat{\omega}(r_j)^{1-\eta}} \int_0^{r_j} \frac{\omega(r)}{(1-r)^{q\eta\left(M-\frac{1}{p}\right)}}\,dr \\ &\lesssim \frac{\widehat{\omega}(r_j)^{\eta}}{(1-r_j)^{q\eta\left(M-\frac{1}{p}\right)}}, \end{aligned} \tag{8.20}$$

while the fourth case of (8.12) implies

$$\int_{r_j}^1 \frac{\widehat{\omega}(r)^{-(1-\eta)}\omega(r)}{(1-r_jr)^{qM\theta-\frac{q}{p}}(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}\,dr \le \frac{1}{(1-r_j)^{Mq\theta-\frac{q}{p}}} \int_{r_j}^1 \frac{\widehat{\omega}(r)^{-(1-\eta)}\omega(r)}{(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}\,dr,$$

where, by an integration by parts and (8.7) together with the sixth case of (8.12),

$$
\begin{aligned}
&\int_{r_j}^{1} \frac{\widehat{\omega}(r)^{-(1-\eta)}}{(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}\omega(r)\,dr \\
&= \frac{\widehat{\omega}(r_j)^{\eta}}{\eta(1-r_j)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} + \frac{1}{\eta q(M(\eta-\theta)+\frac{1-\eta}{p})}\int_{r_j}^{1} \frac{\widehat{\omega}(r)^{\eta}}{(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)+1}}\,dr \\
&\lesssim \frac{\widehat{\omega}(r_j)^{\eta}}{(1-r_j)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} + \frac{\widehat{\omega}(r_j)^{\eta}}{(1-r_j)^{\alpha\eta}}\int_{r_j}^{1} \frac{dr}{(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)+1-\alpha\eta}} \\
&\asymp \frac{\widehat{\omega}(r_j)^{\eta}}{(1-r_j)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}},
\end{aligned}
$$

and thus

$$
\int_{r_j}^{1} \frac{\widehat{\omega}(r)^{-(1-\eta)}\omega(r)}{(1-r_j r)^{qM\theta-\frac{q}{p}}(1-r)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}\,dr \lesssim \frac{\widehat{\omega}(r_j)^{\eta}}{(1-r_j)^{q\eta\left(M-\frac{1}{p}\right)}}.
$$

This together with (8.19) and (8.20) gives (8.3).

Case 2.2: $p>1$ and $0<p<q$. By (8.18) and the fourth case of (8.12),

$$\|F\|^q_{A^{p,q}_\omega} \lesssim \int_0^1 \left(\frac{\widehat{\omega}(r)^{-\frac{p(1-\eta)}{q}}}{(1-r)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j=0}^\infty c_j^p \frac{(1-r_j)^{(pM-1)\eta}\widehat{\omega}(r_j)^{-\frac{p\eta}{q}}}{(1-r_jr)^{pM\theta-1}} \right)^{\frac{q}{p}} \omega(r)\,dr$$

$$= \sum_{k=0}^\infty \int_{r_k}^{r_{k+1}} \left(\frac{\widehat{\omega}(r)^{-\frac{p(1-\eta)}{q}}}{(1-r)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j=0}^\infty c_j^p \frac{(1-r_j)^{(pM-1)\eta}\widehat{\omega}(r_j)^{-\frac{p\eta}{q}}}{(1-r_jr)^{pM\theta-1}} \right)^{\frac{q}{p}} \omega(r)\,dr$$

$$\lesssim \sum_{k=0}^\infty \left(\frac{\widehat{\omega}(r_k)^{-\frac{p(1-\eta)}{q}}}{(1-r_k)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j=0}^\infty c_j^p \frac{(1-r_j)^{(pM-1)\eta}\widehat{\omega}(r_j)^{-\frac{p\eta}{q}}}{(1-r_jr_k)^{pM\theta-1}} \right)^{\frac{q}{p}} (\widehat{\omega}(r_k)-\widehat{\omega}(r_{k+1}))$$

$$\le \sum_{k=0}^\infty \left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j=0}^\infty c_j^p \frac{(1-r_j)^{(pM-1)\eta}}{(1-r_jr_k)^{pM\theta-1}\,\widehat{\omega}(r_j)^{\frac{p\eta}{q}}} \right)^{\frac{q}{p}}$$

$$\lesssim \sum_{k=0}^\infty \left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j=k+1}^\infty c_j^p \frac{(1-r_j)^{(pM-1)\eta}}{(1-r_jr_k)^{pM\theta-1}\,\widehat{\omega}(r_j)^{\frac{p\eta}{q}}} \right)^{\frac{q}{p}}$$

$$+ \sum_{k=0}^\infty \left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j=0}^k c_j^p \frac{(1-r_j)^{(pM-1)\eta}}{(1-r_jr_k)^{pM\theta-1}\,\widehat{\omega}(r_j)^{\frac{p\eta}{q}}} \right)^{\frac{q}{p}}$$

$$\lesssim \sum_{k=0}^\infty \left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{\eta(pM-1)}} \sum_{j=k+1}^\infty c_j^p \frac{(1-r_j)^{(pM-1)\eta}}{\widehat{\omega}(r_j)^{\frac{p\eta}{q}}} \right)^{\frac{q}{p}}$$

$$+ \sum_{k=0}^\infty \left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j=0}^k c_j^p \frac{(1-r_j)^{pM(\eta-\theta)+(1-\eta)}}{\widehat{\omega}(r_j)^{\frac{p\eta}{q}}} \right)^{\frac{q}{p}} = S_5(F)+S_6(F).$$

To prove the estimate $S_5(F)\lesssim \|\lambda\|^q_{\ell^{p,q}}$, define the step functions

$$U(x)=\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{\eta(pM-1)}},\quad x\in[k,k+1),\quad k\in\mathbb{N}\cup\{0\};$$

$$f(x)=c_k^p\frac{(1-r_k)^{\eta(pM-1)}}{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}},\quad x\in[k,k+1),\quad k\in\mathbb{N}\cup\{0\};$$

$$V(x)=\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{\eta(pM-1)}},\quad x\in[k,k+1),\quad k\in\mathbb{N}\cup\{0\}.$$

Then

$$\|\lambda\|^q_{\ell^{p,q}}=\sum_k^\infty c_k^q=\int_0^\infty (V(x)f(x))^{\frac{q}{p}}\,dx$$

and

$$\int_0^\infty \left(U(x)\int_x^\infty f(y)\,dy\right)^{\frac{q}{p}} dx \ge \sum_{k=0}^\infty \left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{\eta(pM-1)}} \sum_{j=k+1}^\infty c_j^p \frac{(1-r_j)^{\eta(pM-1)}}{\widehat{\omega}(r_j)^{\frac{p\eta}{q}}} \right)^{\frac{q}{p}} = S_5(F).$$

Therefore $S_5(F) \lesssim \|\lambda\|_{\ell^{p,q}}^q$ follows by [45, Theorem 2] if

$$\sup_{x\geq 0}\left(\int_0^x U(y)^{\frac{q}{p}}dy\right)^{\frac{p}{q}}\left(\int_x^\infty V(y)^{-\left(\frac{q}{p}\right)'}dy\right)^{\frac{1}{\left(\frac{q}{p}\right)'}} < \infty. \tag{8.21}$$

To prove this, let $x \geq 0$ and take $N = N(x) \in \mathbb{N}\cup\{0\}$ such that $N \leq x < N+1$. Then Lemma E(ii) and the hypothesis (8.2) yield

$$\begin{aligned}\left(\int_0^x U(y)^{\frac{q}{p}}\,dy\right)^{\frac{p}{q}} &\leq \left(\sum_{k=0}^{N+1}|U(k)|^{\frac{q}{p}}\right)^{\frac{p}{q}} = \left(\sum_{k=0}^{N}\left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{\eta(pM-1)}}\right)^{\frac{q}{p}}\right)^{\frac{p}{q}}\\
&= \left(\sum_{k=0}^{N}\frac{\widehat{\omega}(r_k)^{\eta}}{(1-r_k)^{\eta\left(qM-\frac{q}{p}\right)}}\right)^{\frac{p}{q}} \lesssim \left(\frac{\widehat{\omega}(r_N)^{\eta}}{(1-r_N)^{\beta\eta}}\sum_{k=0}^{N}\frac{1}{(1-r_k)^{\eta\left(qM-\frac{q}{p}-\beta\right)}}\right)^{\frac{p}{q}}\\
&\asymp \left(\frac{\widehat{\omega}(r_N)^{\eta}}{(1-r_N)^{\eta\left(qM-\frac{q}{p}\right)}}\right)^{\frac{p}{q}} = \frac{\widehat{\omega}(r_N)^{\frac{p\eta}{q}}}{(1-r_N)^{\eta(pM-1)}}.\end{aligned} \tag{8.22}$$

Another application of Lemma E(ii) and the hypothesis (8.2) give

$$\begin{aligned}\left(\int_x^\infty V(y)^{-\left(\frac{q}{p}\right)'}dy\right)^{\frac{1}{\left(\frac{q}{p}\right)'}} &\leq \left(\sum_{k=N}^{\infty}\left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{\eta(pM-1)}}\right)^{-\frac{q}{q-p}}\right)^{\frac{q-p}{q}}\\
&\lesssim \left(\frac{(1-r_N)^{\frac{p\beta\eta}{q-p}}}{\widehat{\omega}(r_N)^{\frac{p\eta}{q-p}}}\sum_{k=N}^{\infty}\frac{1}{(1-r_k)^{\frac{p\eta}{q-p}(\beta-q(M-\frac{1}{p}))}}\right)^{\frac{q-p}{q}}\\
&\asymp \frac{(1-r_N)^{\eta(pM-1)}}{\widehat{\omega}(r_N)^{\frac{p\eta}{q}}},\end{aligned}$$

which together with (8.22) implies (8.21).

We next prove $S_6(F) \lesssim \|\lambda\|_{\ell^{p,q}}^q$. Define

$$\begin{aligned}U(x) &= \frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}, \quad x\in[k,k+1), \quad k\in\mathbb{N}\cup\{0\};\\
f(x) &= c_k^p\frac{(1-r_k)^{pM(\eta-\theta)+1-\eta}}{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}, \quad x\in[k,k+1), \quad k\in\mathbb{N}\cup\{0\};\\
V(x) &= \frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{pM(\eta-\theta)+1-\eta}}, \quad x\in[k,k+1), \quad k\in\mathbb{N}\cup\{0\}.\end{aligned}$$

Then

$$\|\lambda\|_{\ell^{p,q}}^q = \sum_{k=0}^{\infty}c_k^q = \int_0^\infty (V(x)f(x))^{\frac{q}{p}}\,dx$$

and

$$\int_0^\infty \left(U(x) \int_0^x f(y)\,dy \right)^{\frac{q}{p}} dx \geq \sum_{k=0}^\infty \left(U(k) \sum_{j=0}^{k-1} f(j) \right)^{\frac{q}{p}}$$

$$= \sum_{k=0}^\infty \left(\frac{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}}{(1-r_k)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \sum_{j=0}^{k-1} c_j^p \frac{(1-r_j)^{pM(\eta-\theta)+(1-\eta)}}{\widehat{\omega}(r_j)^{\frac{p\eta}{q}}} \right)^{\frac{q}{p}},$$

and hence

$$S_6(F) \lesssim \int_0^\infty \left(U(x) \int_0^x f(y)\,dy \right)^{\frac{q}{p}} dx + \|\lambda\|^q_{\ell^{p,q}}.$$

Therefore $S_6(F) \lesssim \|\lambda\|^q_{\ell^{p,q}}$ follows by [45, Theorem 1] once we have shown that

$$\sup_{x\geq 0} \left(\int_x^\infty U(y)^{\frac{q}{p}} dy \right)^{\frac{p}{q}} \left(\int_0^x V(y)^{-\left(\frac{q}{p}\right)'} dy \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} < \infty. \tag{8.23}$$

To see this, let $x \geq 0$ and choose $N = N(x) \in \mathbb{N} \cup \{0\}$ such that $N \leq x < N+1$. Then, by (8.7) and the sixth case of (8.12) we deduce

$$\left(\int_x^\infty U(y)^{\frac{q}{p}}\,dy \right)^{\frac{p}{q}} \leq \left(\sum_{k=N}^\infty \frac{\widehat{\omega}(r_k)^\eta}{(1-r_k)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}} \right)^{\frac{p}{q}}$$

$$\lesssim \left(\frac{\widehat{\omega}(r_N)^\eta}{(1-r_N)^{\eta\alpha}} \sum_{k=N}^\infty \frac{1}{(1-r_k)^{q\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)-\eta\alpha}} \right)^{\frac{p}{q}}$$

$$\asymp \frac{\widehat{\omega}(r_N)^{\frac{p\eta}{q}}}{(1-r_N)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}$$

and

$$\left(\int_0^x V(y)^{-\left(\frac{q}{p}\right)'} dy \right)^{\frac{1}{\left(\frac{q}{p}\right)'}} \leq \left(\sum_{k=0}^N \left(\frac{(1-r_k)^{pM(\eta-\theta)+1-\eta}}{\widehat{\omega}(r_k)^{\frac{p\eta}{q}}} \right)^{\frac{q}{q-p}} \right)^{\frac{q-p}{q}}$$

$$\lesssim \left(\frac{(1-r_N)^{\frac{p\alpha\eta}{q-p}}}{\widehat{\omega}(r_N)^{\frac{p\eta}{q-p}}} \sum_{k=0}^N (1-r_k)^{\frac{q}{q-p}\left(pM(\eta-\theta)+1-\eta-\frac{p\eta\alpha}{q}\right)} \right)^{\frac{q-p}{q}}$$

$$\lesssim \frac{(1-r_N)^{p\left(M(\eta-\theta)+\frac{1-\eta}{p}\right)}}{\widehat{\omega}(r_N)^{\frac{p\eta}{q}}},$$

from which (8.23) follows. This finishes the proof of Case 2.2.

Finally, let us prove that there exist η, θ satisfying (8.12). By (8.2), the third and fifth cases in (8.12) are equivalent to

$$\eta \in \left(\frac{\gamma}{q\left(M-\frac{1}{p}\right)}, 1 - \frac{1}{p'\left(M-\frac{1}{p}-\frac{\beta}{q}\right)} \right),$$

where the inequality $\frac{\gamma}{q\left(M-\frac{1}{p}\right)} < 1 - \frac{1}{p'\left(M-\frac{1}{p}-\frac{\beta}{q}\right)}$ follows from (8.2). Let us observe that (8.2) also implies $\frac{\gamma}{q\left(M-\frac{1}{p}\right)} < \frac{M-1}{M-\frac{1}{p}}$. Therefore we may choose an η satisfying

$$\eta \in \left(\frac{\gamma}{q\left(M-\frac{1}{p}\right)}, \min\left\{1-\frac{1}{p'\left(M-\frac{1}{p}-\frac{\beta}{q}\right)}, \frac{M-1}{M-\frac{1}{p}}\right\}\right). \tag{8.24}$$

Next, observe that the first and the fourth cases in (8.12) are equivalent to

$$\theta \in \left(\frac{1}{Mp}, 1-\frac{1}{Mp'}\right),$$

where $\frac{1}{Mp} < 1 - \frac{1}{Mp'}$ by (8.2). Further, by (8.2), the second and the sixth conditions in (8.12) are equivalent to

$$\theta \in \left(\frac{\left(M-\frac{1}{p}-\frac{\alpha}{q}\right)\eta+\frac{1}{p}}{M}, \frac{\left(M-\frac{1}{p}\right)\eta+\frac{1}{p}}{M}\right), \tag{8.25}$$

where trivially $\frac{\left(M-\frac{1}{p}-\frac{\alpha}{q}\right)\eta+\frac{1}{p}}{M} < \frac{\left(M-\frac{1}{p}\right)\eta+\frac{1}{p}}{M}$. It is clear that $\frac{1}{Mp} < \frac{\left(M-\frac{1}{p}-\frac{\alpha}{q}\right)\eta+\frac{1}{p}}{M}$ as $\alpha \leq \beta$, and $\frac{\left(M-\frac{1}{p}\right)\eta+\frac{1}{p}}{M} < 1 - \frac{1}{Mp'}$ by (8.24). Therefore it is enough to choose θ satisfying (8.25). This finishes the proof of (8.12).

Case 3.1: $p = \infty$ and $0 < q \leq 1$. For this we set $\lambda(j) = \sup_l |\lambda_{j,l}|$. Then the estimates in (8.14) imply

$$|F(z)| \lesssim \sum_{j=0}^{\infty} \lambda(j) \frac{(1-r_j)^{M-1}}{(1-r_j|z|)^{M-1}\,\widehat{\omega}(r_j)^{\frac{1}{q}}}, \quad z \in \mathbb{D}, \tag{8.26}$$

for each $M > 1$. Since $0 < q \leq 1$ and $q(M-1) > \gamma$ by the hypothesis (8.2), Lemma E(iii) yields

$$\|F\|^q_{A^{\infty,q}_\omega} \lesssim \sum_{j=0}^{\infty} \lambda(j)^q \int_0^1 \frac{(1-r_j)^{q(M-1)}}{(1-r_jr)^{q(M-1)}\,\widehat{\omega}(r_j)} \omega(r)\,dr \lesssim \sum_{j=0}^{\infty} \lambda(j)^q = \|\lambda\|^q_{\ell^{\infty,q}},$$

and thus this case is proved.

Case 3.2: $p = \infty$ and $1 < q < \infty$. By using (8.26) we deduce

$$\begin{aligned}
\|F\|^q_{A^{\infty,q}_\omega} &\lesssim \int_0^1 \left(\sum_{j=0}^{\infty} \lambda(j) \frac{(1-|z_{j,l}|)^{M-1}}{(1-r_jr)^{M-1}\,\widehat{\omega}(z_{j,l})^{\frac{1}{q}}}\right)^q \omega(r)\,dr \\
&\lesssim \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} \lambda(j) \left(\frac{1-r_j}{1-r_jr_k}\right)^{M-1} \left(\frac{\widehat{\omega}(r_k)}{\widehat{\omega}(r_j)}\right)^{\frac{1}{q}}\right)^q \\
&\lesssim \sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\infty} \lambda(j) \left(\frac{1-r_j}{1-r_k}\right)^{M-1} \left(\frac{\widehat{\omega}(r_k)}{\widehat{\omega}(r_j)}\right)^{\frac{1}{q}}\right)^q \\
&\quad + \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \lambda(j) \left(\frac{\widehat{\omega}(r_k)}{\widehat{\omega}(r_j)}\right)^{\frac{1}{q}}\right)^q = S_7(F) + S_8(F).
\end{aligned}$$

To prove $S_7(F) \lesssim \|\lambda\|^q_{\ell^\infty,q}$, define the step functions

$$U(x) = \frac{\widehat{\omega}(r_k)^{\frac{1}{q}}}{(1-r_k)^{M-1}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\};$$
$$f(x) = \lambda(k) \frac{(1-r_k)^{M-1}}{\widehat{\omega}(r_k)^{\frac{1}{q}}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\};$$
$$V(x) = \frac{\widehat{\omega}(r_k)^{\frac{1}{q}}}{(1-r_k)^{M-1}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\}.$$

Then

$$\|\lambda\|^q_{\ell^\infty,q} = \sum_{k=0}^{\infty} \lambda(k)^q = \int_0^\infty (V(x) f(x))^q \, dx$$

and

$$\int_0^\infty \left(U(x) \int_x^\infty f(y)\, dy \right)^q dx \geq \sum_{k=0}^{\infty} \left(\sum_{j=k+1}^{\infty} \lambda(j) \left(\frac{1-r_j}{1-r_k} \right)^{M-1} \left(\frac{\widehat{\omega}(r_k)}{\widehat{\omega}(r_j)} \right)^{\frac{1}{q}} \right)^q = S_7(F).$$

Therefore $S_7(F) \lesssim \|\lambda\|^q_{\ell^\infty,q}$ follows by [45, Theorem 2], if

$$\sup_{x \geq 0} \left(\int_0^x U(y)^q \, dy \right)^{\frac{1}{q}} \left(\int_x^\infty V(y)^{-q'} \, dy \right)^{\frac{1}{q'}} < \infty. \tag{8.27}$$

To prove this, let $x \geq 0$ and $N \in \mathbb{N} \cup \{0\}$ such that $N \leq x < N+1$. Then Lemma E(ii) and the hypothesis (8.2) imply

$$\begin{aligned} \int_0^x U(y)^q \, dy &\leq \sum_{k=0}^{N} \frac{\widehat{\omega}(r_k)}{(1-r_k)^{q(M-1)}} \\ &\lesssim \frac{\widehat{\omega}(r_N)}{(1-r_N)^{\beta}} \sum_{k=0}^{N} \frac{1}{(1-r_k)^{q(M-1)-\beta}} \asymp \frac{\widehat{\omega}(r_N)}{(1-r_N)^{q(M-1)}} \end{aligned}$$

and

$$\begin{aligned} \int_x^\infty V(y)^{-q'} \, dy &\leq \sum_{k=N}^{\infty} \frac{(1-r_k)^{q'(M-1)}}{\widehat{\omega}(r_k)^{\frac{q'}{q}}} \lesssim \frac{(1-r_N)^{\frac{q'\beta}{q}}}{\widehat{\omega}(r_N)^{\frac{q'}{q}}} \sum_{k=N}^{\infty} (1-r_k)^{q'\left(M-1-\frac{\beta}{q}\right)} \\ &\asymp \frac{(1-r_N)^{q'(M-1)}}{\widehat{\omega}(r_N)^{\frac{q'}{q}}}, \end{aligned}$$

from which (8.27) follows. Thus $S_7(F) \lesssim \|\lambda\|^q_{\ell^\infty,q}$.

To obtain $S_8(F) \lesssim \|\lambda\|^q_{\ell^\infty,q}$, define

$$U(x) = \widehat{\omega}(r_k)^{\frac{1}{q}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\};$$
$$f(x) = \frac{\lambda(k)}{\widehat{\omega}(r_k)^{\frac{1}{q}}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\};$$
$$V(x) = \widehat{\omega}(r_k)^{\frac{1}{q}}, \quad x \in [k, k+1), \quad k \in \mathbb{N} \cup \{0\}.$$

Then

$$\|\lambda\|_{\ell^{\infty,q}}^{q} = \sum_{k=0}^{\infty} \lambda(k)^q = \int_0^\infty (V(x)f(x))^q \, dx$$

and

$$\int_0^\infty \left(U(x)\int_0^x f(y)\,dy\right)^q dx \geq \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k-1}\lambda(j)\left(\frac{\widehat{\omega}(r_k)}{\widehat{\omega}(r_j)}\right)^{\frac{1}{q}}\right)^q,$$

and hence

$$S_8(F) \lesssim \int_0^\infty \left(U(x)\int_0^x f(y)\,dy\right)^q dx + \|\lambda\|_{\ell^{\infty,q}}^{q}.$$

Therefore $S_8(F) \lesssim \|\lambda\|_{\ell^{\infty,q}}^{q}$ holds by [45, Theorem 1], once we have shown that

$$\sup_{x\geq 0}\left(\int_x^\infty U(y)^q\,dy\right)^{\frac{1}{q}}\left(\int_0^x V(y)^{-q'}\,dy\right)^{\frac{1}{q'}} < \infty. \tag{8.28}$$

To see this, let $x \geq 0$ and $N \in \mathbb{N}\cup\{0\}$ such that $N \leq x < N+1$. By using (8.7) we deduce

$$\int_x^\infty U(y)^q\,dy \leq \sum_{k=N}^{\infty}\widehat{\omega}(r_k) \lesssim \frac{\widehat{\omega}(r_N)}{(1-r_N)^\alpha}\sum_{k=N}^{\infty}(1-r_k)^\alpha \asymp \widehat{\omega}(r_N)$$

and

$$\int_0^x V(y)^{-q'}\,dy = \sum_{k=0}^{N}\widehat{\omega}(r_k)^{\frac{-q'}{q}} \lesssim \frac{(1-r_N)^{\frac{\alpha q'}{q}}}{\widehat{\omega}(r_N)^{\frac{q'}{q}}}\sum_{k=0}^{N}(1-r_k)^{-\alpha\frac{q'}{q}} \asymp \widehat{\omega}(r_N)^{-\frac{q'}{q}},$$

from which (8.28) follows. Thus $S_8(F) \lesssim \|\lambda\|_{\ell^{\infty,q}}^{q}$. This finishes the proof of Case 3.2 and the proof of the theorem as well.

8.3 A REPRESENTATION THEOREM FOR FUNCTIONS IN $A^{P,Q}_{\omega}$

To prove Theorem 18 some definitions and lemmas are needed. For each dyadic polar rectangle $Q_{j,l}$ defined in (8.4), consider the set of indexes

$$U_{j,l} = \left\{(i,m) : dist(Q_{i,m},Q_{j,l}) \leq \frac{1}{K^{j+1}}\left(1-\frac{1}{K}\right)\right\}, \quad j\in\mathbb{N}\cup\{0\}, \quad l = 0,1,\dots,K^{j+3}-1,$$

and denote

$$\widehat{Q}_{j,l} = \bigcup_{(i,m)\in U_{j,l}} Q_{i,m}. \tag{8.29}$$

For $f \in \mathcal{H}(\mathbb{D})$, define $f_{j,l} = \sup_{z\in Q_{j,l}}|f(z)|$ and $\widehat{f}_{j,l} = \sup_{\zeta\in\widehat{Q}_{j,l}}|f(\zeta)|$. Then

$$\widehat{f}_{j,l} \leq \sum_{(i,m)\in U_{j,l}} f_{i,m} \lesssim \widehat{f}_{j,l}, \quad j\in\mathbb{N}\cup\{0\}, \quad l = 0,1,\dots,K^{j+3}-1, \tag{8.30}$$

because $\# U_{j,l}$ has a finite uniform bound independent of j, l and K. For $f \in \mathcal{H}(\mathbb{D})$, write $\lambda(f) = \{\lambda(f)_{j,l}\}$, where

$$\lambda(f)_{j,l} = K^{-\frac{j}{p}} \widehat{\omega}(r_j)^{\frac{1}{q}} f_{j,l} \tag{8.31}$$

for all $j \in \mathbb{N} \cup \{0\}$ and $l = 0, 1, \ldots, K^{j+3} - 1$.

Lemma 9. *Let $0 < p, q < \infty$, $\omega \in \mathcal{D}$ and $K \in \mathbb{N} \setminus \{1\}$ such that* (8.1) *holds. Then $\|f\|_{A^{p,q}_\omega} \asymp \|\lambda(f)\|_{\ell^{p,q}}$ for all $f \in \mathcal{H}(\mathbb{D})$.*

Proof. Lemma E(ii) implies

$$\|f\|^q_{A^{p,q}_\omega} = \sum_{j=0}^{\infty} \int_{r_j}^{r_{j+1}} M_p^q(r, f)\omega(r)\,dr \le \sum_{j=0}^{\infty} M_p^q(r_{j+1}, f)\widehat{\omega}(r_j) \lesssim \sum_{j=1}^{\infty} M_p^q(r_j, f)\widehat{\omega}(r_j),$$

where

$$M_p^p(r_j, f) = \int_0^{2\pi} \left| f(r_j e^{i\theta}) \right|^p d\theta = \sum_{l=0}^{K^{j+3}-1} \int_{\frac{2\pi l}{K^{j+3}}}^{\frac{2\pi(l+1)}{K^{j+3}}} \left| f(r_j e^{i\theta}) \right|^p d\theta \lesssim K^{-j} \sum_{l=0}^{K^{j+3}-1} f_{j,l}^p,$$

and hence

$$\|f\|^q_{A^{p,q}_\omega} \lesssim \sum_{j=1}^{\infty} \left(K^{-j} \sum_{l=0}^{K^{j+3}-1} f_{j,l}^p \right)^{\frac{q}{p}} \widehat{\omega}(r_j) = \sum_{j=1}^{\infty} \left(\sum_{l=0}^{K^{j+3}-1} \left(K^{-\frac{j}{p}} \widehat{\omega}(r_j)^{\frac{1}{q}} f_{j,l} \right)^p \right)^{\frac{q}{p}} \le \|\lambda(f)\|^q_{\ell^{p,q}}.$$

To prove the reverse inequality, choose $z^\star_{j,l} \in \overline{Q_{j,l}}$ such that $f_{j,l} = |f(z^\star_{j,l})|$, and $n_0 \in \mathbb{N}$ such that

$$r_{j-1} \le r_j - \frac{\operatorname{diam} Q_{j,l}}{K^{n_0}} < r_{j+1} + \frac{\operatorname{diam} Q_{j,l}}{K^{n_0}} \le r_{j+2}$$

for all j and l. Then the subharmonicity of $|f|^p$ gives

$$\begin{aligned}\sum_{l=0}^{K^{j+3}-1} f_{j,l}^p &\lesssim \sum_{l=0}^{K^{j+3}-1} \frac{1}{|D(z^\star_{j,l}, \frac{\operatorname{diam} Q_{j,l}}{K^{n_0}})|} \int_{D\left(z^\star_{j,l}, \frac{\operatorname{diam} Q_{j,l}}{K^{n_0}}\right)} |f(\zeta)|^p\,dA(\zeta) \\ &\lesssim K^{2j} \int_{A_{j-1} \cup A_j \cup A_{j+1}} |f(\zeta)|^p dA(\zeta) \lesssim K^j M_p^p(r_{j+2}, f), \quad j \in \mathbb{N} \cup \{0\},\end{aligned}$$

with the convenience that $A_{-1} = \emptyset$. Moreover, (8.1) implies $\widehat{\omega}(r_j) \le \frac{C}{C-1} \int_{r_j}^{r_{j+1}} \omega(r)\,dr$ for all $j \in \mathbb{N} \cup \{0\}$. These two estimates together with Lemma E(ii) now yield

$$\begin{aligned}\|\lambda(f)\|^q_{\ell^{p,q}} &= \sum_{j=0}^{\infty} \left(\sum_{l=0}^{K^{j+3}-1} \left(K^{-\frac{j}{p}} \widehat{\omega}(r_j)^{\frac{1}{q}} f_{j,l} \right)^p \right)^{\frac{q}{p}} \lesssim \sum_{j=0}^{\infty} \left(K^{-j} \widehat{\omega}(r_j)^{\frac{p}{q}} K^j M_p^p(r_{j+2}, f) \right)^{\frac{q}{p}} \\ &\lesssim \sum_{j=2}^{\infty} \widehat{\omega}(r_j) M_p^q(r_j, f) \lesssim \sum_{j=0}^{\infty} \int_{r_j}^{r_{j+1}} M_p^q(r, f)\omega(r)\,dr = \|f\|^q_{A^{p,q}_\omega},\end{aligned} \tag{8.32}$$

and therefore the assertion is proved. □

Lemma 10. *Let $0 < p \leq \infty$, $0 < q < \infty$ and $\omega \in \widehat{\mathcal{D}}$, and let $\beta = \beta(\omega) > 0$ be that of Lemma E(ii). Then $A^{p,q}_\omega \subset A^1_\eta$ for all $\eta > \frac{\beta}{q} + \frac{1}{p} - 1$.*

Proof. If $f \in \mathcal{H}(\mathbb{D})$ and $\omega \in \widehat{\mathcal{D}}$, then the well known inequality $M_\infty(r,f) \lesssim M_p(\frac{1+r}{2}, f)(1-r)^{-1/p}$ gives

$$\|f\|^q_{A^{p,q}_\omega} \geq \int_{\frac{1+r}{2}}^1 M_p^q(s,f)\omega(s)\,ds \geq M_p^q\left(\frac{1+r}{2}, f\right)\widehat{\omega}\left(\frac{1+r}{2}\right) \gtrsim M^q_\infty(r,f)\widehat{\omega}(r)(1-r)^{\frac{q}{p}},$$

from which Lemma E(ii) yields

$$\|f\|_{A^1_\eta} \lesssim \|f\|_{A^{p,q}_\omega} \int_0^1 \frac{(1-r)^{\eta-\frac{1}{p}}}{\widehat{\omega}(r)^{\frac{1}{q}}}\,dr \lesssim \frac{\|f\|_{A^{p,q}_\omega}}{\widehat{\omega}(0)^{\frac{1}{q}}} \int_0^1 (1-r)^{\eta-\frac{1}{p}-\frac{\beta}{q}}\,dr \asymp \|f\|_{A^{p,q}_\omega},$$

and the assertion follows. □

Proof ofTheorem 18. The fact that the functions of the form (8.5) with $\lambda(f) = \{\lambda(f)^k_{j,l}\} \in \ell^{p,q}$ belong to $A^{p,q}_\omega$ and the inequality $\left\|\{\lambda(f)^k_{j,l}\}\right\|_{\ell^{p,q}} \lesssim \|f\|_{A^{p,q}_\omega}$ follow from Theorem 5.3.1.

Let us now prove that each function in $A^{p,q}_\omega$ is of the form (8.5), where $\lambda(f) = \{\lambda(f)^k_{j,l}\} \in \ell^{p,q}$, and $\|f\|_{A^{p,q}_\omega} \lesssim \left\|\{\lambda(f)^k_{j,l}\}\right\|_{\ell^{p,q}}$. To do this we use ideas from [63, (1.5) Theorem]. Let $\eta = \eta(p,q,\omega) > 1 + \frac{1}{p} + \frac{\beta+\gamma}{q}$, where $\beta = \beta(\omega) > 0$ and $\gamma = \gamma(\omega) > 0$ are those of Lemma E(ii)(iii). Then η satisfies the condition (8.2) assumed on M in Theorem 5.3.1, and $A^{p,q}_\omega \subset A^1_\eta$ by Lemma 10. Therefore, in particular,

$$P_\eta(f)(z) = (\eta+1)\int_{\mathbb{D}} \frac{f(\zeta)}{(1-\overline{\zeta}z)^{2+\eta}}(1-|\zeta|^2)^\eta\,dA(\zeta) = f(z), \quad z \in \mathbb{D}, \quad f \in A^{p,q}_\omega.$$

Consider the operator S_η defined by

$$\begin{aligned} S_\eta(f)(z) &= (\eta+1)\sum_{j,l,k} f(\zeta^k_{j,l})\frac{(1-|\zeta^k_{j,l}|^2)^\eta}{\left(1-\overline{\zeta^k_{j,l}}z\right)^{\eta+2}}\left|Q^k_{j,l}\right| \\ &= (\eta+1)\sum_{j,l,k} f(\zeta^k_{j,l})(1-|\zeta^k_{j,l}|^2)^{\frac{1}{p}}\widehat{\omega}(r_j)^{\frac{1}{q}}\frac{(1-|\zeta^k_{j,l}|^2)^{\eta-\frac{1}{p}}\widehat{\omega}(r_j)^{-\frac{1}{q}}}{\left(1-\overline{\zeta^k_{j,l}}z\right)^{\eta+2}}\left|Q^k_{j,l}\right| \\ &= (\eta+1)\sum_{j,l,k} a_{j,l,k}(f)\frac{(1-|\zeta^k_{j,l}|^2)^{\eta-\frac{1}{p}}\widehat{\omega}(r_j)^{-\frac{1}{q}}}{\left(1-\overline{\zeta^k_{j,l}}z\right)^{\eta+2}}\left|Q^k_{j,l}\right|, \quad z \in \mathbb{D}, \end{aligned}$$

where

$$a(f)_{j,l,k} = f(\zeta^k_{j,l})(1-|\zeta^k_{j,l}|^2)^{\frac{1}{p}}\widehat{\omega}(r_j)^{\frac{1}{q}}, \quad j \in \mathbb{N}\cup\{0\}, \quad l = 0,\dots,K^j-1, \quad k = 1,\dots,M^2. \tag{8.33}$$

The estimate $M_\infty(r,f) \lesssim \|f\|_{A^{p,q}_\omega}\widehat{\omega}(r)^{-\frac{1}{q}}(1-r)^{-\frac{1}{p}}$, obtained in the proof of Lemma 10, ensures that $S_\eta(f)$ is well defined for each $f \in A^{p,q}_\omega$. Write $a(f) = \{a(f)_{j,l,k}\}$ and observe that

$$\|a(f)\|_{\ell^{p,q}} \lesssim \|\lambda(f)\|_{\ell^{p,q}} \asymp \|f\|_{A^{p,q}_\omega} \tag{8.34}$$

by Lemma 9. It is shown next that for M large enough, S_η satisfies

$$\|f - S_\eta(f)\|_{A^{p,q}_\omega} \le \frac{1}{2}\|f\|_{A^{p,q}_\omega}. \tag{8.35}$$

To see this, note first that

$$\begin{aligned} f(z) - S_\eta(f)(z) &= P_\eta(f)(z) - S_\eta(f)(z) \\ &= (\eta+1)\Bigg(\int_{\mathbb{D}} \frac{f(\zeta)}{(1-\overline{\zeta}z)^{2+\eta}}(1-|\zeta|^2)^\eta \, dA(\zeta) \\ &\quad - \sum_{j,l,k} f(\zeta^k_{j,l}) \frac{\left(1-|\zeta^k_{j,l}|^2\right)^\eta}{\left(1-\overline{\zeta^k_{j,l}}z\right)^{\eta+2}} \left|Q^k_{j,l}\right| \Bigg) \\ &= (\eta+1)\sum_{j,l,k}\int_{Q^k_{j,l}} \left(H_z(\zeta) - H_z(\zeta^k_{j,l})\right) dA(\zeta), \quad z \in \mathbb{D}, \end{aligned} \tag{8.36}$$

where $H_z(\zeta) = f(\zeta)\frac{(1-|\zeta|^2)^\eta}{(1-\overline{\zeta}z)^{\eta+2}}$ for all $z, \zeta \in \mathbb{D}$. It is clear that H_z satisfies

$$\left|H_z(\zeta) - H_z(\zeta^k_{j,l})\right| \le \operatorname{diam} Q^k_{j,l} \sup_{w \in Q^k_{j,l}} |\nabla H_z(w)|, \quad \zeta \in Q^k_{j,l}, \tag{8.37}$$

and also

$$\frac{\partial}{\partial \zeta} H_z(\zeta) = \left(f'(\zeta)(1-|\zeta|^2) - f(\zeta)\eta\overline{\zeta}\right) \frac{(1-|\zeta|^2)^{\eta-1}}{(1-\overline{\zeta}z)^{\eta+2}}, \quad \zeta \in \mathbb{D}, \tag{8.38}$$

and

$$\frac{\partial}{\partial \overline{\zeta}} H_z(\zeta) = f(\zeta)\frac{(1-|\zeta|^2)^{\eta-1}}{(1-\overline{\zeta}z)^{\eta+2}} \left((\eta+2)z\frac{1-|\zeta|^2}{1-\overline{\zeta}z} - \eta\zeta\right), \quad \zeta \in \mathbb{D}. \tag{8.39}$$

Moreover, the Cauchy integral formula implies

$$|f^{(n)}(\zeta)| \lesssim \int_{|\xi-\zeta| = \frac{1}{K^{j+1}\left(1-\frac{1}{K}\right)}} \frac{|f(\xi)|}{|\xi-\zeta|^{n+1}} |d\xi| \lesssim K^{jn}\widehat{f}_{j,l}, \quad \zeta \in Q_{j,l}, \quad n \in \mathbb{N}\cup\{0\}. \tag{8.40}$$

The identities (8.38) and (8.39) together with the estimate (8.40) now give

$$\sup_{w \in Q^k_{j,l}} |\nabla H_z(w)| \lesssim \frac{(1-|\zeta^k_{j,l}|^2)^{\eta-1}}{\left|1-z\overline{\zeta^k_{j,l}}\right|^{\eta+2}} \sup_{w \in \widehat{Q}_{j,l}} |f(w)| = \frac{(1-|\zeta^k_{j,l}|^2)^{\eta-1}}{\left|1-z\overline{\zeta^k_{j,l}}\right|^{\eta+2}} \widehat{f}_{j,l}. \tag{8.41}$$

The identity (8.36) together with the estimates (8.37), (8.41) and (8.30) give

$$
\begin{aligned}
|f(z)-S_\eta(f)(z)| &\lesssim \sum_{j,l,k}\left(\int_{Q^k_{j,l}}\left(\operatorname{diam} Q^k_{j,l}\frac{(1-|\zeta^k_{j,l}|^2)^{\eta-1}}{\left|1-z\overline{\zeta^k_{j,l}}\right|^{\eta+2}}\sup_{w\in\widehat{Q}_{j,l}}|f(w)|\right)dA(\zeta)\right)\\
&\lesssim \sum_{j,l}\sum_{k=1}^{M^2}\left(\left|Q^k_{j,l}\right|\operatorname{diam} Q^k_{j,l}\frac{(1-|\zeta^k_{j,l}|^2)^{\eta-1}}{\left|1-z\overline{\zeta^k_{j,l}}\right|^{\eta+2}}\widehat{f}_{j,l}\right)\\
&\lesssim \sum_{j,l}\frac{1}{M^3}\frac{(1-|\zeta_{j,l}|^2)^{\eta+2}}{\left|1-z\overline{\zeta_{j,l}}\right|^{\eta+2}}\widehat{f}_{j,l}\sum_{k=1}^{M^2}1\\
&\le \frac{1}{M}\sum_{j,l}\frac{(1-|\zeta_{j,l}|^2)^{\eta+2}}{\left|1-z\overline{\zeta_{j,l}}\right|^{\eta+2}}\sum_{(i,m)\in U_{k,j}}f_{i,m}\\
&\lesssim \frac{1}{M}\sum_{j,l}\lambda(f)_{j,l}\frac{(1-|\zeta_{j,l}|^2)^{\eta+2-\frac{1}{p}}\widehat{\omega}(r_j)^{-\frac{1}{q}}}{\left|1-z\overline{\zeta_{j,l}}\right|^{\eta+2}},
\end{aligned}
$$

where $\{\lambda_{j,l}\}$ is that of (8.31). Then, by combining the above estimate with Theorem 5.3.1, with $M=\eta+2$, and Lemma 9 it follows that

$$\|f-S_\eta(f)\|_{A^{p,q}_\omega}\lesssim\frac{1}{M}\|\lambda(f)\|_{\ell^{p,q}}\asymp\frac{1}{M}\|f\|_{A^{p,q}_\omega}.$$

The inequality (8.35) follows by choosing M large enough.

Let $\{f_n\}_{n=1}^\infty$ be defined by $f_1=S_\eta(f)$ and $f_n=S_\eta\left(f-\sum_{m=1}^{n-1}f_m\right)$ for $n\in\mathbb{N}\setminus\{1\}$. Further, let $a(f)^{(1)}_{j,l,k}=a(f)_{j,l,k}$, where $\{a(f)_{j,l,k}\}$ are those defined in (8.33), and

$$a(f)^{(n)}_{j,l,k}=\left(f-\sum_{m=1}^{n-1}f_m\right)(\zeta^k_{j,l})(1-|\zeta^k_{j,l}|^2)^{\frac{1}{p}}\widehat{\omega}(r_j)^{\frac{1}{q}},\quad n\in\mathbb{N}\setminus\{1\}.$$

With this notation

$$f_n(z)=(\eta+1)\sum_{j,l,k}a(f)^{(n)}_{j,l,k}\frac{(1-|\zeta^k_{j,l}|^2)^{\eta-\frac{1}{p}}\widehat{\omega}(r_j)^{-\frac{1}{q}}}{\left(1-\overline{\zeta^k_{j,l}}z\right)^{\eta+2}}\left|Q^k_{j,l}\right|,\quad n\in\mathbb{N},\tag{8.42}$$

by the definition of S_η. Moreover, n applications of (8.35) give

$$
\begin{aligned}
\left\|f-\sum_{m=1}^{n}f_m\right\|_{A^{p,q}_\omega}&=\left\|f-\sum_{m=1}^{n-1}f_m-f_n\right\|_{A^{p,q}_\omega}=\left\|\left(Id-S_\eta\right)\left(f-\sum_{m=1}^{n-1}f_m\right)\right\|_{A^{p,q}_\omega}\\
&\le\frac{1}{2}\left\|\left(f-\sum_{m=1}^{n-1}f_m\right)\right\|_{A^{p,q}_\omega}\le\cdots\le\frac{1}{2^n}\|f\|_{A^{p,q}_\omega}.
\end{aligned}\tag{8.43}
$$

Therefore, by denoting $a(f)^{(n)}=\{a(f)^{(n)}_{j,l,k}\}$, and applying (8.34) to $f-\sum_{m=1}^{n-1}f_m$ yields

$$\|a(f)^{(n)}\|_{\ell^{p,q}}\lesssim\left\|f-\sum_{m=1}^{n-1}f_m\right\|_{A^{p,q}_\omega}\le 2^{-n+1}\|f\|_{A^{p,q}_\omega},\quad n\in\mathbb{N}.\tag{8.44}$$

Finally, set $b(f)_{j,l,k} = \sum_{n=1}^{\infty} a(f)_{j,l,k}^{(n)}$ and

$$g(z) = (\eta+1) \sum_{j,l,k} b(f)_{j,l,k} \frac{(1-|\zeta_{j,l}^k|^2)^{\eta-\frac{1}{p}} \widehat{\omega}(r_j)^{-\frac{1}{q}}}{(1-\overline{\zeta_{j,l}^k} z)^{\eta+2}} \left|Q_{j,l}^k\right|, \quad z \in \mathbb{D}.$$

Then (8.42) yields

$$\begin{aligned} g(z) - \sum_{m=1}^{n} f_m &= (\eta+1) \sum_{j,l,k} \left(b(f)_{j,l,k} - \sum_{m=1}^{n} a(f)_{j,l,k}^{(m)} \right) \frac{(1-|\zeta_{j,l}^k|^2)^{\eta-\frac{1}{p}} \widehat{\omega}(r_j)^{-\frac{1}{q}}}{(1-\overline{\zeta_{j,l}^k} z)^{\eta+2}} \left|Q_{j,l}^k\right| \\ &= \sum_{j,l,k} \left(\sum_{m=n+1}^{\infty} a(f)_{j,l,k}^{(m)} \right) \frac{(1-|\zeta_{j,l}^k|^2)^{\eta-\frac{1}{p}} \widehat{\omega}(r_j)^{-\frac{1}{q}}}{(1-\overline{\zeta_{j,l}^k} z)^{\eta+2}} \left|Q_{j,l}^k\right|, \end{aligned}$$

from which Theorem 5.3.1 and (8.44) give

$$\begin{aligned} \left\| g - \sum_{m=1}^{n} f_m \right\|_{A_\omega^{p,q}} &\lesssim \left\| \left\{ \sum_{m=n+1}^{\infty} a_{j,l,k}^{(m)} \right\}_{j,l,k} \right\|_{\ell^{p,q}} \lesssim \|f\|_{A_\omega^{p,q}} \left(\sum_{m=n+1}^{\infty} 2^{-m\min\{1,p,q\}} \right)^{\frac{1}{\min\{1,p,q\}}} \\ &\asymp 2^{-n} \|f\|_{A_\omega^{p,q}}, \quad n \in \mathbb{N}. \end{aligned}$$

By combining this with (8.43) we deduce

$$\begin{aligned} \|f-g\|_{A_\omega^{p,q}} &\leq \left\| f - \sum_{m=1}^{n} f_m + \sum_{m=1}^{n} f_m - g \right\|_{A_\omega^{p,q}} \\ &\lesssim \left\| f - \sum_{m=1}^{n} f_m \right\|_{A_\omega^{p,q}} + \left\| g - \sum_{m=1}^{n} f_m \right\|_{A_\omega^{p,q}} \\ &\lesssim 2^{-n} \|f\|_{A_\omega^{p,q}}, \quad n \in \mathbb{N} \setminus \{1\}, \end{aligned} \tag{8.45}$$

and it follows that $f = g$. The assertion of the theorem follows for $M = \eta + 2$ and

$$\lambda(f)_{j,l}^k = (\eta+1) b(f)_{j,l,k} \frac{|Q_{j,l}^k|}{(1-|\zeta_{j,l}^k|^2)^2},$$

because $\|\{\lambda(f)_{j,l}^k\}\|_{\ell^{p,q}} \lesssim \|f\|_{A_\omega^{p,q}}$ by (8.44). This finishes the proof. □

8.4 DIFFERENTIATION OPERATORS FROM $A_\omega^{P,Q}$ TO L_μ^S

We recall that the spaces $\ell^{p,q}$ obey the basic inclusion relations $\ell^{p,q} \subset \ell^{r,q}$ for $p \leq r$, and $\ell^{p,q} \subset \ell^{p,s}$ if $q \leq s$. Moreover, it is known that by denoting

$$p' = \begin{cases} \infty, & 0 < p \leq 1, \\ \frac{p}{p-1}, & 1 < p < \infty, \\ 1, & p = \infty, \end{cases}$$

we have

$$\|b\|_{\ell^{p',q'}} = \sup\left\{\left|\sum_{j,l} c_{j,l} b_{j,l}\right| : \|c\|_{\ell^{p,q}} = 1\right\}$$

by [46, Theorem 1]. The following proof uses ideas form the proof of [41, Theorem 2].

Proof ofTheorem 19. Assume first that $D^{(n)} : A^{p,q}_\omega \to L^s_\mu$ is bounded. Let

$$F_t(z) = \sum_{j,l} a_{j,l}(t)\lambda_{j,l} \frac{(1-|z_{j,l}|)^{M-\frac{1}{p}} \widehat{\omega}(z_{j,l})^{-\frac{1}{q}}}{\left(1-\overline{z_{j,l}}z\right)^M}, \quad z \in \mathbb{D},$$

where $\{z_k\}$ is a separated sequence and $a_{j,l}$ are the Rademacher functions [25, Appendix A] and M satisfies the hypothesis (8.2) of Theorem 5.3.1. Then

$$\|F_t^{(n)}\|_{L^s_\mu} \le \|D^{(n)}\|_{A^{p,q}_\omega \to L^s_\mu} \|F_t\|_{A^{p,q}_\omega} \lesssim \|D^{(n)}\|_{A^{p,q}_\omega \to L^s_\mu} \|\lambda\|_{\ell^{p,q}}. \tag{8.46}$$

Moreover, Khinchine's inequality [25, Appendix A] yields

$$\begin{aligned}
&\int_0^1 \|F_t^{(n)}\|^s_{L^s_\mu}\, dt \\
&= \int_{\mathbb{D}} \int_0^1 \left| M(M+1)\cdots(M+n-1) \sum_{j,l} a_{j,l}(t)\lambda_{j,l}\overline{z_{j,l}}^n \frac{(1-|z_{j,l}|)^{M-\frac{1}{p}}\widehat{\omega}(z_{j,l})^{-\frac{1}{q}}}{\left(1-\overline{z_{j,l}}z\right)^{M+n}}\right|^s dt\, d\mu(z) \\
&\gtrsim \int_{\mathbb{D}} \left(\sum_{j,l}\left|\lambda_{j,l}\frac{(1-|z_{j,l}|)^{M-\frac{1}{p}}\widehat{\omega}(z_{j,l})^{-\frac{1}{q}}}{\left(1-\overline{z_{j,l}}z\right)^{M+n}}\right|^2\right)^{\frac{s}{2}} d\mu(z) \\
&\gtrsim \int_{\mathbb{D}} \left(\sum_{j,l}\left|\chi_{Q_{j,l}}(z)\lambda_{j,l}(1-|z_{j,l}|)^{-n-\frac{1}{p}}\widehat{\omega}(z_{j,l})^{-\frac{1}{q}}\right|^2\right)^{\frac{s}{2}} d\mu(z) \\
&= \sum_{j,l} |\lambda_{j,l}|^s \mu(Q_{j,l})(1-|z_{j,l}|)^{-s\left(n+\frac{1}{p}\right)}\widehat{\omega}(z_{j,l})^{-\frac{s}{q}} \\
&\asymp \sum_{j,l} |\lambda_{j,l}|^s \mu(Q_{j,l}) K^{js\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{-\frac{s}{q}}.
\end{aligned} \tag{8.47}$$

By integrating (8.46) with respect to t, using (8.47) and writing $b = \{b_{j,l}\} = \{|\lambda_{j,l}|^s\}$ we obtain

$$\sum_{j,l} b_{j,l}\mu(Q_{j,l})K^{js\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{-\frac{s}{q}} \lesssim \|D^{(n)}\|^s_{A^{p,q}_\omega \to L^s_\mu} \|b\|_{\ell^{\left(\frac{p}{s}\right),\left(\frac{q}{s}\right)}}$$

for all $b \in \ell^{\left(\frac{p}{s}\right),\left(\frac{q}{s}\right)}$ with $b_{j,l} \ge 0$. It follows that

$$\left\{\mu(Q_{j,l})K^{sj(n+\frac{1}{p})}\widehat{\omega}(r_j)^{\frac{-s}{q}}\right\}_{j,l} \in \ell^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}$$

with norm bounded by a constant times $\|D^{(n)}\|^s_{A^{p,q}_\omega\to L^s_\mu}$ by [46, Theorem 1]. Thus (ii) is satisfied.

To see the converse implication, note first that the estimate (8.40) implies $f^{(n)}_{j,l}\lesssim K^{jn}\widehat{f}_{j,l}$. This together with the fact that $\#U_{j,l}$ has a finite uniform bound independent of j, l and K, and Lemma 9 give

$$\left\|\left\{f^{(n)}_{j,l}K^{-j\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{\frac{1}{q}}\right\}_{j,l}\right\|_{\ell^{p,q}}\lesssim\left\|\left\{\widehat{f}_{j,l}K^{-\frac{j}{p}}\widehat{\omega}(r_j)^{\frac{1}{q}}\right\}_{j,l}\right\|_{\ell^{p,q}}\lesssim\left\|\left\{f_{j,l}K^{-\frac{j}{p}}\widehat{\omega}(r_j)^{\frac{1}{q}}\right\}_{j,l}\right\|_{\ell^{p,q}}$$
$$=\|\lambda(f)\|_{\ell^{p,q}}\asymp\|f\|_{A^{p,q}_\omega}.$$

By applying [46, Theorem 1] and the estimate just established, we deduce

$$\begin{aligned}\int_{\mathbb{D}}|f^{(n)}(z)|^s\,d\mu(z)&=\sum_{j,l}\int_{Q_{j,l}}|f^{(n)}(z)|^s\,d\mu(z)\le\sum_{j,l}\left(f^{(n)}_{j,l}\right)^s\mu(Q_{j,l})\\&=\sum_{j,l}\left(f^{(n)}_{j,l}\right)^sK^{-js\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{\frac{s}{q}}\mu(Q_{j,l})K^{js\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{-\frac{s}{q}}\\&\le\left\|\left\{\left(f^{(n)}_{j,l}\right)^sK^{-js\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{\frac{s}{q}}\right\}_{j,l}\right\|_{\ell^{\frac{p}{s},\frac{q}{s}}}\\&\quad\cdot\left\|\left\{\mu(Q_{j,l})K^{js\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{-\frac{s}{q}}\right\}_{j,l}\right\|_{\ell^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}}\\&=\left\|\left\{f^{(n)}_{j,l}K^{-j\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{\frac{1}{q}}\right\}_{j,l}\right\|^s_{\ell^{p,q}}\\&\quad\cdot\left\|\left\{\mu(Q_{j,l})K^{js\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{-\frac{s}{q}}\right\}_{j,l}\right\|_{\ell^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}}\\&\lesssim\|f\|^s_{A^{p,q}_\omega}\left\|\left\{\mu(Q_{j,l})K^{js\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{-\frac{s}{q}}\right\}_{j,l}\right\|_{\ell^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}},\end{aligned}$$

and hence

$$\|D^{(n)}\|^s_{A^{p,q}_\omega\to L^s_\mu}\lesssim\left\|\left\{\mu(Q_{j,l})2^{sj\left(n+\frac{1}{p}\right)}\widehat{\omega}(r_j)^{-\frac{s}{q}}\right\}_{j,l}\right\|_{\ell^{\left(\frac{p}{s}\right)',\left(\frac{q}{s}\right)'}}.$$

It remains to show that (ii) is equivalent to its continuous counterpart (iii). To do this, first define $U^r_{j,l}=\left\{(i,m):\varrho(Q_{j,l},Q_{i,m})<r\right\}$, and note that $\sup_{j,l}\#U^r_{j,l}\le C(r)<\infty$. Further, set $\widehat{Q}^r_{j,l}=\cup_{(i,m)\in U^r_{j,l}}Q_{i,m}$ and write $T_r=T_{r,u,v}$ for short. Assume first $s<\min\{p,q\}$. Then, by choosing $r=r(K)>0$ sufficiently large we have

$$S_r(z)=\sum_j\sum_l\frac{\mu(Q_{j,l})}{K^{-j(sn+1)}\widehat{\omega}(r_j)}\chi_{Q_{j,l}}(z)\lesssim T_r(z)\lesssim\sum_j\sum_l\frac{\mu(\widehat{Q}^r_{j,l})}{K^{-j(sn+1)}\widehat{\omega}(r_j)}\chi_{Q_{j,l}}(z)=B_r(z)$$

for all $z \in \mathbb{D}$. By using $\sup_{j,l} \# U^r_{j,l} < \infty$ we deduce

$$
\begin{aligned}
\|B_r\|^{\frac{q}{q-s}}_{L^{\frac{p}{p-s},\frac{q}{q-s}}_\omega} &= \int_0^1 \left(\int_0^{2\pi} B_r(te^{i\theta})^{\frac{p}{p-s}}\, d\theta \right)^{\frac{q(p-s)}{p(q-s)}} \omega(t)\, dt \\
&\asymp \sum_{k=0}^\infty \int_{r_k}^{r_{k+1}} \left(\sum_{i=0}^{K^{k+3}-1} \mu(\widehat{Q}^r_{k,i})^{\frac{p}{p-s}} K^{k\frac{p}{p-s}(1+sn)} K^{-k} \widehat{\omega}(r_k)^{-\frac{p}{p-s}} \right)^{\frac{q(p-s)}{p(q-s)}} \omega(t)\, dt \\
&\asymp \sum_{k=0}^\infty \left(\sum_{i=0}^{K^{k+3}-1} \left(\mu(\widehat{Q}^r_{k,i}) K^{ks(n+\frac{1}{p})} \widehat{\omega}(r_k)^{-\frac{s}{q}} \right)^{\frac{p}{p-s}} \right)^{\frac{q(p-s)}{p(q-s)}} \\
&\lesssim \sum_{k=0}^\infty \left(\sum_{i=0}^{K^{k+3}-1} \left(\mu(Q_{k,i}) K^{ks(n+\frac{1}{p})} \widehat{\omega}(r_k)^{-\frac{s}{q}} \right)^{\frac{p}{p-s}} \right)^{\frac{q(p-s)}{p(q-s)}},
\end{aligned}
$$

and an essentially identical reasoning gives

$$
\|S_r\|^{\frac{q}{q-s}}_{L^{\frac{p}{p-s},\frac{q}{q-s}}_\omega} \asymp \sum_{k=0}^\infty \left(\sum_{i=0}^{K^{k+3}-1} \left(\mu(Q_{k,i}) K^{ks(n+\frac{1}{p})} \omega(r_k)^{-\frac{s}{q}} \right)^{\frac{p}{p-s}} \right)^{\frac{q(p-s)}{p(q-s)}}.
$$

This finishes the proof of the case $s < \min\{p,q\}$ for $r > 0$ large enough.

In the case $p \le s < q$, we have

$$
S_r(z) = \sum_j \sum_l \frac{\mu(Q_{j,l})}{K^{-js(n+\frac{1}{p})} \widehat{\omega}(r_j)} \chi_{Q_{j,l}}(z) \lesssim T_r(z) \lesssim \sum_j \sum_l \frac{\mu(\widehat{Q}^r_{j,l})}{K^{-js(n+\frac{1}{p})} \widehat{\omega}(r_j)} \chi_{Q_{j,l}}(z) = B_r(z)
$$

for all $z \in \mathbb{D}$. By using again $\sup_{j,l} \# U^r_{j,l} < \infty$ we deduce

$$
\begin{aligned}
\|B_r\|_{L^{\infty,\frac{q}{q-s}}_\omega} &= \left(\int_0^1 (M_\infty(B_r,t))^{\frac{q}{q-s}} \omega(t)\, dt \right)^{\frac{q-s}{q}} \\
&\asymp \left(\sum_{k=0}^\infty \int_{r_k}^{r_{k+1}} \left(K^{ks(n+\frac{1}{p})} \widehat{\omega}(r_k)^{-1} \sup_i \mu(\widehat{Q}^r_{k,i}) \right)^{\frac{q}{q-s}} \omega(t)\, dt \right)^{\frac{q-s}{q}} \\
&\asymp \left(\sum_{k=0}^\infty \left(\sup_i \mu(\widehat{Q}^r_{j,i}) K^{ks(n+\frac{1}{p})} \widehat{\omega}(r_k)^{-\frac{s}{q}} \right)^{\frac{q}{q-s}} \right)^{\frac{q-s}{q}} \\
&\lesssim \left(\sum_{k=0}^\infty \left(\sup_i \mu(Q_{j,i}) K^{ks(n+\frac{1}{p})} \omega(r_k)^{-\frac{s}{q}} \right)^{\frac{q}{q-s}} \right)^{\frac{q-s}{q}},
\end{aligned}
$$

and similarly

$$
\|S_r\|_{L^{\infty,\frac{q}{q-s}}_\omega} \asymp \left(\sum_{k=0}^\infty \left(\sup_i \mu(Q_{j,i}) K^{ks(n+\frac{1}{p})} \omega(r_k)^{-\frac{s}{q}} \right)^{\frac{q}{q-s}} \right)^{\frac{q-s}{q}}.
$$

Hence the case $p \le s < q$ is proved for $r > 0$ large enough.

In the case $q \le s < p$ we have

$$S_r(z) = \sum_j \sum_l \frac{\mu(Q_{j,l})}{K^{-j(sn+1)}\widehat{\omega}(r_j)^{\frac{s}{q}}}\chi_{Q_{j,l}}(z) \lesssim T_r(z) \lesssim \sum_j \sum_l \frac{\mu(\widehat{Q}^r_{j,l})}{K^{-j(sn+1)}\widehat{\omega}(r_j)^{\frac{s}{q}}}\chi_{Q_{j,l}}(z) = B_r(z)$$

for all $z \in \mathbb{D}$. By using again $\sup_{j,l} \# U^r_{j,l} < \infty$ we deduce

$$\begin{aligned}
\|B_r\|_{L_\omega^{\frac{p}{p-s},\infty}} &= \sup_{0<t<1}\left(\int_0^{2\pi} B_r(te^{i\theta})^{\frac{p}{p-s}}\,d\theta\right)^{\frac{p-s}{p}} \\
&\asymp \sup_j \left(\sum_{i=0}^{K^{k+3}-1} \mu(\widehat{Q}^r_{j,i})^{\frac{p}{p-s}} K^{j\frac{p}{p-s}(sn+1)} K^{-j}\widehat{\omega}(r_j)^{-\frac{sp}{q(p-s)}}\right)^{\frac{p-s}{p}} \\
&\asymp \sup_j \left(\sum_{i=0}^{K^{k+3}-1} \left(\mu(\widehat{Q}^r_{j,i})K^{js(n+\frac{1}{p})}\widehat{\omega}(r_j)^{-\frac{s}{q}}\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}} \\
&\lesssim \sup_j \left(\sum_{i=0}^{K^{k+3}-1} \left(\mu(Q_{j,i})K^{js(n+\frac{1}{p})}\omega(r_j)^{-\frac{s}{q}}\right)^{\frac{p}{p-s}}\right)^{\frac{p-s}{p}} \asymp \|S_r\|_{L_\omega^{\frac{p}{p-s},\infty}},
\end{aligned}$$

completing the proof of the case $q \le s < p$ for $r > 0$ large enough.

The remaining case $s \ge \max\{p, q\}$ is the simplest one of all because now

$$S_r(z) = \sum_j \sum_l \frac{\mu(Q_{j,l})}{K^{-js(n+\frac{1}{p})}\widehat{\omega}(r_j)^{\frac{s}{q}}}\chi_{Q_{j,l}}(z) \lesssim T_r(z) \lesssim \sum_j \sum_l \frac{\mu(\widehat{Q}^r_{j,l})}{K^{-js(n+\frac{1}{p})}\widehat{\omega}(r_j)^{\frac{s}{q}}}\chi_{Q_{j,l}}(z) = B_r(z),$$

and hence

$$\begin{aligned}
\left\| \left\{ \frac{\mu(Q_{j,l})}{K^{-js(n+\frac{1}{p})}\widehat{\omega}(r_j)^{\frac{s}{q}}} \right\}_{j,l} \right\|_{\ell^\infty} \lesssim \|T_r\|_{L^\infty} &\lesssim \left\| \left\{ \frac{\mu(\widehat{Q}^r_{j,l})}{K^{-js(n+\frac{1}{p})}\widehat{\omega}(r_j)^{\frac{s}{q}}} \right\}_{j,l} \right\|_{\ell^\infty} \\
&\lesssim \left\| \left\{ \frac{\mu(Q_{j,l})}{K^{-js(n+\frac{1}{p})}\widehat{\omega}(r_j)^{\frac{s}{q}}} \right\}_{j,l} \right\|_{\ell^\infty}.
\end{aligned}$$

This completes the proof of the theorem in the case in which $r > 0$ is sufficiently large, say $r \ge r_0 = r_0(K)$. If $r \in (0, r_0)$, then dividing $Q_{j,l}$ into M^2 rectangles of equal area as in the proof of Theorem 18, and then slightly modifying the proof just presented, the assertion easily follows. The details of this deduction do not offer us anything new and are therefore omitted. □

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