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Results in Mathematics

Conformal Vector Fields and Null Hypersurfaces

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Abstract. We give conditions for a conformal vector field to be tangent to a null hypersurface. We particularize to two important cases: a Killing vector field and a closed and conformal vector field. In the first case, we obtain a result ensuring that a null hypersurface is a Killing horizon. In the second one, the vector field gives rise to a foliation of the manifold by totally umbilical hypersurfaces with constant mean curvature which can be spacelike, timelike or null. We prove several results which ensure that a null hypersurface with constant null mean curvature is a leaf of this foliation.

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1. Introduction

Generalized Robertson–Walker spaces $(I \times F, -dt^2 + f(t)^2 g_0)$, where $I \subset \mathbb{R}$, (F, g_0) is a Riemannian manifold and $f \in C^\infty(I)$ is a positive function, are of great importance since they include the classical cosmological models and they have been widely studied from different points of view. For example, in [1,2,16,17] the authors gave sufficient conditions for a constant mean curvature spacelike hypersurface to be a slice $\{t\} \times F$. Observe that these slices are totally umbilical hypersurfaces and they are the orthogonal leaves of the foliation induced by the timelike, closed and conformal vector field $K = f\partial t$. In fact, a generalized Robertson–Walker space is locally characterized by the existence of a vector field with these properties. This is why in [5] the authors considered directly a Lorentzian manifold furnished with such a vector field, which is locally a generalized Robertson–Walker space but not necessarily a global one, and they obtained sufficient conditions for a constant mean curvature spacelike hypersurface to be an orthogonal leaf.

The causal character of a closed and conformal vector field can change pointwise (see Example 1), so we can not ensure the decomposition as a generalized Robertson–Walker space even locally. This kind of vector field in the semi-Riemannian setting were studied in [14]. Since we let K to be null at some point, then the induced foliation can have leaves which are totally umbilical null hypersurfaces and through them K is tangent and orthogonal to the leaf at the same time.

Null hypersurfaces are more difficult to handle than the spacelike or timelike ones, since they do not inherit a useful structure. Totally umbilical null hypersurfaces in generalized Robertson–Walker spaces were studied and classified in [9], but not much is known if we consider a more general ambient as a Lorentzian manifold furnished with a closed and conformal vector field which changes its causal character.

In this paper, we first consider a conformal vector field and we obtain conditions to ensure that it is tangent to a null hypersurfaces. An interesting particular case is when the vector field is Killing, since in this situation the above result says that the null hypersurface is a Killing horizon.

After that, we give sufficient conditions for a constant mean curvature null hypersurface to be an orthogonal leaf of the foliation induced by a closed and conformal vector field. The obtained results involve more conditions than the usual ones for spacelike hypersurfaces, as is predictable due to the difficulty presented by null hypersurfaces. The main tool to obtain the results is the rigging technique introduced in [13], which allows us to apply systematically the classical maximum principle in a null hypersurface.

2. Null Hypersurfaces

Suppose that L is a hypersurface of a n-dimensional Lorentzian manifold (M, g) and that the inherit metric tensor is degenerate for each point in L, i.e., $Rad(T_pL) = T_pL \cap (T_pL)^{\perp}$ is not zero for all $p \in L$. In this case we say that L is a null hypersurface and it holds that $Rad(T_pL) = (T_pL)^{\perp} \subset T_pL$ and $dim\, Rad(T_pL) = 1$ for all $p \in L$. Moreover, $Rad(T_pL)$ is spanned by a null vector and all other directions in T_pL are spacelike and orthogonal to $Rad(T_pL)$.

The study of this kind of hypersurfaces presents obvious difficulties. A starting point to handle them is to make an arbitrary choice which allows us to induce all the geometric objects that we need.

Definition 1. [13] We say that a vector field ζ defined in a open neighbourhood of a null hypersurface L is a rigging if $\zeta_p \notin T_pL$ for all $p \in L$. If ζ is defined only over L then we call it a restricted rigging.

A rigging may not exist (globally) for an arbitrary null hypersurface, but locally its existence is guaranteed. A rigging gives rise in a natural way to a null vector field $\xi \in \mathfrak{X}(L)$ with $Rad(T_pL) = span(\xi_p)$ and $g(\zeta, \xi) = 1$ called rigged vector field, a spacelike distribution $\mathcal{S}$ on L called the screen distribution and a null vector field transverse to L given by

$$N = \zeta - \frac{1}{2} g(\zeta, \zeta)\xi. \tag{1}$$

Moreover, we have $g(N, \xi) = 1$, $N \perp \mathcal{S}$ and

$$T_pM = T_pL \oplus span(N_p), \tag{2}$$

$$T_pL = \mathcal{S}_p \oplus_{orth} span(\xi_p). \tag{3}$$

So, for each $p \in L$ we can consider $\mathcal{P}_{TL} : T_pM \to T_pL$ and $\mathcal{P}_{\mathcal{S}} : T_pM \to \mathcal{S}_p$ the projections induced by the above decompositions.

The tensors

$$\begin{aligned} B(U, V) &= -g(\nabla_U \xi, V), \\ C(U, X) &= -g(\nabla_U N, X), \\ \tau(U) &= g(\nabla_U \zeta, \xi), \end{aligned} \tag{4}$$

where $U, V \in \mathfrak{X}(L)$ and $X \in \Gamma(\mathcal{S})$, are called second fundamental form, screen second fundamental form and rotation one-form respectively.

According to decompositions (2) and (3) we have

$$\begin{aligned} \nabla_U V &= \nabla^L_U V + B(U, V)N, \\ \nabla_U N &= \tau(U)N - A(U), \\ \nabla_U \xi &= -\tau(U)\xi - A^*(U), \end{aligned} \tag{5}$$

$$\nabla^L_U X = \nabla^*_U X + C(U, X)\xi, \tag{6}$$

where $\nabla^L_U V \in \mathfrak{X}(L)$ and $A(U), A^*(U), \nabla^*_U X \in \Gamma(\mathcal{S})$. B is a symmetric tensor which holds

$$B(U, V) = g(A^*(U), V), \tag{7}$$

$$B(U, \xi) = 0 \tag{8}$$

(therefore ξ is a pre-geodesic vector field) and C holds

$$C(U, X) = g(A(U), X), \tag{9}$$

$$\omega([X, Y]) = C(X, Y) - C(Y, X) \tag{10}$$

for all $X, Y \in \Gamma(\mathcal{S})$, where ω is the rigged one-form given by $\omega(U) = g(\zeta, U)$ for all $U \in \mathfrak{X}(L)$. The null mean curvature of the null hypersurface is the trace of A^*, namely

$$H_p = \sum_{i=1}^{n} B(e_i, e_i),$$

where $\{e_1, \ldots, e_{n-2}\}$ is an orthonormal basis in $\mathcal{S}_p$.

Recall that although the tensor B depends on the chosen rigging (more concretely, on the rigged vector field), some conditions about the null hypersurface as being totally umbilical ($B = \frac{H}{n-2}g$), being totally geodesic ($B = 0$) or having zero null mean curvature are independent on any choice. From equation (8) it follows that B is always degenerate, but it also implies that having a null second fundamental form with the property $B(v, v) \neq 0$ for all $v \in \mathcal{S}$ with $v \neq 0$, is also independent on the chosen rigging. We call this a screen non-degenerate second fundamental form.

Some equations linking the above tensors are

$$-2C(U, X) = d\omega(U, X) + (L_\zeta g)(U, X) + g(\zeta, \zeta)B(U, X) \tag{11}$$

and the Gauss–Codazzi equation

$$\begin{aligned} g(R_{UV}W, \xi) = g(\left(\nabla^L_U A^*\right)(V), W) - g(\left(\nabla^L_V A^*\right)(U), W) \\ + \tau(U)g(A^*(V), W) - \tau(V)g(A^*(U), W). \end{aligned} \tag{12}$$

Recall also the important Raychaudhuri equation :

$$Ric(\xi, \xi) = \xi(H) + \tau(\xi)H - trace(A^{*2}). \tag{13}$$

From this we can deduce that if ξ is geodesic, H is constant and $Ric(\xi, \xi) \geq 0$ then L is totally geodesic. Another basic curvature relation is

$$Ric(v, \xi) = \sum_{i=1}^{n-2} g(R_{e_i v}\xi, e_i) + g(R_{\xi v}\xi, N), \tag{14}$$

where $\{e_1, \ldots, e_{n-2}\}$ is an orthonormal basis of $\mathcal{S}_p$ and $v \in T_pM$.

The rigged metric is a Riemannian metric on the null hypersurface L defined by

$$\widetilde{g} = g + \omega \otimes \omega.$$

This metric declares ξ $\widetilde{g}$-unitary and $\widetilde{g}$-orthogonal to $\mathcal{S}$, therefore $\widetilde{\nabla}_\xi \xi \in \Gamma(\mathcal{S})$. It can be used as an auxiliary tool and its usefulness have been shown in several papers, [3,10,11,18]. An important relation between the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ and the Levi-Civita connection ∇ of g is

$$\begin{aligned} g(\widetilde{\nabla}_U V, W) &= g(\nabla_U V, W) \\ &\quad + \frac{1}{2}\big(\omega(W)(L_\xi \widetilde{g})(U, V) + \omega(U)d\omega(V, W) + \omega(V)d\omega(U, W)\big) \end{aligned} \tag{15}$$

for all $U, V, W \in \mathfrak{X}(L)$. In particular, for all $X, Y, Z \in \Gamma(\mathcal{S})$ it holds

$$\widetilde{g}(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z), \tag{16}$$

$$(L_\xi \widetilde{g})(X, Y) = -2B(X, Y). \tag{17}$$

In the case of being ω closed, $d\omega = 0$, we have that the screen distribution is integrable and we can give an easier relation between ∇ and $\widetilde{\nabla}$ given by

$$\widetilde{g}(\widetilde{\nabla}_U V, W) = g(\nabla_U V, W) + \omega(W)U(\omega(V))$$

for all $U, V, W \in \mathfrak{X}(L)$, [13, Proposition 3.15].

We say that a rigging is distinguished if the induced rotation one-form τ vanishes. On the other hand, we say that it is screen conformal if there is a function $\varphi \in C^\infty(L)$ such that $C = \varphi B$. In this case, C is symmetric and thus from Eq. (10) we have that $\mathcal{S}$ is integrable. Moreover, we have the following.

Lemma 1. [10,18] *Let L be a null hypersurface and ζ a rigging for it.*

1. *$C(\xi, X) + \tau(X) = -g(\widetilde{\nabla}_\xi \xi, X)$ for all $X \in \Gamma(\mathcal{S})$.*
2. *If ζ is distinguished and screen conformal then the rigged one-form is closed, $d\omega = 0$.*
3. *If $d\omega = 0$, then $C(\xi, X) = -\tau(X)$ for all $X \in \Gamma(\mathcal{S})$.*

Proof. Observe that ω is the $\widetilde{g}$-metrically equivalent one-form to ξ, so it holds

$$d\omega(U, V) = \widetilde{g}(\widetilde{\nabla}_U \xi, V) - \widetilde{g}(U, \widetilde{\nabla}_V \xi) \tag{18}$$

for all $U, V \in \mathfrak{X}(L)$. In particular, $d\omega = 0$ if and only if $\mathcal{S}$ is integrable and ξ is $\widetilde{g}$-geodesic. Now, from Eqs. (11) and (18) we have

$$\begin{aligned} -2C(\xi, X) &= d\omega(\xi, X) + (L_\zeta g)(\xi, X) = \widetilde{g}(\widetilde{\nabla}_\xi \xi, X) + g(\nabla_\xi \zeta, X) + g(\xi, \nabla_X \zeta) \\ &= \widetilde{g}(\widetilde{\nabla}_\xi \xi, X) - C(\xi, X) + \tau(X) \end{aligned}$$

and we get the first point. The second and third points follow immediately from the first one. □

In the following lemma we relate the laplacian of a function defined in M and the laplacian with respect to $\widetilde{g}$ of its restriction to L, obtaining an analogous formula to [7, Formula 4]. In the case of a closed rigging, we also relate the laplacian with respect to $\widetilde{g}$ of a function defined in L and the laplacian of its restriction to a leaf of the screen computed with respect to the induced metric from the ambient.

Lemma 2. *Let (M, g) be a Lorentzian manifold and $f \in C^\infty(M)$. Suppose that L is a null hypersurface with a rigging ζ and $i : L \to M$ is the canonical inclusion.*

1. *If we call $\widetilde{f} = f \circ i$ and take $X, Y \in \Gamma(\mathcal{S})$, then*

$$\begin{aligned} \widetilde{Hess}_{\widetilde{f}}(X, Y) &= Hess_f(X, Y) + g(\nabla f, N)B(X, Y) + g(\nabla f, \xi)C(X, Y) \\ &\quad + g(\nabla f, \xi)\widetilde{g}(\widetilde{\nabla}_X \xi, Y), \\ \widetilde{Hess}_{\widetilde{f}}(X, \xi) &= Hess_f(X, \xi) - g(\nabla f, \xi)\tau(X) + \frac{1}{2} d\omega(\mathcal{P}_{\mathcal{S}}(\nabla f), X), \\ \widetilde{Hess}_{\widetilde{f}}(\xi, \xi) &= Hess_f(\xi, \xi) - g(\nabla f, \xi)\tau(\xi) - g(\nabla f, \widetilde{\nabla}_\xi \xi). \end{aligned}$$

2. *If we call* $\triangle_{\mathcal{S}} f = \sum_{i=1}^{n-2} g(\nabla_{e_i}\nabla f, e_i)$*, where* $\{e_1,\ldots,e_{n-2}\}$ *is an orthonormal basis of* $\mathcal{S}_p$*, then*

$$\begin{aligned}&\triangle f = \triangle_{\mathcal{S}} f + 2Hess_f(\xi,N),\\ &\widetilde{\triangle}\widetilde{f} = \triangle_{\mathcal{S}} f + g(\xi,\nabla f)\ trace_{\mathcal{S}}(A) + g(N-\xi,\nabla f)H - g(\nabla f,\widetilde{\nabla}_\xi \xi)\\ &\quad -\tau(\xi)g(\xi,\nabla f) + Hess_f(\xi,\xi).\end{aligned}$$

3. *If* $d\omega = 0$*,* $\phi \in C^\infty(L)$*,* S *is a leaf of the screen distribution* $\mathcal{S}$ *and* $j : S \to L$ *is the canonical inclusion, then*

$$\widetilde{\triangle}\phi = \triangle^S(\phi\circ j) - \xi(\phi)H + \xi(\xi(\phi)),$$

where $\triangle^S$ *is the laplacian computed in the induced metric on the leaf* S*.*

Proof. If we decompose $\nabla f = \mathcal{P}_{\mathcal{S}}(\nabla f) + g(\nabla f,N)\xi + g(\nabla f,\xi)N$ according to decompositions (2) and (3), then it is straightforward to check that

$$\widetilde{\nabla}\widetilde{f} = \mathcal{P}_{\mathcal{S}}(\nabla f) + g(\nabla f,\xi)\xi.$$

Thus, using Eq. (16), we have

$$\begin{aligned}\widetilde{Hess}_{\widetilde{f}}(X,Y) &= g(\nabla_X \mathcal{P}_{\mathcal{S}}(\nabla f),Y) + g(\nabla f,\xi)\widetilde{g}(\widetilde{\nabla}_X\xi,Y)\\ &= Hess_f(X,Y) + g(\nabla f,N)B(X,Y) + g(\nabla f,\xi)C(X,Y)\\ &\quad + g(\nabla f,\xi)\widetilde{g}(\widetilde{\nabla}_X\xi,Y).\end{aligned}$$

On the other hand,

$$\widetilde{Hess}_{\widetilde{f}}(\xi,X) = g(\widetilde{\nabla}_\xi\mathcal{P}_{\mathcal{S}}(\nabla f),X) + g(\nabla f,\xi)g(\widetilde{\nabla}_\xi\xi,X),$$

so using Eq. (15) and Lemma 1 we have

$$\begin{aligned}\widetilde{Hess}_{\widetilde{f}}(\xi,X) &= g(\nabla_\xi\mathcal{P}_{\mathcal{S}}(\nabla f),X) + \frac{1}{2}d\omega(\mathcal{P}_{\mathcal{S}}(\nabla f),X)\\ &\quad - g(\nabla f,\xi)(C(\xi,X)+\tau(X))\\ &= Hess_f(\xi,X) + \frac{1}{2}d\omega(\mathcal{P}_{\mathcal{S}}(\nabla f),X) - g(\nabla f,\xi)\tau(X).\end{aligned}$$

For the third formula of item (1) just note that

$$\begin{aligned}\widetilde{Hess}_{\widetilde{f}}(\xi,\xi) &= \widetilde{g}(\widetilde{\nabla}_\xi\mathcal{P}_{\mathcal{S}}(\nabla f),\xi) + \xi g(\nabla f,\xi)\\ &= -g(\nabla f,\widetilde{\nabla}_\xi\xi) + Hess_f(\xi,\xi) - \tau(\xi)g(\nabla f,\xi).\end{aligned}$$

Now, fix $\{e_1,\ldots,e_{n-2}\}$ an orthonormal basis of $\mathcal{S}_x$. First formula of item (2) easily follows from the fact that $\{e_1,\ldots,e_{n-2},\frac{1}{\sqrt{2}}(N+\xi),\frac{1}{\sqrt{2}}(N-\xi)\}$ is an orthonormal basis of T_xM. The formula for $\widetilde{\triangle}\widetilde{f}$ follows from the Eq. (17) and the formulas for $\widetilde{Hess}_{\widetilde{f}}$.

Finally, we have $\widetilde{\nabla}\phi = \nabla^S(\phi \circ j) + \xi(\phi)\xi$, where ∇^S is the gradient in the induced metric on S. Now, using again Eqs. (16) and (17) we have

$$\begin{aligned}\widetilde{\triangle}\phi &= \sum_{i=1}^{n-2} \widetilde{g}(\widetilde{\nabla}_{e_i}\widetilde{\nabla}\phi, e_i) + \widetilde{g}(\widetilde{\nabla}_{\xi}\widetilde{\nabla}\phi, \xi)\\ &= \triangle^S(\phi\circ j) - \xi(\phi)H + \xi(\xi(\phi)) - g(\widetilde{\nabla}\phi, \widetilde{\nabla}_{\xi}\xi).\end{aligned}$$

Since $d\omega = 0$, from Lemma 1 we get $\widetilde{\nabla}_{\xi}\xi = 0$ and we obtain the desired formula. □

The fundamental tensors of a null hypersurface (B, C and τ) depend on the chosen rigging. However, if we change the rigging, then we can express the new tensors in terms of the old ones. For our purpose, we only consider a very special rigging change.

Lemma 3. [18] *If ζ is a rigging for a null hypersurface L and $\Phi \in C^{\infty}(L)$ is a never vanishing function, then $\zeta' = \Phi\zeta$ is also a rigging for L and*

$$\begin{aligned}\xi' &= \frac{1}{\Phi}\xi, & N' &= \Phi\zeta,\\ A^{*\prime} &= \frac{1}{\Phi}A^*, & A' &= \Phi A,\\ H' &= \frac{1}{\Phi}H, & \tau' &= \tau + \frac{d\Phi}{\Phi}.\end{aligned}$$

3. Conformal Vector Fields

A vector field $K \in \mathfrak{X}(M)$ is conformal if $L_K g = 2\rho g$ for certain $\rho \in C^{\infty}(M)$. If $\rho = 0$, then it is called a Killing vector field.

If we call η the metrically equivalent one-form to K, then $(L_K g)(U,V) + d\eta(U,V) = 2g(\nabla_U K, V)$, so K is conformal if and only if

$$\nabla_U K = \rho U + \varphi(U) \tag{19}$$

for all $U \in \mathfrak{X}(M)$, where φ is characterized by $d\eta(U,V) = 2g(\varphi(U),V)$. If $\varphi = 0$, then η is closed and K is called closed and conformal. If $\rho = 0$ and $\varphi = 0$, then K is called a parallel vector field.

It is immediate that φ is skew-symmetric and so $\nabla_U\varphi$ is also. Moreover,

$$d\eta(\varphi(U), V) = -d\eta(U, \varphi(V))$$

and since $d\eta$ is closed, it also holds

$$g\big((\nabla_U\varphi)(V), W\big) + g\big((\nabla_V\varphi)(W), U\big) + g\big((\nabla_W\varphi)(U), V\big) = 0$$

for all $U, V, W \in \mathfrak{X}(M)$.

In the following lemma we give some basic facts about conformal vector fields. From now on, we call $\lambda = g(K,K)$.

Lemma 4. *Suppose that $K \in \mathfrak{X}(M)$ is a conformal vector field.*

1. $\nabla\lambda = 2\rho K - 2\varphi(K)$.
2. $\triangle\lambda = 2K(\rho) + 2n\rho^2 + 2g(\mathrm{div}\varphi, \mathrm{K}) - 2\,\mathrm{trace}(\varphi^2)$.
3. *If* $X \in \mathfrak{X}(M)$ *with* $X \perp K$, *then*

$$\lambda X(\rho) = -g((\nabla_K\varphi)(K), X) = -\frac{1}{2}\Big(\nabla_K d\eta\Big)(K, X).$$

4. *Given* $U, V \in \mathfrak{X}(M)$, *it holds*

$$\begin{aligned} R_{UV}K &= U(\rho)V - V(\rho)U + (\nabla_U\varphi)(V) - (\nabla_V\varphi)(U), \\ Ric(U, K) &= -(n-1)U(\rho) - g(\mathrm{div}\varphi, \mathrm{U}). \end{aligned}$$

Proof. We get the first point taking derivative in $\lambda = g(K, K)$ and using Eq. (19). For the second one, taking divergence

$$\begin{aligned} \triangle\lambda &= 2K(\rho) + 2n\rho^2 - 2\mathrm{div}\varphi(\mathrm{K}) \\ &= 2K(\rho) + 2n\rho^2 + 2g(\mathrm{div}\varphi, \mathrm{K}) - 2\,\mathrm{trace}(\varphi^2). \end{aligned}$$

For the third point

$$K(\lambda) = g(\nabla\lambda, K) = 2\rho\lambda \tag{20}$$

and if $X \perp K$, then

$$X(\lambda) = g(X, \nabla\lambda) = -2g(X, \varphi(K)).$$

Now,

$$\begin{aligned} X(K(\lambda)) &= Xg(\nabla\lambda, K) = g(\nabla_X\nabla\lambda, K) + g(\nabla\lambda, \nabla_X K) \\ &= g(\nabla_K\nabla\lambda, X) + \rho g(\nabla\lambda, X) + g(\nabla\lambda, \varphi(X)) \\ &= 2\rho g(\varphi(K), X) - 2g(\nabla_K\varphi(K), X) - 2\rho g(\varphi(K), X) \\ &\quad + 2\rho g(K, \varphi(X)) - 2g(\varphi(K), \varphi(X)) \\ &= -2g((\nabla_K\varphi)(K), X) - 2g(\varphi(\nabla_K K), X) + 2\rho g(K, \varphi(X)) \\ &\quad - 2g(\varphi(K), \varphi(X)) \\ &= -2g((\nabla_K\varphi)(K), X) + 4\rho g(K, \varphi(X)) \end{aligned}$$

If we take derivative in Eq. (20), then

$$2X(\rho)\lambda + 2\rho X(\lambda) = X(K(\lambda)) = -2g((\nabla_K\varphi)(K), X) + 4\rho g(K, \varphi(X))$$

and thus $\lambda X(\rho) = -g((\nabla_K\varphi)(K), X)$.

The last point is straightforward. □

Under some suitable conditions, a conformal vector field is parallel, as the following lemma shows.

Lemma 5. *Let* $K \in \mathfrak{X}(M)$ *be a conformal vector field.*

1. *If* K *is never null and it has constant length, then it is Killing.*
2. *If* K *is closed and it has constant length, then it is parallel.*
3. *If* K *is closed and causal,* $Ric(K, K) \leq 0$ *and there is a point* p *with* K_p *null, then* K *is a null parallel vector field.*

Proof. The first point follows from Eq. (20). For the second point, observe that

$$0 = Ug(K,K) = 2\rho g(U,K)$$

for all $U \in \mathfrak{X}(M)$, which implies that $\rho = 0$.

In the third case, the function $\lambda = g(K,K)$ has a maximum at p. Using Lemma 4 we have

$$\triangle\lambda = -\frac{2}{n-1} Ric(K,K) + 2n\rho^2 \geq 0$$

so by the maximum principle we have that $\lambda = 0$. Therefore, K is parallel because it has constant length. □

If we drop the condition $Ric(K,K) \leq 0$ in the third point of the above lemma, then we can not conclude that K is parallel. In fact, the causality of a closed and conformal vector field can be arbitrary, as the following example shows.

Example 1. Take $I \subset \mathbb{R}$, $f \in C^\infty(I)$, (F, g_F) a Riemannian or Lorentzian manifold and $\varepsilon = \pm 1$. The vector field $K = f\partial t$ in the warped product

$$\left(I \times \ F, \varepsilon dt^2 + f(t)^2 g_F\right)$$

is closed and conformal. If $\varepsilon = 1$, then K is spacelike at every point and if $\varepsilon = -1$, then K is timelike at every point.

On the other hand, the position vector field $K = \sum_{i=1}^n x_i \partial x_i$ in the Minkowski space is closed, conformal and its causal character changes pointwise.

From [6], we can also construct an example of a causal, closed and conformal vector field which is null at some point. Take $E(v)$ an arbitrary function and consider the Lorentzian surface $(M,g) = (\mathbb{R}^2, E(v)du^2 + 2dudv)$. The vector field $K = \partial u - E(v)\partial v$ holds $\nabla_U K = -\frac{E_v}{2} U$ for all $U \in \mathfrak{X}(M)$ and therefore it is closed and conformal. Since $g(K,K) = -E(v)$, for a suitable choice of $E(v)$ we get the desired example.

Suppose now that L is a null hypersurface with a rigging ζ and write

$$K = K_0 + \nu\xi + \mu N,$$

where $K_0 = \mathcal{P}_{\mathcal{S}}(K) \in \Gamma(\mathcal{S})$, $\nu = g(K,N)$ and $\mu = g(K,\xi)$ according to decompositions (2) and (3). We need to compute the laplacian of μ and ν with respect to $\widetilde{g}$, but since they are functions defined only on L we can not use Lemma 2. We begin computing the gradient with respect to $\widetilde{g}$.

Lemma 6. *Let $K \in \mathfrak{X}(M)$ be a conformal vector field and L a null hypersurface with rigging ζ. Then for all $v \in T_pL$*

$$\begin{aligned} \widetilde{g}(\widetilde{\nabla}\nu, v) &= \rho\omega(v) + \nu\tau(v) - C(v, K_0) - g(\varphi(N), v), \\ \widetilde{g}(\widetilde{\nabla}\mu, v) &= -\mu\tau(v) - B(K_0, v) - g(\varphi(\xi), v), \\ \nabla^*_v K_0 &= \rho\mathcal{P}_{\mathcal{S}}(v) + \mathcal{P}_{\mathcal{S}}(\varphi(v)) + \nu A^*(v) + \mu A(v). \end{aligned} \tag{21}$$

Proof. Using Eqs. (5), (6) and (19) we have

$$\rho v + \varphi(v) = \nabla_v(K_0 + \nu\xi + \mu N) = \nabla^*_v K_0 - \nu A^*(v) - \mu A(v) \\ + (C(v, K_0) + v(\nu) - \nu\tau(v))\,\xi + (B(v, K_0) + v(\mu) + \mu\tau(v))\,N$$

If we multiply by N and ξ and take into account (1), (7) and (9) we get the first and second equation. Using the projection $\mathcal{P}_{\mathcal{S}}$ we obtain the third one. □

Observe that being φ skew-symmetric, we have that $\varphi(\xi) \in \mathfrak{X}(L)$. Next, we compute the divergence with respect to $\widetilde{g}$ of $\mathcal{P}_{\mathcal{S}}(\varphi(\xi))$ and $A^*(\mathcal{P}_{\mathcal{S}}(K))$

Proposition 1. *If $K \in \mathfrak{X}(M)$ is a conformal vector field and L a null hypersurface with rigging ζ, then*

$$\widetilde{\mathrm{div}}\mathcal{P}_{\mathcal{S}}(\varphi(\xi)) = -g(\mathrm{div}\varphi, \xi) - \xi \mathrm{g}(\varphi(\xi), \mathrm{N}) + \mathrm{g}(\varphi(\xi), \mathrm{N})\mathrm{H},$$

$$\widetilde{\mathrm{div}} A^*(\mathcal{P}_{\mathcal{S}}(K)) = (n-2)\xi(\rho) + \mathcal{P}_{TL}(K)(H) + \mu Ric(\xi, N) - \mu K(span(\xi, N)) \\ + \rho H + \mu\, trace(A^* \circ A) + C\big(\xi, A^*(\mathcal{P}_{\mathcal{S}}(K)) + \mathcal{P}_{\mathcal{S}}(\varphi(\xi))\big) \\ + \tau(A^*(\mathcal{P}_{\mathcal{S}}(K))) + g(\mathrm{div}\varphi, \xi) + \xi \mathrm{g}(\varphi(\xi), \mathrm{N}).$$

Proof. If $\{e_1, \ldots, e_{n-2}\}$ is an orthonormal basis of $\mathcal{S}$ at a point p, then using Eq. (16) we have

$$\begin{aligned} \widetilde{\mathrm{div}}\mathcal{P}_{\mathcal{S}}(\varphi(\xi)) &= \sum_{i=1}^{n-2} \widetilde{g}(\widetilde{\nabla}_{e_i}\mathcal{P}_{\mathcal{S}}(\varphi(\xi)), e_i) + \widetilde{g}(\widetilde{\nabla}_{\xi}\mathcal{P}_{\mathcal{S}}(\varphi(\xi)), \xi) \\ &= \sum_{i=1}^{n-2} g(\nabla_{e_i}\mathcal{P}_{\mathcal{S}}(\varphi(\xi)), e_i) - g(\varphi(\xi), \widetilde{\nabla}_{\xi}\xi) \\ &= \sum_{i=1}^{n-2} g(\nabla_{e_i}\varphi(\xi), e_i) + g(\varphi(\xi), N)H - g(\varphi(\xi), \widetilde{\nabla}_{\xi}\xi) \\ &= \sum_{i=1}^{n-2} \Big(g((\nabla_{e_i}\varphi)(\xi), e_i) - g(\varphi(A^*(e_i)), e_i) - \tau(e_i) g(\varphi(\xi), e_i) \Big) \\ &\quad + g(\varphi(\xi), N)H - g(\varphi(\xi), \widetilde{\nabla}_{\xi}\xi). \end{aligned}$$

We can suppose that e_i are eigenvectors of A^*, so $\sum_{i=1}^{n-2} g(\varphi(A^*(e_i)), e_i) = 0$ because φ is skew-symmetric. Moreover, since $\sum_{i=1}^{n-2} \tau(e_i) g(\varphi(\xi), e_i) = \tau(\mathcal{P}_{\mathcal{S}}(\varphi(\xi)))$, from Lemma 1 we get

$$\widetilde{\mathrm{div}}\mathcal{P}_{\mathcal{S}}(\varphi(\xi)) = \sum_{i=1}^{n-2} g((\nabla_{e_i}\varphi)(\xi), e_i) + g(\varphi(\xi), N)H + C(\xi, \mathcal{P}_{\mathcal{S}}(\varphi(\xi))). \quad (22)$$

On the other hand, since $\{e_1, \ldots, e_{n-2}, \frac{\xi+N}{\sqrt{2}}, \frac{\xi-N}{\sqrt{2}}\}$ is an orthonormal basis, then

$$g(\mathrm{div}\varphi, \xi) = \sum_{i=1}^{n-2} g((\nabla_{e_i}\varphi)(e_i), \xi) - g((\nabla_\xi \varphi)(\xi), N),$$

but

$$\begin{aligned} g((\nabla_\xi\varphi)(\xi), N) &= g(\nabla_\xi \varphi(\xi), N) - g(\varphi(\nabla_\xi \xi), N) \\ &= \xi g(\varphi(\xi), N) - g(\varphi(\xi), \nabla_\xi N) + \tau(\xi) g(\varphi(\xi), N) \\ &= \xi g(\varphi(\xi), N) + C(\xi, \mathcal{P}_{\mathcal{S}}(\varphi(\xi))) \end{aligned}$$

and therefore

$$\sum_{i=1}^{n-2} g((\nabla_{e_i}\varphi)(\xi), e_i) = -g(\mathrm{div}\varphi, \xi) - \xi \mathrm{g}(\varphi(\xi), \mathrm{N}) - \mathrm{C}(\xi, \mathcal{P}_{\mathcal{S}}(\varphi(\xi))). \tag{23}$$

If we replace (23) in the expression (22), then we obtain the first formula.

For the second one, using again Eq. (16), the Gauss–Codazzi Eq. (12) and the formula (21) of Lemma 6 we have

$$\begin{aligned} \widetilde{\mathrm{div}} A^*(K_0) &= \sum_{i=1}^{n-2} \widetilde{g}(\widetilde{\nabla}_{e_i} A^*(K_0), e_i) + \widetilde{g}(\widetilde{\nabla}_\xi A^*(K_0), \xi) \\ &= \sum_{i=1}^{n-2} g(\nabla^L_{e_i} A^*(K_0), e_i) - g(A^*(K_0), \widetilde{\nabla}_\xi \xi) \\ &= \sum_{i=1}^{n-2} g(\left(\nabla^L_{e_i} A^*\right)(K_0), e_i) + g(A^*(\nabla^L_{e_i} K_0), e_i) - g(A^*(K_0), \widetilde{\nabla}_\xi \xi) \\ &= \sum_{i=1}^{n-2} g(R_{\xi e_i} K_0, e_i) + \sum_{i=1}^{n-2} g(\left(\nabla^L_{K_0} A^*\right)(e_i), e_i) \\ &\quad + \rho H + \nu\, trace(A^{*2}) + \mu\, trace(A^* \circ A) - g(A^*(K_0), \widetilde{\nabla}_\xi \xi). \end{aligned}$$

Now, we compute the term $\sum_{i=1}^{n-2} g(\left(\nabla^L_{K_0} A^*\right)(e_i), e_i)$. For this, we can suppose that $\{e_1, \ldots, e_{n-2}\}$ is a basis of eigenvectors of A^*. Extend them to an orthonormal basis $\{E_1, \ldots, E_{n-2}\}$ locally defined in a neighbourhood of p such that $E_i \in \Gamma(\mathcal{S})$ and $E_i(p) = e_i$. Then

$$\begin{aligned} \sum_{i=1}^{n-2} g(\left(\nabla^L_{K_0} A^*\right)(e_i), e_i) &= \sum_{i=1}^{n-2} g(\nabla^L_{K_0} A^*(E_i), E_i) - g(A^*(\nabla^L_{K_0} E_i), e_i) \\ &= K_0(H) - 2\sum_i g(A^*(e_i), \nabla^L_{K_0} E_i) \\ &= K_0(H) - 2\sum \lambda_i g(e_i, \nabla^L_{K_0} E_i) = K_0(H). \end{aligned}$$

On the other hand, using Eq. (14) and Lemma 4,

$$\begin{aligned}\sum_{i=1}^{n-2} g(R_{\xi e_i}K_0, e_i) &= \sum_{i=1}^{n-2} g(R_{\xi e_i}K, e_i) - \nu g(R_{\xi e_i}\xi, e_i) - \mu g(R_{\xi e_i}N, e_i)\\ &= (n-2)\xi(\rho) - \sum_{i=1}^{n-2} g((\nabla_{e_i}\varphi)(\xi), e_i) + \nu Ric(\xi,\xi)\\ &\quad + \mu Ric(\xi, N) - \mu K(span(\xi, N)).\end{aligned}$$

Taking into account equations (13) and (23) we get the second formula. □

Now, we can give a result ensuring that a conformal vector field is tangent to a null hypersurface.

Theorem 1. *Let $K \in \mathfrak{X}(M)$ be a conformal vector field with constant conformal factor ρ. Suppose that L is a null hypersurface with zero null mean curvature and ζ is a rigging for L such that*

1. *$d\tau = 0$.*
2. *$C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.*
3. *$0 \leq trace(A^* \circ A)$.*
4. *$K(span(\xi, N)) \leq Ric(N, \xi)$.*

If $g(K, \xi)$ is signed and there is a point $p \in L$ with $K_p \in T_pL$, then $K_x \in T_xL$ for all $x \in L$.

Proof. We can suppose that there is a positive function f defined in a neighbourhood $\theta \subset L$ of p such that $\tau = d\ln f$. From Lemma 3 we have that for the restricted rigging $\zeta' = \frac{1}{f}\zeta$ the associated rotation one-form vanishes and all the hypotheses in the theorem remain true. Moreover, we can suppose that $\mu = g(K, \xi)$ is non-positive changing the sign of the rigging if necessary. Now, Lemma 6 gives us that

$$\widetilde{\nabla}\mu = -A^*(K_0) - \mathcal{P}_{\mathcal{S}}(\varphi(\xi))$$

and applying Proposition 1 we have

$$\widetilde{\triangle}\mu = \mu\left(K(span(\xi, N) - Ric(\xi, N) - trace(A^* \circ A)\right) \geq 0.$$

Since μ has a local maximum at p, then μ vanishes in θ. By connectedness, μ vanishes on L and so K is tangent to L. □

A null hypersurface L is called a Killing horizon if there is a Killing vector field $K \in \mathfrak{X}(M)$ such that $K_x = \nu(x)\xi_x$ for all $x \in L$, where $\nu \in C^\infty(L)$ is a never vanishing function. In this case, L is necessarily totally geodesic, since $B(U, V) = -g(\nabla_U\xi, V)$ would be symmetric and skew-symmetric. The following corollaries gives us conditions for a null hypersurface to be a Killing horizon.

Corollary 1. *Let $K \in \mathfrak{X}(M)$ be a Killing vector field. Suppose that L is a null hypersurface with zero null mean curvature and ζ is a rigging for L such that*

1. $d\tau = 0$.
2. $C(\xi, X) = 0$ *for all* $X \in \Gamma(\mathcal{S})$.
3. $0 \leq trace(A^* \circ A)$.
4. $K(span(\xi, N)) \leq Ric(N, \xi)$.

If K_x *is causal for all* $x \in L$ *and there is a point* $p \in L$ *with* $K_p \in T_pL$ *(and therefore* K_p *is null), then* L *is totally geodesic and* $K_x = \nu(x)\xi_x$ *for all* $x \in L$ *and certain* $\nu \in C^\infty(L)$.

Example 2. We give an example where the hypotheses of the above corollary are fulfilled. Let ϖ be a positive constant and $Q = \{(u, v) \in \mathbb{R}^2 : -\frac{2\varpi}{e} < uv\}$. Take the functions $F(r) = \frac{8\varpi^2}{r} e^{1-\frac{r}{2\varpi}}$, $f(r) = (r - 2\varpi)e^{\frac{r}{2\varpi}-1}$ for $0 \leq r$ and $r(u, v) = f^{-1}(uv)$ for $(u, v) \in Q$. The Kruskal space is the product $Q \times \mathbb{S}^2$ endowed with the metric

$$2F(r)dudv + r^2 g_0,$$

where g_0 is the standard metric in $\mathbb{S}^2$, [20]. The totally geodesic null hypersurface

$$L = \{(0, v, x) \in M : v > 0, x \in \mathbb{S}^2\}$$

is a Killing horizon for the Killing vector field $K = v\partial v - u\partial u$. If we take the rigging $\zeta = \partial u$, then the rigged vector field is $\xi = \frac{1}{F}\partial v$ and the null transverse vector field is $N = \zeta$. Through L it holds $r = 2\varpi$, so a direct computation shows that $\tau = 0$, $C = -\frac{v}{2\varpi} g$ and in particular $C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$. Clearly, it also holds $K(span(\xi, N)) \leq Ric(N, \xi)$ since both vanish.

Corollary 2. *Let* $K \in \mathfrak{X}(M)$ *be a conformal vector field with constant conformal factor* ρ. *Suppose that* L *is a totally geodesic null hypersurface and* ζ *is a rigging for* L *such that*

1. $d\tau = 0$.
2. $C(\xi, X) = 0$ *for all* $X \in \Gamma(\mathcal{S})$.
3. $K(span(\xi, N)) \leq Ric(N, \xi)$.

If K_x *is causal for all* $x \in L$ *and there is a point* $p \in L$ *with* $K_p \in T_pL$ *(and therefore* K_p *is null), then* K *is a Killing vector field and* $K_x = \nu(x)\xi_x$ *for all* $x \in L$ *and certain* $\nu \in C^\infty(L)$.

Proof. Applying Theorem 1 we have $K_x = \nu(x)\xi_x$ for all $x \in L$, but since L is totally geodesic, then necessarily $\rho = 0$. □

Remark 1. Suppose that L is a Killing horizon for a Killing vector field $K \in \mathfrak{X}(M)$. If we fix a rigging, then $K_x = \nu(x)\xi_x$ for all $x \in L$, so through L we have $\nabla_\xi K = fK$ where $f = \xi(\nu) - \nu\tau(\xi)$. If $f(x) \neq 0$ for some $x \in L$, then the causal character of K changes from spacelike to timelike in a neighborhood of x. Indeed, for a transverse vector $v \in T_pM$ we have

$$v(\lambda) = -2g(\nabla_K K, v) = -2\nu(x)^2 f(x) g(\xi_x, v) \neq 0.$$

The existence of a timelike gradient vector field is incompatible with the existence of compact null hypersurfaces, [13]. We can also give an obstruction in the case of a conformal timelike vector field.

Theorem 2. *Let $K \in \mathfrak{X}(M)$ be a timelike conformal vector field with constant conformal factor ρ. Suppose that L is a totally geodesic null hypersurface and ζ is a rigging for L such that*

1. $\tau = 0$.
2. $C(\xi, X) = 0$ *for all* $X \in \Gamma(\mathcal{S})$.
3. $K(span(\xi_x, N_x)) \neq Ric(N_x, \xi_x)$ *for all* $x \in L$.

Then L can not be compact.

Proof. As before, Lemma 1 and Proposition 1 give us

$$\widetilde{\triangle}\mu = \mu\left(K(span(\xi, N) - Ric(\xi, N)\right),$$

which is signed. If L is compact, then μ is a nonzero constant and integrating with respect to $\widetilde{g}$ we get

$$\int_L (K(\xi, N) - Ric(\xi, N))d\widetilde{g} = 0,$$

which is a contradiction. □

Example 3. We give an example of a compact totally geodesic null hypersurface where the hypotheses of the above theorem are fulfilled except the condition about the curvature. In the Lorentzian flat torus

$$(\mathbb{T}^n, g) = \left(\mathbb{S}^1 \times \cdots \times \mathbb{S}^1, dx_1 dx_2 + dx_3^2 + \cdots + dx_n^2\right)$$

the null hypersurface $L = \{x \in \mathbb{T}^n : x_2 = p\}$ for a fixed $p \in \mathbb{S}^1$ is totally geodesic and $\zeta = \partial x_2$ is a null rigging for it. Since ζ is parallel, we have that $\tau = 0$ and $C = 0$. On the other hand, $K = \partial x_1 - \partial x_2$ is a timelike parallel vector field.

4. Closed and Conformal Vector Fields and Null Hypersurfaces

The orthogonal distribution to a closed vector field is integrable, so it gives rise to a foliation on the manifold. In this case, if $K_p \neq 0$, we call $\mathcal{F}_p$ the orthogonal leaf through $p \in M$. The following lemmas show some properties about the leaves.

Lemma 7. *If $K \in \mathfrak{X}(M)$ is a closed and conformal vector field, then $X(\lambda) = 0$ and $X(\rho) = 0$ for all $X \in \mathfrak{X}(M)$ with $X \perp K$. In particular, λ and ρ are constant through the leaves $\mathcal{F}_p$.*

Proof. From Lemma 4 we have $X(\lambda) = 0$ and $X(\rho)\lambda = 0$. Therefore, if $\lambda(p) \neq 0$ for some point $p \in M$, then $X_p(\rho) = 0$. If $\lambda(p) = 0$ but there is a sequence p_n converging to p with $\lambda(p_n) \neq 0$, then by continuity $X_p(\rho) = 0$. If λ vanishes in a neighbourhood of p, using point 2 of Lemma 5, we also get that $\rho = 0$ in this neighbourhood and so $X_p(\rho) = 0$. □

Lemma 8. *Let $K \in \mathfrak{X}(M)$ be closed and conformal vector field and fix $p \in M$.*

- *If K_p is timelike/spacelike then $\mathcal{F}_p$ is a spacelike/timelike totally umbilical hypersurface with mean curvature vector $H = -\frac{\rho}{|g(K,K)|}K$. Moreover, there is a neighbourhood of p where (M, g) decomposes as*

$$\left((-\delta, \delta) \times \Theta, \varepsilon dt^2 + f(t)^2 g_0\right),$$

  where $\varepsilon = \frac{g(K_p,K_p)}{|g(K_p,K_p)|}$, $f \in C^\infty(-\delta, \delta)$ is a positive function, Θ is a open subset of $\mathcal{F}_p$, g_0 is the induced metric and $f\partial t$ is identified with K.
- *If K_p is null, then the orthogonal leaf $\mathcal{F}_p$ is a totally umbilical null hypersurface, $Rad(T_xL) = span(K_x)$ for all $x \in \mathcal{F}_p$ and the null mean curvature of $\mathcal{F}_p$ with respect to K is the constant $H = -\rho$.*

Proof. From Lemma 7, λ is constant through the orthogonal leaves, so if K_p is timelike, spacelike or null, then $\mathcal{F}_p$ is a spacelike, timelike or null hypersurface respectively.

If K_p is timelike or spacelike, then it will be timelike or spacelike in a neighborhood of p. The one-dimensional foliation given by K is totally geodesic and the orthogonal foliation is spherical, so [21] ensures the local decomposition. □

If K_p is timelike for all $p \in M$, then under some suitable hypotheses we can also ensure the global decomposition of the manifold as a generalized Robertson–Walker space, [12].

Now we can give an analogous result as Theorem 1 but for a closed and conformal vector field.

Theorem 3. *Let $K \in \mathfrak{X}(M)$ be a closed and conformal vector field. Suppose that L is a null hypersurface with zero null mean curvature and ζ is a rigging for L such that*

1. *$d\tau = 0$.*
2. *$C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.*
3. *$0 \leq trace(A^* \circ A)$.*
4. *$K(span(\xi, N)) \leq Ric(N, \xi)$.*

If $g(\xi, K)Ric(\xi, K) \leq 0$, $g(K, \xi)$ is signed and there is a point $p \in L$ with $K_p \in T_pL$, then $K_x \in T_xL$ for all $x \in L$.

Proof. As in Theorem 1, we can suppose that $\mu = g(K, \xi)$ is non-positive and $\tau = 0$. From Lemma 6 and Proposition 1 we have

$$\begin{aligned}\widetilde{\triangle}\mu &= -(n-2)\xi(\rho) + \mu\big(K(span(\xi,N)) - Ric(\xi,N)) - trace(A^*\circ A)\big)\\ &\geq -(n-2)\xi(\rho).\end{aligned}$$

But Lemma 4 ensures that

$$\widetilde{\triangle}\mu \geq \frac{n-2}{n-1}Ric(\xi,K) \geq 0,$$

Since μ has a local maximum at p, then μ vanishes and therefore K is tangent to L. □

Observe that the above theorem does not say that L is an orthogonal leaf of K. Indeed, consider L a degenerate plane passing through the origin in the Minkowski space and K the position vector field. The null cone with vertex at the origin is an orthogonal leaf of K which is tangent to L along a null geodesic. All the hypotheses in the Theorem 3 hold in this case and truely K is tangent to L at every point, but L is not an orthogonal leaf of K.

With a restriction on the causality of K we can get more information about the null leaves.

Lemma 9. *Let $K \in \mathfrak{X}(M)$ be a causal closed and conformal vector field. If K_p is null for some $p \in M$, then $\rho(p) = 0$ and the leaf $\mathcal{F}_p$ is a totally geodesic null hypersuface. In particular, if $\rho(p) \neq 0$ for all $p \in M$, then K is timelike.*

Proof. If K_p is null, then we know that $\mathcal{F}_p$ is a totally umbilical null hypersurface with constant null mean curvature $-\rho(p)$. Suppose that $\rho(p) \neq 0$ and take α a transverse curve to $\mathcal{F}_p$ with $\alpha(0) \in \mathcal{F}_p$. Then $\lambda(\alpha(0)) = 0$ and

$$\frac{d}{dt}\lambda(\alpha(t))|_{t=0} = 2\rho(p)g(\alpha'(0), K_{\alpha(0)}) \neq 0,$$

so K becomes spacelike at some point near $\alpha(0)$, which is a contradiction. Therefore, $\rho(p) = 0$ and $\mathcal{F}_p$ is totally geodesic. □

Now we give conditions to ensure that a null hypersurface is an orthogonal leaf of a closed and conformal vector field.

Corollary 3. *Let $K \in \mathfrak{X}(M)$ be a causal, closed and conformal vector field. Suppose that L is a null hypersurface with zero null mean curvature and ζ is a rigging for L such that*

1. *$d\tau = 0$.*
2. *$C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.*
3. *$0 \leq trace(A^* \circ A)$.*
4. *$K(span(\xi, N)) \leq Ric(N, \xi)$.*

If $g(\xi, K)Ric(\xi, K) \leq 0$ and there is a point $p \in L$ with $K_p \in T_pL$ (and therefore K_p is null), then $K_x = \nu(x)\xi_x$ for all $x \in L$ and L is a totally geodesic orthogonal leaf of K.

Observe that in Example 1 we showed a causal, closed and conformal vector field which is null at some points.

On the other hand, under the conditions of the above corollary, we have a totally geodesic null hypersurface $\mathcal{F}_p$ and a null hypesurface L with zero null mean curvature which are tangent at p, but we can not apply the maximum principle for null hypersurfaces [8, Theorem II.1] because, a priori, we can not ensure that one null hypersurface lies to the future side of the other one.

Remark 2. If we suppose that $\tau = 0$ instead of $d\tau = 0$ in the above corollary, then we can conclude that ν is constant. In fact, by Lemma 8 we have that $\rho = 0$ along L and thus $\nabla_U K = 0$ for all $U \in \mathfrak{X}(L)$. If we take derivative in $K = \nu\xi$ along $U \in \mathfrak{X}(L)$, then

$$0 = g(\nabla_U K, N) = U(\nu) + \nu g(\nabla_U \xi, N) = U(\nu) - \nu\tau(U) = U(\nu)$$

and thus ν is constant.

In a similar way as above we can also prove the following theorem which ensure that a null hypersurface is an orthogonal leaf of a parallel null vector field.

Theorem 4. *Let $K \in \mathfrak{X}(M)$ be a null parallel vector field and L a null hypersurface with rigging ζ such that*

- *H is constant.*
- *$\tau = 0$.*
- *$C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.*
- *$0 \leq trace(A^* \circ A)$.*
- *$K(span(\xi, N)) \leq Ric(\xi, N)$.*

If there is $p \in L$ such that $K_p \in T_pL$, then $K_x = \nu\xi_x$ for all $x \in L$ and a nonzero constant $\nu \in \mathbb{R}$ and L is a totally geodesic orthogonal leaf of K.

Proof. We can suppose that $\mu = g(\xi, K)$ is non-positive and so using Proposition 1 we have $\widetilde{\triangle}\mu \geq 0$. Therefore, since K is causal, we have $K = \nu\xi$ for certain $\mu \in C^\infty(L)$ and L is a totally geodesic orthogonal leaf. We can show as in Remark 2 that ν is necessarily a constant. □

Observe that in the above theorem we can not suppose $d\tau = 0$ as in Theorems 1 and 3. In these theorems we can scale the rigging to get $\tau = 0$ and all the hypotheses still hold. In the case of Theorem 4 if we scale the rigging, then we lost the condition H constant.

In the following corollary, observe that if the null mean curvature of a compact null hypersurface is constant, then it is necessarily zero since it holds $\int_L H\, d\widetilde{g} = 0$, [13].

Corollary 4. *Let $f \in C^\infty(M)$ be a function such that $K = \nabla f$ is a null parallel vector field. Suppose that L is a null compact hypersurface with rigging ζ such that*

– *H is constant.*
– $\tau = 0$.
– *$C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.*
– *$0 \leq trace(A^* \circ A)$.*
– *$K(span(\xi, N)) \leq Ric(\xi, N)$.*

Then $K_x = \nu\xi_x$ for all $x \in L$ and a nonzero constant $\nu \in \mathbb{R}$ and L is a totally geodesic orthogonal leaf of K.

Proof. If $K_p \notin T_pL$ for all $p \in L$, then K is a rigging for L, but this is not possible because it is a gradient and L is compact, [13,18]. Thus $K_p \in T_pL$ for some $p \in L$ an we can apply the above theorem. □

We say that a rigging ζ induces a preferred rigged connection if the Levi-Civita connection induced from the rigged metric $\widetilde{g}$ coincides with the induced connection ∇^L. In some sense, a null hypersurface admitting a preferred rigging connection can be handle formally as a nondegenerate one, [18]. The necessary and sufficient conditions for a rigging to induce a preferred rigging connection are $\tau = 0$ and $B = C$, [4,18,19].

Corollary 5. *Let K be a parallel null vector field, L a null hypersurface and ζ a rigging for it. If H is constant, ζ induces a preferred rigged connection, $K(span(\xi, N)) \leq Ric(\xi, N)$ and there is a point $p \in L$ with $K_p \in T_pL$, then $K_x = \nu\xi_x$ for all $x \in L$ and a nonzero constant $\nu \in \mathbb{R}$ and L is a totally geodesic orthogonal leaf of K.*

Example 4. Take (M_0, g_0) a Riemannian manifold and consider the plane fronted wave $(M, g) = \left(M_0 \times \mathbb{R}^2, g_0 + 2dudv + \phi(x, u)du^2\right)$. We have that $K = \partial_v$ is a parallel null vector field and the orthogonal leaf through a point $p = (x_0, u_0, v_0)$ is given by $\mathcal{F}_p = \{(x, u_0, v) : x \in M_0, v \in \mathbb{R}\}$. This is a totally geodesic null hypersurface and $\zeta = \nabla v = \partial u - \Phi \partial v$ is a rigging for $\mathcal{F}_p$ with rigged $\xi = \partial v$. From Eq. (4) we have that $\tau = 0$ and using that $g(\nabla_X \zeta, Y) = 0$ for all $X, Y \in \mathfrak{X}(M_0)$ and Eq. (11) we also have $C = 0$. Therefore, ζ induced a preferred rigged connection on $\mathcal{F}_p$. Moreover, since $\xi = \partial v$ is parallel, then $K(span(\xi, N)) = Ric(\xi, N) = 0$.

Using the above corollary, the orthogonal leaves of K are the unique null hypersurfaces in (M, g) with these properties.

As we said before Theorem 2, the existence of a timelike gradient prevents the existence of compact null hypersurfaces. More general, if the first De Rham cohomology group is trivial, then the existence of a closed rigging is an obtructition for the compactness of the null hypersurface. We give an obstruction for the compactness in the case of a closed (non necessarily a gradient) conformal vector field.

Theorem 5. *Let $K \in \mathfrak{X}(M)$ be a closed and conformal vector field and L a null hypersurface. Suppose that K is a rigging for L and one of the following holds.*

– *$\rho(x) \neq 0$ for all $x \in L$.*
– *$Ric(K_x, \xi_x) \neq 0$ for all $x \in L$.*

Then L is not compact. Moreover, if ξ is a complete vector field, then L is diffeomorphic to $\mathbb{R} \times S$.

Proof. If we call $\zeta = K$ and $\widetilde{\lambda} = \lambda \circ i$, then $\nabla\lambda = 2\rho\zeta$ and $\widetilde{\nabla}\widetilde{\lambda} = 2\rho\xi$. On the other hand, we have $X(\rho) = 0$ for all $X \perp K$, so if K_x is not null for some $x \in L$, then $\nabla\rho_x = \xi(\rho)K_x$. If K_x is null, then $K_x = N_x$ and since the screen distribution and K_x is orthogonal to K_x itself we get $\nabla\rho_x = \xi(\rho)N_x = \xi(\rho)K_x$. Thus, in any case $\nabla\rho = \xi(\rho)K$ and thus $\widetilde{\nabla}\widetilde{\rho} = \xi(\rho)\xi = -\frac{1}{n-1}Ric(K, \xi)\xi$, where as before $\widetilde{\rho} = \rho \circ i$. If L is compact, then $\widetilde{\lambda}$ and $\widetilde{\rho}$ have a critical point, which contradicts the hypotheses.

For the last part, since K is closed we have that ξ is $\widetilde{g}$-unitary and closed and we can check as in [12, Proposition 2.1] that the flow Φ of ξ gives us a covering map $\Phi : \mathbb{R} \times S \to L$, being S a leaf of the screen. We have that both λ and ρ are constant through the leaves of the screen and by hypotheses, fixed $x \in S$, we have that $\lambda(\Phi_s(x))$ or $\rho(\Phi_s(x))$ are strictly monotone functions. Therefore, $\Phi : \mathbb{R} \times S \to L$ is injective and so a diffeomorphism. □

Observe that from [15, Theorem 18], in the above situation we can scale the rigging to obtain a geodesic rigged vector field, i.e., $\tau(\xi) = 0$. On the other hand, since $\widetilde{g}(\xi, \widetilde{\nabla}\rho) = Ric(K, \xi)$, to ensure that L is not compact under the assumption $Ric(K, \xi) \neq 0$ we do need to suppose that $K_x \notin T_xL$ for all $x \in L$.

We focus now on the case where the closed and conformal vector field is tangent to the null hypersurface and we give sufficient conditions to ensure that the null hypersurface is an orthogonal leaf in this situation.

Proposition 2. *Let $K \in \mathfrak{X}(M)$ be a closed and conformal vector field and L a null hypersurface such that $K_x \in T_xL$ for all $x \in L$. If L has a screen non-degenerate second fundamental form, then $K_x = \nu(x)\xi_x$ for all $x \in L$ and L is an orthogonal leaf of K.*

In particular, if L is totally umbilical with never vanishing null mean curvature, then L is an orthogonal leaf of K.

Proof. Since $g(K, \xi) = 0$, Lemma 6 gives us $A^*(\mathcal{P}_{\mathcal{S}}(K)) = 0$, but being the null second fundamental form of L non-degenerate, we have $\mathcal{P}_{\mathcal{S}}(K) = 0$ and $K = \nu\xi$. Therefore L is an orthogonal leaf of K. □

Theorem 6. *Let $K \in \mathfrak{X}(M)$ be a closed and conformal vector field and L a null hypersurface with zero null mean curvature such that $K_x \in T_xL$ for all $x \in L$. Suppose that ζ is a rigging for L such that.*

– *$d\tau = 0$.*
– *$C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$.*

If $(n-1)(n-2)\rho^2 \leq Ric(K, K)$ and K_p is null for some point $p \in L$, then $K_x = \nu(x)\xi_x$ for all $x \in L$ and L is a totally geodesic orthogonal leaf of K.

Proof. As in Theorem 1 we can take a restricted rigging in a neighbourhood of $p \in \theta \subset L$ such that $\tau = 0$ and $C(\xi, X) = 0$ for all $X \in \Gamma(\mathcal{S})$. Since K is tangent to L, then $g(K, \xi) = 0$. Moreover, from Lemma 4 we know that the function $\lambda = g(K, K)$ holds $\nabla\lambda = 2\rho K$ and so $g(\nabla\lambda, \xi) = 0$.

Using the Lemmas 1 and 2 we get that the laplacian of $\widetilde{\lambda} = \lambda \circ i$ with respect to $\widetilde{g}$ is $\widetilde{\triangle}\widetilde{\lambda} = \triangle\lambda - 2Hess_\lambda(\xi, N) + Hess_\lambda(\xi, \xi)$. Since ξ is orthogonal to K, Lemma 7 ensures $\xi(\rho) = 0$ and so $Hess_\lambda(\xi, \xi) = 0$ and $Hess_\lambda(\xi, N) = 2\rho^2$. Using this jointly with the expression for $\triangle\lambda$ given in Lemma 4 gives us

$$\widetilde{\triangle}\widetilde{\lambda} = -\frac{2}{n-1}Ric(K, K) + (2n - 4)\rho^2 \leq 0.$$

Since λ has a minimum at p, then λ vanishes in θ and by connectedness in the whole L. Therefore, $K = \nu\xi$ for certain $\nu \in C^\infty(L)$ and since $H = 0$ and the orthogonal leaves of K are totally umbilical, then L is a totally geodesic orthogonal leaf of K. □

As in Remark 2, if we suppose in the above theorem that $\tau = 0$ then we can conclude that ν is a constant.

Finally, the following result gives us conditions for a closed and conformal vector field in a null hypersurface to be tangent to the screen distribution.

Theorem 7. *Let $K \in \mathfrak{X}(M)$ be a closed and conformal vector field and L a null hypersurface such that $K_x \in T_xL$ for all $x \in L$. If there is a preferred rigging ζ for L, $K_p \in \mathcal{S}_p$ for some $p \in L$, $g(K, N)$ is signed and $0 \leq g(K, N)\rho H$, then $K_x \in \mathcal{S}_x$ for all $x \in L$.*

Proof. Since $\mu = g(K, \xi) = 0$, from Lemma 6 we have that the gradient with respect to $\widetilde{g}$ of $\nu = g(K, N)$ is $\widetilde{\nabla}\nu = \rho\xi$. Therefore, $\widetilde{\triangle}\nu = \xi(\rho) + \rho\widetilde{\mathrm{div}}\xi$, but from Eq. (17) we get $\widetilde{\mathrm{div}}\xi = -H$ and from Lemma 7 we have $\xi(\rho) = 0$. Applying the maximum principle to ν we get the result. □

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