Journal of Geometry and Physics 178 (2022) 104576

Contents lists available at ScienceDirect

Journal of Geometry and Physics

www.elsevier.com/locate/geomphys

Contact structures on null hypersurfaces

Cyriaque Atindogbé [a], Manuel Gutiérrez [b], Raymond Hounnonkpe [a], Benjamín Olea [c,*]

[a] Department of Mathematics, University of Abomey-Calavi, Benin
[b] Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Spain
[c] Departamento de Matemática Aplicada, Universidad de Málaga, Spain

ARTICLE INFO

Article history:
Received 7 January 2022
Accepted 14 May 2022
Available online 23 May 2022

MSC:
53C50
53C40
53D10
57K33

Keywords:
Null hypersurface
Rigging technique
Contact structure
Sasaki structure

ABSTRACT

The aim of this paper is to show how we can induce contact structures, contact metric structures and Sasaki structures on a null hypersurface from a rigging vector field. We give several explicit examples of this construction and some obstructions to its existence. For example, contact metric structures can be introduced only on zero null mean curvature null hypersurfaces, whereas Sasaki structures can only be introduced in totally geodesic ones. In particular we show how we can construct a Kähler halo around an isolated horizon of a black hole. We also study the stability of the construction. We prove that close enough contact metric (resp. Sasaki) structures can be connected by a one parameter family of contact metric (resp. Sasaki) structures.

1. Introduction

A contact structure in a $2n+1$-dimensional manifold M is a codimension one distribution which is maximally non-integrable. This means that there is an open covering of M such that for each open set U in this covering there is a one-form $\omega \in \Lambda^1(U)$ such that the distribution is given by the kernel of ω and

$$\omega \wedge (d\omega)^n \neq 0 \tag{1}$$

at every point of U. Contact structures have applications to different fields, as thermodynamic, mechanics and optics, and they can be considered as the odd-dimensional analogue to symplectic structures.

We will always consider coorientable contact structures, that is, the codimension one distribution is given by the kernel of a globally defined one-form ω, called contact form. As a consequence M is orientable and there is a distinguished vector field ξ, called Reeb or characteristic vector field, which is the unique such that $\iota_\xi d\omega = 0$ and $\omega(\xi) = 1$. Hereafter, we will use the same letter ξ to denote other (*a priori*) vector fields, but it will turn out that they are always Reeb vector fields.

* Corresponding author.
E-mail addresses: atincyr@gmail.com (C. Atindogbé), m_gutierrez@uma.es (M. Gutiérrez), rhounnonkpe@ymail.com (R. Hounnonkpe), benji@uma.es (B. Olea).

https://doi.org/10.1016/j.geomphys.2022.104576

If there is a field of endomorphisms ϕ, a vector field ξ, a one-form ω and a Riemannian metric g on M such that

$$\phi^2 = -id + \omega \otimes \xi,$$
$$g(\phi(U), \phi(V)) = g(U, V) - \omega(U)\omega(V)$$

for all $U, V \in \mathfrak{X}(M)$, then we say that (ϕ, ξ, ω, g) is an almost contact metric (ACM) structure on M. In this case, it follows that $\phi(\xi) = 0$, ϕ is skew-symmetric and ω is the metrically equivalent one-form to ξ.

A contact manifold (M, ω) always carries an ACM structure (ϕ, ξ, ω, g) such that ξ is the Reeb vector field of the contact form ω and

$$d\omega(U, V) = 2g(U, \phi(V)) \tag{2}$$

for all $U, V \in \mathfrak{X}(M)$ [5, Theorem 4.4], where our convention for the exterior derivative is such that

$$d\omega(U, V) = U\omega(V) - V\omega(U) - \omega([U, V]).$$

An ACM structure (ϕ, ξ, ω, g) is called a contact metric (CM) structure on M if ω is a contact form and Equation (2) holds. If moreover it holds

$$(\nabla_U \phi)(V) = g(U, V)\xi - \omega(V)U$$

for all $U, V \in \mathfrak{X}(M)$, then (ϕ, ξ, ω, g) is called a Sasakian structure on M.

The relationships between contact metric, Sasakian, symplectic and Kählerian manifolds are well-known. Symplectic and Kählerian structures have been used in the geometric quantization program. The existence of a Sasaki structure in the horizon of a kind of black holes and a Kählerian halo around it, as we will see in the present work, deserves further study to understand its significance, if any.

On the other hand, a hypersurface of a Lorentzian manifold is null if the metric tensor is degenerated on it. In some sense, null hypersurfaces of a Lorentzian manifold resemble the Lagrangian and Legendrian submanifolds in symplectic and contact manifolds: the induced structure from the ambient space is degenerated.

The rigging technique as introduced in [11] tries to overcome the difficulties arisen from this degeneration by inducing a screen distribution, a null section and a Riemannian metric on the null hypersurface from a rigging. A rigging for a null hypersurface L is a vector field ζ defined in some open neighbourhood of L such that $\zeta_p \notin T_pL$ for all $p \in L$. If ζ is defined only over L, then we call it a restricted rigging.

If the rigging exists, then we can take the unique null vector field $\xi \in \mathfrak{X}(L)$ such that $g(\zeta, \xi) = 1$ (called rigged vector field) and the screen distribution given by $\mathcal{S}_p = \zeta_p^{\perp} \cap T_pL$ for all $p \in L$. We can also define the rigged metric as the Riemannian metric on L given by $\widetilde{g} = g + \omega \otimes \omega$, where $\omega = i^*\alpha$, α is the g-metrically equivalent one-form to ζ and $i : L \to M$ is the canonical inclusion. This metric makes ξ a $\widetilde{g}$-unitary vector field orthogonal to $\mathcal{S}$. Moreover, ω is $\widetilde{g}$-metrically equivalent to ξ, so it is called the rigged one-form. The rigged data associated to the rigging ζ is the triple $(\xi, \omega, \widetilde{g})$.

The rigging technique presents two main advantages. The first one is that all the geometric objects defined above from the rigging are tuned together in a way that allows us to link properties of the null hypersurface with properties of the ambient space. The second one is the presence of the Riemannian metric $\widetilde{g}$, whose geometry is reasonably well coupled with the ambient geometry in most cases and it allows us to use Riemannian tools for the study of the null hypersurface, [11].

We can define the following three tensors for a fixed rigging for a null hypersurface.

$$B(U, V) = -g(\nabla_U \xi, V), \tag{3}$$
$$\tau(U) = -g(\nabla_U \xi, \zeta), \tag{4}$$
$$C(U, V) = -g(\nabla_U N, P(V)), \tag{5}$$

for all $U, V \in \mathfrak{X}(L)$, where $P : TL \to \mathcal{S}$ is the canonical projection associated to the decomposition

$$TL = span\{\xi\} \oplus \mathcal{S}$$

and $N = \zeta - \frac{1}{2}g(\zeta, \zeta)\xi$ is the unique null vector field defined on L, orthogonal to the screen distribution and such that $g(N, \xi) = 1$.

The tensor B is called the null second fundamental form. It is symmetric and it is related to C by the formula

$$-2C(U, X) = d\omega(U, X) + \left(L_\zeta g\right)(U, X) + g(\zeta, \zeta)B(U, X) \tag{6}$$

for all $U \in \mathfrak{X}(L)$ and $X \in \mathcal{S}$. Another important formula relating the second fundamental form and the rigged metric is

$$(L_\xi \widetilde{g})(X, Y) = -2B(X, Y) \tag{7}$$

for all $X, Y \in \mathcal{S}$, which in particular leads us to

$$H = -\widetilde{\operatorname{div}}\xi, \tag{8}$$

being H the null mean curvature of L. Moreover, Equations (4) and (5) show that

$$C(\xi, X) + \tau(X) = -d\alpha(\xi, X) = -d\omega(\xi, X) = -\widetilde{g}(\widetilde{\nabla}_\xi \xi, X) \tag{9}$$

for all $X \in \mathcal{S}$.

It is said that L is totally geodesic if $B = 0$ and totally umbilic if $B = \rho g$ for certain $\rho \in C^\infty(L)$. Observe that these definitions do not depend on the chosen rigging, although the tensors B, τ and C do depend.

The philosophy of the rigging technique is to select an adapted rigging for each situation in order to endow the associated rigged data with appropriate properties which will allow us to unveil properties of the null hypersurface itself.

Following this philosophy, in this paper we construct a contact, contact metric or Sasaki structure on a null hypersurface starting from a rigging and using its rigged data. We give some illustrative examples of this construction. In Example 2.4 we show that any contact metric or Sasaki structure in a three-dimensional Riemannian manifold can be realized by our construction as the associated structure of a suitable rigging on a null hypersurface.

Some obstructions to get a contact metric or Sasaki structure are also given. For example, if we obtain a contact metric structure, then the null mean curvature is zero and if we get a Sasaki structure, then the null hypersurface is totally geodesic. Moreover, under a curvature assumption we also get a topological obstruction for the existence of a Sasaki structure in a compact null hypersurface.

Finally, we prove that the contact metric structures constructed in this way are stable in the following sense: if a one parameter family of riggings connects two riggings each of which have an associated contact metric structure, then each element of the family has an associated contact metric structure. In particular, the contact structure induced from a timelike contact metric rigging in a compact null hypersurface, if any, is unique up to contactomorphism.

2. Contact metric rigging for null hypersurfaces

Definition 2.1. A rigging for a null hypersurface is a contact rigging if its rigged one-form is a contact form.

Observe that in this case the rigged vector field ξ does not need to be the Reeb vector field of the rigged form ω. A necessary and sufficient condition for this to happen is that ξ is $\widetilde{g}$-geodesic, since

$$\iota_\xi d\omega(U) = \widetilde{g}(\widetilde{\nabla}_\xi \xi, U)$$

for all $U \in \mathfrak{X}(L)$.

Example 2.1. Let $M = \mathbb{R}^4$ be the Minkowski space with metric $g = 2dtdx + dy^2 + dz^2$. The plane L given by $t = 0$ is a null hypersurface and $\zeta = \partial_t + y\partial_z$ is a rigging which induces the rigged vector field $\xi = \partial_x$ and the rigged one-form $\omega = dx + ydz$. Therefore, ζ is a contact rigging for L. The Reeb vector field in this case is just the rigged vector field $\xi = \partial_x$.

If we take the rigging $\zeta' = e^x\zeta$, the rigged vector field is $\xi' = e^{-x}\xi$ and the rigged one form is $\omega' = e^x\omega$, which is also a contact form. But the Reeb vector field of ω' is not ξ', since

$$d\omega' = e^x dx \wedge \omega + e^x d\omega,$$

thus $\iota_{\xi'} d\omega' = ydz \neq 0$.

A contact form ω in a null hypersurface L such that $\omega(u) \neq 0$ for all null vector $u \in TL$ always comes from a contact (restricted) rigging. In fact, we can take a global null section $\xi \in \mathfrak{X}(L)$ with $\omega(\xi) = 1$ which jointly with the screen distribution given by $\ker\omega$ determines a restricted rigging ζ with rigged one-form ω, [8, p. 79]. If moreover L is closed in the topological sense, then we can extend ζ to an open set containing L using the differentiable version of the Tietze extension theorem, [2, Theorem 6.5.9].

A standard way to get a contact form on a hypersurface in a symplectic manifold is to use a transverse Liouville vector field. We can take advantage of this construction in our situation. To be precise, suppose that the Lorentzian manifold is also furnished with a symplectic form σ. If $\widehat{\zeta}$ is a Liouville vector field transverse to a null hypersurface L and we call $\alpha = \iota_{\widehat{\zeta}}\sigma$, then $\omega = i^*(\alpha)$ is a contact form on L, [9, Lemma 1.4.5]. Take ζ the metrically equivalent vector field to α. If ζ is transverse to L, then it is a contact rigging since its rigged one-form is precisely ω. We give an explicit example of this construction.

Example 2.2. Let (F, η) be a contact manifold and $\big(\mathbb{R} \times F, \sigma = d(e^t\eta)\big)$ its symplectization. Call $\widehat{\zeta} = \partial_t$ which is a Liouville vector field and $\alpha = \iota_{\widehat{\zeta}}\sigma = e^t\eta$.

Suppose that g_F is a Riemannian metric on F and consider the Lorentzian metric on $\mathbb{R} \times F$ given by $g = -dt^2 + f(t)^2 g_F$, where $f \in C^\infty(\mathbb{R})$ is a positive function. If $h : \theta \subset F \to \mathbb{R}$ is a function with $||\nabla^F h|| = f \circ h$, then the graph of h given by

$$L = \{(t, x) : t = h(x), x \in \theta\}$$

is a null hypersurface and $(f \circ h)^2 \partial_t + \nabla^F h$ is a null section on L, [10]. Since $g(\widehat{\zeta}, (f \circ h)^2 \partial_t + \nabla^F h) = -(f \circ h)^2 \neq 0$ we have that $\widehat{\zeta}$ is transverse to L, so $\omega = i^*(\alpha)$ is a contact form on L. Moreover, if $\eta(\nabla^F h) \neq 0$, then $\alpha((f \circ h)^2 \partial_t + \nabla^F h) \neq 0$. Thus the g-metrically equivalent vector field to α is transverse to L, so it is a contact rigging for L.

With the same arguments, we can also ensure directly that a rigging is a contact rigging in the following way.

Lemma 2.1. *Let (M, g) be a Lorentzian manifold, L a null hypersurface and ζ a rigging for it with g-equivalent one-form α. Suppose that $d\alpha$ is a symplectic form and take $\widehat{\zeta}$ the unique vector field such that $\iota_{\widehat{\zeta}} d\alpha = \alpha$. If $\widehat{\zeta}$ is also a rigging for L, then ζ is a contact rigging.*

Proof. Just use the Cartan Formula to get that $\widehat{\zeta}$ is a Liouville vector field, so $\omega = i^*(\alpha)$, which is the rigged one-form associated to ζ, is a contact form on L. □

Example 2.3. Let (F, g_F) be a $(2n+1)$-dimensional Riemannian manifold, $\theta \in \Lambda^1(F)$ and $f \in C^\infty(F)$ a positive function such that $df \wedge (d\theta)^n$ is a volume form on F. Consider the static spacetime (M, g) given by

$$\left(\mathbb{R} \times F, -f dt^2 + \theta \otimes dt + dt \otimes \theta + g_F\right).$$

The vector field $\zeta = \partial_t$ is a timelike Killing vector field and $\alpha = -f dt + \theta$ is its metrically equivalent one-form. The 2-form $\sigma = d\alpha = -df \wedge dt + d\theta$ is a symplectic form on M. We want to find a vector field $\widehat{\zeta}$ such that $\iota_{\widehat{\zeta}} \sigma = \alpha$.

Since $df \wedge (d\theta)^n$ is a volume form, $d\theta|_{\ker df}$ is non-degenerate, but $d\theta$ is degenerate because $\dim F$ is odd. This implies that there is a unique vector field $R \in \mathfrak{X}(F)$ with $\iota_R d\theta = 0$ normalized by $df(R) = f$. Using the non degeneracy of $d\theta|_{\ker df}$, there is also a unique vector field $X_0 \in \mathfrak{X}(F)$ with $X_0 \in \ker df$ and $\iota_{X_0} d\theta|_{\ker df} = \theta|_{\ker df}$. In particular, $\theta(X_0) = 0$. Using $\iota_{X_0} d\theta(R) = 0$, it holds

$$\iota_{X_0} d\theta = \theta - \frac{\theta(R)}{f} df.$$

Now, if we define

$$\widehat{\zeta} = \frac{\theta(R)}{f} \partial_t + X_0 + R,$$

then we have

$$\iota_{\widehat{\zeta}} \sigma = -df(\widehat{\zeta}) dt + \frac{\theta(R)}{f} df + \iota_{X_0} d\theta = -f dt + \theta = \alpha.$$

Observe that $\widehat{\zeta}$ and ζ are orthogonal since $g(\zeta, \widehat{\zeta}) = \alpha(\widehat{\zeta}) = \iota_{\widehat{\zeta}} \sigma(\widehat{\zeta}) = 0$, so $\widehat{\zeta}$ is spacelike because ζ is timelike. Now, Lemma 2.1 implies that $\zeta = \partial_t$ is a contact rigging for any null hypersurface transverse to $\widehat{\zeta}$.

As an example of the above situation, consider the future null cone with vertex $p = (0, x_0)$ given by

$$C_p^+ = \exp_p\left(\{u \in T_p M : g(u, u) = 0, g(u, \partial_t) < 0\}\right).$$

We cannot ensure that it is a hypersurface due to the possible existence of null conjugate points or null crossing points, but near the vertex it is a null hypersurface. If we take a null vector $u \in T_p M$ with $g(u, \widehat{\zeta}_p) \neq 0$, then there is a portion of C_p^+ which contains $\exp_p(\varepsilon u)$ for some small $\varepsilon > 0$ and it is a null hypersurface transverse to $\widehat{\zeta}$.

We can choose the manifold F to be more explicit. In fact, let $F = \mathbb{R}^{2n+1}$ be the Euclidean space with standard coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$, $f = e^z$ and $\theta = \sum_{i=1}^n x_i dy_i$. We can check that in this case $R = \partial_z$, $X_0 = \sum_{i=1}^n x_i \partial_{x_i}$ and

$$\widehat{\zeta} = \partial_z + \sum_{i=1}^n x_i \partial_{x_i}.$$

Martinet theorem ensures that every closed three-dimensional manifold admits a contact structure, so in particular it also carries an ACM structure. This is also true for noncompact three-dimensional manifolds as it is shown in [7]. We give here an explicit construction adapted to our purpose.

Proposition 2.1. *If $(L, \widetilde{g})$ is an oriented three-dimensional Riemannian manifold and $\xi \in \mathfrak{X}(L)$ is a unitary vector field, then there exists a unique ACM structure $(\phi, \xi, \omega, \widetilde{g})$ on L such that every orthonormal positive basis $\{\xi_p, X_p, Y_p\}$ is a ϕ-basis for all $p \in L$.*

Proof. Given a local positive basis $\{\xi, X, Y\}$ on L, we define $\phi(\xi) = 0$, $\phi(X) = Y$ and $\phi(Y) = -X$. The endomorphism ϕ does not depend on the chosen local positive basis with first vector field ξ, so we can extend it to the whole L and we can easily check that $\phi^2 = -id + \omega \otimes \xi$, where ω is the metrically equivalent one-form to ξ.

A direct computation leads to $\widetilde{g}(\phi(U), \phi(V)) = \widetilde{g}(U, V) - \omega(U)\omega(V)$ for all $U, V \in \mathfrak{X}(L)$, thus $(\phi, \xi, \omega, \widetilde{g})$ is an ACM structure. □

In the context of Proposition 2.1, we call

$$\varrho_p = d\omega(X_p, Y_p), \tag{10}$$

where $\{\xi_p, X_p, Y_p\}$ is a positive orthonormal basis in T_pL. The function ϱ does depend on ξ_p but not on X_p, Y_p, thus ϱ extends to a differentiable function globally defined on L, once ξ is fixed. By Equation (1), it is immediate that ϱ never vanishes if and only if ω is a contact form, so we call it the contact function. It is straightforward to prove the following.

Proposition 2.2. *Let $(L, \widetilde{g})$ be an oriented three-dimensional Riemannian manifold. The ACM structure constructed in Proposition 2.1 is a CM structure if and only if $\varrho = -2$ and ξ is $\widetilde{g}$-geodesic.*

Remark 2.1. The natural orientation induced from the contact structure provided in Proposition 2.2 is always the opposite of the original orientation of L. It is due to the definition of ϕ we made. Moreover, observe that the sign of ϱ depends on the chosen orientation. Thus, if ξ is $\widetilde{g}$-geodesic and $\varrho = 2$, then we only have to change the orientation to turn the ACM structure constructed in Proposition 2.1 into a CM structure.

We can relate ϱ and the divergence of ξ as in [13, Proposition 5.7.2 1a] (see [1] for an approach using the Newman-Penrose formalism).

Lemma 2.2. *If $(L, \widetilde{g})$ is a three-dimensional oriented Riemannian manifold and ξ is a $\widetilde{g}$-geodesic and unitary vector field, then $\xi(\varrho) = -\varrho \cdot \widetilde{\operatorname{div}}\xi$.*

Proof. Take an orthonormal positive basis $\{\xi_p, e_1, e_2\}$ in T_pL and γ an integral curve of ξ with $\gamma(0) = p$. We translate it parallelly along γ to obtain an orthonormal basis $\{\xi_{\gamma(t)}, X(t), Y(t)\}$ in $T_{\gamma(t)}M$ with $X(0) = e_1$ and $Y(0) = e_2$. Now, at the point p we have

$$\begin{aligned}
\xi(\varrho) &= \widetilde{g}(\widetilde{\nabla}_\xi \widetilde{\nabla}_X \xi, Y) - \widetilde{g}(X, \widetilde{\nabla}_\xi \widetilde{\nabla}_Y \xi) \\
&= \widetilde{g}(\widetilde{R}_{\xi X}\xi, Y) + \widetilde{g}(\widetilde{\nabla}_{[\xi, X]}\xi, Y) - \widetilde{g}(\widetilde{R}_{\xi Y}\xi, X) - \widetilde{g}(\widetilde{\nabla}_{[\xi, Y]}\xi, X) \\
&= -\widetilde{g}(\widetilde{\nabla}_{\widetilde{\nabla}_X \xi}\xi, Y) + \widetilde{g}(\widetilde{\nabla}_{\widetilde{\nabla}_Y \xi}\xi, X).
\end{aligned}$$

Taking into account that ξ is unitary, we set $\widetilde{\nabla}_X \xi = A_{11}X + A_{12}Y$ and $\widetilde{\nabla}_Y \xi = A_{21}X + A_{22}Y$, where A_{ij} are the Fourier coefficient in the above basis. We have

$$\begin{aligned}
\widetilde{\operatorname{div}}\xi &= A_{11} + A_{22}, \\
\varrho &= A_{12} - A_{21}
\end{aligned}$$

and therefore

$$\xi(\varrho) = -A_{11}A_{12} - A_{12}A_{22} + A_{21}A_{11} + A_{22}A_{21} = -\varrho \cdot \widetilde{\operatorname{div}}\xi. \quad \square$$

If there is a rigging ζ for an orientable null hypersurface L in a four-dimensional Lorentzian manifold, then we can apply Proposition 2.1 to $(L, \widetilde{g})$ to determine a unique ACM structure $(\phi, \xi, \omega, \widetilde{g})$ such that a positive basis is a ϕ-basis, being $(\xi, \omega, \widetilde{g})$ the rigged data associated to ζ. We call this structure the rigged ACM structure associated to ζ. This leads us to the following definition.

Definition 2.2. Let L be an oriented null hypersurface in a four-dimensional Lorentzian manifold and ζ a rigging for it. If the rigged ACM structure associated to ζ is a CM or Sasaki structure, then ζ is called a CM or Sasaki rigging respectively.

Using Proposition 2.2, Lemma 2.2 and Equation (8) we get a first obstruction to the existence of a CM rigging.

Proposition 2.3. *Let L be an oriented null hypersurface in a four-dimensional Lorentzian manifold. If there is a CM rigging for L, then L has zero null mean curvature.*

In [12] it is shown how the fundamental tensors associated to a rigging change under a rigging change. In particular, it follows that having zero null mean curvature is independent of the choice of the rigging.

The Weinstein conjecture, which have been proved in dimension 3, see [15], claims that the Reeb vector field of a contact form in a compact manifold has a closed orbit. If ζ is a CM rigging, then its rigged vector field ξ is necessarily the Reeb vector field due to Proposition 2.2. Thus we have another obstruction (this one from causality theory) to the existence of a CM rigging in compact null hypersurfaces.

Proposition 2.4. *Let (M, g) be a causal Lorentzian manifold of dimension four. If L is a compact oriented null hypersurface, then it does not admit a CM rigging.*

Remark 2.2. If we rescale the rigging $\zeta' = f\zeta$, being f a never vanishing function, then $\xi' = \frac{1}{f}\xi$ and the contact function associated to ζ' is $\varrho' = |f|\varrho$. Observe that the absolute value is due to a possible change in the orientation of the rigged vector field. Therefore, if ζ is a contact rigging, then $\zeta' = f\zeta$ is also a contact rigging. In particular, if ζ is a CM rigging, then $-\zeta$ is also a CM rigging and the identity is a metric contactomorphism, that is, an equivalence between these contacts metric structures, see Definition 2.3 below.

Proposition 2.5. *Let L be an oriented null hypersurface in a four-dimensional Lorentzian manifold. If there is a rigging ζ such that $\varrho \neq 0$ and*

$$d\omega(\xi, U) = -d(\ln|\varrho|)(U)$$

for all $U \in \mathcal{S}$, then $\zeta' = \frac{2}{|\varrho|}\zeta$ is a CM rigging. In particular, L has zero null mean curvature.

Proof. From Remark 2.1 we can suppose that $\varrho < 0$ changing the orientation if necessary. We denote $(\xi', \omega', \widetilde{g}')$ and $\mathcal{S}'$ the rigged data and the screen distribution induced by ζ'. We have $\mathcal{S} = \mathcal{S}'$, $\omega' = \frac{2}{|\varrho|}\omega$ and $\xi' = \frac{|\varrho|}{2}\xi$. Using Equation (9), we get

$$\begin{aligned}\widetilde{g}'(\widetilde{\nabla}'_{\xi'}\xi', U) &= d\omega'(\xi', U) = \frac{2}{\varrho^2}(d\varrho \wedge \omega)(\xi', U) - \frac{2}{\varrho}d\omega(\xi', U)\\ &= d\ln|\varrho|(U) + d\omega(\xi, U) = 0\end{aligned}$$

for all $U \in \mathcal{S}$. Therefore, ξ' is $\widetilde{g}'$-geodesic and the contact function induced from ζ' is $\varrho' = \frac{2}{|\varrho|}\varrho = -2$. Proposition 2.2 says that ζ' is a CM rigging. □

Corollary 2.1. *Let L be an oriented null hypersurface in a four-dimensional Lorentzian manifold. If there is a rigging ζ such that ϱ is a nonzero constant and $d\omega(\xi, U) = 0$ for all $U \in \mathcal{S}$, then $\zeta' = \frac{2}{|\varrho|}\zeta$ is a CM rigging.*

We now show stability properties of CM riggings. The most relevant is that they are locally convex.

Let ζ_t be a one parameter family of rigging vector fields and $(\phi_t, \xi_t, \omega_t, \widetilde{g}_t)$ their associated rigged ACM structures with the coincident subindices. Observe that for two contact forms ω_0 and ω_1 in a three-dimensional manifold, it holds

$$1 \leq \dim(\ker\omega_0 \cap \ker\omega_1) \leq 2.$$

We use this fact in the following theorem which is in some sense complementary to the D-deformation of Tanno [14] for dimension three, but in our case the contact distribution does not remain constant.

Theorem 2.1. *Let (M, g) be a four-dimensional Lorentzian manifold and L an oriented null hypersurface in M. Suppose that ζ_0 and ζ_1 are two CM riggings for L. If $\zeta_t = (1-t)\zeta_0 + t\zeta_1$ is a rigging for L for all $t \in [0, 1]$, then ζ_t is a CM rigging for all $t \in [0, 1]$.*

Proof. It is immediate that $\omega_t = (1-t)\omega_0 + t\omega_1$ is the associated rigged one-form to ζ_t, and $\widetilde{g}_t = \omega_t \otimes \omega_t + g$ is its rigged metric.

Fix a point $p \in L$. Since all the rigged vector fields are proportional, there is a function $c : [0, 1] \to \mathbb{R}$ such that

$$\xi_t(p) = c(t)\xi_0(p) \tag{11}$$

for all $t \in [0, 1]$. For simplicity we will omit the point p in the following argument.

If we take a g-unitary vector $v_0 \in \ker\omega_0 \cap \ker\omega_1$, then we can construct a $\widetilde{g}_t$-orthonormal and positive basis $\{\xi_t, X_t, Y_t\}$ at T_pL with $X_t = v_0$ for all $t \in [0, 1]$. Observe that $Y_t \in span\{\xi_0, Y_0\}$, so we can write

$$Y_t = a(t)\xi_0 + b(t)Y_0$$

for certain functions $a, b : [0, 1] \to \mathbb{R}$ with $a(0) = 0$, $b(0) = 1$ and b never vanishing. It is immediate that

$$1 = \widetilde{g}_t(Y_t, Y_t) = b(t)^2,$$

therefore $b(t) = 1$ by continuity. Hence the expression for Y_t simplifies to

$$Y_t = a(t)\xi_0 + Y_0. \tag{12}$$

Now we compute the contact function ϱ_t. Using equation (10) we have

$$\begin{aligned}\varrho_t &= d\omega_t(X_t, Y_t)\\ &= (1-t)d\omega_0(X_0, a(t)\xi_0 + Y_0) + td\omega_1(X_1, a(t)\xi_0 + Y_0)\\ &= -2(1-t) + td\omega_1(X_1, (a(t)-a(1))\xi_0 + Y_1) = -2,\end{aligned}$$

where we have used Proposition 2.2 to ensure that $d\omega_0(X_0, Y_0) = d\omega_1(X_1, Y_1) = -2$ and ξ_0, ξ_1 are $\widetilde{g}_0, \widetilde{g}_1$-geodesic respectively. Moreover, ξ_t is $\widetilde{g}_t$-geodesic since

$$\iota_{\xi_t} d\omega_t = \iota_{\xi_t}((1-t)d\omega_0 + td\omega_1) = 0.$$

Since $p \in L$ is arbitrary, using again Proposition 2.2, ζ_t is a CM rigging for all $t \in [0, 1]$. □

The above theorem says that two close enough contact metric riggings can be connected by a straight segment family of contact metric riggings, that is, the set of contact metric riggings is locally convex. Observe that this is not true in general for CM structures in a three-dimensional manifold. Indeed, if ω_0 and ω_1 are contact forms, then $(1-t)\omega_0 + t\omega_1$ does not even have to be a contact form.

Moreover, since a timelike vector field is a rigging for any null hypersurface, the set of future (past) timelike contact metric riggings in a time orientable four dimensional Lorentzian manifold is convex.

An immediate consequence of Gray's stability theorem [9, Theorem 2.2.2] and Theorem 2.1 above is the following corollary.

Corollary 2.2. *Let (M, g) be a four-dimensional Lorentzian manifold and L an oriented compact null hypersurface in M. Suppose that ζ_0 and ζ_1 are CM riggings for L. If $\zeta_t = (1-t)\zeta_0 + t\zeta_1$ is a rigging for L for all $t \in [0, 1]$, then the contact structures associated to ζ_0 and ζ_1 are contactomorphic.*

Taking into account Remark 2.2 we also have the following.

Corollary 2.3. *Let (M, g) be a four-dimensional Lorentzian manifold and L an oriented compact null hypersurface in M. The contact structures induced by timelike CM rigging vector fields on L, if any, are unique up to contactomorphism.*

Compact null hypersurfaces are not frequent. For example, strongly causal Lorentzian manifolds do not admit them, [4]. On the other hand, Proposition 2.4 says that compact null hypersurfaces in causal spacetimes do not admit CM riggings. Moreover, important null hypersurfaces like black hole horizons are not compact, thus it would be desirable to relax the compact assumption in Corollary 2.2. This is done in the following result.

Theorem 2.2. *Let (M, g) be a four-dimensional Lorentzian manifold and L an oriented null hypersurface in M. Suppose that ζ_0 and ζ_1 are CM riggings for L such that $\zeta_t = (1-t)\zeta_0 + t\zeta_1$ is a rigging for L for all $t \in [0, 1]$. If there is a vector field $X \in \mathfrak{X}(L)$ such that:*

- $X \in \ker\omega_0 \cap \ker\omega_1$.
- *The traces of the integral curves of X is diffeomorphic to $\mathbb{S}^1$.*

Then the induced contact structures from ζ_0 and ζ_1 are contactomorphic.

Proof. As in the proof of Theorem 2.1, we have the family of 1-forms

$$\omega_t = (1-t)\omega_0 + t\omega_1, \tag{13}$$

which are in fact contact forms for all $t \in \mathbb{R}$. According to the proof of the Gray theorem, see [9], for each $t \in \mathbb{R}$ we take the unique vector field $Z_t \in \mathfrak{X}(L)$ such that

$$\omega_t(Z_t) = 0 \tag{14}$$

$$\dot{\omega}_t + \iota_{Z_t} d\omega_t = \mu_t \omega_t \tag{15}$$

From it, we construct the vector field $\hat{Z}_{(t,p)} = \partial_t + Z_t(p) \in \mathfrak{X}(\mathbb{R} \times L)$. If $\hat{Z}$ is a complete vector field, then we can take the isotopy $\psi_s : L \to L$ defined by $\psi_s(p) = \Pi(\Phi_s(0, p))$ for all $s \in \mathbb{R}$, where Φ is the flow of $\hat{Z}$ and Π is the canonical projection onto L. This isotopy holds $\dot{\psi}_s(p) = Z_s(\psi_s(p))$, so, according to the Gray construction, $\psi_s^*(\omega_s) = \lambda_s \omega_0$ for some never vanishing family of functions λ_s. Thus ψ_1 is the contactomorphism we are looking for.

Therefore, all that we have to do is to check that the integral curve of $\hat{Z}$ through $(0, p)$ is defined in the whole $\mathbb{R}$ for all $p \in L$.

In our particular case $\dot{\omega}_t = \omega_1 - \omega_0$, so using Equation (15) we have $\omega_0(Z_t) = \omega_1(Z_t)$ and Equations (13) and (14) imply $Z_t \in \ker \omega_0 \cap \ker \omega_1$ for all $t \in \mathbb{R}$.

It is straightforward to check that for any point $p \in L$ such that $\dim(\ker \omega_0 \cap \ker \omega_1)_p = 2$, it holds $Z_t(p) = 0$. Therefore we can write $Z_t(p) = f(t, p)X(p)$, where $f \in C^\infty(\mathbb{R} \times L)$, since for a suitable choice of f the vector field $f(t, p)X(p)$ holds equations (14) and (15).

If $\alpha : \mathbb{R} \to L$ is an integral curve of X with $\alpha(0) = p$, then $\gamma(s) = (s, \alpha(h(s)))$ is an integral curve of $\hat{Z}$ with $\gamma(0) = (0, p)$ for a suitable function $h(s)$. Since the trace of α is compact, it follows that γ is defined for all $s \in \mathbb{R}$. □

It would be interesting to study not only contactomorphisms but metric contactomorphisms in Corollary 2.2, but at the present we know about their existence but nothing about their properties, so we have not information enough to study them. By the way, we recall the equivalence for contact metric structures and Sasaki structures.

Definition 2.3. Two CM structures on a manifold are metric contactomorphic if there is a contactomorphism which is also an isometry.

If $(\Phi_i, \xi_i, \omega_i, \widetilde{g}_i)$ for $i = 1, 2$ are CM structures on L and $f : L \to L$ is a contactomorphism (with $f^*(\omega_2) = \lambda\omega_1$), then it is an isometry if and only if $f_*(\xi_1) = \lambda\xi_2$ and $\Phi_2 \circ f_* = \lambda f_* \circ \Phi_1$ with $\lambda^2 = 1$. Moreover, if two CM structures are metric contactomorphic and one of them is Sasaki, then the other one is also Sasaki, and the equivalence for the Sasaki structures is just the metric contactomorphism.

We finish this section with an important observation. Any CM (resp. Sasaki) structure in a three-dimensional Riemann manifold can be realized as the associated structure to a timelike CM (resp. Sasaki) rigging for a null hypersurface where the rigged metric is the original Riemannian metric. The following example shows the construction.

Example 2.4. Let F be a three-dimensional manifold admitting a CM (resp. Sasaki) structure (ϕ, ξ, ω, g_F). In $M = \mathbb{R} \times F$ define the Lorentzian metric

$$g_L = dt \otimes \omega + \omega \otimes dt - \omega \otimes \omega + g_F.$$

We have that $g_L(\xi, \xi) = g_L(\partial_t, \partial_t) = 0$ and $g_L(\xi, \partial_t) = 1$. Moreover, F is a null hypersurface in (M, g_L) and $\zeta = \partial_t - \xi$ is a timelike rigging with induced rigged vector field ξ. The rigged metric constructed from ζ on F is just g_F and the rigged one-form is ω. Changing the sign of ζ if necessary, ϕ coincides with the one constructed in Proposition 2.1. Therefore, ζ is a timelike CM (resp. Sasaki) rigging for F as a null hypersurface in M which induces the original structure.

3. Sasaki riggings for null hypersurfaces

In this section we will derive a couple of consequences of the existence of a CM or a Sasaki rigging for a null hypersurface.

Proposition 3.1. *Let (M, g) be a four-dimensional Lorentzian manifold and L an oriented null hypersurface.*

1. *If L admits a Sasaki rigging ζ, then L is totally geodesic.*
2. *If L is totally geodesic, then any CM rigging on L is a Sasaki rigging.*

Proof. It follows from Equation (7) and the fact that a three-dimensional contact metric structure is a Sasakian structure if and only if the Reeb vector field is a Killing vector field, [5, Corollary 6.3, 6.5 and Theorem 6.3]. □

Point 1 above is true in any dimension. This result leads us to give the following corollary of Theorem 2.1 which tell us that the family of Sasaki rigging is locally convex and the family of future (past) timelike Sasaki rigging in a time orientable four-dimensional Lorentzian manifold is convex.

Corollary 3.1. *Let (M, g) be a four-dimensional Lorentzian manifold and L an oriented null hypersurface in M. Suppose that ζ_0 is a Sasaki rigging and ζ_1 is a CM rigging for L. If $\zeta_t = (1-t)\zeta_0 + t\zeta_1$ is a rigging for L for all $t \in [0, 1]$, then ζ_t is a Sasaki rigging for all $t \in [0, 1]$.*

Another consequence of the above proposition is due to the fact that the existence of Sasaki rigging vector fields occurs in totally geodesic null hypersurfaces, precisely a property that share isolated horizons of black holes, so it is interesting to analyze the existence of Sasaki structures on these kind of horizons.

On the other hand, if L is a totally geodesic null hypersurface which admits a Sasaki rigging, then we can construct a Kähler halo around it. Indeed, it exists $\epsilon > 0$ such that $\Phi : (0,\epsilon)\times L \to M$ defined by $\Phi(s,x) = \exp_x s\zeta_x$ is a diffeomorphism (alternatively, we can use the flow of ζ if $(0,\epsilon)\times L$ is contained in its domain). We can furnish $(0,\epsilon)\times L$ with the cone metric $ds^2 + s^2\widetilde{g}$ which make it a Kähler manifold, see [6], and use Φ to define a Kähler structure on $\Phi((0,\epsilon)\times L)$. We can do the same switching $(0,\epsilon)$ with $(-\epsilon,0)$ and the result is that we have a Kähler halo around L. Kähler manifolds are used in the geometric quantization program, so the above construction around a black hole isolated horizon is suggestive.

The following example shows that it exists a Sasaki rigging vector field on an open dense subset of the horizon of a black hole in Kruskal spacetime.

Example 3.1. Let $Q \subset \mathbb{R}^2$ be an open subset containing the origin and $F \in C^\infty(Q)$ a never vanishing function. We consider the Lorentzian metric in Q given by $2F(u,v)dudv$. Suppose that (S, g_S) is an oriented 2-dimensional Riemannian manifold and $\eta \in \Lambda^1(S)$ such that $d\eta$ is a symplectic form. We call J the associated almost complex structure.

Consider the warped product $(M,g) = \left(Q \times S, 2F(u,v)dudv + h(u,v)^2 g_S\right)$, where $h \in C^\infty(Q)$ is a positive function, and the one-form $\alpha = -2e^u du + 2e^u dv + 2\eta$. The g-metrically equivalent vector field to α is

$$\zeta = \frac{2e^u}{F}\partial_u - \frac{2e^u}{F}\partial_v + \frac{2}{h^2}E,$$

where E is the vector field g_S-metrically equivalent to η, and

$$\sigma = d\alpha = 2e^u du \wedge dv + 2d\eta$$

is a symplectic form on M. If we call X_0 the unique vector field with $\iota_{X_0} d\eta = \eta$, then

$$\widehat{\zeta} = \partial_u + \partial_v + X_0$$

holds $\iota_{\widehat{\zeta}}\sigma = \alpha$.

The hypersurface $L = \{(u,0,x) : (u,0) \in Q, x \in S\}$ is a null hypersurface and both ζ and $\widehat{\zeta}$ are rigging for it, so Lemma 2.1 ensures that ζ is a contact rigging.

The rigged vector field is $\xi = \frac{-e^{-u}}{2}\partial_u$ and the rigged one-form is $\omega = -2e^u du + 2\eta$. Therefore, $d\omega = 2d\eta$ and $\iota_\xi d\omega = 0$, which implies that ξ is $\widetilde{g}$-geodesic. The second fundamental form of L respect to ξ is given by $B = \frac{e^{-u}h_u}{h} g$. Thus, if we suppose that h is constant along L, then L is totally geodesic. Without loss of generality, we can also assume that $h(u,0) = 1$ for all $(u,0) \in Q$.

Let $X \in \mathfrak{X}(S)$ be a g_S-unitary vector field and call $Y = J(X)$. Then

$$\widetilde{X} = \eta(X)e^{-u}\partial_u + X,$$
$$\widetilde{Y} = \eta(Y)e^{-u}\partial_u + Y,$$

form a positive orthonormal basis of $\mathcal{S}$. A direct computation gives

$$\varrho = d\omega(\widetilde{X},\widetilde{Y}) = 2d\eta(X,Y) = 2g_S(X, J(Y)) = -2.$$

Using Proposition 2.2 and Proposition 3.1 we have that ζ is a Sasaki rigging for L.

We can try to particularize this example to the Kruskal space (M,g) given by

$$\left(Q \times \mathbb{S}^2, 2F(r)dudv + r^2 g_0\right),$$

being g_0 the standard metric on $\mathbb{S}^2$ and $h = r$ is constant on the null hypersurface L. Unfortunately, we cannot apply Lemma 2.1 because it does not exist a 1-form η in $\mathbb{S}^2$ such that $d\eta$ is a symplectic form due to Stokes theorem. A solution is to take an open dense set in $\mathbb{S}^2$. In fact, if we take $S = \mathbb{S}^2 - K$ where K has cardinal 1 or 2, then it is diffeomorphic to a plane or a cylinder, in both cases they have a global 1-form η such that $d\eta$ is a symplectic form. This 1-form can be used to apply the above construction.

We get another interesting case taking spherical coordinates (φ,ψ) in $\mathbb{S}^2$, where $0 < \varphi < 2\pi$ and $-\frac{\pi}{2} < \psi < \frac{\pi}{2}$. In this case $\eta = \varphi\cos^2\psi\, d\psi$ holds the above conditions in the domain U of the spherical chart, which is an open dense set in $\mathbb{S}^2$. The associated metric is the restriction to U of the standard Euclidean metric on $\mathbb{S}^2$ and the almost complex structure is given by $J(\partial\varphi) = -\partial\psi$ and $J(\partial\psi) = \partial\varphi$.

The following theorem gives us a topological property of compact null hypersurfaces with a Sasaki rigging.

Theorem 3.1. *Let (M, g) be a four-dimensional Lorentzian manifold with positive sectional curvature on spacelike planes. If L is an oriented compact null hypersurface admitting a Sasaki rigging for it, then the first and second Betti numbers hold $b_1(L) = b_2(L) = 0$.*

Proof. From [14, Theorem 1] it is sufficient to prove that the sectional curvature of $(L, \widetilde{g})$ is greater than -3. This is trivially satisfied for a plane containing ξ, the rigged vector field, since its sectional curvature is 1, [5, Theorem 7.2]. On the other hand, if Π is a plane orthogonal to ξ, from Propositions 2.2 and 3.1 we know that the contact function $\varrho = -2$ and the null second fundamental form $B = 0$, so the formula [11, Theorem 4.2] gives $\widetilde{K}(\Pi) > -3$.

Take now an arbitrary tangent plane Π to L. We can suppose that $\Pi = span\{X, U\}$, where $U = Y + a\xi$, $X, Y \in \ker\omega$ and $\widetilde{g}(X, Y) = 0$. Moreover, we suppose that X and U are $\widetilde{g}$-unitary, thus $\widetilde{g}(Y, Y) + a^2 = 1$. The curvature of Π is

$$\widetilde{K}(\Pi) = \widetilde{g}(\widetilde{R}_{XY}Y, X) + a\widetilde{g}(\widetilde{R}_{XY}\xi, X) + a\widetilde{g}(\widetilde{R}_{X\xi}Y, X) + a^2\widetilde{g}(\widetilde{R}_{X\xi}\xi, X),$$

but the second and third terms vanish due to [5, Proposition 7.3]. Therefore

$$\widetilde{K}(\Pi) = \widetilde{K}(span\{X, Y\})\widetilde{g}(Y, Y) + a^2 > -3\widetilde{g}(Y, Y) + a^2 = -3 + 4a^2 \geq -3. \quad \square$$

Corollary 3.2. *Let (M, g) be a four-dimensional Lorentzian manifold with positive sectional curvature on spacelike planes. If there is a closed timelike vector field in M, then it does not exist a Sasaki rigging for any oriented and compact null hypersurface.*

Proof. Suppose L is a compact and oriented null hypersurface and there is a Sasaki rigging for it. Since a timelike vector field is also a rigging for L, applying Theorem 3.1 and [3, Theorem 3.1] we get a contradiction. $\square$

Acknowledgement

Second, third and fourth authors were funded by Universidad de Málaga-Junta de Andalucía-FEDER grant number UMA18-FEDERJA-183 and Ministry of Science and Innovation of Spain (MICINN) I+D+I, grant number PID2020-118452GB-I00.

References

[1] A.B. Aazami, The Newman-Penrose formalism for Riemannian 3-manifolds, J. Geom. Phys. 94 (2015) 1–7.
[2] R. Abraham, J.E. Marsden, T. Ratiu, Manifolds, Tensor Analysis and Applications, Springer, 2002.
[3] C. Atindogbé, M. Gutiérrez, R. Hounnonkpe, New properties on normalized null hypersurfaces, Mediterr. J. Math. 15 (2018) 166.
[4] C. Atindogbé, M. Gutiérrez, R. Hounnonkpe, Compact null hypersurfaces in Lorentzian manifolds, Adv. Geom. 21 (2021) 251–263.
[5] D. Blair, Riemannian Geometry of Contact and Symplectic Manifolds, Birkhäuser, 2010.
[6] C.P. Boyer, K. Galicki, Sasakian Geometry, Oxford University Press, 2008.
[7] J.L. Cabrerizo, M. Fernández, J.S. Gómez, On the existence of contact metric structure and the contact magnetic field, Acta Math. Hung. 125 (2009) 191–199.
[8] K.L. Duggal, A. Bejancu, Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer Academic Publishers, 1996.
[9] H. Geiges, An Introduction to Contact Topology, Cambridge University Press, 2008.
[10] M. Gutiérrez, B. Olea, Totally umbilic null hypersurfaces in generalized Robertson-Walker spaces, Differ. Geom. Appl. 42 (2015) 15–30.
[11] M. Gutiérrez, B. Olea, Induced Riemannian structures on null hypersurfaces, Math. Nachr. 289 (2016) 1219–1236.
[12] F. Ngakeu, H.F. Tetsing, B. Olea, Rigging technique for 1-lightlike submanifolds and preferred rigged, Mediterr. J. Math. 16 (2019) 139.
[13] B. O'Neill, The Geometry of Kerr Black Holes, Wellesley, Massachusetts, 1995.
[14] S. Tanno, The topology of contact Riemannian manifolds, Ill. J. Math. 12 (1968) 700–717.
[15] C.H. Taubes, The Seiberg–Witten equations and the Weinstein conjecture, Geom. Topol. 11 (2007) 2117–2202.