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# Classification of fiber sequences with a prescribed holonomy action

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We define $\mathcal{H}$-fibration sequences as fibrations where the holonomy action of the fundamental group of the base on the fiber lies in a given subgroup $\mathcal{H}$ of $\mathcal{E}(F)$, where $\mathcal{E}(F)$ is the homotopy automorphism group of the fiber. Furthermore, we classify these $\mathcal{H}$-fibration sequences via a universal $\mathcal{H}$-fibration sequence.



## 0. Introduction

From the foundational paper of Stasheff [12] to the recent reference [3] of homotopy theoretical flavor, the literature is splashed with results which let us classify fibrations, sometimes of a certain type (see for instance [7,14]), by means of a suitable classifying space. The most general result in the homotopy category is probably the classical work of May [9] which classifies fibrations with a given *category of fibers*.

In these works, there is a classifying object $B\operatorname{aut} F$, and a universal quasi-fibration sequence

$$F \to B(*, \operatorname{aut} F, F) \to B\operatorname{aut}(F)$$

such that, given a space $B$ of the homotopy type of a CW-complex, there is a bijection

$$\Lambda : [B, B\operatorname{aut} F] \to \mathcal{F}\mathrm{ib}(B, F)$$

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where $[B, B\operatorname{aut} F]$ is the set of homotopy classes of maps from $B$ to $B\operatorname{aut} F$ and $\mathcal{F}\mathrm{ib}(B, F)$ is defined at section 1.

However, this treatment does not readily apply to classify fibrations with a prescribed holonomy action of the fundamental group of the base on the fiber: given a fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$, there is map of groups $\pi_1(B, b_0) \to \mathcal{E}(F) = \pi_0(\operatorname{aut} F)$. Here $\mathcal{E}(F)$ is a discrete space such that, with the operation induced by the composition of maps, it becomes a discrete group. Given a subgroup $\mathcal{H}$ of $\mathcal{E}(F)$, we say that a fibration sequence is an $\mathcal{H}$-fibration sequence if the image of the holonomy action lies in the subgroup $\mathcal{H}$.

The purpose of this text is to define rigorously the concepts above and to construct a classifying object and a universal $\mathcal{H}$-quasi-fibration sequence, of the form

$$F \to B(*, \operatorname{aut}_{\mathcal{H}} F, F) \to B\operatorname{aut}_{\mathcal{H}} F$$

which classifies $\mathcal{H}$-fibration sequences. Here $\operatorname{aut}_{\mathcal{H}} F$ will denote the submonoid of $\operatorname{aut} F$ of self-homotopy equivalences $\varphi$ such that $[\varphi] \in \mathcal{H} \subseteq \mathcal{E}(F)$. Note that when $\mathcal{H} = \mathcal{E}(F)$ is the whole group, then $\operatorname{aut}_{\mathcal{H}} F = \operatorname{aut} F$ and we recover the usual classifying space and the original result of Stasheff [12]. The other extreme case $\mathcal{H} = 1$ makes $B\operatorname{aut}_{\mathcal{H}} F$ weakly equivalent to the universal cover of $B\operatorname{aut} F$; for this choice of subgroup, an $\mathcal{H}$-fibration sequence is a fibration sequence such that its holonomy action on the fiber is trivial. The rational homotopy type of this last space is well-known (see for instance [13, §VII]). Other choices of the subgroup $\mathcal{H}$ yield intermediate situations between the two described above.

Although this classification may be part of the folklore, we are not aware of any detailed reference in the literature and the purpose of this note is to fill this gap in a rigorous way. The resulting classifying space is a well known and interesting object, see for instance [4]. Particularly interesting examples and applications of this classification, in the rational homotopy category, are considered in [5]; there it is shown that for a good choice of the subgroup $\mathcal{H}$, the classifying space has the homotopy type of a rational space. Furthermore, its classifying properties are fundamental for obtaining homotopical information about $B\operatorname{aut}_{\mathcal{H}}(F)$.

As a final remark, I stress that this result differs from the classical classification of topological $G$-bundles for a given topological group $G$ (see for instance [2]) to the same extent as the classification of any class of fibrations differs from the corresponding bundle counterpart.

The text is structured as follows: in the first section, the concepts of holonomy action and $\mathcal{H}$-fibration sequence are introduced. In the second, the geometric-bar construction is used to prove the classification Theorem 2.1. Finally, in the third part, we give a pointed version of these results.

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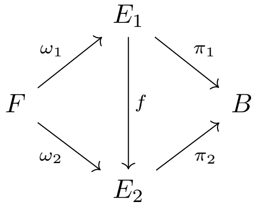
## 1. $\mathcal{H}$-fibration sequences

Throughout this text, any considered topological space is compactly generated and weakly Hausdorff. Also, the fiber $F$ and the base $B$ shall always be considered as connected spaces of the homotopy type of a CW-complex and $b_0$ is a fixed basepoint in $B$ such that its inclusion is a cofibration. By a *fibration sequence*, we mean a sequence

$$F \xrightarrow{\omega} E \xrightarrow{\pi} B$$

where $p$ is a Hurewicz fibration, and the map $\omega\colon F \xrightarrow{\simeq_w} F_0 = \pi^{-1}(b_0)$ is a weak homotopy equivalence. A *map between two fibration sequences*, $F \xrightarrow{\omega_1} E_1 \xrightarrow{\pi_1} B$ and $F \xrightarrow{\omega_2} E_1 \xrightarrow{\pi_2} B$, is a homotopy commutative diagram of the form,




In particular, $f$ is a weak homotopy equivalence and there is no loss of generality if the right triangle is imposed to be strictly commutative. The maps of fibration sequences generate an equivalence relation (by imposing symmetry and associativity) and we denote by $\mathcal{F}\mathrm{ib}(B, F)$ the corresponding quotient set.

**Remark 1.1.** Note that this definition of $\mathcal{F}\mathrm{ib}(B, F)$ (which is the one considered in [1], for example) differs from the one considered in [12] or [9], where a fibration with fiber $F$ is a Hurewicz fibration $\pi : E \to B$ whose fiber is weakly equivalent to $F$, but no explicit weak equivalence is given. These two approaches give rise to different equivalence classes and, therefore, to different classifying spaces. However, both spaces are weakly equivalent (see [10, explanation below Theorem 1.2]). Since we are considering base spaces which are of the homotopy type of a CW-complex, the two approaches coincide in our setting.

**Remark 1.2.** As in [9, Theorem 7.6] we will use quasi-fibrations (defined below) in order to construct a universal fibration sequence. However the set $\mathcal{F}\mathrm{ib}(B, F)$ consists of 'strict' fibrations sequences rather than quasi-fibration sequences.

Recall how the fundamental group of the base acts on the homotopy automorphisms of the fiber. Given a fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ and $\beta \in \pi_1(B, b_0)$, the lifting property of the fibration yields an induced homotopy equivalence $\bar{\beta}\colon F_0 = \pi^{-1}(b_0) \xrightarrow{\simeq} F_0$. Together with the bijection $\omega_*\colon [F, F] \xrightarrow{\cong} [F, F_0]$, the map $\bar{\beta}$ produces the homotopy equivalence $\hat{\beta} = {\omega_*}^{-1}[\bar{\beta} \circ \omega] \in [F, F]$. That is, $\omega \circ \hat{\beta} \simeq \bar{\beta} \circ \omega$. Here $[X, Y]$ indicates free homotopy classes of (non-based) maps from $X$ to $Y$.

Let $\operatorname{aut} F$ be the topological monoid of self homotopy equivalences of $F$ and write $\mathcal{E}(F) = \pi_0(\operatorname{aut} F)$. Fix a subgroup $\mathcal{H} \subseteq \mathcal{E}(F)$ and consider $\operatorname{aut}_{\mathcal{H}} F \subset \operatorname{aut} F$ the submonoid of homotopy equivalences $\varphi$ such that $[\varphi] \in \mathcal{H}$.

In particular, the process described above, gives, for each fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$, a map

$$\pi_1(B, b_0) \to \mathcal{E}(F), \quad [\beta] \mapsto [\hat{\beta}]$$

which will be called the *holonomy action of* $\pi_1(B, b_0)$ *on the fiber*. This is well-known concept (see for example [11, §8 Theorem 12] for a classical reference), but there is not a widely accepted name for this action. The word 'holonomy' is used in the context of Rational Homotopy Theory (see [6]).

**Definition 1.3.** A fibration sequence $F \xrightarrow{\omega} E \xrightarrow{p} B$ is an $\mathcal{H}$*-fibration sequence* if the image of the holonomy action is contained in $\mathcal{H}$. This means that $\hat{\beta} \in \mathcal{H}$ for any $\beta \in \pi_1(B, b_0)$.

The holonomy action of two fibration sequences, connected by a map of fibrations, are closely related. Then, a direct inspection shows the following proposition.

**Proposition 1.4.** *Given two fibration sequences over $B$ with fiber $F$ and a map of fibration sequences between them, if one of them is an $\mathcal{H}$-fibration sequence, then the other one also is.*

In particular, we can consider $\mathcal{F}\mathrm{ib}_{\mathcal{H}}(B, F)$ the set of equivalence classes of $\mathcal{H}$-fibration sequences over $B$ with fiber $F$, which is a subset of $\mathcal{F}\mathrm{ib}(B, F)$.

**Example 1.5.** In the extreme case, $\mathcal{H} = \mathcal{E}(F)$, we recover the usual notion of fibration sequence, since $\mathcal{F}\text{ib}_{\mathcal{H}}(B, F) = \mathcal{F}\text{ib}(B, F)$. In the other extreme case, $\mathcal{H} = 1$, an $\mathcal{H}$-fibration sequence is a fibration sequence with a trivial holonomy action. As an intermediate case, we can take the homology functor (or more generally, any functor from the homotopy category of topological spaces), and take $\mathcal{H}$ as those automorphisms which are the identity on the homology.

We finish this section recalling two usual constructions to fix the notation.

**Definition 1.6.** Given a fiber sequence $F \xrightarrow{\omega} E \xrightarrow{p} B$ and a based map $f : A \to B$, the *pullback fibration sequence* is defined as

$$F \xrightarrow{f^*\omega} f^*E = \{(x, a) \in E \times A \mid f(a) = \pi(x)\} \xrightarrow{f^*\pi} A$$

where $f^*\omega(x) = (\omega(x), a_0)$ and $f^*\pi(x, a) = a$.

It is immediate to check that the pullback fibration sequence of an $\mathcal{H}$-fibration sequence is an $\mathcal{H}$-fibration sequence.

**Remark 1.7.** If $\{a_0\} \hookrightarrow A$ is a cofibration, an arbitrary map $f : (A, a_0) \to (B, b_0)$ is (free) homotopy equivalent to a based map, so we can always assume that the maps between base spaces are based.

**Definition 1.8.** For an arbitrary map $\pi : E \to B$, we write

$$\Gamma E = \{(x, \beta) \in E \times B^I \mid \beta(0) = \pi(x)\}$$

$$\Gamma\pi : \Gamma E \to B, \quad (x, \beta) \mapsto \beta(1)$$

where $B^I$ is the space of paths on $B$. The map $\Gamma\pi$ is a fibration with *homotopy fiber*

$$\mathcal{F} = (\Gamma\pi)^{-1}(b_0) = \{(x, \beta) \in E \times B^I \mid \beta(0) = \pi(x),\ \beta(1) = b_0\}.$$

If the natural homotopy equivalence $i : E \to \Gamma E,\ i(x) = (x, c_{\pi(x)})$, where $c_{\pi(x)}$ is the constant path at $\pi(x)$, induces a weak homotopy equivalence $\pi^{-1}(b) \to (\Gamma\pi)^{-1}(b)$ for all $b \in B$, we say that $\pi : E \to B$ is a *quasi-fibration*. As explained in Remark 1.2 this generalization is necessary for the construction of a universal fibration sequence.

By a *quasi-fibration sequence* we mean a sequence

$$F \xrightarrow{\omega} E \xrightarrow{\pi} B$$

such that $\pi$ is a quasi-fibration and $i \circ \omega : F \to \mathcal{F}$ is a weak homotopy equivalence. The *associated fibration sequence* is the fibration sequence

$$F \xrightarrow{i\circ\omega} \Gamma E \xrightarrow{\Gamma\pi} B$$

Finally, we define an *$\mathcal{H}$-quasi-fibration sequence* as a quasi-fibration sequence whose associated fibration sequence is an $\mathcal{H}$-fibration sequence.

## 2. The classification theorem

We strongly rely on the classical reference [9] from which we recall some facts. Let $G$ be a topological monoid with the identity element $e$ a strongly nondegenerate basepoint, and let $X$ and $Y$ be left and right $G$-spaces respectively. The *geometric bar construction* $B(Y, G, X)$ is the geometric realization of the simplicial topological space whose space of $j$ simplices is $Y \times G^j \times X$ and the face and degeneracy operators are given by,

$$d_i(y, g_1, \dots, g_j, x) = \begin{cases} (y \cdot g_1, g_2, \dots, g_j, x) & \text{if } i = 0, \\ (y, g_1, \dots, g_{i-1}, g_i \cdot g_{i+1}, g_{i+2}, \dots, g_j, x) & \text{if } 1 \leq i < j, \\ (y, g_1, \dots, g_{j-1}, g_j \cdot x) & \text{if } i = j, \end{cases}$$
$$s_i(y, g_1, \dots, g_j, x) = (y, g_1, \dots, g_i, e, g_{i+1}, \dots, g_j, x).$$

This is a functor from the category of triples $(Y, G, X)$ as above, whose morphisms are triples $(g, f, h)$ where $f\colon G \to G'$ is a map of monoids and $g\colon Y \to Y'$ and $h : X \to X'$ are $f$-equivariant maps. Of particular interest is the map

$$p\colon B(*, G, X) \to B(*, G, *) = BG$$

induced by the trivial $G$-map $X \to *$. Then if $G$ is a grouplike topological monoid, i.e. $\pi_0(G)$ is a group, then [9, Theorem 7.6] asserts that $p$ is a quasi-fibration with fiber $X$. In particular, the inclusion of the homotopy fiber

$$i : X \xrightarrow{\simeq_w} \mathcal{F} = (\Gamma p)^{-1}(*) = \{(z, \beta) \mid z \in B(*, G, X), \beta : [0, 1] \to BG, p(z) = \beta(0), \beta(1) = *\}$$

is a weak homotopy equivalence, where $i(x) = (x, c_*)$, $* \in BG$ is the unique point in the unique 0-simplex and $x \in B(*, G, X)$ lies in the 0-simplex of the simplicial topological space, which agrees with $X$.

In particular, whenever $F$ is of the homotopy type of a CW-complex, the *universal fibration sequence*

$$F \xrightarrow{i} \Gamma B(*, \operatorname{aut} F, F) \xrightarrow{\Gamma p} B \operatorname{aut} F$$

classifies fibrations with fiber $F$. Explicitly, see [9, Theorem 9.2], for any space $B$ of the homotopy type of a CW-complex, the map,

$$\Lambda\colon [B, B \operatorname{aut} F] \xrightarrow{\cong} \mathcal{F}\mathrm{ib}(B, F), \quad \Lambda[f] = f^* \Gamma p$$

is a natural bijection. Given $\mathcal{H} \subseteq \mathcal{E}(F)$ a subgroup, we denote by $p_{\mathcal{H}}$ the map

$$p_{\mathcal{H}}\colon B(*, \operatorname{aut}_{\mathcal{H}} F, F) \to B(*, \operatorname{aut}_{\mathcal{H}} F, *) = B \operatorname{aut}_{\mathcal{H}} F$$

induced by $F \to *$.

Finally, let's recall that whenever $G$ is grouplike, then there is a natural weak homotopy equivalence (see [9, Proposition 8.7])

$$\zeta : G \xrightarrow{\simeq_w} \Omega BG, \quad g \mapsto \gamma_g$$

where $\gamma_g : [0, 1] \to BG$ is a path such that $\gamma_g(t) = (g, t)$ where this element lies in the 1-skeleton of $BG$. This is, in the realization of the simplicial topological space, the subspace $G \times [0, 1]/\sim$ of $BG$.

Lastly, denote by $j : \operatorname{aut}_{\mathcal{H}} F \hookrightarrow \operatorname{aut} F$, the inclusion of monoids.

With these elements we can finally prove the classification theorem.

**Theorem 2.1.** *The map,*

$$\Lambda_{\mathcal{H}}\colon [B, B\operatorname{aut}_{\mathcal{H}} F] \xrightarrow{\cong} \mathcal{F}\mathrm{ib}_{\mathcal{H}}(B, F), \quad \Lambda_{\mathcal{H}}[f] = f^{*}\Gamma p_{\mathcal{H}},$$

*is a natural bijection which fits in the following commutative diagram*

$$\begin{array}{ccc} [B, B\operatorname{aut} F] & \xrightarrow[\cong]{\Lambda} & \mathcal{F}\mathrm{ib}(B, F) \\ {\scriptstyle (Bj)_*}\uparrow & & \uparrow \\ [B, B\operatorname{aut}_{\mathcal{H}} F] & \xrightarrow[\cong]{\Lambda_{\mathcal{H}}} & \mathcal{F}\mathrm{ib}_{\mathcal{H}}(B, F). \end{array} \tag{1}$$

**Proof.** *(i)* For any $[f] \in [X, B\operatorname{aut}_{\mathcal{H}} F]$, $\Lambda_{\mathcal{H}}[f]$ is indeed an $\mathcal{H}$-fibration sequence: for each $g \in \operatorname{aut}_{\mathcal{H}} F$, lift the path $\gamma_g = \zeta(g)$, in the fibration

$$\Gamma p_{\mathcal{H}} : \Gamma B(*, \operatorname{aut}_{\mathcal{H}} F, F) \to B\operatorname{aut}_{\mathcal{H}} F$$

to obtain a homotopy equivalence $\bar{\gamma}_g : \mathcal{F} \to \mathcal{F}$ of the homotopy fiber. Then a straightforward inspection shows that the following diagram commutes up to homotopy,

$$\begin{array}{ccc} F & \xrightarrow{\quad g \quad} & F \\ {\scriptstyle \simeq_w}\downarrow{\scriptstyle i} & & {\scriptstyle \simeq_w}\downarrow{\scriptstyle i} \\ \mathcal{F} & \xrightarrow{\quad \bar{\gamma}_g \quad} & \mathcal{F}. \end{array}$$

In particular, we obtain that $[\hat{\gamma}_g] = [g] \in \mathcal{E}(F)$. Since $\pi_1(B\operatorname{aut}_{\mathcal{H}} F) \cong \pi_0(\operatorname{aut}_{\mathcal{H}} F) \cong \mathcal{H}$, we have that the holonomy action factors through

$$\pi_1(B\operatorname{aut}_{\mathcal{H}}(F)) \cong \mathcal{H} \xrightarrow{\mathrm{id}} \mathcal{H} \subset \mathcal{E}(F)$$

so its image lies in $\mathcal{H}$. Then $F \xrightarrow{i} B(*, \operatorname{aut}_{\mathcal{H}} F, *) \xrightarrow{\Gamma p_{\mathcal{H}}} B\operatorname{aut}_{\mathcal{H}} F$ is an $\mathcal{H}$-fibration sequence. Thus, the pullback $\Lambda_{\mathcal{H}}[f]$ is an $\mathcal{H}$-fibration sequence.

*(ii)* The diagram (1) commutes:

It is enough to show that the two fibrations sequences with fiber $F$ and fibrations

$$\Gamma p_{\mathcal{H}}\colon \Gamma B(*, \operatorname{aut}_{\mathcal{H}} F, F) \to B\operatorname{aut}_{\mathcal{H}} F$$

and

$$(Bj)^{*}\Gamma p\colon (Bj)^{*}\Gamma B(*, \operatorname{aut} F, F) \to B\operatorname{aut}_{\mathcal{H}} F$$

respectively, are equivalent. In other words, we have to see that there is a map of fibration sequences between them.

For it, define the map

$$\begin{array}{rccc} \mu : & \Gamma B(*, \operatorname{aut}_{\mathcal{H}} F, F) & \to & \Gamma B(*, \operatorname{aut} F, F) \\ & (z, \beta) & \mapsto & (B(*, j, \mathrm{id}_F)(z), Bj \circ \beta). \end{array}$$

It can be verified that this is a well-defined map. Furthermore, applying [9, Proposition 7.8] to the map $j : \operatorname{aut}_{\mathcal{H}} F \to \operatorname{aut} F$, we get that $\mu$ is a map of fibrations, this means that, restricted to each fiber, $\mu$ is a weak homotopy equivalence.

Now consider the following commutative diagram:

$$\begin{array}{ccccc}
\Gamma B(*, \mathrm{aut}_{\mathcal{H}} F, F) & \dashrightarrow & (Bj)^* \Gamma B(*, \mathrm{aut}\, F, F) & \longrightarrow & \Gamma B(*, \mathrm{aut}\, F, F) \\
 & {\scriptstyle \Gamma p_{\mathcal{H}}} \searrow & \downarrow {\scriptstyle (Bj)^*\Gamma p} & & \downarrow {\scriptstyle \Gamma p} \\
 & & B\, \mathrm{aut}_{\mathcal{H}} F & \xrightarrow{Bj} & B\, \mathrm{aut}\, F,
\end{array}
\qquad \mu : \Gamma B(*, \mathrm{aut}_{\mathcal{H}} F, F) \to \Gamma B(*, \mathrm{aut}\, F, F)$$

where the dashed arrow exists because of the universal property of the pullback. Both $\mu$ and the upper horizontal arrow are maps of fibrations, so also is the dashed arrow. Therefore we have obtained a map of fibrations between our two initial fibrations.

*(iii)* The map $\Lambda_{\mathcal{H}}$ is injective:

Equivalently, we will show that $(Bj)_* : [B, B\,\mathrm{aut}_{\mathcal{H}} F] \to [B, B\,\mathrm{aut}\, F]$ is injective. By [9, Remark 8.9] the map $Bj \colon B\,\mathrm{aut}_{\mathcal{H}} F \to B\,\mathrm{aut}\, F$ is equivalent to a quasi-fibration with fiber $B(\mathrm{aut}\, F, \mathrm{aut}_{\mathcal{H}} F, *)$. Since $j$ is an inclusion of some connected components of $\mathrm{aut}\, F$, $\pi_n(j) : \pi_n(\mathrm{aut}_{\mathcal{H}} F) \to \pi_n(\mathrm{aut}\, F)$ is an isomorphism for $n \geq 1$ and injective for $n = 0$; then $\pi_n(Bj)$ is an isomorphism for $n \geq 2$ and injective for $n = 1$. A long exact sequence argument shows that $B(\mathrm{aut}\, F, \mathrm{aut}_{\mathcal{H}} F, *)$ is weakly equivalent to the discrete space $\mathcal{E}(F)/\mathcal{H}$. In [9, Definition 8.6] this space is defined as $\mathrm{aut}\, F / \mathrm{aut}_{\mathcal{H}} F$, and as this notation suggests, its homotopy groups are the quotients of the homotopy groups of the monoids. In particular, up to weak homotopy equivalences, $Bj$ is a covering map. Let's formalize this intuition.

Using the CW approximation theorem, we can find CW-complexes $Y, Z$ and weak homotopy equivalences $\kappa \colon Y \to B\,\mathrm{aut}_{\mathcal{H}} F$ and $\lambda \colon Z \to B\,\mathrm{aut}\, F$ such that

$$\begin{array}{ccc}
Y & \xrightarrow{\kappa} & B\,\mathrm{aut}_{\mathcal{H}} F \\
\downarrow {\scriptstyle \xi} & & \downarrow {\scriptstyle Bj} \\
Z & \xrightarrow{\lambda} & B\,\mathrm{aut}\, F
\end{array}$$

commutes up to homotopy, $\xi$ is a covering map and $\kappa$ and $\lambda$ are weak homotopy equivalences. The associated subgroup of the covering map is given by the image of $\mathcal{H} \subseteq \mathcal{E}(F)$ under the isomorphisms $\mathcal{E}(F) \cong \pi_1(B\,\mathrm{aut}\, F)$ and $\pi_1(\lambda) \colon \pi_1(Z) \xrightarrow{\cong} \pi_1(B\,\mathrm{aut}\, F)$. Therefore we have a commutative diagram

$$\begin{array}{ccc}
[B, Z] & \xrightarrow[\cong]{\lambda_*} & [B, B\,\mathrm{aut}\, F] \\
{\scriptstyle \xi_*} \uparrow & & {\scriptstyle (Bj)_*} \uparrow \\
[B, Y] & \xrightarrow[\cong]{\kappa_*} & [B, B\,\mathrm{aut}_{\mathcal{H}} F]
\end{array} \tag{2}$$

in which, by the lifting property of covering maps, $\xi_* \colon [B, Y] \to [B, Z]$ is injective, so is $(Bj)_* : [X, B\,\mathrm{aut}_{\mathcal{H}} F] \to [X, B\,\mathrm{aut}\, F]$.

*(iv)* The map $\Lambda_{\mathcal{H}}$ is surjective:

We start with an $\mathcal{H}$-fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$; via the inverse of $\Lambda$ we get a map $f \colon B \to B\,\mathrm{aut}\, F$ which, with the notation in (2), produces another map $f' \colon B \to Z$ with $\lambda \circ f' \simeq f$. It is then enough to show that $\mathrm{Im}\, \pi_1(f) \subset \mathcal{H}$. In this case, by the lifting property of covering maps, there exists $\tilde{f} \colon B \to Y$ such that $\xi \circ \tilde{f} = f'$. Therefore, $(Bj) \circ \kappa \circ \tilde{f} \simeq f$ and $\Lambda_{\mathcal{H}}[\kappa \circ \tilde{f}]$ is the original $\mathcal{H}$-fibration sequence.

To finish, we show that in fact, $\operatorname{Im}\pi_1(f) \subset \mathcal{H}$. To give an explicit description of $f$, recall from [9, p.49] the existence of a commutative diagram of fibrations with fiber $\operatorname{aut} F$

$$\begin{array}{ccccc} PE & \longleftarrow & B(PE, \operatorname{aut} F, \operatorname{aut} F) & \longrightarrow & B(*, \operatorname{aut} F, \operatorname{aut} F) \\ \downarrow{\scriptstyle P\pi} & & \downarrow & & \downarrow \\ B & \underset{\varphi}{\overset{\longleftarrow}{\dashrightarrow}} & B(PE, \operatorname{aut} F, *) & \xrightarrow{\quad q \quad} & B\operatorname{aut} F \end{array} \tag{3}$$

where:

- The space $PE$ is the space of maps $\psi\colon F \to E$ such that $\psi(F) \subset \pi^{-1}(b)$ for some $b \in B$, and $P\pi$ is defined by $P\pi(\psi) = \pi(\psi(F))$, see [9, Definition 4.3] for more details.
- $B(*, \operatorname{aut} F, \operatorname{aut} F)$ is contractible and all the maps in the right square are induced by the functor $B(-,-,-)$.
- All the arrows pointing left are weak homotopy equivalences, so, since $B$ is of the homotopy type of a CW-complex, there exists $\varphi$, a right homotopy inverse.

Then, $\Lambda^{-1}[\pi] = [f]$ with $f = q \circ \varphi$. To check that applying the fundamental group this map is sent into $\mathcal{H}$, we first choose a basepoint $B(PE, \operatorname{aut} F, \operatorname{aut} F)$ which, in turn, determines basepoints in any of the spaces in (3). Among the space $PE \times \operatorname{aut} F$ of 0-simplices of $B(PE, \operatorname{aut} F, \operatorname{aut} F)$ we fix $(\omega, \mathrm{id}_F)$ with $\omega\colon F \overset{\simeq_w}{\to} F_0 = \pi^{-1}(b_0)$ the weak homotopy equivalence given in the data of the fibration sequence. With this choice, the fiber of $P\pi$ is the space $\mathcal{F}$ of all weak homotopy equivalences $F \overset{\simeq_w}{\to} F_0$.

With this choice of basepoints, the long homotopy exact sequences associated to the $\operatorname{aut} F$-fibrations in (3) yield a commutative diagram,

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ \pi_1(PE) & \longleftarrow & \pi_1 B(PE, \operatorname{aut} F, \operatorname{aut} F) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \pi_1(P\pi)} & & \downarrow & & \downarrow \\ \pi_1(B) & \xleftarrow{\cong} & \pi_1(B(PE, \operatorname{aut} F, *)) & \xrightarrow{\pi_1(q)} & \pi_1(B\operatorname{aut} F) \\ \downarrow{\scriptstyle \delta} & & \downarrow & & \downarrow{\scriptstyle \cong} \\ \pi_0(\mathcal{F}) & \xleftarrow{\quad \omega_* \quad} & \mathcal{E}(F) & \xrightarrow{\quad \mathrm{id} \quad} & \mathcal{E}(F) \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \end{array}$$

Hence, $\operatorname{Im}\pi_1(f) = \operatorname{Im}\pi_1(q \circ \varphi) \in \mathcal{H}$ if and only if $\operatorname{Im}(w_*^{-1} \circ \delta) \in \mathcal{H}$. However, an easy inspection shows that for any $\beta \in \pi_1(B)$ the map $\delta$ is such that $\delta(\beta) = \bar{\beta} \circ \omega \simeq \omega \circ \hat{\beta}$. Since we have started with an $\mathcal{H}$-fibration sequence, then $\hat{\beta} \in \mathcal{H}$, so $\omega_*^{-1} \circ \delta$ sends a path $\beta$ to an element in $\mathcal{H}$, which concludes the proof. $\square$

**Remark 2.2.** As in the ordinary case, a more convenient expression of the universal $\mathcal{H}$-quasi-fibration sequence can be given. Fix a base point $x_0 \in F_0$ whose inclusion is a cofibration and consider the *evaluation fibration*

$$\operatorname{aut}^*_{\mathcal{H}} F \hookrightarrow \operatorname{aut}_{\mathcal{H}} F \xrightarrow{\mathrm{ev}} F$$

where $\operatorname{aut}^*_{\mathcal{H}} F$ is the submonoid of $\operatorname{aut}_{\mathcal{H}} F$ of self homotopy equivalences which fix the base point and $\operatorname{ev}(g) = g(x_0)$. This yields a weak homotopy equivalence $F \simeq_w \operatorname{aut}_{\mathcal{H}} F / \operatorname{aut}^*_{\mathcal{H}} F = B(\operatorname{aut}_{\mathcal{H}}, \operatorname{aut}^*_{\mathcal{H}}, *)$ which, in turn, produces equivalent fibrations with fiber $F$,

$$\Gamma B(*, \operatorname{aut}_{\mathcal{H}} F, F) \to B \operatorname{aut}_{\mathcal{H}} F \quad \text{and} \quad \Gamma B(*, \operatorname{aut}_{\mathcal{H}} F, \operatorname{aut}_{\mathcal{H}} F / \operatorname{aut}^*_{\mathcal{H}} F) \to B \operatorname{aut}_{\mathcal{H}} F.$$

On the other hand, by [9, Remark 8.9], the quasi-fibrations

$$B(*, \operatorname{aut}_{\mathcal{H}} F, \operatorname{aut}_{\mathcal{H}} F / \operatorname{aut}^*_{\mathcal{H}} F) \to B \operatorname{aut}_{\mathcal{H}} F \quad \text{and} \quad B \operatorname{aut}^*_{\mathcal{H}} F \to B \operatorname{aut}_{\mathcal{H}} F$$

are equivalent. That is, the fibration sequences,

$$F \to B \operatorname{aut}^*_{\mathcal{H}} F \to B \operatorname{aut}_{\mathcal{H}} F \quad \text{and} \quad F \to B(*, \operatorname{aut}_{\mathcal{H}} F, F) \to B \operatorname{aut}_{\mathcal{H}} F$$

are equivalent.

Therefore, Theorem 2.1 can be reformulated in the following way:

$$F \to B \operatorname{aut}^*_{\mathcal{H}} F \to B \operatorname{aut}_{\mathcal{H}} F$$

is the universal $\mathcal{H}$-quasi-fibration sequence which classifies $\mathcal{H}$-fibration sequences.

## 3. Based fibrations

The previous results can be adapted to the based case. Let's give a brief exposition of the concepts needed for a rigorous definition.

A fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ is *based* if $\pi$ has a section $\sigma\colon B \to E$ and the weak equivalence $\omega$ sends the basepoint of $F$ to $\sigma(b_0)$ (see [8]). Consider the 'whisker construction' (see [9, Addenda]),

$$\widetilde{\pi}\colon \widetilde{E} \longrightarrow B \quad \text{where} \quad \widetilde{E} = \frac{E \sqcup (B \times [0,1])}{\sigma(b) \sim (b,0), \forall b \in X}, \quad \widetilde{\pi}(y) = \pi(y), \quad \widetilde{\pi}(b,t) = b,$$

for $y \in E, b \in F$ and $t \in [0,1]$. This is a fibration with fiber $\widetilde{\pi}^{-1}(x_0) = \widetilde{F}_0 = F_0 \vee [0,1]$. Given $\beta \in \pi_1(B, b_0)$ the induced homotopy equivalence $\widetilde{\beta}\colon \widetilde{F}_0 \xrightarrow{\simeq} \widetilde{F}_0$ can be taken to send 1 to 1. Since the inclusion of the basepoint in $F$ is a cofibration, there is a pointed weak homotopy equivalence $\tilde{\omega} : F \to \tilde{F}_0$, sending $x_0$ to 1, which induces a bijection $\widetilde{\omega}_*\colon [F, F]^* \xrightarrow{\cong} [F, \tilde{F}_0]^*$ between pointed homotopy classes of base-preserving maps. This yields the pointed homotopy equivalence $\dot{\beta} = \widetilde{\omega}_*^{-1}[\widetilde{\beta} f]^* \in [F, F]^*$.

Let $\operatorname{aut}^* F$ be the topological monoid of pointed self homotopy equivalences of $F$ and denote by $\mathcal{E}^*(F) = \pi_0(\operatorname{aut}^* F)$. Fix a subgroup $\mathcal{H} \subseteq \mathcal{E}^*(F)$ and consider $\operatorname{aut}^*_{\mathcal{H}} F \subset \operatorname{aut}^* F$ the submonoid of pointed homotopy equivalences $\varphi$ such that their (pointed) homotopy classes $[\varphi]^*$ belong to $\mathcal{H}$.

**Definition 3.1.** A based fibration sequence $F \xrightarrow{\omega} E \xrightarrow{\pi} B$ is a *based $\mathcal{H}$-fibration sequence* if $\dot{\beta} \in \mathcal{H}$ for any $\beta \in \pi_1(B, b_0)$. Denote by $\mathcal{F}\mathrm{ib}^*_{\mathcal{H}}(B, F)$ the set of equivalence classes of based $\mathcal{H}$-fibrations sequences over $B$ with fiber $F$.

Again, pullbacks preserve based $\mathcal{H}$-fibration sequences and this definition is independent of the equivalence class of the given based fibration sequences

Consider the maps

$$p\colon B(*, \operatorname{aut}^* F, F) \to B(*, \operatorname{aut}^* F, *) = B \operatorname{aut}^* F$$

and

$$p_{\mathcal{H}}\colon B(*,\operatorname{aut}^*_{\mathcal{H}} F, F) \to B(*,\operatorname{aut}^*_{\mathcal{H}} F, *) = B\operatorname{aut}^*_{\mathcal{H}} F,$$

both endowed with the section induced by the inclusion of the basepoint in $F$, and let $\Lambda\colon [B, B\operatorname{aut}^* F] \xrightarrow{\cong} \mathcal{F}\mathrm{ib}^*(B,F)$ be the natural bijection given in [9, Theorem 9.2 (b)].

Then, by requiring all the maps and fibrations involved in the proof of Theorem 2.1 to be basepoint preserving and based respectively, we obtain:

**Theorem 3.2.** *The map,*

$$\Lambda_{\mathcal{H}}\colon [B, B\operatorname{aut}^*_{\mathcal{H}} F] \xrightarrow{\cong} \mathcal{F}\mathrm{ib}^*_{\mathcal{H}}(B,F), \quad \Lambda_{\mathcal{H}}[f] = f^*\widetilde{\Gamma p_{\mathcal{H}}},$$

*is a natural bijection which fits in the following commutative diagram*

$$\begin{array}{ccc} [B, B\operatorname{aut}^* F] & \xrightarrow[\cong]{\Lambda} & \mathcal{F}\mathrm{ib}^*(B,F) \\ \uparrow & & \uparrow \\ [B, B\operatorname{aut}^*_{\mathcal{H}} F] & \xrightarrow[\cong]{\Lambda_{\mathcal{H}}} & \mathcal{F}\mathrm{ib}^*_{\mathcal{H}}(B,F). \end{array}$$

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