



Inner Ideals of Real Simple Lie Algebras

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Abstract

A classification up to automorphism of the inner ideals of the real finite-dimensional simple Lie algebras is given, jointly with precise descriptions in the case of the exceptional Lie algebras.

Keywords Inner ideals · Extremal elements · Exceptional Lie algebra · Simple Lie algebra · Structurable algebras · Models · Incidence geometries

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Dedicated to the memory of Professor Georgia Benkart

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1 Introduction

Inner ideals in Lie algebras were introduced in [24], where Faulkner defines inner ideals of modules for Lie algebras. This notion came from the world of Jordan algebras. Faulkner's aim was not to develop a general theory of inner ideals, but he used inner ideals to reconstruct in some way the geometry: a hexagonal geometry with certain collineations is coordinatized by a Jordan division algebra which permits to construct the Lie algebra. We come back later to this idea.

Georgia Benkart in her doctoral thesis [4] began a systematic research of the role of the inner ideals, as well as of the ad-nilpotent elements, in the study of Lie algebras. This topic was suggested to her by Seligman, taking into account the relevance for the theory of Jordan algebras of studying those ones satisfying the minimum condition on Jordan inner ideals, obviously inspired by associative algebras satisfying the descending chain condition on their left or right ideals. Her following papers [5, 6] laid the groundwork for the development of an Artinian theory for Lie algebras.

A considerable amount of material of inner ideals of Lie algebras, including the results in [6], can be found in the recent monograph by the AMS [25], which tries to show how Jordan theory can be applied to the study of Lie algebras not necessarily of finite dimension. Precisely the book contains a whole chapter about an Artinian theory for Lie algebras [25, Chapter 12]. After introducing the notion of a complemented inner ideal, the following key result is stated: every abelian inner ideal B of finite length of a non-degenerate Lie algebra L yields a finite \mathbb{Z} -grading $L = L_{-n} \oplus \cdots \oplus L_n$ such that $B = L_n$, so that in particular B is a complemented inner ideal. In fact, a Lie algebra is complemented (meaning that every inner ideal is so) if and only if it is a direct sum of simple non-degenerate Artinian Lie algebras. Note that the algebras considered here not only have arbitrary dimension, they can even be considered over a ring of scalars with very few restrictions.

It is well known the rich interplay of inner ideals with ad-nilpotent elements. Using inner ideals, some generalizations of Kostrikin's lemma have been achieved in [27], which finds ad-nilpotent elements of index 3 from ad-nilpotent elements of greater index. These kind of elements play a fundamental role in the classification of simple modular Lie algebras. They have strong implications on the structure of the algebra. For instance, for algebraically closed fields (of characteristic greater than 5) the existence of an ad-nilpotent element implies that the finite-dimensional non-degenerate simple Lie algebra is necessarily classical [6]. Thus, we have a criterion for distinguishing some concrete Lie algebras from other ones using inner ideals or ad-nilpotent elements. Similarly, [8] obtained a new characterization of infinite locally finite simple diagonal Lie algebras (characteristic zero) in terms of inner ideals.

Apart from this algebraic line of study of inner ideals, from which we have only included a small sample of the contributions, there is a second line of study, focused on the more geometric Faulkner's approach, which follows being topical. It is based on incidence geometries. The story begins with Tits, who introduced buildings in [36] in order to study algebraic groups. Some point-line spaces were associated to these buildings, called root shadows [10, Definition 3]. Many of these ones coincide with the following point-line spaces associated to Lie algebras: the so-called *extremal geometry* of a Lie algebra L has as points the points in the projective space $P(L)$

spanned by extremal elements in L , and as lines the lines in $P(L)$ such that all their points are spanned by extremal elements. In the case that L is finite-dimensional, simple and generated (as an algebra) by pure extremal elements, if the set of lines is non-empty, then the extremal geometry is a root shadow space (for instance see [11]). Very recently, [13] extends Faulkner's results, showing that the correspondence between inner ideals of the Lie algebra of a simple algebraic group and shadows on the set of long root groups of the building associated with the algebraic group holds for fields of characteristic different from two too.

This raises the question whether one can recover the Lie algebra from its extremal geometry. Is the algebra characterized by its geometry? A work in this direction is [9], which proves that the simple Lie algebra is uniquely determined (up to isomorphism) when the extremal geometry is the root shadow space of a spherical building of rank at least 3. (This requires that the algebra is generated by its set of extremal elements, but, for characteristic distinct from 2 and from 3 and finite dimension, it is enough that it contains an extremal element that is not a sandwich [12]).

Other authors try to focus on direct connections between Lie algebras and buildings, without the intermediate step of considering algebraic groups, see for instance, [14, 15]. The idea is to use structurable algebras, a class of non-associative algebras with involution generalizing Jordan algebras, introduced in [2], which is usually employed to construct 5-graded Lie algebras via the Tits–Kantor–Koecher construction. So, Moufang polygons are constructed in [15] using inner ideals of Lie algebras obtained from structurable algebras via such construction. The considered geometry in [14] is not the extremal one but a generalization called the *inner ideal geometry* of a Lie algebra, where the points are the 1-dimensional inner ideals but the lines are the minimal proper inner ideals containing at least two points. It turns out to be a Moufang spherical building of rank at least 2, or a Moufang set in case there are no lines.

The main purpose in this paper is to deal with the case of the real algebras. Note that a great amount of the mentioned works aim to avoid restrictions on the characteristic of the field. So, in a sense the real case is one of the easiest cases, after the complex one. Concretely, real simple finite-dimensional Lie algebras are completely classified and they are very well known. But on the other hand, they are very important cases, that are worth a specific study. We cannot forget their fundamental role in Differential Geometry and in Physics, which not even needs references. We concentrate on two objectives. First, in this work, on the classification up to isomorphism of the inner ideals of the real simple finite-dimensional Lie algebras, which is achieved in Theorem 3.4, and is followed by a case by case description in Sect. 4. Second, in a forthcoming paper, what are we able to say about the incidence geometries related to these inner ideals? To study the real case will allow us to gain intuition about the inner ideal geometries, often difficult to visualize. Besides, our study will provide some concrete realizations in Corollary 5.7 of the exceptional real Lie algebras obtained from structurable algebras related to composition algebras, joint with other realizations focused on finding point line spaces in Sect. 5.4.

Our first tool to address the classification is our knowledge on the classification of inner ideals in the complex case. Both cases, real and complex, are obviously related, since an inner ideal of a real Lie algebra will be, after complexification, an inner ideal of the complexified algebra. Then, we use the complete description of the abelian inner

ideals of any simple Lie algebra L over \mathbb{C} obtained in [18] (of course it can also be consulted in [25], which collects many results). Namely, take H a Cartan subalgebra of L , and fix $\Delta = \{\alpha_1, \dots, \alpha_l\}$ a basis of simple roots of the root system Φ relative to H . Denote by $\tilde{\alpha} = \sum_{i=1}^l m_i \alpha_i$ the maximal root in Φ . Then, for any nonzero abelian inner ideal B of L , there is an automorphism $\varphi \in \text{Aut}(L)$ and a subset $I \subseteq \{1, \dots, l\}$ such that $\varphi(B) = B_I$, where,

$$B_I := \bigoplus_{\alpha \in \Phi} \{L_\alpha : \alpha = \sum_{1 \leq i \leq l} p_i \alpha_i \text{ with } p_j = m_j \text{ for all } j \in I\}.$$

In order to relate the inner ideals of the real Lie algebras with the inner ideals of the complex ones, we will use the Satake diagrams of the corresponding real forms. We will obtain a result similar to that of the complex case, that is, that every abelian inner ideal is conjugated to some B_I , but only for I a subset of indices *adapted to the Satake diagram* (Definition 3.1). In some sense, it can be considered as a generalization of [16], which proves that there is a bijection between isomorphism classes of real semisimple \mathbb{Z} -graded Lie algebras and weighted (with certain restrictions) Satake diagrams.

The structure of the paper is as follows. Preliminaries are recalled in Sect. 2: the Satake diagrams of the simple real Lie algebras and the classification of the inner ideals of the (also simple) complex Lie algebras. The main result in the real case is Theorem 3.4 in Sect. 3. This theorem is easily applied in Sect. 4 to obtain a detailed classification of the inner ideals of the real Lie algebras up to automorphism. This turns to be a combinatorial description, so we have added Sect. 5, which tries to describe the inner ideals of the exceptional Lie algebras without using roots, but in terms of some constructions of these algebras. To be precise, we use the Tits–Kantor–Koecher construction applied to the tensor product of composition algebras and to Albert algebras, as well as some nice models of the split exceptional algebras based entirely on linear and multilinear algebra, which allow us to obtain some remarkable inner ideals formed completely by extremal elements. As a by-product, all the non-compact exceptional Lie algebras other than those of type G_2 are constructed by TKK-construction of the tensor product of real composition algebras in Corollary 5.7.

2 Preliminaries

All the algebras considered throughout this paper are finite-dimensional (over a field \mathbb{F} , mainly \mathbb{R} and \mathbb{C}).

2.1 Background on Satake Diagrams

We recall some very well-known facts on Satake diagrams in order to fix notation, mainly extracted from [28, Chapter III, §7] and from the summary in [19].

Let \mathfrak{g} be a simple Lie algebra over \mathbb{R} and $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$ its Killing form.

Take any maximal abelian subalgebra \mathfrak{a} such that $\kappa|_{\mathfrak{a} \times \mathfrak{a}}$ is positive definite. (So there exists a Cartan involution $\theta: \mathfrak{g} \rightarrow \mathfrak{g}$ such that $\theta|_{\mathfrak{a}} = -\text{id}_{\mathfrak{a}}$.) For each λ in the dual space \mathfrak{a}^* of \mathfrak{a} , let $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} : [h, x] = \lambda(h)x, \forall h \in \mathfrak{a}\}$. Then, λ is called a *restricted root* if $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq 0$. Denote by Σ the set of restricted roots, which is an abstract root system (not necessarily reduced), and by $m_\lambda = \dim \mathfrak{g}_\lambda$ the *multiplicity* of the restricted root. Note that the simultaneous diagonalization of $\text{ad}_{\mathfrak{g}} \mathfrak{a}$ gives the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus (\oplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda)$, for $\mathfrak{g}_0 = \mathfrak{a} \oplus \text{Cent}_{\mathfrak{t}}(\mathfrak{a})$, with $\mathfrak{t} = \text{Fix}(\theta)$.

Now take \mathfrak{h} any maximal abelian subalgebra of \mathfrak{g} containing \mathfrak{a} . Then, \mathfrak{h} is a Cartan subalgebra of \mathfrak{g} (that is, $\mathfrak{h}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$). Denote by Φ the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$ and by $\mathfrak{g}_\alpha^{\mathbb{C}}$ the one-dimensional root space for any $\alpha \in \Phi$. If $\alpha \in \Phi$, denote by $\bar{\alpha} := \alpha|_{\mathfrak{a}}: \mathfrak{a} \rightarrow \mathbb{R}$. The roots in $\Phi_0 = \{\alpha \in \Phi : \bar{\alpha} = 0\}$ are called the *compact* roots and those in $\Phi \setminus \Phi_0$ the *non-compact* roots. Note that $\alpha \in \Phi_0$ if and only if $\alpha(\mathfrak{h}) \subseteq i\mathbb{R}$. The restricted roots are exactly the nonzero restrictions of roots to $\mathfrak{a} \subseteq \mathfrak{h}^{\mathbb{C}}$, that is, $\Sigma = \{\bar{\alpha} : \alpha \in \Phi \setminus \Phi_0\}$. Moreover, for any $\lambda \in \Sigma$,

$$\mathfrak{g}_\lambda = (\oplus \{\mathfrak{g}_\alpha^{\mathbb{C}} : \bar{\alpha} = \lambda\}) \cap \mathfrak{g}, \tag{1}$$

where we understand \mathfrak{g} to be naturally contained in $\mathfrak{g}^{\mathbb{C}}$; and m_λ coincides with the number of roots $\alpha \in \Phi$ satisfying $\bar{\alpha} = \lambda$. Moreover, it is possible to choose a basis Δ of the root system Φ in such a way that $\Delta_0 = \Delta \cap \Phi_0 = \{\alpha \in \Delta : \bar{\alpha} = 0\}$ is a basis of Φ_0 , also a root system.

The Satake diagram of the real Lie algebra \mathfrak{g} is defined as follows. In the Dynkin diagram associated to such basis Δ , the roots in Δ_0 are denoted by a black circle \bullet and the roots in $\Delta \setminus \Delta_0$ are denoted by a white circle \circ . If $\alpha, \beta \in \Delta \setminus \Delta_0$ are such that $\bar{\alpha} = \bar{\beta}$, then α and β are joined by a curved arrow. The *real rank* of \mathfrak{g} is defined as $\dim \mathfrak{a}$, which coincides the number of white nodes in the Satake diagram minus the number of arrows.

Remark 2.1 If \mathfrak{g} is compact, it turns out that $\mathfrak{a} = 0$, so $\Phi_0 = \Phi$ and necessarily its Satake diagram is the Dynkin diagram (of $\mathfrak{g}^{\mathbb{C}}$) with all the nodes colored in black. If \mathfrak{g} is split, it turns out that \mathfrak{a} is a Cartan subalgebra ($\mathfrak{a} = \mathfrak{h}$), so $\Phi_0 = \emptyset$ and necessarily its Satake diagram is the Dynkin diagram (of $\mathfrak{g}^{\mathbb{C}}$) with all the nodes in white.

2.2 Satake Diagrams of the Simple Real Lie Algebras

We recall the classification in order to unify notation and labelings. Here, I_n denotes the identity matrix, $I_{p,q} := \text{diag}(I_p, -I_q)$ and $J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$.

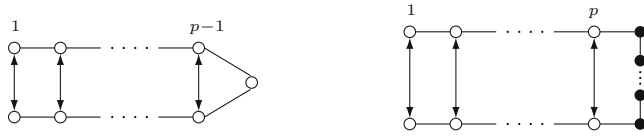
• *Special type.* The real forms of the special linear algebra $\mathfrak{sl}_{n+1}(\mathbb{C})$ of the traceless matrices are

- the split Lie algebra, $\mathfrak{sl}_{n+1}(\mathbb{R})$, of real rank n ;
- $\mathfrak{sl}_m(\mathbb{H}) = \{x \in \mathfrak{gl}_m(\mathbb{H}) : \text{Re}(\text{tr}(x)) = 0\}$, only for odd $n = 2m - 1 > 1$, with real rank $m - 1$ and Satake diagram



- $\mathfrak{su}_{p,q} = \{x \in \mathfrak{sl}_{n+1}(\mathbb{C}) : I_{p,q}x + \bar{x}^t I_{p,q} = 0\}$, with $p + q = n + 1, p \leq q$,

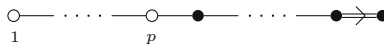
with real rank p and Satake diagram



if $p = q$ and $p < q$, respectively. Here, we assume $p \geq 1$, for $p = 0$ the algebra $\mathfrak{su}_{0,n+1} \equiv \mathfrak{su}_{n+1}$ is the compact one.

• *Orthogonal type.* The real forms of the orthogonal algebras $\mathfrak{so}_{2n+1}(\mathbb{C})$ and $\mathfrak{so}_{2n}(\mathbb{C})$ of skew-symmetric matrices are

- $\mathfrak{so}_{p,q}(\mathbb{R}) \equiv \mathfrak{so}_{p,q} = \{x \in \mathfrak{gl}_{p+q}(\mathbb{R}) : I_{p,q}x + x^t I_{p,q} = 0\}$, with $p \leq q$. If $p + q = 2n + 1$ (type B_n), the Satake diagram is:



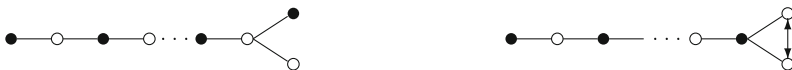
Here, there are $0 \leq p \leq n$ white nodes (p is the real rank). For $p = n$, we have the split real form, and if $p = 0$ the compact one.

If $p + q = 2n$ (type D_n), the Satake diagrams are



if $p \leq n - 2$ and $p = n - 1$, respectively, and again the real rank is p . For $p = n$, we have the split real form, and if $p = 0$, the compact one.

- $\mathfrak{so}_{2n}^*(\mathbb{R}) \equiv \mathfrak{u}_n^*(\mathbb{H}) = \{x \in \mathfrak{gl}_n(\mathbb{H}) : x^t h + h \bar{x} = 0\}$, where $h = \text{diag}(\mathbf{i}, \dots, \mathbf{i}) = \mathbf{i}I_n$. This case (type D_n) only happens when $n > 4$. The Satake diagrams are



if n is even and odd, respectively. (The real rank is the integer part of $n/2$.)

• *Symplectic type.* The real forms of $\mathfrak{sp}_{2n}(\mathbb{C})$ are

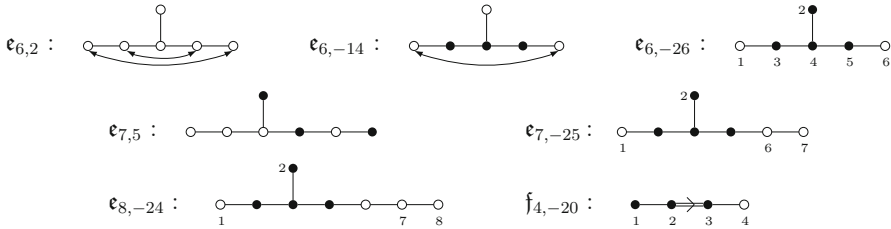
- The split Lie algebra $\mathfrak{sp}_{2n}(\mathbb{R}) = \{x \in \mathfrak{gl}_{2n}(\mathbb{R}) : J_n x + x^t J_n = 0\}$, of real rank n ;
- $\mathfrak{sp}_{p,q}(\mathbb{H}) \equiv \mathfrak{sp}_{p,q} = \{x \in \mathfrak{gl}_n(\mathbb{H}) : I_{p,q}x + \bar{x}^t I_{p,q} = 0\}$, for $p \leq q$ and $p + q = n$,

has real rank p and its Satake diagram is



if $1 \leq p < q$ or $p = q$, respectively ($p = 0$ is the compact one).

• *Exceptional type.* The real forms of the exceptional complex Lie algebras, apart from the split and compact cases, have the next Satake diagrams



Here, the second subindex, after the rank, indicates the signature of the Killing form. The real ranks are 4, 2 and 2, for the real forms of E_6 -type (respectively), 4 and 3, for the real forms of E_7 -type, $e_{8,-24}$ has real rank 4 and $f_{4,-20}$ equal to 1. These real ranks will be related with the maximal length of a chain of proper inner ideals as well as with the rank of the incidence geometries related to their inner ideals.

2.3 Inner Ideals of Complex Simple Lie Algebras

Usually the next definitions are considered over arbitrary fields, in spite that in this work we are mainly interested in the real field as an aim and in the complex field as a tool.

Definition 2.2 A nonzero element e of a Lie algebra L is called an *extremal element* if $[e, [e, L]] \subseteq \langle e \rangle = \mathbb{F}e$. Also, e is called an *absolute zero divisor* or a *sandwich element* if $[e, [e, L]] = 0$. An extremal element in L is called *pure* if it is not an absolute zero divisor. A Lie algebra is *non-degenerate* if it has no absolute zero divisors.

Example 2.3 The elements $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ in the special Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ are both pure extremal elements.

Definition 2.4 A vector subspace B of a Lie algebra L is called an *inner ideal* if $[B, [B, L]] \subseteq B$. An inner ideal B is called *proper* if B is neither $\{0\}$ nor L . Such B is called a *minimal inner ideal* if it does not contain properly any nonzero inner ideal.

It is clear, independently of the characteristic of the field, that for any $0 \neq e \in L$, then $\langle e \rangle$ is a minimal abelian inner ideal if and only if e is an extremal element. These extremal elements play a key role in the incidence geometries related to inner ideals.

It follows from [6, Lemma 1.13] and the fact that a simple Lie algebra over a field of characteristic 0 is non-degenerate [25, Corollary 3.24], that every proper inner ideal of a simple finite-dimensional Lie algebra over a field of characteristic 0 is abelian.

An important source of abelian inner ideals is the ad-nilpotent elements. On the one hand, any element e of an abelian inner ideal B is ad-nilpotent of index at most 3: $[e, [e, [e, L]]] \subseteq [e, B] \subseteq [B, B] = 0$ for each $0 \neq e \in B$. On the other hand, if e is an ad-nilpotent element of index at most 3, then $[e, [e, L]]$ is an inner ideal [6, Lemma 1.8]. We can go a little bit further and recall a well-known but key result.

Lemma 2.5 *Let B be a proper inner ideal of a simple (finite-dimensional) Lie algebra L over a field of zero characteristic. Every $0 \neq e \in B$ is ad-nilpotent of index equal to 3 and there exist f and h in L such that $\{e, h, f\}$ is a \mathfrak{sl}_2 -triple, that is,*

$$[h, e] = 2e, \quad [e, f] = h, \quad [h, f] = -2f;$$

and such that $\text{ad } h$ diagonalizes L with eigenvalues $\pm 2, \pm 1$ and 0 .

Proof As above, e is ad-nilpotent, since B is abelian. By the Jacobson–Morozov Lemma (for instance, consult the version in [31, Chapter III, §11, Theorem 17]), we can embed e in a \mathfrak{sl}_2 -triple of L . In particular the nilpotency index of e is precisely 3 since $[e, [e, f]] = -2e \neq 0$. Now, [29, Lemma 1] gives the result on the eigenvalues of $\text{ad } h$ and the corresponding diagonalization. \square

Clearly every extremal element belongs to a minimal abelian inner ideal, but the converse is not true.

Next we focus in the real case. If B is an inner ideal of a real Lie algebra \mathfrak{g} , then the complexification $B^{\mathbb{C}} := B \otimes_{\mathbb{R}} \mathbb{C}$ is an inner ideal of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$. So, the first step in order to classify inner ideals of the real Lie algebras is to know the classification of the inner ideals of the complex Lie algebras. This classification is well known for complex finite-dimensional simple Lie algebras. In such case, every abelian inner ideal coincides with the *corner* L_n of some \mathbb{Z} -grading $L = \bigoplus_{m=-n}^n L_m$ of the simple \mathbb{C} -algebra L . A very concrete description of the abelian inner ideals of L is obtained in [18] in terms of roots (and also related to Jordan pairs, see Sect. 5.5). Namely, take H a Cartan subalgebra of L , and fix $\Delta = \{\alpha_1, \dots, \alpha_l\}$ a basis of the root system Φ relative to H . Denote by $\tilde{\alpha} = \sum_{i=1}^l m_i \alpha_i$ the maximal root in Φ relative to the fixed ordering. Consider, for any subset of indices $I \subseteq \{1, \dots, l\}$, the set of roots

$$\Phi_I := \left\{ \alpha = \sum_{1 \leq i \leq l} p_i \alpha_i \in \Phi : p_j = m_j \text{ for all } j \in I \right\}, \quad (2)$$

and the corresponding sum of root subspaces

$$B_I := \bigoplus_{\alpha \in \Phi_I} L_{\alpha} = \bigoplus_{\alpha \in \Phi} \left\{ L_{\alpha} : \alpha = \sum_{1 \leq i \leq l} p_i \alpha_i \text{ with } p_j = m_j \text{ for all } j \in I \right\}. \quad (3)$$

It is easy to check that B_I is always an abelian inner ideal of L , but, as in [18, Theorem 3.1 or Theorem 4.4], the converse is also true: for any nonzero abelian inner ideal B of L , there is an automorphism $\varphi \in \text{Aut}(L)$ and a subset $I \subseteq \{1, \dots, l\}$ such that $\varphi(B) = B_I$.

Note that, if $I \subseteq J \subseteq \{1, \dots, l\}$, then $B_J \subseteq B_I$. In particular, the maximal abelian inner ideals of L are conjugate to $B_{\{i\}}$ for some $i \in \{1, \dots, l\}$. It is straightforward to describe these inner ideals by means of combinatorial arguments, as well as the lattice $\{B_I : I \subseteq \{1, \dots, l\}\}$ (for fixed H and Δ). The reader can find the related Hasse diagrams in [18].

3 Results on the Real Case

Assume we are still in the above setting: \mathfrak{g} is a real simple Lie algebra with complexification $\mathfrak{g}^{\mathbb{C}} = L$, $\Delta = \{\alpha_1, \dots, \alpha_l\}$ is a set of simple roots of the root system Φ relative

to a Cartan subalgebra H of L , and $\tilde{\alpha} = \sum_{i=1}^l m_i \alpha_i$ is the maximal root in Φ relative to the ordering given by Δ .

The following result provides us a source for finding abelian inner ideals of \mathfrak{g} whose complexification is just B_I for a suitable choice of the indices in I .

Definition 3.1 We will say that a non-empty set $I \subseteq \{1, \dots, l\}$ is *adapted to the Satake diagram* of \mathfrak{g} if it satisfies

- (i) If $i \in I$, the related node in the Satake diagram of \mathfrak{g} is white (that is, $\alpha_i|_{\mathfrak{a}} \neq 0$);
- (ii) If $i \in I, j \in \{1, \dots, l\}$ and $\alpha_i|_{\mathfrak{a}} = \alpha_j|_{\mathfrak{a}}$, then $j \in I$.

Proposition 3.2 For any $I \subseteq \{1, \dots, l\}$ adapted to the Satake diagram of \mathfrak{g} , we have

- (a) If $\alpha \in \Phi_I$, then $\alpha|_{\mathfrak{a}} \in \Sigma$;
- (b) If $\alpha \in \Phi_I$ and $\beta \in \Phi$ satisfies $\alpha|_{\mathfrak{a}} = \beta|_{\mathfrak{a}}$, then $\beta \in \Phi_I$;
- (c) $\mathcal{B}_I := \oplus\{\mathfrak{g}_{\tilde{\alpha}} : \alpha \in \Phi_I\}$ is an abelian inner ideal of \mathfrak{g} such that $(\mathcal{B}_I)^{\mathbb{C}} = B_I$.

Proof Recall that, if $r = \dim \mathfrak{a}$ (the real rank of \mathfrak{g}), the set $\{\tilde{\alpha}_i : \alpha_i \in \Delta \setminus \Delta_0\}$ is a set of r linearly independent elements in \mathfrak{a}^* . Note that

$$r = l - |\Delta_0| - \frac{1}{2}r_0,$$

with r_0 the number of nodes connected by an arrow in the Satake diagram. Indeed, this follows by taking into account that $\tilde{\alpha}_i = \tilde{\alpha}_j$ if α_i and α_j are connected by an arrow.

Part a) is clear, since if $\alpha = \sum_{i=1}^l p_i \alpha_i \in \Phi_I$, then the coefficient of $\tilde{\alpha}_j$ in $\tilde{\alpha} = \sum_{\alpha_i \notin \Delta_0} p_i \tilde{\alpha}_i$ for $j \in I$ is $p_j = m_j \neq 0$. For b), take roots $\alpha = \sum_{i=1}^l p_i \alpha_i, \beta = \sum_{i=1}^l q_i \alpha_i \in \Phi$ with $\tilde{\alpha} = \tilde{\beta}$ and $p_j = m_j$ for all $j \in I$, and let us check that also $q_j = m_j$ whenever $j \in I$. Fixed $j \in I$, then $\alpha_j \notin \Delta_0$ by i). If α_j is not joined by an arrow to any other node, then $0 = \sum_{i=1}^l (q_i - p_i) \tilde{\alpha}_i$ implies $q_j - p_j = 0$. And, if α_j is joined to α_k (which is necessarily unique, namely $k = \mu(j)$ for μ the involution of the Dynkin diagram), the linear independence of $\{\tilde{\alpha}_i : \alpha_i \in \Delta \setminus \Delta_0\}$ implies that $q_j - p_j + q_k - p_k = 0$. We also have $p_k = m_k$, since ii) yields $k \in I$. As $\tilde{\alpha}$ is the maximal root, $q_i \leq m_i$ for any index i and so $m_j + m_k = p_j + p_k = q_j + q_k \leq m_j + m_k$, forcing $q_j = m_j$. For part c), let us check that $\mathcal{B}_I = B_I \cap \mathfrak{g}$. On the one hand, if $\alpha \in \Phi_I$, then

$$\mathfrak{g}_{\tilde{\alpha}} \stackrel{(1)}{=} (\oplus\{\mathfrak{g}_{\tilde{\gamma}}^{\mathbb{C}} : \gamma \in \Phi, \tilde{\gamma} = \tilde{\alpha}\}) \cap \mathfrak{g} \stackrel{b)}{=} (\oplus\{\mathfrak{g}_{\tilde{\gamma}}^{\mathbb{C}} : \gamma \in \Phi_I, \tilde{\gamma} = \tilde{\alpha}\}) \cap \mathfrak{g} \subseteq B_I \cap \mathfrak{g}.$$

This means that $\mathcal{B}_I \subseteq B_I \cap \mathfrak{g}$. On the other hand, $(\mathfrak{g}^{\mathbb{C}})_{\tilde{\gamma}} \subseteq (\mathfrak{g}_{\tilde{\gamma}})^{\mathbb{C}}$ for any $\tilde{\gamma} \in \Phi$, so

$$B_I = \oplus\{(\mathfrak{g}^{\mathbb{C}})_{\tilde{\gamma}} : \tilde{\gamma} \in \Phi_I\} \subseteq \oplus\{(\mathfrak{g}_{\tilde{\gamma}})^{\mathbb{C}} : \tilde{\gamma} \in \Phi_I\} \subseteq (\mathcal{B}_I)^{\mathbb{C}} \subseteq (B_I \cap \mathfrak{g})^{\mathbb{C}} \subseteq B_I,$$

which implies that $(\mathcal{B}_I)^{\mathbb{C}} = (B_I \cap \mathfrak{g})^{\mathbb{C}}$ and \mathcal{B}_I is a subset of $B_I \cap \mathfrak{g}$ with the same dimension, so necessarily equal. Finally, $[\mathcal{B}_I, [\mathcal{B}_I, \mathfrak{g}]] \subseteq [B_I, [B_I, \mathfrak{g}]] \subseteq B_I$ and is also contained in \mathfrak{g} , so that it is contained in \mathcal{B}_I , showing that \mathcal{B}_I is an inner ideal. \square

Our objective is to prove that every abelian inner ideal of \mathfrak{g} is conjugated to some B_I as in Proposition 3.2. First, we adapt [18, Lemma 4.1] in order to prove that, given any abelian inner ideal B of \mathfrak{g} , there is a decomposition as above such that B is homogeneous and $B \cap \mathfrak{g}_0 = 0$, that is, $B = \bigoplus_{\lambda \in \Sigma} B \cap \mathfrak{g}_\lambda$. Moreover, it is possible to prove something slightly stronger, namely:

Lemma 3.3 *If B is an abelian inner ideal of a simple real Lie algebra \mathfrak{g} , there is a maximal abelian subalgebra \mathfrak{a} satisfying that $\kappa|_{\mathfrak{a} \times \mathfrak{a}}$ is a positive definite symmetric bilinear form, and that, if $\Gamma : \mathfrak{g} = \mathfrak{g}_0 \oplus (\bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda)$ is the decomposition in restricted root spaces $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} : [h, x] = \lambda(h)x \quad \forall h \in \mathfrak{a}\}$ as above, then B is homogeneous and $\mathfrak{g}_\lambda \subseteq B$ for each $\lambda \in \Sigma$ such that $\mathfrak{g}_\lambda \cap B \neq 0$. Besides, $\mathfrak{g}_0 \cap B = 0$.*

Proof Take $0 \neq e \in B$. By Lemma 2.5, we can find an \mathfrak{sl}_2 -triple $\{e, h, f\} \subseteq \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

is the diagonalization relative to $\text{ad } h$, i.e., $\mathfrak{g}_m = \{x \in \mathfrak{g} : [h, x] = mx\}$. Of course B is homogeneous for this grading, since $[h, B] = [[e, f], B] \subseteq [[e, B], f] + [e, [f, B]] = [e, [f, B]] \subseteq [B, [B, \mathfrak{g}]] \subseteq B$. Let us check that $\mathfrak{g}_2 \subseteq B$. Indeed, for any $x \in \mathfrak{g}_2$, $[h, x] = 2x$. As $[e, x] = 0$, the Jacobi identity tells that $[e, [f, x]] = 2x$. Now $[e, [f, [f, x]]] = [h, [f, x]] + [f, 2x] = [f, 2x]$ and

$$4x = [h, 2x] = [e, [f, 2x]] = [e, [e, [f, [f, x]]]] \in [e, [e, \mathfrak{g}_{-2}]] \subseteq [B, [B, \mathfrak{g}]] \subseteq B.$$

If we take a second element $e' \in \mathfrak{g}_2$, again there is an \mathfrak{sl}_2 -triple $\{e', h', f'\}$ in \mathfrak{g} . As $[e', [e', f']] = -2e'$, then [26, Proposition 5.2] (see also the proof of [18, Lemma 4.1]) says that we can replace f' with another element in \mathfrak{g}_{-2} . Thus, $h' = [e', f'] \in \mathfrak{g}_0$ commutes with h , and we can consider the simultaneous diagonalization. In this way, we can take a collection of elements $\{e_1, \dots, e_k\} \subseteq B \setminus \{0\}$ such that the related \mathfrak{sl}_2 -triples $\{e_i, h_i, f_i\}_{i=1}^k$ satisfy $[h_i, h_j] = 0$ for any i, j and the \mathbb{Z}^k -induced grading on \mathfrak{g} ,

$$\Gamma_0 : \mathfrak{g} = \bigoplus \mathfrak{g}_{(n_1, \dots, n_k)}, \quad \mathfrak{g}_{(n_1, \dots, n_k)} = \{x \in \mathfrak{g} : [h_i, x] = n_i x \quad \forall i\}$$

has a maximal number of homogeneous components. This grading satisfies

- (a) If $\mathfrak{g}_{(n_1, \dots, n_k)} \cap B \neq 0$, then $\mathfrak{g}_{(n_1, \dots, n_k)} \subseteq B$;
- (b) $\mathfrak{g}_{(0, \dots, 0)} \cap B = 0$;
- (c) the restriction of the Killing form $\kappa|_{(h_1, \dots, h_k)}$ is positive definite.

Indeed, if $0 \neq e_{k+1} \in \mathfrak{g}_{(n_1, \dots, n_k)} \cap B$, we can take as before an \mathfrak{sl}_2 -triple $\{e_{k+1}, h_{k+1}, f_{k+1}\}$ with $f_{k+1} \in \mathfrak{g}_{(-n_1, \dots, -n_k)}$. As the new grading is not a proper refinement of Γ_0 , $e_{k+1} \in \mathfrak{g}_{(n_1, \dots, n_k)} = \mathfrak{g}_{(n_1, \dots, n_k)} \cap \{x \in \mathfrak{g} : [h_{k+1}, x] = 2x\}$ which is contained in B as above (the eigenspace of $\text{ad } h_{k+1}$ of eigenvalue 2 is). Also, if $\mathfrak{g}_{(0, \dots, 0)} \subseteq B$, then $2e_1 = [h_1, e_1]$ would belong to $[B, B] = 0$, a contradiction. Finally, $h = \sum_{i=1}^k \omega_i h_i$ satisfies $\kappa(h, h) = \sum_{(n_1, \dots, n_k) \in \mathbb{Z}^k} (\sum \omega_i n_i)^2 \dim \mathfrak{g}_{(n_1, \dots, n_k)} \geq 0$. And if $\kappa(h, h) = 0$, then $[h, \mathfrak{g}] = 0$ and $h \in Z(\mathfrak{g}) = 0$.

Now we can take an abelian subalgebra \mathfrak{a} containing $\langle h_1, \dots, h_k \rangle$ maximal with the property that $\kappa|_{\mathfrak{a} \times \mathfrak{a}}$ is positive definite. Let Γ be the simultaneous diagonalization relative to $\{\text{ad } h : h \in \mathfrak{a}\}$, which is a refinement of Γ_0 , so that it still satisfies properties a) and b). (In particular this implies the homogeneity of B for the grading Γ .) \square

Now, we can prove the converse of Proposition 3.2 to describe, up to conjugation, all the abelian inner ideals of \mathfrak{g} .

Theorem 3.4 *If B is an abelian inner ideal of \mathfrak{g} , there is $\emptyset \neq I \subseteq \{1, \dots, l\}$ adapted to the Satake diagram of \mathfrak{g} such that B is conjugated to B_I .*

Proof First we apply Lemma 3.3 to find a maximal abelian subalgebra \mathfrak{a} with $\kappa|_{\mathfrak{a} \times \mathfrak{a}}$ positive definite and such that the restricted root space $\mathfrak{g}_\lambda = \{x \in \mathfrak{g} : [h, x] = \lambda(h)x \ \forall h \in \mathfrak{a}\}$ is contained in B for any $\lambda \in \Sigma$ such that $\mathfrak{g}_\lambda \cap B \neq 0$. Again consider \mathfrak{h} any maximal abelian subalgebra of \mathfrak{g} containing \mathfrak{a} . Let Φ denote the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ and let $(\mathfrak{g}^{\mathbb{C}})_\alpha$ denote the complex one-dimensional root space for any $\alpha \in \Phi$. The root space decomposition of $\mathfrak{g}^{\mathbb{C}}$ is a grading $\tilde{\Gamma}$ which is a refinement of $\Gamma^{\mathbb{C}}$, for $\Gamma : \mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\lambda \in \Sigma} \mathfrak{g}_\lambda$. Now note that $B^{\mathbb{C}}$ is an abelian inner ideal of $\mathfrak{g}^{\mathbb{C}}$ homogeneous for $\tilde{\Gamma}$. A slight adaptation of [18, Theorem 4.4] implies there are a basis $\Delta = \{\alpha_1, \dots, \alpha_l\}$ of Φ and a non-empty subset $I \subseteq \{1, \dots, l\}$ such that $B^{\mathbb{C}} = B_{\mathfrak{h}^{\mathbb{C}}, \Delta, I} = B_I$ as in (3). To be precise, [18, Theorem 4.4] provides the Cartan subalgebra, while we have started with the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$, but the proof in [18] only uses that $B^{\mathbb{C}}$ is homogeneous for $\tilde{\Gamma}$. Let us prove that

$$B = B_I = \oplus \{ \mathfrak{g}_{\bar{\alpha}} : \alpha \in \Phi_I \}.$$

Recall that $B = \oplus_{\lambda \in \Sigma} B \cap \mathfrak{g}_\lambda$, but $B \cap \mathfrak{g}_0 = 0$ so there is $\Sigma' \subseteq \Sigma$ such that $B = \oplus_{\lambda \in \Sigma'} \mathfrak{g}_\lambda$. Our aim is to check that $\Sigma' = \Sigma^I$, where $\Sigma^I := \{ \bar{\alpha} : \alpha \in \Phi_I \}$. First take $\lambda \in \Sigma'$ and $\beta \in \Phi$ such that $\bar{\beta} = \lambda$. Since $\mathfrak{g}_\lambda = (\oplus_{\alpha \in \Phi} \{ (\mathfrak{g}^{\mathbb{C}})_\alpha : \bar{\alpha} = \lambda \}) \cap \mathfrak{g} \subseteq B$, we have

$$(\mathfrak{g}^{\mathbb{C}})_\beta \subseteq (\mathfrak{g}_\lambda)^{\mathbb{C}} \subseteq B^{\mathbb{C}} = B_I = \oplus_{\alpha \in \Phi_I} (\mathfrak{g}^{\mathbb{C}})_\alpha.$$

But $\dim_{\mathbb{C}} (\mathfrak{g}^{\mathbb{C}})_\beta = 1$, so that there is $\alpha \in \Phi_I$ such that $\alpha = \beta$ and in fact $\beta \in \Phi_I$. Hence not only $\Sigma' \subseteq \Sigma^I$, but we have proved that

$$\text{if } \beta \in \Phi \text{ such that } \bar{\beta} \in \Sigma' \text{ then } \beta \in \Phi_I. \tag{4}$$

Second, take $\alpha \in \Phi_I$ and let us see that $\bar{\alpha} \in \Sigma'$. Again the one-dimensional space $(\mathfrak{g}^{\mathbb{C}})_\alpha$ is contained in $B_I = B^{\mathbb{C}} = \oplus_{\lambda \in \Sigma'} (\mathfrak{g}_\lambda)^{\mathbb{C}}$, so that there is $\lambda \in \Sigma' \subseteq \mathfrak{a}^*$ such that $(\mathfrak{g}^{\mathbb{C}})_\alpha \subseteq (\mathfrak{g}_\lambda)^{\mathbb{C}} = \oplus_{\beta \in \Phi} \{ (\mathfrak{g}^{\mathbb{C}})_\beta : \bar{\beta} = \lambda \}$. Then, $\bar{\alpha} = \lambda$, which belongs to Σ' . We have proved $\Sigma' = \Sigma^I$.

Note that the chosen subset I may not be the required one, because it is not necessarily adapted to the Satake diagram of \mathfrak{g} . But we can replace I with a convenient subset: Take $\tilde{I} \subseteq I$ minimal with the property that $\Phi_{\tilde{I}} = \Phi_I$. This can be done since $J \subseteq I$ implies $\Phi_J \subseteq \Phi_I$. The new \tilde{I} satisfies i) in Definition 3.1. Indeed, assume there

is $j \in \tilde{I}$ such that $\bar{\alpha}_j = 0$. By minimality, $\Phi_{\tilde{I} \setminus \{j\}} \neq \Phi_{\tilde{I}}$. Then take $\alpha = \sum_{i=1}^l p_i \alpha_i$ with $p_j \neq m_j$ but $p_i = m_i$ for any $i \in \tilde{I} \setminus \{j\}$. Taking into account the combinatorial properties of the roots, there is a path connecting α with the maximal root $\bar{\alpha} = \sum_{i=1}^l m_i \alpha_i$, that is, there are $\{i_1, \dots, i_s\} \subseteq \{1, \dots, l\}$ (not necessarily different indices) such that every element in the list

$$\alpha, \alpha + \alpha_{i_1}, \alpha + \alpha_{i_1} + \alpha_{i_2}, \dots, \alpha + \alpha_{i_1} + \dots + \alpha_{i_s} = \bar{\alpha} \tag{5}$$

is a root. Note that $j \in \{i_1, \dots, i_s\}$ and that every element in (5) belongs to $\Phi_{\tilde{I} \setminus \{j\}}$. Choose β the last element in the list (5) such that $\beta + \alpha_j$ belongs to (5) too. Hence, $\beta + \alpha_j \in \Phi_{\tilde{I}}$ and $\beta \notin \Phi_{\tilde{I}}$. This means that $\bar{\beta} + \bar{\alpha}_j \in \Sigma'$ but $\bar{\beta} \notin \Sigma'$ by (4), which is a contradiction since $\bar{\beta} + \bar{\alpha}_j = \bar{\beta} + \bar{\alpha}_j = \bar{\beta}$. This finishes the proof in case that the Satake diagram does not contain an arrow. Otherwise, let μ be the order 2 automorphism of the Dynkin diagram fixing the Satake diagram, that is, $\bar{\alpha}_i = \overline{\mu(\alpha_i)}$ for any $\alpha_i \in \Delta$. (This only happens for some real forms of type A, D or E_6 .) Think of μ as an automorphism of the set of indices $\{1, \dots, l\}$, i.e., $\alpha_{\mu(i)} = \mu(\alpha_i)$. Now replace \tilde{I} with $\tilde{I} \cup \mu(\tilde{I})$, which of course satisfies property ii) and it still satisfies i). There is no problem with this replacement since $\Sigma' = \Sigma^{\tilde{I}} = \Sigma^{\tilde{I} \cup \mu(\tilde{I})}$. Indeed, choose $j \in \tilde{I}$ (so that α_j corresponds to a white node) and let us see that $\Sigma^{\tilde{I}} = \Sigma^{\tilde{I} \cup \mu(j)}$. Take $\lambda \in \Sigma^{\tilde{I}}$. This means that $\lambda = \bar{\alpha}$ with $\alpha = \sum_{i=1}^l p_i \alpha_i \in \Phi$ such that $p_i = m_i$ for any $i \in \tilde{I}$ (in particular, $p_j = m_j$). As $\overline{\mu(\alpha)} = \bar{\alpha}$ (μ acts \mathbb{Z} -linearly in Φ and fixes the Satake diagram), also $\lambda = \overline{\mu(\alpha)}$ and $\mu(\alpha) \in \Phi_{\tilde{I}}$ by (4). In particular the coefficient of the root α_j in $\mu(\alpha)$ is m_j . But $\mu(\alpha) = p_1 \mu(\alpha_1) + \dots + p_l \mu(\alpha_l) = p_1 \alpha_{\mu(1)} + \dots + p_l \alpha_{\mu(l)}$ has as coefficient of the root $\alpha_j = \alpha_{\mu(\mu(j))}$ just $p_{\mu(j)}$, so that $p_{\mu(j)} = m_j = m_{\mu(j)}$ and then $\alpha \in \Phi_{\tilde{I} \cup \mu(j)}$. \square

Remark 3.5 Some comments are in order. First, note that $\{\bar{\alpha}_i : 1 \leq i \leq l\} \setminus \{0_{\mathfrak{a}^*}\}$ is a basis of the abstract root system Σ (of course taking out the elements appearing twice), because every element in Σ is the restriction of an integral linear combination of elements in Δ with all the coefficients having the same sign. The positive roots Σ^+ are the restrictions to \mathfrak{a} of the positive non-compact roots in $\Phi^+ \setminus \Phi_0$, and $\bar{\beta} := \bar{\alpha}|_{\mathfrak{a}}$ is the maximal root of Σ for this choice of basis. The only caution is that the basis of Σ obtained in this natural way is not ordered in the usual way. In any case, let us denote by $\beta_i = \bar{\alpha}_i$ for each $\alpha_i \notin \Delta_0$ (also $\beta_i = \overline{\mu(\alpha_i)}$ if the Satake diagram has arrows). So the set of simple roots of Σ is $\{\beta_i : i \in K\}$ for certain $K \subseteq \{1, \dots, l\}$ of cardinality $r = \dim \mathfrak{a}$. Then, the maximal root $\bar{\beta} = \sum_{i \in K} M_i \beta_i$ of Σ has coefficients $M_i = 2m_i$ if the Satake diagram has arrows and $i \neq \mu(i)$, and $M_i = m_i$ otherwise. Note now that the set related to \mathcal{B}_I defined by $\Sigma^I = \{\bar{\alpha} : \alpha \in \Phi_I\}$, coincides with $\Sigma_I = \{\sum_{i \in K} q_i \beta_i \in \Sigma : q_i = M_i \forall i \in I\}$ when the subset I is adapted to the Satake diagram of \mathfrak{g} , i.e., I satisfies i) and ii).

Both viewpoints are useful for describing the inner ideals of a concrete simple Lie algebra \mathfrak{g} . We will follow mainly that one in Theorem 3.4, which allows us to determine the dimensions of the inner ideals of \mathfrak{g} by counting roots in $\mathfrak{g}^{\mathbb{C}}$ or simply using the tables and lattices in [18]. This is quite easy since $\dim \mathcal{B}_I = |\Phi_I|$ (in general $\neq |\Sigma_I|$). The advantage of the second approach is that the root system Σ is

considerably smaller than Φ , and the corresponding sets Σ_I are equally listed (at least for the reduced root systems). But $\dim \mathcal{B}_I = \sum_{\lambda \in \Sigma_I} m_\lambda$, so that the knowledge of the restricted multiplicities is necessary. An example is shown in Remark 4.6.

4 Classification of Inner Ideals of Real Simple Lie Algebras

Through this section, we will assume we have fixed a Cartan subalgebra of a simple split (real or complex) Lie algebra and a set of simple roots of the related root system, labeled as in Sect. 2.2.

Recall that it is possible that $\mathcal{B}_I = \mathcal{B}_J$ happens for some $I \neq J$. We will say that a subset $I \subseteq \{1, \dots, l\}$ is *maximal* describing an inner ideal B if $B = \mathcal{B}_I$ and whenever there is another $J \subseteq \{1, \dots, l\}$ with $B = \mathcal{B}_J$ then $J \subseteq I$. Sometimes it is useful to have this maximal I since, only in that case we can assure that $\mathcal{B}_I \subseteq \mathcal{B}_J$ implies $J \subseteq I$ (recall that the converse was always true). We will use this to construct the lattices of inner ideals. Similarly, we will say that a subset $I \subseteq \{1, \dots, l\}$ is *minimal* describing an inner ideal B if $B = \mathcal{B}_I$ and whenever there is another $J \subseteq \{1, \dots, l\}$ with $B = \mathcal{B}_J$ then $J \supseteq I$. We can always find both a maximal set and a minimal set describing any proper inner ideal. The minimal choice is useful in order to apply Theorem 3.4, since we have only to check whether or not the minimal set representing a determined inner ideal is adapted to the Satake diagram.

Our description and classification of the inner ideals will be up to conjugation, although we will not explicitly recall this every time. So “the nonzero abelian inner ideals are” should be read as “up to conjugation, the nonzero abelian inner ideals are.”

4.1 Type A_l

Consider first the split Lie algebra, whose inner ideals are determined analogously to those ones of its complexified algebra. Recall that the set of positive roots is

$$\Phi^+ = \{\alpha_i + \alpha_{i+1} + \dots + \alpha_j : 1 \leq i \leq j \leq l\}$$

and the maximal root is $\tilde{\alpha} = \alpha_1 + \dots + \alpha_l$. The nonzero abelian inner ideals of $\mathfrak{sl}_{l+1}(\mathbb{R})$ are

$$\{\mathcal{B}_{\{s,t\}} : 1 \leq s \leq t \leq l\},$$

because $\mathcal{B}_I = \mathcal{B}_{\{\min I, \max I\}}$. Here we enclose the possibility $\mathcal{B}_{\{s,s\}} \equiv \mathcal{B}_{\{s\}}$. The corresponding roots are

$$\Phi_{\{s,t\}} = \{\alpha_i + \alpha_{i+1} + \dots + \alpha_s + \alpha_{s+1} + \dots + \alpha_t + \alpha_{t+1} + \dots + \alpha_k : 1 \leq i \leq s \leq t \leq k \leq l\},$$

so that $\mathcal{B}_{\{s,t\}}$ has dimension $s(l + 1 - t)$. Since it is conjugated to $\mathcal{B}_{\{l+1-t, l+1-s\}}$, the set of nonzero abelian inner ideals up to conjugation coincides with $\{\mathcal{B}_{\{s,t\}} : 1 \leq s \leq t \leq l, s + t \leq l + 1\}$. (This set has size k^2 if $l = 2k - 1$ and $k^2 + k$ if $l = 2k$.) The

lattice is described by taking into account that

$$\mathcal{B}_{\{s,t\}} \subseteq \mathcal{B}_{\{s',t'\}} \Leftrightarrow s \leq s', t \geq t'.$$

The only minimal abelian inner ideal (unique if we assume it adapted to the Cartan subalgebra and to the fixed ordering) is $\mathcal{B}_{\{1,l\}} = \mathfrak{g}_{\tilde{\alpha}}$, while the maximal ones are $\mathcal{B}_{\{k\}}$, with $k \in \{1, \dots, l\}$ (although $\mathcal{B}_{\{k\}} \cong \mathcal{B}_{\{l+1-k\}}$). For the other non-compact real forms of $\mathfrak{sl}_{l+1}(\mathbb{C})$, we use the results in the previous section. Thus,

Proposition 4.1 • *There are p nonzero abelian inner ideals of $\mathfrak{su}_{p,l+1-p}$, namely*

$$\begin{cases} \mathcal{B}_{\{1,l\}} \subset \mathcal{B}_{\{2,l-1\}} \subset \dots \subset \mathcal{B}_{\{p,l+1-p\}} & \text{if } 2p \leq l, \\ \mathcal{B}_{\{1,l\}} \subset \mathcal{B}_{\{2,l-1\}} \subset \dots \subset \mathcal{B}_{\{p-1,p+1\}} \subset \mathcal{B}_{\{p\}} & \text{if } 2p = l + 1. \end{cases}$$

• *If l is odd, the nonzero abelian inner ideals of $\mathfrak{sl}_m(\mathbb{H})$ for $m = \frac{l+1}{2}$ are:*

$$\{\mathcal{B}_{\{2s,2t\}} : 1 \leq s \leq t \leq m - 1\}.$$

Proof Simply take into account that the Satake diagram of $\mathfrak{su}_{p,l+1-p}$ ($p \geq 1$) has white nodes $\{1, \dots, p\} \cup \{l + 1 - p, \dots, l\}$ and arrows connecting the i th node with the $(l + 1 - i)$ th node for all $i \leq p$; while $\mathfrak{sl}_m(\mathbb{H})$ has white nodes $\{2, 4, \dots, l - 1\}$ and no arrows. (The compact form is $\mathfrak{su}_{0,l+1}$, without abelian inner ideals, corresponding to $p = 0$.) □

In particular, $\mathfrak{sl}_m(\mathbb{H})$ has no extremal points (more precisely, its minimal inner ideals have dimension 4) and the longest chain of proper inner ideals has length $m - 1$ (coinciding with the real rank of the algebra), for instance, $\mathcal{B}_{\{2,l-1\}} \subset \mathcal{B}_{\{4,l-1\}} \subset \dots \subset \mathcal{B}_{\{l-1,l-1\}}$. In the list of the abelian inner ideals of $\mathfrak{sl}_m(\mathbb{H})$ above, there are $\binom{m}{2}$ inner ideals, but again some of them are conjugated.

4.2 Type $B_l, l \geq 2$

There are $2l - 2$ nonzero abelian inner ideals of the split Lie algebra $\mathfrak{so}_{l,l+1}$ (as in the complex case). Recall that the sets of short and long positive roots are, respectively,

$$\begin{aligned} \Phi_c^+ &= \{\alpha_i + \dots + \alpha_l : 1 \leq i \leq l\}, \\ \Phi_l^+ &= \{\alpha_i + \dots + \alpha_{j-1}, \alpha_i + \dots + \alpha_{j-1} + 2\alpha_j + \dots + 2\alpha_l : 1 \leq i < j \leq l\}. \end{aligned}$$

The maximal root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 \dots + 2\alpha_l$. The abelian inner ideals are

$$\{\mathcal{B}_{\{k\}} : 1 \leq k \leq l\} \cup \{\mathcal{B}_{\{1,k\}} : 3 \leq k \leq l\}, \tag{6}$$

because $\mathcal{B}_I = \mathcal{B}_{\{2\}} = \mathfrak{g}_{\tilde{\alpha}}$ if $2 \in I$, $\mathcal{B}_I = \mathcal{B}_{\{\min I\}}$ if $1 \notin I$ and $\mathcal{B}_I = \mathcal{B}_{\{1, \min(I \setminus \{1\})\}}$ if $1 \in I$ and $2 \notin I$. These are precisely the minimal choices for representing subsets.

The dimensions are $\dim \mathcal{B}_{\{1,k\}} = k - 1$ if $k \neq 1$, $\dim \mathcal{B}_{\{1\}} = 2l - 1$ and $\dim \mathcal{B}_{\{k\}} = \binom{k}{2}$ if $k \geq 2$, because, for $k \neq 1$,

$$\begin{aligned} \Phi_{\{k\}} &= \{\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_l : 1 \leq i < j \leq k\} = \Phi_{\{k,k+1,\dots,l\}}, \\ \Phi_{\{1\}} &= \{\alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_l, \alpha_1 + \cdots + \alpha_j, \alpha_1 : 1 < j \leq l\}, \\ \Phi_{\{1,k\}} &= \{\alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_l : 1 < j \leq k\} = \Phi_{\{1,k,k+1,\dots,l\}}. \end{aligned}$$

The other non-compact real forms of $\mathfrak{so}_{2l+1}(\mathbb{C})$ have the following inner ideals.

Proposition 4.2 • *There is only one (up to conjugation) nonzero abelian inner ideal of $\mathfrak{so}_{1,2l}$, namely $\mathcal{B}_{\{1\}}$ (of dimension $2l - 1$). In particular it has no extremal elements.*

- *There are $2p - 2$ nonzero abelian inner ideals of $\mathfrak{so}_{p,2l+1-p}$ if $2 \leq p \leq l$, namely $\mathcal{B}_{\{2\}} \subset \mathcal{B}_{\{1\}}$ if $p = 2$, and*

$$\{\mathcal{B}_{\{k\}} : 1 \leq k \leq p\} \cup \{\mathcal{B}_{\{1,k\}} : 3 \leq k \leq p\}$$

if $p \geq 3$. Thus, a chain of maximal length of nonzero abelian inner ideals has length p (coinciding with the real rank of the algebra)

$$\mathcal{B}_{\{2\}} \subset \mathcal{B}_{\{1,3\}} \subset \mathcal{B}_{\{1,4\}} \subset \cdots \subset \mathcal{B}_{\{1,p\}} \subset \mathcal{B}_{\{1\}}.$$

Proof The Satake diagram of $\mathfrak{so}_{p,2l+1-p}$ has white nodes just $\{1, \dots, p\}$. □

4.3 Type $C_l, l \geq 3$

There are l nonzero abelian inner ideals of the split Lie algebra $\mathfrak{sp}_{2l}(\mathbb{R})$. Recall that the sets of short and long positive roots are, respectively,

$$\begin{aligned} \Phi_c^+ &= \{\alpha_i + \cdots + \alpha_{j-1}, \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-1} + \alpha_l : i < j \leq l\}, \\ \Phi_l^+ &= \{2\alpha_i + \cdots + 2\alpha_{l-1} + \alpha_l : 1 \leq i \leq l\}. \end{aligned}$$

(The notation here is slightly confusing, for instance the long root for $i = l$ is α_l .) The maximal root is $\tilde{\alpha} = 2\alpha_1 + \cdots + 2\alpha_{l-1} + \alpha_l$. It is clear that $\mathcal{B}_l = \mathcal{B}_{\{\min l\}}$, so that the abelian inner ideals are just $\{\mathcal{B}_{\{k\}} : 1 \leq k \leq l\}$, where $\mathcal{B}_{\{k\}}$ has dimension $\binom{k+1}{2}$, since

$$\begin{aligned} \Phi_{\{k\}} &= \{2\alpha_i + \cdots + 2\alpha_{l-1} + \alpha_l : 1 \leq i \leq k\} \cup \\ &\quad \{\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-1} + \alpha_l : 1 \leq i < j \leq k\}. \end{aligned}$$

Thus, we have a chain $\mathcal{B}_{\{1\}} = \mathfrak{g}_{\tilde{\alpha}} \subset \mathcal{B}_{\{2\}} \subset \cdots \subset \mathcal{B}_{\{l\}}$ of inner ideals of maximal length, l . In particular $\mathfrak{sp}_{2l}(\mathbb{R})$ has extremal elements (but no inner ideals of dimension 2). Let us apply the results in the above section to the remaining symplectic real algebras.

Proposition 4.3 *There are p nonzero abelian inner ideals of the Lie algebra $\mathfrak{sp}_{p,l-p}$ ($2p \leq l$), which in fact form a (maximal) chain, namely*

$$\mathcal{B}_{\{2\}} \subset \mathcal{B}_{\{4\}} \subset \cdots \subset \mathcal{B}_{\{2p\}}.$$

Proof The Satake diagram of $\mathfrak{sp}_{p,l-p}$ has white nodes $\{2, 4, \dots, 2p\}$ if $p \neq 0$ (none if $p = 0$). □

The word *maximal* here simply means of maximal length. In particular $\mathfrak{sp}_{p,l-p}$ does not possess extremal elements for any p , and $\mathfrak{sp}_{1,l-1}$ only has one nonzero abelian inner ideal up to conjugation.

4.4 Type $D_l, l \geq 4$

4.4.1 Type $D_l, l > 4$

There are $2l - 2$ nonzero abelian inner ideals of the split Lie algebra $\mathfrak{so}_{l,l}$, with $l > 4$. Recall that the set of positive roots is

$$\begin{aligned} \Phi^+ = & \{\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l : 1 \leq i < j \leq l-2\} \cup \{\alpha_l\} \\ & \cup \{\alpha_i + \cdots + \alpha_j : 1 \leq i \leq j \leq l-1\} \cup \{\alpha_i + \cdots + \alpha_l, \alpha_i + \cdots + \alpha_{l-2} + \alpha_l : 1 \leq i \leq l-2\} \end{aligned}$$

and the maximal root is $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$. A set of $2l - 2$ representatives of the conjugacy classes of the nonzero abelian inner ideals is

$$\{\mathcal{B}_{\{k\}}, \mathcal{B}_{\{1,k\}} : 3 \leq k \leq l-1\} \cup \{\mathcal{B}_{\{1\}}, \mathcal{B}_{\{2\}}, \mathcal{B}_{\{l-1,l\}}, \mathcal{B}_{\{1,l-1,l\}}\}. \tag{7}$$

To be precise, for $2 \leq k \leq l-2$,

- $\Phi_{\{k\}} = \Phi_{\{k,\dots,l\}} = \{\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l : 1 \leq i < j \leq k\}$ has $\binom{k}{2}$ elements, (coinciding with $\mathfrak{g}_{\tilde{\alpha}}$ for $k = 2$),
- $\Phi_{\{1,k\}} = \Phi_{\{1,k,\dots,l\}} = \{\alpha_1 + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l : 1 < j \leq k\}$ has $k - 1$ elements,
- $\Phi_{\{1\}} = \Phi_{\{1,l-2\}} \cup \{\alpha_1 + \cdots + \alpha_j : 1 \leq j \leq l\} \cup \{\alpha_1 + \cdots + \alpha_{l-2} + \alpha_l\}$ has $2l - 2$ elements and it corresponds with a maximal inner ideal,
- $\Phi_{\{1,l-1\}} = \Phi_{\{1,l-2\}} \cup \{\alpha_1 + \cdots + \alpha_{l-1}, \alpha_1 + \cdots + \alpha_l\}$ has $l - 1$ elements,
- $\Phi_{\{1,l-1,l\}} = \Phi_{\{1,l-2\}} \cup \{\alpha_1 + \cdots + \alpha_l\}$ has $l - 2$ elements,
- $\Phi_{\{l-1,l\}} = \Phi_{\{l-2\}} \cup \{\alpha_i + \cdots + \alpha_l : i \leq l-2\}$ has $\binom{l-1}{2}$ elements, and
- $\Phi_{\{l-1\}} = \Phi_{\{l-1,l\}} \cup \{\alpha_i + \cdots + \alpha_{l-1} : i \leq l-1\}$ has $\binom{l}{2}$ elements.

Here, $\mathcal{B}_{\{1\}}$ and $\mathcal{B}_{\{l-1\}} \cong \mathcal{B}_{\{l\}}$ are the maximal inner ideals, while $\mathcal{B}_{\{1,2\}} \cong \mathcal{B}_{\{2\}} = \mathfrak{g}_{\tilde{\alpha}}$ is the minimal one. Note that also $\mathcal{B}_{\{1,l-1\}} \cong \mathcal{B}_{\{1,l\}}$. The longest chain of proper inner ideals has length l . Note that, for $3 \leq k \leq k' \leq l-2$, we have $\mathcal{B}_{\{1,k\}} \subseteq \mathcal{B}_{\{1,k'\}}$, $\mathcal{B}_{\{1,k\}} \subseteq \mathcal{B}_{\{k'\}}$ and $\mathcal{B}_{\{k\}} \subseteq \mathcal{B}_{\{k'\}}$.

Proposition 4.4 • *There are $2p - 2$ nonzero abelian inner ideals of $\mathfrak{so}_{p,2l-p}$ if $2 \leq p \leq l$ and 1 if $p = 1$, namely*

- $\{\mathcal{B}_{\{1\}}\}$ if $p = 1$;
- $\{\mathcal{B}_{\{1\}}, \mathcal{B}_{\{2\}}\}$ if $p = 2$;
- $\{\mathcal{B}_{\{k\}} : 1 \leq k \leq p\} \cup \{\mathcal{B}_{\{1,k\}} : 3 \leq k \leq p\}$ if $3 \leq p \leq l - 2$;
- $\{\mathcal{B}_{\{k\}} : 1 \leq k \leq l - 2\} \cup \{\mathcal{B}_{\{1,k\}} : 3 \leq k \leq l - 2\} \cup \{\mathcal{B}_{\{l-1,l\}}, \mathcal{B}_{\{1,l-1,l\}}\}$ if $p = l - 1$.

(If $p = 0$ the algebra is compact and it does not contain any nonzero abelian inner ideal, and, if $p = l$ the algebra is the above considered split algebra).

- If l is even (respectively, odd), there are $\frac{l}{2}$ (respectively, $\frac{l-1}{2}$) nonzero abelian inner ideals of $\mathfrak{u}_l^*(\mathbb{H})$, namely:
 - $\mathcal{B}_{\{2\}} \subset \mathcal{B}_{\{4\}} \subset \dots \subset \mathcal{B}_{\{l-2\}} \subset \mathcal{B}_{\{l\}}$ if l is even;
 - $\mathcal{B}_{\{2\}} \subset \mathcal{B}_{\{4\}} \subset \dots \subset \mathcal{B}_{\{l-3\}} \subset \mathcal{B}_{\{l-1,l\}}$ if l is odd.

Proof The Satake diagram of $\mathfrak{so}_{p,2l-p}$ has

- white nodes $\{1, \dots, p\}$ and no arrows, if $1 \leq p \leq l - 2$,
- white nodes $\{1, \dots, l\}$ and an arrow connecting the $l - 1$ th node with the l th node, if $p = l - 1$;

while the Satake diagram of $\mathfrak{u}_l^*(\mathbb{H})$ has

- white nodes $\{2, 4, \dots, l\}$ and no arrows, if l is even;
- white nodes $\{2, 4, \dots, l - 1\} \cup \{l\}$ and an arrow connecting the $l - 1$ th node with the l th node, if l is odd.

□

In particular, neither $\mathfrak{u}_l^*(\mathbb{H})$ nor $\mathfrak{so}_{p,2l-p}$ has extremal elements.

4.4.2 Type D_4

As there are more automorphisms of the algebra of type D_4 , there are less inner ideals. To be precise, there are 4 inner ideals up to conjugation:

$$\mathcal{B}_{\{1,2,3,4\}} \subset \mathcal{B}_{\{1,3,4\}} \subset \mathcal{B}_{\{1,4\}} \subset \mathcal{B}_{\{1\}},$$

of dimensions 1, 2, 3 and 6, respectively. (The map $\alpha_1 \mapsto \alpha_3 \mapsto \alpha_4 \mapsto \alpha_1$ fixing the 2nd-node is a diagram automorphism.) The other non-compact real forms work as in Proposition 4.4, namely $\mathfrak{so}_{p,8-p}$ with a chain of maximal length p ($p = 0$ compact, and $p = 4$ split):

- Only $\mathcal{B}_{\{1\}}$ up to conjugation for $\mathfrak{so}_{1,7}$, an inner ideal of dimension 6;
- $\mathcal{B}_{\{1,2,3,4\}} \subset \mathcal{B}_{\{1\}}$ for $\mathfrak{so}_{2,6}$, inner ideals of dimensions 1 and 6;
- $\mathcal{B}_{\{1,2,3,4\}} \subset \mathcal{B}_{\{1,3,4\}} \subset \mathcal{B}_{\{1\}}$ for $\mathfrak{so}_{3,5}$, inner ideals of dimensions 1, 2 and 6.

4.5 Type E_6

The maximal root is $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$. There are 7 nonzero abelian inner ideals of the split algebra $\mathfrak{e}_{6,6}$, given by

$$\begin{aligned} \mathcal{B}_{\{1,2,3,4,5,6\}} &= \mathcal{B}_{\{2\}} = \mathfrak{g}_{\bar{\alpha}}, & \mathcal{B}_{\{1,3,4,5,6\}} &= \mathcal{B}_{\{4\}}, & \mathcal{B}_{\{1,3,5,6\}} &= \mathcal{B}_{\{3,5\}}, \\ \mathcal{B}_{\{1,3,6\}} &= \mathcal{B}_{\{3,6\}} \cong \mathcal{B}_{\{1,5\}}, & \mathcal{B}_{\{1,3\}} &= \mathcal{B}_{\{3\}} \cong \mathcal{B}_{\{5\}}, & \mathcal{B}_{\{1,6\}}, & \mathcal{B}_{\{1\}} \cong \mathcal{B}_{\{6\}}; \end{aligned} \quad (8)$$

with dimensions 1, 2, 3, 4, 5, 8 and 16, respectively. Here, we have to take care with the order 2 automorphism of the Dynkin diagram. We have chosen, for each inner ideal \mathcal{B} , the set $I \subseteq \{1, \dots, 6\}$ to the left (right, respectively) maximal (minimal, respectively) among the ones satisfying $\mathcal{B} = \mathcal{B}_I$. Recall that the choice of a maximal I helps us to find a chain of inner ideals of maximal length, while a minimal I gives us our main tool to know if a concrete real form possesses such inner ideal: it reduces to check if I is or is not adapted to the Satake diagram.

- Proposition 4.5** • *There are 4 nonzero abelian inner ideals of the Lie algebra $\mathfrak{e}_{6,2}$, of dimensions 1, 2, 3 and 8, namely $\mathcal{B}_{\{2\}} \subset \mathcal{B}_{\{4\}} \subset \mathcal{B}_{\{3,5\}} \subset \mathcal{B}_{\{1,6\}}$.*
- *There are 2 nonzero abelian inner ideals of the Lie algebra $\mathfrak{e}_{6,-14}$, of dimensions 1 and 8, namely $\mathcal{B}_{\{2\}} \subset \mathcal{B}_{\{1,6\}}$.*
 - *There are 2 nonzero abelian inner ideals of the Lie algebra $\mathfrak{e}_{6,-26}$, of dimensions 8 and 16, namely $\mathcal{B}_{\{1,6\}} \subset \mathcal{B}_{\{1\}}$.*

Proof The Satake diagram of the real form \mathfrak{g} has

- white nodes $\{1, 2, 3, 4, 5, 6\}$ and arrows connecting the nodes 3 and 5 and the nodes 1 and 6, if $\mathfrak{g} = \mathfrak{e}_{6,2}$;
- white nodes $\{1, 2, 6\}$ and arrows connecting the nodes 1 and 6, if $\mathfrak{g} = \mathfrak{e}_{6,-14}$;
- white nodes $\{1, 6\}$ and no arrows, if $\mathfrak{g} = \mathfrak{e}_{6,-26}$.

□

Hence, $\mathfrak{e}_{6,-26}$ has no pure extremal elements (there are no \mathbb{Z} -gradings with a corner of dimension 1).

Remark 4.6 According to Remark 3.5, there is an alternative method to arrive at the same results. Note that $\Phi \setminus \Phi_0$ is a root system of type F_4 , BC_2 and A_2 if the signature of the real form is 2, -14 or -26 , respectively. For instance, in the last case, the inner ideals of a split algebra of type A_2 are the root spaces $B^1 = \mathfrak{g}_{\alpha'_1}$ and $B^2 = \mathfrak{g}_{\alpha'_1} \oplus \mathfrak{g}_{\alpha'_1 + \alpha'_2}$, where $\alpha'_1 = \alpha_1|_{\mathfrak{a}}$ and $\alpha'_2 = \alpha_6|_{\mathfrak{a}}$. It is well known that the restricted multiplicities are $m_{\bar{\alpha}_1} = 8 = m_{\bar{\alpha}_6}$ [28] (see also [19], with explicit constructions of the Satake diagrams of these real forms), so that $\dim B^1 = 8$ and $\dim B^2 = 16$. Similarly, the structure of the inner ideals of the complex Lie algebras of type F_4 and B_2 tells us that for $\mathfrak{e}_{6,2}$ there will be 4 abelian inner ideals, and for $\mathfrak{e}_{6,-14}$ there will be only 2. In order to compute the explicit dimensions, we again need the restricted multiplicities.

4.6 Type E_7

The maximal root is $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$. There are 10 nonzero abelian inner ideals of the split algebra $\mathfrak{e}_{7,7}$, given by

$$\begin{aligned} \mathcal{B}_{\{1,2,3,4,5,6,7\}} &= \mathcal{B}_{\{1\}}, & \mathcal{B}_{\{2,3,4,5,6,7\}} &= \mathcal{B}_{\{3\}}, & \mathcal{B}_{\{2,4,5,6,7\}} &= \mathcal{B}_{\{4\}}, \\ \mathcal{B}_{\{2,5,6,7\}} &= \mathcal{B}_{\{2,5\}}, & \mathcal{B}_{\{2,6,7\}} &= \mathcal{B}_{\{2,6\}}, & \mathcal{B}_{\{5,6,7\}} &= \mathcal{B}_{\{5\}}, \\ & & \mathcal{B}_{\{2,7\}} &= \mathcal{B}_{\{2,7\}}, & \mathcal{B}_{\{6,7\}} &= \mathcal{B}_{\{6\}}, & \mathcal{B}_{\{2\}} &= \mathcal{B}_{\{7\}}; \end{aligned}$$

of dimensions 1, 2, 3, 4, 5, 5, 6, 10, 7 and 27, respectively.

- Proposition 4.7** • *There are 4 nonzero abelian inner ideals of the Lie algebra $\mathfrak{e}_{7,5}$, of dimensions 1, 2, 3 and 10, namely $\mathcal{B}_{\{1\}} \subset \mathcal{B}_{\{3\}} \subset \mathcal{B}_{\{4\}} \subset \mathcal{B}_{\{6\}}$.*
- *There are 3 nonzero abelian inner ideals of the Lie algebra $\mathfrak{e}_{7,-25}$, of dimensions 1, 10 and 27, namely $\mathcal{B}_{\{1\}} \subset \mathcal{B}_{\{6\}} \subset \mathcal{B}_{\{7\}}$.*

Proof The Satake diagram of L has white nodes $\{1, 3, 4, 6\}$ if $L = \mathfrak{e}_{7,5}$; and white nodes $\{1, 6, 7\}$ if $L = \mathfrak{e}_{7,-25}$. □

4.7 Type E_8

The maximal root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$. There are 10 nonzero abelian inner ideals of the split algebra $\mathfrak{e}_{8,8}$, given by

$$\begin{aligned} \mathcal{B}_{\{1,2,3,4,5,6,7,8\}} &= \mathcal{B}_{\{8\}}, & \mathcal{B}_{\{1,2,3,4,5,6,7\}} &= \mathcal{B}_{\{7\}}, & \mathcal{B}_{\{1,2,3,4,5,6\}} &= \mathcal{B}_{\{6\}}, \\ \mathcal{B}_{\{1,2,3,4,5\}} &= \mathcal{B}_{\{5\}}, & \mathcal{B}_{\{1,2,3,4\}} &= \mathcal{B}_{\{4\}}, & \mathcal{B}_{\{1,2,3\}} &= \mathcal{B}_{\{2,3\}}, \\ & & \mathcal{B}_{\{1,2\}} &= \mathcal{B}_{\{1,2\}}, & \mathcal{B}_{\{2\}} &= \mathcal{B}_{\{1\}}; \end{aligned}$$

with dimensions 1, 2, 3, 4, 5, 6, 7, 7, 8 and 14. Again we have chosen, for each inner ideal, the set I to the left (right, respectively) maximal (minimal, respectively) among the ones representing the inner ideal.

- Proposition 4.8** *There are 4 nonzero abelian inner ideals of the Lie algebra $\mathfrak{e}_{8,-24}$, of dimensions 1, 2, 3 and 14, namely $\mathcal{B}_{\{8\}} \subset \mathcal{B}_{\{7\}} \subset \mathcal{B}_{\{6\}} \subset \mathcal{B}_{\{1\}}$.*

Proof The Satake diagram of $\mathfrak{e}_{8,-24}$ has white nodes $\{1, 6, 7, 8\}$. □

4.8 Type F_4

The maximal root is $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4$. There are 4 nonzero abelian inner ideals of the split algebra $\mathfrak{f}_{4,4}$, given by

$$\begin{aligned} \mathcal{B}_{\{1,2,3,4\}} &= \mathcal{B}_{\{1\}}, & \mathcal{B}_{\{2,3,4\}} &= \mathcal{B}_{\{2\}}, \\ \mathcal{B}_{\{3,4\}} &= \mathcal{B}_{\{3\}}, & \mathcal{B}_{\{4\}} &= \mathcal{B}_{\{4\}}, \end{aligned}$$

of dimensions 1, 2, 3 and 7, respectively. (Again we use the same convention for the maximal and minimal subsets.)

Proposition 4.9 *There is only one nonzero abelian inner ideal of the Lie algebra $\mathfrak{f}_{4,-20}$, of dimension 7, namely $\mathcal{B}_{\{4\}}$. That is, all the proper inner ideals are conjugated.*

Proof The Satake diagram of $\mathfrak{f}_{4,-20}$ has only one white node, $\{4\}$. □

4.9 Type G_2

There are 2 nonzero abelian inner ideals of the split algebra $\mathfrak{g}_{2,2}$, given by $\mathcal{B}_{\{1\}}$ and $\mathcal{B}_{\{2\}}$, of dimensions 1 and 2, respectively. Of course, the compact algebra $\mathfrak{g}_{2,-14}$ has no nonzero abelian inner ideals.

5 More on Inner Ideals of Real Exceptional Lie Algebras

We consider that our combinatorial description of the inner ideals falls short of providing an insight on the essential nature of these objects. Thus, our next objective is, at least in the case of the exceptional Lie algebras, to give concrete descriptions and realizations of all these inner ideals.

5.1 Preliminaries on Structurable Algebras

It is well known that any Jordan algebra allows to define a 3-graded Lie algebra by means of the Tits–Koecher construction. Structurable algebras are a class of non-associative algebras with involution introduced in [1], containing the class of Jordan algebras, which also permits to construct a \mathbb{Z} -graded Lie algebra starting from any algebra in this class, in general with 5 pieces, by means of a generalized *Tits–Kantor–Koecher construction*. Moreover, [2, Theorem 10] also proves that every finite-dimensional central simple Lie algebra over a field of characteristic zero containing some nonzero ad-nilpotent element (for instance, with proper inner ideals) is obtained from a (central simple) structurable algebra by using this construction. This is the case of all the non-compact real (finite-dimensional) simple Lie algebras.

A recent work by De Medts and Meulewaeter [15] studies the set of inner ideals of the Lie algebras constructed from some structurable algebras. With these inner ideals, they construct Moufang sets, Moufang triangles and Moufang hexagons.

Definition 5.1 Let $(\mathcal{A}, -)$ be an algebra with involution (an order 2 anti-homomorphism). For each $x, y \in \mathcal{A}$, denote by $V_{x,y}: \mathcal{A} \rightarrow \mathcal{A}$ the linear operator given by

$$V_{x,y}(z) = (x\bar{y})z + (z\bar{y})x - (z\bar{x})y,$$

for any $z \in \mathcal{A}$. The algebra \mathcal{A} is called a *structurable algebra* if, for any $x, y, z, w \in \mathcal{A}$,

$$[V_{x,y}, V_{z,w}] = V_{V_{x,y}(z),w} - V_{z,V_{y,x}(w)}.$$

In particular $\text{Instr}(\mathcal{A}) := \{\sum V_{x_i,y_i} : x_i, y_i \in \mathcal{A}\} \equiv V_{\mathcal{A},\mathcal{A}}$ is a Lie algebra.

Example 5.2 Some examples, extracted from [1, §8], are:

- (i) If $(\mathcal{A}, -)$ is a unital associative algebra with involution, then it is structurable.
- (ii) If J is a unital Jordan algebra, that is, commutative and satisfying the Jordan identity $((x^2y)x = x^2(yx))$, then J is structurable for the involution given by the identity.
- (iii) If (C, n) is a unital composition algebra, that is, C is endowed with a non-degenerate multiplicative quadratic form $n : C \rightarrow \mathbb{F}$ (that is, $n(xy) = n(x)n(y)$), then C has an involution $-$ such that $n(x) = x\bar{x}$ and it turns to be a structurable algebra. Moreover, if C_1 and C_2 are two unital composition algebras, then the tensor product $C_1 \otimes C_2$ is structurable for the involution given by $\overline{x_1 \otimes x_2} = \overline{x_1} \otimes \overline{x_2}$.

Definition 5.3 If $(\mathcal{A}, -)$ is a structurable algebra, we consider the sets of *skew-symmetric* elements and of *Hermitian* elements defined, respectively, by

$$\mathcal{S} = \text{Skew}(\mathcal{A}, -) := \{s \in \mathcal{A} : \bar{s} = -s\}, \quad \mathcal{H} := \{h \in \mathcal{A} : \bar{h} = h\}.$$

Clearly $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$. The dimension of \mathcal{S} is called the *skew-dimension* of \mathcal{A} .

As mentioned, the interesting point on structurable algebras is that they provide a 5-graded Lie algebra by means of a construction which generalizes the Tits–Koecher construction for Jordan algebras. And conversely, the isotropic finite-dimensional simple Lie algebras over fields of characteristic zero can be constructed by the TKK-construction (abbreviation of Tits, Kantor and Koecher) applied to some structurable algebra:

Definition 5.4 ([2, §3]) Let $(\mathcal{A}, -)$ be a structurable algebra with $\mathcal{S} = \text{Skew}(\mathcal{A}, -)$ and consider the \mathbb{Z} -graded vector space $\mathcal{K}(\mathcal{A}) = \mathcal{K}_{-2} \oplus \mathcal{K}_{-1} \oplus \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$, for

$$\begin{aligned} \mathcal{K}_2 &:= \{(0, s) : s \in \mathcal{S}\}, & \mathcal{K}_1 &:= \{(a, 0) : a \in \mathcal{A}\}, & \mathcal{K}_0 &:= \text{Instr}(\mathcal{A}, -) = V_{\mathcal{A}, \mathcal{A}}, \\ \mathcal{K}_{-2} &:= \{(0, s)^\sim : s \in \mathcal{S}\}, & \mathcal{K}_{-1} &:= \{(a, 0)^\sim : a \in \mathcal{A}\}, \end{aligned}$$

where the tilde \sim simply denotes a copy of the pair. Then, $\mathcal{K}(\mathcal{A})$ is a (graded) Lie algebra for the product such that $\text{Instr}(\mathcal{A})$ is a Lie subalgebra and the following conditions hold:

- ★ $[(a, r), (b, s)] = (0, a\bar{b} - b\bar{a}),$
- ★ $[(a, r)^\sim, (b, s)^\sim] = (0, a\bar{b} - b\bar{a})^\sim,$
- ★ $[(a, r), (b, s)^\sim] = (-sa, 0)^\sim + V_{a,b} + L_r L_s + (rb, 0),$ (9)
- ★ $[T, (a, r)] = (Ta, T^\delta r),$
- ★ $[T, (a, r)^\sim] = (T^\epsilon a, (T^\epsilon)^\delta r)^\sim,$

for any $T \in \text{Instr}(\mathcal{A}, -)$, $a, b \in \mathcal{A}$, $r, s \in \mathcal{S}$, where $L_s, R_s : \mathcal{A} \rightarrow \mathcal{A}$ denote the left and right multiplication operators by $s \in \mathcal{S}$, $T^\epsilon := T - L_{T(1)+\overline{T(1)}}$ and $T^\delta := T + R_{\overline{T(1)}}$. This Lie algebra will be called the *Kantor construction* or *TKK-construction* attached to the structurable algebra \mathcal{A} .

In the complex case, every simple Lie algebra is obtained in this way, while in the real case every simple non-compact Lie algebra is obtained in this way.

5.2 Inner Ideals of $\mathcal{K}(\mathcal{A})$ for \mathcal{A} Tensor Product of Composition Algebras

It is clear that \mathcal{K}_2 is an inner ideal of $\mathcal{K}(\mathcal{A})$ for any structurable algebra \mathcal{A} . The purpose now is to collect information about the remaining inner ideals of the Lie algebras $\mathcal{K}(\mathcal{A})$ when \mathcal{A} is a tensor product of composition algebras. We are interested in them because all the exceptional complex Lie algebras not of type G_2 arise from the case iii) in Example 5.2: If C_1 and C_2 are two unital composition algebras over the complex numbers, with C_2 of dimension 8 (usually called a Cayley algebra), then $\mathcal{K}(C_1 \otimes C_2)$ is the exceptional algebra of type F_4 , E_6 , E_7 and E_8 according to the dimension of C_1 being 1, 2, 4 and 8, respectively. Hence, if (C_1, n_1) and (C_2, n_2) are two unital composition algebras over the real numbers, with C_2 a Cayley algebra, thus $\mathcal{K}(C_1 \otimes C_2)$ is a real form of an exceptional Lie algebra, $\mathcal{K}(C_1^{\mathbb{C}} \otimes C_2^{\mathbb{C}})$. In these cases, we will describe some inner ideals, and we will use the provided information to recognize the concrete real form obtained.

Note that the set of skew-symmetric elements of $\mathcal{A} = C_1 \otimes C_2$, for C_1 and C_2 arbitrary unital real composition algebras, coincides with $\mathcal{S} = (\mathcal{S}_1 \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \mathcal{S}_2)$, where $\mathcal{S}_i = \{x \in C_i : \bar{x} = -x\}$ denotes the set of skew-symmetric elements of the corresponding composition algebra. A key tool to study this type of structurable algebras is to consider the so-called *Albert form* defined in [3,§3] by

$$q_{\mathcal{A}}: \mathcal{S} \rightarrow \mathbb{R}, \quad q_{\mathcal{A}}(s_1 \otimes 1 + 1 \otimes s_2) = n_1(s_1) - n_2(s_2).$$

This allows us to find some of the inner ideals of $\mathcal{K}(\mathcal{A})$, besides the homogeneous component \mathcal{K}_2 , which of course it is always an inner ideal that can be identified with \mathcal{S} .

Proposition 5.5 *The only inner ideals of $\mathcal{K}(C_1 \otimes C_2)$ contained in the component \mathcal{K}_2 are:*

- *The whole $\mathcal{K}_2 = \{(0, s) : s \in \mathcal{S}\}$;*
- *$\{(0, s) : s \in I\}$ for any subspace I of \mathcal{S} such that $q_{\mathcal{A}}(I) = 0$.*

In consequence, $\mathcal{K}(C_1 \otimes C_2)$ has inner ideals of each dimension $1, \dots, m$ for m the Witt index of $q_{\mathcal{A}}$ (the dimension of the maximal isotropic subspace), and also an inner ideal of dimension equal to $\dim \mathcal{S} = \dim C_1 + \dim C_2 - 2$.

Proof Of course \mathcal{K}_2 is an inner ideal. Now take $B = \{(0, s) : s \in I\} \subseteq \mathcal{K}_2$ for a subspace I of \mathcal{S} such that $q_{\mathcal{A}}(I) = 0$, and let us check that $[B, [B, \mathcal{K}]] \subseteq B$, or equivalently, due to the grading, that $[B, [B, \mathcal{K}_{-2}]] \subseteq B$. That is, we have to see, for any $s, s' \in I$ and $t \in \mathcal{S}$, if $[(0, s), [(0, s'), (0, t)]] = (0, -(L_{s'}L_t)^{\delta}s) = -(0, s'(ts) + s(ts'))$ belongs or not to B . By using [3,Equation 3.7], we have that $s(ts) \in \langle s \rangle$ for any $s, t \in \mathcal{S}$ such that $q_{\mathcal{A}}(s) = 0$. In our case $s, s', s + s' \in I$; so that $q_{\mathcal{A}}(s) = q_{\mathcal{A}}(s') = q_{\mathcal{A}}(s + s') = 0$, so $-(s'(ts) + s(ts')) = s(ts) + s'(ts') - (s + s')(t(s + s')) \in \langle s, s' \rangle \subseteq I$, as we needed.

Conversely, if $B \subseteq \mathcal{K}_2$ is an inner ideal, there is a vector subspace $I \leq \mathcal{S}$ with $B = \{(0, s) : s \in I\}$. Let us see that, if there exists $s \in I$ with $q_{\mathcal{A}}(s) \neq 0$, then I coincides with the whole \mathcal{S} . Take into account that $[(0, s), [(0, s), (0, t)]] \in [B, [B, \mathcal{K}]] \subseteq B$ for any $t \in \mathcal{S}$, which means that $s(ts) \in I$. Again [3,Equation 3.7] says that $q_{\mathcal{A}}(s)t^{\delta} =$

$s(ts) + q_{\mathcal{A}}(s, t^{\natural})s \in I$, where $(s_1 \otimes 1 + 1 \otimes s_2)^{\natural} := s_1 \otimes 1 - 1 \otimes s_2$. Hence, $t^{\natural} \in I$ for any $t \in \mathcal{S}$ and $I = \mathcal{S}$. □

Another inner ideal of the Lie algebra $\mathcal{K}(C_1 \otimes C_2)$, which is not contained in \mathcal{K}_2 , appears in case $C_2 = \mathbb{R} \oplus \mathbb{R}$. This is a composition algebra with product componentwise and exchange involution. So $1_{C_2} = (1, 1) = e_1 + e_2$ for $e_1 = (1, 0)$ and $e_2 = (0, 1)$ orthogonal idempotents with $\overline{e_1} = e_2$ and $\overline{e_2} = e_1$. Note that $C_2 e_i \subseteq \mathbb{R} e_i$ for any $i = 1, 2$, and $e_i \overline{e_i} = 0$. Besides, it is easy to prove that $C_1 \otimes C_2 \cong C_1 \oplus C_1^{op}$ with $\overline{(x, y)} = (y, x)$.

Proposition 5.6 (See [15, Lemma 6.10]) *If C_1 is a composition algebra and $C_2 = \mathbb{R} \oplus \mathbb{R}$, then*

$$B = \{(a, 0) : a \in C_1 \otimes (1, 0)\} \oplus \{(0, s) : s \in \mathcal{S} = (\mathcal{S}_1 \otimes 1) \oplus (\mathbb{R}1 \otimes (1, -1))\}$$

is an inner ideal of $\mathcal{K}(C_1 \otimes C_2)$ of dimension equal to $2 \dim \mathcal{S} = 2 \dim C_1$, which contains the homogeneous component \mathcal{K}_2 .

Proof Note that $B = B_1 \oplus \mathcal{K}_2$ with $B_1 = \{(a, 0) : a \in C_1 \otimes e_1\} \subseteq \mathcal{K}_1$. Taking into account the \mathbb{Z} -grading on the Lie algebra, then checking $[B, [B, \mathcal{K}(C_1 \otimes C_2)]] \subseteq B$ is equivalent to checking the following conditions:

- (a) $[B_1, [\mathcal{K}_2, \mathcal{K}_{-2}]] \subseteq B_1$ and $[\mathcal{K}_2, [B_1, \mathcal{K}_{-2}]] \subseteq B_1$,
- (b) $[B_1, [B_1, \mathcal{K}_{-2}]] = 0$ and
- (c) $[B_1, [B_1, \mathcal{K}_{-1}]] \subseteq B_1$.

Take $a, b \in C_1 \otimes e_1, c \in C_1 \otimes C_2$ and $s, t \in \mathcal{S}$. For item a), $[(a, 0), [(0, s), (0, t)^{\natural}]] = (-s(ta), 0) \in I_1$ since $(C_1 \otimes C_2)(C_1 \otimes e_1) \subseteq C_1 \otimes e_1$. The other expression in a) holds by applying the Jacobi identity. For b), let us see that $[(a, 0), [(b, 0), (0, t)^{\natural}]] = -V_{a, tb}$ is the zero map. It is enough to see that $V_{a, b}(c) = 0$, since $c \in \mathcal{A}$ is arbitrary and $tb \in C_1 \otimes e_1$. Write $a = a_1 \otimes e_1, b = b_1 \otimes e_1$ and $c = c_1 \otimes c_2$ (the argument goes equally if $c = \sum_i c_i^1 \otimes c_i^2$). Then, $V_{a, b}(c) = (a\overline{b})c + (c\overline{b})a - (c\overline{a})b = 0$ since $a\overline{b} = 0, (c\overline{b})a = (c_1\overline{b_1})a_1 \otimes (c_2 e_2)e_1 = 0$ and similarly changing a and b . For c), we compute $[(a, 0), [(b, 0), (c, 0)^{\natural}]] = -(V_{b, c}(a), 0)$ but $V_{b, c}(a) = (b\overline{c})a + (a\overline{c})b - 0 \in C_1 \otimes C_2 e_1 \subseteq C_1 \otimes e_1$. □

Note that the above two propositions are also valid for a field \mathbb{F} and $C_2 = \mathbb{F} \oplus \mathbb{F}$. But in the real case we get some useful conclusions, taking advantage of our descriptions of the inner ideals in Sect. 4. In particular, all the non-compact real forms of E -type appear:

Corollary 5.7 *Denote by \mathbb{O} and \mathbb{O}_s the octonion division algebra and the split octonion algebra, respectively. Denote by $\mathbb{C}_s = \mathbb{R} \oplus \mathbb{R}$. Then, the non-compact real exceptional Lie algebras of E_6 -type can be obtained as follows:*

$$\mathcal{K}(\mathbb{O}_s \otimes \mathbb{C}_s) \cong \mathfrak{e}_{6,6}; \quad \mathcal{K}(\mathbb{O}_s \otimes \mathbb{C}) \cong \mathfrak{e}_{6,2}; \quad \mathcal{K}(\mathbb{O} \otimes \mathbb{C}) \cong \mathfrak{e}_{6,-14}; \quad \mathcal{K}(\mathbb{O} \otimes \mathbb{C}_s) \cong \mathfrak{e}_{6,-26}.$$

In the same way, denoting by \mathbb{H} and $\mathbb{H}_s \cong \text{Mat}_{2 \times 2}(\mathbb{R})$ the quaternion division algebra and the split quaternion algebra, respectively, then the non-compact real forms of E_7 -type are

$$\mathcal{K}(\mathbb{O}_s \otimes \mathbb{H}_s) \cong \epsilon_{7,7}; \quad \mathcal{K}(\mathbb{O}_s \otimes \mathbb{H}) \cong \epsilon_{7,5} \cong \mathcal{K}(\mathbb{O} \otimes \mathbb{H}); \quad \mathcal{K}(\mathbb{O} \otimes \mathbb{H}_s) \cong \epsilon_{7,-25};$$

while the non-compact real forms of E_8 -type are

$$\mathcal{K}(\mathbb{O}_s \otimes \mathbb{O}_s) \cong \epsilon_{8,8} \cong \mathcal{K}(\mathbb{O} \otimes \mathbb{O}); \quad \mathcal{K}(\mathbb{O} \otimes \mathbb{O}_s) \cong \epsilon_{8,-24}.$$

Proof By Proposition 4.5, the inner ideals of $\epsilon_{6,-26}$ have dimensions 8 and 16, the inner ideals of $\epsilon_{6,-14}$ have dimensions 1 and 8 and the inner ideals of $\epsilon_{6,2}$ have dimensions 1, 2, 3 and 8. The inner ideals of $\epsilon_{6,6}$ have dimensions 1, 2, 3, 4, 5, 8 and 16 by (8).

Take $\mathcal{A} = \mathbb{O} \otimes \mathbb{C}_s$, with skew-dimension 8. By Proposition 5.6, the algebra $\mathcal{K}(\mathcal{A})$ has an inner ideal of dimension 16, so that its signature is either 6 or -26 . But it is not 6 since the Witt index of $q_{\mathcal{A}}$ is 0, so Proposition 5.5 says that $\mathcal{K}(\mathcal{A})$ has not nonzero inner ideals contained properly in that one of dimension 8 (the split algebra has, on the contrary).

If $\mathcal{A} = \mathbb{O}_s \otimes \mathbb{C}_s$, the Witt index of $q_{\mathcal{A}}$ is 4, so $\mathcal{K}(\mathcal{A})$ has inner ideals of dimensions 1, 2, 3, 4 contained in \mathcal{K}_2 by Proposition 5.5. Only the split real form satisfies that condition.

If $\mathcal{A} = \mathbb{O} \otimes \mathbb{C}$, the Witt index of $q_{\mathcal{A}}$ is 1 and $\mathcal{K}(\mathcal{A})$ cannot have inner ideals contained properly in \mathcal{K}_2 (of dimension 8) up to those of dimension 1. This forces the signature -14 .

Now, the Witt index of the Albert form of $\mathbb{O}_s \otimes \mathbb{C}$ is 3, and so its Kantor construction gives a real form of signature 2, by similar arguments. This finishes the E_6 -case.

For E_7 , we only have to note that the Albert form has $I_{5,5}$ as a related matrix (in a suitable basis) for $\mathcal{A} = \mathbb{O}_s \otimes \mathbb{H}_s$, $I_{3,7}$ for $\mathcal{A} = \mathbb{O}_s \otimes \mathbb{H}$, $I_{7,3}$ for $\mathcal{A} = \mathbb{O} \otimes \mathbb{H}$ and $I_{9,1}$ for $\mathcal{A} = \mathbb{O} \otimes \mathbb{H}_s$, with Witt indices 5, 3, 3 and 1, respectively. Proposition 5.5 gives immediately the signature of the Lie algebras constructed from these structurable algebras, once we recall that $\epsilon_{7,-25}$ has only inner ideals of dimensions 1, 10 and 27, $\epsilon_{7,5}$ has inner ideals of dimensions 1, 2, 3 and 10, while the split real form has an inner ideal of dimension 5. Finally, the E_8 -case is immediate once one notes that the Witt indices of $C_1 \otimes C_2$, with $C_1, C_2 \in \{\mathbb{O}, \mathbb{O}_s\}$, are 7 if $C_1 = C_2$ and 3 otherwise. \square

Remark 5.8 Having these realizations of the real forms of E_6 helps to explain some of their fine gradings, described in [21] and [22]. For instance, an immediate consequence is that all the non-compact real forms of ϵ_6 , namely $\epsilon_{6,6}$, $\epsilon_{6,2}$, $\epsilon_{6,-14}$ and $\epsilon_{6,-26}$, possess a $\mathbb{Z} \times \mathbb{Z}_2^4$ -grading which does not admit proper refinements. Similarly, all the non-compact real algebras of type E_7 have a fine $\mathbb{Z} \times \mathbb{Z}_2^5$ -grading and all the non-compact real algebras of type E_8 have a fine $\mathbb{Z} \times \mathbb{Z}_2^6$ -grading. This is clear from two facts: the complexifications of these gradings are fine [23], and the \mathbb{Z} -grading given by the Kantor construction is compatible with the gradings on the composition algebras, where the real composition algebras of dimension 2^k are always \mathbb{Z}_2^k -graded. Note that, in spite of the active work trying to find all the fine gradings on the simple Lie

algebras, see, for instance, the monograph [23], there is still quite some work to do in the exceptional real Lie algebras.

Many of the inner ideals of the real exceptional Lie algebras have appeared as particular cases of Propositions 5.5 and 5.6, as we have seen through the above proof. In fact, most of them appear in this way, according to the classification obtained in Sect. 4. We would like to find also descriptions of the remaining inner ideals which are not in terms of roots, but in terms of a model or explicit construction of the algebra, similarly to the descriptions above. This will follow being our philosophy through the next subsections.

5.3 Inner ideals coming from Jordan algebras

If \mathcal{J} is a Jordan algebra, then $\mathcal{K}_1 = \mathcal{J}$ is an abelian inner ideal of $\mathcal{K}(\mathcal{J}) = \mathcal{K}_{-1} \oplus \mathcal{K}_0 \oplus \mathcal{K}_1$, since it is the corner of a \mathbb{Z} -grading (the involution is the identity and $\mathcal{S} = 0$). Moreover, the inner ideals of the Jordan algebra provide inner ideals of the related Lie algebra. Recall that a subspace $B \leq \mathcal{J}$ is said an *inner ideal* if $U_B(\mathcal{J}) \subseteq B$, for the quadratic operator $U_x = U_{x,x} = 2L_x^2 - L_{x^2}$.

Lemma 5.9 *The only inner ideals of $\mathcal{K}(\mathcal{J})$ contained in the component \mathcal{K}_1 are the sets $\{(b, 0) : b \in B\}$ for some inner ideal B of \mathcal{J} .*

Proof This is evident: $I = \{(b, 0) : b \in B\}$ is an inner ideal of $\mathcal{K}(\mathcal{J})$ if and only if $[I, [I, \mathcal{K}_{-1}]] \subseteq I$ holds, taking the grading into consideration. But

$$[(a, 0), [(b, 0), (c, 0)]] = (-V_{b,c}(a), 0) = (-U_{b,a}(c), 0)$$

belongs to I if and only if $U_{B,B}(\mathcal{J}) \subseteq B$. □

Take $\mathcal{J} = H_3(C)$ the Jordan algebra of Hermitian matrices of size 3 with coefficients in the complex Cayley algebra C with the symmetrized product $a \circ b = \frac{ab+ba}{2}$, the so-called *Albert algebra*. The Lie algebra obtained when applying the TKK-construction is the complex Lie algebra of type E_7 [30, Proposition 16]. Thus, an inner ideal of dimension 27 of the complex Lie algebra \mathfrak{e}_7 is given by the 1-part of the 3-grading, which can be identified with this Albert algebra.

If we consider any real form of the Albert algebra, its TKK-construction will give a real form of the complex algebra \mathfrak{e}_7 , also with an inner ideal of dimension 27. In particular, the TKK-construction applied to $H_3(\mathbb{O}_s)$ and $H_3(\mathbb{O})$ has to give the real forms of signature either 7 or -25 according to Sect. 4.6, because these real forms are the only ones with inner ideals of dimension 27. In order to distinguish the obtained signature, we could check that the construction used by Jacobson in [30, Table (143), row b)] coincides with ours, what would imply $\mathcal{K}(H_3(\mathbb{O}_s)) \cong \mathfrak{e}_{7,7}$ and $\mathcal{K}(H_3(\mathbb{O})) \cong \mathfrak{e}_{7,-25}$. But note that we can alternatively use our knowledge on the remaining inner ideals. Take an idempotent $e \in \mathbb{O}_s$ and, as in [34, Main Theorem, iv)],

consider

$$B = \left\{ \begin{pmatrix} \lambda & a & ve \\ \bar{a} & 0 & 0 \\ v\bar{e} & 0 & 0 \end{pmatrix} : \lambda, v \in \mathbb{R}, a \in e\mathbb{O}_s \right\}.$$

Then, B is an inner ideal of dimension 6 of the Jordan algebra $H_3(\mathbb{O}_s)$ which provides a 6-dimensional inner ideal of $\mathcal{K}(H_3(\mathbb{O}_s))$. Consequently, $\mathcal{K}(H_3(\mathbb{O}_s))$ is the split real form $\mathfrak{e}_{7,7}$. This 6-dimensional inner ideal is formed uniquely by extremal elements (clear from [18], since an element is extremal if it is after complexification). But $\mathcal{K}(H_3(\mathbb{O}))$ cannot be the split real form, because [34, Theorem 8] says that the inner ideals of $H_3(\mathbb{O})$ formed by extremal elements are necessarily one-dimensional. By Lemma 5.9, $\mathcal{K}(H_3(\mathbb{O}))$ does not have a six-dimensional inner ideal. Hence, its signature is -25 .

Now we deal with the inner ideals not appeared so far, which all satisfy the following property: all their nonzero elements are extremal.

5.4 Point Line Spaces (and Some More Linear Models)

In order to describe the remaining inner ideals, we will use some ad-hoc constructions of the exceptional (split) Lie algebras based on linear and multilinear algebra, following the philosophy of [20].

Definition 5.10 A nonzero subspace P of a Lie algebra \mathfrak{g} is said to be a *point space* if $[P, P] = 0$ and every nonzero element in P is an extremal element. (Recall Definition 2.2.)

Point spaces were introduced in [7], in order to complete the missing cases in [5]. The following is hence well known.

Lemma 5.11 *If P is a point space of \mathfrak{g} , then P is an abelian inner ideal and every subspace of P so is.*

Proof Let us take $0 \neq x, y \in P$ and $z \in \mathfrak{g}$ and let us check that $[x, [y, z]] \in P$. As $[x, y] = 0$, the Jacobi identity implies that $[x, [y, z]] = [y, [x, z]]$ so that

$$[x, [y, z]] = \frac{1}{2}([x, [y, z]] + [y, [x, z]]) = \frac{1}{2}([x + y, [x + y, z]] - [x, [x, z]] - [y, [y, z]]),$$

which belongs to $\langle x, y \rangle \subseteq P$ because x, y and $x + y$ are extremal elements (except for $x + y = 0$, in which case the result is clear too). Thus $[P, [P, \mathfrak{g}]] \subseteq P$.

Moreover, it is evident that every nonzero subspace of P is also a point space, and hence an abelian inner ideal. \square

Now we will describe a point space of dimension 8 of $\mathfrak{e}_{8,8}$, a point space of dimension 7 of $\mathfrak{e}_{7,7}$ and a point space of dimension 5 of $\mathfrak{e}_{6,6}$. The fact that they are point spaces appears explicitly in [18]. These point spaces and their subspaces will exhaust all the abelian inner ideals of the split exceptional Lie algebras not appeared in Sects. 5.2 and 5.3.

5.4.1 A Linear Model of $\mathfrak{e}_{8,8}$

Take U a complex or real vector space of dimension 8 and consider the graded vector space

$$\mathfrak{g} = \mathfrak{g}_{-3} \oplus \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$$

given by

$$\mathfrak{g}_3 = U; \mathfrak{g}_2 = \Lambda^6 U; \mathfrak{g}_1 = \Lambda^3 U; \mathfrak{g}_0 = \mathfrak{gl}(U); \mathfrak{g}_{-1} = \Lambda^5 U; \mathfrak{g}_{-2} = \Lambda^2 U; \mathfrak{g}_{-3} = \Lambda^7 U. \tag{10}$$

Now \mathfrak{g} can be endowed with a Lie algebra structure of type E_8 (over \mathbb{C} or \mathbb{R} , respectively). Indeed, think first of the complex Lie algebra L of type E_8 , H a Cartan subalgebra, $L = H \oplus (\oplus_{\alpha \in \Phi} L_\alpha)$ the root decomposition relative to H , and $\{\alpha_1, \dots, \alpha_8\}$ a set of simple roots (with labeling as in Sect. 2.2). There is a \mathbb{Z} -grading on L with homogeneous components given by:

$$\begin{aligned} L_n &= \oplus \{L_\alpha : \alpha = \sum_{i=1}^8 p_i \alpha_i, p_2 = n\}, \\ L_0 &= H \oplus (\oplus \{L_\alpha : \alpha = \sum_{i=1}^8 p_i \alpha_i, p_2 = 0\}). \end{aligned}$$

A glimpse of the roots gives us that this grading has 7 pieces and the related dimensions are 8, 28, 56, 64, 56, 28 and 8, respectively (necessarily L_n and L_{-n} are dual one of each other through the Killing form κ). This fits with the fact that L_0 is a reductive subalgebra which is the sum of a one-dimensional center and a semisimple subalgebra whose Dynkin diagram is obtained when removing the 2nd-node of the Dynkin diagram of E_8 , that is, a simple Lie algebra of type A_7 . Thus, L_0 is a Lie algebra isomorphic to $\mathfrak{gl}(8, \mathbb{C})$ and to $\mathfrak{gl}(U) = \mathfrak{g}_0$ when our 8-dimensional vector space is considered over the complex numbers. Also, according to [32, Chapter 8], L_n is an irreducible L_0 -module. By dimension count, for each n there is an obvious identification among L_n and either \mathfrak{g}_n or \mathfrak{g}_{-n} compatible with the actions of L_0 and \mathfrak{g}_0 , respectively. Changing U by its dual if necessary, we can assume that the L_0 -module L_1 is isomorphic to the $\mathfrak{gl}(U)$ -module $\Lambda^3 U = \mathfrak{g}_1$. The fact that $\dim_{\mathbb{C}} \text{Hom}(L_n \otimes L_m, L_{n+m}) = 1$ ([32, Chapter 8] too) gives our choices for the remaining exterior powers of U . We mean, the L_0 -modules L_n are now isomorphic to the $\mathfrak{gl}(U)$ -modules $\mathfrak{g}_n = \Lambda^{[3n]_8} U$ where $[n]_8$ denotes here n modulo 8. Now, the isomorphism of vector spaces between L and \mathfrak{g} induces a structure of (complex) Lie algebra on \mathfrak{g} . It is clear that the bracket $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$ is determined up to scalar by the only $\mathfrak{gl}(U)$ -invariant map $\Lambda^{[3n]_8} U \times \Lambda^{[3m]_8} U \rightarrow \Lambda^{[3(n+m)]_8} U$ if $n + m \neq 0$. And, if $n + m = 0$, determined up to scalar by the only $\mathfrak{gl}(U)$ -invariant map $\Lambda^{[3n]_8} U \times \Lambda^{[3n]_8} U^* \rightarrow \mathfrak{gl}(U)$. The concrete scalars can be determined (once we have fixed such invariant maps) by imposing the Jacobi identity to $(\mathfrak{g}, [,])$ instead of passing through the explicit isomorphism, as in [20]. In any case, it is possible to have more information on the pieces of this grading without computing the concrete scalars.

Lemma 5.12 $\mathfrak{g}_3 = U$ is a point space of \mathfrak{g} .

As we know from [18] that any inner ideal of dimension 8 should be a point space, and \mathfrak{g}_3 is an inner ideal because it is the corner of a \mathbb{Z} -grading, this lemma would not require a proof. But we think that a direct argument provides deeper insight.

Proof If we think of \mathfrak{g}_{-3} as U^* , take $0 \neq u \in U$ and $\alpha \in U^*$ and let us check that u is an extremal element: $[u, [u, \mathfrak{g}]] \subseteq \mathbb{C}u$. Taking the grading into consideration, it is enough to prove that $[u, [u, \mathfrak{g}_{-3}]] \subseteq \mathbb{C}u$, since $[u, [u, \mathfrak{g}_n]] \subseteq \mathfrak{g}_{n+6}$ vanishes when $n \neq -3$. Up to scalar, $[u, \alpha]$ corresponds to the map in $\mathfrak{gl}(U)$ given by $v \mapsto \alpha(v)u$ for any $v \in U$. As the action of $\mathfrak{gl}(U)$ on $\mathfrak{g}_3 = U$ is the natural action, also up to scalar $[u, [u, \alpha]]$ should coincide with $-\alpha(u)u \in \mathbb{C}u$. \square

If we now consider U as a real vector space, the same construction (10) gives a real Lie algebra of type E_8 which possesses an inner ideal of dimension 8 (and also inner ideals of each dimension less than 8), so that the description of inner ideals in Sect. 4.7 implies that the constructed real form \mathfrak{g} is necessarily of split type, $\epsilon_{8,8}$.

5.4.2 A Linear Model of $\epsilon_{7,7}$

Similarly to the previous case, the \mathbb{Z} -grading on the complex Lie algebra L of type E_7 (and also, on the real Lie algebra $\epsilon_{7,7}$) produced by choosing the second node is

$$L_n = \oplus \left\{ L_\alpha : \alpha = \sum_{i=1}^7 p_i \alpha_i, p_2 = n \right\},$$

$$L_0 = H \oplus \left(\oplus \{ L_\alpha : \alpha = \sum_{i=1}^7 p_i \alpha_i, p_2 = 0 \} \right).$$

This is a 5-grading, since the maximal root $2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ has 2 as the coefficient of α_2 . Looking at how many roots have coefficient of α_2 equal to $\pm 2, \pm 1, 0$, we get dimensions of the homogeneous components equal to 7, 35 and 49, respectively. Also, looking at the Dynkin diagram obtained when removing the second node, we know $L_0 \cong \mathfrak{gl}(7, \mathbb{C})$. So there is a Lie algebra structure on the following \mathbb{Z} -graded vector space. Take U a vector space of dimension 7, over the complex numbers or over the real ones, and consider

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

given by

$$\mathfrak{g}_2 = U; \quad \mathfrak{g}_1 = \Lambda^4 U; \quad \mathfrak{g}_0 = \mathfrak{gl}(U); \quad \mathfrak{g}_{-1} = \Lambda^3 U; \quad \mathfrak{g}_{-2} = \Lambda^6 U,$$

with the above explained Lie algebra structure induced by the isomorphism with L . The same arguments as above yield

Lemma 5.13 *The vector space of dimension 7 given by $\mathfrak{g}_2 = U$ is a point space of \mathfrak{g} .*

This corresponds with the inner ideal named $\mathcal{B}_{\{2\}}$ of $\epsilon_{7,7}$, and their proper subspaces of dimension at least 4 correspond to the inner ideals $\mathcal{B}_{\{2,7\}} \supseteq \mathcal{B}_{\{2,6,7\}} \supseteq \mathcal{B}_{\{2,5,6,7\}}$, so that this completes, joint with the inner ideals appeared above, the list of inner ideals of $\epsilon_{7,7}$.

Remark 5.14 In this case, we can give an alternative description of an inner ideal of $\mathfrak{e}_{7,7}$ which is a point space of dimension 7. Recall that, as the \mathbb{Z} -grading has 5 pieces, it could be the TKK-construction of some structurable algebra. This is indeed the case, the structurable algebra of dimension 35 is the Smirnov algebra [35]. Thus, the point space of dimension 7 can be understood as the set of the skew-symmetric elements of the Smirnov algebra.

5.4.3 A Linear Model of $\mathfrak{e}_{6,6}$

In this case, the considered \mathbb{Z} -grading on the complex Lie algebra of type E_6 (and also, on the real Lie algebra $\mathfrak{e}_{6,6}$) is produced by choosing the third labeled node. So, after choosing a Cartan subalgebra and a set of simple roots $\{\alpha_i\}_{i=1}^6$ of the related root system, the homogeneous components are

$$\begin{aligned} L_n &= \oplus\{L_\alpha : \alpha = \sum_{i=1}^6 p_i\alpha_i, p_3 = n\}, \\ L_0 &= H \oplus \left(\oplus\{L_\alpha : \alpha = \sum_{i=1}^6 p_i\alpha_i, p_3 = 0\} \right). \end{aligned}$$

This decomposition $L = \oplus_{n \in \mathbb{Z}} L_n$ is again a 5-grading, since the coefficient of α_3 of the maximal root $\alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ is 2. The 0-part is the sum of a one-dimensional center and a semisimple algebra, which is the sum of two simple ideals of types A_4 and A_1 . The dimensions of the non-neutral homogeneous components, by counting roots, are 5 and 20. The considerations about their irreducibility as L_0 -modules work as in the previous cases, although in this case an irreducible module for a semisimple algebra (the derived algebra $[L_0, L_0]$) has to be a tensor product of irreducible modules for each simple component. A model adapted to this situation is the following.

Take U a real vector space of dimension 5 (not 6) and V a real vector space of dimension 2. Take

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

given by

$$\begin{aligned} \mathfrak{g}_2 &= U \otimes \mathbb{R}; & \mathfrak{g}_1 &= \Lambda^3 U \otimes V; & \mathfrak{g}_0 &= \mathfrak{gl}(U) \oplus \mathfrak{sl}(V); & \mathfrak{g}_{-1} &= \Lambda^2 U \otimes V; \\ \mathfrak{g}_{-2} &= \Lambda^4 U \otimes \mathbb{R}, \end{aligned}$$

with the Lie algebra structure such that \mathfrak{g} is a simple real Lie algebra (the uniqueness of the bracket between components is again obtained by passing to the complexification and using then the results in [32, Chapter 8]). We prefer to write $\mathfrak{g}_2 = U \otimes \mathbb{R}$ instead the nicer but equivalent expression $\mathfrak{g}_2 = U$ for emphasizing that the action of $\mathfrak{sl}(V)$ on the homogeneous component \mathfrak{g}_2 is trivial. Once more we have

Lemma 5.15 *The vector space of dimension 5 given by $\mathfrak{g}_2 = U \otimes \mathbb{R}$ is a point space of \mathfrak{g} .*

The argument follows being that the only invariant map $U \otimes U^* \rightarrow \mathfrak{gl}(U)$ is given by $u \otimes \alpha \mapsto (v \mapsto \alpha(v)u)$, up to scalar multiple. Alternatively, \mathfrak{g}_2 is an inner ideal since it is a corner of a \mathbb{Z} -grading, which should be a point space because every inner ideal of dimension 5 of E_6 is a point space.

This point space cannot be described as the set of skew-symmetric elements of a structurable algebra, since there is no central simple structurable algebra with skew-dimension 5 (a simple case-by-case computation from [2, Theorem 11]).

One of the applications of this viewpoint is to provide models to describe the gradings of the exceptional Lie algebras, so, in some sense, a method to look for symmetries. Consult [17] in this line, where Kantor's construction is applied to a structurable algebra of dimension 20 to get a \mathbb{Z} -grading on \mathfrak{e}_6 and on some of its real forms, but such algebra structurable algebra has skew-dimension 1 (not 5).

5.5 Some Comments on Jordan Pairs

For completeness, we relate our results with Jordan pairs. Real simple Jordan pairs are classified in [33, §11.4], where the study of real bounded symmetric domains is reduced to that one of real Jordan pairs. As is explained in [25, §12.5], abelian inner ideals of Lie algebras and inner ideals of Jordan pairs are essentially the same mathematical object. More precisely, if B is an abelian inner ideal of a Lie algebra L , then $\text{Sub}_L B := V = (V^+, V^-)$, for $V^+ = B$ and $V^- = L/\text{Ker}_L B$, is a Jordan pair called the *subquotient* of B . Recall that $\text{Ker}_L B := \{x \in L : [B, [B, x]] = 0\}$. Now the inner ideals of L contained in B are the inner ideals of V contained in V^+ [25, Proposition 11.47i)]. We look at the exceptional cases in the light of Jordan pairs. Recall that any Jordan pair has attached a Lie algebra by a construction also called TKK-construction.

In the complex case, there is (up to conjugation) only one maximal inner ideal of the complex Lie algebra of type E_6 , with dimension 16. It has as subquotient the *Bi-Cayley pair* $V = (\text{Mat}_{1 \times 2}(C), \text{Mat}_{2 \times 1}(C))$ for C the complex Cayley algebra, where the Q -operator is $Q_a b = (ab)a$. Moreover, the TKK-construction applied to the Bi-Cayley pair is again the Lie algebra of type E_6 . In the real case, there are two real forms of the Bi-Cayley pair. The TKK-construction of the Jordan pair $V = (\text{Mat}_{1 \times 2}(\mathbb{O}), \text{Mat}_{2 \times 1}(\mathbb{O}))$ gives $\mathfrak{e}_{6, -26}$, since $\mathfrak{e}_{6, 2}$ and $\mathfrak{e}_{6, -14}$ have no inner ideals of dimension 16, while $\mathfrak{e}_{6, 6}$ has inner ideals of dimensions from 1 to 5, in particular it has point spaces of dimension 1, which is not the case for V . On the other hand, it is clear that the TKK-construction of the Jordan pair $(\text{Mat}_{1 \times 2}(\mathbb{O}_s), \text{Mat}_{2 \times 1}(\mathbb{O}_s))$ gives $\mathfrak{e}_{6, 6}$, again taking into account the lattice of inner ideals. For the other real forms, recall that if $q : X \rightarrow \mathbb{F}$ is a non-degenerate quadratic form on a vector space X over a field \mathbb{F} , then (X, X) is Jordan pair called *Clifford pair*, where the Q -operator is given by $Q_x y = q(x, y)x - q(x)y$. We refer to this Jordan pair as \mathcal{Q}_n ($n = \dim X$) when there is no ambiguity in the signature of q . The lattice of the proper inner ideals of $\mathfrak{e}_{6, 2}$ (resp. $\mathfrak{e}_{6, -14}$) coincides with the lattice of inner ideals of the Clifford Jordan pair defined by a quadratic form of Witt index 3 (resp. 1) on a real vector space of dimension 8.

The complex Lie algebra of type E_7 can be seen as the TKK-construction of the *Albert pair* $V = (\mathcal{J}, \mathcal{J})$ for $\mathcal{J} = H_3(C)$ and C the complex Cayley algebra. There

are two maximal inner ideals, one with dimension 27 and the other one a point space of dimension 7. The subquotient of the first one is the Albert pair. Similar arguments to Sect. 5.3 allow us to conclude that $\epsilon_{7,-25}$ (resp. $\epsilon_{7,7}$) can be obtained as the TKK-construction of the Albert pair given by $H_3(\mathbb{O})$ (resp. $H_3(\mathbb{O}_s)$). Lastly, the lattice of the proper inner ideals of $\epsilon_{7,5}$ coincides with that one of the Clifford Jordan pair defined by a quadratic form of Witt index 3 on a real vector space of dimension 10.

The complex Lie algebra of type E_8 has two maximal inner ideals up to conjugation, one whose subquotient is the Clifford pair Q_{14} and the other one a point space of dimension 8. This time the TKK-construction applied to this Clifford pair Q_{14} does not give the full Lie algebra of type E_8 but only a subalgebra of type D_8 . The lattice of $\epsilon_{8,-24}$ coincides with the lattice of inner ideals of the Clifford Jordan pairs defined by a quadratic form of Witt index 3 on a real vector space of dimension 14.

5.6 Conclusions

In the above subsections, we have found explicit descriptions of the inner ideals of the exceptional real Lie algebras, with the exception of those of type G_2 , which are better understood. These inner ideals, closely related to the different constructions of the exceptional Lie algebras, permit us to have a taste of the general behavior of the inner ideals. In a next paper, we will deal with some incidence geometries related to the inner ideals of the simple real algebras. We will take advantage of the very precise information obtained here. There will appear polar spaces, root shadows and some kinds of partial linear spaces. In some cases, the minimal inner ideals will constitute a Moufang set. These nice geometries will explain, in some way, the more involved structure of the inner ideals or of the extremal elements instead of considering them *up to automorphism*, which often provide an insufficient answer. We mean, sometimes the *vertical* distribution of the inner ideals, once we have chosen them compatible with a determined Cartan subalgebra, hides in some way a more intricate structure, which is far from being completely described only by saying that they are conjugated by an automorphism.

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