



Rigidity results for complete spacelike submanifolds in plane fronted waves

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Abstract

New rigidity results for complete non-compact spacelike submanifolds of arbitrary codimension in plane fronted waves are obtained. Under appropriate assumptions, we prove that a complete spacelike submanifold in these spacetimes is contained in a characteristic light-like hypersurface. Moreover, for a complete codimension two extremal submanifold in a plane fronted wave we show sufficient conditions to guarantee that it is a (totally geodesic) wavefront.

Keywords Extremal submanifold · Weakly trapped submanifold · Plane fronted wave

Mathematics Subject Classification 53C42 · 53C80 · 83C35

1 Introduction

In the search for exact solutions to Einstein's field equation it is usually assumed the existence of a certain symmetry. This symmetry is usually provided by a globally defined causal conformal vector field [18]. In particular, when this vector field is lightlike and parallel, the resulting solution is called a Brinkmann spacetime [5]. In this article, we will focus on a distinguished subfamily of Brinkmann spacetimes, namely, plane fronted waves [9]. A plane fronted wave is a Lorentzian manifold $(\overline{M}^{n+2}, \langle \cdot, \cdot \rangle)$ where $\overline{M}^{n+2} = \mathbb{R}^2 \times M^n$ with (M^n, g_M) a (connected) Riemannian manifold and the Lorentzian metric is

$$\langle \cdot, \cdot \rangle = \mathcal{H}(u, x)du \otimes du + du \otimes dv + dv \otimes du + g_M, \quad (1)$$

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where (u, v) are the natural coordinates on \mathbb{R}^2 and $\mathcal{H}(u, x)$ is a (independent of v) smooth function on $\mathbb{R}^2 \times M^n$. The coordinate vector field $\partial_v := \partial/\partial v$ is lightlike and parallel. Endowing \overline{M}^{n+2} with the time orientation defined by ∂_v , it becomes a spacetime. As $\overline{\nabla}u = \partial_v$, the coordinate $u : \overline{M}^{n+2} \rightarrow \mathbb{R}$ plays the role of a quasi-time function [4, Def. 13.4] i.e., its gradient is everywhere causal and any causal geodesic segment γ such that $u \circ \gamma$ constant is injective. In particular, the spacetime is causal [4, p. 490]. A plane fronted wave is foliated by the (characteristic) lightlike hypersurfaces $u = u_0, u_0 \in \mathbb{R}$.

When $M^n = \mathbb{R}^2$ and g_M is the usual Euclidean metric, the spacetime \overline{M}^4 is called a pp-wave (plane fronted wave with parallel propagating rays) [4]. Plane fronted waves model (electromagnetic or gravitational) radiation propagating at the speed of light. Despite the fact that the study of gravitational waves goes back to Einstein and Rosen [8], the experimental detection of gravitational waves [1] has aroused widespread interest in plane fronted waves.

In these ambient spacetimes we will study spacelike submanifolds of arbitrary codimension. Spacelike submanifolds of codimension greater than one became interesting from a physical viewpoint when Penrose introduced the notion of trapped surface to study spacetimes' singularities [17]. Namely, the existence of a trapped surface is a sign of the presence of a black hole. The original definition of trapped surface was given in terms of null expansions. This is related to the causal orientation of the mean curvature vector field, which allows the extension of the concept of trapped submanifold to arbitrary codimension [14]. Indeed, let $\psi : \Sigma^k \rightarrow \overline{M}^{n+2}$ be a spacelike submanifold of arbitrary codimension in an arbitrary spacetime. Denoting by \vec{H} the mean curvature vector field of the submanifold and following the standard terminology in General Relativity (see [14, 15]), Σ^k is said to be future trapped if \vec{H} is timelike and future pointing everywhere (similarly for past trapped); weakly future trapped if \vec{H} is causal and future pointing everywhere (similarly for weakly past trapped) and extremal if $\vec{H} = 0$.

In a plane fronted wave, each codimension two totally geodesic (hence, extremal) spacelike submanifold defined by $u = u_0, v = v_0, u_0, v_0 \in \mathbb{R}$, is called a wavefront [9]. Physically, a wavefront in a plane fronted wave can be interpreted as a perturbation that is being propagated at the speed of light without experiencing any change (e.g., weakening) [12]. For instance, if we consider in the 4-dimensional Lorentz-Minkowski spacetime \mathbb{L}^4 the coordinates $(u := (z - t)/\sqrt{2}, v := (z + t)/\sqrt{2}, x, y)$, where t, x, y, z are the standard coordinates, we can see how the wavefronts are given by $\{u_0, v_0\} \times \mathbb{R}^2, u_0, v_0 \in \mathbb{R}$.

From a geometrical point of view, it is natural to wonder under which conditions a complete codimension two extremal submanifold in a plane fronted wave is a wavefront. In [7], the authors answer this question for the compact case. Thus, in this paper we will focus on the non-compact case. Our main tools will be certain maximum principles as well as the parabolicity of the spacelike submanifold.

Let us recall that a complete (non-compact) Riemannian manifold is parabolic if the only superharmonic functions bounded from below that it admits are the constants (see, for instance, [13]). From a physical perspective, parabolicity is equivalent to the recurrence of the Brownian motion on a Riemannian manifold [10]. From a mathematical standpoint, complete Riemannian surfaces with non-negative Gaussian curvature are parabolic [11]. In arbitrary dimension there is no clear relation between parabolicity and sectional curvature. Nevertheless, there exist sufficient conditions to ensure the parabolicity of a Riemannian manifold of arbitrary dimension based on its geodesic balls' volume growth [3]. In this article we will use that parabolicity is invariant under quasi-isometries [10, Cor. 5.3].

This paper is organized as follows. Section 2 introduces general notions on plane fronted waves and their spacelike submanifolds that will later allow us to prove our main results. Section 3 contains the main results of this paper, we first state several conditions that guarantee that a non-compact spacelike submanifold is contained in a lightlike hypersurface $u = u_0$ (Theorems 7 and 11). From these theorems we can deduce certain non-existence results for spacelike hypersurfaces (Corollaries 12 and 13) as well as for weakly trapped submanifolds (Theorem 14). Finally, we answer our initial question providing in Theorems 15, 16 and 17 sufficient conditions for a complete non-compact codimension two extremal submanifold to be a wavefront. Indeed, Theorem 15 states that in a plane fronted wave the only codimension two parabolic extremal submanifolds with bounded (either from above or below) $u \circ \psi$ and $v \circ \psi$ are the (necessarily parabolic) wavefronts. Furthermore, Theorem 16 asserts, under the same bound, that a complete codimension two extremal submanifold whose Ricci curvature verifies $\text{Ric} \geq 0$ is also a wavefront.

2 Set up

2.1 Plane fronted waves

Let us consider a plane fronted wave $\overline{M}^{n+2} = \mathbb{R}^2 \times M^n$ and let $\mathcal{L}(M^n)$ be the subspace of $\mathfrak{X}(\overline{M}^{n+2})$ consisting of the lifts to \overline{M}^{n+2} of all vector fields on M^n . From now on, we will use the the same symbol to denote a vector field in $\mathfrak{X}(M^n)$ and its corresponding lift to $\mathcal{L}(M^n)$. In a similar way, we simplify the notation using the same symbol for a function on M^n and its corresponding lift to \overline{M}^{n+2} . The Levi-Civita connection $\overline{\nabla}$ of a plane fronted wave was given in [6] as follows,

Lemma 1 *For any $V, W \in \mathcal{L}(M^n) \subset \mathfrak{X}(\overline{M}^{n+2})$, we have*

- (i) $\overline{\nabla}_{\partial_u} \partial_u = \frac{1}{2}(\partial_u \mathcal{H} \partial_v - \widetilde{\nabla} \mathcal{H}_u) = \frac{1}{2}(\overline{\nabla} \mathcal{H} - \widetilde{\nabla} \mathcal{H}_u)$,
- (ii) $\overline{\nabla}_V \partial_u = \overline{\nabla}_{\partial_u} V = \frac{1}{2} g_M(\widetilde{\nabla} \mathcal{H}_u, V) \partial_v$,
- (iii) $\overline{\nabla}_V W = \widetilde{\nabla}_V W$,
- (iv) $\overline{\nabla}_{\partial_v} \partial_v = \overline{\nabla}_{\partial_v} \partial_u = \overline{\nabla}_{\partial_u} \partial_v = \overline{\nabla}_V \partial_v = \overline{\nabla}_{\partial_v} V = 0$,

where $\widetilde{\nabla}$ denotes the Levi-Civita connection of (M^n, g_M) , $\widetilde{\nabla} \mathcal{H}_u$ is the gradient of \mathcal{H}_u on M^n , $\mathcal{H}_u(x) := \mathcal{H}(u, x)$, and $\overline{\nabla} \mathcal{H}$ is the gradient of \mathcal{H} on \overline{M}^{n+2} .

Every tangent vector $Z \in T_{(u,v,x)} \overline{M}^{n+2}$ admits the decomposition

$$Z = \langle Z, \partial_v \rangle \partial_u + (\langle Z, \partial_u \rangle - \langle Z, \partial_v \rangle \mathcal{H}) \partial_v + d\pi_M(Z) \tag{2}$$

where $\pi_M : \overline{M}^{n+2} \rightarrow M^n$ is the natural projection. Thus, Lemma 1 implies that

$$\overline{\nabla}_Z \partial_u = \frac{1}{2} \langle Z, \partial_v \rangle (\overline{\nabla} \mathcal{H} - \widetilde{\nabla} \mathcal{H}_u) + \frac{1}{2} g_M(\widetilde{\nabla} \mathcal{H}_u, d\pi_M(Z)) \partial_v. \tag{3}$$

The non-necessarily vanishing components of the Riemann curvature tensor \overline{R} of the metric (1) are

$$\begin{aligned} \overline{R}(U, V)W &= R_M(U, V)W, \quad \overline{R}(V, \partial_u) \partial_u = -\frac{1}{2} \overline{\nabla}_V \overline{\nabla} \mathcal{H}_u \quad \text{and} \\ \overline{R}(V, \partial_u)W &= \frac{1}{2} \widetilde{\text{Hess}}(\mathcal{H}_u)(V, W) \partial_v, \end{aligned}$$

where R_M and $\widetilde{\text{Hess}}$ stand, respectively, for the Riemann and Hessian tensors on M^n . Denoting by $\overline{\text{Ric}}$ and Ric_M the Ricci tensors of \overline{M}^{n+2} and M^n , respectively, we obtain that the non-necessarily vanishing components of $\overline{\text{Ric}}$ are

$$\overline{\text{Ric}}(V, W) = \text{Ric}_M(V, W) \quad \text{and} \quad \overline{\text{Ric}}(\partial_u, \partial_u) = -\frac{1}{2} \widetilde{\Delta} \mathcal{H}_u,$$

where $\widetilde{\Delta}$ is the Laplacian on M^n . It can be easily seen that a plane fronted wave $(\overline{M}^{n+2}, \langle \cdot, \cdot \rangle)$ satisfies the timelike convergence condition (TCC), i.e., $\overline{\text{Ric}}(T, T) \geq 0$ for every timelike vector T , if and only if

$$\widetilde{\Delta} \mathcal{H}_u \leq 0 \quad \text{and} \quad \text{Ric}_M \geq 0. \tag{4}$$

2.2 Spacelike submanifolds

Let us now consider $\psi : \Sigma^k \rightarrow \overline{M}^{n+2}$ a (connected) spacelike submanifold in a plane fronted wave $(\overline{M}^{n+2}, \langle \cdot, \cdot \rangle)$. That is, ψ is an immersion and the induced metric on Σ^k is Riemannian. We will also denote the induced metric on Σ^k by $\langle \cdot, \cdot \rangle$. For any vector field V along the immersion ψ , we write V^\top and V^\perp for the tangent and normal parts along ψ , respectively. If we denote by $\mu := u \circ \psi$ and $\nu := v \circ \psi$, the restrictions of the coordinate functions u and v to Σ^k , we obtain that their gradients are given by

$$\nabla \mu = \partial_v^\top \tag{5}$$

and

$$\nabla \nu = \partial_u^\top - (\mathcal{H} \circ \psi) \partial_v^\top. \tag{6}$$

Therefore, from these formulas we see how wavefronts (i.e., codimension two spacelike submanifolds given by $u = u_0, v = v_0$, for $u_0, v_0 \in \mathbb{R}$) satisfy that the vector fields ∂_u and ∂_v are normal at every point.

Let us recall the Gauss and Weingarten’s formulas, given respectively by

$$\overline{\nabla}_X Y = \nabla_X Y + \text{II}(X, Y), \tag{7}$$

and

$$\overline{\nabla}_X N = -A_N X + \nabla_X^\perp N, \tag{8}$$

for $X, Y \in \mathfrak{X}(\Sigma^k)$ and $N \in \mathfrak{X}^\perp(\Sigma^k)$, where $\overline{\nabla}$ and ∇ are the Levi–Civita connections of \overline{M}^{n+2} and Σ^k , respectively, II is the second fundamental form and A_N is the shape operator with respect to N and ∇^\perp is the normal connection. The mean curvature vector field of the spacelike submanifold Σ^k is

$$\vec{H} = \frac{1}{k} \text{trace}_{\langle \cdot, \cdot \rangle}(\text{II}). \tag{9}$$

Since ∂_v is a parallel vector field, we obtain using (7) and (8) for $X \in \mathfrak{X}(\Sigma^k)$

$$0 = \overline{\nabla}_X \partial_v = \overline{\nabla}_X \partial_v^\top + \overline{\nabla}_X \partial_v^\perp = \nabla_X \partial_v^\top + \text{II}(X, \partial_v^\top) - A_{\partial_v^\perp} X + \nabla_X^\perp \partial_v^\perp. \tag{10}$$

Taking tangent and normal parts in (10) we get

$$\nabla_X \partial_v^\top = A_{\partial_v^\perp} X \tag{11}$$

and

$$\nabla_X^\perp \partial_v^\perp = -\Pi(X, \partial_v^\top). \tag{12}$$

In particular, we have $\operatorname{div}(\partial_v^\top) = \operatorname{trace}(A_{\partial_v^\perp})$. Now, we can use (5), (6) and (11) to obtain that the Laplacians of μ and ν on Σ^k are

$$\Delta\mu = k\langle \vec{H}, \partial_v \rangle \tag{13}$$

and

$$\Delta\nu = \operatorname{div}(\partial_u^\top) - \partial_v^\top(\mathcal{H} \circ \psi) - k\langle \mathcal{H} \circ \psi, \vec{H} \rangle \langle \vec{H}, \partial_v \rangle. \tag{14}$$

For a codimension two spacelike submanifold Σ^n contained in a lightlike hypersurface $u = u_0$, the normal bundle is generated by the normal vector fields ∂_v and $\eta = -\partial_u^\perp + \frac{1}{2}(\partial_u^\perp, \partial_u^\perp)\partial_v$ which satisfy $\langle \eta, \eta \rangle = 0$ and $\langle \eta, \partial_v \rangle = -1$. From (11) and (12), we have

$$A_{\partial_v} = 0 \quad \text{and} \quad \nabla^\perp \partial_v = 0. \tag{15}$$

From (15), the following formula holds for the second fundamental form

$$\Pi(X, Y) = \langle A_{\partial_u^\perp}(X), Y \rangle \partial_v, \tag{16}$$

where $X, Y \in \mathfrak{X}(\Sigma^n)$. Now, decomposing ∂_u into its tangent and normal components and using (11) and (12), we get from (3)

$$\nabla_W \partial_u^\top = A_{\partial_u^\perp}(W), \quad \nabla_W^\perp \partial_u^\perp = -\Pi(W, \partial_u^\top) + \frac{1}{2} \langle \nabla \mathcal{H}_u, d(\pi_M \circ \Psi)(W) \rangle \partial_v. \tag{17}$$

Hence, from (14), (16) and (17), the mean curvature vector of a spacelike submanifold Σ^n contained in a lightlike hypersurface $u = u_0$ is given by (see [7])

$$\vec{H} = \frac{1}{n} \operatorname{trace}(A_{\partial_u^\perp}) \partial_v = \frac{1}{n} \operatorname{div}(\partial_u) \partial_v = \frac{1}{n} \Delta\nu \partial_v. \tag{18}$$

From (15) and (17) we have that each wavefront is totally geodesic and is also clearly contained in a lightlike hypersurface $u = u_0$. As shown in [7, Lemma 4.2], any codimension two spacelike submanifold of \overline{M}^{n+2} contained in a lightlike hypersurface $u = u_0$ is locally isometric to M^n . Indeed, the following lemma extends this result.

Lemma 2 *Let $\psi : \Sigma^n \rightarrow \overline{M}^{n+2}$ be a codimension two spacelike submanifold in a plane fronted wave \overline{M}^{n+2} . If $\psi(\Sigma^n)$ is contained in a lightlike hypersurface $u = u_0$ or the lightlike vector field $2\partial_u - (\mathcal{H} \circ \psi) \partial_v$ is normal to Σ^n at any point, then, $\pi_M \circ \psi : \Sigma^n \rightarrow M^n$ is a local isometry.*

Proof From (2), for any $Z \in T_p \Sigma^n$ we obtain

$$\langle d\psi_p(Z), d\psi_p(Z) \rangle = \langle Z, \partial_v \rangle \langle Z, 2\partial_u - (\mathcal{H} \circ \psi)_p \partial_v \rangle + g_M(d(\pi_M \circ \psi)_p(Z), d(\pi_M \circ \psi)_p(Z)).$$

Hence, the first addend on the right hand side vanishes under our assumptions. □

Remark 3 By means of Eqs. (5) and (6) we obtain that $2\partial_u - (\mathcal{H} \circ \psi) \partial_v$ being normal to Σ^n is equivalent to

$$2\nabla\nu + (\mathcal{H} \circ \psi) \nabla\mu = 0.$$

Therefore, if $2\partial_u - (\mathcal{H} \circ \psi) \partial_v$ is normal to Σ^n and $\psi(\Sigma^n)$ is contained in a lightlike hypersurface $u = u_0$, then $\psi(\Sigma^n)$ is contained in a wavefront.

3 Main results

First, let us obtain the next result for the restriction of the quasi-time function u to any spacelike submanifold.

Proposition 4 *Let $\psi : \Sigma^k \longrightarrow \overline{M}^{n+2}$ be a spacelike submanifold in a plane fronted wave.*

- (i) *If $\langle \vec{H}, \partial_v \rangle > 0$, then the function μ attains no local maximum.*
- (ii) *If $\langle \vec{H}, \partial_v \rangle < 0$, then μ attains no local minimum.*

Proof Assume μ attains a local maximum at p_{max} . Therefore, at p_{max} from (13) we have $\Delta\mu(p_{max}) = k\langle \vec{H}, \partial_v \rangle(p_{max}) \leq 0$, contradicting the fact that $\langle \vec{H}, \partial_v \rangle > 0$. The proof of the second statement is analogous. \square

As a direct consequence of Proposition 4, we have the following result.

Corollary 5 *Let $\psi : \Sigma^k \longrightarrow \overline{M}^{n+2}$ be a spacelike submanifold in a plane fronted wave with $\langle \vec{H}, \partial_v \rangle > 0$ (resp. $\langle \vec{H}, \partial_v \rangle < 0$). Then, there is no (non-empty) open subset $U \subset \Sigma^k$ such that $\psi(U)$ is contained in a lightlike hypersurface $u = u_0$.*

Another immediate consequence of Proposition 4 is

Corollary 6 *Let $\psi : \Sigma^k \longrightarrow \overline{M}^{n+2}$ be a spacelike submanifold in a plane fronted wave. Then,*

- (i) *If μ attains a local maximum, then $\inf_{\Sigma^k} \langle \vec{H}, \partial_v \rangle \leq 0$.*
- (ii) *If μ attains a local minimum, then $\sup_{\Sigma^k} \langle \vec{H}, \partial_v \rangle \geq 0$.*

Note that, as an application of Proposition 4, we reobtain the well known non-existence of compact trapped submanifolds in a plane fronted wave [7, 15].

Our next result ensures the constancy of the quasi-time function’s restriction to the spacelike submanifold under certain assumptions, equivalently, under which conditions the spacelike submanifold factorizes through the lightlike hypersurface $u = u_0$, for $u_0 \in \mathbb{R}$.

Theorem 7 *Let $\psi : \Sigma^k \longrightarrow \overline{M}^{n+2}$ be a complete, orientable, non-compact spacelike submanifold in a plane fronted wave.*

- (i) *If $\langle \vec{H}, \partial_v \rangle \geq 0$, $\mu \geq u_0$, for $u_0 \in \mathbb{R}$ and*

$$\lim_{r(p) \rightarrow \infty} \mu = u_0,$$

where $r(p) = d(p, o)$ is the Riemannian distance function on Σ^k from a fixed point $o \in \Sigma^k$, then, $\mu = u_0$.

- (ii) *If $\langle \vec{H}, \partial_v \rangle \leq 0$, $\mu \leq u_0$ for $u_0 \in \mathbb{R}$ and*

$$\lim_{r(p) \rightarrow \infty} \mu = u_0,$$

then, $\mu = u_0$.

Proof To prove the first statement, let us define on Σ^k the function $\widehat{\mu} := \mu - u_0$. From our assumptions we have that $\widehat{\mu} \geq 0$ and

$$\lim_{r(p) \rightarrow \infty} \widehat{\mu} = 0.$$

Furthermore, using (5) we obtain

$$\langle \nabla \widehat{\mu}, \partial_v^\top \rangle = |\partial_v^\top \widehat{\mu}|^2 = |\nabla \mu|^2 \geq 0.$$

Moreover, from our assumptions and (13) we have

$$\operatorname{div}(\partial_v^\top) = \Delta \mu = k \langle \vec{H}, \partial_v \rangle \geq 0.$$

Reasoning by contradiction, suppose that $\mu \neq u_0$. Hence, for some $p \in \Sigma^k$ we have $\widehat{\mu}(p) > 0$, so that $\widehat{\mu} \neq 0$. Thus, we can apply [2, Thm. 2.2] to conclude that

$$\langle \nabla \widehat{\mu}, \partial_v^\top \rangle = |\nabla \mu|^2 = 0.$$

Therefore, μ must be constant on Σ^k and since μ is asymptotic to u_0 we obtain $\mu = u_0$, reaching a contradiction. The proof of (ii) is analogous defining on Σ^k the function $\widetilde{\mu} := u_0 - \mu$ and using the vector field $-\partial_v^\top$ instead of ∂_v^\top in the computations. \square

In particular, for a spacelike submanifold $\psi : \Sigma^k \rightarrow \mathbb{L}^{n+2}$ in the Lorentz–Minkowski spacetime \mathbb{L}^{n+2} , using Beltrami’s equation $\Delta \psi = k \vec{H}$, we derive from Theorem 7 the next result.

Corollary 8 *Let $\psi : \Sigma^k \rightarrow \mathbb{L}^{n+2}$ be a complete, orientable, non-compact spacelike submanifold in \mathbb{L}^{n+2} with $\psi = (\psi_0, \psi_1, \dots, \psi_{n+1})$. Assume that there is $j \in \{1, \dots, n + 1\}$ such that $\Delta(\psi_j - \psi_0) \geq 0$ and $\psi_j - \psi_0 \geq u_0$, for $u_0 \in \mathbb{R}$. If*

$$\lim_{r(p) \rightarrow \infty} (\psi_j - \psi_0) = u_0,$$

where $r(p) = d(p, o)$ is the Riemannian distance function on Σ^k from a fixed point $o \in \Sigma^k$, then ψ factorizes through the lightlike hyperplane $x_j - x_0 = u_0$.

Remark 9 For the case of a spacelike hypersurface ($k = n + 1$), the orientability assumption in Corollary 8 can be dropped. Indeed, the orientation and time orientation of \mathbb{L}^{n+2} guarantees the orientability of Σ^{n+1} . In general, if the ambient spacetime \overline{M}^{n+2} is orientable, then every spacelike hypersurface in \overline{M}^{n+2} is also orientable.

Remark 10 Note that for the codimension two case, under the assumptions of Lemma 2, Σ^n is locally isometric to M^n . Therefore, if the plane fronted wave verifies the TCC, then the Ricci curvature of Σ^n will be non-negative (4). In the particular case where $n = 2$, this fact means that the Gaussian curvature of Σ^2 is non-negative. Hence, if Σ^2 is also complete, it must be parabolic by [11]. More generally, in a plane fronted wave $\overline{M}^{n+2} = \mathbb{R}^2 \times M^n$ with M^n parabolic, a simply connected complete codimension two spacelike submanifold with $\mu = u_0$ or everywhere orthogonal to $2\partial_u - \mathcal{H}\partial_v$ is also parabolic due to Lemma 2 and the fact that parabolicity is invariant under quasi-isometries [10, Cor. 5.3].

Furthermore, for parabolic spacelike submanifolds we have the next theorem.

Theorem 11 Let $\psi : \Sigma^k \rightarrow \overline{M}^{n+2}$ be a parabolic spacelike submanifold in a plane fronted wave. If $\langle \vec{H}, \partial_\nu \rangle \geq 0$ (resp., $\langle \vec{H}, \partial_\nu \rangle \leq 0$) and $\sup_{\Sigma^k} \mu < +\infty$ (resp., $\inf_{\Sigma^k} \mu > -\infty$), then $\mu = u_0$. In addition, if Σ^n is a simply-connected codimension two submanifold, then the wavefront is parabolic.

Proof From (13), if $\langle \vec{H}, \partial_\nu \rangle \geq 0$ holds, μ would be a superharmonic function on a parabolic manifold. Thus, if it is bounded from below, it must be constant. We can prove the other case in a similar way.

To conclude, the last statement is a consequence of the fact that in the codimension two case, if μ is constant and Σ^n is simply connected, then it is globally isometric to the wavefront [7, Prop. 4.6]. \square

As an immediate consequence of Theorem 11 we deduce that plane fronted waves do not admit compact spacelike hypersurfaces which are everywhere non-contracting or non-expanding.

Corollary 12 In a plane fronted wave there are no compact spacelike hypersurfaces with signed mean curvature.

Moreover, if we consider a spacelike hypersurface in a plane fronted wave and choose its normal unitary vector N such that $\langle N, \partial_\nu \rangle < 0$, we obtain from Theorem 7 the next result for parabolic spacelike hypersurfaces (compare with [16, Thm. 4] and [19, Thm. 4]).

Corollary 13 In a plane fronted wave there are no parabolic spacelike hypersurfaces with non-positive mean curvature and μ bounded from above nor parabolic spacelike hypersurfaces with non-negative mean curvature and μ bounded from below.

Also, by means of Theorem 11 we can deduce the following non-existence result for parabolic weakly trapped submanifolds in a plane fronted wave.

Theorem 14 In a plane fronted wave there are no codimension two parabolic weakly future trapped (resp., weakly past trapped) submanifolds with $\inf_{\Sigma^n} \mu > -\infty$ and $\sup_{\Sigma^n} \nu < +\infty$ (resp., $\sup_{\Sigma^n} \mu < +\infty$ and $\inf_{\Sigma^n} \nu > -\infty$).

Proof Let us assume the existence of a codimension two parabolic weakly future trapped submanifold in a plane fronted wave with μ bounded from below and ν bounded from above. From Theorem 11 we obtain that it satisfies $\mu = u_0$. Moreover, from (18) we obtain $\Delta \nu > 0$, which combined with the upper bound of ν and the parabolicity of the submanifold implies that ν is constant and therefore, the submanifold would be extremal, reaching a contradiction. We can prove the other case in an analogous manner. \square

Using these ideas we get the next rigidity result for the extremal case in codimension two, which extends [7, Thm. 4.10] to the non-compact case.

Theorem 15 In a plane fronted wave the only codimension two parabolic extremal submanifolds with μ and ν bounded (either from above or below) are the (necessarily parabolic) wavefronts.

Proof From (13) we obtain that μ is a bounded harmonic function in a parabolic Riemannian manifold. Thus, $\mu = u_0$ on Σ^n . In addition, from (18) we have that since ν is also a bounded harmonic function $\nu = v_0$. \square

In addition, using Yau’s result obtained in [20, Cor. 1] we can extend this result to codimension two spacelike submanifolds with non-negative Ricci curvature.

Theorem 16 *In a plane fronted wave the only complete codimension two extremal submanifolds with μ and ν bounded (either from above or below) and whose Ricci curvature verifies $\text{Ric} \geq 0$ are the wavefronts.*

To conclude, we can provide another extension to the non-parabolic case by means of Theorem 7 as follows.

Theorem 17 *Let $\psi : \Sigma^n \rightarrow \overline{M}^{n+2}$ be a complete, orientable, non-compact extremal submanifold in a plane fronted wave that satisfies the TCC. If μ and ν are bounded (either from above or below) by $u_0, v_0 \in \mathbb{R}$, respectively, and*

$$\lim_{r(p) \rightarrow \infty} \mu = u_0,$$

then Σ^n is a wavefront.

Proof From Theorem 7, we obtain that $\mu = u_0$. Moreover, if \overline{M}^{n+2} satisfies the TCC, [7, Prop.12] guarantees that Σ^n has non-negative Ricci curvature. Now, using (18) and our assumptions we obtain that ν is a bounded harmonic function on a Riemannian manifold with non-negative Ricci curvature. Thus, it is constant by [20, Cor. 1]. \square

Example 18 A remarkable family of codimension two spacelike submanifolds in a plane fronted wave \overline{M}^{n+2} satisfying $\mu = u_0$ are the spacelike graphs introduced in [7, Sec. 4]. Namely, given a smooth function $h \in C^\infty(\Omega)$, where Ω is an open domain in M^n and $u_0 \in \mathbb{R}$, the graph

$$\Sigma_{u_0}^n(h) = \{ (u_0, h(x), x) \in \mathbb{R} \times \mathbb{R} \times M^n : x \in \Omega \}, \tag{19}$$

defines a codimension two spacelike submanifold, for any h and u_0 .

Note that, as a direct consequence of (18), the mean curvature of the graph $\Sigma_{u_0}^n(h)$ is $\vec{H} = (\Delta h/n) \partial_\nu$. Clearly, each of these graphs is a wavefront if and only if h is constant. Thus, the assumptions in Theorems 15, 16 and 17 cannot be weakened, since we would easily find counterexamples within this family of spacelike graphs.

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