A Jordan Canonical Form for nilpotent elements in arbitrary ring.

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Abstract:
In this paper we give an inductive new proof of the Jordan canonical form of a nilpotent element in an arbitrary ring. If $a \in R$ is a nilpotent element of index $n+1$ with von Neumann regular $a^n$, we decompose $a=a_1+a_2(1-a_1)$ with $a_1 e R e A_n$, a Jordan block of size $n+1$ over a corner $S$ of $R$, and $(1-a_1)a^k$ nilpotent of index $\leq n+1$ for an idempotent $e$ of $R$ commuting with $a$. This result makes it possible to characterize prime rings of bounded index $n$ with a nilpotent element $a \in R$ of index $n$ and von Neumann regular $a^{n-1}$ as a matrix ring over a unital domain.

Introduction:

Von Neumann regular elements: An element $a \in R$ is said to be von Neumann regular if there exists $b \in R$ such that $aba=a$.

Nilpotent last regular element: A nilpotent element $a \in R$ of index $n+1$ is said to be last regular if $a^n$ is von Neumann regular.

Rus-inverse: Given a nilpotent last regular element $a \in R$ of index $n+1$, we said that $b \in R$ is a Rus-inverse of $a$ if

$$a^n b a^n = a^n, \quad b a^n b = b, \quad \text{and} \quad b a^n b = 0$$

for every $0 \leq k \leq n-1$.

Lemma[1]: Let $R$ be a ring and let $a \in R$ be a nilpotent last regular element of index $n+1$. Then there exists $b \in R$ a Rus-inverse of $a$.

Theorem[2]: Let $R$ be a ring and let $a \in R$ be a nilpotent last regular element of index $n+1$. Then $b$ is a Rus-inverse of $a$.

Definition[3]: A nilpotent element $a$ of index $n$ is called a last regular element if $a^n$ is von Neumann regular.

Remark: In general, neither the Rus-inverse nor the associated idempotent in [1] and [2] are unique: if $M_n(F)$, and $a=e_{1,2}$ the element $b=e_{2,1}$ is a Rus-inverse for $a$ and $a$ is a block-element with associated idempotent $e=a_{1,1}, e_{2,2}$ the element $b'=e_{1,2}, e_{2,1}$ is another Rus-inverse for $a$ and $aS$ is a block-element with associated idempotent $e'=e_{1,1}, e_{2,2}+e_{1,2}$.

Definition: We say that a nilpotent last regular element $a \in R$ of index $n+1$ is block-maximal if one of its associated block-idempotents belongs to the center of $R$, i.e., if there exists a Rus-inverse of such a that the idempotent block constructed in [1] is central.

Proposition: If $a \in R$ has maximal index of nilpotence, then it is block-maximal.

References:

OTHER NOTIONS

Von Neumann regular: A ring $R$ is said to be von Neumann regular if every element of $R$ is von Neumann regular.

Abelian Regular: A ring $R$ is said to be an abelian regular ring if $R$ is von Neumann regular and every idempotent of $R$ is contained in the center of $R$. 