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A Jordan Canonical Form for nilpotent elements in arbitrary ring.

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Abstract:

In this paper we give an inductive new proof of the Jordan canonical form of a nilpotent element in an arbitrary ring. If $a \in R$ is a nilpotent element of index n+1 with von Neumann regular a^n, we decompose $a=ea+(1-e)a$ with $ea \in eRe \approx M_n(S)$ a Jordan block of size n+1 over a corner S of R, and $(1-e)a$ nilpotent of index <n+1 for an idempotent e of R commuting with a. This result makes it possible to characterize prime rings of bounded index n with a nilpotent element $a \in R$ of index n and von Neumann regular a^{n-1} as a matrix ring over a unital domain.

Introduction:

Von Neumann regular elements: An element $a \in R$ is said to be von Neumann regular if there exists $b \in R$ such that $aba=a$.

Nilpotent Last regular element: A nilpotent element $a \in R$ of index n+1 is said to be last regular if a^n is von Neumann regular.

Rus-inverse: Given a nilpotent last regular element $a \in R$ of index n+1, we said that $b \in R$ is a Rus inverse of a if $a^n b a^n = a^n$, $ba^n b=b$ and $ba^k b=0$ for every $0 \leq k \leq n-1$.

Lemma[1]: Let R be a ring and let $a \in R$ be a nilpotent last regular element of index n+1. Then there exists $b \in R$ a Rus-inverse of a.

Theorem[2]: Let R be a ring and let $a \in R$ be a nilpotent last regular element of index n+1. Let b be a Rus-inverse of a. Then there exists n+1 nonzero orthogonal idempotents e_k, with k=1,…,n+1 in R (depending on b) such that if $e= \sum e_k$

- $ea=ae$,
- $a^n e= a^n$,
- $(1-e)a$ is nilpotent of index less than n+1,
- $eRe \approx M_n(e_1 R e_1)$, and if e_{ij} are the matrix units of the matrix ring eRe, $ea=eae =\sum e_{k+1,k}$ (a Jordan Block)
- For every $s\in\{1,2,\dots, n\}$, a^s is unit-regular in eRe: taking $d= \sum e_{k,k+1} + e_{n+1,1}$, then $(ea)^s d^s (ea)^s = (ea)^s$ and d^s is invertible in eRe.

In the matrix representation of ea on item (d) we have that $e_{1,n+1}$=eb=be is a Rus-inverse for ea with associated idempotent e.

Theorem: Let R be a ring and $a \in R$ be a nonzero nilpotent element of index n such that for every $s \in N$, a^s is von Neumann regular. Then there exists a family u_i with i=1,…k, of nonzero orthogonal idempotents that commute with a and such that $a = \sum u_i a$, and every $u_i a$ is a nilpotent block-element of index n_i with $n= n_1 > n_2 > \dots > n_k$ associated to the block-idempotent u_i.

Theorem: Let A be a unital algebra over a field F and let a be an algebraic element in A such that its minimal polynomial $m_a(X)$ totally decomposes in F[X] as $\prod (X- \lambda_i)^{k_i}$ where $\lambda_i\neq \lambda_j$ if $i \neq j$. Suppose that $(a- \lambda_i)^{k_i}$ is von Neumann regular for every $k_i \leq n_i$, i=1,…, k. Then there exists a family of orthogonal idempotents $\{v_s\}$ where s=1,…,k and families of orthogonal idempotents $\{u_{s,i}\} \subseteq v_s R v_s$, all commuting with a, such that $a \sum u_{s,i} a$, where each $u_{s,i} a- \lambda_s u_{s,i} \in u_{s,i} R u_{s,i}$ is a nilpotent block-element of index $n_{s,i} \leq n_s$ associated to the block-idempotent $u_{s,i.}$

Remark: In general, neither the Rus-inverse nor the associated idempotent in [1] and [2] are unique: If $M_n(F)$, and $a= e_{1,2}$ the element $b= e_{2,1}$ is a Rus-inverse for a and a is a block-element with associated block-idempotent $e= e_{1,1} + e_{2,2}$ the element $b'= e_{2,1} + e_{3,1}$ is another Rus-inverse for a and a is a block-element with associated block-idempotent $e'= e_{1,1} + e_{2,2} +e_{3,2}$

Definition: We say that a nilpotent last regular element $a \in R$ of index n+1 is block-maximal if one of its associated block-idempotents belongs to the center of R, i.e., if there exists a Rus-inverse of a such that the idempotent built in [1] is central.

Proposition: Al central block-idempotents associated to a block-maximal element coincide. If $a \in R$ has maximal index of nilpotence, then it is block-maximal index n. Then

- Any Rus-inverse of a gives rise to the same (central) idempotent e and $R=eRe \oplus(1-e)R(1-e)$. Moreover, $eRe \approx M_n(S)$, where S is a ring without nonzero nilpotent elements.
- If R is von Neumann regular, $eRe \approx M_n(S)$, where S is abelian regular.
- If R is indecomposable, $eRe \approx M_n(S)$, where S is a unital ring without nilpotent elements.
- If R is prime, $eRe \approx M_n(S)$, where S is a unital domain.
- If R is indecomposable and von Neumann regular, $eRe \approx M_n(S)$, where S is a division ring.

In any case, as soon as R is indecomposable, e=1 and a is a block-element.

The proof of this theorem is algorithmic, so it provides a method to compute the Jordan canonical form of any nilpotent matrix:

Let $A \in M_n(F)$ nilpotent of index m.

- A^{m-1} is von Neumann regular and there exists $B \in M_n(F)$ with
$$A^{m-1}B\, A^{m-1} =A^{m-1}$$
- $B'= B A^{m-1} B$ satisfies $A^{m-1}B' A^{m-1} =A^{m-1}$ and $B' A^{m-1} B'=B'$
- If consider $D=1-B A^{m-1}$ and $B''=DB'$ we have
$$A^{m-1}B'' A^{m-1} =A^{m-1},\ B'' A^{m-1} B''=B'' \text{ and } B'' A^{m-2} B''=0$$
- If we consider B with
$$A^{m-1}B\, A^{m-1} =A^{m-1},\ B A^{m-1} B=B \text{ and } B A^k B=0 \text{ for } k= s+1,\dots,m-2$$

And consider $D=1- A^{m-1-s}B A^s$ and $B'=DB$ we have
$$A^{m-1}B' A^{m-1} =A^{m-1},\ B'A^{m-1} B'=B' \text{ and } B' A^k B'=0 \text{ for } k= s,\dots,m-2$$

Note that: If B is a Rus inverse for A, the dimension of $Im(e_{1,1})$ where $e_{1,1}$ is the idempotent $B A^{m-1}$ is the number of Jordan Block of size m and for any element of a base $\{v_1,\dots, v_k\}$ of $Im(e_{1,1})$

$\{v_s, A v_s ,\dots, A^{m-1} v_s\}$ is a Base of each Jordan Block of size m

References:

[1] E. García, M. Gómez Lozano R. Munóz and G. vera de Salas, " A Jordan canonical form for nilpotent elements in an arbitrary ring" Linear Algebra and its Applications, Volume 581, Pages 324-335

OTHER NOTIONS

Von Neumann regular: A ring R is said to be an von Neumann regular if every element of R is von Neumann regular.

Abelian Regular: A ring R is said to be an abelian regular ring if R is von Neumann regular and every idempotent of R is contained in the center of R.