Associative and Lie algebras of quotients. Zero product determined matrix algebras.

TESIS DOCTORAL

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Mercedes Siles Molina, Profesora Titular de Álgebra de la Universidad de Málaga informa: Que ha dirigido la Tesis Doctoral titulada "Associative and Lie algebras of quotients. Zero product determined matrix algebras" ("Álgebras de cocientes asociativos y de Lie. Álgebras de matrices determinadas por un producto nulo"), realizada por la Licenciada Juana Sánchez Ortega.

Finalizada la investigación que ha llevado a la conclusión de la citada Tesis Doctoral, y de acuerdo con el artículo 8.1 del Real Decreto 778/98 de 30 de abril BOE de 1 de mayo del 98, autoriza su presentación por considerar que reúne los requisitos formales y científicos legalmente establecidos para la obtención del título de Doctora en Matemáticas.

Y para que así conste y surta los efectos oportunos expide y firma el presente informe en Málaga a 11 de diciembre de 2007.

Fdo.: Mercedes Siles Molina

A mis padres
y a mi hermana

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Introduction

The notion of ring of quotients has played an important role in the development of the theories of associative and commutative rings. Its origin can be placed between 1930 and 1940 in the works of Ore and Osano on the construction of the total ring of fractions.

In order to ease the difficult task of finding a ring of quotients of a given ring, Ore proved that a necessary and sufficient condition (the well-known **right Ore condition**) for a ring R to have a classical right ring of quotients is that for any regular element $a \in R$ and any $b \in R$ there exist a regular element $c \in R$ and $d \in R$ such that $bc = ad$.

At the end of 50's, Goldie, Lesieur and Croisot characterized the rings that are classical right orders in semiprime and right artinian rings. This result is nowadays known as **Goldie's Theorem**. (See [55, Chapter IV].) In 1956, Utumi introduced in [82] a more general notion of right quotient ring, that would generalize the others quotients: an overring Q of a ring R is said to be a **(general) right quotient ring of** R if given p, $q \in Q$, with $p \neq 0$, there exists $a \in R$ such that $pa \neq 0$ and $qa \in R$.

In his paper, Utumi proved that every ring R without total left zero divisor (it happens for example when R is semiprime) has a maximal right ring of quotients $Q^r_{max}(R)$ and constructed it. Maximal in the sense that every right quotient ring of R can be embedded into $Q^r_{max}(R)$ via a monomorphism which is the identity when restricted to R.

The fact that, in the associative case, rings of quotients allow a deeper understanding of certain classes of rings motivated to several authors to extend these notions and results to the non-associative setting.

The study of Jordan algebras of quotients has its origin in the question raised by Jacobson [47, p. 426] of whether it would be possible to imbed a Jordan domain in a Jordan division algebra, emulating Ore's construction in the associative setting. This problem inspired to many authors to study suitable algebras of quotients for Jordan algebras and also to try to adapt Goldie's theory in the Jordan setting.

The search of a Jordan version of Goldie's Theorem was solved in the case of special Jordan algebras $J = H(A, *)$ by D. J. Britten and S. Montgomery. (See [20, 21, 22, 69].) A definitive solution for linear Jordan algebras was given by E. Zelmanov in [85, 86] making use of his fundamental result on the structure theory of strongly prime Jordan algebras. Later on A. Fernández López, E. García Rus and F. Montaner extended [37] this result to quadratic Jordan algebras.

Recently, C. Martínez [25] solved the original problem of finding analogues of Ore's ring of fractions by a different approach. In her work, she gave an Ore type condition for a Jordan algebra to have a classical algebra of fractions. Moreover, making use of the Tits-Kantor-Koecher construction that relates the Jordan and Lie structures, she built a maximal Jordan algebra of quotients considering partial derivations.

The study of algebras of quotients for Lie algebras was initiated by M. Siles Molina in [79]. She introduced, following the original pattern of Utumi, the notion of a general (abstract) algebra of quotients of a Lie algebra: an overalgebra Q of a Lie algebra L is said to be an **algebra of quotients of** L if given $p, q \in Q$, with $p \neq 0$, there exists $a \in L$ such that $[a, p] \neq 0$ and

$[a, \operatorname{ad} x_1 \operatorname{ad} x_2 \ldots \operatorname{ad} x_n q]$, for every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in L$.

In keeping with Martínez's idea of considering equivalence classes of partial derivations, M. Siles Molina built the maximal algebra of quotients of a semiprime Lie algebra. (See [79, Section 3].)

Using Siles Molina's construction of maximal Lie algebras of quotients and inspired by Martínez's idea of moving from a Jordan setting to a Lie one through the Tits-Kantor-Koecher construction, E. García and M. A. Gómez Lozano gave in [39] the notion of maximal Jordan system of quotients for non-degenerate Jordan systems.

All these ideas have been the starting point for our work. Concretely, one can regard the main part of this thesis as a development of the theory of algebras of quotients of Lie algebras and of Jordan systems of quotients. We will show that Lie algebras of quotients, in particular graded Lie algebras of quotients, which will be introduced in Chapter 2, are the natural framework were to settle the different quotients for Jordan systems that we have just mentioned.

We describe now the organization of this thesis, namely, the content of the chapters and their sections. The first chapter is devoted to the study of Lie and graded Lie algebras of quotients; the original results can be found in the papers [29, 78]. We start by collecting, in Sections 1.1, 1.2 and 1.3, the main definitions and results that will be needed throughout the chapter and even the thesis. The bulk of Section 1.4 is to extend the notion of (weak) algebra of quotients of Lie algebras for graded Lie ones. We define:

Definitions 1.4.12. Let $L = \oplus_{\sigma \in G} L_\sigma$ be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma \in G} Q_\sigma$.

– We say that Q is a **graded algebra of quotients of** L if given $0 \neq p_\sigma \in Q_\sigma$ and $q_\tau \in Q_\tau$, there exists $x_\alpha \in L_\alpha$ such that $[x_\alpha, p_\sigma] \neq 0$ and $[x_\alpha, {}_L(q_\tau)] \subseteq$

L. The algebra L will be called a **graded subalgebra of quotients of** Q.

– If for any nonzero $p_\sigma \in Q_\sigma$ there exists $x_\alpha \in L_\alpha$ such that $0 \neq [x_\alpha, p_\sigma] \in L$, then we say that Q is a **graded weak algebra of quotients of** L, and L is called a **graded weak subalgebra of quotients of** Q.

A necessary and sufficient condition for a graded Lie algebra to have a graded (weak) algebra of quotients is given.

We show (see Proposition 1.4.18) that as it happens in the non-graded case, graded algebras of quotients of graded Lie algebras inherit primeness, semiprimeness and strongly non-degeneracy.

The relationship between Lie and associative quotients has studied by F. Perera and M. Siles Molina in [75] being one of their main results the following:

Theorem. ([75, Theorem 2.12 and Proposition 3.5]). Let A be a semiprime associative algebra and Q a subalgebra of $Q_s(A)$ that contains A. Then

(i) $A^-/Z_A \subseteq Q^-/Z_Q$ and $[A, A]/Z_{[A,A]} \subseteq [Q, Q]/Z_{[Q,Q]}$ are dense extension.

(ii) Q^-/Z_Q is an algebra of quotients of A^-/Z_A and $[Q, Q]/Z_{[Q,Q]}$ is an algebra of quotients of $[A, A]/Z_{[A,A]}$.

Our target in Section 1.5 is to extend the result above to skew Lie algebras. We will prove the following theorem.

Theorem 1.5.19. Let A be a semiprime associative algebra with an involution $*$ and let Q be a $*$-subalgebra of $Q_s(A)$ containing A. Then the following conditions are satisfied:

(i) K_A is a dense subalgebra of K_Q, and $[K_A, K_A]$ is a dense subalgebra of $[K_Q, K_Q]$.

(ii) K_A/Z_{K_A} is a dense subalgebra of K_Q/Z_{K_Q}, and $[K_A, K_A]/Z_{[K_A, K_A]}$ is a dense subalgebra of $[K_Q, K_Q]/Z_{[K_Q, K_Q]}$.

(iii) K_Q/Z_{K_Q} is an algebra of quotients of K_A/Z_{K_A}, and $[K_Q, K_Q]/Z_{[K_Q, K_Q]}$ is an algebra of quotients of $[K_A, K_A]/Z_{[K_A, K_A]}$.

To conclude the chapter we analyze, in Section 1.6, the relationship between graded (weak) algebras of quotients and (weak) algebras of quotients.

Following the construction given by Siles Molina in [79] of the maximal algebra of quotients $Q_m(L)$ of a semiprime Lie algebra L, we build, in Chapter 2, a maximal graded algebra of quotients for every graded semiprime Lie algebra. Taking into account that the elements of $Q_m(L)$ arise from partial derivations defined on essential ideals, our ingredients now are graded partial derivations and graded essential ideals. With this idea in mind we introduce in **Construction 2.2.3** a new Lie algebra denoted by $Q_{gr-m}(L)$. In Section 2.2 we show that $Q_{gr-m}(L)$ has good properties; let us point out here that it is a graded algebra of quotients of L and cannot be enlarged.

Our objective in the rest of the chapter is to compute $Q_m(L)$ for some Lie algebras. Specifically, we are interested in Lie algebras of the form $L = A^-/Z$, where A^- is the Lie algebra associated to a prime associative algebra A with center Z, and in Lie algebras of the form $L = K/Z_K$, where K is the Lie algebra of skew elements of a prime associative algebra with involution and Z_K its center. More concretely:

in Section 2.3 we compute $Q_m(A^-/Z)$; it turns out that (under a very mild technical assumption) it is equal to a certain Lie algebra that arises from derivations from nonzero ideals of A into A. Its definition is a bit technical to be stated here; let us just mention that this Lie algebra lies between $\mathrm{Der}(A)$ and $\mathrm{Der}(Q_s(A))$, where $Q_s(\,.\,)$ denotes the symmetric Martindale algebra of quotients of A. Section 2.4 yields similar results for K/Z_K (the analogy with the A^-/Z case is perfect, the only difference is that we have to deal only with derivations δ that preserve $*$; in the sense that $\delta(x^*) = \delta(x)^*$).

The purpose of Chapter 3 is to determine when some important properties of associative algebras of quotients remain true in the context of Lie algebras of quotients. The problem of whether $Q_m(I)$, where I is an essential ideal of a semiprime Lie algebra L, is equal to $Q_m(L)$ is considered in Section 3.1. It is well-known that this result is true in the associative case (see e.g. [15, Proposition 2.1.10]). In the Lie setting, we will give a positive answer provided that L is strongly semiprime: a Lie algebra L is said to be **strongly semiprime** (respectively, **strongly prime**) if:

(i) L is semiprime (respectively, prime).

(ii) For each n, given $0 \neq U_n \lhd \ldots \lhd U_2 \lhd U_1 \lhd L$ there exists $0 \neq W \lhd L$ such that $W \subseteq U_n$.

Theorem 3.1.7. Let I be an essential ideal of a strongly semiprime Lie algebra L. Then $Q_m(I)$ is the maximal algebra of quotients of L, i. e. $Q_m(I) \cong Q_m(L)$.

Once we have built the maximal graded algebra of quotients, it is natural to ask, as we have just made in section above, whether $Q_{gr-m}(I)$ will be isomorphic to $Q_{gr-m}(L)$, for a graded essential ideal I of a graded semiprime Lie algebra L. This question will be treated in Section 3.2.

Finally, Section 3.3 is devoted to the question of whether $Q_m((Q_m(L))$ is equal to $Q_m(L)$. While in the associative setting in which the answer to this question is positive (see [15, Theorem 2.1.11]); we show that in certain special situations this holds true, namely, if L is a simple algebra or if $L = A^-/Z$, where A is either a simple associative algebra (satisfying a minor technical assumption) or an affine PI prime algebra (i.e. a finitely generated prime algebra which satisfies a polynomial identity). In general, however, it is not true that $Q_m((Q_m(L))$ agrees with $Q_m(L)$; we give an example (see Example 3.3.7) by using the example that Passman gave in [73] to show that take $Q_s(.)$ is not a closure operation.

The relationship between Lie algebras of quotients and Jordan systems of quotients in the sense of [39] is studied in Chapter 4. It is divided in two sections. The first one deals with 3-graded Lie algebras; we prove that for a 3-graded semiprime Lie algebra L, the maximal algebra of quotients of L is 3-graded too and coincides with the maximal graded algebra of quotients of L. Namely, the result is the following:

Theorem 4.1.2. Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a semiprime 3-graded Lie algebra. Then:

(i) $Q_m(L)$ is graded isomorphic to $Q_{gr-m}(L)$.

(ii) If L is strongly non-degenerate and Φ is 2 and 3-torsion free, then $Q_m(L)$ is a 3-graded strongly non-degenerate Lie algebra.

Finally, in Section 4.2, and making use of the Tits-Kantor-Koecher construction, we relate maximal Jordan systems of quotients to maximal Lie algebras of quotients. Our main results are the following:

Theorem 4.2.11. Assume that $\frac{1}{6} \in \Phi$.

(i) Let V be a strongly non-degenerate Jordan pair. Then

$$Q_m(V) = \Big((Q_m(\mathrm{TKK}(V)))_1,\, (Q_m(\mathrm{TKK}(V)))_{-1}\Big)$$

is the maximal Jordan pair of $\mathfrak{M}$-quotients of V.

(ii) If $L = L_{-1} \oplus L_0 \oplus L_1$ is a strongly non-degenerate Jordan 3-graded Lie algebra satisfying that $Q_m(L)$ is Jordan 3-graded, then

$$Q_m(L) \cong Q_m(\mathrm{TKK}(V)) \cong \mathrm{TKK}(Q_m(V)),$$

where $V = (L_1, L_{-1})$ is the associated Jordan pair of L.

Theorem 4.2.22. Let T be a strongly non-degenerate Jordan triple system over a ring of scalars Φ containing $\frac{1}{6}$. Then the maximal Jordan triple system of $\mathfrak{M}$-quotients of T is the first component of the maximal algebra of quotients of the TKK-algebra of the double Jordan pair $V(T) = (T, T)$ associated to T, i.e.,

$$Q_m(T) = (Q_m(\mathrm{TKK}(V(T))))_1.$$

Theorem 4.2.27. Let J be a strongly non-degenerate Jordan algebra over a ring of scalars Φ containing $\frac{1}{6}$. Then

$$Q_m(J) = Q_m(J_T) = (Q_m(\mathrm{TKK}(V(J_T))))_1,$$

is the maximal Jordan algebra of quotients of J, where J_T denotes the Jordan triple system associated to J and $V(J_T) = (J_T, J_T)$ is the double Jordan pair associated to J_T.

The reader can find the original results of Chapters 2, 3 and 4 in [19, 78].

During the author's stay in the University of Maribor (Slovenia), she was working, jointly M. Brešar and M. Grašič, in the problem of whether the matrices $\mathbb{M}_n(B)$, where B is any unital algebra (over a fixed commutative ring C), are zero product determined. We close this thesis with the results obtained in [17].

The most important motivation to study this problem is the connection to the thoroughly studied problems of describing zero (associative, Lie, Jordan) product preserving linear maps (see e.g. [3, 8, 30, 31, 32, 33, 45, 46, 83]).

S. Banach can be considered the encourager of this mathematical research area. He was the first one who described isometries on $L^p([0,\,1])$ with $p \neq 2$ (see [9]). Although Banach did not give the full proof for this case (this was provided by J. Lamperti [58]), he pointed out that isometries must take functions with disjoint support into functions with disjoint support. This property arises in a variety of situations and was considered by several authors. For example, in the theory of Banach lattices there is an extensive literature about linear maps $T : X \to Y$, where X and Y are Banach lattices, with the property that

$$\mid T(x) \mid \wedge \mid T(y) \mid = 0$$

whenever $x,\, y \in X$ are such that $\mid x \mid \wedge \mid y \mid = 0$. Such maps are called **disjointness preserving operators** or **d-homomorphisms**. We refer the reader to the monograph [1]. The notion of a disjointness preserving operator was exported to function algebras by E. Beckenstein and L. Narici (see [12] for general information). Let A and B be function algebras; linear operators $T : A \to B$ with the property $T(a)T(b) = 0$ are called **Lamperti operators** or **separating maps**. They have been studied over many years and by many authors; this concept of separating maps can be extended to pure algebra. The most common and natural way is to consider literally the same condition, that is, a linear map T from an algebra A into an algebra B is called a **zero product preserving map** if

$$x,\, y \in A, \quad xy = 0 \Rightarrow T(x)T(y) = 0.$$

In the recent paper [18], M. Brešar and P. Šemrl consider one of the most studied linear preserver problems, that is, the problem of describing commutativity preserving linear maps. It is said that a linear map $S : A \to B$

preserves commutativity if

$$S(x)S(y) = S(y)S(x) \quad \text{whenever} \quad x,\, y \in A, \;\; xy = yx.$$

The assumption of preserving commutativity can be reformulated as the condition of preserving zero Lie product, as follows:

if $S : A \to B$ is a linear map which preserves commutativity then the bilinear map $T : A \times A \to B$ defined by $T(x, y) = [S(x), S(y)]$ clearly satisfies

$$T(x, y) = 0 \;\; \text{whenever} \;\; [x, y] = 0,$$

which means that T **preserves zero Lie product**.

Brešar and Šemrl have proved (see [18, Theorem 2.1]) that in the simplest case where $B = C$ the matrices $\mathbb{M}_n(C)$ are zero Lie product determined. We will obtain it as a consequence of our results. In Section 4.2 we show that for the ordinary product $\mathbb{M}_n(B)$ are zero product determined for every unital algebra B and every $n \geq 2$, and in Section 4.3 we prove the same for the Jordan product; however we will have to assume that $n \geq 3$ and 2 invertible in B. The behavior of the Lie product is very different; this case will be treated in Section 4.4.

Resumen en español
Spanish abstract

La noción de anillo de cocientes jugó un papel crucial en el desarrollo de la teoría de los anillos asociativos conmutativos. Podemos situar sus orígenes en los años 30 y 40, en los trabajos de Ore y Osano acerca de la construcción de un anillo total de fracciones.

La tarea de encontrar un anillo de cocientes de un anillo dado no es nada sencilla, por lo que para facilitarla Ore provó que una condición necesaria y suficiente (la hoy conocida por todos como **condición de Ore por la derecha**) para que un anillo R tenga un anillo clásico de cocientes por la derecha es que para todo elemento regular $a \in R$ y todo $b \in R$ exista un elemento regular $c \in R$ y un elemento $d \in R$ tales que $bc = ad$.

A finales de los años 50, Goldie, Lesieur y Croisot caracterizaron los anillos que son órdenes por la derecha clásicos en anillos semiprimos artinianos por la derecha, resultado actualmente conocido bajo el nombre de **Teoremas de Goldie**. (Ver [55, Chapter IV].) En 1956, Utumi introdujo en [82] una noción más general de anillo de cocientes por la derecha que generalizaría a los demás cocientes: Se dice que $Q \supseteq R$ es un **anillo de cocientes (general) por la derecha de** R si dados $p, q \in Q$, con $p \neq 0$, existe $a \in R$ tal que $pa \neq 0$, $qa \in R$.

En su artículo, Utumi probó que todo anillo R sin divisores totales de cero por la izquierda (esto se tiene, por ejemplo, cuando R es semiprimo) tiene un

anillo de cocientes por la derecha maximal $Q^r_{max}(R)$ y dio su construcción. Maximal en el sentido de que cualquier otro anillo de cocientes por la derecha de R puede sumergirse en $Q^r_{max}(R)$ vía un monomorfismo que restringido a R es la identidad en R.

El hecho de que, en el caso asociativo, el uso de anillos de cocientes permitiera un profundo estudio de ciertas clases de anillos motivó a diversos autores a extender estas nociones a un contexto no asociativo.

El estudio de álgebras de cocientes de álgebras de Jordan comenzó a raíz de la pregunta planteada por Jacobson [47, p. 426] acerca de cuándo es posible sumergir un dominio de Jordan en un álgebra de Jordan de división, imitando la construcción de Ore del caso asociativo. Este problema fue la fuente de inspiración de varios autores para introducir nociones de álgebras de cocientes para álgebras de Jordan en un intento de adaptar la teoría de Goldie al ambiente Jordan.

D. J. Britten y S. Montgomery [20, 21, 22, 69] dieron, para álgebras de Jordan de la forma $J = H(A, *)$, una versión de los Teoremas de Goldie. La solución definitiva para álgebras de Jordan lineales fue dada por E. Zelmanov [85, 86], haciendo uso de su resultado fundamental en la teoría de estructuras de las álgebras de Jordan fuertemente primas. Este resultado fue extendido a álgebras de Jordan cuadráticas por A. Fernández López, E. García Rus y F. Montaner en [37].

Más recientemente, C. Martínez [25] resolvió el problema original de encontrar un análogo al anillo de fracciones de Ore, desde un punto de vista totalmente distinto. En su trabajo, dio una condición de tipo Ore para que toda álgebra de Jordan que la satisfaga tenga un álgebra clásica de fracciones. Es más, haciendo uso de la construcción de Tits-Kantor-Koecher, la cual relaciona las estructuras de Jordan y de Lie, construyó un álgebra de Jordan de cocientes maximal, considerando derivaciones parciales.

En el ambiente Lie, siguiendo el modelo original de Utumi, M. Siles Molina inició en [79] el estudio de las álgebras de cocientes para álgebras de Lie introduciendo la siguiente noción: Se dice que un álgebra de Lie $Q \supseteq L$ es un **álgebra de cocientes de** L si dados p, $q \in Q$, con $p \neq 0$, existe $a \in L$ tal que $[a, p] \neq 0$, $[a, \operatorname{ad} x_1 \operatorname{ad} x_2 \ldots \operatorname{ad} x_n q] \subseteq L$, para cualesquiera $n \in \mathbb{N}$, $x_1, \ldots, x_n \in L$.

Basándose en la idea de Martínez de considerar clases de equivalencia de derivaciones parciales, Siles Molina construyó en [79, Sección 3] el álgebra de cocientes maximal de un álgebra de Lie semiprima. Usando esta construcción e inspirándose en la idea de Martínez de pasar del contexto Jordan al Lie a través de la construcción de Tits-Kantor-Koecher, E. García y M. A. Gómez Lozano introdujeron en [39] nociones de sistemas de cocientes maximales para sistemas de Jordan no degenerados.

Estas ideas constituyeron el punto de partida de nuestro trabajo. Concretamente, podemos ver parte de esta tesis como una contribución al progreso de la teoría de las álgebras de cocientes de álgebras de Lie y de sistemas de cocientes de sistemas de Jordan. Veremos que los cocientes Lie, en particular los cocientes de álgebras graduadas de Lie, que introduciremos en el Capítulo 2, constituyen el marco perfecto en el que situar los cocientes Jordan que acabamos de mencionar.

A continuación, describiremos cómo está organizada la tesis, es decir, el contenido de los capítulos y de las secciones. El primero de ellos se centra en el estudio de álgebras de cocientes de álgebras de Lie (graduadas); los resultados originales pueden verse en los trabajos [29, 78]. Para hacer autocontenida esta memoria empezaremos recordando, en las Secciones 1.1, 1.2 y 1.3 las principales definiciones y resultados que usaremos a lo largo de la misma. El objetivo de la Sección 1.4 es extender la noción de álgebra (débil) de cocientes de álgebras de Lie a álgebras de Lie graduadas. Definiremos:

Definiciones 1.4.12. Sea $L = \oplus_{\sigma \in G} L_\sigma$ una subálgebra graduada de un álgebra de Lie graduada $Q = \oplus_{\sigma \in G} Q_\sigma$.

– Diremos que Q es un **álgebra graduada de cocientes de** L o que L es una **subálgebra graduada de cocientes de** Q si para cualesquiera $0 \neq p_\sigma \in Q_\sigma$, $q_\tau \in Q_\tau$, existen $x_\alpha \in L_\alpha$ tales que $[x_\alpha,\, p_\sigma] \neq 0$, $[x_\alpha,\, {}_L(q_\tau)] \subseteq L$.

– Si para todo $p_\sigma \in Q_\sigma$ existe $x_\alpha \in L_\alpha$ tal que $0 \neq [x_\alpha,\, p_\sigma] \in L$, diremos que Q es un **álgebra graduada débil de cocientes de** L, o que L es una **subálgebra graduada débil de cocientes de** Q.

Daremos una condición necesaria y suficiente para que un álgebra de Lie graduada tenga un álgebra graduada (débil) de cocientes.

Probaremos (véase Proposición 1.4.18) que, al igual que sucede en el caso no graduado, las álgebras graduadas de cocientes de álgebras de Lie graduadas heredan la primidad, la semiprimidad y el carácter no degenerado.

F. Perera y M. Siles Molina estudiaron en [75] la relación que existe entre los cocientes asociativos y los cocientes Lie, siendo el siguiente teorema uno de sus principales resultados:

Teorema. ([75, Theorem 2.12 and Proposition 3.5]). Sea A un álgebra asociativa semiprima y sea Q una subálgebra de $Q_s(A)$ que contiene a A. Entonces

(i) $A^-/Z_A \subseteq Q^-/Z_Q$ y $[A,\, A]/Z_{[A,\,A]} \subseteq [Q,\, Q]/Z_{[Q,\,Q]}$ son extensiones densas.

(ii) Q^-/Z_Q es un álgebra de cocientes de A^-/Z_A y $[Q,\, Q]/Z_{[Q,\,Q]}$ lo es de $[A,\, A]/Z_{[A,\,A]}$.

Nuestra tarea en la Sección 1.5 será extender el resultado anterior a las álgebras de Lie de tipo skew. Obtendremos el siguiente teorema:

Teorema 1.5.19. Sea A un álgebra asociativa semiprima con involución $*$ y sea Q una $*$-subálgebra de $Q_s(A)$ que contiene a A. Entonces se satisfacen las siguientes condiciones:

(i) K_A es una subálgebra densa de K_Q, y $[K_A, K_A]$ lo es de $[K_Q, K_Q]$.

(ii) K_A/Z_{K_A} es una subálgebra densa de K_Q/Z_{K_Q}, y $[K_A, K_A]/Z_{[K_A, K_A]}$ lo es de $[K_Q, K_Q]/Z_{[K_Q, K_Q]}$.

(iii) K_Q/Z_{K_Q} es un álgebra de cocientes de K_A/Z_{K_A}, y $[K_Q, K_Q]/Z_{[K_Q, K_Q]}$ lo es de $[K_A, K_A]/Z_{[K_A, K_A]}$.

Acabamos el capítulo analizando en la Sección 1.6 la relación entre álgebras graduadas (débiles) de cocientes y álgebras (débiles) de cocientes.

Siguiendo la construcción, dada por Siles Molina en [79], del álgebra de cocientes maximal $Q_m(L)$ de un álgebra de Lie semiprima L construiremos, en el Capítulo 2, un álgebra de cocientes maximal graduada para cada álgebra de Lie graduada semiprima. Teniendo en cuenta que los elementos de $Q_m(L)$ provienen de derivaciones parciales definidas en ideales esenciales, nuestros ingredientes serán ahora derivaciones parciales graduadas e ideales esenciales graduados. Con esta idea en mente, introduciremos en **Construcción 2.2.3** una nueva álgebra de Lie, que denotaremos por $Q_{gr-m}(L)$. En la Sección 2.2 probaremos que $Q_{gr-m}(L)$ tiene buenas propiedades.

Nuestro objetivo en el resto del capítulo será calcular $Q_m(L)$ para ciertas álgebras de Lie. Concretamente, nos centraremos en las álgebras de Lie de la forma $L = A^-/Z$, donde A^- es el álgebra de Lie asociada a un álgebra asociativa prima A de centro Z, y en las de la forma $L = K/Z_K$, donde K es el álgebra de Lie de los elementos skew de un álgebra asociativa prima con involución y Z_K su centro. En la Sección 2.3, calcularemos $Q_m(A^-/Z)$; se tiene que (bajo ciertas hipótesis) $Q_m(A^-/Z)$ coincide con una cierta álgebra de Lie que proviene de derivaciones parciales de A. Debido a que la definición es un poco técnica para darla aquí, diremos que dicha álgebra vive entre $\mathrm{Der}(A)$ y $\mathrm{Der}(Q_s(A))$, donde $Q_s(\,.\,)$ denota el álgebra de cocientes simétricos de Martindale de A. En la Sección 2.4, obtendremos resultados similares para K/Z_K (este caso es totalmente análogo al de A^-/Z, la única diferencia es que tendremos que considerar derivaciones δ que preserven $*$, en el sentido de que $\delta(x^*) = \delta(x)^*$).

El propósito del Capítulo 3 es determinar bajo qué condiciones algunas propiedades importantes de los cocientes asociativos continúan siendo ciertas para los cocientes Lie. En la Sección 3.1, estudiaremos el problema de cuándo $Q_m(I)$, donde I es un ideal esencial de un álgebra de Lie semiprima L, coincide con a $Q_m(L)$. Se sabe que este resultado es cierto en el caso asociativo (ver, por ejemplo [15, Proposition 2.1.10]); en el contexto Lie, podremos dar una respuesta afirmativa imponiendo que L sea fuertemente semiprima: se dice que un álgebra de Lie L es **fuertemente semiprima** (respectivamente, **fuertemente prima**) si:

(i) L es semiprima (respectivamente, prima).

(ii) Para cada n, dados $0 \neq U_n \lhd \ldots \lhd U_2 \lhd U_1 \lhd L$ existe $0 \neq W \lhd L$ tal que $W \subseteq U_n$.

Teorema 3.1.7. Sea I un ideal esencial de un álgebra de Lie fuertemente semiprima L. Entonces $Q_m(I)$ es el álgebra de cocientes maximal de L, es decir $Q_m(I) \cong Q_m(L)$.

Una vez que hemos construido el álgebra graduada de cocientes maximal, es natural preguntarse, al igual que hicimos en el caso no graduado, cuándo

$Q_{gr-m}(I)$ será isomorfo a $Q_{gr-m}(L)$, para I un ideal graduado esencial de un álgebra graduada semiprima L; trataremos esta cuestión en la Sección 3.2.

Cerramos este tercer capítulo analizando en la Sección 3.3 cuándo $Q_m((Q_m(L))$ es igual a $Q_m(L)$. A diferencia del caso asociativo en el que siempre se tiene esta igualdad (ver [15, Theorem 2.1.11]), probaremos que en ciertas situaciones especiales, a saber, si L es un álgebra de Lie simple o si $L = A^-/Z$ donde A es un álgebra asociativa simple (satisfaciendo ciertas hipótesis técnicas) o A es un álgebra finitamente generada que satisface una identidad polinómica, la respuesta sigue siendo positiva, pero que en general $Q_m((Q_m(L))$ no coincide con $Q_m(L)$; damos un ejemplo (veáse Ejemplo 3.3.7) haciendo uso del ejemplo dado por Passman en [73] con el que mostró que tomar $Q_s(.)$ no es una operación cerrada.

En el Capítulo 4, estudiaremos la relación existente entre los cocientes Lie y los cocientes Jordan en el sentido de [39]. Consta de dos secciones; en la primera de ellas probaremos que para las álgebras de Lie 3-graduadas semiprimas, el álgebra de cocientes maximal es de nuevo 3-graduada y coincide con el álgebra graduada de cocientes maximal. Concretamente, el resultado obtenido es el siguiente:

Teorema 4.1.2. Sea $L = L_{-1} \oplus L_0 \oplus L_1$ un álgebra de Lie 3-graduada semiprima. Entonces:

(i) $Q_m(L)$ es isomorfa graduada a $Q_{gr-m}(L)$.

(ii) Si L es fuertemente no degenerada y Φ es 2 y 3 libre de torsión, entonces $Q_m(L)$ es un álgebra de Lie 3-graduada fuertemente no degenerada.

En la Sección 4.2, haciendo uso de la construcción de Tits-Kantor-Koecher, relacionaremos los sistemas de Jordan de cocientes maximales con las álgebras de Lie de cocientes maximales. Los principales resultados son:

Teorema 4.2.11. Supongamos que $\frac{1}{6} \in \Phi$.

(i) Sea V un par de Jordan fuertemente no degenerado. Entonces

$$Q_m(V) = \Big((Q_m(\mathrm{TKK}(V)))_1,\, (Q_m(\mathrm{TKK}(V)))_{-1}\Big)$$

es el par de Jordan de $\mathfrak{M}$-cocientes maximal de V.

(ii) Si $L = L_{-1} \oplus L_0 \oplus L_1$ es un álgebra de Lie de Jordan 3-graduada fuertemente no degenerada tal que $Q_m(L)$ es de Jordan 3-graduada, entonces

$$Q_m(L) \cong Q_m(\mathrm{TKK}(V)) \cong \mathrm{TKK}(Q_m(V)),$$

donde $V = (L_1, L_{-1})$ es el par de Jordan asociado a L.

Teorema 4.2.22. Sea T un sistema triple de Jordan fuertemente no degenerado sobre un anillo de escalares Φ que contiene a $\frac{1}{6}$. Entonces el sistema triple de Jordan de $\mathfrak{M}$-cocientes maximal de T es la primera componente del álgebra de cocientes maximal de la TKK-álgebra del par de Jordan doble $V(T) = (T, T)$ asociado a T, es decir,

$$Q_m(T) = (Q_m(\mathrm{TKK}(V(T))))_1.$$

Teorema 4.2.27. Sea J un álgebra de Jordan fuertemente no degenerada sobre un anillo de escalares Φ que contiene a $\frac{1}{6}$. Entonces

$$Q_m(J) = Q_m(J_T) = (Q_m(\mathrm{TKK}(V(J_T))))_1,$$

es el álgebra de Jordan de cocientes maximal de J, donde J_T denota el sistema triple de Jordan asociado a J y $V(J_T) = (J_T, J_T)$ es el par doble par de Jordan asociado a J_T.

El lector puede encontrar los resultados originales de los Capítulos 2, 3 y 4 en [19, 78].

Durante la estancia de la autora en la Universidad de Maribor (Eslovenia) trabajó, junto a M. Brešar y M. Grašič, en el problema de estudiar cuándo los anillos de matrices $\mathbb{M}_n(B)$, donde B es cualquier álgebra unitaria (sobre un anillo conmutativo fijado C) quedan determinadas por un producto nulo. Concluiremos esta tesis con los resultados obtenidos en [17].

Como motivación para estudiar este problema, destacaremos la conexión con los ampliamente estudiados problemas de describir las aplicaciones lineales que conservan los productos nulos (ver, por ejemplo [3, 8, 30, 31, 32, 33, 45, 46, 83]).

Podemos considerar a S. Banach el propulsor de esta área de investigación matemática; fue el primero en describir las isometrías de $L^p([0,1])$ para $p \neq 2$ (ver [9]). Banach no dio la prueba completa para este caso (que fue dada más tarde por J. Lamperti [58]) pero hizo hincapié en el hecho de que las isometrías deben aplicar funciones con soporte disjunto en funciones con soporte disjunto. Esta propiedad surge de manera natural en una gran cantidad de situaciones y ha sido considerada por diversos autores. Por ejemplo, en la teoría de los retículos de Banach hay una extensa literatura acerca de aplicaciones lineales $T : X \to Y$, donde X e Y son retículos de Banach, satisfaciendo la propiedad de que

$$| T(x) | \wedge | T(y) | = 0$$

siempre que $x, y \in X$ sean tales que $| x | \wedge | y | = 0$. A estas aplicaciones se las llama **operadores que preservan la"disjunción"** o **d-homomorfismos**. Para más información, referimos al lector a [1].

La noción de operadores que preservan la"disjunción" fue trasladada por E. Beckenstein y L. Narici [12] a las álgebras de funciones. Si A, B son álgebras de funciones, a los operadores lineales $T : A \to B$ que satisfacen $T(a)T(b) = 0$ se les llama **operadores de Lamperti** o **aplicaciones separadoras**. Este concepto de aplicaciones separadoras puede trasladarse al álgebra pura, siendo la manera más natural el considerar literalmente la misma condición, o sea, se dice que una aplicación lineal T de un álgebra A en otra B **conserva el producto nulo** si

$$x, y \in A, \quad xy = 0 \Rightarrow T(x)T(y) = 0.$$

En el reciente artículo [18], M. Brešar y P. Šemrl se ocupan de uno de los problemas más estudiados, a saber, del problema de describir las aplicaciones lineales que conservan la conmutatividad. Se dice que una aplicación lineal $S : A \to B$ **conserva la conmutatividad** si

$$S(x)S(y) = S(y)S(x) \quad \text{siempre que} \quad x, y \in A, \quad xy = yx.$$

El suponer que la conmutatividad se conserva se puede reformular en términos de la condición de conservar el producto de Lie nulo, del siguiente modo:

Si $S : A \to B$ es una aplicación lineal que conserva la conmutatividad, entonces la aplicación bilineal $T : A \times A \to B$ dada por $T(x, y) = [S(x), S(y)]$ satisface claramente que

$$T(x, y) = 0 \quad \text{siempre que} \quad [x, y] = 0,$$

lo que se expresa diciendo que T **conserva el producto nulo de Lie**.

Brešar y Šemrl probaron (ver [18, Theorem 2.1]) que en el caso más simple, o sea, cuando $B = C$ las matrices $\mathbb{M}_n(C)$ quedan determinadas por el producto de Lie nulo. Dicho resultado se podrá obtener como consecuencia de los aquí expuestos. En la Sección 5.2 probaremos que para el producto ordinario, las matrices $\mathbb{M}_n(B)$ quedan determinadas por el producto nulo para cualquier álgebra unitaria B y todo $n \geq 2$, y en la Sección 5.3 mostraremos lo mismo para el producto Jordan; pero añadiendo las hipótesis de que $n \geq 3$ y de que 2 es inversible en B. Veremos que el comportamiento del producto de Lie, del que nos ocuparemos en la Sección 5.4, es muy distinto.

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Chapter 1

Algebras and graded algebras of quotients of Lie algebras

The theory of associative algebras of quotients has a rich history and is still an active research area. In recent years, there has been a trend to extend notions and results of associative settings to the non-associative ones. In the paper [79] M. Siles Molina initiated the study of algebras of quotients of Lie algebras.

In this chapter we will study algebras of quotients of skew Lie algebras and we will also introduce the notion of graded algebras of quotients of graded Lie algebras.

1.1 Introduction

Throughout the chapter and in the rest of the work we will consider Lie and associative algebras, and we will tacitly assume that all of them are algebras over a fixed commutative unital ring of scalars Φ. Lie algebras will be usually denoted by L, and associative ones by A. For convenience we will assume that all our algebras are **2-torsion-free** (i. e. $2x \neq 0$ for every nonzero x in an algebra), although this assumption is not always necessary; we will use it without further mention. For associative algebras we will not assume that they must be unital.

Let us start by introducing the basic notation and recalling some definitions and results. We will omit the proofs of some of these preliminary well-known facts.

Definitions 1.1.1. Let L be a Φ-module together with a bilinear map $[\,,\,] : L \times L \to L$, denoted by $(x, y) \mapsto [x, y]$ (called the **bracket of** x **and** y). We say that L is a **Lie algebra over** Φ if the following axioms are satisfied:

(i) $[x, x] = 0$, and

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (the so-called **Jacobi identity**).

Let X be a subset of an (associative or not) algebra A. The set

$$\mathrm{Ann}(X) = \mathrm{Ann}_A(X) = \{a \in A \mid ax = 0 = xa \text{ for every } x \in X\}$$

is called the **annihilator** of X in A. It is easy to check that $\mathrm{Ann}(X)$ is an ideal of A when X is also an ideal of A. In the special situation that $X = A$, $\mathrm{Ann}(A)$ is called the **center** of A and will be denoted by $Z = Z_A$.

In case L is a Lie algebra and X is a subset of L, the **annihilator** of X in L is defined as

$$\mathrm{Ann}(X) = \mathrm{Ann}_L(X) = \{a \in L \mid [a, x] = 0 \text{ for every } x \in X\}.$$

Every element of $\mathrm{Ann}(L)$ will be called a **total zero divisor**. It will be clear from the context whether $\mathrm{Ann}(X)$ denotes the annihilator in the associative or in the Lie algebra setting.

Lie algebras abound in the mathematical literature. The following examples are well-known.

Example 1.1.2. Lie algebras that arise from associative ones. Let A be an associative algebra A; we can obtain a Lie algebra A^- by considering the same module structure of A and bracket given by

$$[x, y] = xy - yx \quad \text{for every } x,\, y \in A.$$

Ideals of A^- will be called **Lie ideals** of A, so, a Φ-submodule U of A is a Lie ideal of A if it satisfies $[U, A] \subseteq U$.

Remarks 1.1.3. Clearly, if I is an ideal of A, then it is also a Lie ideal of A; however, the converse is not true. For example, consider $[I, A]$, the linear span of all $[y, x]$ with $y \in I$ and $x \in A$, which is a Lie ideal but not necessarily an ideal of A.

Note that Z_{A^-}, the center of the Lie algebra A^-, agrees with the associative center Z of A, and is clearly a Lie ideal of A. So we can form the Lie algebra A^-/Z. In the sequel, we will write $\overline{I}$ to the denote the ideal $(I + Z)/Z$ of A^-/Z, for a Lie ideal I of A.

Example 1.1.4. Skew Lie algebras. Let A be an associative algebra with involution $*$; then the set of its **skew elements**

$$K = K_A = \{x \in A \mid x^* = -x\}$$

is a subalgebra of A^-. The ideal $[K,\, K]$ of K is particularly important, since sometimes its use allows to avoid exceptional situations (see [64, 24]). The Lie algebras K/Z_K and $[K,\, K]/Z_{[K,\,K]}$ are called **algebras of skew type** or **skew Lie algebras**.

These kinds of algebras involving commutators are of great interest since they appear in Zelmanov's classification of simple M-graded Lie algebras over fields of characteristic at least $2d + 1$, where d is the width of M (see [84]).

Example 1.1.5. The Lie algebra of derivations. Let A be an associative algebra. A linear map $\delta : A \to A$ is called a **derivation of** A if

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. For example, if x is an element of A, the map $\mathrm{ad}\, x \colon A \to A$ defined by $\mathrm{ad}\, x(y) = [x, y]$ is a derivation of A.

We denote by $\mathrm{Der}(A)$ the set of all derivations of A. Clearly, $\mathrm{Der}(A)$ is a Φ-module if we define the operations in the natural way and, moreover, it becomes a Lie algebra if we define the bracket by

$$[\delta, \mu] = \delta\mu - \mu\delta,$$

for every $\delta, \mu \in \mathrm{Der}(A)$.

Here, we shall give the notions of semiprimeness, primeness, and essentiality for Lie algebras; in case of associative algebras these concepts can be defined in exactly the same way as those for Lie algebras, just by replacing the bracket by the associative product.

Definitions 1.1.6. Let L be a Lie algebra.

(i) We say that L is **semiprime** if for very nonzero ideal I of L, $[I, I] \neq 0$. In the sequel we shall usually denote $[I, I]$ by I^2.

(ii) L is said to be **prime** if for every nonzero ideals I, J of L, $[I, J] \neq 0$.

There are a several examples of semiprime and prime Lie algebras. The most interesting for us are the following ones:

Example 1.1.7. The Lie algebra of derivations, $\mathrm{Der}(A)$ of a semiprime (prime) associative algebra A, is semiprime (prime). It was proved by C. R. Jordan and D. A. Jordan in [49, Theorem 4 (Theorem 2)].

Example 1.1.8. ([64, Theorem 6.1]). If A is a semiprime associative algebra with involution $*$ then the skew Lie algebra K/Z_K is semiprime.

The following remark asserts that every Lie algebra of the form A^-/Z, where A is an associative algebra with center Z, can be seen as a skew Lie algebra. This fact constitutes a very useful tool.

Remark 1.1.9. Note that if A is an associative algebra, then the Lie algebra A^- is isomorphic to $K_{A\oplus A^0}$ and hence A^-/Z is isomorphic to $K_{A\oplus A^0}/Z_{K_{A\oplus A^0}}$, where A^0 denotes the opposite algebra of A, and $A \oplus A^0$ is endowed with the exchange involution.

A first application of this remark is written below.

Example 1.1.10. If A is a semiprime associative algebra then the Lie algebra A^-/Z is also semiprime. It is obtained from Example 1.1.8 taking into account the remark above. Another proof of this fact can be found in [75, Theorem 2.12]; concretely:

Proof. Let $\overline{U}$ be a Lie ideal of A^-/Z such that $[\overline{U}, \overline{U}] = 0$. Then $\overline{U}$ is the image of the Lie ideal U of A via the natural map $A^- \to A^-/Z$. The condition on $\overline{U}$ implies that $[U, U] \subseteq Z$. Applying [43, Lemma 1] we obtain $U \subseteq Z$, that is, $\overline{U} = 0$. □

Definition 1.1.11. We say that an ideal I of a Lie algebra L is **essential**, and write $I \lhd_e L$ to denote it, if I cuts in a nontrivial way every nonzero ideal of L, i.e., $I \cap J \neq 0$ for every nonzero ideal J of L. We denote by $\mathcal{I}_e(L)$ the set of all essential ideals of L.

Some examples and properties of essential ideals are collected in the following result.

Lemma 1.1.12. *Let L be a Lie algebra.*

(i) *If* $I,\ J \in \mathcal{I}_e(L)$ *then* $I \cap J \in \mathcal{I}_e(L)$, *that is, the intersection of essential ideals is again an essential ideal.*

(ii) *If* L *is semiprime and* $I \in \mathcal{I}_e(L)$ *then* $I^2 \in \mathcal{I}_e(L)$.

(iii) *If* L *is prime, then every nonzero ideal of* L *is essential.*

In case of semiprime algebras, essential ideals can be characterized in terms of their annihilators as follows:

Lemma 1.1.13. ([79, Lemma 1.2]). *Let* I *be an essential ideal of a semiprime Lie algebra* L. *Then:*

(i) $I \cap \mathrm{Ann}(I) = 0$.

(ii) I *is an essential ideal of* L *if and only if* $\mathrm{Ann}(I) = 0$.

The proof of the following lemma is included in the proof of [75, Theorem 2.12].

Lemma 1.1.14. *Let* A *be a semiprime algebra. Then for every essential ideal* I *of* A, *the ideal* $\overline{I}$ *is essential ideal in* A^-/Z.

Proof. Let I be an essential ideal of A; we claim that $\overline{I} = (I + Z)/Z$ has zero annihilator in A^-/Z (which implies that it is an essential ideal of A^-/Z). If $x \in A$ is such that $[\overline{x},\ \overline{I}] = 0$, then $[x,\ I] \subseteq Z$. Therefore $[[x,\ I],\ I] = 0$ and making use of the Jacobi identity we have $[x,\ [I,\ I]] = 0$. From [43, Lemma 2] it follows that $[x,\ I] = 0$. Note that A is an algebra of quotients of I since it is an essential ideal of A. Hence, using [79, Lemma 1.3 (iv)] we obtain $[x,\ A] = 0$. Thus $\overline{x} = 0$ in A^-/Z. □

Remark 1.1.15. By the previous lemma, an essential ideal I of a noncommutative semiprime algebra A cannot be central.

1.2 Associative algebras of quotients

The well-known construction of the field of fractions of an integral domain is a particular case of the notion of right (left) order.

Definition 1.2.1. Let $R \subseteq Q$ be rings with Q unital. The ring R is said to be a **right (left) order** in Q, or Q is called a **classical right (left) quotient ring** of R if

(i) Every regular element of R is invertible in Q.

(ii) Every element $q \in Q$ has the form $q = ba^{-1}$ $(q = a^{-1}b)$ for some regular element a of R and $b \in R$.

As mentioned, the field of fractions of an integral domain is always a classical right and left quotient ring of that integral domain but however, not every example of a classical right (or left) quotient ring comes from the field of fractions of an integral domain.

Example 1.2.2. Consider the rings

$$\mathbb{M}_n(D) \subseteq \mathbb{M}_n(F),$$

where D is an integral domain and F its field of fractions. Then $\mathbb{M}_n(D)$ is a right order in $\mathbb{M}_n(F)$, but neither $M_n(D)$ is an integral domain nor $M_n(F)$ is a field.

The notion of "being a classical right (or left quotient ring)" has a restriction: we need to consider unital rings; what can we do if our ring Q does not have a unit element? In such a case, we couldn't consider regular elements. Y. Utumi [82] solved satisfactorily this question introducing a suitable notion of ring of quotients for this setting.

Definition 1.2.3. ([82]). An overring Q of a ring R is said to be a **(general) right quotient ring** of R if given p, $q \in Q$, with $p \neq 0$, there exists $r \in R$ such $pr \neq 0$ and $qr \in R$. Left quotients rings are similarly defined.

Again, any classical right quotient ring Q of a ring R is also a right quotient ring of it:

Given p, $q \in Q$, with $p \neq 0$, there exists a regular element $a \in R$ and an element $b \in R$ such that $q = ba^{-1}$ which implies $qa = b \in R$; note, that $pa \neq 0$, since $p \neq 0$ and a is a regular element of R and hence, it is invertible in Q.

At this point, we may ask if there exists a right quotient ring of R such that any other right quotient ring of R can be embedded into it. Y. Utumi answered affirmatively this question provided R is left faithful. First, let us recall some definitions.

Definitions 1.2.4. An element $x \in R$ is a **total right zero divisor** if $xR = 0$. A ring R is **left faithful** if it has no nonzero total right zero divisors, that is, $xR = 0$ implies $x = 0$.

A right ideal I of R is said to be **dense** if given any x, $y \in R$, with $x \neq 0$, there exists $a \in R$ such that $xa \neq 0$ and $ya \in I$, i.e., R is a right quotient ring of I. The collection of all dense right ideals of R will be denoted by $\mathcal{I}_{dr}(R)$.

One defines total right zero divisors, right faithfulness and dense left ideals in an analogous fashion.

We pause to mention the notion of essential right ideal.

Definition 1.2.5. A right ideal I of R is **essential** if it cuts in a nontrivial way every nonzero right ideal of R.

A discussion of the relationship between essential and dense right ideals can be found in [15, Section 2.1]. By now we have selected two remarks.

Remarks 1.2.6. (See [15, Remarks 2.1.3 and 2.1.4].) Every dense right ideal of a semiprime ring is also an essential right ideal. There are special cases where these two notions coincide, for example, when the ring is left nonsingular.

Let I be an ideal of a semiprime ring R. Then the following conditions are equivalent:

(i) I is a dense right ideal.

(ii) I is an essential right ideal.

(iii) I is essential as an ideal.

We are now in a position to explain Utumi's construction.

Construction 1.2.7. ([82]). Let R be a left faithful ring. We say that two pairs (f, I), (g, J), where $I, J \in \mathcal{I}_{dr}(R)$ and $f : I_R \to R_R$, $g : J_R \to R_R$ are right R-module homomorphisms, **are equivalent** if and only if there exists $K \in \mathcal{I}_{dr}(R)$ contained in $I \cap J$ and such that $f = g$ on K. This is an equivalence relation. Denote by f_I the equivalence class determined by (f, I). The set of all such classes becomes a ring if we define addition and multiplication as follows:

$$f_I + g_J = (f + g)_{I\cap J}, \quad f_I g_J = fg_{g^{-1}(I)}.$$

Definition 1.2.8. For a left faithful ring R, the ring constructed above will be called the **maximal right ring of quotients** of R and will be denoted by $Q^r_{max}(R)$.

An example of maximal right quotient ring is the following:

Examples 1.2.9. If D is an integral domain and F is its field of fractions, then $Q^r_{max}(D) = F$. In particular,

$$Q^r_{max}(\mathbb{Z}) = \mathbb{Q} \quad \text{and} \quad Q^r_{max}(F[x]) = F(x).$$

Y. Utumi was the first that constructed this maximal ring of quotients; there are others “homological” constructions of it available (see, e.g., [56]). Let us point out that Utumi’s construction is more natural in the sense that speaking very loosely, given a right R-module homomorphism $f : I_R \to R_R$ and $fa = r$, for $a \in I$ and $r \in R$, we may “solve” for f and get “$f = ra^{-1}$”, which says that f is like a fraction.

In a similar fashion, using the set of dense left ideals of a right faithful ring R, one can construct the **maximal left ring of quotients** of R, denoted by $Q^l_{max}(R)$. Of course the maximal left and maximal right quotient rings need not coincide.

Example 1.2.10. Consider the ring

$$R = \begin{pmatrix} F & F & F \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix},$$

where F is a field. As it is shown in [55, p. 372], $Q^l_{max}(R) \cong \mathbb{M}_3(F)$ while $Q^r_{max}(R) \cong \mathbb{M}_2(F) \times \mathbb{M}_2(F)$, and they are obviously not isomorphic.

Proposition 1.2.11. (See [15, Proposition 2.1.7].) *Let R be a semiprime ring. Then $Q^r_{max}(R)$ satisfies:*

(i) *R is a subring of $Q^r_{max}(R)$.*

(ii) *For all $q \in Q^r_{max}(R)$, there exists $J \in \mathcal{I}_{dr}(R)$ such that $qJ \subseteq R$.*

(iii) *For all $q \in Q^r_{max}(R)$ and $J \in \mathcal{I}_{dr}(R)$, $qJ = 0$ if and only if $q = 0$.*

(iv) *For any ideal $J \in \mathcal{I}_{dr}(R)$ and any right R-module homomorphism $f : J_R \to R_R$ there exists $q \in Q^r_{max}(R)$ such that $f(x) = qx$ for every $x \in J$.*

Furthermore, properties (i)-(iv) *characterize $Q^r_{max}(R)$ up to isomorphism.*

Remark 1.2.12. It can be proved, by using [57, Lemma 4.3.2], that conditions (i) and (ii) in Proposition 1.2.11 are equivalent to saying that S is a right quotient ring of R.

The notion of two-sided ring of quotients was introduced by W. S. Martindale III in [63] for prime rings (and extended to semiprime ones by S. A. Amitsur [4]). We are going to describe the construction of the two-sided ring of quotients for semiprime rings; let us point out here that in case of prime rings this construction has an especially simple form since every nonzero ideal of a prime ring is dense (it has zero annihilator).

Construction 1.2.13. ([82]). Let R be a semiprime ring. Denote by $\mathcal{I}(R)$ the collection of all ideals of R having zero annihilator. Note that $\mathcal{I}(R)$ is closed under sums and finite intersection; we also mention that any $I \in \mathcal{I}(R)$ is dense and essential as a right (or left) ideal accordingly we shall call such ideals **dense**.

We define that two pairs (f, I), (g, J), where $I, J \in \mathcal{I}(R)$ and $f : I_R \to R_R$, $g : J_R \to R_R$ are right R-module homomorphisms, **are equivalent** if and only if there exists $K \in \mathcal{I}(R)$ contained in $I \cap J$ and such that $f = g$ on K. This is an equivalence relation. Denote by f_I the equivalence class determined by (f, I). The set of all such classes becomes a ring if we define addition and multiplication as follows:

$$f_I + g_J = (f + g)_{I \cap J}, \quad f_I g_J = fg_{JI}.$$

Definition 1.2.14. For a semiprime ring R, the ring constructed above will be called the **two-sided right ring of quotients** of R and will be denoted by $Q_r(R)$.

The following result collects the principal properties of $Q_r(R)$.

Proposition 1.2.15. (See [15, Proposition 2.2.1].) *Let R be a semiprime ring. Then $Q_r(R)$ satisfies:*

(i) *R is a subring of $Q_r(R)$.*

(ii) *For all $q \in Q_r(R)$, there exists $J \in \mathcal{I}(R)$ such that $qJ \subseteq R$.*

(iii) *For all $q \in Q_r(R)$ and $J \in \mathcal{I}(R)$, $qJ = 0$ if and only if $q = 0$.*

(iv) *For any ideal $J \in \mathcal{I}(R)$ and any right R-module homomorphism $f : J_R \to R_R$ there exists $q \in Q_r(R)$ such that $f(x) = qx$ for every $x \in J$.*

Furthermore, properties (i)-(iv) *characterize $Q_r(R)$ up to isomorphism.*

The next proposition describes the relationship between $Q^r_{max}(R)$ and $Q_r(R)$.

Proposition 1.2.16. (See [15, Proposition 2.2.2].) *Given a semiprime ring R, there exists a unique ring monomorphism $\sigma : Q_r(R) \to Q^r_{max}(R)$ such that $\sigma(r) = r$ for all $r \in R$. Further,*

$$\mathrm{Im}(\sigma) = \{q \in Q^r_{max}(R) \mid qJ \subseteq R \ \ \textit{for some} \ \ J \in \mathcal{I}(R)\}.$$

Definition 1.2.17. Let R be a semiprime ring. The set

$$Q_s(R) = \{q \in Q_r(R) \mid qJ \cup Jq \subseteq R \ \ \text{for some} \ \ J \in \mathcal{I}(R)\}$$

is called the **symmetric Martindale ring of quotients of** R.

As noted by D. S. Passman (see [73, Proposition 1.4]), $Q_s(R)$ may be characterized by four properties analogous to those which characterize $Q^r_{max}(R)$ (see Proposition 1.2.11).

Proposition 1.2.18. (See [15, Proposition 2.2.3].) *Let R be a semiprime ring. Then $Q_s(R)$ satisfies:*

(i) *R is a subring of $Q_s(R)$.*

(ii) *For all $q \in Q_s(R)$, there exists $J \in \mathcal{I}(R)$ such that $qJ \cup Jq \subseteq R$.*

(iii) *For all $q \in Q_s(R)$ and $J \in \mathcal{I}(R)$, $qJ = 0$ (or $Jq = 0$) if and only if $q = 0$.*

(iv) *Given $J \in \mathcal{I}(R)$, $f : J_R \to R_R$ and $g : {}_RJ \to {}_RR$ right and left, respectively R-module homomorphisms such that $xf(y) = g(x)y$ for all $x, y \in J$ there exists $q \in Q_s(R)$ such that $f(x) = qx$, $g(x) = xq$ for every $x \in J$.*

Furthermore, properties (i)-(iv) *characterize $Q_s(R)$ up to isomorphism.*

Remark 1.2.19. (See [15, Remark 2.2.4].) We have defined $Q_s(R)$ as a subring of $Q_r(R) \subseteq Q^r_{max}(R)$ and so, more accurately, we should have called $Q_s(R)$ the **right symmetric Martindale ring of quotients** of R. Analogously, $\tilde{Q}_s(R)$ the **left symmetric Martindale ring of quotients** of R may be defined as a subring of $Q_l(R) \subseteq Q^l_{max}(R)$. For $q \in Q_s(R)$ we define $\tilde{q} = J$ and $g \in \tilde{Q}_s(R)$, where $g(x) = xq$ for all $x \in J$. Then the map $q \mapsto \tilde{q}$ is an isomorphism of $Q_s(R)$ onto $\tilde{Q}_s(R)$.

Definition 1.2.20. We call the center $\mathcal{C} = Z(Q_r(R))$ of the two-sided ring of quotients of a semiprime ring R the **extended centroid of** R.

Some important properties of the extended centroid are the following.

Lemma 1.2.21. ([15, 2.3]) *Let R be a semiprime ring. Then*

$$Z(Q_s(R)) = \mathcal{C} = Z(Q^r_{max}(R)) = \{q \in Q^r_{max}(R) \mid qr = rq \ \textit{for all} \ r \in R\}.$$

Moreover, if R is prime then $\mathcal{C}$ is a field.

The following result will play an important role in our computations.

Proposition 1.2.22. See ([15, Proposition 2.5.1].) *Let R be a semiprime ring. Any derivation δ of R can be extended uniquely to a derivation of $Q^r_{max}(R)$ also denoted by δ. Furthermore $\delta(Q_r(R)) \subseteq Q_r(R)$ and $\delta(Q_s(R)) \subseteq Q_s(R)$.*

We conclude this section by recalling the notion of generalized polynomial identity with involution. Let A be a semiprime associative algebra with an involution $*$. It is easy to see that $*$ can be lifted to an involution, also denoted by $*$, of $Q_s(A)$. Moreover, the extended centroid $\mathcal{C}$ of A, remains $*$-invariant.

Definition 1.2.23. Let $\mathbf{X}$ be a countably infinite set (of "formal variables") and let $\mathbf{X}^*$ be a disjoint copy of $\mathbf{X}$. Denote, as usual, by $C\langle \mathbf{X} \cup \mathbf{X}^* \rangle$ the free associative algebra over C generated by $\mathbf{X} \cup \mathbf{X}^*$, and by $Q_s(A)_C\langle \mathbf{X} \cup \mathbf{X}^* \rangle$ the coproduct of the C-algebras $Q_s(A)$ and $C\langle \mathbf{X} \cup \mathbf{X}^* \rangle$. An element $\phi = \phi(\mathbf{x_1}, \cdots, \mathbf{x_n}, \mathbf{x_1}^*, \cdots, \mathbf{x_n}^*)$ of $Q_s(A)_C\langle \mathbf{X} \cup \mathbf{X}^* \rangle$ is said to be a **generalized polynomial identity with involution** (in short $*$-GPI) on a nonzero $*$-ideal (i.e. ideal invariant under $*$) U of A if $s(\phi) = 0$ for all ($*$-substitution) C-algebra homomorphisms $s : Q_s(A)_C\langle \mathbf{X} \cup \mathbf{X}^* \rangle \to Q_s(A)$ such that $s(\mathbf{X}) \subseteq U$, $s(\mathbf{x}^*) = s(\mathbf{x})^*$ for all $\mathbf{x} \in \mathbf{X}$ and $s(q) = q$ for all q in $Q_s(A)$.

1.3 Lie algebras of quotients

Let $L \subseteq Q$ be Lie algebras. For any $q \in Q$, we denote by ${}_L(q)$ the linear span in Q of q and the elements of the form $\operatorname{ad} x_1 \operatorname{ad} x_2 \ldots \operatorname{ad} x_n q$, where $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in L$. In particular, if $q \in L$, note that then ${}_L(q)$ is just the ideal of L generated by q.

Definition 1.3.1. ([79, Definition 2.1].) Let $L \subseteq Q$ be Lie algebras. We say that Q is an **algebra of quotients of** L (or also that L is a **subalgebra of quotients of Q**) if given p and q in Q with $p \neq 0$, there exists x in L such that $[x, p] \neq 0$ and $[x, {}_L(q)] \subseteq L$.

A Lie algebra L has an algebra of quotients if and only if it has no nonzero total zero divisors, or, equivalently, $\mathrm{Ann}(L) = 0$ (see [79, Remark 2.3]).

Certain properties of a Lie algebra are inherited by each of its algebras of quotients (see Proposition 1.3.4 below). In fact, these results remain valid under a weaker hypothesis, that of "being a weak algebra of quotients", notion that we proceed to introduce.

Definition 1.3.2. ([79, Definition 2.5].) Let $L \subseteq Q$ be Lie algebras. We say that Q is a **weak algebra of quotients of** L if for every nonzero element $q \in Q$ there exists $x \in L$ such that $0 \neq [x, q] \in L$.

Remark 1.3.3. Let us point out that every algebra of quotients of a Lie algebra L is a weak algebra of quotients, but as it was shown in [79, Remark 2.6] the converse is not true.

Proposition 1.3.4. *Let Q be a weak algebra of quotients of a subalgebra L.*

(i) *For every nonzero ideal I of Q, $I \cap L$ is a nonzero ideal of L.*

(ii) *L semiprime (prime) implies Q semiprime (prime).*

(iii) *If Φ is two and three-torsion free and L is strongly non-degenerate, then Q is strongly non-degenerate.*

One can prove that the definition of algebra of quotients of a Lie algebra L can be expressed in terms of ideals of L with zero annihilator. (See [79, Proposition 2.15].)

Definition 1.3.5. ([79, Definition 2.9].) Let $L \subseteq Q$ be Lie algebras. We say that Q is **ideally absorbed into** L, that is, for every nonzero element $q \in Q$ there exists an ideal I of L with $\mathrm{Ann}_L(I) = 0$ such that $0 \neq [I, q] \subseteq L$.

1.4 Graded Lie algebras of quotients

Following the idea of M. Siles Molina [79] of introducing a notion of algebra of quotients for Lie algebras and taking into account the success obtained by G. Aranda Pino and M. Siles Molina [7] in the context of graded associative setting, our aim in this section is to extend such a notion to the more general case of graded Lie algebras.

We begin by introducing the main definitions and some basic results derived from them.

Definitions 1.4.1. Let G be an abelian group (whose neutral element will be denoted by e); a Lie algebra L is called G**-graded** if $L = \oplus_{\sigma \in G} L_\sigma$, where L_σ is a Φ-subspace of L and $[L_\sigma, L_\tau] \subseteq L_{\sigma\tau}$ for every $\sigma, \tau \in G$. In the sequel, we sometimes use the term "graded" instead of "G-graded" when the group is understood.

The set of **homogeneous elements** is $\bigcup_{\sigma \in G} L_\sigma$. Elements of L_σ are called **homogeneous of degree** σ.

For any subset X of L, its **support** is defined as

$$\mathrm{Supp}(X) = \{\sigma \in G \mid x_\sigma \neq 0 \text{ for some } x \in X\}.$$

The grading on L is called **finite** if $\mathrm{Supp}(L)$ is a finite set and it is said **trivial** if $L = L_e$ and $L_\sigma = 0$ for every $\sigma \in G$ with $\sigma \neq e$. In the particular case of having L a finite $\mathbb{Z}$-grading, we may write the Lie algebra L as a finite direct sum $L = L_{-n} \oplus \ldots \oplus L_n$, and we say that L has a $(2n+1)$**-grading**.

Example 1.4.2. Every Lie algebra L becomes a graded Lie algebra over any abelian group G, by considering the trivial grading, that is, by doing $L_e = L$ and $L_\sigma = 0$ for $\sigma \neq e$.

There are several examples of graded associative algebras. Let us point out that if A is a G-graded associative algebra then the Lie algebra A^- associated to A is automatically a G-graded Lie algebra. Keeping this fact in mind, other examples of graded Lie algebras are the following:

Example 1.4.3. The algebra of polynomials, $R = A[x]$, where A is a noncommutative algebra, is a $\mathbb{Z}$-graded algebra with grading given by

$$R_n = \begin{cases} Ax^n & \text{if} \quad n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus R^- becomes a $\mathbb{Z}$-graded Lie algebra.

Example 1.4.4. Matrix algebras, $R = \mathbb{M}_n(A)$ are $(2n-1)$-graded with

$$R_k = \sum_{\{i,j \in \{1,\ldots,n\} \mid \; i-j=k\}} Ae_{i,j}$$

for $k < n$ and $R_k = 0$ otherwise. It turns out that the Lie algebra R^- has a $(2n-1)$-grading.

Definitions 1.4.5. Given G-graded Lie algebras L and Q, with L a subalgebra of Q, we say that L is a **graded subalgebra of** Q if $L_\sigma \subseteq Q_\sigma$ for every $\sigma \in G$.

A Lie algebra homomorphism $\varphi : L \to Q$ is a **graded homomorphism of degree** $\tau \in G$ if $\varphi(L_\sigma) \subseteq Q_{\sigma\tau}$ for all $\sigma \in G$. Graded monomorphisms, graded epimorphisms and graded isomorphisms are defined in the natural way.

Definitions 1.4.6. Let $L = \oplus_{\sigma \in G} L_\sigma$ be a graded Lie algebra. An ideal I of L is called a **graded ideal** if whenever $y = \sum y_\sigma \in I$ we have $y_\sigma \in I$, for every $\sigma \in G$.

A graded ideal I of L is said to be **graded essential** if every nonzero graded ideal of L hits I, i.e., $I \cap J \neq 0$ for every nonzero graded ideal J of L.

We will use the following lemma without further mention.

Lemma 1.4.7. *Let $L = \oplus_{\sigma \in G} L_\sigma$ be a graded Lie algebra and I, J two graded ideals of L. Then*

(i) *$I + J$ and $I \cap J$ are graded ideals of L. Further, if I and J are graded essential then $I \cap J$ is again a graded essential ideal.*

(ii) *$[I, J]$ is a graded ideal of L.*

Proof. It is well-known that all of the sets considered in the statements are ideals of L. It only remains to prove that they are indeed graded.

(i). The case of the sum and the intersection are similar. For example, to show that $I + J$ is graded, consider $x \in I$ and $y \in J$. Decomposing x and y into their homogeneous components and taking into account that I and J are graded ideals we have $x = \sum_\sigma x_\sigma \in I$, with $x_\sigma \in I$, and $y = \sum_\sigma y_\sigma$, with $y_\sigma \in J$. Hence the σ-homogeneous component of $x + y$ is $x_\sigma + y_\sigma$, which lives inside $I + J$.

(ii). Take $x \in I$ and $y \in J$ and consider the element $z := [x, y] \in [I, J]$. In order to check that $[I, J]$ is graded, it is enough to show that the homogeneous components of z belong to $[I, J]$. Decomposing x and y into their homogeneous components and applying that I and J are graded ideals we may write $x = \sum_\sigma x_\sigma$, with $x_\sigma \in I$, and $y = \sum_\sigma y_\sigma$, with $y_\sigma \in J$. Note that the homogeneous components of z are

$$z_\sigma = [x, y]_\sigma = \sum_\tau [x_\tau, y_{\tau^{-1}\sigma}],$$

all of them living inside $[I, J]$. □

Following the definitions of primeness and semiprimeness for Lie algebras (see Definitions 1.1.6), one can now introduce the notions of graded semiprimeness and graded primeness for graded Lie algebras; concretely:

Definitions 1.4.8. Let L be a graded Lie algebra. We say that L is **graded semiprime** if for every nonzero graded ideal I of L, $[I, I] \neq 0$. In the sequel we shall usually denote $[I, I]$ by I^2.

The Lie algebra L is said to be **graded prime** if for nonzero graded ideals I and J of L, $[I, J] \neq 0$.

An (homogeneous) element x of L is called an **(homogeneous) absolute zero divisor** if $(\operatorname{ad} x)^2 = 0$. The algebra L is said to be **(graded) strongly non-degenerate** if it does not contain nonzero (homogeneous) absolute zero divisors.

Remark 1.4.9. It is obvious from the definitions that (graded) strongly non-degenerate Lie algebras are (graded) semiprime, but the converse does not hold. (See [79, Remark 1.1].)

As we have in the non-graded case, we can characterize the graded essential ideals in terms of their annihilators.

Lemma 1.4.10. *Let I be a graded ideal of a graded Lie algebra $L = \oplus_{\sigma \in G} L_\sigma$. Then $\operatorname{Ann}(I)$ is a graded Lie ideal of L. In particular, Z_L, the center of L, is a graded ideal of L. If moreover L is graded semiprime, then:*

(i) *I^2 is a graded essential ideal of L if I is so.*

(ii) *$I \cap \operatorname{Ann}(I) = 0$.*

(iii) *I is a graded essential ideal of L if and only if $\operatorname{Ann}(I) = 0$.*

Proof. It is straightforward to check, by using the Jacobi identity that, $\operatorname{Ann}(I)$ is an ideal of L; so the only thing we are going to show is that every homogeneous component of any element $x \in \operatorname{Ann}(I)$ is again in $\operatorname{Ann}(I)$. Fix $\tau \in G$. Note that $[x_\sigma, I_\tau] = 0$ for every $\sigma \in G$ because otherwise there would exist $y_\tau \in I_\tau$ such that $[x_\sigma, y_\tau] \neq 0$ for some $\sigma \in G$; this would imply $0 \neq [x, I_\tau] \subseteq [x, I] = 0$, a contradiction. Hence $[x_\sigma, I] = \oplus_{\tau \in G}[x_\sigma, I_\tau] = 0$.

Assume now that L is graded semiprime.

(i). Let I be a graded essential ideal of L. Apply Lemma 1.4.7 (ii) to obtain that I^2 is also a graded ideal of L; the graded essentiality of L follows now from the essentiality of I.

To obtain (ii) and (iii), see the proofs of conditions (i) and (ii) in [79, Lemma 1.2]. $\square$

Recall that the elements of the center were called total zero divisors. An use of the lemma above gives:

Lemma 1.4.11. *A graded Lie algebra L has no nonzero homogeneous total zero divisors if and only if it has no nonzero total zero divisors.*

Proof. Suppose first that L has no homogeneous total zero divisors, and consider $x \in \mathrm{Ann}(L)$. Then, by Lemma 1.4.10 (i), we have $[x_\sigma, L] = 0$ for every $\sigma \in G$. This implies $x_\sigma = 0$ and so $x = 0$. The reverse implication is obvious. $\square$

We are now ready to introduce which may be considered the main objects of this section.

Definitions 1.4.12. Let $L = \oplus_{\sigma \in G} L_\sigma$ be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma \in G} Q_\sigma$.

– We say that Q is a **graded algebra of quotients** of L if given $0 \neq p_\sigma \in Q_\sigma$ and $q_\tau \in Q_\tau$, there exists $x_\alpha \in L_\alpha$ such that $[x_\alpha, p_\sigma] \neq 0$ and $[x_\alpha, {}_L(q_\tau)] \subseteq L$. The algebra L will be called a **graded subalgebra of quotients of** Q.

– If for any nonzero $p_\sigma \in Q_\sigma$ there exists $x_\alpha \in L_\alpha$ such that $0 \neq [x_\alpha, p_\sigma] \in L$, then we say that Q is a **graded weak algebra of quotients** of L, and L is called a **graded weak subalgebra of quotients of** Q.

Remark 1.4.13. These definitions are consistent with the non-graded ones (see Definitions 1.3.1 and 1.3.2) in the sense that if Q is a (weak) algebra of quotients of a Lie algebra L, then it is also a graded (weak) algebra of L when considering the trivial gradings on L and Q.

The necessary and sufficient condition for a graded Lie algebra to have a graded (weak) algebra of quotients is the absence of homogeneous total zero divisors different from zero, condition that turns out to be equivalent to have zero center. More concretely, the result is the following:

Lemma 1.4.14. *Let L be a graded Lie algebra. The following conditions are equivalent:*

(i) *L is a graded algebra of quotients of itself.*

(ii) *L has a graded algebra of quotients.*

(iii) *L has no nonzero homogeneous total zero divisors.*

(iv) *L has no nonzero total zero divisors.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (i). Let Q be a graded overalgebra of L with Q being a graded algebra of quotients of L; take $0 \neq p_\sigma \in L_\sigma \subseteq Q_\sigma$ and $q_\tau \in L_\tau \subseteq Q_\tau$. Applying the hypothesis on Q we find $x_\alpha \in L_\alpha$ such that $[x_\alpha, p_\sigma] \neq 0$ and $[x_\alpha, {}_L(q_\tau)] \subseteq L$. This means that L is a graded algebra of quotients of itself, as desired.

(i) $\Rightarrow$ (iii). Take $x_\sigma \in L_\sigma$, with $[L, x_\sigma] = 0$. If $x_\sigma \neq 0$ we would find $x_\mu \in L_\mu$ such that $[x_\mu, x_\sigma] \neq 0$, a contradiction; so necessarily $x_\sigma = 0$.

(iii) $\Rightarrow$ (i). Given $0 \neq p_\sigma \in L_\sigma$ and $q_\tau \in L_\tau$, by (iii) there exists $x_\alpha \in L_\alpha$ such that $[x_\alpha, p_\sigma] \neq 0$. It is obvious that x_α satisfies that $[x_\alpha, {}_L(q_\tau)] \subseteq L$.

Finally (iii) $\Leftrightarrow$ (iv) is Lemma 1.4.11. □

Although every graded algebra of quotients is a graded weak algebra of quotients, the converse is not true, as shown in the following example (see [79, Remark 2.6]).

Example 1.4.15. Consider the $\mathbb{C}$-module P of all polynomials $\sum_{r=0}^{m} \alpha_r x^r$, with $\alpha_i \in \mathbb{C}$ and $m \in \mathbb{N}$, with the natural $\mathbb{Z}$-grading. Denote by $\sigma : \mathbb{C} \to \mathbb{C}$ the complex conjugation. Then the following product makes P into a $\mathbb{Z}$-graded Lie algebra:

$$\left[\sum_{r=0}^{m} \alpha_r x^r, \sum_{s=0}^{n} \beta_s x^s \right] = \sum_{r,s} (\alpha_r \beta_s^\sigma - \beta_s \alpha_r^\sigma) x^{r+s}.$$

Let Q be the $\mathbb{Z}$-graded Lie algebra P/I, where I denotes the $\mathbb{Z}$-graded ideal of P consisting of all polynomials whose first nonzero term has degree at least 4, and let L be the following graded subalgebra of Q:

$$L = \{\overline{\alpha}_0 + \overline{\alpha}_2 \overline{x}^2 + \overline{\alpha}_3 \overline{x}^3 \mid \alpha_0, \alpha_2, \alpha_3 \in \mathbb{C}\},$$

where $\overline{y}$ denotes the class of an element $y \in P$ in P/I. Then Q is a graded weak algebra of quotients of L, but Q is not a graded quotient algebra of L since no $l \in L$ satisfies $[l, \overline{x}] \in L$ and $[l, \ \overline{x}^3] \neq 0$.

Parallel to what happened in the non-graded case (see Proposition 1.3.4), we are going to show how certain properties of a graded Lie algebra L are inherited by each of its algebras of quotients. First, we need some definitions and results.

Definition 1.4.16. Let X and Y be two subsets of a Lie algebra L. The set

$$\mathrm{QAnn}_X(Y) := \{x \in X \mid [x, [x, y]] = 0 \text{ for every } y \in Y\}$$

is called the **quadratic annihilator of Y in X**.

Note that the quadratic annihilator of an ideal needs not be an ideal.

Examples 1.4.17. ([76, Examples 1.1]) **1.** Consider a field F and the Lie algebra $\mathfrak{t}(3, F)$. Then

$$\begin{aligned} \mathrm{QAnn}(L) = \quad & \{a(e_{11}+e_{22}+e_{33})+be_{13}+ce_{23} \mid a,\, b,\, c \in F\} \cup \\ & \{a(e_{11}+e_{22}+e_{33})+be_{12}+ce_{13} \mid a,\, b,\, c \in F\}, \end{aligned}$$

where, as usual, e_{ij} denotes the matrix in $\mathbb{M}_3(F)$ whose entries are all zero except the one in row i and column j which is 1. Then $\mathrm{QAnn}(L)$ is not closed under sums.

2. Now, consider the Lie algebra $\bar{L} := L/Z$, for L as before. Then

$$\mathrm{QAnn}(\bar{L}) = \{\overline{ae_{13}+be_{23}} \mid a,\, b \in F\} \cup \{\overline{ae_{12}+be_{13}} \mid a,\, b \in F\},$$

where $\bar{x}$ denotes the class of an element x in L. Again we have the quadratic annihilator of this algebra $\bar{L}$ is not closed under sums.

Proposition 1.4.18. *Let $Q = \oplus_{\sigma\in G} Q_\sigma$ be a graded weak algebra of quotients of a graded subalgebra L. Then:*

(i) *For every nonzero graded ideal I of Q, $I \cap L$ is a nonzero graded ideal of L.*

(ii) *L graded semiprime (graded prime) implies Q graded semiprime (graded prime).*

(iii) *Suppose that Φ is 2 and 3-torsion free. Then L graded strongly non-degenerate implies Q graded strongly non-degenerate.*

Proof. (i). Let I be a nonzero graded ideal of Q and take a nonzero $y_\tau \in I_\tau$, for some $\tau \in G$. By the hypothesis, there exists $x_\alpha \in L_\alpha$ satisfying that $0 \neq [x_\alpha, y_\tau] \in I \cap L$.

(ii). Suppose that L is graded prime and take nonzero graded ideals I and J of Q. Applying (i) we obtain that $\widetilde{I} := I \cap L$ and $\widetilde{J} := J \cap L$ are nonzero graded ideals of L, while the graded primeness of L implies $0 \neq [\widetilde{I},\, \widetilde{J}] \subseteq [I,\, J]$, which proves that Q is graded prime. The graded semiprimeness of Q can be shown in a similar way.

(iii). Suppose that there exists an element $0 \neq q_\tau$ in Q_τ such that $(\operatorname{ad} q_\tau)^2 = 0$. Since Q is a graded weak algebra of quotients of L, $0 \neq y := [q_\tau,\, x_\sigma] \in L$ for some $x_\sigma \in L_\sigma$. As q_τ is in $\operatorname{QAnn}_Q(Q) \subseteq \operatorname{QAnn}_Q(L)$ we have, by [76, Theorem 2.1], that $[y,\, [y,\, u]] \in \operatorname{QAnn}_L(L)$ for every $u \in L$ (observe that the map $u \mapsto \operatorname{ad} u$ gives an isomorphism between L and its image inside $A(Q)$, the Lie subalgebra of $\operatorname{End}(Q)$ generated by the elements $\operatorname{ad} x$ for x in Q; this allows to apply the result in [76]). But $\operatorname{QAnn}_L(L)$ is zero, because L is graded strongly non-degenerate, therefore $[y,\, [y,\, u]] = 0$ for every $u \in L$. Again the same reasoning leads to $y = 0$, a contradiction. This shows the statement. $\square$

Definition 1.4.19. Let L be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma \in G} Q_\sigma$. We say that Q is **graded ideally absorbed into** L if for every nonzero element $q_\tau \in Q_\tau$ there exists a nonzero graded ideal I of L with $\operatorname{Ann}_L(I) = 0$ and such that $0 \neq [I,\, q_\tau] \subseteq L$.

It is immediate to see that "being graded ideally absorbed" implies "being a graded weak algebra of quotients". Our following aim will be to show that the notions of graded algebra of quotients and of absorption by graded ideals are equivalent. First we gather together several lemmas.

Recall that given a subalgebra L of a Lie algebra Q and an element $q \in Q$, the set

$$(L : q) := \{x \in L \mid [x,\, {}_L(q)] \subseteq L\}$$

is an ideal of L. (See [79, Lemma 2.10 (i)].)

Lemma 1.4.20. *Let L be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma\in G} Q_\sigma$ and consider $q_\tau \in Q_\tau$. Then:*

(i) *$(L : q_\tau)$ is a graded ideal of L.*

(ii) *If Q is a graded algebra of quotients of L then $\mathrm{Ann}_L((L : q_\tau)) = 0$. In particular, $(L : q_\tau)$ is a graded essential ideal of L.*

(iii) *If Q is graded ideally absorbed into L then $\mathrm{Ann}_L((L : q_\tau)) = 0$. In particular, $(L : q_\tau)$ is a graded essential ideal of L.*

Proof. (i). To prove that $(L : q_\tau)$ is graded, take $x = \sum_{\sigma\in G} x_\sigma \in (L : q_\tau)$. Given $n \in \mathbb{N}$, $\nu_i \in G$ and $y_i \in L_{\nu_i}$ $(i = 1, \ldots, n)$,

$$[x,\, \mathrm{ad}\, y_1 \ldots \mathrm{ad}\, y_n q_\tau] = \sum_\sigma [x_\sigma,\, \mathrm{ad}\, y_1 \ldots \mathrm{ad}\, y_n q_\tau] \in \oplus_{\sigma\in G} Q_{\sigma\nu_1\ldots\nu_n\tau}.$$

Note that each $[x_\sigma,\, \mathrm{ad}\, y_1 \ldots \mathrm{ad}\, y_n q_\tau]$ is one of the homogeneous components of the element $[x,\, \mathrm{ad}\, y_1 \ldots \mathrm{ad}\, y_n q_\tau] \in L$, which yields

$$[x_\sigma,\, \mathrm{ad}\, y_1 \ldots \mathrm{ad}\, y_n q_\tau] \in L_{\sigma\nu_1\ldots\nu_n\tau} \subseteq L. \tag{1.1}$$

Now, consider arbitrary (and not necessarily homogeneous) elements $z_1, \ldots, z_n \in L$. Since $[x_\sigma,\, \mathrm{ad}\, z_1 \ldots \mathrm{ad}\, z_n q_\tau]$ is a sum of elements as in (1.1), this same result implies $[x_\sigma,\, \mathrm{ad}\, z_1 \ldots \mathrm{ad}\, z_n q_\tau] \in L$, that is, $x_\sigma \in (L : q_\tau)$ for every $\sigma \in G$.

(ii). Suppose that Q is a graded algebra of quotients of L. We prove first that $(L : q_\tau)$ is a graded essential ideal of L. Let I be a nonzero graded ideal of L, and pick $0 \neq y_\mu \in I_\mu$, for some $\mu \in G$. Apply that Q is a graded algebra of quotients of L to find $x_\alpha \in L_\alpha$ satisfying $[x_\alpha,\, y_\mu] \neq 0$ and $[x_\alpha,\, {}_L(q_\tau)] \subseteq L$, i.e., $x_\alpha \in (L : q_\tau)$. As $(L : q_\tau)$ is an ideal, $0 \neq [x_\alpha,\, y_\mu] \in I \cap (L : q_\tau)$; in other words, $(L : q_\tau)$ is a graded essential ideal.

If $\mathrm{Ann}_L((L : q_\tau)) \neq 0$, the essentiality of $(L : q_\tau)$ would imply the existence of a nonzero homogeneous element $u_\alpha \in \mathrm{Ann}_L((L : q_\tau)) \cap (L : q_\tau)$. Applying that Q is a graded algebra of quotients of L we would find $x_\mu \in L_\mu$ (for some $\mu \in G$) satisfying $[x_\mu, u_\alpha] \neq 0$ and $[x_\mu, {}_L(q_\tau)] \subseteq L$. This would mean that x_μ is an element in $(L : q_\tau)$ which does not annihilate $u_\alpha \in \mathrm{Ann}_L((L : q_\tau))$, a contradiction. Consequently, $\mathrm{Ann}_L((L : q_\tau)) = 0$.

(iii). Suppose now that Q is graded ideally absorbed into L. Take a graded ideal I of L such that $\mathrm{Ann}_L(I) = 0$ (in particular I is a graded essential ideal) and $0 \neq [I, q_\tau] \subseteq L$. We are going to show that $I \subseteq (L : q_\tau)$, in which case $\mathrm{Ann}_L((L : q_\tau)) \subseteq \mathrm{Ann}_L(I) = 0$ and the proof will be complete.

We will prove, by induction on n, that $[I, \mathrm{ad}\, y_1 \ldots \mathrm{ad}\, y_n q_\tau] \subseteq L$ for every $n \in \mathbb{N}$ and $y_i \in L$ $(i = 1, \ldots, n)$. For $n = 1$,

$$[I, [y_1, q_\tau]] \subseteq [[I, y_1], q_\tau] + [y_1, [I, q_\tau]] \subseteq [I, q_\tau] + [L, [I, q_\tau]] \subseteq L.$$

Suppose the result true for $n-1$ and consider $y_i \in L$, with $i = 1, \ldots, n$. Then we have

$$\begin{aligned} [I, \mathrm{ad}\, y_1 \ldots \mathrm{ad}\, y_n q_\tau] &\subseteq [[I, y_1], \mathrm{ad}\, y_2 \ldots \mathrm{ad}\, y_n q_\tau] + [y_1, [I, \mathrm{ad}\, y_2 \ldots \mathrm{ad}\, y_n q_\tau]] \\ &\subseteq [I, \mathrm{ad}\, y_2 \ldots \mathrm{ad}\, y_n q_\tau] + [L, [I, \mathrm{ad}\, y_2 \ldots \mathrm{ad}\, y_n q_\tau]] \subseteq L \end{aligned}$$

by the induction hypothesis. This shows our claim. □

Lemma 1.4.21. *Let $Q = \oplus_{\sigma \in G} Q_\sigma$ be a graded weak algebra of quotients of L. Then, for every graded ideal I of L, $\mathrm{Ann}_L(I) = 0$ implies $\mathrm{Ann}_Q(I) = 0$.*

Proof. Let I be a graded ideal of L with $\mathrm{Ann}_L(I) = 0$. Suppose on the contrary that $\mathrm{Ann}_Q(I) \neq 0$. Reasoning as in the proof of condition (i) in Lemma 1.4.10, it can be shown that $\mathrm{Ann}_Q(I)$ contains every homogeneous component of each of its elements, hence we may choose $0 \neq q_\tau \in \mathrm{Ann}_Q(I)$, for some $\tau \in G$. By the hypothesis, there exist $\alpha \in G$ and $x_\alpha \in L_\alpha$ such that $0 \neq [x_\alpha, q_\tau] \in L$. Since $\mathrm{Ann}_L(I) = 0$ we find $y \in I$ satisfying

$$0 \neq [y, [x_\alpha, q_\tau]] = [[y, x_\alpha], q_\tau] + [x_\alpha, [y, q_\tau]] \in [I, q_\tau] + [L, [I, q_\tau]] = 0,$$

which is a contradiction. □

With these lemmas in mind we are now ready to prove the announced equivalency of the notions of "being a graded algebra of quotients" and "being graded ideally absorbed". The result is the following:

Proposition 1.4.22. *Let L be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma \in G} Q_\sigma$. Then Q is a graded algebra of quotients of L if and only if Q is graded ideally absorbed into L.*

Proof. Suppose that Q is a graded algebra of quotients of L and consider $0 \neq q_\tau \in Q_\tau$. Applying Lemma 1.4.20 (ii) we have that $(L : q_\tau)$ is a graded ideal of L with zero annihilator in L, and by Lemma 1.4.21 it has also zero annihilator in Q, so $0 \neq [(L : q_\tau), q_\tau] \subseteq L$.

Conversely, assume that Q is graded ideally absorbed into L, and take $0 \neq p_\sigma \in Q_\sigma$ and $q_\tau \in Q_\tau$. By Lemma 1.4.20 (iii) and Lemma 1.4.21, $[(L : q_\tau), p_\sigma] \neq 0$, so there exist $x \in (L : q_\tau)$ and $\alpha \in G$ such that $[x_\alpha, p_\sigma] \neq 0$. Since $(L : q_\tau)$ is a graded ideal of L, we have $x_\alpha \in (L : q_\tau)$, that is, $[x_\alpha, {}_L(q_\tau)] \subseteq L$, which completes the proof. □

The following result is a first application of the characterization below.

Corollary 1.4.23. *Let $Q = \oplus_{\sigma \in G} Q_\sigma$ be a graded algebra of quotients of a graded semiprime Lie algebra L. Then for every graded essential ideal I of L we have that Q is a graded algebra of quotients of I.*

Proof. Let I be a graded essential ideal of L. We will show that Q is graded ideally absorbed into I and the conclusion will follow from Proposition 1.4.22.

Take $0 \neq q_\tau \in Q_\tau$; by Proposition 1.4.22 there exists a graded ideal J of L with $\mathrm{Ann}_L(J) = 0$ satisfying $0 \neq [J, q_\tau] \subseteq L$. As J and I are graded essential ideals of L, the graded semiprimeness of L implies that $(I \cap J)^2$ is also a graded essential ideal of L, equivalently (condition (iii) in Lemma 1.4.10) $\mathrm{Ann}_L((I \cap J)^2) = 0$; in particular $\mathrm{Ann}_I((I \cap J)^2) = 0$.

On the other hand, it follows from Lemma 1.4.21 that $\mathrm{Ann}_Q((I \cap J)^2) = 0$, so that $[(I \cap J)^2, q_\tau] \neq 0$. Finally, $[J, q_\tau] \subseteq L$ and the Jacobi identity yield $0 \neq [(I \cap J)^2, q_\tau] \subseteq I$, which completes the proof. □

We continue by studying the relationship between graded (weak) algebras of quotients and (weak) algebras of quotients, a useful tool that (combined with other results) will provide with examples of graded algebras of quotients.

Lemma 1.4.24. *Let L be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma \in G} Q_\sigma$. If Q is a weak algebra of quotients of L then Q is also a graded weak algebra of quotients of L.*

Proof. For $0 \neq q_\tau \in Q_\tau$, apply the hypothesis to find $x \in L$ such that $0 \neq [x, q_\tau] \in L$; in particular, $0 \neq [x_\alpha, q_\tau] \in L_{\alpha\tau}$ for some $\alpha \in G$. □

The following lemma is a graded Lie version of the generalized common denominator property for associative setting.

Lemma 1.4.25. *Let $Q = \oplus_{\sigma \in G} Q_\sigma$ be a graded algebra of quotients of a graded semiprime Lie algebra L. Then, given $0 \neq p_\sigma \in Q_\sigma$ and $q_{\tau_i} \in Q_{\tau_i}$, with $\tau_i \in G$ and $i = 1, \ldots, n$ (for any $n \in \mathbb{N}$), there exist $\alpha \in G$ and $x_\alpha \in L_\alpha$ such that $[x_\alpha, p_\sigma] \neq 0$ and $[x_\alpha, {}_L(q_{\tau_i})] \subseteq L$ for every $i = 1, \ldots, n$.*

Proof. Consider $0 \neq p_\sigma \in Q_\sigma$ and $q_{\tau_i} \in Q_{\tau_i}$, with $i = 1, \ldots, n$. By Lemma 1.4.20 (i), $(L : q_{\tau_i})$ is a graded essential ideal of L for every i, hence $I =$

$\cap_{i=1}^{n}(L : q_{\tau_i})$ is again a graded essential ideal of L. Condition (iii) in Lemma 1.4.10 implies $\mathrm{Ann}_L(I) = 0$ and by Lemma 1.4.21 we obtain $\mathrm{Ann}_Q(I) = 0$. So, there exists $x \in I$ such that $[x, p_\sigma] \neq 0$, and if we decompose x into its homogeneous components we find some $\alpha \in G$ satisfying $[x_\alpha, p_\sigma] \neq 0$. Now the proof is complete because $x_\alpha \in I$ as I is a graded ideal and $x \in I$. □

Proposition 1.4.26. *Let L be a graded subalgebra of a graded Lie algebra $Q = \oplus_{\sigma\in G} Q_\sigma$. Consider the following conditions:*

(i) *Q is an algebra of quotients of L.*

(ii) *Q is a graded algebra of quotients of L.*

Then (i) implies (ii). Moreover, if L is graded semiprime then (ii) implies (i).

Proof. (i) ⇒ (ii). Given $0 \neq p_\sigma \in Q_\sigma$ and $q_\tau \in Q_\tau$, by the hypothesis there exists $x \in L$ satisfying $[x, p_\sigma] \neq 0$ and $[x, {}_L(q_\tau)] \subseteq L$, that is, $x \in (L : q_\tau)$. This means, by Lemma 1.4.20 (i), that $x_\alpha \in (L : q_\tau)$.

(ii) ⇒ (i). Suppose now that Q is a graded algebra of quotients of L, with L graded semiprime. Take p, q in Q, with $p \neq 0$; let $\sigma \in G$ be such that $p_\sigma \neq 0$ and write $\tau_1, \tau_2, \ldots, \tau_n$ to denote the elements of $\mathrm{Supp}(q)$.

By Lemma 1.4.25 it is possible to find an element $x_\alpha \in L_\alpha$ satisfying $[x_\alpha, p_\sigma] \neq 0$ and $[x_\alpha, {}_L(q_{\tau_i})] \subseteq L$ for every $i = 1, \ldots, n$, hence $[x_\alpha, p] \neq 0$ and $[x_\alpha, {}_L(q)] \subseteq L$; this shows that Q is an algebra of quotients of L. □

We conclude the section with an important example of graded algebras of quotients of graded Lie algebras. We refer the reader to [35, 5.4] to see the definitions involved in it. However, in Section 4.2 (more concretely in 4.2.5) we will explain that the TKK-algebra of a Jordan pair is. Recall that any strongly prime hermitian Jordan pair V is sandwiched as follows (see [35, 5.4]):

$$H(R, *) \lhd V \leq H(Q(R), *),$$

where R is a $*$-prime associative pair with involution and $Q(R)$ is its associative Martindale pair of symmetric quotients.

Example 1.4.27. Let R be a $*$-prime associative pair with involution, and $Q(R)$ its Martindale pair of symmetric quotients. Then $\mathrm{TKK}(H(Q(R), *))$ is a 3-graded algebra of quotients of $\mathrm{TKK}(H(R, *))$.

Proof. From [38, Proposition 4.2 and Corollary 4.3], $Q := \mathrm{TKK}(H(Q(R), *))$ is ideally absorbed into the strongly prime Lie algebra $L := \mathrm{TKK}(H(R, *))$; use [79, Proposition 2.15] to obtain that Q is an algebra of quotients of L and Proposition 1.4.26 to reach the conclusion. □

1.5 Lie algebras of quotients for skew Lie algebras

In order to foresee the importance of the concept of algebra of quotients in the non-associative setting, F. Perera and M. Siles Molina undertook in [75] a study of the relationship between the Lie and associative quotients. It is mentioned (see the previous comments to Lemma 3.6 in [75]) that similar results to [75, Theorem 2.12 and Proposition 3.5] should be available for skew Lie algebras. In what follows, our goal will be to prove that this is in fact the case.

Our tools to reach it will be the theory of generalized polynomial identities, for which our basic reference will be [15], Herstein's Lie theory, as treated in [64] and [11], dense extensions and also multiplicative semiprime algebras.

Throughout this section we will consider algebras over a field with characteristic different from 2.

Definition 1.5.1. Let A be an (associative or not) algebra; denote by $L(A)$ the algebra of all linear mappings from A into A. For $a \in A$, L_a and R_a will stand for the left and right, respectively, multiplication operators by a on A. The **multiplication algebra** $M(A)$ **of** A is the subalgebra of $L(A)$ generated by the identity operator Id_A and the set $\{L_a, R_a \mid a \in A\}$.

Remark 1.5.2. Let A be an associative algebra. From the fact that $\mathrm{ad}\, a = L_a - R_a$, where $\mathrm{ad}\, a$ denotes the multiplication operator by a on the Lie algebra A^-, it follows that $M(A^-)$ is a subalgebra of $M(A)$. By the way, note that if A^1 denotes the unital envelope of A, and, for $a, b \in A^1$, we denote by $M_{a,b}$ the two-sided multiplication operator on A defined by $M_{a,b}(x) = axb$ for all $x \in A$, then

$$M(A) = \left\{ \sum_{i=1}^{n} M_{a_i, b_i} \ : \ n \in \mathbb{N},\ a_i, b_i \in A^1\ (1 \leq i \leq n) \right\}.$$

A motivation to introduce multiplicative semiprime algebras is that the multiplication algebra $M(A)$ of a not necessarily associative semiprime algebra A needs not be semiprime. An example of this is the following:

Example 1.5.3. (Albert, 1942). Consider the three-dimensional unital algebra A over a field F with generators $\{1,\ u,\ v\}$ given by the relations

$$u^2 = 1, \quad uv = v^2 = v, \quad vu = 0$$

It is easy to verify that the only nonzero proper ideals of A are

$$Fv, \quad Fv + F(1+u) \text{ and } \quad Fv + F(1-u).$$

Hence, it follows that A is in fact prime. However $M(A)$ is not semiprime: $L_v R_u \neq 0$ but $L_v R_u M(A) L_v R_u = 0$.

Definition 1.5.4. An algebra A is **multiplicative semiprime** (**prime**) whenever A and its multiplication algebra $M(A)$ are semiprime (prime).

Several examples of multiplicative semiprime (prime) algebras are the following:

Example 1.5.5. Semiprime (prime) associative algebras are multiplicative semiprime (prime). It was shown by M. Cabrera and A. A. Mohammed in [27].

This fact suggests that the same must be true for algebras that are nearly associative. It was corroborated by M. Cabrera and A. R. Villena [28]. They proved that

Example 1.5.6. Strongly non-degenerate (non-degenerate in their terminology) alternative and strongly non-degenerate Jordan algebras are multiplicative semiprime.

J. C. Cabello, M. Cabrera, G. López, W. S. Martindale III studied in [24] the multiplicative semiprimeness of skew Lie algebras. Let A be a semiprime (prime) algebra over a field of characteristic not 2, then

Example 1.5.7. The Lie algebra A^-/Z is multiplicative semiprime (prime) in some important cases that are covered in [24, Corollary 2.4], but not in general (see [24, Theorem 2.1]). This contrasts with the case of $[A, A]/Z_{[A, A]}$, which is always multiplicative semiprime (prime) provided A is semiprime (prime) (see [24, Corollary 2.4]).

Furthermore, the same results hold for skew Lie algebras; if our algebra A is endowed with an involution $*$, it turns out that

Example 1.5.8. The skew Lie algebras K/Z_K and $[K, K]/Z_{[K, K]}$ are multiplicative semiprime. (See [24, Theorems 2.3 and 3.4].)

The notion of dense subalgebra was introduced by M. Cabrera in [25] and, as we will explain below, corresponds to the concept of ε-density for the ε-closure in the terminology of [23].

Definitions 1.5.9. ([25]). Let $B \subseteq A$ be an **extension of algebras**, which means that B is a subalgebra of A; the **annihilator of** B **in** $M(A)$ is defined by $B^{ann} := \{T \in M(A) \mid T(b) = 0 \text{ for every } b \in B\}$.

We say that it is a **dense extension** (or also that B is a **dense subalgebra of** A) if every nonzero element in $M(A)$ remains nonzero when restricted to B, i.e., $B^{ann} = 0$.

The first examples of dense extensions were given in the context of multiplicative semiprime algebras.

Example 1.5.10. M. Cabrera has proved in [25] that every essential ideal of a multiplicative semiprime algebra is dense.

F. Perera and M. Siles Molina found in [75] new and significant instances where dense extensions naturally appear. More concretely:

Example 1.5.11. (See [75, Lemma 3.4 and Proposition 3.5].) Let A be a semiprime associative algebra and Q a subalgebra of $Q^r_{max}(A)$ that contains A. Then the extensions $A \subseteq Q$, $A^-/Z_A \subseteq Q^-/Z_Q$ and $[A, A]/Z_{[A,A]} \subseteq [Q, Q]/Z_{[Q,Q]}$ are dense.

The following elemental result asserts that, for an extension of algebras $B \subseteq A$, the multiplication operators of B can be extended to multiplication operators of A.

Proposition 1.5.12. *Let $B \subseteq A$ be an extension of algebras. Then for each $F \in M(B)$ there exists $T \in M(A)$ such that $T|_B = F$.*

Proof. It is easy to see that the set $\mathcal{S} = \{F \in M(B) \mid \text{there exists } T \in M(A) \text{ such that } T|_B = F\}$ is a subalgebra of $M(B)$. Moreover, it is clear that $\mathrm{Id}_A(x) = \mathrm{Id}_B(x)$, $L^A_b(x) = L_b(x)$, and $R^A_b(x) = R_b(x)$ for all $x, b \in B$, where, to avoid any confusion, we have denoted by L^A_b and R^A_b the left and right (respectively) multiplication operators by b on A. Therefore, $\mathrm{Id}_B, L_b, R_b \in \mathcal{S}$ for all $b \in B$, and hence $\mathcal{S} = M(B)$ and the proof is complete. □

As a consequence we deduce that dense subalgebras are just those in which the multiplication operators have the **unique extension property** (this is condition (ii) in the corollary below).

Corollary 1.5.13. *Let $B \subseteq A$ be an extension of algebras. Then the following assertions are equivalent:*

(i) *B is a dense subalgebra of A.*

(ii) *For each $F \in M(B)$ there exists a unique $T \in M(A)$ such that $T|_B = F$.*

Proof. Assume that B is a dense subalgebra of A. Taking into account Proposition 1.5.12, it only remains to prove uniqueness. Assume that, for $F \in M(B)$, there exist T_1, $T_2 \in M(A)$ satisfying $T_1(x) = T_2(x) = F(x)$ for all $x \in B$. Then $T_1 - T_2 \in B^{ann} = 0$ by the density of B. Hence, $T_1 = T_2$.

To prove the converse, it is enough to note that each $T \in B^{ann}$ is an extension of the zero operator in $M(B)$, and consequently $T = 0$. □

Corollary 1.5.14. *Let $B \subseteq A$ be a dense extension of algebras. Then $M(B)$ can be regarded as a subalgebra of $M(A)$.*

Proof. The map $\varphi : M(B) \to M(A)$ given by $\varphi(F) = F'$, where F' is the unique extension of F which exists by Corollary 1.5.13, is a well-defined monomorphism that allows us to consider $M(B)$ as a subalgebra of $M(A)$. □

Definitions 1.5.15. ([23]). Let A be an (associative or not) algebra A.

1. For each subspace $\mathcal{N}$ of $M(A)$, we define

$$\mathcal{N}_{ann} = \{a \in A \, : \, \mathcal{N}(a) = 0\}.$$

2. The ε-**closure** of a subspace S of A is defined by $S^\wedge = (S^{ann})_{ann}$.

   It is easy to check that $U^\wedge$ is an ideal of A whenever U is so.

3. A subspace S of A is said to be ε-**closed** whenever $S^\wedge = S$ and ε-**dense** if it satisfies that $S^\wedge = A$.

   Trivial examples of ε-closed ideals of A are 0, $\mathrm{Ann}(A)$ and A.

Remark 1.5.16. Note that this notion of ε-density coincides with the density give in Definition 1.5.9. In fact, an extension of algebras $B \subseteq A$ is dense if and only if the annihilator of B in $M(A)$ is equal to zero, that is, $B^{ann} = 0$, which is equivalent to say that $B^\wedge = A$, i.e., B is a ε-**dense** subalgebra of A.

The behavior of the ε-closure with respect to the action of evaluation was determined in [23, Proposition 1.8] obtaining the **continuity property**:

If $T \in M(A)$ and if S is a subspace of A, then $T(S^\wedge) \subseteq T(S)^\wedge$.

As a consequence, we have $S_1^\wedge S_2^\wedge \subseteq (S_1S_2)^\wedge$ for all subspaces S_1, S_2 of A.

Now, we will show that, for a general algebra A, the density condition behaves properly with respect to the actions of passing to $A/\mathrm{Ann}(A)$ and A^2. First, we study the annihilator of a dense subalgebra.

Proposition 1.5.17. *Let $B \subseteq A$ be a dense extension of algebras. Then*

(i) $\mathrm{Ann}(B) = \mathrm{Ann}(A) \cap B$ *and the correspondence*

$$x + \mathrm{Ann}(B) \mapsto x + \mathrm{Ann}(A)$$

is a well-defined monomorphism from $B/Ann(B)$ into $A/\mathrm{Ann}(A)$ that allows us to regard $B/Ann(B)$ as a subalgebra of $A/\mathrm{Ann}(A)$.

(ii) $B/\mathrm{Ann}(B)$ *is a dense subalgebra of $A/Ann(A)$.*

Proof. (i). It is clear that $\mathrm{Ann}(A) \cap B \subseteq \mathrm{Ann}(B)$. To prove the converse inclusion take $x \in \mathrm{Ann}(B)$ and note that $L_x^A(B) = R_x^A(B) = 0$, from which it follows, taking into account that B is dense in A, that $L_x^A = R_x^A = 0$ and, as a result, $x \in \mathrm{Ann}(A)$. Thus, we have proved the equality $\mathrm{Ann}(B) = \mathrm{Ann}(A) \cap B$, from which it immediately follows that the correspondence $x + \mathrm{Ann}(B) \mapsto x + \mathrm{Ann}(A)$ is a well-defined monomorphism from $B/\mathrm{Ann}(B)$ into $A/\mathrm{Ann}(A)$.

(ii). Regarding $B/\mathrm{Ann}(B)$ as a subalgebra of $A/\mathrm{Ann}(A)$, we will show that $B/\mathrm{Ann}(B)$ is dense in $A/\mathrm{Ann}(A)$. To this end, we will consider the quotient map $\varrho : A \to A/\mathrm{Ann}(A)$, as well as the map

$$\varrho' : M(A) \to M(A/\mathrm{Ann}(A)),$$

which is uniquely determined by the condition $\varrho'(T) \circ \varrho = \varrho \circ T$ for all $T \in M(A)$. It is straightforward to verify that ϱ' is an epimorphism from $M(A)$ onto $M(A/\mathrm{Ann}(A))$ with kernel

$$[\mathrm{Ann}(A) : A] := \{T \in M(A) \ : \ T(A) \subseteq \mathrm{Ann}(A)\}.$$

Suppose that $F \in M(A/\mathrm{Ann}(A))$ satisfies $F(B/\mathrm{Ann}(B)) = 0$ and take $T \in M(A)$ such that $\varrho'(T) = F$. Then, for each $b \in B$ we have

$$\varrho(T(b)) = \varrho'(T)(\varrho(b)) = F(\varrho(b)) = 0,$$

and hence $T(B) \subseteq \mathrm{Ann}(A)$. Using this fact, the continuity property and that $B^\wedge = A$ (because the density of B), we obtain that

$$T(A) = T(B^\wedge) \subseteq T(B)^\wedge \subseteq \mathrm{Ann}(A)^\wedge = \mathrm{Ann}(A).$$

Therefore, $T \in [\mathrm{Ann}(A) : A]$ and so $F = \varrho'(T) = 0$, which concludes the proof. □

The following result was proved for Lie algebras in [75, Lemma 3.3].

Proposition 1.5.18. *Let A be an algebra. If B is a dense subalgebra of A, then B^2 is a dense subalgebra of A^2.*

Proof. Let B be a dense subalgebra of A. Assume that $F \in M(A^2)$ satisfies $F(B^2) = 0$, and choose $T \in M(A)$ such that $T(x) = F(x)$ for all $x \in A^2$, which is possible by Proposition 1.5.12. From the continuity property and taking into account the density of B in A we deduce that

$$T(A^2) = T((B^\wedge)^2) \subseteq T((B^2)^\wedge) \subseteq T(B^2)^\wedge = F(B^2)^\wedge = 0.$$

Therefore $T(A^2) = 0$ and hence $F = 0$, as desired. □

As we have said, our aim here is to extend [75, Theorem 2.12 and Proposition 3.5] to skew Lie algebras. We are now in a position to show it.

Theorem 1.5.19. *Let A be a semiprime associative algebra with an involution $*$ and let Q be a $*$-subalgebra of $Q_s(A)$ containing A. Then the following conditions are satisfied:*

(i) *K_A is a dense subalgebra of K_Q, and $[K_A, K_A]$ is a dense subalgebra of $[K_Q, K_Q]$.*

(ii) *K_A/Z_{K_A} is a dense subalgebra of K_Q/Z_{K_Q}, and $[K_A, K_A]/Z_{[K_A,K_A]}$ is a dense subalgebra of $[K_Q, K_Q]/Z_{[K_Q,K_Q]}$.*

(iii) *K_Q/Z_{K_Q} is an algebra of quotients of K_A/Z_{K_A}, and $[K_Q, K_Q]/Z_{[K_Q,K_Q]}$ is an algebra of quotients of $[K_A, K_A]/Z_{[K_A,K_A]}$.*

Proof. Let A be a semiprime associative algebra with an involution $*$ and let Q be a $*$-subalgebra of $Q_s(A)$ containing A.

(i). Assume that $F \in M(K_Q)$ satisfies $F(K_A) = 0$. Regarding K_A as a subalgebra of $Q_s(A)^-$, and keeping in mind Proposition 1.5.12, we can choose $T \in M(Q_s(A)^-)$ such that $T(q) = F(q)$ for all $q \in K_Q$. Let $n \in \mathbb{N}$ and

$p_i, q_i \in Q_s(A)$ $(i = 1, \ldots, n)$ be such that $T(q) = \sum_{i=1}^{n} p_i q q_i$ for all $q \in Q_s(A)$. Since $T(a - a^*) = F(a - a^*) = 0$ for each $a \in A$, it follows that

$$\psi(\mathbf{x}, \mathbf{x}^*) = \sum_{i=1}^{n} p_i \mathbf{x} q_i - \sum_{i=1}^{n} p_i \mathbf{x}^* q_i$$

is a $*$-GPI on A. By [15, Theorem 6.4.7] ψ also is a $*$-GPI on $Q_s(A)$, hence $T|_{K_{Q_s(A)}} = 0$, and so $F = 0$. Thus, we have proved that K_A is a dense subalgebra of K_Q. Now, by Proposition 1.5.18 (or, alternatively, [75, Lemma 3.3]), $[K_A, K_A]$ is a dense subalgebra of $[K_Q, K_Q]$.

(ii). Since K_A is a dense subalgebra of K_Q, applying Proposition 1.5.17 we obtain that

$$Z_{K_A} = Z_{K_Q} \cap K_A,$$

and K_A/Z_{K_A} can be regarded as a dense subalgebra of K_Q/Z_{K_Q}. Analogously, since $[K_A, K_A]$ is a dense subalgebra of $[K_Q, K_Q]$, again by Proposition 1.5.17 we have that

$$Z_{[K_A, K_A]} = Z_{[K_Q, K_Q]} \cap [K_A, K_A],$$

and $[K_A, K_A]/Z_{[K_A, K_A]}$ can be also regarded as a dense subalgebra of the algebra $[K_Q, K_Q]/Z_{[K_Q, K_Q]}$.

(iii). First we note that, for an essential $*$-ideal U of A, the inclusion map from U into A can be extended to an $*$-isomorphism from $Q_s(U)$ onto $Q_s(A)$ (see [66, Theorem 4.1]). Hence, keeping in mind conclusion (ii) in the statement, K_U/Z_{K_U} can be seen as a dense subalgebra of K_Q/Z_{K_Q}, and also as a dense ideal of K_A/Z_{K_A}. From this it follows that

$$\mathrm{Ann}_{K_A/Z_{K_A}}(K_U/Z_{K_U}) \subseteq \mathrm{Ann}(K_A/Z_{K_A}) \tag{1.2}$$

and

$$\mathrm{Ann}_{K_Q/Z_{K_Q}}(K_U/Z_{K_U}) \subseteq \mathrm{Ann}(K_Q/Z_{K_Q}). \tag{1.3}$$

Since A is semiprime, and so is Q (by [15, Lemma 2.1.9 (i)]), it follows from [64, Theorem 6.1] that K_A/Z_{K_A} and K_Q/Z_{K_Q} are semiprime Lie algebras. In particular,

$$\mathrm{Ann}(K_A/Z_{K_A}) = 0 \text{ and } \mathrm{Ann}(K_Q/Z_{K_Q}) = 0.$$

Thus, (1.2) and (1.3) allow us to conclude that

$$\mathrm{Ann}_{K_A/Z_{K_A}}(K_U/Z_{K_U}) = 0 \text{ and } \mathrm{Ann}_{K_Q/Z_{K_Q}}(K_U/Z_{K_U}) = 0 \qquad (1.4)$$

for any essential $*$-ideal U of A.

Now, let $q \in K_Q \backslash Z_{K_Q}$ and choose an essential $*$-ideal U of A such that $qU + Uq \subseteq A$. Then $0 \neq [q + Z_{K_Q}, K_U/Z_{K_U}] \subseteq K_A/Z_{K_A}$ by (1.4). Thus K_Q/Z_{K_Q} is an algebra of quotients of K_A/Z_{K_A}.

To verify that $[K_Q, K_Q]/Z_{[K_Q, K_Q]}$ is an algebra of quotients of the Lie algebra $[K_A, K_A]/Z_{[K_A, K_A]}$, we will consider the map $\varphi = \varrho \circ \iota$, where ι is the inclusion map from $[K_A, K_A]$ into K_A and ϱ is the quotient map from K_A onto K_A/Z_{K_A}. It is clear that φ is a homomorphism from $[K_A, K_A]$ into K_A/Z_{K_A} such that $\varphi([K_A, K_A]) = [K_A/Z_{K_A}, K_A/Z_{K_A}]$. Since, by [11, Lemma 2.14], $Z_{[K_A, K_A]} = [K_A, K_A] \cap Z_{K_A}$, it follows that $\ker(\varphi) = Z_{[K_A, K_A]}$. Thus, we have an isomorphism

$$[K_A/Z_{K_A}, K_A/Z_{K_A}] \cong [K_A, K_A]/Z_{[K_A, K_A]}. \qquad (1.5)$$

Analogously, we also have that

$$[K_Q/Z_{K_Q}, K_Q/Z_{K_Q}] \cong [K_Q, K_Q]/Z_{[K_Q, K_Q]}. \qquad (1.6)$$

Taking into account that K_A/Z_{K_A} is a semiprime Lie algebra and K_Q/Z_{K_Q} is an algebra of quotients of K_A/Z_{K_A}, it follows from [75, Lemma 2.13] that $[K_Q/Z_{K_Q}, K_Q/Z_{K_Q}]$ is an algebra of quotients of $[K_A/Z_{K_A}, K_A/Z_{K_A}]$. The isomorphisms (1.5) and (1.6) allow us to conclude now that $[K_Q, K_Q]/Z_{[K_Q, K_Q]}$ is an algebra of quotients of $[K_A, K_A]/Z_{[K_A, K_A]}$. □

We close this section by showing that [75, Theorem 2.12 and Proposition 3.5] can be obtained as a corollary of the theorem above.

Corollary 1.5.20. *Let A be a semiprime associative algebra and Q be a subalgebra of $Q_s(A)$ containing A. Then*

(i) *A^- is a dense subalgebra of Q^-, and $[A, A]$ is a dense subalgebra of $[Q, Q]$.*

(ii) *A^-/Z_A is a dense subalgebra of Q^-/Z_Q, and $[A, A]/Z_{[A, A]}$ is a dense subalgebra of $[Q, Q]/Z_{[Q, Q]}$.*

(iii) *Q^-/Z_Q is an algebra of quotients of A^-/Z_A, and $[Q, Q]/Z_{[Q, Q]}$ is an algebra of quotients of $[A, A]/Z_{[A, A]}$.*

Proof. Consider the semiprime associative algebra $A \oplus A^0$ endowed with the exchange involution. It is easy to see that the inclusion map from $A \oplus A^0$ into $Q_s(A \oplus A^0)$ can be extended to a $*$-isomorphism from $Q_s(A) \oplus Q_s(A)^0$ onto $Q_s(A \oplus A^0)$. In this way $Q \oplus Q^0$ can be seen as a $*$-subalgebra of $Q_s(A \oplus A^0)$ containing $A \oplus A^0$. Keeping in mind that A^- is isomorphic to $K_{A \oplus A^0}$ and Q^- is isomorphic to $K_{Q \oplus Q^0}$, the conclusions follow directly from Theorem 1.5.19. □

1.6 Graded Lie algebras of quotients for skew graded Lie algebras

Let A be a G-graded associative algebra with an involution $*$ satisfying that $A_\sigma^* = A_\sigma$, for all $\sigma \in G$. Then the Lie algebras K_A and K_A/Z_{K_A} are G-graded Lie algebras too.

Theorem 1.6.1. *Let A be a semiprime G-graded associative algebra with an involution $*$ such that $A_\sigma^* = A_\sigma$, for every $\sigma \in G$, and let $Q = \oplus_{\sigma \in G} Q_\sigma$ be* *a G-graded overalgebra of A contained in $Q_s(A)$ and satisfying $Q_\sigma^* = Q_\sigma$ for every $\sigma \in G$. Then:*

(i) K_Q/Z_{K_Q} *is a graded algebra of quotients of* K_A/Z_{K_A}.

(ii) $[K_Q, K_Q]/Z_{[K_Q,K_Q]}$ *is a graded algebra of quotients of* $[K_A, K_A]/Z_{[K_A,K_A]}$.

Proof. By Theorem 1.5.19 (iii), K_Q/Z_{K_Q} and $[K_Q, K_Q]/Z_{[K_Q,K_Q]}$ are algebras of quotients of K_A/Z_{K_A} and $[K_A, K_A]/Z_{[K_A,K_A]}$, respectively. Since A is semiprime, and so is Q (by [15, Lemma 2.1.9 (i)]) which imply that K_A/Z_{K_A} and K_Q/Z_{K_Q} are semiprime Lie algebras. In particular, they are graded semiprime, hence Proposition 1.4.26 applies to get the result. □

As a consequence we have:

Corollary 1.6.2. *Let A be a semiprime graded associative algebra and Q be a graded subalgebra of $Q_s(A)$ containing A. Then*

(i) Q^-/Z_Q *is a graded algebra of quotients of* A^-/Z_A.

(ii) $[Q, Q]/Z_{[Q,Q]}$ *is a graded algebra of quotients of* $[A, A]/Z_{[A,A]}$.

Proof. It is enough to note that for an arbitrary graded associative algebra A, the graded Lie algebra A^- is graded isomorphic to $K_{A\oplus A^0}$ and hence A^-/Z_A is graded isomorphic to $K_{A\oplus A^0}/Z_{K_{A\oplus A^0}}$, where A^0 denotes the opposite algebra of A, and $A \oplus A^0$ is endowed with the exchange involution and apply the theorem above. □

Chapter 2

Maximal and maximal graded algebras of quotients of Lie algebras

Following the original pattern of Y. Utumi and adapting some ideas coming from the Jordan setting [65], M. Siles Molina introduced in [79] the notion of the maximal algebra of quotients $Q_m(L)$ of a semiprime Lie algebra L. The reason for this name is that every algebra of quotients of L can be embedded into $Q_m(L)$.

Inspired by M. Siles Molina's construction, our first target in this second chapter will be to build a maximal algebra of quotients for every graded semiprime Lie algebra. Secondly, while the preceding chapter mostly considered abstract properties of algebras of quotients, in this one our target will be to compute $Q_m(L)$ for some Lie algebras. Specifically, we are interested in Lie algebras of the form $L = A^-/Z$, where A^- is the Lie algebra associated to a prime associative algebra A with center Z, and in Lie algebras of the form $L = K/Z_K$, where K is the Lie algebra of skew elements of a prime associative algebra with involution and Z_K its center.

2.1 The maximal algebra of quotients of a semiprime Lie algebra

Definitions 2.1.1. Let B be a subalgebra of an algebra A. A linear map $\delta : B \to A$ is called a **partial derivation** if

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in B$. Let us denote by $\mathrm{PDer}(B,\ A)$ the set of all partial derivations from B to A.

Any element x of A determines a map $\mathrm{ad}\, x \colon A \to A$ defined by $\mathrm{ad}\, x(y) = [x, y]$ which is a derivation of A. For every Lie ideal U of A, the restriction of the map $\mathrm{ad} \colon A \to \mathrm{Der}(A)$ to U,

$$\begin{array}{ccc} U & \to & \mathrm{Der}(A) \\ y & \mapsto & \mathrm{ad}\, y \end{array}$$

defines a Lie algebra homomorphism with kernel $\mathrm{Ann}(U)$, which allows us to identify $U/\mathrm{Ann}(U)$ with the subalgebra $\mathrm{ad}\,(U)$ of $\mathrm{Der}(A)$. For any $y \in U$ and $\delta \in \mathrm{Der}(A)$, we have

$$[\delta, \mathrm{ad}\, y] = \mathrm{ad}\, \delta(y),$$

hence $\mathrm{ad}\,(U)$ is an ideal of $\mathrm{Der}(A)$ whenever $\delta(U) \subseteq U$ for every $\delta \in \mathrm{Der}(A)$. We denote by $\mathrm{Inn}(A)$ the ideal $\mathrm{ad}\,(A)$ of $\mathrm{Der}(A)$ and we call the elements of $\mathrm{Inn}(A)$ **inner derivations of** A. Note that $A^-/Z \cong \mathrm{Inn}(A)$.

Partial derivations are defined analogously in the Lie algebra context.

Definitions 2.1.2. Let M be a subalgebra of a Lie algebra L; a linear map $\delta : M \to L$ is called a **partial derivation** if

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]$$

for all $x, y \in M$. By $\mathrm{PDer}(M,\ L)$ we will denote the set of all partial derivations from M to L and by $\mathrm{Der}(L)$ we will mean the Lie algebra of all derivations from L into L.

Remark 2.1.3. Incidentally, if δ is a derivation of an associative algebra A, then it is also a derivation of the Lie algebra A^-. The converse is not true in general: for example, every linear map from A into the center of A that vanishes on $[A, A]$ is a derivation of A^-. We call derivations of A^- **Lie derivations of** A.

We also have to define the concept of **degree** of a prime algebra A. The reason for this is that algebras of certain low degrees must be excluded in the results on Lie derivations [13, 14] that we are going to apply. In the case of having an involution, we shall need to use results that appear in [16, 36, 59, 60], which also require restrictions on the degree which can not be eliminated.

Definition 2.1.4. Let A be a prime algebra. For every $x \in A$ we define $\deg(x)$, the **degree of** x, as the degree of algebraicity of x over the extended centroid $\mathcal{C}$, provided x is algebraic over $\mathcal{C}$. If x is not algebraic over $\mathcal{C}$, then we write $\deg(x) = \infty$. The **degree of** A is defined as,

$$\deg(A) = \sup\{\deg(x) \mid x \in A\}.$$

Remark 2.1.5. Note that $\deg(A) < \infty$ if and only if A is a PI algebra. Furthermore, it is known that $\deg(A) = n < \infty$ if and only if A satisfies the standard polynomial identity of degree $2n$, but does not satisfy any polynomial identity of degree $< 2n$, and this is further equivalent to the condition that A can be embedded into the matrix algebra $\mathbb{M}_n(F)$ for some field F (say, one can take F as the algebraic closure of $\mathcal{C}$), but cannot be embedded into $\mathbb{M}_{n-1}(F)$ for any commutative algebra F.

We have now all the ingredients to explain the construction of the maximal algebra of quotients of a Lie algebra L. We have to confine ourselves to the case where L is semiprime. The definition is based on partial derivations defined on essential ideals of L.

Construction 2.1.6. (See [79, Lemma 3.2 and Theorem 3.4].) Let L be a semiprime Lie algebra. We say that two pairs (δ, I), (μ, J), where I, J are essential ideals of L and $\delta : I \to L$, $\mu : J \to L$ are partial derivations, **are equivalent** if δ and μ agree on some essential ideal contained in $I \cap J$.

This is an equivalence relation. Denote by δ_I the equivalence class determined by (δ, I). The set of all such classes becomes a Lie algebra if we define addition, scalar multiplication, and bracket as follows:

$$\delta_I + \mu_J = (\delta + \mu)_{I\cap J}, \quad \alpha(\delta_I) = (\alpha\delta)_I, \quad [\delta_I, \mu_J] = (\delta\mu - \mu\delta)_{(I\cap J)^2}.$$

This Lie algebra is called the **maximal algebra of quotients of** L, and will be denoted by $Q_m(L)$. One may identify L with a subalgebra of $Q_m(L)$ via the embedding $x \mapsto \operatorname{ad} x_L$. The maximality of $Q_m(L)$ is shown in the next result.

Proposition 2.1.7. (See [79, Proposition 3.6].) *Let L be a semiprime Lie algebra. Then $Q_m(L)$ is semiprime and an algebra of quotients of L. Moreover, $Q_m(L)$ is maximal among the algebras of quotients of L, in the sense that if Q is an algebra of quotients of L, then there exists a Lie monomorphism $\psi : Q \to Q_m(L)$ which is the identity on L. In particular, the map*

$$\begin{array}{cccc} \psi : & Q & \to & Q_m(L) \\ & x & \mapsto & \operatorname{ad} x_{(L:\, x)} \end{array}$$

is a Lie monomorphism which is the identity when restricted to L.

The axiomatic characterization of the symmetric Martindale rings of quotients (see Proposition 1.2.18) inspired to M. Siles Molina to give the following description of the maximal algebra of quotients of a semiprime Lie algebra.

Theorem 2.1.8. (See [79, Theorem 3.8].) *Let L be a semiprime Lie algebra and consider an overalgebra Q of L. Then Q is isomorphic to $Q_m(L)$, under an isomorphism which is the identity on L if and only if Q satisfies the following properties:*

(i) for every $q \in Q$ there exists an essential ideal I of L such that $[I, q] \subseteq L$,

(ii) $[q, I] \neq 0$ for every nonzero $q \in Q$ and every essential ideal I of L, and

(iii) for every essential ideal I of L and any derivation $\delta : I \to L$ there exists $q \in Q$ such that $\delta(x) = [q, x]$ for all $x \in I$.

2.2 The maximal graded algebra of quotients of a graded semiprime Lie algebra

Taking into account that, for a semiprime Lie algebra L, the elements of $Q_m(L)$, the maximal algebra of quotients of L, arise from partial derivations defined on essential ideals, it seems natural to consider instead graded partial derivations and graded essential ideals. With this idea in mind, we proceed to introduce a new graded algebra. First, we recall and introduce some definitions.

Definitions 2.2.1. (See Definitions 2.1.2.) Let L be a Lie algebra graded by an abelian group G, and I a graded ideal of L. We say that a partial derivation $\delta : I \to L$ has degree $\sigma \in G$ if it satisfies $\delta(I_\tau) \subseteq L_{\tau\sigma}$ for every $\tau \in G$. In this case, δ is called a **graded partial derivation of degree** σ.

Denote by $\mathrm{PDer}_{\mathrm{gr}}(I,\, L)_\sigma$ the set of all graded partial derivations of degree σ. Clearly, it becomes a Φ-module by defining operations in the natural way and, consequently,

$$\mathrm{PDer}_{\mathrm{gr}}(I,\, L) := \oplus_{\sigma\in G}\mathrm{PDer}_{\mathrm{gr}}(I,\, L)_\sigma$$

is also a Φ-module.

Example 2.2.2. If L is a G-graded Lie algebra and $x \in L$ is an homogeneous element of degree σ, then $\operatorname{ad} x$ is a partial derivation of degree σ. In general, for any x in the graded Lie algebra L,

$$\operatorname{ad} x = \sum_{\sigma \in G} \operatorname{ad} x_\sigma \in \oplus_{\sigma \in G} \mathrm{PDer}_{\mathrm{gr}}(I,\, L)_\sigma = \mathrm{PDer}_{\mathrm{gr}}(I,\, L).$$

In order to ease the notation, denote by $\mathcal{I}_{gr-e}(L)$ the set of all graded essential ideals of a graded Lie algebra L.

Construction 2.2.3. Let $L = \oplus_{\sigma \in G} L_\sigma$ be a G-graded semiprime Lie algebra over Φ. Consider the set

$$\mathcal{D}_{gr} := \{(\delta,\, I) \mid I \in \mathcal{I}_{gr-e}(L),\, \delta \in \mathrm{PDer}_{\mathrm{gr}}(I,\, L)\},$$

and define on $\mathcal{D}_{gr}$ the following relation: $(\delta,\, I) \equiv (\mu,\, J)$ if and only if there exists $K \in \mathcal{I}_{gr-e}(L)$ such that $K \subseteq I \cap J$ and $\delta\mid_K = \mu\mid_K$. It is easy to see that $\equiv$ is an equivalence relation.

Denote by $Q_{gr-m}(L)$ the quotient set $\mathcal{D}_{gr}/\equiv$ and by δ_I the equivalence class of $(\delta,\, I)$ in $Q_{gr-m}(L)$, for $\delta \in \mathrm{PDer}_{\mathrm{gr}}(I,\, L)$ and $I \in \mathcal{I}_{gr-e}(L)$. Then $Q_{gr-m}(L)$, with the following operations:

$$\delta_I + \mu_J = (\delta + \mu)_{I \cap J}$$

$$\alpha(\delta_I) = (\alpha\delta)_I$$

$$[\delta_I, \mu_J] = (\delta\mu - \mu\delta)_{(I \cap J)^2}$$

(for any $\delta_I, \mu_J \in Q_{gr-m}(L) = \oplus_{\sigma \in G} Q_\sigma$ and $\alpha \in \Phi$) becomes a G-graded Lie algebra over Φ, where

$$Q_\sigma := \{(\delta_\sigma)_I \mid \delta_\sigma \in \mathrm{PDer}_{\mathrm{gr}}(I,\, L)_\sigma,\, I \in \mathcal{I}_{gr-e}(L)\}.$$

Following the proof of [79, Theorem 3.4] one can see that $Q_{gr-m}(L)$ is a Lie algebra. At this point, we remark that given a finite family

$$\{(\delta_{\sigma_1})_{I_1}, \ldots, (\delta_{\sigma_n})_{I_n}\}$$

of elements of $Q_{gr-m}(L)$, it is always possible to find a graded ideal I of L satisfying that

$$(\delta_{\sigma_i})_{I_i} = (\delta_{\sigma_i})_I, \text{ for every } i = 1, \ldots, n.$$

Take, for example, $I = \cap_{i=1}^n I_i$. In the sequel, we will use this fact without an explicit mention.

Now it is easy to see that $Q_{gr-m}(L)$ is indeed graded:

Consider δ_I in $Q_{gr-m}(L)$ and write $\delta = \sum_{\sigma\in G} \delta_\sigma$, with $\delta_\sigma \in \mathrm{PDer_{gr}}(I,\, L)_\sigma$. As $\mathrm{Supp}(\delta)$ is finite, it is possible to write $\delta_I = \sum_\sigma (\delta_\sigma)_I$ with $(\delta_\sigma)_I \in Q_\sigma$; this shows that $\delta_I \in \sum_{\sigma\in G} Q_\sigma$ and consequently that $Q_{gr-m}(L) = \sum_{\sigma\in G} Q_\sigma$. We claim that this sum is direct: suppose on the contrary that

$$Q_\tau \cap (\sum_{\sigma\neq\tau} Q_\sigma) \neq 0 \text{ for some } \tau \in G,$$

and take $0 \neq \sum_{\sigma\neq\tau} (\delta_\sigma)_I \in Q_\tau$; in particular, $\sum_{\sigma\neq\tau} \delta_\sigma \neq 0$ on I and therefore it is nonzero on I_ν for some $\nu \in G$. On the other hand, $\delta_\sigma(I_\nu) \subseteq L_{\sigma\nu}$ and so $\sum_{\sigma\neq\tau} \delta_\sigma(I_\nu) \subseteq (\sum_{\sigma\neq\tau} L_{\sigma\nu}) \cap L_{\tau\nu} = 0$, a contradiction.

The following result shows how good is the graded Lie algebra that we have just built. Let us point out here that it is a graded algebra of quotients and cannot be enlarged.

Theorem 2.2.4. *Let $L = \oplus_{\sigma\in G} L_\sigma$ be a G-graded semiprime Lie algebra. Then:*

(i) *$Q_{gr-m}(L)$ contains L as a graded subalgebra, via the following graded Lie monomorphism:*

$$\begin{array}{rccc} \varphi: & L & \to & Q_{gr-m}(L) \\ & x & \mapsto & (\mathrm{ad}\, x)_L \end{array}$$

(ii) *$Q_{gr-m}(L)$ is graded semiprime and a graded algebra of quotients of L.*

(iii) *$Q_{gr-m}(L)$ is maximal among the graded algebras of quotients of L, in the sense that if S is a graded algebra of quotients of L, then there exists a graded Lie monomorphism $\psi : S \to Q_{gr-m}(L)$ which is the identity on L. In particular, the map*

$$\begin{array}{rccc} \psi : & S & \to & Q_{gr-m}(L) \\ & x & \mapsto & \sum_{\sigma\in G}(\mathrm{ad}\, x_\sigma)_{(L:\, x_\sigma)} \end{array}$$

where $x = \sum_{\sigma\in G} x_\sigma$, is a graded Lie monomorphism which is the identity when restricted to L.

Proof. (i). The map φ is well-defined: for x in L we have $\mathrm{ad}\, x = \sum_{\sigma\in G} \mathrm{ad}\, x_\sigma \in \mathrm{PDer}_{\mathrm{gr}}(I,\, L)$, which implies $(\mathrm{ad}\, x)_L = \sum_{\sigma\in G}(\mathrm{ad}\, x_\sigma)_L \in Q_{gr-m}(L)$.

The more "difficult" point in proving that φ is a graded Lie homomorphism is to see that $\varphi([x,\, y]) = [\varphi(x),\, \varphi(y)]$ for every $x,\, y \in L$. So, consider $x = \sum_\sigma x_\sigma$ and $y = \sum_\sigma y_\sigma$ in L. Note that the homogeneous components of $[x,\, y]$ are $[x,\, y]_\sigma = \sum_\tau [x_\tau,\, y_{\tau^{-1}\sigma}]$, with $\sigma \in G$. Then

$$\varphi([x,\, y]) = \sum_\sigma (\mathrm{ad}\, [x,\, y]_\sigma)_L = \sum_\sigma (\mathrm{ad} \sum_\tau [x_\tau,\, y_{\tau^{-1}\sigma}])_L.$$

and $\varphi([x,\, y])_\sigma = (\mathrm{ad} \sum_\tau [x_\tau,\, y_{\tau^{-1}\sigma}])_L$. On the other hand,

$$[\varphi(x),\, \varphi(y)] = \left[\sum_\sigma (\mathrm{ad}\, x_\sigma)_L,\, \sum_\sigma (\mathrm{ad}\, y_\sigma)_L\right]$$

implies

$$[\varphi(x),\, \varphi(y)]_\sigma = \sum_\tau [(\mathrm{ad}\, x_\tau)_L,\, (\mathrm{ad}\, y_{\tau^{-1}\sigma})_L] = \sum_\tau (\mathrm{ad}\, [x_\tau,\, y_{\tau^{-1}\sigma}])_L = \varphi([x,\, y])_\sigma,$$

for every $\sigma \in G$ and hence $\varphi([x,\, y]) = [\varphi(x),\, \varphi(y)]$ as desired.

Injectivity of φ: suppose $\varphi(x) = (\mathrm{ad}\, x)_L = 0$ for some $x \in L$. This means $\mathrm{ad}\, x(I) = 0$ for some $I \in \mathcal{I}_{gr-e}(L)$, that is, $x \in \mathrm{Ann}_L(I) = 0$ (apply Lemma 1.4.10 (iii)).

Identifying L with its image L^φ via the graded Lie monomorphism φ, we can regard L as a graded subalgebra of $Q_{gr-m}(L)$.

In what follows, for any $X \subseteq L$, write X^φ to denote the image of X inside $Q_{gr-m}(L)$ via the graded Lie monomorphism φ above. Let us stop, for a moment, the proof of the theorem in order to obtain a useful tool for our computations.

Remark 2.2.5. For every $\delta_I \in Q_{gr-m}(L)$ and $(\operatorname{ad} x)_L \in I^\varphi$, with $x \in I$, we have:

$$[\delta_I,\, (\operatorname{ad} x)_L] = (\operatorname{ad} \delta x)_L \in L^\varphi.$$

In fact, for any $y \in I$, $[\delta,\, \operatorname{ad} x]y = \delta([x,\, y]) - [x,\, \delta y] = [\delta x,\, y] + [x,\, \delta y] - [x,\, \delta y] = [\delta x,\, y] = (\operatorname{ad} \delta x)y$ and so $[\delta_I,\, (\operatorname{ad} x)_L] = (\operatorname{ad} \delta x)_L \in L^\varphi$.

Keeping this remark in mind, let us continue with the proof.

(ii). Show first that $Q_{gr-m}(L)$ is a graded algebra of quotients of L. Consider $0 \neq (\delta_\sigma)_I \in Q_\sigma$ and $(\mu_\tau)_I \in Q_\tau$. Choose $y_\alpha \in I_\alpha$ satisfying $\delta_\sigma(y_\alpha) \neq 0$ (it is possible because I is a graded ideal). Then $(\operatorname{ad} y_\alpha)_L \in L_\alpha$ satisfies:

$$[(\delta_\sigma)_I,\, (\operatorname{ad} y_\alpha)_L] = (\text{by Remark 2.2.5})\ (\operatorname{ad} \delta_\sigma(y_\alpha))_L \neq 0:$$

Otherwise, $0 = \operatorname{ad} \delta_\sigma(y_\alpha)(J) = [\delta_\sigma(y_\alpha),\, J]$ for some $J \in \mathcal{I}_{gr-e}(L)$, that is, $\delta_\sigma(y_\alpha) \in \operatorname{Ann}_L(J) = 0$ (by Lemma 1.4.10 (iii)), a contradiction.

Moreover, given $(\operatorname{ad} x_1)_L \in L_{\nu_1}, \ldots, (\operatorname{ad} x_n)_L \in L_{\nu_n}$ (for $n \in \mathbb{N}$ and $\nu_i \in G$) we have

$$[(\operatorname{ad} y_\alpha)_L,\, \operatorname{ad}((\operatorname{ad} x_1)_L) \ldots \operatorname{ad}((\operatorname{ad} x_n)_L)((\mu_\tau)_I)] \in L_{\alpha\nu_1\ldots\nu_n\tau}.$$

Indeed, as

$$\operatorname{ad}((\operatorname{ad} x_1)_L) \ldots (\operatorname{ad}(\operatorname{ad} x_n)_L)((\mu_\tau)_I) = \operatorname{ad}(\operatorname{ad} x_1)_L \ldots \operatorname{ad}(\operatorname{ad} x_{n-1})_L$$

$$[\operatorname{ad} x_n,\, \mu_\tau]_I = [\operatorname{ad} x_1,\, [\operatorname{ad} x_2, \ldots [\operatorname{ad} x_n,\, \mu_\tau] \ldots]_I \in Q_{\nu_1\nu_2\ldots\nu_n\tau},$$

if we define $\gamma := [\mathrm{ad}\, x_1,\, [\mathrm{ad}\, x_2, \ldots [\mathrm{ad}\, x_n,\, \mu_\tau] \ldots]$, then

$$[\mathrm{ad}\,(y_\alpha)_L,\, \mathrm{ad}\,((\mathrm{ad}\, x_1)_L) \ldots \mathrm{ad}\,((\mathrm{ad}\, x_n)_L)((\mu_\tau)_I)] = (\text{by Remark}(2.2.5))$$
$$[(\mathrm{ad}\, y_\alpha)_L,\, \gamma_I] \in L_{\alpha\nu_1\nu_2\ldots\nu_n\tau}.$$

Note that the result follows now immediately. The graded semiprimeness of $Q_{gr-m}(L)$ is obtained from Proposition 1.4.18 (ii).

(iii). Suppose that S is a graded Lie algebra of quotients of L and consider the map ψ given in the statement. It is well-defined by Lemma 1.4.20 (ii). The more "difficult" point in proving that ψ is a graded Lie homomorphism is to show that $\psi([x,\, y]) = [\psi(x),\, \psi(y)]$ for every $x,\, y \in S$. Take $x = \sum_\sigma x_\sigma$, $y = \sum_\sigma y_\sigma \in S$. Note that the homogeneous components of $[x,\, y]$ are $[x,\, y]_\sigma = \sum_\tau [x_\tau,\, y_{\tau^{-1}\sigma}]$, with $\sigma \in G$. On the other hand, as $\mathrm{Supp}([x,\, y])$ is a finite set, denote its elements by $\sigma_1,\, \sigma_2, \ldots, \sigma_n$, then $I := \cap_{i=1}^n (L :\, [x,\, y]_{\sigma_i})$ is a graded essential ideal of L. Then:

$$\begin{aligned}
\psi([x,\, y]) &= \sum_{i=1}^n (\mathrm{ad}\,[x,\, y]_{\sigma_i})_I = \sum_{i=1}^n (\mathrm{ad}\, \sum_\tau [x_\tau, y_{\tau^{-1}\sigma_i}])_I \\
&= \sum_{i=1}^n \left(\sum_\tau [\mathrm{ad}\, x_\tau,\, \mathrm{ad}\, y_{\tau^{-1}\sigma_i}] \right)_I = \sum_{i=1}^n \sum_\tau [(\mathrm{ad}\, x_\tau)_I,\, (\mathrm{ad}\, y_{\tau^{-1}\sigma_i})_I] \\
&= [\psi(x),\, \psi(y)]
\end{aligned}$$

To prove the injectivity, take $x \in S$ such that $\psi(x) = 0$. Then $(\mathrm{ad}\, x_\sigma)_{(L:\, x_\sigma)} = 0$ for every $\sigma \in G$. This means that for every $\sigma \in G$ there exists a graded essential ideal I^σ of L, contained in $(L : x_\sigma)$, such that $(\mathrm{ad}\, x_\sigma)(I^\sigma) = 0$. Hence $x_\sigma \in \mathrm{Ann}_L(I^\sigma) = 0$ (by Lemma 1.4.10 (iii)) for every $\sigma \in G$, whence $x = 0$ as desired. □

Definition 2.2.6. For a graded semiprime Lie algebra L, the graded algebra $Q_{gr-m}(L)$ constructed in (2.2.3) will be called the **maximal graded algebra of quotients** of L.

Once we have shown the existence of a maximal graded algebra of quotients for any graded semiprime Lie algebra, we proceed to its characterization. A consequence of the result that follows is the uniqueness of the maximal graded algebra of quotients (up to graded isomorphism).

Theorem 2.2.7. *Let L be a graded semiprime Lie algebra and consider a graded overalgebra S of L. Then S is graded isomorphic to $Q_{gr-m}(L)$, under an isomorphism which is the identity on L, if and only if S satisfies the following properties:*

(i) *For any $s_\sigma \in S_\sigma$ $(\sigma \in G)$ there exists $I \in \mathcal{I}_{gr-e}(L)$ such that $[I, s_\sigma] \subseteq L$.*

(ii) *For $s_\sigma \in S_\sigma$ $(\sigma \in G)$ and $I \in \mathcal{I}_{gr-e}(L)$, $[I, s_\sigma] = 0$ implies $s_\sigma = 0$.*

(iii) *For $I \in \mathcal{I}_{gr-e}(L)$ and $\delta \in \mathrm{PDer}_{\mathrm{gr}}(I, L)_\sigma$ $(\sigma \in G)$ there exists $s_\sigma \in S_\sigma$ such that $\delta(x) = [s_\sigma, x]$ for every $x \in I$.*

Proof. Define

$$\begin{array}{rccc} \psi: & S & \to & Q_{gr-m}(L) \\ & s & \mapsto & \sum_{\sigma\in G}(\mathrm{ad}\, s_\sigma)_I \end{array}$$

where $s = \sum_{\sigma\in G} s_\sigma$ and I is a graded essential ideal of L satisfying that $[I, s_\sigma] \subseteq L$ for all $\sigma \in G$.

The map ψ is well-defined: take $s \in S$ and denote by $\sigma_1, \sigma_2, \ldots, \sigma_n$ the elements of $\mathrm{Supp}(s)$. By (i) it is possible to find, for each $i = 1, \ldots, n$, $I_i \in \mathcal{I}_{gr-e}(L)$ such that $[I_i, s_{\sigma_i}] \subseteq L$. Then $I = \cap_{i=1}^n I_i \in \mathcal{I}_{gr-e}(L)$ satisfies $[I, s_{\sigma_i}] \subseteq L$ for all $i = 1, \ldots, n$.

Moreover, ψ is a graded monomorphism: given $s, t \in S$, apply again (i) and a reasoning similar to the described in the paragraph above to find $I, J, K \in \mathcal{I}_{gr-e}(L)$ such that $[I, s_\sigma] \subseteq L$, $[J, t_\sigma] \subseteq L$ and $[K, [s, t]_\sigma] \subseteq L$

for all $\sigma \in G$, where $s = \sum_\sigma s_\sigma$, $t = \sum_\sigma t_\sigma$ and $[s, t] = \sum_\sigma [s, t]_\sigma$ are the decompositions of s, t, $[s, t]$, respectively, into their homogeneous components. Take $U = I \cap J \cap K \in \mathcal{I}_{gr-e}(L)$. Then $U^2 \in \mathcal{I}_{gr-e}(L)$ and we have:

$$\psi([s, t]) = \sum_\sigma (\mathrm{ad}\,[s, t]_\sigma)_{U^2} = \sum_\sigma (\mathrm{ad} \sum_\tau [s_\tau, t_{\tau^{-1}\sigma}])_{U^2}.$$

This implies

$$\psi([s, t])_\sigma = (\mathrm{ad} \sum_\tau [s_\tau, t_{\tau^{-1}\sigma}])_{U^2}.$$

On the other hand, as

$$[\psi(s), \psi(t)] = [\sum_\sigma (\mathrm{ad}\, s_\sigma)_{U^2}, \sum_\sigma (\mathrm{ad}\, t_\sigma)_{U^2}]$$

we have

$$\begin{aligned} [\psi(s), \psi(t)]_\sigma &= \sum_\tau [(\mathrm{ad}\, s_\tau)_{U^2}, (\mathrm{ad}\, t_{\tau^{-1}\sigma})_{U^2}] = \sum_\tau (\mathrm{ad}\,[s_\tau, t_{\tau^{-1}\sigma}])_{U^2} \\ &= (\sum_\tau \mathrm{ad}\,[s_\tau, t_{\tau^{-1}\sigma}])_{U^2}, \end{aligned}$$

which shows $\psi([s, t])_\sigma = [\psi(s), \psi(t)]_\sigma$ for all $\sigma \in G$, hence $\psi([s, t]) = [\psi(s), \psi(t)]$.

Injectivity of ψ: if $\psi(s) = 0$ for some $s \in S$, then $\sum_\sigma (\mathrm{ad}\, s_\sigma)_I = 0$, where $I \in \mathcal{I}_{gr-e}(L)$ satisfies $[I, s_\sigma] \subseteq L$ for all $\sigma \in G$, hence $(\mathrm{ad}\, s_\sigma)_I = 0$. This means $0 = (\mathrm{ad}\, s_\sigma)(J) = [s_\sigma, J]$ for some graded essential ideal J (of L) contained in I and every $\sigma \in G$. By (ii), $s_\sigma = 0$ for all $\sigma \in G$, that is, $s = 0$.

Surjectivity of ψ: given $\sum_\sigma (\delta_\sigma)_I \in Q_{gr-m}(L)$, by (iii) there exists $s_\sigma \in S_\sigma$ such that δ_σ and $\mathrm{ad}\, s_\sigma$ coincide on the graded essential ideal I of L, hence

$$\sum_\sigma (\delta_\sigma)_I = \sum_\sigma (\mathrm{ad}\, s_\sigma)_I = \psi(\sum_\sigma s_\sigma).$$

Finally, to see that ψ is the identity on L, identify L with L^φ, where φ is the map defined in Theorem 2.2.4.

Conversely, we will show that $Q_{gr-m}(L)$ satisfies the three conditions in the statement.

(i). For $q_\sigma \in Q_\sigma$ we have $(L : q_\sigma) \in \mathcal{I}_{gr-e}(L)$, by Theorem 2.2.4 and Lemma 1.4.20 (ii), and by definition $[(L : q_\sigma), q_\sigma] \subseteq L$.

(ii). Consider $q_\sigma \in Q_\sigma$ and $I \in \mathcal{I}_{gr-e}(L)$ such that $[I, q_\sigma] = 0$. Then $q_\sigma \in \mathrm{Ann}_{Q_{gr-m}(L)}(I) = 0$ by Theorem 2.2.4 and Lemma 1.4.21.

(iii). Given $I^\varphi \in \mathcal{I}_{gr-e}(L^\varphi)$ and $\bar{\delta} \in \mathrm{PDer}_{\mathrm{gr}}(I^\varphi, L^\varphi)_\sigma$, we have to find $q_\sigma \in Q_\sigma$ such that $\bar{\delta} = \mathrm{ad}\, q_\sigma$ on I^φ. Consider $\delta : I \to L$ defined by $\delta(x) = \varphi^{-1}(\bar{\delta}((\mathrm{ad}\, x)_L))$. Then $q_\sigma := \delta_I \in Q_\sigma$ satisfies $[q_\sigma, x^\varphi] = [\delta_I, (\mathrm{ad}\, x)_L]$ (by Remark 2.2.5) $(\mathrm{ad}\, \delta x)_L = (\delta x)^\varphi = \bar{\delta}((\mathrm{ad}\, x)_L) = \bar{\delta}(x^\varphi)$. □

Remark 2.2.8. Conditions (i) and (ii) in the theorem above are equivalent to the following one:

(ii)′ S is a graded algebra of quotients of L.

Proof. If S is a graded algebra of quotients of L, condition (i) is satisfied by Proposition 1.4.22. On the other hand, (ii) follows immediately by the graded semiprimeness of L and Lemma 1.4.21.

Conversely, assume that S satisfies conditions (i) and (ii). We are going to show that S is graded ideally absorbed into L, in which case (ii)′ will follow by Proposition 1.4.22. Consider $0 \neq s_\sigma \in S_\sigma$; by (i) there exists $I \in \mathcal{I}_{gr-e}(L)$ such that $[I, s_\sigma] \subseteq L$ and $[I, s_\sigma] \neq 0$ by (ii). Note that since L is graded semiprime, Lemma 1.4.10 (iii) implies $\mathrm{Ann}_L(I) = 0$. □

Let us finish the section by showing that the notion of maximal graded algebra of quotients is a good generalization of that of maximal algebra of quotients, as the maximal graded algebra of quotients and the maximal algebra of quotients of a semiprime Lie algebra coincide when considering the trivial grading over such an algebra.

Lemma 2.2.9. *Let L be a semiprime Lie algebra, then the algebras $Q_{gr-m}(L)$ and $Q_m(L)$ are isomorphic, considering L as a G-graded algebra with the trivial G-grading.*

Proof. Note that in this particular case, every ideal I of L is a graded ideal. It easily implies $(Q_{gr-m}(L))_e = Q_m(L)$. On the other hand, given $\sigma \in G$ with $\sigma \neq e$ and $\delta \in \mathrm{PDer}_{\mathrm{gr}}(I,\, L)_\sigma$ then

$$\delta(I) = \delta(I_e) \subseteq L_{e\sigma} = L_\sigma = 0,$$

that is, $\delta = 0$ and hence $(Q_{gr-m}(L))_\sigma = 0$ for every $e \neq \sigma \in G$. Taking into account these considerations it is now clear that the algebras above are isomorphic. □

2.3 The maximal Lie algebra of quotients of A^-/Z

Our aim in this section is to give a description of $Q_m(A^-/Z)$, where A is a (semi)prime associative algebra. Since the elements of the maximal algebra of quotients of a Lie algebra arise from partial derivations defined on essential Lie ideals and our Lie algebra A^-/Z comes from an associative algebra A, it seems natural to consider instead associative derivations that are defined on essential associative ideals. With this idea in mind we proceed to introduce a new Lie algebra.

Construction 2.3.1. Let A be a semiprime associative algebra over Φ. Consider the set

$$\mathcal{D} := \{(\delta,\, I) \mid I \in \mathcal{I}_e(A),\, \delta \in \mathrm{PDer}(I,\, A)\},$$

and define on $\mathcal{D}$ the following relation: $(\delta,\, I) \equiv (\mu,\, J)$ if and only if δ and μ agree on some essential ideal of A contained in $I \cap J$. One can easily show that $\equiv$ is an equivalence relation.

Denote by $\mathrm{Der}_{\mathrm{m}}(A)$ the quotient set $\mathcal{D}/\equiv$ and by δ_I the equivalence class of $(\delta,\, I)$ in $\mathrm{Der}_{\mathrm{m}}(A)$, for $\delta \in \mathrm{PDer}(I,\, A)$ and $I \in \mathcal{I}_e(A)$. Then $\mathrm{Der}_{\mathrm{m}}(A)$, with the following operations:

$$\delta_I + \mu_J = (\delta + \mu)_{I\cap J}$$

$$\alpha(\delta_I) = (\alpha\delta)_I$$

$$[\delta_I, \mu_J] = (\delta\mu - \mu\delta)_{(I\cap J)^2}$$

for any $\delta_I, \mu_J \in \mathrm{Der}_{\mathrm{m}}(A)$ and $\alpha \in \Phi$ becomes a Lie algebra over Φ.

The only not entirely obvious part in proving that $\mathrm{Der}_{\mathrm{m}}(A)$ is a Lie algebra is to show that the Lie bracket is well defined on $\mathrm{Der}_{\mathrm{m}}(A)$. Let $\delta_I,\ \mu_J \in \mathrm{Der}_{\mathrm{m}}(A)$; for every $u, v \in I \cap J$ we have $[\delta, \mu](uv) = \delta\mu(uv) - \mu\delta(uv) = \delta((\mu u)v + u(\mu v)) - \mu((\delta u)v + u(\delta v))$, which makes sense because $(\mu u)v, u(\mu v), (\delta u)v, u(\delta v) \in I \cap J$. Since δ and μ are partial derivations, $[\delta, \mu] : (I \cap J)^2 \to A$ is a partial derivation too.

It turns out that (under a very mild technical assumption) the algebra we have just built coincides with $Q_m(A^-/Z)$. Let us start by showing that this Lie algebra is sandwiched between $\mathrm{Der}(A)$ and $\mathrm{Der}(Q_s(A))$. First, we need a lemma.

Lemma 2.3.2. *Let A be a semiprime algebra and let Q be a subalgebra of $Q_s(A)$ that contains A. If $\delta : Q \to Q_s(A)$ is a derivation such that $\delta|_A = 0$, then $\delta = 0$.*

Proof. Suppose on the contrary that $\delta(q) \neq 0$ for some $q \in Q$. Since $Q_s(A)$ is a left quotient algebra of A, there exists $a \in A$ satisfying $aq \in A$ and $a\delta(q) \neq 0$. By the hypothesis, $0 = \delta(aq) = \delta(a)q + a\delta(q) = a\delta(q)$, which is a contradiction. □

Lemma 2.3.3. *If A is a prime algebra, then*

$$\mathrm{Der}(A) \subseteq \mathrm{Der}_{\mathrm{m}}(A) \subseteq \mathrm{Der}(Q_s(A)).$$

Proof. Define

$$\begin{array}{rccc} \phi: & \mathrm{Der}(A) & \to & \mathrm{Der}_{\mathrm{m}}(A) \\ & \delta & \mapsto & \delta_A \end{array}$$

It is straightforward to verify that ϕ is a well-defined Lie algebra homomorphism. To prove the injectivity, take $\delta \in \mathrm{Der}(A)$ such that $\delta_A = 0$; this means that $\delta(I) = 0$ for some nonzero ideal I of A. Since $Q_s(I) = Q_s(A)$ (see [15, Proposition 2.1.10]) applying Lemma 2.3.2 to $I \subseteq A \subseteq Q_s(I)$ we obtain that $\delta = 0$, as desired.

Let δ_I be in $\mathrm{Der}_{\mathrm{m}}(A)$, with I a nonzero ideal of A and $\delta : I \to A$ a derivation. Apply Proposition 1.2.22 to extend δ uniquely to a derivation δ' of $Q_s(A)$. Consider

$$\begin{array}{rccc} \varphi: & \mathrm{Der}_{\mathrm{m}}(A) & \to & \mathrm{Der}(Q_s(A)) \\ & \delta_I & \mapsto & \delta' \end{array}$$

If $\delta_I = \mu_J$, then there exists a nonzero ideal U of A contained in $I \cap J$ such that $\delta|_U = \mu|_U$. Extend δ and μ to derivations δ' and μ', respectively, of $Q_s(A)$. Since $\delta'|_U = \delta|_U = \mu|_U = \mu'|_U$ and $Q_s(A) = Q_s(U)$ (see [15, Proposition 2.1.10]), by Lemma 2.3.2 applied to $U \subseteq A \subseteq Q_s(U)$ we obtain that $\delta'|_A = \mu'|_A$, and again by Lemma 2.3.2 it follows that $\delta' = \mu'$, which proves that φ is well-defined. Finally, note that φ is a Lie algebra monomorphism. □

We come back to the announced problem, namely, whether one can say that $\mathrm{Der}_{\mathrm{m}}(A)$ and $Q_m(A^-/Z)$ are isomorphic. Let us denote by $\langle X \rangle$ the subalgebra of an algebra A generated by a set X.

Lemma 2.3.4. *Let U be a Lie ideal of a semiprime algebra A such that $\overline{U}$ is an essential ideal of A^-/Z. Then $\langle U \rangle$ contains an essential ideal of A.*

Proof. First we show that $\langle U\rangle$ contains a nonzero ideal of A. It is clear that $[\langle U\rangle, U] \subseteq \langle U\rangle$. Moreover, $[\langle U\rangle, U] \neq 0$; otherwise $[x, U] = 0$ for every $x \in U$. This would imply, by [42, Sublemma, p. 5], $x \in Z$ and, consequently, $U \subseteq Z$, a contradiction. Therefore [43, Theorem 3] yields our claim.

Hence, let I be a nonzero ideal contained in $\langle U\rangle$. Since the sum of all ideals contained in $\langle U\rangle$ is again an ideal contained in U, there is no loss of generality in assuming that I is the largest ideal contained in $\langle U\rangle$.

We will show that I is an essential ideal of A. First, we see that $\mathrm{Ann}(I)\cap U \subseteq Z$. Otherwise, by [43, Theorem 3], there exists a nonzero ideal J of A contained in $\langle \mathrm{Ann}(I)\cap U\rangle \subseteq \mathrm{Ann}(I)\cap\langle U\rangle$. Since $I\cap \mathrm{Ann}(I) = 0$ because A is semiprime, $I \subsetneq I\oplus J \subseteq \langle U\rangle$, which contradicts the maximality of I. Since $\overline{U}$ is an essential ideal of A^-/Z, $\mathrm{Ann}_{A^-/Z}(\overline{U}) = 0$. Note that $[\mathrm{Ann}(I), U] \subseteq \mathrm{Ann}(I)\cap U \subseteq Z$ implies $(\mathrm{Ann}(I)+Z)/Z = 0$, that is, $\mathrm{Ann}(I) \subseteq Z \subseteq U$. Now, $I \subseteq I\oplus \mathrm{Ann}(I) \subseteq \langle U\rangle$ and the maximality of I imply $\mathrm{Ann}(I) = 0$, hence I is an essential ideal of A. □

Proposition 2.3.5. *Let A be a semiprime algebra. Define*

$$\begin{array}{cccc} \varphi: & \mathrm{Der}_{\mathrm{m}}(A) & \to & Q_m(A^-/Z) \\ & \delta_I & \mapsto & \bar{\delta}_{\overline{I}} \end{array}$$

where

$$\begin{array}{cccc} \bar{\delta}: & \overline{I} & \to & A^-/Z \\ & \bar{y} & \mapsto & \overline{\delta(y)} \end{array}$$

Then φ is a Lie algebra homomorphism with kernel

$$\{\delta_I \in \mathrm{Der}_{\mathrm{m}}(A) \mid \delta(I) \subseteq Z\}.$$

Proof. The map $\bar{\delta}$ is well-defined. Indeed, taking into account Lemma 1.1.14 we see that it is enough to show that for I an essential ideal of A, and $\delta \in \mathrm{Der}(I, A)$, $y \in I\cap Z$ implies $\delta(y) \in Z$. Note that for every $x \in I$ we have $[\delta(y), x]\delta([y, x]) - [y, \delta(x)] = 0$. But this yields $\delta(y) \in Z$. Namely, only central elements can commute with every element from an essential ideal. Indeed, $[a, u] = 0$ for every $u \in I$ yields $[a, x]u = [a, xu] = 0$ for all $x \in A$ and $u \in I$, and hence $[a, x] = 0$ since I is essential.

It is easy to see that φ is a well-defined Lie algebra homomorphism. Let us now compute its kernel. First we show that if $\delta_I \in \mathrm{Der}_m(A)$ is such that $\delta(J) \subseteq Z$ for some essential ideal J of A contained in I, then $\delta(I) \subseteq Z$. For $x \in I$ and $u \in J$ we have $xu \in J$, and so $\delta(u), \delta(xu) \in Z$. Accordingly, $\delta(x)u = \delta(xu) - x\delta(u)$ commutes with x, that is, $\delta(x)ux = x\delta(x)u$. Replacing u by uy, where $u \in J$ and $y \in A$, it follows that $\delta(x)uyx = x\delta(x)uy = \delta(x)uxy$. Thus, $\delta(x)u[x, y] = 0$ for all $x \in I$, $y \in A$ and $u \in J$. Linearizing this identity we get $\delta(x)u[z, y] + \delta(z)u[x, y] = 0$ for all $x, z \in I$, $y \in A$, $u \in J$. Consequently, for $u, v \in J$, $x, z \in I$ and $y \in A$ we have

$$\delta(x)u[z, y]v\delta(x)u[z, y] = -\delta(x)u[z, y]v\delta(z)u[x, y] \in \delta(x)J[x, y] = 0.$$

Since J is essential, $aJa = 0$ with $a \in A$ implies $a = 0$. Therefore, $\delta(x)u[z, y] = 0$ for all $u \in J$, $x, z \in I$, $y \in A$. In particular, $[\delta(x), z]J[\delta(x), z] = 0$ for all $x, z \in I$, which yields $[\delta(x), z] = 0$. Since elements commuting with all elements from an essential ideal of A must lie in the center of A, it follows that $\delta(x) \in Z$, as desired.

Denote by T the set $\{\delta_I \in \mathrm{Der_m}(A) \mid \delta(I) \subseteq Z\}$. Clearly, T is contained in the kernel of φ. For the converse containment, suppose $\bar{\delta}_{\bar{I}} = 0$ for δ_I an element in $\mathrm{Der_m}(A)$. Then there exists an essential ideal $\overline{U}$ of A^-/Z, contained in $\bar{I}$, such that $\bar{\delta}(\overline{U}) = 0$. Consider $V := \pi^{-1}(\overline{U}) \cap I$, for $\pi : A \to A^-/Z$ the canonical projection. The ideal $\overline{V}$ is essential because $\overline{U}$ and $\bar{I}$ are. By Lemma 2.3.4, there is an essential ideal J of A such that $J \subseteq \langle V + Z\rangle \subseteq I + \langle Z\rangle$. For an element x in the essential ideal $I \cap J$ of A, $\bar{\delta}(\bar{x}) \in \bar{\delta}(\overline{U}) = 0$, that is, $\delta(I \cap J) \subseteq Z$. By what was proved in the preceding paragraph it follows that $\delta_I \in T$. □

Lemma 2.3.6. *Let A be a prime noncommutative algebra, I an ideal of A and $\delta : I \to A$ a partial derivation. If $\delta(I) \subseteq Z$ then $\delta = 0$.*

Proof. Suppose that $\delta(I) \subseteq Z$ and let $u \in I$. Then $u^2 \in I$, so $\delta(u^2) \in Z$, that is, $2u\delta(u) \in Z$. Given $x \in A$ we have that $0 = [2u\delta(u), x] = 2\delta(u)[u, x]$ and since A is prime, this implies that $u \in Z$ or $\delta(u) = 0$. Thus, for every $u \in I$ we have either $u \in Z$ or $\delta(u) = 0$. Taking into account Remark 1.1.15 there exists $y \in I \setminus Z$ and so $\delta(y) = 0$. Now take $v \in I$. If $v \notin Z$, then $\delta(v) = 0$, and if $v \in Z$, then $v + y \notin Z$ whence $\delta(v + y) = 0$. Therefore $\delta(v) = 0$ in any case. □

Theorem 2.3.7. *Let A be a prime algebra such that either* $\deg(A) \neq 3$ *or* $\mathrm{char}(A) \neq 3$. *Then* $\mathrm{Der_m}(A) \cong Q_m(A^-/Z)$.

Proof. Consider the map φ in Proposition 2.3.5. Its injectivity is proved by Lemma 2.3.6. Let us prove the surjectivity. Let $\bar{\delta}_{\overline{J}}$ be in $Q_m(A^-/Z)$, with $\overline{J}$ a nonzero ideal of A^-/Z and $\bar{\delta} : \overline{J} \to A^-/Z$ a derivation. Let $\pi : A \to A^-/Z$ be the canonical projection. Note that $\overline{J}$ can be represented as J/Z where $J = \pi^{-1}(\overline{J})$ is a noncentral Lie ideal of A. Define $\delta : J \to A^-/Z$ by $\delta = \bar{\delta}\pi$. It is clear that δ is a derivation in the sense of [14]. We are now in a position to apply [14, Theorem 1.3]. Picking any set-theoretic map $\gamma : J \to A$ such that $\overline{\gamma(x)} = \delta(x)$ for every $x \in J$, it follows that there exists a derivation $d : \langle J\rangle \to \langle J \cup \gamma(J)\rangle\mathcal{C} + \mathcal{C}$, where $\mathcal{C}$ is the extended centroid of A, and a map $\mu : J \to \mathcal{C}$ such that $d(x) = \gamma(x) + \mu(x)$ for all $x \in J$. As above, here $\langle S\rangle$ denotes the subalgebra generated by the set S.

For $x, y \in J$ we have $d([x, y]) = [d(x), y] + [x, d(y)] = [\gamma(x), y] + [x, \gamma(y)]$ since $\mu(J) \subseteq \mathcal{C}$. This shows that $d([J, J]) \subseteq J$, which in turn implies $d(\langle [J, J]\rangle) \subseteq \langle J\rangle \subseteq A$. As $[J, J]$ is a noncentral Lie ideal of A, there exists a nonzero ideal I of A contained in $\langle [J, J]\rangle$ (cf. the first step of the proof of Lemma 2.3.4). Note that d_I is an element of $\mathrm{Der_m}(A)$, and that $\varphi(d_I) = \overline{\delta}_{\overline{J}}$. This concludes the proof. □

As a consequence, we have:

Corollary 2.3.8. *Let A be a prime algebra such that either $\deg(A) \neq 3$ or $\mathrm{char}(A) \neq 3$. If $A = Q_s(A)$, then*

$$Q_m(A^-/Z) \cong \mathrm{Der}(A).$$

Proof. By Lemma 2.3.3 we obtain that $\mathrm{Der}(A) \cong \mathrm{Der_m}(A)$ and applying Theorem 2.3.7 it follows that $\mathrm{Der_m}(A) \cong Q_m(A^-/Z)$, as desired. □

In our final corollary we will extend Corollary 2.3.8 by considering prime algebras A such that $Q_s(A) = AZ^{-1}$, i. e., every element in $Q_s(A)$ is of the form $\frac{a}{\lambda}$, where $a \in A$ and $\lambda \in Z$. However, we have to add the assumption that A is affine which means generated by a finite number of elements.

Corollary 2.3.9. *Let A be an affine prime algebra such that $Q_s(A) = AZ^{-1}$ and either $\deg(A) \neq 3$ or $\mathrm{char}(A) \neq 3$. Then*

$$Q_m(A^-/Z) \cong \mathrm{Der}(Q_s(A)).$$

Proof. Consider the monomorphism $\varphi : \mathrm{Der_m}(A) \to \mathrm{Der}(Q_s(A))$ in the proof of Lemma 2.3.3. In order to check that φ is surjective it is enough to show that given δ in $\mathrm{Der}(Q_s(A))$ there exists a nonzero ideal I of A such that $\delta(I) \subseteq A$. Indeed, if this were true, then we could consider $\delta_I \in \mathrm{Der_m}(A)$ and then applying Lemma 2.3.2 for the case $I \subseteq A \subseteq Q_s(A) = Q_s(I)$ would conclude that $\delta = \varphi(\delta_I)$.

So pick $\delta \in \mathrm{Der}(Q_s(A))$. Let $x_1, \ldots, x_n$ be generators of A. According to our assumption, for each $i = 1, \ldots, n$ we have $\delta(x_i) = \frac{y_i}{\lambda_i}$ for some $y_i \in A$, $\lambda_i \in Z$. Set $\lambda = \prod_{i=1}^n \lambda_i \in Z$. It is clear that $\delta(A) \subseteq \sum_{i=1}^n A\delta(x_i)A$, which in turn implies that $\lambda\delta(A) \subseteq A$. Accordingly, $\delta(\lambda^2 x) = 2\lambda\delta(\lambda)x + \lambda^2\delta(x) \in A$ for every $x \in A$. That is, δ maps the ideal $I = \lambda^2 A \neq 0$ of A into A. □

2.4 The maximal Lie algebra of quotients of K/Z_K

The purpose of the current section is to obtain results on the maximal algebra of quotients of the skew Lie algebra K/Z_K that arises from an associative algebra with involution. Our line of argument benefits from the approach developed in the previous sections, although the proofs do not carry over verbatim.

In particular, we have to take into account whether the involution is of the first kind or of the second kind (see Definition 2.4.1 below). It is also natural to restrict our attention to the Lie algebra $\mathrm{SDer}(A)$ of those derivations that commute with the involution $*$ and to construct a Lie algebra $\mathrm{SDer}_m(A)$ similar to $\mathrm{Der}_m(A)$ as in Section 2.3 (see Construction 2.3.1), considering partial derivations defined on $*$-ideals. The main result is then parallel to Theorem 2.3.7.

Definition 2.4.1. Let A be a semiprime algebra with involution $*$. Then $*$ induces an involution on $\mathcal{C}$, the extended centroid of A. It is said that the involution on A is **of the first kind** if $\mathcal{C} \cap K = 0$; otherwise it is said to be **of the second kind**, that is, $\mathcal{C} \cap K \neq 0$.

The set $\mathrm{SDer}(A) := \{\delta \in \mathrm{Der}(A) \mid \delta(x^*) = \delta(x)^* \text{ for all } x \in A\}$ is a Lie subalgebra of $\mathrm{Der}(A)$. As usual, we will denote by $\mathrm{ad}\,(K)$ the Lie algebra of derivations $\mathrm{ad}\, x : A \to A$ with x in K.

Here, we collect some very useful properties of $\mathrm{SDer}(A)$.

Lemma 2.4.2. *Let A be a semiprime algebra with involution $*$. Then:*

(i) $\mathrm{ad}\,(K) \subseteq \mathrm{Inn}(A) \cap \mathrm{SDer}(A)$.

(ii) $\delta(K) \subseteq K$ *for every* $\delta \in \mathrm{SDer}(A)$.

(iii) $\mathrm{ad}\,(K)$ *is an ideal of* $\mathrm{SDer}(A)$.

Proof. (i). For every $a \in K$ and $x \in A$, $((\mathrm{ad}\, a)x)^* = [a, x]^* = [x^*, a^*] = [a, x^*] = (\mathrm{ad}\, a)(x^*)$. This implies $\mathrm{ad}\, a \in \mathrm{SDer}(A)$.

(ii). Let δ be in $\mathrm{SDer}(A)$. For every $x \in K$, $\delta(x)^* = \delta(x^*) = \delta(-x) = -\delta(x)$. This shows $\delta(K) \subseteq K$.

(iii). For $a \in K$ and $\delta \in \mathrm{SDer}(A)$ we have $[\delta, \mathrm{ad}\, a] = \mathrm{ad}\, \delta(a)$, which, together with condition (ii), implies (iii). □

The following result is a generalization of [16, Lemma 2.9].

Lemma 2.4.3. *Let A be a prime algebra with involution $*$ of the first kind such that* $\deg(A) > 2$. *If $t \in K$ and $[t, K] = 0$, then $t = 0$.*

Proof. By [60, Lemma 2], the subalgebra generated by $[K, K]$ contains a nonzero ideal I of A. For $t \in K$ satisfying $[t, K] = 0$, use induction and the identity $[a, bc] = [a, b]c + b[a, c]$, which holds for all $a, b, c \in A$, to show that $[t, I] = 0$. Now, apply [16, Lemma 2.5] to obtain $t \in K \cap Z = 0$, as desired. □

The next result tell us what is the center Z_K of K. We have to distinguish if the involution $*$ is of the first or second kind.

Lemma 2.4.4. *Let A be a prime algebra with involution $*$. Then:*

(i) *If $*$ is of the second kind, then $Z_K = Z \cap K$ and $\delta(Z_K) \subseteq Z_K$ for every* $\delta \in \mathrm{SDer}(A)$.

(ii) *If $*$ is of the first kind and* $\deg(A) > 2$, *then $Z_K = Z \cap K = 0$.*

Proof. (i) follows taking into account Lemma 2.4.2 (ii) and applying [49, Lemma 2 (ii)] and [16, Theorem 2.13]. To prove (ii) it is enough to apply Lemma 2.4.3. □

Lemma 2.4.5. *Let* A *be a prime algebra with involution* $*$ *with* $\deg(A) > 2$. *Then* $[I \cap K, K] \neq 0$ *for every nonzero* $*$*-ideal* I *of* A*.*

Proof. Consider a nonzero $*$-ideal I of A and suppose $I \cap K \subseteq Z_K = Z \cap K$ by Lemma 2.4.4. From Remark 1.1.15 we have $I \nsubseteq Z$. Hence, there exists $x \in I$ such that $x \notin Z$. By the hypothesis, $[x, I \cap K] \subseteq Z$ and taking into account [59, Theorem 2] if $*$ is of the second kind or [59, Theorem 3] if it is of the first kind we obtain $I \subseteq Z$, which is a contradiction. □

We now turn to the question of having a good description of the Lie algebra $Q_m(K/Z_K)$, in the case of being A prime with an involution. As already mentioned, to this end we shall introduce a new Lie algebra whose definition is based on partial $*$-preserving derivations.

Denote by $\mathcal{I}^*(A)$ the collection of all nonzero $*$-ideals of A and by $\mathrm{PSDer}(I,\, A)$ the set of all partial derivations from I to A which commutes with the involution $*$.

Construction 2.4.6. Let A be a prime associative algebra with involution $*$ over Φ. Consider the set

$$\mathcal{SD} := \{(\delta,\, I) \mid I \in \mathcal{I}^*(A),\, \delta \in \mathrm{PSDer}(I,\, A)\},$$

and define on $\mathcal{SD}$ the following relation: $(\delta,\, I) \equiv (\mu,\, J)$ if and only if δ and μ agree on some nonzero $*$-ideal of A contained in $I \cap J$. One can easily show that $\equiv$ is an equivalence relation.

Denote by $\mathrm{SDer_m}(A)$ the quotient set $\mathcal{SD}/\equiv$ and by δ_I the equivalence class of $(\delta,\, I)$ in $\mathrm{SDer_m}(A)$, for $\delta \in \mathrm{PSDer}(I,\, A)$ and $I \in \mathcal{I}^*(A)$. Then

$\mathrm{SDer}_{\mathrm{m}}(A)$, with the following operations:

$$\delta_I + \mu_J = (\delta + \mu)_{I\cap J}$$

$$\alpha(\delta_I) = (\alpha\delta)_I$$

$$[\delta_I, \mu_J] = (\delta\mu - \mu\delta)_{(I\cap J)^2}$$

for any $\delta_I, \mu_J \in \mathrm{SDer}_{\mathrm{m}}(A)$ and $\alpha \in \Phi$ becomes a Lie algebra over Φ.

The following result is analogous to Lemma 2.3.3. To prove it, it is enough to show that every $\delta \in \mathrm{SDer}(A)$ can be uniquely extended to a derivation δ' in $\mathrm{SDer}(Q_s(A))$. Basically, it follows from the fact that $Q_s(A)$ is an algebra of left quotients of A, coupled with the fact that every derivation on A can be extended uniquely to a derivation of $Q_s(A)$. (See Proposition 1.2.22.)

Lemma 2.4.7. *If A is a prime algebra with involution, then*

$$\mathrm{SDer}(A) \subseteq \mathrm{SDer}_{\mathrm{m}}(A) \subseteq \mathrm{SDer}(Q_s(A)).$$

Our aim now is to construct an isomorphism between the Lie algebra $\mathrm{SDer}_{\mathrm{m}}(A)$ defined in Construction 2.4.6 and the maximal algebra of quotients of the Lie algebra K/Z_K.

Lemma 2.4.8. *Let A be a prime algebra with involution $*$ such that* $\deg(A) > 4$*, and let U be an ideal of K such that $U \nsubseteq Z_K$. Then the algebra $\langle U\rangle$ contains a nonzero $*$-ideal of A.*

Proof. Clearly $\langle U\rangle^* = \langle U\rangle$, and $\langle U\rangle \nsubseteq Z$ since $U \nsubseteq Z_K$. On the other hand, note that $[\langle U\rangle, K] \subseteq \langle U\rangle$. This follows by an induction argument using the identity $[uv, x] = u[v, x] + [u, x]v$, for every $u, v, x \in A$. Next, apply [60, Theorem 2] to obtain the desired conclusion. □

Lemma 2.4.9. *Let A be a prime algebra with involution $*$ such that* $\deg(\mathrm{A}) > 2$*. If δ_I is an element of* $\mathrm{SDer}_{\mathrm{m}}(A)$ *such that $\delta(I \cap K) \subseteq Z_K$ then $\delta_I = 0$.*

Proof. It is well known that $\deg(I) = \deg(A)$ (because A is prime, see, e.g. [15, Theorem 6.4.1]). Therefore [61, Theorem 3] applies to show that, since $[x, \delta(x)] = 0$ for any $y \in K_I$ by our assumption and $\deg(I) > 2$, necessarily $\delta_I = 0$. □

Theorem 2.4.10. *Let A be a prime algebra with involution $*$ such that $\deg(A) > 4$. Then $\mathrm{SDer}_m(A) \cong Q_m(K/Z_K)$.*

Proof. Consider

$$\begin{array}{rccc} \varphi: & \mathrm{SDer}_m(A) & \to & Q_m(K/Z_K) \\ & \delta_I & \mapsto & \bar{\delta}_{\overline{I}} \end{array}$$

where $\overline{I} = ((I \cap K) + Z_K)/Z_K$ and

$$\begin{array}{rccc} \bar{\delta}: & \overline{I} & \to & K/Z_K \\ & \bar{y} & \mapsto & \overline{\delta(y)} \end{array}$$

The map $\bar{\delta}$ is well-defined. To see this, it is enough to check, by Lemma 2.4.5, that $\delta((I \cap K) \cap Z_K) \subseteq Z_K$, whenever I is a nonzero $*$-ideal of A and $\delta \in \mathrm{SDer}(I, A)$. By Lemma 2.4.2, if $y \in I \cap Z_K$ we have $y \in Z$ and arguing as in the proof of Proposition 2.3.5 we obtain $\delta(y) \in Z$. Consequently $\delta(y) \in \delta(K) \cap Z \subseteq Z_K$.

It is easy to see that φ is a well-defined Lie algebra homomorphism. We first prove it is one-to-one. Let δ_I be an element in $\mathrm{SDer}_m(A)$ such that $\bar{\delta}_{\overline{I}} = 0$. Then there exists a nonzero ideal $\overline{J} := J/Z_K$ of K/Z_K contained in $\overline{I}$ such that $\bar{\delta}(\overline{J}) = 0$. Consider $J_1 := \pi^{-1}(\overline{J}) \cap I$, where $\pi : K \to K/Z_K$ is the canonical projection, and note that the ideal $(J_1 + Z_K)/Z_K$ is nonzero because $\overline{J}$ and $\overline{I}$ are nonzero. By Lemma 2.4.8, there is a nonzero $*$-ideal U of A such that $U \subseteq \langle J_1 + Z_K \rangle \subseteq (I \cap K) + \langle Z_K \rangle$. Since $\bar{\delta}(\bar{u}) \in \bar{\delta}(\overline{J}) = 0$ for any element u in $(U \cap I) \cap K$, we see that $\delta((U \cap I) \cap K) \subseteq Z_K$ and, by Lemma 2.4.9, we conclude $\delta_I = 0$.

Now we show that φ is surjective. Let $\bar{\delta}_{\overline{J}}$ be in $Q_m(K/Z_K)$, with $\overline{J}$ a nonzero ideal of K/Z_K and $\bar{\delta} : \overline{J} \to K/Z_K$ a derivation. Note that $\overline{J}$ can be represented as J/Z_K where $J = \pi^{-1}(\overline{J})$ is a noncentral ideal of K. Define $\delta : J \to A^-/Z$ by $\delta = i\bar{\delta}\pi$, where $i : K/Z_K \to A^-/Z$ is given by $i(\bar{x}) = \bar{x} \in A^-/Z$. Since $Z_K = Z \cap K$ (see Lemma 2.4.2) it is straightforward to verify that i is a Lie algebra monomorphism. On the other hand, it is clear that δ is a Lie derivation in the sense of [13] and that K satisfies the conditions in [13, Theorem 3.2]. Therefore, take any set-theoretic map $\gamma : J \to K$ such that $\overline{\gamma(x)} = \delta(x)$ for every $x \in J$ (note that we may actually choose γ with image contained in K because $\delta(J) \subseteq K/Z_K$), and then it follows that there exists a derivation $d : \langle J\rangle \to \langle J \cup \gamma(J)\rangle\mathcal{C} + \mathcal{C}$, where $\mathcal{C}$ is the extended centroid of A, and a map $\mu : J \to \mathcal{C}$ such that $d(x) = \gamma(x) + \mu(x)$ for all $x \in J$.

For $x, y \in J$ we have

$$d([x, y]) = [d(x), y] + [x, d(y)] = [\gamma(x), y] + [x, \gamma(y)]$$

since $\mu(J) \subseteq \mathcal{C}$. This shows that $d([J, J]) \subseteq [K, J] \subseteq J$, which in turn implies $d(\langle[J, J]\rangle) \subseteq \langle J\rangle \subseteq A$. Apply Lemma 2.4.8 to the ideal $[J, J]$ of K (which is not contained in Z_K) to find a nonzero $*$-ideal I of A contained in $\langle[J, J]\rangle$. Note that d_I is an element of $\mathrm{SDer_m}(A)$. Finally, since $\mu(I) \subseteq K \cap \mathcal{C} = K \cap A \cap \mathcal{C} = K \cap Z = Z_K$ (by using [79, Lemma 1.3 (i)] and Lemma 2.4.2) it follows that $\varphi(d_I) = \bar{\delta}_{\overline{J}}$. This concludes the proof. □

Corollary 2.4.11. *Let A be a prime algebra with involution $*$ such that* $\deg(A) > 4$. *If $A = Q_s(A)$, then*

$$Q_m(K/Z_K) \cong \mathrm{SDer}(A).$$

Proof. By Lemma 2.4.7 we obtain that $\mathrm{SDer}(A) \cong \mathrm{SDer_m}(A)$ and applying Theorem 2.4.10 it follows that $\mathrm{SDer_m}(A) \cong Q_m(K/Z_K)$, as desired. □

Chapter 3

Natural questions concerning Lie algebras of quotients

Two of the most important properties of the maximal right algebra of quotients of a semiprime associative algebra A, are the following:

1. The maximal right algebra of quotients of A, coincides with the maximal right algebra of quotients of any essential ideal of A, i.e., $Q^r_{max}(A) = Q^r_{max}(I)$ for every essential ideal I of A. (See e.g. [15, Proposition 2.1.10].)

2. If one compute the maximal right algebra of quotients of the maximal right algebra of quotients of A the obtained result is the maximal right algebra of quotients of A, that is, taking the maximal right algebra of quotients is a closure operation, i.e., $Q^r_{max}(Q^r_{max}(A))) = Q^r_{max}(A)$. (See e.g. [15, Theorem 2.1.1].)

The main target in this chapter will be to determine the conditions under which the analogous results are valid in the context of maximal Lie algebras of quotients introduced in the preceding chapters. As we will see, the answer is not as easy as in the associative case.

3.1 The maximal Lie algebra of quotients of an essential ideal

The purpose of this section is to consider the problem of whether $Q_m(I)$ is isomorphic to $Q_m(L)$, for an essential ideal I of a semiprime Lie algebra L. Of course, this question only makes sense if we assume that I itself is a semiprime algebra, so that $Q_m(I)$ exists at all. Under this assumption we will give a positive answer provided that L satisfies a certain additional condition.

Definition 3.1.1. ([64]). We say that a Lie algebra L is **strongly semiprime** (respectively, **strongly prime**) if:

(i) L is semiprime (respectively, prime).

(ii) For each n, given $0 \neq U_n \triangleleft \ldots \triangleleft U_2 \triangleleft U_1 \triangleleft L$ there exists $0 \neq W \triangleleft L$ such that $W \subseteq U_n$.

We shall use SSP (or SP) as a shorthand for strong semiprimeness (respectively, strong primeness). We will also say that U_n as in the definition above is an n-**subideal**. Of course, 1-subideals are just ideals.

The notion of strongly semiprime (respectively, strongly prime) algebras was introduced by W. S. Martindale III and C. R. Miers in [64] for non-associative algebras; we are interested in Lie algebras. In this context they proved that skew Lie algebras are SSP (SP); specifically:

Example 3.1.2. (See [64, Theorems 6.1 and 6.2]). If A is a semiprime (prime) associative algebra with involution then the Lie algebra K/Z_K is SSP (SP).

Example 3.1.3. If A is a semiprime (prime) associative algebra then A^-/Z is SSP (SP).

Proof. We have already mentioned (see Examples 1.1.10) that $\mathrm{Inn}(A)$ is a semiprime (prime) Lie algebra. The example above jointly with Remark 1.1.9 yield that $A^-/Z \cong \mathrm{Inn}(A)$ is an SSP (SP) Lie algebra. □

The proof of the following lemma is included in the proof of [64, Theorem 6.2].

Lemma 3.1.4. *A Lie algebra L is SSP (SP) if and only if*

(i) *L is semiprime (prime), and*

(ii) *given $0 \neq U_2 \lhd U_1 \lhd L$, there exists $0 \neq W \lhd L$ such that $W \subseteq U_2$.*

Proof. Assume that L satisfies conditions (i) and (ii) in the statement. We proceed by induction on n; for $n = 2$, there is nothing to prove. Suppose the result true for $n \geq 2$ and consider

$$0 \neq U_{n+1} \lhd U_n \lhd \ldots \lhd U_2 \lhd U_1 \lhd L.$$

Define $V_{n+1} = U_{n+1} + [U_{n+1}, U_{n-1}] + [[U_{n+1}, U_{n-1}], U_{n-1}] + \ldots \lhd U_{n-1}$ and note that $U_{n+1} \subseteq V_{n+1} \subseteq U_n$, so, applying the induction hypothesis to the chain $0 \neq V_{n+1} \lhd U_{n-1} \lhd \ldots \lhd U_2 \lhd U_1 \lhd L$ we find a nonzero ideal V of L such that V is contained in V_{n+1}. We claim that $[V_{n+1}, V]$ is a nonzero ideal of V (i.e. a subideal of L) contained in U_{n+1}; otherwise $[V_{n+1}, V] = 0$ would imply, by using the Jacobi identity, $[V_{n+1}, V] = 0$, and hence $[V, V] = 0$, which contradicts (i). Therefore $[V_{n+1}, V] \neq 0$ and applying condition (ii) to the chain $0 \neq [V_{n+1}, V] \lhd V \lhd L$ we obtain a nonzero ideal W of L satisfying that $W \subseteq [V_{n+1}, V] \subseteq U_{n+1}$. This shows that L is SSP (SP).

The converse holds trivially. □

Another characterization of strong semiprimeness (resp. primeness) is the following:

Lemma 3.1.5. ([64, Remark 1.2]). *A Lie algebra L is SSP (SP) if and only if L is semiprime (prime), and for each n, given $0 \neq U_n \lhd \ldots \lhd U_2 \lhd U_1 \lhd L$ there exists an ideal W of L such that W, viewed as an ideal of U_n, is essential.*

Lemma 3.1.6. *Let L be an SSP Lie algebra. Then, for any n-subideal U_n of L there exists an ideal $\widetilde{U}_n$ of L, which is the largest ideal of L contained in U_n. If U_i is essential in U_{i-1}, $i = 2, \ldots, n$, and U_1 is essential in L, then $\widetilde{U}_n$ is an essential ideal of L.*

Proof. The first assertion is obvious: one just defines $\widetilde{U}_n$ as the sum of all ideals of L contained in U_n. Assume now that U_i is essential in U_{i-1} and U_1 is essential in L. This implies that $I \cap U_n \neq 0$ for every nonzero ideal I of L. Suppose that $I \cap \widetilde{U}_n = 0$. Since L is an SSP Lie algebra, $I \cap U_n$ contains a nonzero ideal J of L. By hypothesis, $J \cap \widetilde{U}_n = 0$, and $\widetilde{U}_n + J$ is an ideal of L bigger than $\widetilde{U}_n$ and contained in U_n, which contradicts the maximality of the ideal $\widetilde{U}_n$. □

We have included below a different proof of Lemma 3.1.6 which makes use of Lemma 3.1.5 and [79, Lemma 2.11].

Proof. Define, as above, $\widetilde{U}_n$ as the sum of all ideals of L contained in U_n. Apply Lemma 3.1.5 to find an ideal U of L such that $U \lhd_e U_n$. By the definition of $\widetilde{U}_n$ we have $U \subseteq \widetilde{U}_n$. Since U_n is semiprime (viewed as a Lie algebra, see [64, Remark 1.1]) from Lemma 1.1.13 (ii) it follows $\mathrm{Ann}_{U_n}(U) = 0$, which implies, by using [79, Lemma 2.11], that $\mathrm{Ann}_{U_{n-1}}(U) = 0$. Applying $n-1$ times [79, Lemma 2.11] we obtain $\mathrm{Ann}_L(U) = 0$, that is, U is an essential ideal of L and hence, $\widetilde{U}_n$ is so. □

Theorem 3.1.7. *Let I be an essential ideal of an SSP Lie algebra L. Then $Q_m(I)$ is the maximal algebra of quotients of L, i. e. $Q_m(I) \cong Q_m(L)$.*

Proof. Notice that I viewed as an algebra is SSP (see [64, Remark 2.11]), so we can consider $Q_m(I)$. Define

$$\begin{array}{cccc} \varphi: & Q_m(L) & \to & Q_m(I) \\ & \delta_J & \mapsto & \delta_{(J\cap I)^2} \end{array}$$

The map φ is well-defined: Since $\mathrm{Ann}_I(J\cap I) \subseteq \mathrm{Ann}_L(J\cap I) = 0$, we have that $J\cap I$ is an essential ideal of I. Hence $(J\cap I)^2$ is also an essential ideal of I. Finally, note that δ maps $(J\cap I)^2$ into I.

It is straightforward to verify that φ is a Lie algebra monomorphism. To see the surjectivity take $\gamma_{I'} \in Q_m(I)$, with I' an essential ideal of I. By Lemma 3.1.6 there exists an essential ideal J of L contained in I'. Then, for $\gamma_J \in Q_m(L)$ we have $\varphi(\gamma_J) = \gamma_{(I\cap J)^2} = \gamma_{I'}$ and the proof is complete. □

Remark 3.1.8. Let A be a semiprime algebra. For every Lie ideal I of A, $Z_I = I\cap Z$ since $[y, I] = 0$, with $y \in I$, implies $y \in Z$; indeed, this follows from [42, Sublemma, p. 5] which states that an element y in a semiprime algebra satisfying that $[y, [y, x]] = 0$ for all element x in the algebra must lie in its center. Moreover, we have the following isomorphism

$$I/Z_I = I/(I\cap Z) \cong (I + Z)/Z.$$

Corollary 3.1.9. *Let A be a semiprime algebra. Then:*

$$Q_m([A,\, A]/Z_{[A,\, A]}) \cong Q_m(A^-/Z).$$

Proof. Applying Remark 3.1.8 we have $Z_{[A,\, A]} = [A,\, A]\cap Z$ from which it immediately follows that the map determined by $[x,\, y] + Z_{[A,\, A]} \mapsto [x,\, y] + Z$ is a well-defined Lie algebra monomorphism from $[A,\, A]/Z_{[A,\, A]}$ into A^-/Z. Identifying $[A,\, A]/Z_{[A,\, A]}$ with its image, we can regard it as an ideal of A^-/Z. We will prove now that $[A,\, A]/Z_{[A,\, A]}$ is essential in A^-/Z. To this end, given $a \in A \setminus Z$ it is enough to show that $[a,\, A] \not\subseteq Z_{[A,\, A]}$ (see Lemma 1.1.13 (ii)).

Since $a \notin Z$ by [42, Sublemma, p. 5] it follows that $[a, [a, A]] \neq 0$; this means that $[a, A] \nsubseteq Z$ which implies that $[a, A] \nsubseteq Z_{[A, A]}$, as desired. Keeping in mind that A^-/Z is an SSP Lie algebra, the conclusion follows directly from Theorem 3.1.7. □

Corollary 3.1.10. *Let A be a prime algebra. If* $\mathrm{Der}(A)$ *is SP then*

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)).$$

Proof. The result follows from Theorem 3.1.7. □

The previous result needs the assumption that $\mathrm{Der}(A)$ is strongly prime. It does not seem clear how to verify whether this condition is fulfilled. In what follows we give a criterion based only on the ideal lattice of A. An useful tool to obtain it is the fact of $\mathrm{Inn}(A)$ is strongly prime. To make use of it we pause to reduce the study of the strong primeness of $\mathrm{Der}(A)$ to the case

$$0 \neq \mathrm{ad}\,([I, A]) \lhd \mathrm{Inn}(A) \lhd \mathrm{Der}(A),$$

where I is an ideal of A.

Lemma 3.1.11. *Let A be a semiprime algebra and let $\widetilde{I}$ be a nonzero ideal of* $\mathrm{Inn}(A)$. *Then there exists an ideal U of A such that* $0 \neq \mathrm{ad}\,([U, A]) \subseteq \widetilde{I}$.

Proof. It is easy to see that $I = \{x \in A \mid \mathrm{ad}\, x \in \widetilde{I}\,\}$ is a noncentral Lie ideal of A (use [42, Sublemma, p. 5]). Apply [43, Theorem 5] to find a nonzero ideal U of A satisfying $0 \neq [U, A] \subseteq I$, that is, U is an ideal of A such that $0 \neq \mathrm{ad}\,([U, A]) \subseteq \widetilde{I}$ and the lemma is proved. □

Lemma 3.1.12. *Let A be a prime algebra. Assume that for every nonzero ideal U of A there exists a nonzero ideal $\widetilde{U}$ of* $\mathrm{Der}(A)$ *such that* $\widetilde{U} \subseteq \mathrm{ad}\,([U, A])$. *Then* $\mathrm{Der}(A)$ *is an SP Lie algebra.*

Proof. Let $0 \neq \widetilde{I} \lhd \widetilde{J} \lhd \mathrm{Der}(A)$. Apply [79, Lemma 2.13] to obtain that $\mathrm{Der}(A)$ is an algebra of quotients of $\widetilde{J} \cap \mathrm{Inn}(A)$. Hence, given $0 \neq \delta \in \widetilde{I} \subseteq \mathrm{Der}(A)$ there exists $x \in A$ satisfying $0 \neq \mathrm{ad}\, x \in \widetilde{J} \cap \mathrm{Inn}(A)$ and $[\delta, \mathrm{ad}\, x] \neq 0$. Since $\widetilde{I}$ is an ideal of $\widetilde{J}$, $[\delta, \mathrm{ad}\, x] \in \widetilde{I}$ and $[\delta, \mathrm{ad}\, x] = \mathrm{ad}\, \delta(x) \in \mathrm{Inn}(A)$, therefore $\widetilde{I} \cap \mathrm{Inn}(A) \neq 0$. Consider $0 \neq \widetilde{I} \cap \mathrm{Inn}(A) \lhd \widetilde{J} \cap \mathrm{Inn}(A) \lhd \mathrm{Inn}(A)$. Since $\mathrm{Inn}(A)$ is an SP Lie algebra, there exists a nonzero ideal $\widetilde{K}$ of $\mathrm{Inn}(A)$ contained in $\widetilde{I} \cap \mathrm{Inn}(A)$. Apply Lemma 3.1.11 to find a nonzero ideal U of A such that $0 \neq \mathrm{ad}\,([U, A]) \subseteq \widetilde{K}$. Now, by the hypothesis there exists a nonzero ideal $\widetilde{U}$ of $\mathrm{Der}(A)$ satisfying $\widetilde{U} \subseteq \mathrm{ad}\,([U, A]) \subseteq \widetilde{I}$, as desired. □

With the lemma above in hand, the proof of the announced criterion is now very easy.

Theorem 3.1.13. *Let A be a prime algebra. Then the following conditions are equivalent:*

(i) $\mathrm{Der}(A)$ *is SP.*

(ii) *Every nonzero ideal of A contains a nonzero ideal of A invariant under every element of* $\mathrm{Der}(A)$.

Moreover, if these conditions hold, then

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)).$$

Proof. Identify A^-/Z with $\mathrm{Inn}(A)$.

(i) $\Rightarrow$ (ii). Let I be a nonzero ideal of A. By Remark 1.1.15, $\mathrm{ad}\,(I)$ is a nonzero ideal of $\mathrm{Inn}(A)$. Consider

$$0 \neq \mathrm{ad}\,(I) \lhd \mathrm{Inn}(A) \lhd \mathrm{Der}(A);$$

by the hypothesis there exists $0 \neq \widetilde{I} \lhd \mathrm{Der}(A)$ contained in $\mathrm{ad}\,(I)$. It is clear that $J := \sum_{\delta \in \widetilde{I}} A\delta(A)A$ is a nonzero ideal of A. Moreover, J is indeed invariant under every element of $\mathrm{Der}(A)$. In fact, for $x, y, z \in A$, $\delta \in \widetilde{I}$, $\mu \in \mathrm{Der}(A)$ we have

$$\begin{aligned}\mu(x\delta(y)z) &= \mu(x)\delta(y)z + x\mu\delta(y)z + x\delta(y)\mu(z) \\ &= \mu(x)\delta(y)z + x[\mu,\delta](y)z + x\delta\mu(y)z + x\delta(y)\mu(z) \in J\end{aligned}$$

since $\delta, [\mu, \delta] \in \widetilde{I}$. This shows that $\mu(J) \subseteq J$ for every $\mu \in \mathrm{Der}(A)$. Finally, taking into account that $\widetilde{I} \subseteq \mathrm{ad}\,(I)$ we have $J \subseteq A\widetilde{I}(A)A \subseteq A[I, A]A \subseteq I$.

(ii) $\Rightarrow$ (i). To prove the strong primeness of $\mathrm{Der}(A)$ we will use Lemma 3.1.12. Let us therefore consider

$$0 \neq \mathrm{ad}\,([U, A]) \lhd \mathrm{Inn}(A) \lhd \mathrm{Der}(A),$$

for U an ideal of A. By the hypothesis, there exists a nonzero ideal J of A, which is contained in U and is invariant under every element of $\mathrm{Der}(A)$. Since $\mathrm{ad}\,([J, A])$ is contained in $\mathrm{ad}\,([U, A])$, the proof will be complete by showing that $\mathrm{ad}\,([J, A])$ is a nonzero ideal of $\mathrm{Der}(A)$. It is straightforward to verify that $\mathrm{ad}\,([J, A])$ is an ideal of $\mathrm{Der}(A)$. The containment $\mathrm{ad}\,([J, A]) \subseteq \mathrm{ad}\,([U, A])$ is obvious. The ideal $[J, A]$ is noncentral; otherwise, apply that Z is a prime ideal of A^- (see [49, Lemma 4]) to obtain $J \subseteq Z$, which is impossible by Remark 1.1.15. Thus, $[J, A] \nsubseteq Z$ and therefore $\mathrm{ad}\,([J, A]) \neq 0$.

The last assertion follows directly from Corollary 3.1.10. □

Example 3.1.14. If A is a prime algebra such that every nonzero ideal I of A contains a nonzero idempotent ideal J, then $\mathrm{Der}(A)$ is SP. This follows from Theorem 3.1.13 together with the fact that $J = J^2$ implies $\delta(J) = \delta(J^2) \subseteq J$ for every $\delta \in \mathrm{Der}(A)$. In particular, this holds if A is a prime von Neumann regular algebra or, more generally, if A is an exchange algebra with zero Jacobson radical. (In case of rings with unit, the definition of exhange rings can be found in [34]; the notion of exchange rings for rings without unit was introduced by P. Ara in [6].)

Corollary 3.1.15. *Let A be a simple algebra such that either* $\deg(A) \neq 3$ *or* $\mathrm{char}(A) \neq 3$. *Then*

$$Q_m(A^-/Z) \cong Q_m(\mathrm{Der}(A)) \cong \mathrm{Der}(A).$$

Proof. Apply Theorem 3.1.13 to show that $Q_m(\mathrm{Der}(A)) \cong Q_m(A^-/Z)$ and Corollary 2.3.8 to have that $Q_m(A^-/Z) \cong \mathrm{Der}(A)$, which completes the proof. □

The second part of this section is devoted to the study of similar questions when the associative algebra A has an involution.

Remark 3.1.16. Let A be a prime algebra with involution $*$. The map

$$\begin{array}{ccc} K & \to & \mathrm{ad}\,(K) \\ x & \mapsto & \mathrm{ad}\,x \end{array}$$

is a Lie algebra epimorphism with kernel Z_K; this allows to identify K/Z_K with the ideal $\mathrm{ad}\,(K)$ of $\mathrm{SDer}(A)$. If the involution is of the first kind and $\deg(A) > 2$, it is in fact an isomorphism, by Lemma 2.4.4 (ii).

On the other hand, for every ideal I of K, the restriction of the map above to I, that is,

$$\begin{array}{ccc} I & \to & \mathrm{ad}\,(K) \\ y & \mapsto & \mathrm{ad}\,y \end{array}$$

is a Lie algebra homomorphism with kernel $Z_I = I \cap Z_K$, if the involution is of the second kind, or zero, if it is of the first kind and $\deg(A) > 2$. Indeed, $[y, I] = 0$, with $y \in I$, implies $[y, [y, K]] = 0$. Then apply [16, Theorem 2.13] or Lemma 2.4.3 to have $y \in Z_K$ or $y = 0$. Moreover,

$$I/Z_I = I/(I \cap Z_K) \cong (I + Z_K)/Z_K \lhd K/Z_K.$$

Corollary 3.1.17. *Let A be a prime algebra with involution. Then:*

$$Q_m([K,\, K]/Z_{[K,\,K]}) \cong Q_m(K/Z_K).$$

Proof. Applying Remark 3.1.16 we can regard $[K,\, K]/Z_{[K,\, K]}$ as an ideal of K/Z_K. Keeping in mind that K/Z_K is an SP Lie algebra, the conclusion follows directly from Theorem 3.1.7. □

The following result is a straightforward corollary to Theorem 3.1.7.

Corollary 3.1.18. *Let A be a prime algebra with involution. If* $\mathrm{SDer}(A)$ *is SP then*

$$Q_m(K/Z_K) \cong Q_m(\mathrm{SDer}(A)).$$

Let A be a prime algebra with involution. From [48, Theorem 2] we obtain that the Lie algebra $\mathrm{SDer}(A)$ is prime. The question of whether $\mathrm{SDer}(A)$ is SP is again more delicate and is related to the ideal structure of A. Reasoning as above, the first step is to reduce the study of the strong primeness of $\mathrm{SDer}(A)$ to a particular case of chains:

$$0 \neq \mathrm{ad}\,([U \cap K, K]) \lhd \mathrm{ad}\,(K) \lhd \mathrm{SDer}(A),$$

where U is a nonzero $*$-ideal of A.

Lemma 3.1.19. *Let A be a prime algebra with involution $*$ with* $\deg(A) > 4$. *Then, for every nonzero ideal $\widetilde{I}$ of* $\mathrm{ad}\,(K)$ *there exists a $*$-ideal U of A such that* $0 \neq \mathrm{ad}\,([U \cap K, K]) \subseteq \widetilde{I}$.

Proof. The set $I := \{x \in K \mid \mathrm{ad}\, x \in \widetilde{I}\,\}$ is an ideal of K not contained in Z_K and, therefore, it is not contained in Z. Apply [36, Theorem 1] if $*$ is of the second kind or [36, Theorem 5 and Lemma 7] if it is of the first kind to find a nonzero $*$-ideal U of A satisfying $[U \cap K, K] \subseteq I$, that is, $\mathrm{ad}\,([U \cap K, K]) \subseteq \widetilde{I}$. Note that $[U \cap K, K] \nsubseteq Z_K$ as otherwise, $U \cap K \subseteq Z_K$, which contradicts Lemma 2.4.5. □

Lemma 3.1.20. *Let A be a prime algebra with involution $*$ and such that* $\deg(A) > 4$. *Assume that for every $*$-ideal U of A there exists a nonzero* *ideal* $\widetilde{U}$ *of* $\mathrm{SDer}(A)$ *such that* $\widetilde{U} \subseteq \mathrm{ad}\,[U \cap K, K]$. *Then* $\mathrm{SDer}(A)$ *is an SP Lie algebra.*

Proof. Let $0 \neq \widetilde{I} \lhd \widetilde{J} \lhd \mathrm{SDer}(A)$ by [79, Lemma 2.13] we obtain that $\mathrm{SDer}(A)$ is an algebra of quotients of $\widetilde{J} \cap \mathrm{ad}\,(K)$. Hence, given $0 \neq \delta \in \widetilde{I} \subseteq \mathrm{SDer}(A)$ there exists $x \in K$ satisfying $0 \neq \mathrm{ad}\, x \in \widetilde{J} \cap \mathrm{ad}\,(K)$ and $[\delta, \mathrm{ad}\, x] \neq 0$. Since $\widetilde{I}$ is an ideal of $\widetilde{J}$, $[\delta, \mathrm{ad}\, x] \in \widetilde{I}$ and $[\delta, \mathrm{ad}\, x] = \mathrm{ad}\ \delta(x) \in \mathrm{ad}\,(K)$, therefore $\widetilde{I} \cap \mathrm{ad}\,(K) \neq 0$. Consider $0 \neq \widetilde{I} \cap \mathrm{ad}\,(K) \lhd \widetilde{J} \cap \mathrm{ad}\,(K) \lhd \mathrm{ad}\,(K)$. Since $\mathrm{ad}\,(K)$ is an SP Lie algebra, there exists a nonzero ideal $\widetilde{V}$ of $\mathrm{ad}\,(K)$ contained in $\widetilde{I} \cap \mathrm{ad}\,(K)$. Apply Lemma 3.1.19 to find a nonzero $*$-ideal U of A such that $0 \neq \mathrm{ad}\,([U \cap K, K]) \subseteq \widetilde{V}$. Now, by the hypothesis there exists a nonzero ideal $\widetilde{U}$ of $\mathrm{SDer}(A)$ satisfying $\widetilde{U} \subseteq \mathrm{ad}\,([U \cap K, K]) \subseteq \widetilde{I}$, as desired. □

Theorem 3.1.21. *Let A be a prime algebra with involution $*$ and such that* $\deg(A) > 4$. *Then the following conditions are equivalent:*

(i) $\mathrm{SDer}(A)$ *is SP.*

(ii) *Every nonzero $*$-ideal of A contains a nonzero $*$-ideal of A invariant under every element of* $\mathrm{SDer}(A)$.

Moreover, if the previous conditions are satisfied we have

$$Q_m(K/Z_K) \cong Q_m(\mathrm{SDer}(A)).$$

Proof. Taking into account Remark 3.1.16, the skew Lie algebra K^-/Z_K can be identified with $\mathrm{ad}\,(K)$.

(i) $\Rightarrow$ (ii). Let I be a nonzero $*$-ideal of A. By Lemma 2.4.5, $\mathrm{ad}\,(I \cap K)$ is a nonzero ideal of $\mathrm{ad}\,(K)$. Apply the hypothesis to the chain

$$0 \neq \mathrm{ad}\,(I \cap K) \lhd \mathrm{ad}\,(K) \lhd \mathrm{SDer}(A)$$

in order to find $0 \neq \widetilde{I} \lhd \mathrm{SDer}(A)$ contained in $\mathrm{ad}\,(I \cap K)$. It is straightforward to check that $J := \sum_{\delta \in \widetilde{I}} A\delta(A)A$ is a nonzero $*$-ideal of $\mathrm{SDer}(A)$. Moreover, J is in fact invariant under every element of $\mathrm{SDer}(A)$. To prove it take $x, y, z \in A$, $\delta \in \widetilde{I}$, $\mu \in \mathrm{SDer}(A)$ and compute

$$\begin{aligned} \mu(x\delta(y)z) &= \mu(x)\delta(y)z + x\mu\delta(y)z + x\delta(y)\mu(z) \\ &= \mu(x)\delta(y)z + x[\mu, \delta](y)z + x\delta\mu(y)z + x\delta(y)\mu(z) \in J \end{aligned}$$

since $\delta, [\mu, \delta] \in \widetilde{I}$. This shows that $\mu(J) \subseteq J$ for every $\mu \in \mathrm{SDer}(A)$. Finally, taking into account that $\widetilde{I} \subseteq \mathrm{ad}\,(I \cap K)$ we have $J \subseteq A\widetilde{I}(A)A \subseteq A[I, A]A \subseteq I$.

(ii) $\Rightarrow$ (i). To prove the strong primeness of $\mathrm{SDer}(A)$ we will use Lemma 3.1.20. Let us therefore consider

$$0 \neq \mathrm{ad}\,([U \cap K,\, K]) \lhd \mathrm{ad}\,(K) \lhd \mathrm{SDer}(A),$$

for U an $*$-ideal of A. By the hypothesis, there exists a nonzero $*$-ideal J of A, which is contained in U and is invariant under every element of $\mathrm{SDer}(A)$. Since $\mathrm{ad}\,([J \cap K,\, K])$ is contained in $\mathrm{ad}\,([U \cap K,\, K])$, the proof will be complete by showing that $\mathrm{ad}\,([J \cap K,\, K])$ is a nonzero ideal of $\mathrm{SDer}(A)$. It is straightforward to verify that $\mathrm{ad}\,([J \cap K,\, K])$ is an ideal of $\mathrm{SDer}(A)$. On the other hand, $[J \cap K,\, K] \nsubseteq Z$; otherwise, apply Lemma 2.4.4 and the fact of Z_K is a prime ideal of K to obtain that $J \cap K \subseteq Z_K$, which is impossible by Lemma 2.4.5. Thus $[J \cap K, K] \nsubseteq Z$ and therefore $\mathrm{ad}\,([J \cap K, K]) \neq 0$.

The last assertion follows directly from Corollary 3.1.10. □

As consequences we have:

Corollary 3.1.22. *Let A be a prime algebra with involution $*$ such that $\deg(A) > 4$. If A is a $*$-simple algebra, then:*

$$Q_m(K/Z_K) \cong Q_m(\mathrm{SDer}(A)).$$

Corollary 3.1.23. *Let A be a simple algebra with involution such that* $\deg(A) > 4$. *Then:*

$$Q_m(K/Z_K) \cong Q_m(\mathrm{SDer}(A)) \cong \mathrm{SDer}(A).$$

Proof. Apply Corollary 3.1.22 to obtain that $Q_m(\mathrm{SDer}(A)) \cong Q_m(K/Z_K)$. Corollary 2.4.11 implies $Q_m(K/Z_K) \cong \mathrm{SDer}(A)$. □

3.2 The maximal graded algebra of quotients of a graded essential ideal

Once we have built the maximal graded algebra of quotients, and we have study what happens in the non-graded case, it is natural to ask whether $Q_{gr-m}(I)$ will be isomorphic to $Q_{gr-m}(L)$, for a graded essential ideal I of a graded semiprime Lie algebra L. Of course, as in the non-graded setting this question only makes sense if we assume that I itself is a graded semiprime Lie algebra, so that $Q_{gr-m}(I)$ exists at all.

Let us start by introducing the main ingredient in Section 3.2, that is, the notion of graded strongly semiprimeness (primeness) for graded Lie algebras. In order to ease the notation, if L is a graded Lie algebra, we shall write $I \lhd_{gr} L$ to denote that I is a graded ideal of L.

Definition 3.2.1. We say that a graded Lie algebra L is **graded strongly semiprime** (**graded strongly prime**) if:

(i) L is graded semiprime (graded prime).

(ii) For each n, given $0 \neq U_n \lhd_{gr} \dots \lhd_{gr} U_2 \lhd_{gr} U_1 \lhd_{gr} L$ there exists $0 \neq W \lhd_{gr} L$ such that $W \subseteq U_n$.

We shall use graded SSP (or graded SP) as a shorthand for graded strong semiprimeness (primeness). We will also say that U_n as in the definition above is an n-**graded subideal**. Of course, 1-graded subideals are just graded ideals.

The proof of the following result is analogous to the non-graded one which is Lemma 3.1.4.

Lemma 3.2.2. *A Lie algebra L is graded SSP (graded SP) if and only if*

(i) *L is graded semiprime (graded prime), and*

(ii) *given $0 \neq U_2 \lhd_{gr} U_1 \lhd_{gr} L$, there exists $0 \neq W \lhd_{gr} L$ such that $W \subseteq U_2$.*

Lemma 3.2.3. *Let L be a graded SSP Lie algebra. Then, for any n-graded subideal U_n of L there exists a graded ideal $\widetilde{U}_n$ of L, which is the largest graded ideal $\widetilde{U}_n$ of L contained in U_n. If U_i is graded essential in U_{i-1}, $i = 2, \ldots, n$, and U_1 is graded essential in L, then $\widetilde{U}_n$ is a graded essential ideal of L.*

The proof of the previous lemma is practically the same of Lemma 3.1.6 but considering now $\widetilde{U}_n$ as the sum of all graded ideals of L contained in U_n.

Theorem 3.2.4. *Let I be a graded essential ideal of a graded SSP Lie algebra L. Then $Q_{gr-m}(I)$ is the maximal graded algebra of quotients of L, i. e. $Q_{gr-m}(I) \cong Q_{gr-m}(L)$.*

Proof. Notice that I viewed as a graded Lie algebra is graded SSP (see [64, Remark 2.11]), so we can consider $Q_{gr-m}(I)$. One can show as in the proof of Theorem 3.1.7 (using now Lemma 3.2.3) that the map $\varphi : Q_m(L) \to Q_m(I)$ defined by

$$\varphi(\delta_J) = \varphi\left(\sum_\sigma (\delta_\sigma)_J\right) = \sum_\sigma (\delta_\sigma)_{(J\cap I)^2} = \delta_{(J\cap I)^2}$$

is a graded Lie isomorphism. □

3.3 Max-closed algebras

This final section is devoted to the problem of whether taking the maximal algebra of quotients is a closure operation, that is, if $Q_m(Q_m(L)) = Q_m(L)$

holds for every semiprime Lie algebra L. Notice that this question makes sense since $Q_m(L)$ is also semiprime ([79, Proposition 2.7 (ii)]). Although in some interesting special cases the answer is positive, we will prove that the containment $Q_m(L) \subseteq Q_m(Q_m(L))$ can be strict.

Definition 3.3.1. We say that a semiprime Lie algebra L is **max-closed** if $Q_m(Q_m(L)) = Q_m(L)$.

In the next results we introduce various examples of max-closed Lie algebras.

Corollary 3.3.2. *Let A be a simple algebra such that either* $\deg(A) \neq 3$ *or* $\operatorname{char}(A) \neq 3$. *Then A^-/Z is max-closed.*

Proof. By Corollary 3.1.15 we have $Q_m(A^-/Z) \cong Q_m(\operatorname{Der}(A)) \cong \operatorname{Der}(A)$; hence taking $Q_m(\,.\,)$ into the isomorphisms above, we obtain

$$Q_m(Q_m(A^-/Z)) \cong Q_m(\operatorname{Der}(A)) \cong Q_m(A^-/Z),$$

which proves that A^-/Z is max-closed. □

Corollary 3.3.3. *Let A be a simple algebra with involution $*$ such that* $\deg(A) > 4$. *Then K/Z_K is max-closed.*

Proof. The following isomorphisms $Q_m(K/Z_K) \cong Q_m(\operatorname{SDer}(A)) \cong \operatorname{SDer}(A)$ hold by Corollary 3.1.23. Take $Q_m(\,.\,)$ to obtain

$$Q_m(K/Z_K) \cong Q_m(\operatorname{SDer}(A)) \cong Q_m(K/Z_K).$$

which concludes the proof. □

Theorem 3.3.4. *If L is a simple Lie algebra, then $Q_m(L) \cong \operatorname{Der}(L)$ is an SP Lie algebra and L is max-closed.*

Proof. In view of the simplicity of L we clearly have $Q_m(L) \cong \mathrm{Der}(L)$. Moreover, these two Lie algebras are prime by [79, Proposition 2.7 (ii)].

We claim that L is isomorphic to the smallest nonzero ideal of $\mathrm{Der}(L)$. Indeed, since $Z_L = 0$ we have $L \cong \mathrm{ad}(L) \lhd \mathrm{Der}(L)$. Identify L with $\mathrm{ad}(L)$ and consider $0 \neq \widetilde{U} \lhd \mathrm{Der}(L)$. Taking into account the simplicity of L and that $0 \neq \widetilde{U} \cap L \lhd L$ we obtain $\widetilde{U} \cap L = L$, which implies $L \subseteq \widetilde{U}$.

For $0 \neq \widetilde{I} \lhd \widetilde{J} \lhd \mathrm{Der}(L)$ apply what we have proved to obtain $L \subseteq \widetilde{J}$. We claim that $U = \widetilde{I} \cap L$ is a nonzero ideal of L. In fact, $[U, L] \subseteq L$ and $[U, L] \subseteq [U, \widetilde{J}] \subseteq \widetilde{I}$, which implies $[U, L] \subseteq \widetilde{I} \cap L = U$. To show that $U \neq 0$, consider $0 \neq \delta \in \widetilde{I}$. Since $Z_L = 0$ there exists $x \in L$ such that $0 \neq \mathrm{ad}\,\delta(x) \in L$. Moreover, $\mathrm{ad}\,\delta(x) = [\delta, \mathrm{ad}\,x] \in \widetilde{I}$; hence, $0 \neq \mathrm{ad}\,\delta(x) \in U$. Thus, U is a nonzero ideal of a simple Lie algebra L, so that $L = U \subseteq \widetilde{I}$. From Lemma 3.1.4 we now see that $\mathrm{Der}(L)$ is an SP Lie algebra.

It remains to show that L is max-closed. We have $Q_m(Q_m(L)) \cong Q_m(\mathrm{Der}(L))$. Since L is a nonzero ideal of an SP Lie algebra $\mathrm{Der}(L)$, it follows from Theorem 3.1.7 that $Q_m(L) \cong Q_m(\mathrm{Der}(L))$. □

The following example was the motivation for Example 3.3.7.

Proposition 3.3.5. *Let F be a field and $A = M_2(F[x])$. Then the Lie algebra $L = A^-/Z$ is max-closed.*

Proof. We first observe that $Q_s(Q_s(A)) \cong Q_s(A) \cong M_2(F(x))$ (see, e.g. [80, p. 61 exercise 9 (i)]). Secondly, we claim that $\mathrm{Der_m}(A) \cong \mathrm{Der}(Q_s(A))$; the map $\varphi : \mathrm{Der_m}(A) \to \mathrm{Der}(Q_s(A))$ which sends $\delta_I \in \mathrm{Der_m}(A)$ into the unique extension $\delta' \in \mathrm{Der}(Q_s(A))$ of the partial derivation $\delta : I \to A$, is a well-defined Lie algebra monomorphism. (See the proof of Lemma 2.3.3.) Moreover, in our particular case, φ is in indeed an isomorphism. To prove that, given $\delta \in \mathrm{Der}(Q_s(A))$ it is enough to find a nonzero ideal I of A such that $\delta(I) \subseteq A$.

In fact, if we would have showed that we could consider $\delta_I \in \mathrm{Der_m}(A)$, then, applying Lemma 2.3.2 to $I \subseteq A \subseteq Q_s(A) = Q_s(I)$ we would obtain $\delta = \varphi(\delta_I)$, which would complete the proof. In order to check this, let $\delta \in \mathrm{Der}(Q_s(A))$; taking into account the form of the associative derivations of $Q_s(A)$, we obtain that $\delta(a) = (\mathrm{ad}\, u)a + \frac{p}{q}\, a'$ where $u \in Q_s(A)$, $p \in F[x]$, $\frac{1}{q} \in F(x)$ and a' is the matrix whose elements are the derivative of the elements of $a \in A$.

Denote by J the ideal of $F[x]$ generated by $q^2 \left(\prod_{i,\, j=1,2} v_{ij}\right)$, where $u = \left(\frac{u_{ij}}{v_{ij}}\right)$, and set the ideal $I := M_2(J)$ of $Q_s(A)$. We now claim that $\delta(I) \subseteq A$. In fact, let $a = (a_{ij})$ be in I; we have:

$$\delta(a) = [u, a] + \frac{p}{q}\, a' = (ua - au) + \frac{p}{q}\, \left(a'_{ij}\right).$$

Note that $ua \in A$ because $ua = \left(\sum_k \frac{u_{ik}}{v_{ik}}\, a_{kj}\right)$ and by the definition of J the elements $\frac{a_{kj}}{v_{ik}}$ are in A. Analogously, we see that $au \in A$ and whence $[u, a] \in A$. To check that $\frac{p}{q}\, a' \in A$, write the elements of a as a product $a_{ij} = q^2 f_{ij}(x)$ where $f_{ij}(x) \in F[x]$, and compute:

$$\frac{p}{q}\, a'_{ij} = \frac{p}{q}\left(2qq' f_{ij}(x) + q^2 f'_{ij}(x)\right) \in F[x].$$

Hence $\frac{p}{q}\, a' \in A$ and therefore $\delta(a) \in I$, as wanted.

To resume, we have just proved that $\mathrm{Der_m}(A) \cong \mathrm{Der}(Q_s(A))$; then by Theorem 2.3.5 we have $Q_m(L) \cong \mathrm{Der}(Q_s(A))$. On the other hand the simplicity of $Q_s(A)$ jointly Corollaries 3.1.15 and 3.3.2 allow us to say that $L_s = Q_s(A)^-/Z_{Q_s(A)}$ is max-closed and it satisfies $Q_m(L_s) \cong \mathrm{Der}(Q_s(A))$. Hence:

$$Q_m(Q_m(L)) \cong Q_m(\mathrm{Der}(Q_s(A)) \cong Q_m(Q_m(L_s)) \cong Q_m(L_s) \cong$$
$$\mathrm{Der}(Q_s(A)) \cong Q_m(L),$$

which concludes the proof. □

Another class of algebras that provides examples of max-closed algebras is that of prime affine PI algebra:

Theorem 3.3.6. *Let A be a prime affine PI algebra such that either* $\deg(A) \neq 3$ *or* $\operatorname{char}(A) \neq 3$, *and let J be a noncentral Lie ideal of A. Then the Lie algebra $J/(J \cap Z)$ is max-closed.*

Proof. Recall that $A^-/Z \cong \operatorname{Inn}(A)$ is an SP Lie algebra (see Example 3.1.3). Accordingly, applying Theorems 2.3.7 and 3.1.7 it follows that $Q_m(J/(J \cap Z)) \cong Q_m(A^-/Z) \cong \operatorname{Der_m}(A)$. It is well-known that A, as a prime PI algebra, satisfies $Q_s(A) = AZ^{-1}$, and moreover, that $Q_s(A)$ is a simple algebra (see e.g. [77, Theorem 1.7.9] or [44, Theorem 1.4.3] from which this can be easily derived). Therefore we infer from Corollary 2.3.9 that $\operatorname{Der_m}(A) \cong \operatorname{Der}(Q_s(A))$. On the other hand, Corollary 3.1.15 shows that $Q_m(\operatorname{Der}(Q_s(A))) \cong \operatorname{Der}(Q_s(A))$, and the proof is thereby complete. □

We will finish this chapter by finding an example of a Lie algebra which is not max-closed. The algebra A we shall deal with is the one that Passman used in [73] to show that $Q_s(\,.\,)$ is not a closure operation.

Example 3.3.7. (See [73, Lemma 4.1 (ii), Theorem 4.4 and Proposition 4.5].) Let F be a field and let

$$A = F[\,t\,][x, y \mid xy = tyx].$$

Then we have:

(i) A is a domain with center $Z = F[\,t\,]$;

(ii) $Q_s(A) = F(t)[x, y \mid xy = tyx]$;

(iii) $Q_s(Q_s(A)) = F(t)[x^{-1}, x, y^{-1}, y \mid xy = tyx]$.

We shall make use of (iii) in the proof below, but not in an explicit way.

Theorem 3.3.8. *Let $A = F[t][x, y \mid xy = tyx]$. Then the Lie algebra A^-/Z is not max-closed.*

Proof. We shall write Q for $Q_s(A)$. Note that the conditions of Corollary 2.3.9 are again fulfilled. Therefore, this corollary together with Theorem 2.3.7 shows that

$$Q_m(A^-/Z) \cong \mathrm{Der_m}(A) \cong \mathrm{Der}(Q).$$

Therefore it is enough to prove that $Q_m(\mathrm{Der}(Q)) \supsetneq \mathrm{Der}(Q)$.

Note that $Qx = xQ = QxQ$; this will be frequently used in the sequel without mention. We also remark that Q is the vector space direct sum of Qx and $\sum_{i=0}^{\infty} F(t)y^i$.

Let δ be a derivation of Q. Since $xy = tyx$ it follows that

$$\delta(x)y + x\delta(y) = \delta(t)yx + t\delta(y)x + ty\delta(x),$$

and hence $\delta(x)y - ty\delta(x) \in Qx$. Writing

$$\delta(x) = qx + \sum_{i=0}^{m} \lambda_i(t)y^i, \quad \text{where} \quad q \in Q \quad \text{and} \quad \lambda_i(t) \in F(t),$$

it follows that

$$\sum_{i=0}^{m} \lambda_i(t)y^{i+1} - \sum_{i=0}^{m} t\lambda_i(t)y^{i+1} \in Qx.$$

That is, $\sum_{i=0}^{m}(1-t)\lambda_i(t)y^{i+1} \in Qx$. But then $\sum_{i=0}^{m}(1-t)\lambda_i(t)y^{i+1} = 0$ and hence $\lambda_i(t) = 0$ for each i. This proves that $\delta(x) \in Qx$, which in turn implies $\delta(Qx) \subseteq Qx$. Thus, Qx is invariant under every derivation of Q.

Let I be the linear span of all inner derivations of the form $\mathrm{ad}\,(\delta_1 \ldots \delta_n(x))$, where $n \in \mathbb{N}$ and $\delta_1, \ldots, \delta_n \in \mathrm{Der}(Q)$. We claim that I is a nonzero Lie ideal of $\mathrm{Der}(Q)$. Indeed, for every $\delta \in \mathrm{Der}(Q)$ we have

$$[\delta, \mathrm{ad}\,(\delta_1 \ldots \delta_n(x))] = \mathrm{ad}\,(\delta\delta_1 \ldots \delta_n(x)) \in I,$$

showing that I is an ideal, and $\mathrm{ad}\,(\mathrm{ad}\, y(x)) = \mathrm{ad}\,[y, x]$, and so $I \neq 0$. Define $\Delta : I \to \mathrm{Der}(Q)$ by $\Delta(d) = [\mathrm{ad}\, x^{-1}, d]$ for every $d \in I$, so that

$$\Delta(\mathrm{ad}\,(\delta_1 \ldots \delta_n(x))) = [\mathrm{ad}\, x^{-1}, \mathrm{ad}\,(\delta_1 \ldots \delta_n(x))] = \mathrm{ad}\,[x^{-1}, \delta_1 \ldots \delta_n(x)];$$

this makes sense since $\delta_1 \ldots \delta_n(x) \in Qx$ by what was proved in the preceding paragraph. Clearly Δ is a derivation. This allows us to consider $\Delta_I \in Q_m(\mathrm{Der}(Q))$. We claim that Δ_I is not in $\mathrm{Der}(Q)$. Suppose this was not true. Then $\Delta_I = \mathrm{ad}\, \delta_{\mathrm{Der}(Q)}$ for some $\delta \in \mathrm{Der}(Q)$. This means that there exists a nonzero ideal J of $\mathrm{Der}(Q)$ contained in I and such that $\Delta|_J = (\mathrm{ad}\, \delta)|_J$. It is easy to see that derivations defined on I which agree on a nonzero ideal J contained in I, must agree on the entire I. Thus, $\Delta = (\mathrm{ad}\, \delta)|_I$. That is,

$$[\mathrm{ad}\, x^{-1}, \mathrm{ad}\,(\delta_1 \ldots \delta_n(x))] = [\delta, \mathrm{ad}\,(\delta_1 \ldots \delta_n(x))] = \mathrm{ad}\,(\delta\delta_1 \ldots \delta_n)(x)$$

for all $\delta_1, \ldots, \delta_n \in \mathrm{Der}(Q)$. In particular,

$$\mathrm{ad}\,[x^{-1}, [y, x]\,] = [\mathrm{ad}\, x^{-1}, \mathrm{ad}\,[y, x]\,] = \mathrm{ad}\,(\delta([y, x])),$$

which implies $[x^{-1}, [y, x]\,] - \delta([y, x]) \in Z_Q = F(t)$. Since δ, as a derivation of Q, leaves Qx invariant, it follows that $[x^{-1}, [y, x]\,] \in Qx + F(t)$. However,

$$[x^{-1}, [y, x]\,] = x^{-1}(yx - xy) - (yx - xy)x^{-1} = t^{-1}y - y - y + ty = (t^{-1} + t - 2)y,$$

a contradiction. $\square$

Chapter 4

Jordan systems of quotients versus Lie algebras of quotients

In recent years, there have appeared different quotients for Jordan systems. In [65], C. Martínez constructed an algebra of fractions for a linear Jordan algebra. A notion of quotients for Jordan systems with respect to filters of ideals was given by E. García and M. A. Gómez Lozano in [39]. On the other hand, a Jordan version of Utumi's rings of quotients was obtained by F. Montaner in [68] for non-degenerate Jordan algebras. His notion includes that of E. García and M. A. Gómez Lozano in the case of algebras.

In this chapter, we will show that graded Lie algebras of quotients are the natural framework were to settle quotients for Jordan systems introduced by E. García and M. A. Gómez Lozano.

4.1 Maximal graded algebras of quotients of 3-graded Lie algebras

Let L be a $\mathbb{Z}$-graded Lie algebra with a finite grading. Recall that we may write $L = \oplus_{k=-n}^{n} L_k$ and it is said that L has a $(2n+1)$-grading. In what follows, we will deal with 3-graded Lie algebras.

In this section we will show that for a 3-graded semiprime Lie algebra L, the maximal graded algebra of quotients of L is 3-graded too and coincides with the maximal graded algebra of quotients of L, as defined in Section 2.2. First, we need a lemma.

Lemma 4.1.1. *Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a 3-graded Lie algebra and I an ideal of L. Denote by π_i the canonical projection from L into L_i (with $i \in \{-1, 0, 1\}$) and consider $\tilde{I} := J + \pi_{-1}(J) + \pi_1(J)$, where $J := [[I, I], [I, I]]$. Then:*

(i) *$\tilde{I}$ is a graded ideal of L contained in I.*

If moreover L is semiprime, then:

(ii) *I is an essential ideal of L if and only if $\tilde{I}$ is an essential ideal of L.*

(iii) *Suppose that I is a graded ideal. Then I is an essential ideal of L if and only if it is a graded essential ideal of L.*

Proof. (i). Note that $\pi_0(J) \subseteq \tilde{I}$ since $\pi_0 = \mathrm{Id} - \pi_{-1} - \pi_1$. Show first that $\tilde{I}$ is an ideal of L: take $x \in \tilde{I}$ and $y \in L$ and write $x = u + z_{-1} + t_1$, where u and the elements $z = z_{-1} + z_0 + z_1$ and $t = t_{-1} + t_0 + t_1$ are in J. We have

$$[x, y] = [u, y] + [z_{-1}, y] + [t_1, y]. \tag{4.1}$$

Now, since u is in J, which is an ideal of L, we obtain $[u, y] \in J \subseteq \tilde{I}$. On the other hand, writing $y = y_{-1} + y_0 + y_1$ we have $[z_{-1}, y] = [z_{-1}, y_0] + [z_{-1}, y_1]$; apply again that J is an ideal to obtain $[z, y_1], [z, y_0] \in J$, which implies that the elements $[z, y_1]_0 = [z_{-1}, y_1]$ and $[z, y_0]_{-1} = [z_{-1}, y_0]$ are in $\tilde{I}$. Hence, $[z_{-1}, y] \in \tilde{I}$. Analogously, it can be shown $[t_1, y] \in \tilde{I}$. Put together (4.1) and this to obtain $[x, y] \in \tilde{I}$, as desired.

We claim that $\tilde{I}$ is in fact a graded ideal: consider $x = x_{-1} + x_0 + x_1 \in \tilde{I}$ and write, as above, $x = u + z_{-1} + t_1$, with u, $z = z_{-1} + z_0 + z_1$ and $t = t_{-1} + t_0 + t_1$

elements in J. Then $x_{-1} = u_{-1} + z_{-1}$, $x_0 = u_0$ and $x_1 = u_1 + t_1$. Thus, taking into account the definition of $\tilde{I}$ we obtain that $x_i \in \tilde{I}$ for $i \in \{-1, 0, 1\}$.

Finally, we prove that $\tilde{I}$ is contained in I by showing that $\pi_{-1}(J)$ and $\pi_1(J)$ are contained in I. Define $\delta := \pi_1 - \pi_{-1}$. Then $\delta^2 = \pi_{-1} + \pi_1$ implies $2\pi_1 = \delta^2 + \delta$ and $2\pi_{-1} = \delta^2 - \delta$. Hence, to prove that $\pi_{-1}(J)$ and $\pi_1(J)$ are contained in I, it is enough to check that $\delta^2(J)$ and $\delta(J)$ are contained in I. Take $x, y \in I$ and write $x = x_{-1} + x_0 + x_1$ and $y = y_{-1} + y_0 + y_1$ where $x_i, y_i \in L_i$ for $i \in \{-1, 0, 1\}$. A computation gives

$$[x, y]_{-1} = [x_{-1}, y_0] + [x_0, y_{-1}] = [x_{-1}, y] - [x_{-1}, y_1] + [x, y_{-1}] - [x_1, y_{-1}]$$

$$[x, y]_1 = [x_0, y_1] + [x_1, y_0] = [x_1, y] - [x_1, y_{-1}] + [x, y_1] - [x_{-1}, y_1].$$

Hence,

$$\delta([x, y]) = [x, y]_1 - [x, y]_{-1} = [x_1, y] + [x, y_1] - [x_{-1}, y] - [x, y_{-1}] \in I,$$

that is, $\delta([I, I]) \subseteq I$; it can be proved analogously $\delta(J) \subseteq [I, I] \subseteq I$, therefore $\delta^2(J) \subseteq \delta([I, I]) \subseteq I$.

(ii). Consider I as an essential ideal of L. Note that the semiprimeness of L implies that J is also an essential ideal of L. Hence, $J \cap K \neq 0$ for any nonzero ideal K of L and so $\tilde{I} \cap K \neq 0$. This shows that $\tilde{I}$ is an essential ideal of L.

To prove the converse, suppose that $\tilde{I}$ is an essential ideal of L. As $\tilde{I} \subseteq I$ (by (i)), the ideal I must be essential too.

(iii). It is trivial that I essential as an ideal implies I essential as a graded ideal. Suppose now that I is a graded essential ideal and let U be a nonzero ideal of L. Being L semiprime, $K := [[U, U], [U, U]]$ is a nonzero ideal of L. Apply (i) to obtain that $\tilde{U} := K + \pi_{-1}(K) + \pi_1(K)$ is a graded ideal of L contained in U. As I is a graded essential ideal, $I \cap \tilde{U} \neq 0$ and hence $I \cap U \neq 0$. □

Theorem 4.1.2. *Let $L = L_{-1} \oplus L_0 \oplus L_1$ be a 3-graded semiprime Lie algebra. Then:*

(i) *$Q_m(L)$ is graded isomorphic to $Q_{gr-m}(L)$.*

(ii) *If L is strongly non-degenerate and Φ is 2 and 3-torsion free, then $Q_m(L)$ is a 3-graded strongly non-degenerate Lie algebra.*

Proof. (i). Observe that L, viewed as a 3-graded Lie algebra, is graded semiprime (since L is semiprime), so it has sense to consider $Q_{gr-m}(L)$. Define

$$\begin{array}{cccc} \varphi: & Q_m(L) & \to & Q_{gr-m}(L) \\ & \delta_I & \mapsto & \delta_{\tilde{I}} \end{array}$$

where for an essential ideal I of L, $\tilde{I} \subseteq I$ is the graded essential ideal defined in Lemma 4.1.1.

The map φ is well-defined:

let $J = J_{-1} \oplus J_0 \oplus J_1$ be a 3-graded ideal of L; it is easy to check, by considering the canonical projections onto the subspaces J_i ($i \in \{-1, 0, 1\}$), that $\mathrm{PDer}(J, L)$ is just $\oplus_{i=-2}^{2} \mathrm{PDer}_{\mathrm{gr}}(J, L)_i$, which coincides, by definition, with $\mathrm{PDer}_{\mathrm{gr}}(J, L)$. This fact jointly with the considerations above and the definitions of $Q_m(L)$ and $Q_{gr-m}(L)$ allow us to conclude that φ is well-defined.

It is straightforward to verify that φ is a graded Lie algebra homomorphism. Finally, the bijectivity of φ is obtained from Lemma 4.1.1 (iii).

(ii). Apply (i) and [39, Proposition 1.7]. □

4.2 Quotients of Jordan systems and of 3-graded Lie algebras

Inspired by the characterization of algebras of Lie algebras of quotients in terms of absorption by ideals given by M. Siles Molina in [79, Proposition 2.15], E. García and M. A. Gómez Lozano introduced in [39] a notion of quotients for Jordan systems (algebra, pair or triple system).

Our first target in this section will be to analyze the relationship between the notion of Jordan pairs of quotients in the sense of E. García and M. A. Gómez Lozano and of (graded) Lie algebra of quotients, via the Tits-Kantor-Koecher construction.

Definition 4.2.1. A **Jordan pair over** Φ is a pair $V = (V^+, V^-)$ of Φ-modules together with a pair (Q^+, Q^-) of quadratic maps

$$Q^\sigma : V^\sigma \to \mathrm{Hom}(V^{-\sigma}, V^\sigma) \quad (\text{for } \sigma = \pm)$$

with linearizations denoted by

$$Q^\sigma_{x,z}\, y = \{x, y, z\} = D^\sigma_{x,y} z,$$

where $Q^\sigma_{x,z} = Q^\sigma_{x+z} - Q^\sigma_x - Q^\sigma_z$, satisfying the following identities in all the scalar extensions of Φ:

(i) $D^\sigma_{x,y}\, Q^\sigma_x = Q^\sigma_x\, D^{-\sigma}_{y,x}$

(ii) $D^\sigma_{Q^\sigma_x y,\, y} = D^\sigma_{x,\, Q^{-\sigma}_y x}$

(iii) $Q^\sigma_{Q^\sigma_x y} = Q^\sigma_x Q^{-\sigma}_y Q^\sigma_x$

for every $x \in V^\sigma$ and $y \in V^{-\sigma}$.

From now on, we shall deal with Jordan pairs $V = (V^+, V^-)$ over a ring of scalars Φ containing $\frac{1}{2}$. In order to ease the notation, Jordan products will be denoted by $Q_x y$, for any $x \in V^\sigma$, $y \in V^{-\sigma}$.

Notice that $\{x, y, z\} = \{z, x, y\}$ and $\{x, y, x\} = 2Q_x y$ for every $x, z \in V^\sigma$, $y \in V^{-\sigma}$ and $\sigma = \pm$; we will be using these facts even without an explicit reference to them.

We refer the reader to [62] for basic results, notation and terminology on Jordan pairs. Nevertheless, we recall here some notions and basic properties.

Definitions 4.2.2. Let $V = (V^+, V^-)$ be a Jordan pair.

1. An element $x \in V^\sigma$ is called an **absolute zero divisor if** $Q_x = 0$**.**

2. We say that V is **strongly non-degenerate** (**non-degenerate** in the terminology of [39]) if it has no nonzero absolute zero divisors.

3. A pair $I = (I^+, I^-)$ of submodules of V is called an **ideal of** V if it satisfies $Q_{I^\sigma} V^{-\sigma} + Q_{V^\sigma} I^{-\sigma} + \{V^\sigma, V^{-\sigma}, I^\sigma\} \subseteq I^\sigma$ or equivalently, $\{I^\sigma, V^{-\sigma}, V^\sigma\} + \{V^\sigma, I^{-\sigma}, V^\sigma\} \subseteq I^\sigma$ for $\sigma = \pm$.

4. The pair V is said **semiprime** if $Q_{I^\pm} I^\mp = 0$ imply $I = 0$, being I an ideal of V, and is called **prime** if $Q_{I^\pm} J^\mp = 0$ imply $I = 0$ or $J = 0$, for I and J ideals of V. A **strongly prime** pair is a prime and strongly non-degenerate pair.

5. For a subset $X = (X^+, X^-)$ of V, the **annihilator of** X **in** V is $\mathrm{Ann}_V(X) = (\mathrm{Ann}_V(X)^+, \mathrm{Ann}_V(X)^-)$, where, for $\sigma = \pm$

$$\begin{aligned} \mathrm{Ann}_V(X)^\sigma &= \{z \in V^\sigma \mid \{z, X^{-\sigma}, V^\sigma\} = \{z, V^{-\sigma}, X^\sigma\} \\ &= \{V^{-\sigma}, z, X^{-\sigma}\} = 0\}. \end{aligned}$$

   One can check that $\mathrm{Ann}_V(I)$ is an ideal of V if I is so.

Ideals of Lie algebras having zero annihilator are essentials and when the Lie algebra where they live is semiprime, the reverse holds, i.e., every essential ideal has zero annihilator. (See Lemma 1.1.13.) In the context of Jordan pairs, a similar result can be shown.

Lemma 4.2.3. *Let* $I = (I^+, I^-)$ *be an ideal of a semiprime Jordan pair* $V = (V^+, V^-)$*. Then:*

(i) $I \cap \mathrm{Ann}_V(I) = 0$.

(ii) I *is an essential ideal of* V *if and only if* $\mathrm{Ann}_V(I) = 0$.

Proof. (i). If we show that the ideal $K := I \cap \mathrm{Ann}_V(I)$ satisfies that $Q_{K^\pm}K^\mp = 0$, for $K^\sigma = I^\sigma \cap \mathrm{Ann}_V(I)^\sigma$, $\sigma = \pm$, the result follows by the semiprimeness of V. Given $x \in K^\sigma \subseteq \mathrm{Ann}_V(I)^\sigma$ for $\sigma = \pm$ we have $\{x, K^{-\sigma}, V^\sigma\} = 0$ since $K^{-\sigma} \subseteq I^{-\sigma}$. So $\{K^\sigma, K^{-\sigma}, V^\sigma\} = 0$ for $\sigma = \pm$ and hence $Q_{K^\pm}K^\mp = 0$, as desired.

(ii). Consider an essential ideal $I = (I^+, I^-)$ of V; then $I \cap \mathrm{Ann}_V(I) = 0$ by (i), and by the essentiality, $\mathrm{Ann}_V(I) = 0$. Conversely, suppose that $\mathrm{Ann}_V(I) = 0$ and consider an ideal $K = (K^+, K^-)$ of V satisfying $I \cap K = 0$.

For $x \in K^\sigma$, with $\sigma = \pm$, and taking into account that I and K are ideals of V, we obtain

$$\{x, I^{-\sigma}, V^\sigma\}, \{x, V^{-\sigma}, I^\sigma\}, \{V^{-\sigma}, x, I^{-\sigma}\} \subseteq I \cap K = 0,$$

hence, $K \subseteq \mathrm{Ann}_V(I) = 0$. This shows that I is an essential ideal of V. □

Let us recall the connection between Jordan 3-graded Lie algebras and Jordan pairs. First we give some definitions.

Definitions 4.2.4. A 3-graded Lie algebra $L = L_{-1} \oplus L_0 \oplus L_1$ is called **Jordan 3-graded** if $[L_1, L_{-1}] = L_0$ and there exists a Jordan pair structure on (L_1, L_{-1}) whose Jordan product is related to the Lie product by

$$\{x, y, z\} = [[x, y], z],$$

for any $x, z \in L_\sigma$, $y \in L_{-\sigma}$, $\sigma = \pm$. In this case, $V = (L_1, L_{-1})$ is called the **associated Jordan pair**.

Since $\frac{1}{2} \in \Phi$, the product on the associated Jordan pair is unique and given by

$$Q_x y = \frac{1}{2}\{x, y, x\} = \frac{1}{2}[[x, y], x].$$

Conversely, for any 3-graded Lie algebra, the formula above defines a pair structure on (L_1, L_{-1}) whenever $\frac{1}{6} \in \Phi$ (see [72, 1.2]).

One important example of a Jordan 3-graded Lie is the TKK-**algebra of a Jordan pair**. It is built as follows:

Construction 4.2.5. Let $V = (V^+, V^-)$ be a Jordan pair; a pair $(\delta^+, \delta^-) \in \mathrm{End}_\Phi(V^+) \times \mathrm{End}_\Phi(V^-)$ is a **derivation of** V if it satisfies

$$\delta^\sigma(\{x, y, z\}) = \{\delta^\sigma(x), y, z\} + \{x, \delta^{-\sigma}(y), z\} + \{x, y, \delta^\sigma(z)\}$$

for any $x, z \in V^\sigma$ and $y \in V^{-\sigma}$, $\sigma = \pm$. For $(x, y) \in V$ the map $\delta(x, y) := (D_{x,y}, -D_{y,x})$ is a derivation of V (by the identity (JP12) in [62]) called **inner derivation**. Denote by $\mathrm{IDer}(V)$ the Φ-module spanned by all inner derivations of V and define on the Φ-module

$$\mathrm{TKK}(V) := V^+ \oplus \mathrm{IDer}(V) \oplus V^-$$

the following product:

$$\begin{aligned}[x^+ \oplus \gamma \oplus x^-, y^+ \oplus \mu \oplus y^-] &= (\gamma_+ x^+ - \mu_+ x^+) \oplus ([\gamma, \mu] + \delta(x^+, y^-) \\ &- \delta(y^+, x^-)) \oplus (\gamma_- y^- \mu_- x^-),\end{aligned}$$

where $x^\sigma, y^\sigma \in V^\sigma$ and $\gamma = (\gamma_+, \gamma_-)$, $\mu = (\mu_+, \mu_-) \in \mathrm{IDer}(V)$. Then, it can be proved that $\mathrm{TKK}(V)$ becomes a Lie algebra (see e.g. [67]). As this construction has its origin in the fundamental papers [50, 51, 52] by Kantor, in [53, 54] by Koecher and in [81] by Tits, $\mathrm{TKK}(V)$ is called the **Tits-Kantor-Koecher algebra of** V or the TKK-**algebra** for short. It is easy to check that the following provides $\mathrm{TKK}(V)$ with a 3-grading:

$$\mathrm{TKK}(V)_1 = V^+, \ \mathrm{TKK}(V)_0 = \mathrm{IDer}(V), \ \mathrm{TKK}(V)_{-1} = V^-.$$

Moreover, $\mathrm{TKK}(V)$ is a Jordan 3-graded Lie algebra with V as associated Jordan pair.

If L is a Jordan 3-graded Lie algebra with associated Jordan pair V, then the TKK-algebra associated to V is not in general isomorphic to L. Rather, we have:

Lemma 4.2.6. ([71, 2.8]). *Let L be a Jordan 3-graded Lie algebra with associated Jordan pair V. Then* $\mathrm{TKK}(V) \cong L/C_V$, *where* $C_V = \{x \in L_0 \mid [x, L_1] = 0 = [x, L_{-1}]\} = Z(L) \cap L_0$.

The following lemma gives us a lot of information about the relationship between ideals of Jordan pairs and certain ideals of their respective TKK-algebras.

Lemma 4.2.7. *Let V be a semiprime Jordan pair, and $I = (I^+, I^-)$ an ideal of V. Define by* $\mathrm{Id}_{\mathrm{TKK}(V)}(I) = I^+ \oplus ([I^+, V^-] + [V^+, I^-]) \oplus I^-$ *the graded ideal of* $\mathrm{TKK}(V)$ *generated by I. Then* $\mathrm{Ann}_{\mathrm{TKK}(V)}(\mathrm{Id}_{\mathrm{TKK}(V)}(I)) = 0$ *if and only if* $\mathrm{Ann}_V(I) = 0$.

Proof. See [39, Lemma 2.9]. □

Let us recall what may be the main definition of this section.

Definition 4.2.8. (See [39, 2.5].) Let V be a semiprime Jordan pair contained in a Jordan pair W. It is said that W is a **pair of $\mathfrak{M}$-quotients of V** if for every $0 \neq q \in W^\sigma$ (with $\sigma = \pm$) there exists an ideal I of V with $\mathrm{Ann}_V(I) = 0$ such that

$$\{q, I^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I^\sigma\} \subseteq V^\sigma \quad \text{and} \quad \{I^{-\sigma}, q, V^{-\sigma}\} \subseteq V^{-\sigma},$$

with either

$$\{q, I^{-\sigma}, V^\sigma\} + \{q, V^{-\sigma}, I^\sigma\} \neq 0 \quad \text{or} \quad \{I^{-\sigma}, q, V^{-\sigma}\} \neq 0.$$

We are now in a position to show the equivalence between Jordan pairs of quotients and Lie algebras of quotients of their respective TKK-algebras.

Theorem 4.2.9. *Let V be a semiprime subpair of a Jordan pair W. Then the following conditions are equivalent:*

(i) *W is a pair of $\mathfrak{M}$-quotients of V.*

(ii) $\mathrm{TKK}(W)$ *is an algebra of quotients of* $\mathrm{TKK}(V)$.

Proof. (i) $\Rightarrow$ (ii) is [39, Theorem 2.10].

(ii) $\Rightarrow$ (i). Take $0 \neq q^\sigma \in W^\sigma$ $(\sigma = \pm)$ and apply Propositions 1.4.26 and 1.4.22 to find a 3-graded ideal I of $\mathrm{TKK}(V)$ with $\mathrm{Ann}_{\mathrm{TKK}(V)}(I) = 0$ and such that $0 \neq [I, q^\sigma] \subseteq \mathrm{TKK}(V)$. We claim that $I_V := I_1 \oplus ([I_1, V^-] + [V^+, I_{-1}]) \oplus I_{-1}$ is an essential ideal of $\mathrm{TKK}(V)$, where $I = I_1 \oplus I_0 \oplus I_{-1}$. Let $K = K_1 \oplus K_0 \oplus K_{-1}$ be a nonzero 3-graded ideal of $\mathrm{TKK}(V)$; the semiprimeness of V implies that either $I_1 \cap K_1 \neq 0$ or $I_{-1} \cap K_{-1} \neq 0$ (see the proof of [41, Proposition 2.6]) and therefore $I_V \cap K \neq 0$. By Lemma and 1.4.10 (iii), $\mathrm{Ann}_{\mathrm{TKK}(V)}(I_V) = 0$, and by Lemma 4.2.7, $\mathrm{Ann}_V((I_1, I_{-1})) = 0$.

Denote I_1 and I_{-1} by I^+ and I^-, respectively. Then, for $\sigma = \pm$ we have:

$$\begin{aligned} \{q^\sigma, I^{-\sigma}, V^\sigma\} &\subseteq [[q^\sigma, I^{-\sigma}], V^\sigma] \subseteq V^\sigma \\ \{I^{-\sigma}, q^\sigma, V^{-\sigma}\} &\subseteq [[I^{-\sigma}, q^\sigma], V^{-\sigma}] \subseteq V^{-\sigma} \\ \{q^\sigma, V^{-\sigma}, I^\sigma\} &\subseteq [[q^\sigma, V^{-\sigma}], I^\sigma] \subseteq [[V^{-\sigma}, I^\sigma], q^\sigma] \subseteq V^\sigma \end{aligned}$$

To complete the proof we have to check that either $\{q^\sigma, I^{-\sigma}, V^\sigma\} + \{q^\sigma, V^{-\sigma}, I^\sigma\} \neq 0$ or $\{I^{-\sigma}, q^\sigma, V^{-\sigma}\} \neq 0$. We have just showed that $\mathrm{Ann}_{\mathrm{TKK}(V)}(I_V) = 0$; using [79, Lemma 2.11] we obtain that $\mathrm{Ann}_{\mathrm{TKK}(W)}(I_V) = 0$ and hence $0 \neq [I_V, q^\sigma] \subseteq [I, q^\sigma] \subseteq \mathrm{TKK}(V)$ which implies that either $[(I_V)_0, q^\sigma] \neq 0$ or $[I^{-\sigma}, q^\sigma] \neq 0$. In the first case, we have:

$$\begin{aligned} 0 \neq [(I_V)_0, q^\sigma] &= [[I^\sigma, V^{-\sigma}], q^\sigma] + [[V^\sigma, I^{-\sigma}], q^\sigma] \\ &\subseteq \{I^\sigma, V^{-\sigma}, q^\sigma\} + \{V^\sigma, I^{-\sigma}, q^\sigma\}. \end{aligned}$$

In the second case, apply that the representation of $\mathrm{IDer}(V)$ on V is faithful to obtain

$$\begin{aligned} 0 \neq [\,[I^{-\sigma}, q^{\sigma}], V^{-\sigma}] &= \{I^{-\sigma}, q^{\sigma}, V^{-\sigma}\} \quad \text{or} \\ 0 \neq [\,[I^{-\sigma}, q^{\sigma}], V^{\sigma}] \subseteq [\,[V^{\sigma}, I^{-\sigma}], q^{\sigma}] &= \{V^{\sigma}, I^{-\sigma}, q^{\sigma}\} = \{q^{\sigma}, I^{-\sigma}, V^{\sigma}\}. \end{aligned}$$

□

Inspired by C. Martínez's idea [65] of moving from a Jordan setting to a Lie one through the TKK-construction and using the construction of the maximal algebra of quotients of a semiprime Lie algebra given by M. Siles Molina [79], E. García and M. A. Gómez Lozano built [39] the maximal Jordan system of quotients of a strongly non-degenerate Jordan system.

Our next objective is to examine the relationship between maximal Jordan pairs of $\mathfrak{M}$-quotients (see [39, 3.1 and Theorem 3.2] for precise definition) and maximal algebras of quotients of Jordan 3-graded Lie algebras.

Lemma 4.2.10. *Let $V = (V^+, V^-)$ be a strongly non-degenerate Jordan pair. If I is an essential ideal of* $\mathrm{TKK}(V)$*, then there exists an essential ideal $\hat{I}$ of V such that $\mathrm{Id}_{\mathrm{TKK}(V)}(\hat{I})$ is contained in I.*

Proof. Consider an essential ideal I of $\mathrm{TKK}(V)$, which is a strongly non-degenerate Lie algebra (by [38, Proposition 2.6]); in particular, it is semiprime. Therefore, we may apply Lemma 4.1.1 (i) and (ii) to find an essential graded ideal $\hat{I}_{-1} \oplus \hat{I}_0 \oplus \hat{I}_1$ of $\mathrm{TKK}(V)$ contained in I. It can be shown, as in the proof of Theorem 4.2.9, that

$$\hat{I}_{-1} \oplus (\,[\hat{I}_1, V^-] + [V^+, \hat{I}_{-1}]\,) \oplus \hat{I}_1 \subseteq I$$

is an essential ideal of $\mathrm{TKK}(V)$ and, by means of Lemma 4.2.7, $\hat{I} := (\hat{I}_1, \hat{I}_{-1})$ is an essential ideal of V. □

For a strongly non-degenerate Jordan pair V, denote its maximal Jordan pair of $\mathfrak{M}$-quotients by $Q_m(V)$. (See [39, 3.1] for its construction.)

Theorem 4.2.11. *Assume that* $\frac{1}{6} \in \Phi$.

(i) *Let V be a strongly non-degenerate Jordan pair. Then*

$$Q_m(V) = \Big((Q_m(\mathrm{TKK}(V)))_1,\, (Q_m(\mathrm{TKK}(V)))_{-1} \Big)$$

is the maximal Jordan pair of $\mathfrak{M}$-quotients of V.

(ii) *If $L = L_{-1} \oplus L_0 \oplus L_1$ is a strongly non-degenerate Jordan 3-graded Lie algebra satisfying that $Q_m(L)$ is Jordan 3-graded, then*

$$Q_m(L) \cong Q_m(\mathrm{TKK}(V)) \cong \mathrm{TKK}(Q_m(V)),$$

where $V = (L_1, L_{-1})$ is the associated Jordan pair of L.

Proof. (i). The Lie algebras $Q_{\mathcal{F}_{\mathrm{TKK}}}(\mathrm{TKK}(V))$ and $Q_m(\mathrm{TKK}(V))$ are isomorphic by Lemmas 4.2.7 and 4.2.10 (see [39] for the definition of $Q_{\mathcal{F}_{\mathrm{TKK}}}(\mathrm{TKK}(V))$. On the other hand, Theorem 4.1.2 (i) implies that they are isomorphic to $Q_{gr-m}(\mathrm{TKK}(V))$. (Note that $\mathrm{TKK}(V)$ is a strongly non-degenerate Lie algebra by [39, Proposition 2.6] so, it has sense to consider its maximal graded algebra of quotients.) Now, the result follows by [39, Theorem 3.2].

(ii). The Lie algebra L has zero center because it is strongly non-degenerate, hence $L \cong \mathrm{TKK}(V)$ (use Lemma 4.2.6) and, obviously, $Q_m(L) \cong Q_m(\mathrm{TKK}(V))$. This one is a strongly non-degenerate Lie algebra (by [79, Proposition 2.7 (iii)]) and has a 3-grading (by (i)) with associated Jordan pair $Q_m(V)$. The hypothesis on $Q_m(L)$ allows us to use again Lemma 4.2.6 obtaining $Q_m(L) \cong \mathrm{TKK}(Q_m(V))$. □

The following is an example of a strongly non-degenerate Jordan 3-graded Lie algebra L such that its maximal (graded) algebra of quotients $Q_m(L)$ is not Jordan 3-graded. If we denote by V the associated Jordan pair of L, we obtain that $\mathrm{TKK}(Q_m(V))$ is not (graded) isomorphic to $Q_m(L)$) (since $\mathrm{TKK}(Q_m(V))$ is Jordan 3-graded) which thereby means that the condition on L in Theorem 4.2.11 (ii) is necessary.

Example 4.2.12. Denote by $\mathbb{M}_\infty(\mathbb{R}) = \cup_{n=1}^{\infty}\mathbb{M}_n(\mathbb{R})$ the algebra of infinite matrices with a finite number of nonzero entries and consider

$$L := \mathfrak{sl}_\infty(\mathbb{R}) = \{x \in \mathbb{M}_\infty(\mathbb{R}) \mid \mathrm{tr}(x) = 0\},$$

which is a simple Lie algebra of countable dimension (see [10, Theorem 1.4]).

Denote by e_{ij} the matrix whose entries are all zero except for the one in row i and column j and consider the orthogonal idempotents $e := e_{11}$ and $f := \mathrm{diag}(0, 1, 1, \ldots)$ (note that $f \notin \mathbb{M}_\infty(\mathbb{R})$); we can see L as a 3-graded Lie algebra by doing $L = L_{-1} \oplus L_0 \oplus L_1$, where $L_{-1} = eLf$, $L_0 = \{exe + fxf \mid x \in L\}$ and $L_1 = fLe$.

Let $exe + fxf$ be an element of L_0 with $x = (x_{ij}) \in \mathbb{M}_n(\mathbb{R})$ for some $n \in \mathbb{N}$. Taking into account that $\mathrm{tr}(x) = 0$, we obtain:

$$exe + fxf = \sum_{i=2}^{n}\sum_{j=2}^{n}[-e_{1j},\, x_{ij}e_{i1}] \in [L_{-1},\, L_1],$$

This shows that $L_0 = [L_{-1},\, L_1]$, i.e., L is Jordan 3-graded.

In what follows, we will prove that $\mathrm{Der}(L)$ is not Jordan 3-graded. The simplicity of L implies that $Q_m(L) \cong \mathrm{Der}(L)$; on the other hand, the strongly non-degeneracy of L allows us to apply [39, Proposition 1.7] obtaining that $\mathrm{Der}(L)$ is 3-graded. Now, take, $\delta := \mathrm{ad}\, e$; one can easily check that:

$$\delta(L_{-1}) \subseteq L_{-1},\ \delta(L_0) = 0 \ \text{ and } \ \delta(L_1) \subseteq L_1,$$

which means that $\delta \in \mathrm{Der}(L)_0$. But note that $\delta \notin [\mathrm{Der}(L)_{-1}, \mathrm{Der}(L)_1]$ since the elements of $[\mathrm{Der}(L)_{-1}, \mathrm{Der}(L)_1]$ have zero trace on every finite dimensional subspace of L while the trace of δ is always nonzero. Therefore, $[\mathrm{Der}(L)_{-1}, \mathrm{Der}(L)_1] \subsetneq \mathrm{Der}(L)_0$, i.e., $\mathrm{Der}(L)$ is not Jordan 3-graded.

Remark 4.2.13. Note that there exist non-trivial Jordan 3-graded Lie algebras such that their maximal (graded) algebra of quotients are also Jordan 3-graded Lie algebras. For example:

Let F be a field and consider the Lie algebra

$$L := \mathfrak{sl}_2(F) = \{x \in \mathbb{M}_2(F) \mid \mathrm{tr}(x) = 0\}.$$

We have that L is a Jordan 3-graded Lie algebra with the grading $L = L_{-1} \oplus L_0 \oplus L_1$, where

$$L_{-1} = Fe_{21},\ L_0 = F(e_{11} - e_{22}) \ \text{ and } \ L_1 = Fe_{12}.$$

Moreover, L is a finite dimensional semisimple Lie algebra and applying [79, Lemma 3.9] we obtain that $L \cong Q_m(L)$.

We want to obtain now an analogue to Theorem 4.2.11 for Jordan triple systems and Jordan algebras. Let us first start with Jordan triple systems.

Definition 4.2.14. A **Jordan triple system over** Φ is a Φ-module T together with a quadratic map $P : T \to \mathrm{End}_\Phi(T)$ with linearizations denoted by

$$P_{x,z}\, y = \{x,\, y,\, z\} = L_{x,y}\, z,$$

where $P_{x,z} = P_{x+z} - P_x - P_z$, satisfying the following identities in all the scalar extensions of Φ:

(i) $L_{x,y}\, P_x = P_x\, L_{y,x}$

(ii) $L_{P_xy,\,y} = L_{x,\,P_yx}$

(iii) $P_{P_xy} = P_xP_yP_x$

for every $x,\, y \in T$.

As in case of Jordan pairs, we shall deal with Jordan triple systems T over a ring of scalars Φ containing $\frac{1}{2}$. Notice that $\{x,\, y,\, z\} = \{z,\, x,\, y\}$ and $\{x,\, y,\, x\} = 2P_xy$ for every $x,\, y,\, z \in T$.

We refer the reader to [62, 70, 67] for basic results, notation and terminology on Jordan triple systems. Again, we give here some definitions and properties.

Definitions 4.2.15. Let T be a Jordan triple system.

1. An element $x \in T$ is called an **absolute zero divisor** if $P_x = 0$.

2. We say that T is **strongly non-degenerate** (**non-degenerate** in the terminology of [39]) if it has no nonzero absolute zero divisors.

3. A submodule I of T is called an **ideal of** T if it satisfies $P_IT + P_TI + \{T,\, T,\, I\} \subseteq I$ or equivalently, $\{I,\, T,\, T\} + \{T,\, I,\, T\} \subseteq I$.

4. The triple T is said **semiprime** if $P_II = 0$ imply $I = 0$, being I an ideal of T, and is called **prime** if $P_IJ = 0$ imply $I = 0$ or $J = 0$, for I and J ideals of T. A **strongly prime** triple is a prime and strongly non-degenerate triple.

5. For a subset X of T, the **annihilator of** X **in** V is

$$\mathrm{Ann}_T(X) = \{z \in T \mid \{z,\, X,\, T\} = \{z,\, T,\, X\} = \{T,\, z,\, X\} = 0\}.$$

   One can check that $\mathrm{Ann}_T(I)$ is an ideal of T if I is so.

The following remark describe the connection between Jordan pairs and Jordan triple systems; it will be a useful tool for our purpose.

Remark 4.2.16. ([62, 1.13]). Every Jordan triple system T gives rise to a Jordan pair (T, T) with quadratic maps $Q_x = P_x$ for every $x \in T$. It is called the **double Jordan pair associated to** T and denoted by $V(T)$.

From the definitions, it is obvious that a Jordan triple system T is strongly non-degenerate if and only if its double Jordan pair $V(T)$ is so.

After the definition of the double Jordan pair $V(T)$ of a Jordan triple system T, a natural question rises: is there any relationship between ideals of T and ideals of $V(T)$? In the subsequent lemma we answer easily this question. We will use it without further mention.

Lemma 4.2.17. *Let T be a Jordan triple system and $V(T)$ its double Jordan pair. Then*

(i) *If I is an ideal of T the pair $V(I) = (I, I)$ is an ideal of $V(T)$.*

(ii) *If $\hat{I} = (\hat{I}^+, \hat{I}^-)$ is an ideal of $V(T)$ the components $\hat{I}^+$ and $\hat{I}^-$ of $\hat{I}$ are ideals of T.*

(iii) *Every ideal $\hat{I} = (\hat{I}^+, \hat{I}^-)$ of $V(T)$ contains an ideal of the form $V(I)$ for some ideal I of T.*

Proof. Keeping in mind the definition of ideals for Jordan pairs and the definition of ideals for Jordan triples (i) and (ii) follow directly from the construction of the double Jordan pair of a Jordan triple system.

If $\hat{I} = (\hat{I}^+, \hat{I}^-)$ is an ideal of $V(T)$, (iii) follows from (ii) taking the ideal $I = \hat{I}^+ \cap \hat{I}^-$ of T. □

We now examine the behavior of essentiality and annihilators of ideals of a Jordan triple system with respect to ideals of its double Jordan pair.

Lemma 4.2.18. *Let T be a Jordan triple system, I an ideal of T and $V(T)$ the double Jordan pair associated to T. Then:*

(i) *I is essential in T if and only if $V(I)$ is so in $V(T)$.*

(ii) $\mathrm{Ann}_{V(T)}(V(I)) = (\mathrm{Ann}_T(I), \mathrm{Ann}_T(I))$.

(iii) *If $\hat{I} = (\hat{I}^+, \hat{I}^-)$ is an ideal of $V(T)$ having zero annihilator, then the ideal $I = \hat{I}^+ + \hat{I}^-$ of T has zero annihilator.*

Proof. (i). Suppose first that I is an essential ideal of T and take a nonzero ideal $\hat{I} = (\hat{I}^+, \hat{I}^-)$ of $V(T)$; so either $\hat{I}^+ \neq 0$ or $\hat{I}^- \neq 0$ and applying the essentiality of I we obtain that either $I \cap \hat{I}^+ \neq 0$ or $I \cap \hat{I}^- \neq 0$. In any case, $V(I) \cap \hat{I} \neq 0$ which means that $V(I)$ is an essential ideal of $V(T)$.

To prove the converse, consider a nonzero ideal U of T. Since we are assuming $V(I)$ essential in $V(T)$ we obtain $V(I) \cap V(U) \neq 0$ and hence $I \cap U \neq 0$ which concludes the proof.

(ii). By definition $\mathrm{Ann}_{V(T)}(V(I)) = (\mathrm{Ann}_{V(T)}(V(I))^+, \mathrm{Ann}_{V(T)}(V(I))^-)$, where for $\sigma = \pm$, we have

$$\begin{aligned}\mathrm{Ann}_{V(T)}(V(I))^\sigma &= \{x \in V(T)^\sigma \mid \{x, V(I)^{-\sigma}, V(T)^\sigma\} = \\ &\{x, V(T)^{-\sigma}, V(I)^\sigma\} = \{V(T)^{-\sigma}, x, V(I)^{-\sigma}\} = 0\} = \\ &\{x \in T \mid \{x, I, T\} = \{x, T, I\} = \{T, x, I\} = 0\} = \\ &\mathrm{Ann}_T(I).\end{aligned}$$

(iii). Given $\hat{I} = (\hat{I}^+, \hat{I}^-)$ an ideal of $V(T)$ with $\mathrm{Ann}_{V(T)}(\hat{I}) = 0$. Consider the ideal $I = \hat{I}^+ + \hat{I}^-$ of T; then $V(I) = (I, I)$ is an ideal of $V(T)$ contained on $\hat{I}$ and, hence satisfying that $\mathrm{Ann}_{V(T)}(V(I)) = 0$. Thus, by (ii) it follows $\mathrm{Ann}_T(I) = 0$. □

As it happened in the Lie and Jordan pair contexts, one can identify the essential ideals of a Jordan triple system T with the ideals of T having zero annihilator, provided T is semiprime.

Lemma 4.2.19. *Let I be an ideal of a semiprime Jordan triple system T. Then:*

(i) $I \cap \mathrm{Ann}_T(I) = 0$.

(ii) *I is an essential ideal of T if and only if* $\mathrm{Ann}_T(I) = 0$.

Proof. (i). If we show that the ideal $K := I \cap \mathrm{Ann}_T(I)$ satisfies that $P_K K = 0$, the conclusion holds by the semiprimeness of T. Given $x \in K \subseteq \mathrm{Ann}_T(I)$ we have $\{x, K, T\} = 0$ since $K \subseteq I$. So $\{K, K, T\} = 0$ and hence $P_K K = 0$, as desired.

(ii). Let I be an essential ideal of T; then $I \cap \mathrm{Ann}_T(I) = 0$ by (i), and by the essentiality, $\mathrm{Ann}_T(I) = 0$. Conversely, suppose that $\mathrm{Ann}_T(I) = 0$ and take an ideal K of T satisfying $I \cap K = 0$. Taking into account that I and K are ideals of T, we obtain for $x \in K$ that

$$\{x, I, T\}, \{x, T, I\}, \{T, x, I\} \subseteq I \cap K = 0,$$

hence, $K \subseteq \mathrm{Ann}_T(I) = 0$. This shows that I is an essential ideal of T. □

Definition 4.2.20. (See [39, 4.1]). Let T be a semiprime Jordan triple system contained in a Jordan triple system Q. We say that Q is a **triple system of $\mathfrak{M}$-quotients of T** if for each $0 \neq q \in Q$ there exists an ideal I of T with $\mathrm{Ann}_T(I) = 0$ such that

$$0 \neq \{q, I, T\} + \{q, T, I\} + \{I, q, T\} \subseteq T.$$

Theorem 4.2.21. *Let T be a strongly non-degenerate subtriple system of a Jordan triple system Q. Then the following conditions are equivalent:*

(i) *Q is a triple system of $\mathfrak{M}$-quotients of T.*

(ii) *$V(Q)$ is a pair of $\mathfrak{M}$-quotients of $V(T)$.*

(iii) *$\mathrm{TKK}(V(Q))$ is an algebra of quotients of $\mathrm{TKK}(V(T))$.*

Proof. Note that $V(T)$ is a strongly non-degenerate Jordan pair by the strongly non-degeneracy of T. So $V(T)$ is semiprime and it has sense to speak about Jordan pairs of $\mathfrak{M}$-quotients of it.

(ii) $\Leftrightarrow$ (iii) follows from Theorem 4.2.9.

(i) $\Rightarrow$ (ii). Take $0 \neq q \in V(Q)^\sigma = Q$ $(\sigma = \pm)$ from (i) we find an ideal I of T with $\mathrm{Ann}_T(I) = 0$ satisfying that

$$0 \neq \{q,\, I,\, T\} + \{q,\, T,\, I\} + \{I,\, q,\, T\} \subseteq T.$$

The conclusion follows now by applying Lemma 4.2.18 (ii) to the ideal $V(I) = (I,\, I)$ of $V(T)$.

(ii) $\Rightarrow$ (i). Taking into account that $V(T) = (T,\, T)$, given $0 \neq q \in Q = V(Q)^\sigma$, by (ii) we can find an ideal $\hat{I} = (\hat{I}^+,\, \hat{I}^-)$ of $V(T)$ with $\mathrm{Ann}_{V(T)}(\hat{I}) = 0$ such that $0 \neq \{q,\, \hat{I}^{-\sigma},\, T\} + \{q,\, T,\, \hat{I}^\sigma\} + \{\hat{I}^{-\sigma},\, q,\, T\} \subseteq T$. By Lemma 4.2.18 (iii) the ideal $I = \hat{I}^+ + \hat{I}^-$ of T satisfies $\mathrm{Ann}_T(I) = 0$ and

$$0 \neq \{q,\, I,\, T\} + \{q,\, T,\, I\} + \{I,\, q,\, T\} \subseteq T,$$

which concludes the proof. □

For a strongly non-degenerate Jordan triple system T, denote its maximal Jordan triple system of $\mathfrak{M}$-quotients by $Q_m(T)$. (See [39, 4.5] for its construction.)

Theorem 4.2.22. *Let T be a strongly non-degenerate Jordan triple system over a ring of scalars Φ containing $\frac{1}{6}$. Then the maximal Jordan triple system* *of $\mathfrak{M}$-quotients of T is the first component of the maximal algebra of quotients of the TKK-algebra of the double Jordan pair $V(T) = (T, T)$ associated to T, i.e.,*

$$Q_m(T) = (Q_m(\mathrm{TKK}(V(T))))_1.$$

Proof. The Jordan pair $V(T) = (T, T)$ is strongly non-degenerate since T is so. By Theorem 4.2.11 (i) we have

$$Q_m(V(T)) = \Big((Q_m(\mathrm{TKK}(V(T))))_1, (Q_m(\mathrm{TKK}(V(T))))_{-1}\Big).$$

The conclusion follows now from Lemma 4.2.18 (i) and (iii), Lemma 4.2.19 (ii) and from [39, 4.5 and Theorem 4.6]. □

We will finish the chapter with an analogue to Theorem 4.2.11 for Jordan algebras. We first recall some definitions.

Definition 4.2.23. ([62]). A **Jordan algebra over** Φ is a Φ-module J together with quadratic maps

$$U : J \to \mathrm{End}_\Phi(J) \quad \text{and} \quad {}^2 : J \to J \ \ (\textbf{square})$$

with linearizations denoted by $x \circ y$, $U_{x,z}\, y = \{x, y, z\} = V_{x,y}\, z$, where $U_{x,z} = U_{x+z} - U_x - U_z$ and $x \circ y = (x+y)^2 - x^2 - y^2$ satisfying the following identities in all the scalar extensions of Φ:

(i) $V_{x,x} y = x^2 \circ y$

(ii) $U_x(x \circ y) = x \circ U_x y$

(iii) $U_x x^2 = (x^2)^2$

(iv) $U_x U_y x^2 = (U_x y)^2$

(v) $U_{x^2} = U_x^2$

(vi) $U_{U_x y} = U_x U_y U_x$

for every $x, y \in J$.

Since we will assume that $\frac{1}{2} \in \Phi$, it is enough to consider the linearization of the square, $\circ$, because it is related with the linear triple product by $2\{x, y, z\} = (x \circ y) \circ z - (x \circ z) \circ y + (y \circ z) \circ x$.

Definition 4.2.24. An element of a Jordan algebra J is said to be an **absolute zero divisor** if $U_x = 0$. The algebra J is called **strongly non-degenerate** (**non-degenerate** in the terminology of [39]) if it has no nonzero absolute zero divisors.

Definition 4.2.25. ([39, 5.1]). A Jordan overalgebra Q of a Jordan algebra J is said to be a **Jordan algebra of $\mathfrak{M}$-quotients of** J if for every $0 \neq q \in Q$ there exists an ideal I of J having zero annihilator such that $0 \neq q \circ I \subseteq J$.

Remark 4.2.26. ([62, 1.13]). Note that a Jordan algebra J gives rise to a Jordan triple system J_T by simply forgetting the squaring and taking $P = U$. From definitions, it is clear that J is non-degenerate if and only if J_T is so. We call J_T the **Jordan triple system associated to J.**

Denoting by $Q_m(J)$ the maximal Jordan algebra of $\mathfrak{M}$-quotients of a strongly non-degenerate Jordan algebra J (see [39, 5.4] for its construction), the announced result is the following:

Theorem 4.2.27. *Let J be a strongly non-degenerate Jordan algebra over a ring of scalars Φ containing $\frac{1}{6}$. Then*

$$Q_m(J) = Q_m(J_T) = (Q_m(\mathrm{TKK}(V(J_T))))_1,$$

is the maximal Jordan algebra of quotients of J, where J_T denotes the Jordan triple system associated to J and $V(J_T) = (J_T, J_T)$ is the double Jordan pair associated to J_T.

Proof. Note that J_T is a strongly non-degenerate Jordan triple system by the strongly non-degeneracy of the Jordan algebra J. From [39, 5.4 and Theorem 5.5] it follows that the maximal Jordan algebra of quotients $Q_m(J)$ is $Q_m(J_T)$. Finally apply Theorem 4.2.22 to reach the conclusion. □

Chapter 5

Zero product determined matrix algebras

The bulk of this chapter is devoted to the problem of whether the algebra $\mathbb{M}_n(B)$ of $n \times n$ matrices over a unital algebra B is zero (Lie, Jordan) product determined. In Section 4.2 we prove that for the ordinary product the answer is "yes" for every algebra B and every $n \geq 2$, and in Section 4.3 we show the same for the Jordan product - however, for $n \geq 3$ and additionally assuming that B contains the element $\frac{1}{2}$ (i.e., 2 is invertible in B). The Lie product case, treated in Section 4.4, is more entangled. We will see that $\mathbb{M}_n(B)$ is not zero Lie product for all unital algebras B. However, if B is zero Lie product determined, then $\mathbb{M}_n(B)$ is so.

5.1 Introduction and definitions

Throughout the chapter we shall consider algebras over a fixed commutative unital ring C.

Let us start by introducing the basic definitions and results related with the problems that we will study in this chapter.

Definition 5.1.1. Let A be an algebra over C. By A^2 we will denote the C-linear span of all elements of the form xy where $x, y \in A$. Let X be a

C-module and let $\{.,.\} : A \times A \to X$ be a C-bilinear map. Consider the following conditions:

(a) for all $x, y \in A$ such that $xy = 0$ we have $\{x, y\} = 0$;

(b) there exists a C-linear map $T : A^2 \to X$ such that $\{x, y\} = T(xy)$ for all $x, y \in A$.

Trivially, (b) implies (a). We shall say that A is a **zero product determined algebra** if for every C-module X and every C-bilinear map $\{.,.\} : A \times A \to X$, (a) implies (b).

So far A could be any nonassociative algebra. Assume now that A is associative. Recall that A becomes a Lie algebra, usually denoted by A^-, if we replace the original product by the so-called Lie product given by $[x, y] = xy - yx$. Similarly, A becomes a Jordan algebra, denoted by A^+, by replacing the original product by the Jordan product given by $x \circ y = xy + yx$.

Definition 5.1.2. We say that A is a **zero Lie product determined algebra** if A^- is a zero product determined algebra. That is to say, for every C-bilinear map $\{.,.\} : A \times A \to X$, where X is any C-module, we have that $\{.,.\}$ must be of the form $\{x, y\} = T([x, y])$ for some C-linear map $T : [A, A] \to X$ provided that $[x, y] = 0$ implies $\{x, y\} = 0$.

Definition 5.1.3. The algebra A is said to be a **zero Jordan product determined algebra** if A^+ is a zero product determined algebra, that is, $\{.,.\}$ must be of the form $\{x, y\} = T(x \circ y)$ for some C-linear map $T : A \circ A \to X$ provided that $x \circ y = 0$ implies $\{x, y\} = 0$.

There are various reasons for introducing these concepts. Let us mention one important motivation which can most be easily explained. This is **the connection to the thoroughly studied problems of describing zero (associative, Lie, Jordan) product preserving linear maps**.

Motivation 5.1.4. We say that a linear map S from an algebra A into an algebra B **preserves zero products** if for all $x, y \in A$, $xy = 0$ implies $S(x)S(y) = 0$.

The standard goal is to show that, roughly speaking, S is "close" to a homomorphism. Defining

$$\{.\,,.\} : A \times A \to B \ \text{ by } \ \{x, y\} = S(x)S(y)$$

we see that $\{.\,,.\}$ satisfies (a); now if A is zero product determined, then it follows that

$$S(x)S(y) = T(xy) \ \text{ for all } x,\, y \in A,$$

for some linear map T which brings us quite close to our goal. For example, if we further assume that A and B are unital and $S(1) = 1$, then it follows that

$$T(x) = T(x \cdot 1) = S(x)S(1) = S(x) \ \text{ for every } x \in A.$$

Hence $S = T$ is a homomorphism; let us point out that without this assumption the problem remains nontrivial.

Similar remarks can be stated for zero Lie product preserving maps (also known as commutativity preserving maps) and zero Jordan product preserving maps. The approach that we have just outlined was used in recent papers [2] (for zero product preservers) and [18] (for zero Lie product preservers).

We point out now two general facts about the problem of showing that a bilinear map $\{.\,,.\} : A \times A \to X$ satisfies the condition (b).

Remarks 5.1.5. It is clear that the only possible way of defining $T : A^2 \to X$ is given by $T(\sum_t x_t y_t) = \sum_t \{x_t, y_t\}$. The problem, however, is to show that T is well-defined. Accordingly, (b) is equivalent to the condition

(b') if $x_t,\, y_t \in A,\, t = 1, \ldots, m$, are such that $\sum_{t=1}^{m} x_t y_t = 0$, then $\sum_{t=1}^{m} \{x_t, y_t\} = 0$.

Secondly, if A is a unital algebra, then (b) is equivalent to

(b") if $x_1, x_2, y_1, y_2 \in A$, are such that $\sum_{t=1}^{2} x_t y_t = 0$, then $\sum_{t=1}^{2} \{x_t, y_t\} = 0$.

Indeed, if (b") is fulfilled, then we infer from $x \cdot y - xy \cdot 1 = 0$ that $\{x, y\} - \{xy, 1\} = 0$. Thus $\{x, y\} = T(xy)$ where $T : A^2 \to X$ is defined by $T(z) = \{z, 1\}$.

Incidentally, Lemma 5.4.6 below shows that the assumption that A is unital cannot be omitted. This lemma actually considers the case when A is a Lie algebra. Let us say that the two remarks above hold for algebras that may be nonassociative. In what follows, however, by an **algebra** we will always mean an **associative algebra**.

5.2 Zero (associative) product determined matrix algebras

In what follows, we will consider the matrix algebra $\mathbb{M}_n(B)$ where B is a unital algebra (associative, but not necessarily commutative). As usual, a matrix unit will be denoted by e_{ij}. By be_{ij}, where $b \in B$, we denote the matrix whose (i, j) entry is b and all other entries are 0.

Theorem 5.2.1. *If B is a unital algebra, then $\mathbb{M}_n(B)$ is a zero product determined algebra for every $n \geq 2$.*

Proof. Set $A = \mathbb{M}_n(B)$. Let X be a C-module and let $\{.\,,.\} : A \times A \to X$ be a bilinear map such that for all $x, y \in A$, $xy = 0$ implies $\{x, y\} = 0$. Throughout the proof, a and b will denote arbitrary elements in B and i, j, k, l will denote arbitrary indices.

We begin by noticing that

$$\{ae_{ij}, be_{kl}\} = 0 \quad \text{if } j \neq k, \tag{5.1}$$

since $ae_{ij}\, be_{kl} = 0$. Further, we claim that

$$\{ae_{ij}, be_{jl}\} = \{ab\, e_{ik}, e_{kl}\} \quad \text{if } j \neq k. \tag{5.2}$$

Indeed, as $k \neq j$ we have $(ae_{ij} + ab\, e_{ik})(be_{jl} - e_{kl}) = 0$, which implies $\{ae_{ij} + ab\, e_{ik}, be_{jl} - e_{kl}\} = 0$. Apply (5.1) and (5.2) follows.

Replacing a by ab and b by 1 in (5.2) we get

$$\{abe_{ij}, e_{jl}\} = \{ab\, e_{ik}, e_{kl}\}. \tag{5.3}$$

Together with (5.2) this yields

$$\{ae_{ij}, be_{jl}\} = \{ab\, e_{ij}, e_{jl}\}. \tag{5.4}$$

Let $x_t, y_t \in A$ be such that $\sum_{t=1}^{m} x_t\, y_t = 0$, and let us show that $\sum_{t=1}^{m} \{x_t, y_t\} = 0$ (as pointed out above, we could assume that $m = 2$, but this does not simplify our proof). Writing

$$x_t = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^t\, e_{ij} \quad \text{and} \quad y_t = \sum_{k=1}^{n} \sum_{l=1}^{n} b_{kl}^t\, e_{kl}$$

it follows, by examining the (i, l) entry of $x_t y_t$, that for all i and l we have

$$\sum_{t=1}^{m} \sum_{j=1}^{n} a_{ij}^t\, b_{jl}^t = 0. \tag{5.5}$$

Note that

$$\sum_{t=1}^{m} \{x_t, y_t\} = \sum_{t=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \{a_{ij}^t\, e_{ij}, b_{kl}^t\, e_{kl}\}.$$

By (5.1) this summation reduces to

$$\sum_{t=1}^{m} \{x_t, y_t\} = \sum_{t=1}^{m} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{l=1}^{n} \{a_{ij}^t\, e_{ij}, b_{jl}^t\, e_{jl}\}.$$

Using first (5.4) and then (5.3) we see that

$$\{a_{ij}^t\, e_{ij}, b_{jl}^t\, e_{jl}\} = \{a_{ij}^t b_{jl}^t\, e_{ij}, e_{jl}\} = \{a_{ij}^t b_{jl}^t\, e_{i1}, e_{1l}\}.$$

Therefore

$$\sum_{t=1}^{m}\{x_t,y_t\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{l=1}^{n}\{a_{ij}^t b_{jl}^t\, e_{i1}, e_{1l}\} = \sum_{i=1}^{n}\sum_{l=1}^{n}\left\{\Big(\sum_{t=1}^{m}\sum_{j=1}^{n} a_{ij}^t b_{jl}^t\Big)e_{i1}, e_{1l}\right\} = 0 \quad \text{by } (5.5).$$

□

5.3 Zero Jordan product determined matrix algebras

In the recent paper [33] M. A. Chebotar, W.-F. Ke, P.-H. Lee and R.-B. Zhang have considered zero Jordan product preserving maps on matrix algebras. Fortunately, some arguments from this paper are almost directly applicable to the more general situation treated here. There is one problem, however, which we have to face: unlike in [33], where the map $\{x, y\} = S(x) \circ S(y)$ is studied, we cannot assume in advance that our map $\{.,.\}$ treated below is symmetric (in the sense that $\{x, y\} = \{y, x\}$ for all x and y).

Theorem 5.3.1. *If B is a unital algebra containing the element $\frac{1}{2}$, then $\mathbb{M}_n(B)$ is a zero Jordan product determined algebra for every $n \geq 3$.*

Proof. Let $A = \mathbb{M}_n(B)$, let X be a C-module, and let $\{.,.\} : A \times A \to X$ be a bilinear map such that for all $x, y \in A$, $x \circ y = 0$ implies $\{x, y\} = 0$. Let a and b denote arbitrary elements from B and let i, j, k, l denote arbitrary indices.

First, since $ae_{ij} \circ be_{kl} = 0$ if $i \neq l$ and $j \neq k$, it is clear that

$$\{ae_{ij}, be_{kl}\} = 0 \text{ if } i \neq l \text{ and } j \neq k. \tag{5.6}$$

Let $i \neq k$. Then $ae_{ik} \circ (e_{kk} - e_{ii}) = 0$ and so

$$\{ae_{ik}, e_{kk}\} = \{ae_{ik}, e_{ii}\}. \tag{5.7}$$

Similarly,

$$\{e_{kk}, ae_{ik}\} = \{e_{ii}, ae_{ik}\}. \tag{5.8}$$

From $(ae_{ik} - e_{ii}) \circ (ae_{ik} + e_{kk}) = 0$, $i \neq k$, we derive $\{ae_{ik} - e_{ii}, ae_{ik} + e_{kk}\} = 0$. Since $\{ae_{ik}, ae_{ik}\} = 0$ and $\{e_{ii}, e_{kk}\} = 0$ by (5.6), it follows that $\{ae_{ik}, e_{kk}\} = \{e_{ii}, ae_{ik}\}$. This identity together with (5.7) and (5.8) yields

$$\{ae_{ik}, e_{ii}\} = \{ae_{ik}, e_{kk}\} = \{e_{ii}, ae_{ik}\} = \{e_{kk}, ae_{ik}\}. \tag{5.9}$$

Now let $i \neq k$ and $j \neq k$. Then $(ae_{ij} + abe_{ik}) \circ (be_{jk} - e_{kk}) = 0$, and hence $\{ae_{ij} + ab\, e_{ik}, be_{jk} - e_{kk}\} = 0$. By (5.6) this reduces to $\{ae_{ij}, be_{jk}\} = \{ab\, e_{ik}, e_{kk}\}$. On the other hand, we also have $(be_{jk} - e_{kk}) \circ (ae_{ij} + abe_{ik}) = 0$, and so $\{be_{jk} - e_{kk}, ae_{ij} + abe_{ik}\} = 0$. By (5.6) this reduces to $\{be_{jk}, ae_{ij}\} = \{e_{kk}, abe_{ik}\}$. Since $\{ab\, e_{ik}, e_{kk}\} = \{e_{kk}, abe_{ik}\}$ by (5.9), it follows that

$$\{ae_{ij}, be_{jk}\} = \{ab\, e_{ik}, e_{kk}\} = \{be_{jk}, ae_{ij}\} \quad \text{if } i \neq k \text{ and } j \neq k. \tag{5.10}$$

If $i \neq k$, then $(ae_{ik} - e_{ii}) \circ (abe_{ik} + be_{kk}) = 0$ and $(abe_{ik} + be_{kk}) \circ (ae_{ik} - e_{ii}) = 0$. By a similar argument as before this yields

$$\{ae_{ik}, be_{kk}\} = \{ab\, e_{ik}, e_{kk}\} = \{be_{kk}, ae_{ik}\} \quad \text{if } i \neq k. \tag{5.11}$$

Setting $i = j$ in (5.10) we get

$$\{ae_{ii}, be_{ik}\} = \{abe_{ik}, e_{kk}\} = \{be_{ik}, ae_{ii}\} \quad \text{if} \quad i \neq k.$$

Further, $\{abe_{ik}, e_{kk}\} = \{abe_{ik}, e_{ii}\}$ by (5.9), and so we have

$$\{ae_{ii}, be_{ik}\} = \{abe_{ik}, e_{ii}\} = \{be_{ik}, ae_{ii}\}.$$

For our purposes it is more convenient to rewrite this identity so that the roles of i and k, and the roles of a and b are replaced. Hence we have

$$\{be_{kk}, a_{ki}\} = \{bae_{ki}, e_{kk}\} = \{ae_{ki}, be_{kk}\} \quad \text{if } i \neq k. \tag{5.12}$$

Further, we claim that

$$\{ae_{ij}, be_{ji}\} = \frac{1}{2}\left(\{abe_{ii}, e_{ii}\} + \{bae_{jj}, e_{jj}\}\right). \tag{5.13}$$

If $i \neq j$, then $\left(\frac{1}{2}abe_{ii} + ae_{ij} - \frac{1}{2}bae_{jj}\right) \circ (be_{ji} - e_{ii} + e_{jj}) = 0$ and consequently

$$\left\{\frac{1}{2}ab\, e_{ii} + ae_{ij} - \frac{1}{2}ba\, e_{jj},\, be_{ji} - e_{ii} + e_{jj}\right\} = 0.$$

Using (5.6), (5.9), (5.10), (5.11) and (5.12) this yields

$$\{ae_{ij}, be_{ji}\} = \frac{1}{2}\left(\{abe_{ii}, e_{ii}\} + \{bae_{jj}, e_{jj}\}\right).$$

We still have to prove (5.13) for $i = j$.

Let $i \neq k$. Then $(ae_{ii} - be_{ik} + be_{ki} - ae_{kk}) \circ (be_{ii} - ae_{ik} + ae_{ki} - be_{kk}) = 0$ and this gives $\{ae_{ii} - be_{ik} + be_{ki} - ae_{kk},\, be_{ii} - ae_{ik} + ae_{ki} - be_{kk}\} = 0$. By (5.6), (5.9), (5.10) and (5.11) this can be reduced to

$$\{ae_{ii}, be_{ii}\} + \{ae_{jj}, be_{jj}\} = \frac{1}{2}\left(\{(a \circ b)e_{ii}, e_{ii}\} + \{(a \circ b)e_{jj}, e_{jj}\}\right). \tag{5.14}$$

Since $n \geq 3$, we can choose l such that $l \notin \{i, k\}$. Applying (5.14) we get

$$\begin{aligned}
&(\{ae_{ii}, be_{ii}\} + \{ae_{kk}, be_{kk}\}) + (\{ae_{ii}, be_{ii}\} + \{ae_{ll}, be_{ll}\}) = \\
&\frac{1}{2}\left(\{(a \circ b)e_{ii}, e_{ii}\}) + \{(a \circ b)e_{kk}, e_{kk}\}\right) + \\
&\frac{1}{2}\left(\{(a \circ b)e_{ii}, e_{ii}\} + \{(a \circ b)e_{ll}, e_{ll}\}\right) = \\
&\{(a \circ b)e_{ii}, e_{ii}\} + \frac{1}{2}\left(\{(a \circ b)e_{kk}, e_{kk}\} + \{(a \circ b)e_{ll}, e_{ll}\}\right) = \\
&\{(a \circ b)e_{ii}, e_{ii}\} + \{ae_{kk}, be_{kk}\} + \{ae_{ll}, be_{ll}\}.
\end{aligned}$$

Consequently, $\{ae_{ii}, be_{ii}\} = \frac{1}{2}\{(a \circ b)e_{ii}, e_{ii}\}$ which proves the $i = j$ case of (5.13).

Let $x_t, y_t \in A$ be such that $\sum_{t=1}^{m} x_t \circ y_t = 0$. We have to prove that $\sum_{t=1}^{m}\{x_t, y_t\} = 0$. Writing

$$x_t = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^t\, e_{ij} \quad \text{and} \quad y_t = \sum_{k=1}^{n}\sum_{l=1}^{n} b_{kl}^t\, e_{kl}$$

it follows that for all i and l we have

$$\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^{t}b_{jl}^{t}+b_{ij}^{t}a_{jl}^{t})=0. \tag{5.15}$$

First notice that

$$\sum_{t=1}^{m}\{x_t,y_t\}=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\{a_{ij}^{t}\,e_{ij},b_{kl}^{t}\,e_{kl}\}$$

and by (5.6) this summation reduces to

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{a_{ij}^{t}\,e_{ij},b_{jl}^{t}\,e_{jl}\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\ k\neq j}}^{n}\{a_{ij}^{t}\,e_{ij},b_{ki}^{t}\,e_{ki}\}+$$
$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^{t}\,e_{ij},b_{ji}^{t}\,e_{ji}\}.$$

Using (5.10) and (5.11) in the first two summations and (5.13) in the third summation, we see that this is further equal to

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{a_{ij}^{t}b_{jl}^{t}\,e_{il},e_{ll}\}+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\ k\neq j}}^{n}\{b_{ki}^{t}a_{ij}^{t}\,e_{kj},\,e_{jj}\}+$$
$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{1}{2}\left(\{a_{ij}^{t}b_{ji}^{t}\,e_{ii},e_{ii}\}+\{b_{ji}^{t}a_{ij}^{t}\,e_{jj},e_{jj}\}\right)\right).$$

Rewriting the second summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{b_{ij}^{t}a_{jl}^{t}\,e_{il},e_{ll}\},$$

and the third summation as

$$\frac{1}{2}\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}(\{a_{ij}^{t}b_{ji}^{t}\,e_{ii},e_{ii}\}+\{b_{ij}^{t}a_{ji}^{t}\,e_{ii},e_{ii}\}),$$

it follows that

$$\begin{aligned}\sum_{t=1}^{m}\{x_t, y_t\} &= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{(a_{ij}^t b_{jl}^t + b_{ij}^t a_{jl}^t)\, e_{il}, e_{ll}\} + \\ &\frac{1}{2}\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{(a_{ij}^t b_{ji}^t + b_{ij}^t a_{ji}^t)\, e_{ii}, e_{ii}\} = \\ &\sum_{i=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\left\{\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^t b_{jl}^t + b_{ij}^t a_{jl}^t)\, e_{il},\, e_{ll}\right\} + \\ &\frac{1}{2}\sum_{i=1}^{n}\left\{\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^t b_{ji}^t + b_{ij}^t a_{ji}^t)\, e_{ii},\, e_{ii}\right\};\end{aligned}$$

each of these two summations is 0 by (5.15). □

We were unable to find out whether or not Theorem 5.3.1 also holds for $n = 2$; therefore we leave this as an open problem.

5.4 Zero Lie product determined matrix algebras

In the preceding sections, we have shown that (square) matrix algebras (over unital algebras) are always zero product determined, and under some technical restrictions they are also zero Jordan product determined. At this point, natural questions arise:

What can we say about the Lie product? Are $\mathbb{M}_n(B)$ zero Lie product determined?

We will show that this does not hold true for every unital algebra B. However, we will prove that $\mathbb{M}_n(B)$ is zero Lie product determined provided B is so.

Theorem 5.4.1. *If B is a zero Lie product determined unital algebra, then $\mathbb{M}_n(B)$ is a zero Lie product determined algebra for every $n \geq 2$.*

Proof. Let $A = \mathbb{M}_n(B)$, let X a C-module, and let $\{.\,,.\}: A \times A \to X$ be a bilinear map such that $\{x,\, y\} = 0$ whenever $x,\, y \in A$ are such that $[x, y] = 0$.

First notice that $\{x,\, x\} = 0$ for all $x \in A$, and hence $\{x,\, y\} = -\{y,\, x\}$ for all $x,\, y \in A$. Further, the equality $\{x^2,\, x\} = 0$ holds for all $x \in A$, and linearizing it we get

$$\{x \circ y,\, z\} + \{z \circ x,\, y\} + \{y \circ z,\, x\} = 0$$

for all $x,\, y,\, z \in A$. We shall use these identities without mention.

Our first goal is to derive various identities involving elements of the form ae_{ij}. In what follows a and b will be arbitrary elements in B and i, j, k, l will be arbitrary indices.

First, it is clear that

$$\{ae_{ij},\, be_{kl}\} = 0 \quad \text{if } j \neq k \text{ and } i \neq l \tag{5.16}$$

since $[ae_{ij},\, be_{kl}] = 0$. Similarly,

$$\{ae_{ii}, e_{ii}\} = 0. \tag{5.17}$$

Also, if $i \neq j$, then $[a\, e_{ij} + a\, e_{ji},\, e_{ij} + e_{ji}] = 0$, and so

$$\{a\, e_{ij} + a\, e_{ji},\, e_{ij} + e_{ji}\} = 0.$$

As $\{a\, e_{ij},\, e_{ij}\} = 0$ and $\{a\, e_{ji},\, e_{ji}\} = 0$ by (5.16), it follows that

$$\{a\, e_{ij},\, e_{ji}\} = -\{a\, e_{ji},\, e_{ij}\} \quad \text{if } i \neq j. \tag{5.18}$$

Next, we claim that

$$\{ae_{ij},\, be_{jk}\} = \{ab\, e_{ik},\, e_{kk}\} = -\{ab\, e_{ik},\, e_{ii}\} \quad \text{if } i \neq k. \tag{5.19}$$

Indeed, since $[ab\, e_{ik},\, e_{ii} + e_{kk}] = 0$ we have $\{ab\, e_{ik},\, e_{ii} + e_{kk}\} = 0$, and so $\{ab\, e_{ik},\, e_{kk}\} = -\{ab\, e_{ik},\, e_{ii}\}$. We now consider two cases, when $j \neq k$ and when $j = k$. In the first case we have, since also $i \neq k$,

$$[ae_{ij} + ab\, e_{ik},\, be_{jk} - e_{kk}] = 0, \quad \text{and hence } \{ae_{ij} + ab\, e_{ik},\, be_{jk} - e_{kk}\} = 0.$$

From (5.16) it follows that $\{ae_{ij},\, e_{kk}\} = 0$ and $\{ab\, e_{ik},\, be_{jk}\} = 0$, and so the identity above reduces to

$$\{ae_{ij},\, be_{jk}\} = \{ab\, e_{ik},\, e_{kk}\}.$$

In the second case, when $j = k$, we have $[ae_{ik} - e_{ii},\, ab\, e_{ik} + be_{kk}] = 0$, which implies $\{ae_{ik} - e_{ii},\, ab\, e_{ik} + be_{kk}\} = 0$. Since $\{ae_{ik},\, ab\, e_{ik}\} = 0$ and $\{e_{ii}, be_{kk}\} = 0$ by (5.16), it follows that

$$\{ae_{ik},\, be_{kk}\} = \{e_{ii},\, ab\, e_{ik}\} = -\{ab\, e_{ik},\, e_{ii}\},$$

and (5.19) is thereby proved.

Let us prove that

$$\{ae_{ij},\, be_{ji}\} = \{ab\, e_{ij},\, e_{ji}\} + \{a\, e_{jj},\, b\, e_{jj}\}. \tag{5.20}$$

In view of (5.17) we may assume that $i \neq j$. Then we have

$$\{ae_{ij}, be_{ji}\} = \{e_{ij} \circ ae_{jj},\, be_{ji}\} = -\{be_{ji} \circ e_{ij},\, ae_{jj}\} - \{a\, e_{jj} \circ be_{ji},\, e_{ij}\}.$$

Since $\{be_{ii},\, ae_{jj}\} = 0$ by (5.16) and $\{ab\, e_{ij},\, e_{ji}\} = -\{ab\, e_{ji},\, e_{ij}\}$ by (5.18), (5.20) follows.

Finally, we claim that

$$\{ae_{ij},\, be_{ji}\} = \{ab\, e_{ik},\, e_{ki}\} - \{ba\, e_{jk},\, e_{kj}\} + \{a\, e_{kk},\, b\, e_{kk}\}. \tag{5.21}$$

Assume first that $i \neq j$. Taking into account (5.17) and (5.20) we see that (5.21) holds if $k = i$ or $k = j$. If $k \neq i$ and $k \neq j$, then

$$\{a\, e_{ij},\, b\, e_{ji}\} = \{a\, e_{ik} \circ e_{kj},\, be_{ji}\} = -\{ba\, e_{jk},\, e_{kj}\} + \{a\, e_{ik},\, b\, e_{ki}\},$$

and so applying (5.20) we get (5.21). Now suppose that $i = j$. Then

$$\{a\, e_{ii},\, b\, e_{ii}\} = \{a\, e_{ik} \circ e_{ki},\, b\, e_{ii}\} = -\{ba\, e_{ik},\, e_{ki}\} + \{a\, e_{ik},\, b\, e_{ki}\}.$$

From (5.20) it follows that

$$\{a\,e_{ii},\, b\,e_{ii}\} = \{ab\,e_{ik},\, e_{ki}\} - \{ba\,e_{ik},\, e_{ki}\} + \{a\,e_{kk},\, b\,e_{kk}\},$$

and so (5.21) holds is this case as well.

Now pick $x_t,\, y_t \in A$ such that $\sum_{t=1}^{m}[x_t,\, y_t] = 0$. The theorem will be proved by showing that $\sum_{t=1}^{m}\{x_t,\, y_t\} = 0$. Write

$$x_t = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^t e_{ij} \quad \text{and} \quad y_t = \sum_{k=1}^{n}\sum_{l=1}^{n} b_{kl}^t e_{kl}$$

where $a_{ij}^t,\, b_{kl}^t \in B$. Computing the (i,l) entry of $[x_t, y_t]$ we see that

$$\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^t b_{jl}^t - b_{ij}^t a_{jl}^t) = 0 \text{ for all } i,\, l. \tag{5.22}$$

By (5.16) we have

$$\begin{aligned}\sum_{t=1}^{m}\{x_t,\, y_t\} \;&=\; \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n}\{a_{ij}^t e_{ij},\, b_{kl}^t e_{kl}\} = \\ &\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{a_{ij}^t e_{ij},\, b_{jl}^t e_{jl}\} + \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{k=1\\ k\neq j}}^{n}\{a_{ij}^t e_{ij},\, b_{ki}^t e_{ki}\} \\ &+\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^t e_{ij},\, b_{ji}^t e_{ji}\}.\end{aligned}$$

Rewriting the second summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{a_{jl}^t e_{jl},\, b_{ij}^t e_{ij}\} = -\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{b_{ij}^t e_{ij},\, a_{jl}^t e_{jl}\},$$

and using (5.19) we see that the sum of the first and the second summation is equal to

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}(\{a_{ij}^{t}b_{jl}^{t}e_{il},\, e_{ll}\}-\{b_{ij}^{t}a_{jl}^{t}e_{il},\, e_{ll}\})=$$
$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\{(a_{ij}^{t}b_{jl}^{t}-b_{ij}^{t}a_{jl}^{t})e_{il},\, e_{ll}\}=$$
$$\sum_{i=1}^{n}\sum_{\substack{l=1\\ l\neq i}}^{n}\left\{\left(\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^{t}b_{jl}^{t}-b_{ij}^{t}a_{jl}^{t})\right)e_{il},\, e_{ll}\right\}=0$$

by (5.22). Hence

$$\sum_{t=1}^{m}\{x_t,\, y_t\}=\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^{t}e_{ij},\, b_{ji}^{t}e_{ji}\}.$$

We claim that this sum is equal to zero. Applying (5.21) we have that

$$\{a_{ij}^{t}e_{ij},\, b_{ji}^{t}e_{ji}\}=\{a_{ij}^{t}b_{ji}^{t}e_{i1},\, e_{1i}\}-\{b_{ji}^{t}a_{ij}^{t}e_{j1},\, e_{1j}\}+\{a_{ij}^{t}\, e_{11},\, b_{ji}^{t}\, e_{11}\}.$$

Therefore

$$\begin{aligned}\sum_{t=1}^{m}\{x_t,\, y_t\} &= \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^{t}b_{ji}^{t}e_{i1},\, e_{1i}\}-\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{b_{ji}^{t}a_{ij}^{t}e_{j1},\, e_{1j}\}+\\ &\quad \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^{t}\, e_{11},\, b_{ji}^{t}\, e_{11}\}.\end{aligned}$$

Rewriting the second summation as

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{b_{ij}^{t}a_{ji}^{t}e_{i1},\, e_{1i}\}$$

and applying (5.22), we obtain

$$\sum_{t=1}^{m}\{x_t,\, y_t\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{(a_{ij}^{t}b_{ji}^{t}-b_{ij}^{t}a_{ji}^{t})e_{i1},\, e_{1i}\}+$$

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^t\, e_{11},\, b_{ji}^t\, e_{11}\} =$$
$$\sum_{i=1}^{n}\left\{\left(\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^t b_{ji}^t - b_{ij}^t a_{ji}^t)\right) e_{i1},\, e_{1i}\right\} +$$
$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^t\, e_{11},\, b_{ji}^t\, e_{11}\} = \sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^t\, e_{11},\, b_{ji}^t\, e_{11}\}.$$

Thus, the proof will be complete by showing that

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\{a_{ij}^t\, e_{11},\, b_{ji}^t\, e_{11}\} = 0. \tag{5.23}$$

Consider the map $\langle\, .\, ,.\rangle : B \times B \to X$ defined by $\langle a,\, b\rangle = \{a\, e_{11},\, b\, e_{11}\}$ for all $a, b \in B$. It is clear that $\langle\, .\, ,.\rangle$ is bilinear and has the property that $[a, b] = 0$ implies $\langle a, b\rangle = 0$. Since B is a zero Lie product determined algebra, $\langle\, .\, ,.\rangle$ also satisfies the condition that $\sum_{t=1}^{m}[a_t,\, b_t] = 0$ implies $\sum_{t=1}^{m}\langle a_t,\, b_t\rangle = 0$. Taking $l = i$ in (5.22) we have that

$$\sum_{t=1}^{m}\sum_{j=1}^{n}(a_{ij}^t b_{ji}^t - b_{ij}^t a_{ji}^t) = 0$$

for every i, and hence

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}[a_{ij}^t,\, b_{ji}^t] = 0.$$

This implies

$$\sum_{t=1}^{m}\sum_{i=1}^{n}\sum_{j=1}^{n}\langle a_{ij}^t,\, b_{ji}^t\rangle = 0,$$

which is of course equivalent to (5.23). □

As consequence, we have

Corollary 5.4.2. *If B is a commutative unital algebra, then $\mathbb{M}_n(B)$ is a zero Lie product determined algebra for every $n \geq 2$.*

Proof. Note that commutative algebras are trivially zero Lie product determined. Thus the conclusion follows from Theorem 5.4.1. □

Remark 5.4.3. In the simplest case where $B = C$ this corollary was proved in [18]. In fact, for this case [18, Theorem 2.1] tells us more than Corollary 5.4.2. In particular it states that for a C-bilinear map $\{.,.\} : A \times A \to X$, where $A = M_n(C)$ and X is a C-module, the following conditions are equivalent:

(a) if $x, y \in A$ are such that $[x, y] = 0$, then $\{x,\, y\} = 0$;

(b) there is a C-linear map $T : [A, A] \to X$ such that $\{x,\, y\} = T([x, y])$ for all $x, y \in A$;

(c) $\{x, x\} = \{x^2, x\} = 0$ for all $x \in A$;

(d) $\{x, x\} = \{xy, z\} + \{zx, y\} + \{yz, x\} = 0$ for all $x, y, z \in A$.

The condition (c) has proved to be important because of the applications to the commutativity preserving map problem. So it is tempting to try to show that these conditions are equivalent in some more general algebras A.

We remark that trivially (b) implies (c) and (d), (a) implies (c), and also (d) implies (c) as long as A is 3-torsion-free (just set $x = y = z$ in (d)). In the next example we show that in the algebra $\mathbb{M}_2(C[x, y])$ neither (c) nor (d) implies (a), and so [18, Theorem 2.1] cannot be generalized to matrix algebras over commutative algebras.

Example 5.4.4. Let $A = \mathbb{M}_2(C[x, y])$. We define a C-bilinear map

$$\{.,.\} : A \times A \to C$$

as follows:

$$\{xe_{11}, ye_{11}\} = \{xe_{22}, ye_{22}\} = 1, \quad \{ye_{11}, xe_{11}\} = \{ye_{22}, xe_{22}\} = -1,$$
$$\{xe_{12}, ye_{21}\} = \{xe_{21}, ye_{12}\} = 1, \quad \{ye_{21}, xe_{12}\} = \{ye_{12}, xe_{21}\} = -1,$$

and

$$\{ue_{ij}, ve_{kl}\} = 0$$

in all other cases, that is, for all remaining choices of monomials u and v and $i, j, k, l \in \{1, 2\}$. Since $[xe_{11}, ye_{11}] = 0$ and $\{xe_{11}, ye_{11}\} = 1$, $\{.,.\}$ does not satisfy (a) (or (b)). However, as we check below the map $\{.,.\}$ satisfies (c) and (d).

Proof. In order to show (c); take $X = ae_{11} + be_{12} + ce_{21} + de_{22}$ be in A and notice that the only coefficients of a, b, c and d involved in our computations are the respective ones to 1, x and y so let us write

$$a = \alpha_0 + \alpha_1 x + \alpha_2 y + \ldots$$

$$b = \beta_0 + \beta_1 x + \beta_2 y + \ldots$$

$$c = \gamma_0 + \gamma_1 x + \gamma_2 y + \ldots$$

$$d = \delta_0 + \delta_1 x + \delta_2 y + \ldots$$

where α_i, β_i, γ_i, $\delta_i \in C$ for $i = 0, 1, 2$. Then we compute:

$$\begin{aligned}\{X, X\} &= \alpha_1\alpha_2\{xe_{11}, ye_{11}\} + \alpha_2\alpha_1\{ye_{11}, xe_{11}\} + \beta_1\gamma_2\{xe_{12}, ye_{21}\} + \\ &\quad \beta_2\gamma_1\{ye_{12}, xe_{21}\} + \gamma_1\beta_2\{xe_{21}, ye_{12}\} + \gamma_2\beta_1\{ye_{21}, xe_{12}\} + \\ &\quad \delta_1\delta_2\{xe_{22}, ye_{22}\} + \delta_2\delta_1\{ye_{22}, xe_{22}\} = \alpha_1\alpha_2 - \alpha_2\alpha_1 + \beta_1\gamma_2 - \\ &\quad \beta_2\gamma_1 + \gamma_1\beta_2 - \gamma_2\beta_1 + \delta_1\delta_2 - \delta_2\delta_1 = 0.\end{aligned}$$

Since $X^2 = (a^2 + bc)e_{11} + (ab + bd)e_{12} + (ac + cd)e_{21} + (bc + d^2)e_{22}$ we may write

$$\begin{aligned}a^2 + bc &= \alpha_0^2 + \beta_0\gamma_0 + (2\alpha_0\alpha_1 + \beta_0\gamma_1 + \beta_1\gamma_0)\,x + \\ &\quad (2\alpha_0\alpha_2 + \beta_0\gamma_2 + \beta_2\gamma_0)\,y + \ldots \\ ab + bd &= \alpha_0\beta_0 + \beta_0\delta_0 + (\alpha_0\beta_1 + \alpha_1\beta_0 + \beta_0\delta_1 + \beta_1\delta_0)\,x +\end{aligned}$$

$$
\begin{aligned}
& (\alpha_0\beta_2 + \alpha_2\beta_0 + \beta_0\delta_2 + \beta_2\delta_0)\, y + \ldots \\
ac + cd &= \alpha_0\gamma_0 + \gamma_0\delta_0 + (\alpha_0\gamma_1 + \alpha_1\gamma_0 + \gamma_0\delta_1 + \gamma_1\delta_0)\, x + \\
& (\alpha_0\gamma_2 + \alpha_2\gamma_0 + \gamma_0\delta_2 + \gamma_2\delta_0)\, y + \ldots \\
bc + d^2 &= \beta_0\gamma_0 + \delta_0^2 + (\beta_0\gamma_1 + \beta_1\gamma_0 + 2\delta_0\delta_1)\, x + \\
& (\beta_0\gamma_2 + \beta_2\gamma_0 + 2\delta_0\delta_2)\, y + \ldots
\end{aligned}
$$

So, we have

$$
\begin{aligned}
\{X^2, X\} &= (2\alpha_0\alpha_1\alpha_2 + \beta_0\gamma_1\alpha_2 + \beta_1\gamma_0\alpha_2)\{xe_{11}, ye_{11}\} + \\
& (2\alpha_1\alpha_0\alpha_2 + \alpha_1\beta_0\gamma_2 + \alpha_1\beta_2\gamma_0)\{ye_{11}, xe_{11}\} + \\
& (\alpha_0\beta_1\gamma_2 + \alpha_1\beta_0\gamma_2 + \beta_0\delta_1\gamma_2 + \beta_1\delta_0\gamma_2)\{xe_{12}, ye_{21}\} + \\
& (\alpha_0\gamma_2\beta_1 + \alpha_2\gamma_0\beta_1 + \gamma_0\delta_2\beta_1 + \gamma_2\delta_0\beta_1)\{ye_{21}, xe_{12}\} + \\
& (\alpha_0\gamma_1\beta_2 + \alpha_1\gamma_0\beta_2 + \gamma_0\delta_1\beta_2 + \gamma_1\delta_0\beta_2)\{xe_{21}, ye_{12}\} + \\
& (\alpha_0\beta_2\gamma_1 + \alpha_2\beta_0\gamma_1 + \beta_0\delta_2\gamma_1 + \beta_2\delta_0\gamma_1)\{ye_{12}, xe_{21}\} + \\
& (\beta_0\gamma_1\delta_2 + \beta_1\gamma_0\delta_2 + 2\delta_0\delta_1\delta_2)\{xe_{22}, ye_{22}\} + \\
& (\beta_0\gamma_2\delta_1 + \beta_2\gamma_0\delta_1 + 2\delta_0\delta_2\delta_1)\{ye_{22}, xe_{22}\} = 2\alpha_0\alpha_1\alpha_2 + \beta_0\gamma_1\alpha_2 + \\
& \beta_1\gamma_0\alpha_2 - 2\alpha_1\alpha_0\alpha_2 - \alpha_1\beta_0\gamma_2 - \alpha_1\beta_2\gamma_0 + \alpha_0\beta_1\gamma_2 + \alpha_1\beta_0\gamma_2 + \\
& \beta_0\delta_1\gamma_2 + \beta_1\delta_0\gamma_2 - \alpha_0\gamma_2\beta_1 - \alpha_2\gamma_0\beta_1 - \gamma_0\delta_2\beta_1 - \gamma_2\delta_0\beta_1 + \\
& \alpha_0\gamma_1\beta_2 + \alpha_1\gamma_0\beta_2 + \gamma_0\delta_1\beta_2 + \gamma_1\delta_0\beta_2 - \alpha_0\beta_2\gamma_1 - \alpha_2\beta_0\gamma_1 - \\
& \beta_0\delta_2\gamma_1 - \beta_2\delta_0\gamma_1 + \beta_0\gamma_1\delta_2 + \beta_1\gamma_0\delta_2 + 2\delta_0\delta_1\delta_2 - \beta_0\gamma_2\delta_1 - \\
& \beta_2\gamma_0\delta_1 - 2\delta_0\delta_2\delta_1 = 0.
\end{aligned}
$$

Hence, (c) is satisfied. □

We will finish this chapter by finding an example of a unital algebra B such that $\mathbb{M}_n(B)$ is not a zero Lie product determined algebra, which thereby shows that indeed one has to impose some condition on B in Theorem 5.4.1.

For this we need two preliminary results which are of independent interest. The first one, however, is not really surprising, and possibly it is already known. Anyway, the following proof, which was suggested to us by M. A. Chebotar, is very short.

Until the end of this section we are going to assume that C is a field. Let us denote it by F.

Lemma 5.4.5. *Let $A = F\langle x_1, x_2, \ldots, x_{2n}\rangle$ be a free algebra in $2n$ non-commuting indeterminates. Then $[x_1, x_2] + [x_3, x_4] + \ldots + [x_{2n-1}, x_{2n}]$ cannot be written as a sum of less than n commutators of elements in A.*

Proof. Let $a_i, b_i \in A$, $i = 1, \ldots, m$, be such that

$$[a_1, b_1] + [a_2, b_2] + \ldots + [a_m, b_m] = [x_1, x_2] + [x_3, x_4] + \ldots + [x_{2n-1}, x_{2n}]. \quad (5.24)$$

We have to show that $m \geq n$. We proceed by induction on n. The case when $n = 1$ is trivial, so we may assume that $n > 1$. Considering the degrees of monomials appearing in (5.24) we see that we may assume that all a_i's and b_i's are linear combinations of the x_i's. In particular, $b_m = \sum_{j=1}^{2n} \mu_j x_j$ with $\mu_j \in F$. Without loss of generality we may assume that $\mu_{2n} \neq 0$. Of course, we may replace any indeterminate x_i by any element in A in the identity (5.24). So, let us substitute 0 for x_{2n-1} and $-\sum_{j=1}^{2n-2} \mu_{2n}^{-1}\mu_j x_j$ for x_{2n}. Then we get

$$[c_1, d_1] + \ldots + [c_{m-1}, d_{m-1}] = [x_1, x_2] + \ldots + [x_{2n-3}, x_{2n-2}]$$

where all c_i's and d_i's are linear combinations of $x_1, \ldots, x_{2n-2}$. By induction assumption we thus have $m - 1 \geq n - 1$, and so $m \geq n$. □

For any $n \geq 2$, let B_n denote the unital F-algebra generated by $1, u_1, \ldots, u_{2n}$ with the relation $[u_1, u_2] + [u_3, u_4] + \ldots + [u_{2n-1}, u_{2n}] = 0$. That is,

$$B_n = F\langle x_1, x_2, \dots, x_{2n}\rangle / I$$

where I is the ideal of $F\langle x_1, x_2, \dots, x_{2n}\rangle$ generated by

$$[x_1, x_2] + [x_3, x_4] + \dots + [x_{2n-1}, x_{2n}],$$

and $u_i = x_i + I$.

Lemma 5.4.6. *There exists a bilinear map* $\langle .,. \rangle : B_n \times B_n \to F$ *such that for all* $v_t, w_t \in B_n$, $\sum_{t=1}^{n-1}[v_t, w_t] = 0$ *implies* $\sum_{t=1}^{n-1}\langle v_t, w_t\rangle = 0$, *but* $\langle u_1, u_2\rangle + \langle u_3, u_4\rangle + \dots + \langle u_{2n-1}, u_{2n}\rangle \neq 0$. *Moreover, there is no linear map* $T : [B_n, B_n] \to F$ *such that* $\langle x, y\rangle = T([x, y])$.

Proof. The set S consisting of 1 and all possible products $u_{i_1} \dots u_{i_k}$ of the u_i's spans the linear space B_n, and the elements u_1, u_2 are linearly independent. Therefore we can define a bilinear map $\langle .,. \rangle : B_n \times B_n \to F$ such that $\langle u_1, u_2\rangle = -\langle u_2, u_1\rangle = 1$ and $\langle s, t\rangle = 0$ for all other possible choices of $s, t \in S$. In particular, $\langle u_1, u_2\rangle + \langle u_3, u_4\rangle + \dots + \langle u_{2n-1}, u_{2n}\rangle = 1$.

Assume now that $v_t, w_t \in B_n$ are such that $\sum_{t=1}^{n-1}[v_t, w_t] = 0$. We can write

$$v_t = \lambda_t u_1 + \mu_t u_2 + p_t \quad \text{and} \quad w_t = \alpha_t u_1 + \beta_t u_2 + q_t,$$

where $\lambda_t, \mu_t, \alpha_t, \beta_t \in F$ and p_t, q_t lie in the linear span of $S \setminus \{u_1, u_2\}$. Note that

$$\sum_{t=1}^{n-1}\langle v_t, w_t\rangle = \sum_{t=1}^{n-1}(\lambda_t\beta_t - \mu_t\alpha_t).$$

Thus, the lemma will be proved by showing that $\sum_{t=1}^{n-1}(\lambda_t\beta_t - \mu_t\alpha_t) = 0$.

Let us write

$$v_t = \lambda_t x_1 + \mu_t x_2 + l_t + f_t + I \quad \text{and} \quad w_t = \alpha_t x_1 + \beta_t x_2 + m_t + g_t + I,$$

where $\lambda_t, \mu_t, \alpha_t, \beta_t \in F$, l_t, m_t are linear combinations of $x_3, \ldots, x_{2n}$ and f_t, g_t are linear combinations of monomials of degrees 0 or at least 2. Since $\sum_{t=1}^{n-1}[v_t, w_t] = 0$, it follows that

$$\sum_{t=1}^{n-1} [\lambda_t x_1 + \mu_t x_2 + l_t + f_t, \alpha_t x_1 + \beta_t x_2 + m_t + g_t] \in I.$$

Therefore,

$$\sum_{t=1}^{n-1}[\lambda_t x_1 + \mu_t x_2 + l_t + f_t, \alpha_t x_1 + \beta_t x_2 + m_t + g_t] = \omega\Big([x_1, x_2] + [x_3, x_4] + \ldots + [x_{2n-1}, x_{2n}]\Big) + h,$$

where $\omega \in F$ and $h \in I$ is a linear combination of monomials of degree at least 3. Considering the degrees of monomials involved in this identity it clearly follows that

$$\sum_{t=1}^{n-1}[\lambda_t x_1+\mu_t x_2+l_t, \alpha_t x_1+\beta_t x_2+m_t] = \omega\Big([x_1, x_2]+[x_3, x_4]+\ldots+[x_{2n-1}, x_{2n}]\Big).$$

We may now apply Lemma 5.4.5 and conclude that $\omega = 0$. Thus,

$$0 = \sum_{t=1}^{n-1}[\lambda_t x_1 + \mu_t x_2 + l_t, \alpha_t x_1 + \beta_t x_2 + m_t] = \Big(\sum_{t=1}^{n-1}(\lambda_t\beta_t - \mu_t\alpha_t)\Big)[x_1, x_2] + f,$$

where f is a linear combination of monomials different from x_1x_2 and x_2x_1. Consequently, $\sum_{t=1}^{n-1}(\lambda_t\beta_t - \mu_t\alpha_t) = 0$.

Finally, if $T : [B_n, B_n] \to F$ is a linear map satisfying that

$$\langle x, y\rangle = T([x, y])$$

for every x, $y \in B_n$, we would have

$$\begin{aligned} 0 &= T([u_1, u_2] + [u_3, u_4] + \ldots + [u_{2n-1}, u_{2n}]) = \\ &\quad T([u_1, u_2]) + T([u_3, u_4]) + \ldots + T([u_{2n-1}, u_{2n}]) = \\ &\quad \langle u_1, u_2\rangle + \langle u_3, u_4\rangle + \ldots + \langle u_{2n-1}, u_{2n}\rangle, \text{ a contradiction.} \end{aligned}$$

□

Remarks 5.4.7. Note that Lemma 5.4.6 in particular shows that B_n is not a zero Lie product determined algebra for every $n \geq 2$.

We remark in this context that it is very easy to find examples of algebras that are not zero product determined or zero Jordan product determined, simply because there are algebras without nonzero zero divisors (domains), as well as such that the Jordan product of any of their two nonzero elements is always nonzero. On the contrary, finding algebras that are not zero Lie product determined is more difficult since in every algebra we have plenty of elements commuting with each other.

We are now in a position to show that matrix algebras are not always zero Lie product determined.

Theorem 5.4.8. *For every $n \geq 1$, the algebra $\mathbb{M}_n(B_{n^2+1})$ is not zero Lie product determined.*

Proof. By Lemma 5.4.6 there exists a bilinear map $\langle .\,,.\rangle : B_{n^2+1} \times B_{n^2+1} \to F$ such that $\sum_{t=1}^{n^2}[v_t, w_t] = 0$ implies $\sum_{t=1}^{n^2}\langle v_t, w_t\rangle = 0$, but there are $u_t \in B_{n^2+1}$, $t = 1, \ldots, 2n^2+2$, such that

$$\sum_{t=1}^{n^2+1}[u_{2t-1}, u_{2t}] = 0 \quad \text{while} \quad \sum_{t=1}^{n^2+1}\langle u_{2t-1}, u_{2t}\rangle \neq 0.$$

Set $A = M_n(B_{n^2+1})$, and define $\{.\,,.\} : A \times A \to F$ according to

$$\{v, w\} = \sum_{i=1}^{n}\sum_{j=1}^{n}\langle v_{ij}, w_{ji}\rangle,$$

where v_{ij} and w_{ij} are entries of the matrices v and w, respectively. We claim that $\{.\,,.\}$ satisfies the condition

$$\text{“}[v, w] = 0 \Rightarrow \{v, w\} = 0\text{”},$$

but does not satisfy the condition

$$\text{“}\sum_t [v_t, w_t] = 0 \Rightarrow \sum_t \{v_t, w_t\} = 0\text{”}.$$

The latter is obvious, since we may take $v_t = u_{2t-1}e_{11}$ and $w_t = u_{2t}e_{11}$, $t = 1, \ldots, n^2+1$. Now pick v and w in A such that $[v,w] = 0$, i.e. $vw = wv$. Considering just the diagonal entries of matrices on both sides of this identity we see that

$$\sum_{j=1}^{n} v_{ij}w_{ji} = \sum_{j=1}^{n} w_{ij}v_{ji}$$

for every $i = 1, \ldots, n$. Accordingly,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} v_{ij}w_{ji} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}v_{ji}.$$

Rewriting $\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ij}v_{ji}$ as $\sum_{i=1}^{n}\sum_{j=1}^{n} w_{ji}v_{ij}$ we thus see that $\sum_{i=1}^{n}\sum_{j=1}^{n}[v_{ij}, w_{ji}] = 0$. However, this implies $\sum_{i=1}^{n}\sum_{j=1}^{n}\langle v_{ij}, w_{ji}\rangle = 0$, that is, $\{v,w\} = 0$. □

Conclusiones

Cerramos la tesis analizando los objetivos que hemos alcanzando.

En el **Capítulo 1** nos planteábamos esencialmente dos metas, a saber: dar una noción de álgebra de cocientes para álgebras de Lie graduadas que generalizara la dada en [79] por Siles Molina para álgebras de Lie no graduadas y extender a álgebras de Lie de tipo “skew” uno de los principales resultados que Perera y Siles Molina probaron en [75] acerca de la relación que hay entre las álgebras de cocientes de álgebras asociativas y de Lie.

La primera de ellas fue alcanzada en la Sección 1.4 en la que introdujimos este nuevo concepto (ver Definiciones 1.4.12); seguidamente nos cercioramos de que efectivamente ésta era una buena generalización del caso no graduado (ver Observación 1.4.13). Hecho esto y siguiendo el modelo de [79] para el caso no graduado, estudiamos las principales propiedades (ver Proposiciones 1.4.18, 1.4.22) así como la relación con el caso no graduado (ver Lema 1.4.24 y Proposición 1.4.26).

Nuestro segundo objetivo en el Capítulo 1 surgió a raíz de la reflexión que Perera y Siles Molina hacían en [75] acerca de que los resultados [75, Teorema 2.12 y Proposición 3.5] deberían tenerse también para álgebras de Lie de tipo “skew”; efectivamente, como probamos en Teorema 1.5.19, estaban en lo cierto.

En el **Capítulo 2**, continuando con la idea de extender las nociones de álgebras de cocientes de álgebras de Lie a álgebras de Lie graduadas construimos en la Sección 2.2, tras un breve repaso a la construcción de [79] (ver Construcción 2.1.6) para el caso no graduado, el álgebra graduada de cocientes maximal de un álgebra de Lie graduada semiprima (ver Construcción 2.2.3 y Teorema 2.2.4).

El objetivo del resto del Capítulo 2 fue calcular $Q_m(L)$ para ciertas álgebras de Lie. Concretamente, en la Sección 2.3 calculamos $Q_m(A^-/Z)$ para un álgebra asociativa prima A con centro Z; teniendo en cuenta que los elementos del álgebra de cocientes maximal de un álgebra de Lie son clases de derivaciones parciales definidas en ideales esenciales y que nuestra álgebra de Lie A^-/Z proviene de un álgebra asociativa, construimos (ver Construcción 2.3.1) una nueva álgebra de Lie que denotamos por $\mathrm{Der}_{\mathrm{m}}(A)$ y probamos que, bajo ciertas hipótesis técnicas, $Q_m(A^-/Z)$ coincide con $\mathrm{Der}_{\mathrm{m}}(A)$ (ver Teorema 2.3.7). En la Sección 2.4 obtuvimos resultados similares (ver Teorema 2.4.10) para el cálculo de $Q_m(K/Z_K)$, donde K era el álgebra de los elementos skew de un álgebra asociativa prima con involución, considerando ahora en la construcción de $\mathrm{SDer}_{\mathrm{m}}(A)$ las derivaciones de A que conmutan con la involución.

En el **Capítulo 3** estudiamos si dos de las más importantes propiedades de las álgebras de cocientes asociativas continuaban siendo ciertas en el contexto de álgebras de Lie; concretamente:

1. Si el álgebra de cocientes maximal $Q_m(L)$ de un álgebra de Lie semiprima L, coincide con el álgebra de cocientes maximal $Q_m(I)$ para todo ideal esencial I de L.

2. Si tomar $Q_m(.)$ era una operación cerrada, esto es, si $Q_m(Q_m(L)) = Q_m(L)$ para toda álgebra de Lie semiprima L.

Como vimos en la Sección 3.1, la respuesta a la primera pregunta era afirmativa suponiendo que nuestra álgebra de Lie L fuese fuertemente semiprima (ver Definición 3.1.1 y Teorema 3.1.7). En la Sección 3.2 estudiamos la misma cuestión para álgebras de Lie graduadas haciendo uso del álgebra graduada de cocientes maximal construida en el capítulo anterior. En cuanto a la segunda pregunta, introdujimos en la Sección 3.3 la noción de álgebra de Lie maximal-cerrada que es aquélla para la que $Q_m(Q_m(L)) = Q_m(L)$; dimos ejemplos de álgebras de Lie maximal-cerradas (ver Corolarios 3.3.2 , 3.3.3, Proposición 3.3.5 y Teorema 3.3.6) y haciendo uso del ejemplo dado por Passman (ver Ejemplo 3.3.7) probamos (ver Teorema 3.3.8) que hay álgebras de Lie que no son maximal-cerradas, o sea, que la respuesta a la segunda pregunta es negativa en general.

En el **Capítulo 4** probamos que las álgebras graduadas de cocientes de álgebras de Lie graduadas constituyen el marco perfecto para situar los cocientes de sistemas de Jordan introducidos por García y Gómez Lozano en [39]. Comenzamos probando (ver Teorema 4.1.2) en la Sección 4.1 que el álgebra de cocientes maximal de un álgebra de Lie 3-graduada semiprima es de nuevo 3-graduada y además coincide con el álgebra graduada de cocientes maximal. Tras esto y haciendo uso de la construcción de Tits-Kantor-Koecher obtuvimos relaciones, en la Sección 4.2, entre los sistemas de cocientes maximales de sistemas de Jordan y las álgebras de Lie de cocientes maximales (ver Teoremas 4.2.11, 4.2.22 y 4.2.27).

En el **Capítulo 5** tratamos el problema de averiguar cuándo las matrices $\mathbb{M}_n(B)$ quedan determinadas por un producto nulo; en la Sección 5.2 probamos (Teorema 5.2.1) que, para el producto ordinario, las matrices $\mathbb{M}_n(B)$ quedan determinadas por el producto nulo para cualquier álgebra unitaria B y todo $n \geq 2$; en la Sección 5.3 demostramos (Teorema 5.3.1) lo mismo para el producto Jordan pero suponiendo que $n \geq 3$ y que 2 es inversible en B. Vimos en la Sección 5.4 que el caso del producto de Lie requiere la hipótesis adicional de que el álgebra B quede determinada por el producto de Lie nulo (Teorema 5.4.1). Concluimos la sección mostrando (Teorema 5.4.8) que esta hipótesis es necesaria.

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Notation

$\emptyset$	empty set
$\mathbb{N}$	positive integers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$F[x]$	algebra of polynomials
$F(x)$	field of fractions of polynomials
$\mathrm{Reg}(R)$	set of regular elements
$\mathbb{M}_n(R)$	matrix ring
$\mathbb{M}_n(A)$	matrix algebra
e_{ij}	matrix whose entries all are zero except for the one in row i and column j
$\mathrm{tr}(x)$	trace of a matrix
$*$	involution
$\cong$	isomorphism
$\leq$	substructure
$\cup$	union
$\cap$	intersection
$\subseteq \ \subset$	subset
$\subsetneq$	proper subset
$\triangleleft$	ideals
$\triangleleft_{gr}$	graded ideals
$\triangleleft_e$	essential ideals
$\mathcal{I}_e(A)$	family of essential ideals
$\mathcal{I}_{dr}(R)$	family of dense right ideals
Supp(X)	support of a subset of a graded algebra
$\mathrm{Ann}(X)$ $\mathrm{Ann}_Y(X)$	annihilator of X in Y
$\mathrm{QAnn}(X)$ $\mathrm{QAnn}_Y(X)$	quadratic annihilator of X in Y
Z Z_A	center of an algebra
$\mathrm{char}(A)$	characteristic of an algebra

$\deg(A)$	degree of an algebra
A^0	opposite algebra
$M(A)$	multiplication algebra
L	Lie algebra
A^-	Lie algebra that arises from an associative algebra
$K \;\; K_A$	skew elements of an algebra with *
$K/Z_K \;\; [K, K]/Z_{[K, K]}$	skew Lie algebras
$\mathrm{Der}(A)$	Lie algebra of derivations
$\mathrm{Inn}(A)$	Lie algebra of inner derivations
$\mathrm{SDer}(A)$	Lie algebra of derivations that commute with *
$Q^r_{max}(R)$	maximal right ring of quotients
$Q_r(R)$	two-sided right ring of quotients
$Q_s(R)$	symmetric Martindale ring of quotients
$Q_m(L)$	maximal Lie algebra of quotients
$Q_{gr-m}(L)$	maximal Lie graded algebra of quotients
$V = (V^+, V^-)$	Jordan pair
T	Jordan triple system
J	Jordan algebra
$\mathrm{TKK}(V)$	TKK algebra of Jordan pair of V
$V(T)$	double Jordan pair of a Jordan triple system T
J_T	Jordan triple system associated to a Jordan algebra J
$Q_m(V)$	maximal Jordan pair of $\mathfrak{M}$-quotients
$Q_m(T)$	maximal Jordan triple system of $\mathfrak{M}$-quotients
$Q_m(J)$	maximal Jordan algebra of $\mathfrak{M}$-quotients

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