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Weighted weak-type iterated and bilinear Hardy inequalities ☆

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ABSTRACT

We characterize the good weights for a certain weighted weak-type iterated Hardy inequality to hold. As a consequence, we get the characterizations of some weighted weak-type bilinear Hardy inequalities.

1. Introduction and results

G. H. Hardy proved in 1920 (see [14]) that if $p > 1$ then the inequality

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f \right)^p dx \le \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p$$

holds for all positive measurable function f. He also proved ([15], Theorem 330) that if $p > 1$ and $\varepsilon < p - 1$, then

$$\int_0^\infty \left(\int_0^t f(x)dx \right)^p t^{\varepsilon - p} dt \le \left(\frac{p}{p-1-\varepsilon} \right)^p \int_0^\infty f^p(t) t^\varepsilon dt.$$

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These inequalities, which are known as Hardy inequalities, were the starting point for the theory of Hardy inequalities with weights, in which the problem of characterizing the positive functions w, v, the weights, such that

$$\left(\int_a^b \left(\int_a^x f\right)^q w(x)dx\right)^{\frac{1}{q}} \leq C\left(\int_a^b f^p v\right)^{\frac{1}{p}} \tag{1.1}$$

holds for all positive measurable function f with a positive constant C independent of f was intensively studied. Observe that inequality (1.1) means that the Hardy operator $Tf(x) = \int_a^x f$ is bounded from $L^p(v)$ to $L^q(w)$.

This problem was solved by Talenti ([39]), Muckenhoupt ([30]) and Bradley ([4]) in the case $p \leq q$ and by Mazja ([29]), Sinnamon ([36]) in the case $0 < q < 1 < p < \infty$ and Sinnamon and Stepanov ([37]) for $0 < q < 1 = p$. Recently, a new elementary and universal proof of weighted Hardy inequality was found ([13]). The following theorem collects the results.

Theorem A *([4], [13], [29], [30], [36], [37], [39]). Let $0 < q < \infty$, $1 \leq p < \infty$ and let w, v be positive measurable function on (a, b), where $-\infty \leq a < b \leq \infty$. Then there exists a positive constant C such that inequality (1.1) holds for all nonnegative functions f if and only if*

(i) *in the case $p \leq q$,*

$$B_1 \equiv \sup_{s\in(a,b)} \left(\int_s^b w\right)^{\frac{1}{q}} \|\chi_{(a,s)} v^{-\frac{1}{p}}\|_{p'} < \infty,$$

and the best constant C in inequality (1.1) verifies $B_1 \leq C \leq K(q,p)B_1$, where $K(q,p) = \left(1+\frac{q}{p'}\right)^{\frac{1}{q}}\left(1+\frac{p'}{q}\right)^{\frac{1}{p'}}$ if $p > 1$ and $K(q, 1) = 1$;

(ii) *in the case $q < p$,*

$$B_2 \equiv \left(\int_a^b \left(\int_t^b w\right)^{\frac{r}{p}} \|\chi_{(a,t)} v^{-\frac{1}{p}}\|_{p'}^r w(t)dt\right)^{\frac{1}{r}} < \infty,$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, and the best constant C in inequality (1.1) verifies $(p')^{\frac{1}{p'}} q^{\frac{1}{p}}\left(1-\frac{q}{p}\right) B_2 \leq C \leq \left(\frac{r}{q}\right)^{\frac{1}{r}} p^{\frac{1}{p}} (p')^{\frac{1}{p'}} B_2$.

Weighted weak-type inequalities were also studied, that is, the boundedness of T from $L^p(v)$ to $L^{q,\infty}(w)$, where

$$L^{q,\infty}(w) = \left\{ f : \|f\|_{q,\infty;w} = \sup_{\lambda>0} \lambda \left(\int_{\{x:|f(x)|>\lambda\}} w\right)^{\frac{1}{q}} < \infty \right\}.$$

Really, weighted weak-type inequalities have been studied for the modified Hardy operators $T_\beta f(x) = \beta(x)\int_a^x f$, where β is a fixed positive function. It is very easy to characterize the weighted strong type inequalities for T_β, since the function β becomes part of the weight and it is possible to apply Theorem A.

This does not happen with weak-type inequalities, which are technically more difficult to deal with. In fact, the problem of characterizing the boundedness of T_β from $L^p(v)$ to $L^{q,\infty}(w)$ in the case $q < p$ is not completely solved yet.

Weighted weak-type inequalities for modified linear or sublinear operators are included in the topic of weighted mixed weak-type inequalities, which goes back to the work of Andersen and Muckenhoupt [2] and has subsequently received contributions from several authors (see [34], [26], [5], [21], [22], [24], [20]).

The first results on weighted weak-type inequalities for modified Hardy operators are due to Andersen and Muckenhoupt ([2]), who worked with $\beta(x) = x^\alpha$, $\alpha \in \mathbb{R}$, on $(0, \infty)$. The weighted weak-type inequalities with more general functions β were characterized in [6], [26] and [25]. The following two theorems contain such characterizations.

Theorem B *([6], [26]). Let $1 \le p \le q < \infty$ and β, v and w be positive measurable functions on (a, b), where $-\infty \le a < b \le \infty$. Then there exists a positive constant C such that inequality*

$$\left\| \beta(x) \left(\int_a^x f \right) \right\|_{q,\infty;w} \le C \|f\|_{p,v} \tag{1.2}$$

holds for all nonnegative functions f if and only if

$$B_3 \equiv \sup_{a<s<b} \|\beta \chi_{(s,b)}\|_{q,\infty;w} \|\chi_{(a,s)} v^{-\frac{1}{p}}\|_{p'} < \infty, \tag{1.3}$$

and the best constant C in inequality (1.2) verifies $B_3 \le C \le 4B_3$.

Theorem C *([25]). Let $0 < q < p < \infty$ with $p \ge 1$ and β, v and w be positive measurable functions on (a, b), where $-\infty \le a < b \le \infty$ and β is a monotone function. Then there exists a positive constant C such that inequality (1.2) holds for all nonnegative functions f if and only if the function Ψ defined on (a, b) by*

$$\Psi(x) = \sup_{b>c>x} \left[\left(\inf_{y \in (x,c)} \beta(y) \right) \left(\int_x^c w \right)^{\frac{1}{p}} \right] \|\chi_{(a,x)} v^{-\frac{1}{p}}\|_{p'}$$

belongs to $L^{r,\infty}(w)$, where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. In this case, the best constant C in inequality (1.2) verifies $2^{-\frac{1}{p}} \|\Psi\|_{r,\infty;w} \le C \le (1 + 4^p)^{\frac{1}{q}} \|\Psi\|_{r,\infty;w}$.

The theory of Hardy inequalities with weights has been developed in many directions. For a general perspective of the topic, the reader can consult the monographs [33] and [18].

In recent years, bilinear Hardy inequalities and iterated Hardy inequalities have been studied. By a weighted strong type bilinear Hardy inequality we mean an inequality of the form

$$\left\| \left(\int_a^x f \right) \left(\int_a^x g \right) \right\|_{q,w} \le C \|f\|_{p_1,v_1} \|g\|_{p_2,v_2}. \tag{1.4}$$

The characterization of the weights for (1.4) to hold is due to Aguilar, Ortega and Ramírez ([1]). Subsequently, a theory of bilinear Hardy inequalities has been developed with contributions from several authors (see, for instance, [16], [17], [38], [8]).

At about the same time as, but independently of, bilinear Hardy inequalities, weighted strong type iterated Hardy inequalities have been studied. By this kind of inequalities we mean

$$\left\| \left(\int_a^x \left(\int_a^t f \right)^r u(t)dt \right)^{\frac{1}{r}} \right\|_{q,w} \le C\|f\|_{p,v} \tag{1.5}$$

or

$$\left\| \left(\int_a^x \left(\int_t^x f \right)^r u(t)dt \right)^{\frac{1}{r}} \right\|_{q,w} \le C\|f\|_{p,v}. \tag{1.6}$$

The good weights for the iterated Hardy inequalities to hold have been characterized in several works, as [31], [32], [9], [10], [11], [3], [12] and [38].

Recently, Krepela has observed in [16] that there exists a relationship between iterated and bilinear Hardy inequalities, in such a way that to prove a bilinear Hardy inequality can be reduced to prove an iterated one.

The main purpose of this work is to characterize the weights for the weak-type bilinear Hardy inequality

$$\left\| \beta(x) \left(\int_a^x f \right) \left(\int_a^x g \right) \right\|_{q,\infty;w} \le C\|f\|_{p_1,v_1}\|g\|_{p_2,v_2} \tag{1.7}$$

to hold. In the simplest cases, specifically when $1 \le p_1, p_2 \le q$ or $p_1 \le q < p_2$, it is possible to reduce the problem to the linear case by iteration and then to apply Theorems B and C. The first of these two cases was studied in [7], lemma 4, even in a more general setting. The result was the next one.

Theorem D *([7]). Let $1 \le p_1, p_2 \le q < \infty$ and β, v_1, v_2 and w be positive measurable functions on (a,b), where $-\infty \le a < b \le \infty$. Then there exists a positive constant C such that inequality (1.7) holds for all nonnegative functions f and g if and only if*

$$B_4 \equiv \sup_{a<s<b} \|\beta\chi_{(s,b)}\|_{q,\infty;w}\|\chi_{(a,s)}v_1^{-\frac{1}{p_1}}\|_{p_1'}\|\chi_{(a,s)}v_2^{-\frac{1}{p_2}}\|_{p_2'}, < \infty. \tag{1.8}$$

The other simple case is $p_1 \le q < p_2$. We include the statement and proof of the corresponding theorem for completeness. The result reads as follows.

Theorem 1. *Let $1 \le p_1 \le q < p_2 < \infty$ and let β, v_1, v_2 and w be positive measurable functions on (a,b), where $-\infty \le a < b \le \infty$ and β is a monotone function. Then there exists a positive constant C such that inequality (1.7) holds for all nonnegative functions f and g if and only if (1.8) holds and*

$$C_1 \equiv \sup_{a<s<b} \|\Psi_s\|_{r_2,\infty;w}\|\chi_{(a,s)}v_1^{-\frac{1}{p_1}}\|_{p_1'} < \infty, \tag{1.9}$$

where Ψ_s is the function defined for $x \in (s,b)$ by

$$\Psi_s(x) = \sup_{x<c<b} \left[(\inf_{y\in(x,c)} \beta(y)) \left(\int_x^c w \right)^{\frac{1}{p_2}} \right] \|\chi_{(a,x)}v_2^{-\frac{1}{p_2}}\|_{p_2'}$$

and $\frac{1}{r_2} = \frac{1}{q} - \frac{1}{p_2}$. The best constant C in (1.7) verifies $C \approx \max\{C_1, B_4\}$.

However, if $q < p_1$ and $q < p_2$, the fact that the weighted weak-type linear inequality considered in Theorem C is only characterized for monotone β makes it difficult to reduce (1.7) to a linear inequality, especially when β decreases. In this case, we are able to reduce the bilinear problem to characterize a weighted weak-type iterated Hardy inequality of the form

$$\left\| \alpha(x) \left\| \chi_{(a,x)}(t)u(t) \int_a^t f \right\|_r \right\|_{q,\infty;w} \leq C\|f\|_{p,v}, \tag{1.10}$$

which, as far as we know, has not been studied yet. Our result on inequality (1.10) is the next one.

Theorem 2. *Let $1 \leq p \leq q < \infty$, $1 < r \leq \infty$ and $\frac{1}{\theta} = \frac{1}{r} - \frac{1}{p}$ if $r < p$. Let α, u, v and w be positive measurable functions on (a,b). Then there exists a positive constant C such that inequality (1.10) holds for all nonnegative functions f on (a,b) if and only if*

(i) *in the case $p \leq r$,*

$$C_2 \equiv \sup_{a<s<b} \left\| \chi_{(s,b)}(x)\alpha(x)\|\chi_{(s,x)}u\|_r \right\|_{q,\infty;w} \|\chi_{(a,s)}v^{-\frac{1}{p}}\|_{p'} < \infty, \tag{1.11}$$

and the best constant C in inequality (1.10) verifies $C_2 \leq C \leq \max\{2 \cdot 4^{\frac{1}{r}} K(r,p), 2 \cdot 4^{1+\frac{1}{r}}\}C_2$ if $r < \infty$ and $C_2 \leq C \leq 16C_2$ if $r = \infty$;

(ii) *in the case $r < p$, (1.11) and*

$$C_3 \equiv \sup_{a<s<b} \|\chi_{(s,b)}\alpha\|_{q,\infty;w} \left(\int_a^s \left(\int_t^s u^r \right)^{\frac{\theta}{p}} \left(\int_a^t v^{1-p'} \right)^{\frac{\theta}{p'}} u^r(t)dt \right)^{\frac{1}{\theta}} < \infty. \tag{1.12}$$

In this case, the best constant C in inequality (1.10) verifies

$$\max\left\{ C_2, (p')^{\frac{1}{p'}} r^{\frac{1}{p}} \left(1 - \frac{r}{p}\right) C_3 \right\} \leq C \leq \max\left\{ 2 \cdot 4^{\frac{1}{r}} \left(\frac{\theta}{r}\right)^{\frac{1}{\theta}} p^{\frac{1}{p}} (p')^{\frac{1}{p'}} C_3, 2 \cdot 4^{1+\frac{1}{r}} C_2 \right\}.$$

It is worth noting that we include the case $r = \infty$, which is of great interest in the application to the weighted weak-type bilinear Hardy inequalities. The proof of this case will be carried out by means of an approximation argument that requires using some of the estimates of the best constants that appear in the statement of the theorem. This is the reason why we include such estimates.

Next, as a consequence of Theorem 2 we get our result for the weighted weak-type bilinear Hardy inequality (1.7). We are specially interested in the case $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$, but we can really cover all the cases $\frac{1}{q} \leq \frac{1}{p_1} + \frac{1}{p_2}$ without extra work. The result is the next one.

Theorem 3. *Let $1 \leq p_1, p_2 < \infty$, $q > 0$ with $q < p_1$, $q < p_2$ and $\frac{1}{q} \leq \frac{1}{p_1} + \frac{1}{p_2}$. Let β be a positive monotone function on (a,b) and let w, v_1, v_2 be positive measurable functions on (a,b). Let $\alpha_i(x) = \sup_{c>x} \left(\left(\inf_{y\in(x,c)} \beta(y)\right) \left(\int_x^c w\right)^{\frac{1}{p_i}} \right)$, $i = 1, 2$. Then there exists a positive constant C such that inequality (1.7) holds for all nonnegative functions f and g if and only if*

(i) *in the case $p_2 \leq p_1'$,*

$$D_1 \equiv \sup_{a<s<b} \left\| \chi_{(s,b)}(x)\alpha_1(x)\|\chi_{(s,x)}v_1^{-\frac{1}{p_1}}\|_{p_1'} \right\|_{r_1,\infty;w} \|\chi_{(a,s)}v_2^{-\frac{1}{p_2}}\|_{p_2'} < \infty \tag{1.13}$$

and

$$D_2 \equiv \sup_{a<s<b} \left\| \chi_{(s,b)}(x)\alpha_2(x)\|\chi_{(s,x)}v_2^{-\frac{1}{p_2}}\|_{p_2'} \right\|_{r_2,\infty;w} \|\chi_{(a,s)}v_1^{-\frac{1}{p_1}}\|_{p_1'} < \infty, \tag{1.14}$$

where $\frac{1}{r_i} = \frac{1}{q} - \frac{1}{p_i}$ *and the best constant* C *in (1.7) verifies* $C \approx \max\{D_1, D_2\}$*;*

(ii) *in the case* $p_1' < p_2$*, (1.13), (1.14),*

$$D_3 \equiv \sup_{a<s<b} \|\chi_{(s,b)}\alpha_1\|_{r_1,\infty;w} \left(\int_a^s \left(\int_t^s v_1^{1-p_1'} \right)^{\frac{\eta}{p_1'}} \left(\int_a^t v_2^{1-p_2'} \right)^{\frac{\eta}{p_1}} v_2^{1-p_2'}(t)dt \right)^{\frac{1}{\eta}} < \infty \tag{1.15}$$

and

$$D_4 \equiv \sup_{a<s<b} \|\chi_{(s,b)}\alpha_2\|_{r_2,\infty;w} \left(\int_a^s \left(\int_t^s v_2^{1-p_2'} \right)^{\frac{\eta}{p_2'}} \left(\int_a^t v_1^{1-p_1'} \right)^{\frac{\eta}{p_2}} v_1^{1-p_1'}(t)dt \right)^{\frac{1}{\eta}} < \infty, \tag{1.16}$$

where $\frac{1}{\eta} = \frac{1}{p_1'} - \frac{1}{p_2} = \frac{1}{p_2'} - \frac{1}{p_1}$ *and the best constant* C *verifies* $C \approx \max\{D_1, D_2, D_3, D_4\}$.

Finally, we will point out the relationship between our results and the problem of characterizing the good weights for the one-sided bi-sublinear maximal operator $M^=$ defined on pairs of measurable functions of one real variable by

$$M^=(f,g)(x) = \sup_{a<x} \frac{1}{(x-a)^2} \left(\int_a^x |f| \right) \left(\int_a^x |g| \right).$$

This is an open problem that presents many technical difficulties. Probably, the main one is to establish the right relationship between the level sets of the maximal operator and the ones of the averaging (Hardy) operators. In the linear case, this relationship is established by means of Riesz rising sun lemma (see [35], [27], [23] and [28]). Also in the linear case, it is shown that the weighted weak-type inequality for the one-sided maximal operator is equivalent to the uniform weighted weak-type boundedness of the averaging operators. This is not yet known for the one-sided bi-sublinear maximal operator.

Although we are not able to solve the problem for the bi-sublinear one-sided maximal operator, Theorem 3 allows to characterize immediately the uniform weighted weak-type boundedness for the bilinear averaging operators. The result is the next one. In the statement we focus on the case $p_2 \le p_1'$.

Theorem 4. *Let* $1 \le p_1, p_2 < \infty$ *with* $p_2 \le p_1'$ *and* $q > 0$ *with* $\frac{1}{q} = \frac{1}{p_1} + \frac{1}{p_2}$*. Let* w, v_1, v_2 *be positive measurable functions on the real line. Then there exists a positive constant* C *such that*

$$\sup_{a\in\mathbb{R}} \left\| \chi_{(a,\infty)}(x) \frac{1}{(x-a)^2} \left(\int_a^x |f| \right) \left(\int_a^x |g| \right) \right\|_{q,\infty;w} \le C\|f\|_{p_1,v_1}\|g\|_{p_2,v_2}$$

for all f *and* g *if and only if*

$$M_1 \equiv \sup_{-\infty<a<s<\infty} \left\| \chi_{(s,\infty)}(x)\alpha_{1,a}(x)\|\chi_{(s,x)}v_1^{-\frac{1}{p_1}}\|_{p_1'} \right\|_{p_2,\infty;w} \|\chi_{(a,s)}v_2^{-\frac{1}{p_2}}\|_{p_2'} < \infty$$

and

$$M_2 \equiv \sup_{-\infty<a<s<\infty} \left\| \chi_{(s,\infty)}(x)\alpha_{2,a}(x) \|\chi_{(s,x)} v_2^{-\frac{1}{p_2}}\|_{p_2'} \right\|_{p_1,\infty;w} \|\chi_{(a,s)} v_1^{-\frac{1}{p_1}}\|_{p_1'} < \infty$$

where $\alpha_{i,a}(x) = \chi_{(a,\infty)} \sup_{c>x} \left(\frac{1}{(c-a)^2} \left(\int_x^c w\right)^{\frac{1}{p_i}} \right)$.

The proofs of the main results, Theorems 2 and 3, are contained in sections 2 and 3 respectively, while section 4 is devoted to the proof of Theorem 1.

2. Proof of Theorem 2

Assume that (1.10) holds and $p > 1$. Let $a < s < b$ and $f = \chi_{(a,s)} v^{1-p'}$. Then

$$\left\| \alpha(x) \left\| \chi_{(a,x)}(t) u(t) \int_a^t f \right\|_r \right\|_{q,\infty;w} \geq \left\| \chi_{(s,b)}(x)\alpha(x) \|\chi_{(s,x)} u\|_r \right\|_{q,\infty;w} \int_a^s v^{1-p'}$$

and (1.10) gives

$$\left\| \chi_{(s,b)}(x)\alpha(x) \|\chi_{(s,x)} u\|_r \right\|_{q,\infty;w} \int_a^s v^{1-p'} \leq C \left(\int_a^s v^{1-p'} \right)^{\frac{1}{p}},$$

which shows that $C_2 \leq C$. Assume now that $p = 1$. Let $\{a_n\}$ be a strictly increasing sequence of positive numbers which converges to $\|\chi_{(a,s)} v^{-1}\|_\infty$. For each n, let $R_n = \{x \in (a,s) : v^{-1}(x) > a_n\}$. The set R_n has positive measure. Let M_n be a subset of R_n such that $0 < |M_n| < \infty$. Let $f_n = |M_n|^{-1} \chi_{M_n} v^{-1}$. Then

$$\left\| \alpha(x) \left\| \chi_{(a,x)}(t) u(t) \int_a^t f_n \right\|_r \right\|_{q,\infty;w} \geq \left\| \chi_{(s,b)}(x)\alpha(x) \|\chi_{(s,x)} u\|_r \right\|_{q,\infty;w} a_n$$

and by (1.10) we have

$$\left\| \chi_{(s,b)}(x)\alpha(x) \|\chi_{(s,x)} u\|_r \right\|_{q,\infty;w} a_n \leq C.$$

Letting n tend to infinity, we get that $C_2 \leq C$.

It is also clear, in the case $r < p$, that for all f

$$\left\| \alpha(x) \left(\int_a^x \left(\int_a^t f \right)^r u^r(t)dt \right)^{\frac{1}{r}} \right\|_{q,\infty;w} \geq \|\chi_{(s,b)}\alpha\|_{q,\infty;w} \left(\int_a^s \left(\int_a^t f \right)^r u^r(t)dt \right)^{\frac{1}{r}}.$$

Then, since (1.10) holds for all f supported on (a,s), we have

$$\|\chi_{(s,b)}\alpha\|_{q,\infty;w} \left(\int_a^s \left(\int_a^t f \right)^r u^r(t)dt \right)^{\frac{1}{r}} \leq C \left(\int_a^s f^p v \right)^{\frac{1}{p}}$$

and, by Theorem A (ii), this inequality implies $C \geq (p')^{\frac{1}{p'}} r^{\frac{1}{p}} \left(1 - \frac{r}{p}\right) C_3$.

Let us prove now the sufficiency of the conditions. Assume first that $r<\infty$. Let f be a positive function on (a,b) such that $\int_a^b f<\infty$ and $\int_a^b \left(\int_a^t f\right)^r u^r(t)dt<\infty$. Let $\{x_k\}$ be the sequence defined by $x_0=b$ and

$$\int_a^{x_{k+1}} \left(\int_a^t f\right)^r u^r(t)dt = \int_{x_{k+1}}^{x_k} \left(\int_a^t f\right)^r u^r(t)dt.$$

The sequence $\{x_k\}$ decreases to a and verifies

$$\int_a^{x_k} \left(\int_a^t f\right)^r u^r(t)dt = 4\int_{x_{k+2}}^{x_{k+1}} \left(\int_a^t f\right)^r u^r(t)dt \tag{2.1}$$

for all k.

Let $\lambda>0$ and $O_\lambda=\{x\in(a,b):\alpha(x)\left(\int_a^x \left(\int_a^t f\right)^r u^r(t)dt\right)^{\frac{1}{r}}>\lambda\}$. For each k, let $E_k=O_\lambda\cap(x_{k+1},x_k)$. If $x\in E_k$, we have

$$\begin{aligned}\lambda &< \alpha(x)\left(\int_a^x \left(\int_a^t f\right)^r u^r(t)dt\right)^{\frac{1}{r}} \le \alpha(x)\left(\int_a^{x_k} \left(\int_a^t f\right)^r u^r(t)dt\right)^{\frac{1}{r}}\\ &= 4^{\frac{1}{r}}\alpha(x)\left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_a^t f\right)^r u^r(t)dt\right)^{\frac{1}{r}} \le 4^{\frac{1}{r}}\alpha(x)\left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{x_{k+2}}^t f\right)^r u^r(t)dt\right)^{\frac{1}{r}}\\ &+4^{\frac{1}{r}}\alpha(x)\left(\int_{x_{k+2}}^{x_{k+1}} u^r\right)^{\frac{1}{r}}\int_a^{x_{k+2}} f.\end{aligned} \tag{2.2}$$

It is clear that, for each k, $E_k\subset E_{k,1}\cup E_{k,2}$, where

$$E_{k,1}=\left\{x\in(x_{k+1},x_k):\alpha(x)\left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{x_{k+2}}^t f\right)^r u^r(t)dt\right)^{\frac{1}{r}} > \frac{\lambda}{2\cdot 4^{\frac{1}{r}}}\right\}$$

and

$$E_{k,2}=\left\{x\in(x_{k+1},x_k):\alpha(x)\left(\int_{x_{k+2}}^{x_{k+1}} u^r\right)^{\frac{1}{r}}\int_a^{x_{k+2}} f > \frac{\lambda}{2\cdot 4^{\frac{1}{r}}}\right\}.$$

Assume that $p\le r$. Then by Theorem A (i), we have that

$$\begin{aligned}&\left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{x_{k+2}}^t f\right)^r u^r(t)dt\right)^{\frac{1}{r}}\\ &\le K(r,p)\sup_{x_{k+2}<\gamma<x_{k+1}}\left(\int_\gamma^{x_{k+1}} u^r\right)^{\frac{1}{r}}\|\chi_{(x_{k+2},\gamma)}v^{-\frac{1}{p}}\|_{p'}\left(\int_{x_{k+2}}^{x_{k+1}} f^p v\right)^{\frac{1}{p}}\end{aligned} \tag{2.3}$$

Then, every $x \in E_{k,1}$ verifies

$$\lambda < 2 \cdot 4^{\frac{1}{r}} K(r,p)\alpha(x) \sup_{x_{k+2}<\gamma<x_{k+1}} \left(\int_{\gamma}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \|\chi_{(x_{k+2},\gamma)} v^{-\frac{1}{p}}\|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}},$$

which implies

$$\lambda \leq 2 \cdot 4^{\frac{1}{r}} K(r,p)(\inf_{x\in E_{k,1}} \alpha(x)) \sup_{x_{k+2}<\gamma<x_{k+1}} \left(\int_{\gamma}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \|\chi_{(x_{k+2},\gamma)} v^{-\frac{1}{p}}\|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}}. \tag{2.4}$$

Multiplying both sides of (2.4) by $\left(\int_{E_{k,1}} w\right)^{\frac{1}{q}}$, having into account the definition of the norm in $L^{q,\infty}(w)$ and applying condition (1.11), we get

$$\begin{aligned} &\left(\int_{E_{k,1}} w \right)^{\frac{1}{q}} \\ &\leq \frac{2 \cdot 4^{\frac{1}{r}} K(r,p)}{\lambda} \sup_{x_{k+2}<\gamma<x_{k+1}} \|\chi_{(x_{k+1},b)}\alpha\|_{q,\infty;w} \left(\int_{\gamma}^{x_{k+1}} u^r \right)^{\frac{1}{r}} \|\chi_{(x_{k+2},\gamma)} v^{-\frac{1}{p}}\|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}} \\ &\leq \frac{2 \cdot 4^{\frac{1}{r}} K(r,p)}{\lambda} \sup_{a<\gamma<b} \left\| \chi_{(\gamma,b)}(x)\alpha(x) \left(\int_{\gamma}^{x} u^r \right)^{\frac{1}{r}} \right\|_{q,\infty;w} \|\chi_{(a,\gamma)} v^{-\frac{1}{p}}\|_{p'} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}} \\ &\leq \frac{2 \cdot 4^{\frac{1}{r}} K(r,p) C_2}{\lambda} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}} \end{aligned} \tag{2.5}$$

for all k. Raising to the q in (2.5) and summing up in k we get

$$\int_{\cup_k E_{k,1}} w \leq \frac{(2 \cdot 4^{\frac{1}{r}} K(r,p) C_2)^q}{\lambda^q} \left(\int_a^b f^p v \right)^{\frac{q}{p}}. \tag{2.6}$$

If $r < p$, then by Theorem A (ii), we have that

$$\begin{aligned} &\left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{x_{k+2}}^{t} f \right)^r u^r(t)dt \right)^{\frac{1}{r}} \\ &\leq \left(\frac{\theta}{r}\right)^{\frac{1}{\theta}} p^{\frac{1}{p}} (p')^{\frac{1}{p'}} \left(\int_{x_{k+2}}^{x_{k+1}} \left(\int_{t}^{x_{k+1}} u^r \right)^{\frac{\theta}{p}} \left(\int_{x_{k+2}}^{t} v^{1-p'} \right)^{\frac{\theta}{p'}} u^r(t)dt \right)^{\frac{1}{\theta}} \left(\int_{x_{k+2}}^{x_{k+1}} f^p v \right)^{\frac{1}{p}}. \end{aligned} \tag{2.7}$$

Then, every $x \in E_{k,1}$ verifies

$$\frac{\lambda}{2\cdot 4^{\frac{1}{r}}} < \left(\frac{\theta}{r}\right)^{\frac{1}{\theta}} p^{\frac{1}{p}}(p')^{\frac{1}{p'}}\alpha(x)\left(\int_{x_{k+2}}^{x_{k+1}}\left(\int_t^{x_{k+1}} u^r\right)^{\frac{\theta}{p}}\left(\int_{x_{k+2}}^{t} v^{1-p'}\right)^{\frac{\theta}{p'}} u^r(t)dt\right)^{\frac{1}{\theta}}\left(\int_{x_{k+2}}^{x_{k+1}} f^p v\right)^{\frac{1}{p}},$$

which implies

$$\begin{aligned}\frac{\lambda}{2\cdot 4^{\frac{1}{r}}} \le\ & \left(\frac{\theta}{r}\right)^{\frac{1}{\theta}} p^{\frac{1}{p}}(p')^{\frac{1}{p'}}\big(\inf_{x\in E_{k,1}}\alpha(x)\big)\\ &\times\left(\int_{x_{k+2}}^{x_{k+1}}\left(\int_t^{x_{k+1}} u^r\right)^{\frac{\theta}{p}}\left(\int_{x_{k+2}}^{t} v^{1-p'}\right)^{\frac{\theta}{p'}} u^r(t)dt\right)^{\frac{1}{\theta}}\left(\int_{x_{k+2}}^{x_{k+1}} f^p v\right)^{\frac{1}{p}}.\end{aligned} \tag{2.8}$$

Multiplying both sides of (2.8) by $\left(\int_{E_{k,1}} w\right)^{\frac{1}{q}}$, having into account the definition of the norm in $L^{q,\infty}(w)$ and applying condition (1.12), we get

$$\left(\int_{E_{k,1}} w\right)^{\frac{1}{q}} \le \frac{2\cdot 4^{\frac{1}{r}}\left(\frac{\theta}{r}\right)^{\frac{1}{\theta}} p^{\frac{1}{p}}(p')^{\frac{1}{p'}} C_3}{\lambda}\left(\int_{x_{k+2}}^{x_{k+1}} f^p v\right)^{\frac{1}{p}} \tag{2.9}$$

for all k. Raising to the q in (2.9) and summing up in k yields

$$\int_{\cup_k E_{k,1}} w \le \frac{(2\cdot 4^{\frac{1}{r}}\left(\frac{\theta}{r}\right)^{\frac{1}{\theta}} p^{\frac{1}{p}}(p')^{\frac{1}{p'}} C_3)^q}{\lambda^q}\left(\int_a^b f^p v\right)^{\frac{q}{p}}. \tag{2.10}$$

This finishes the estimation of $\int_{\cup_k E_{k,1}} w$.

In order to estimate $\int_{\cup_k E_{k,2}} w$, we will use a technique due to Q. Lai ([19]). Let us define the sequence $\{y'_m\}$ as $y'_0 = b$ and $\int_a^{y'_{m+1}} f = \int_{y'_{m+1}}^{y'_m} f$. Let $\{y_n\}$ be the subsequence of $\{y'_m\}$ defined by $y_0 = y'_0$ and by deleting y'_{m+1} if $[y'_{m+1}, y'_m) \cap \{x_k\} = \emptyset$. In this way, if $y'_{m+1} = y_{n+1} \le x_{k+2} < y_n$, then $x_{k+2} \le y'_m$ and $y_{n+2} \le y'_{m+2}$, which yields

$$\int_a^{x_{k+2}} f \le \int_a^{y'_m} f = 4\int_{y'_{m+2}}^{y'_{m+1}} f \le 4\int_{y_{n+2}}^{y_{n+1}} f. \tag{2.11}$$

Let $E_2^n = \cup_{\{k: y_{n+1}\le x_{k+2} < y_n\}} E_{k,2}$. If $x \in E_2^n$, there exists k with $y_{n+1} \le x_{k+2} < y_n$ such that $x \in E_{k,2}$ and then, by (2.11),

$$\frac{\lambda}{2\cdot 4^{\frac{1}{r}}} < \alpha(x)\left(\int_{x_{k+2}}^{x_{k+1}} u^r\right)^{\frac{1}{r}}\int_a^{x_{k+2}} f \le 4\alpha(x)\left(\int_{y_{n+1}}^{x} u^r\right)^{\frac{1}{r}}\int_{y_{n+2}}^{y_{n+1}} f. \tag{2.12}$$

Since (2.12) holds for all $x \in E_2^n$, we have

$$\frac{\lambda}{2 \cdot 4^{1+\frac{1}{r}}} \leq \inf_{x \in E_2^n} \left[\alpha(x) \left(\int_{y_{n+1}}^{x} u^r \right)^{\frac{1}{r}} \right] \int_{y_{n+2}}^{y_{n+1}} f. \tag{2.13}$$

Multiplying both sides of (2.13) by $\left(\int_{E_2^n} w\right)^{\frac{1}{q}}$ and applying Holder's inequality and condition (1.11), we get

$$\begin{aligned} \frac{\lambda}{2 \cdot 4^{1+\frac{1}{r}}} \left(\int_{E_2^n} w \right)^{\frac{1}{q}} &< \inf_{x \in E_2^n} \left[\alpha(x) \left(\int_{y_{n+1}}^{x} u^r \right)^{\frac{1}{r}} \right] \left(\int_{E_2^n} w \right)^{\frac{1}{q}} \int_{y_{n+2}}^{y_{n+1}} f \\ &\leq \left\| \chi_{(y_{n+1},b)}(x) \alpha(x) \left(\int_{y_{n+1}}^{x} u^r \right)^{\frac{1}{r}} \right\|_{q,\infty;w} \|\chi_{(y_{n+2},y_{n+1})} v^{-\frac{1}{p}}\|_{p'} \left(\int_{y_{n+2}}^{y_{n+1}} f^p v \right)^{\frac{1}{p}} \\ &\leq C_2 \left(\int_{y_{n+2}}^{y_{n+1}} f^p v \right)^{\frac{1}{p}}. \end{aligned} \tag{2.14}$$

Then, raising to the q in (2.14), summing and taking into account that $p \leq q$, we obtain

$$\int_{\cup_k E_{k,2}} w = \sum_{k=0}^{\infty} \int_{E_{k,2}} w = \sum_{n=0}^{\infty} \sum_{\{k: y_{n+1} \leq x_{k+2} < y_n\}} \int_{E_{k,2}} w = \sum_{n=0}^{\infty} \int_{E_2^n} w \leq \frac{(2 \cdot 4^{1+\frac{1}{r}} C_2)^q}{\lambda^q} \left(\int_a^b f^p v \right)^{\frac{q}{p}}. \tag{2.15}$$

Finally, from (2.6) (or (2.10)) and (2.15) we get

$$\left(\int_{O_\lambda} w \right)^{\frac{1}{q}} \leq \frac{C}{\lambda} \left(\int_a^b f^p v \right)^{\frac{1}{p}},$$

where $C = \max\{2 \cdot 4^{\frac{1}{r}} K(r,p), 2 \cdot 4^{1+\frac{1}{r}}\} C_2$ if $p \leq r$ and $C = \max\{2 \cdot 4^{\frac{1}{r}} \left(\frac{\theta}{r}\right)^{\frac{1}{\theta}} p^{\frac{1}{p}} (p')^{\frac{1}{p'}} C_3, 2 \cdot 4^{1+\frac{1}{r}} C_2\}$ if $r < p$. This finishes the proof for the case $r < \infty$.

Assume now that $r = \infty$. We will make an approximation argument which reduces the problem to the case $r < \infty$. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\{a_n\}$ decreases to a and $\{b_n\}$ increases to b. Condition (1.11) implies

$$\sup_{a_n < s < b_n} \left\| \chi_{(s,b_n)}(x) \alpha(x) \|\chi_{(s,x)} u\|_\infty \right\|_{q,\infty;w} \|\chi_{(a_n,s)} v^{-\frac{1}{p}}\|_{p'} \leq C_2. \tag{2.16}$$

For fixed n, there exists $r_0 > p$ such that $(b_n - a_n)^{\frac{1}{r}} \leq 2$ for all r with $r_0 \leq r < \infty$. Then, if $r \geq r_0$ and $a_n < s < x < b_n$, we have

$$\|\chi_{(s,x)} u\|_r \leq \|\chi_{(s,x)} u\|_\infty (b_n - a_n)^{\frac{1}{r}} \leq 2 \|\chi_{(s,x)} u\|_\infty. \tag{2.17}$$

From (2.16) and (2.17) we get

$$\sup_{a_n < s < b_n} \left\| \chi_{(s,b_n)}(x) \alpha(x) \|\chi_{(s,x)} u\|_r \right\|_{q,\infty;w} \|\chi_{(a_n,s)} v^{-\frac{1}{p}}\|_{p'} \leq 2C_2, \tag{2.18}$$

for all r with $r_0 \le r < \infty$.

Condition (2.18) implies that for all $r \ge r_0$ and all f the inequality

$$\left\| \chi_{(a_n,b_n)}(x)\alpha(x) \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^{t} f \right\|_r \right\|_{q,\infty;w} \le 2H(r,p)C_2 \|\chi_{(a_n,b_n)} f\|_{p,v} \tag{2.19}$$

holds, where $H(r,p) = \max\{2 \cdot 4^{\frac{1}{r}} K(r,p), 2 \cdot 4^{1+\frac{1}{r}}\}$. Since

$$\left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^{t} f \right\|_\infty = \lim_{r\to\infty} \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^{t} f \right\|_r$$

for every x, by Fatou's lemma we have

$$\begin{aligned} &\left\| \chi_{(a_n,b_n)}(x)\alpha(x) \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^{t} f \right\|_\infty \right\|_{q,\infty;w} \\ &\le \liminf_{r\to\infty} \left\| \chi_{(a_n,b_n)}(x)\alpha(x) \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^{t} f \right\|_r \right\|_{q,\infty;w} \end{aligned} \tag{2.20}$$

and having into account that $\lim_{r\to\infty} H(r,p) = 8$, from (2.20) and (2.19) we get

$$\left\| \chi_{(a_n,b_n)}(x)\alpha(x) \left\| \chi_{(a_n,x)}(t)u(t) \int_{a_n}^{t} f \right\|_\infty \right\|_{q,\infty;w} \le 16C_2 \|\chi_{(a_n,b_n)} f\|_{p,v}. \tag{2.21}$$

Finally, since (2.21) holds for all n with a constant independent of n, letting n tend to infinity and applying the monotone convergence theorem we get (1.10) in the case $r = \infty$, as we wished to prove.

3. Proof of Theorem 3

First of all, observe that

$$\left(\int_a^x f\right)\left(\int_a^x g\right) = \int_a^x f(t)\left(\int_a^t g\right) dt + \int_a^x g(t)\left(\int_a^t f\right) dt. \tag{3.1}$$

Therefore, we have to characterize the weighted weak-type inequalities

$$\left\| \beta(x) \int_a^x f(t) \left(\int_a^t g\right) dt \right\|_{q,\infty;w} \le C\|f\|_{p_1,v_1}\|g\|_{p_2,v_2} \tag{3.2}$$

and

$$\left\| \beta(x) \int_a^x g(t) \left(\int_a^t f\right) dt \right\|_{q,\infty;w} \le C\|f\|_{p_1,v_1}\|g\|_{p_2,v_2}. \tag{3.3}$$

Inequality (3.2) is equivalent to

$$\left\| \beta(x) \int_a^x h \right\|_{q,\infty;w} \leq C \|h\|_{p_1, \tilde{v}_1^g}, \tag{3.4}$$

where $\tilde{v}_1^g(x) = v_1(x) \left(\int_a^x \frac{g}{\|g\|_{p_2,v_2}} \right)^{-p_1}$ and the constant C does not depend on g.

Since $q < p_1$ and β is a monotone function, by Theorem C inequality (3.4) holds if and only if there exists $C > 0$ such that

$$\|\Psi_g\|_{r_1,\infty;w} \leq C \tag{3.5}$$

for all g, where $\frac{1}{r_1} = \frac{1}{q} - \frac{1}{p_1}$ and

$$\Psi_g(x) = \sup_{c>x} \left((\inf_{y\in(x,c)} \beta(y)) \left(\int_x^c w \right)^{\frac{1}{p_1}} \right) \|\chi_{(a,x)} (\tilde{v}_1^g)^{-\frac{1}{p_1}} \|_{p_1'} = \alpha_1(x) \|\chi_{(a,x)} (\tilde{v}_1^g)^{-\frac{1}{p_1}} \|_{p_1'}.$$

Then (3.5) can be written as

$$\left\| \alpha_1(x) \left\| \chi_{(a,x)}(t) \left(v_1^{-\frac{1}{p_1}}(t) \int_a^t g \right) \right\|_{p_1'} \right\|_{r_1,\infty;w} \leq C \|g\|_{p_2,v_2}. \tag{3.6}$$

Therefore, inequality (3.2) holds if and only if inequality (3.6) holds. Since $p_2 \leq r_1$, by Theorem 2, (3.6) holds if and only if (1.13) holds in the case $p_2 \leq p_1'$ and (1.13) and (1.15) hold in the case $p_1' < p_2$.

In the same way, we see that (3.3) holds if and only if (1.14) holds in the case $p_2 \leq p_1'$ and (1.14) and (1.16) hold in the case $p_1' < p_2$.

4. Proof of Theorem 1

Since $p_1 \leq q$, by Theorem B, inequality (1.7) is equivalent to the existence of a positive constant C such that

$$\left\| \chi_{(s,b)}(x)\beta(x) \int_a^x \frac{g}{\|g\|_{p_2,v_2}} \right\|_{q,\infty;w} \|\chi_{(a,s)} v_1^{-\frac{1}{p_1}} \|_{p_1'} \leq C \tag{4.1}$$

for all $s \in (a,b)$ and all g.

Inequality (4.1) is equivalent to

$$\left\| \chi_{(s,b)}(x)\beta(x) \int_s^x g \right\|_{q,\infty;w} \leq C \|\chi_{(a,s)} v_1^{-\frac{1}{p_1}} \|_{p_1'}^{-1} \|g\chi_{(s,b)}\|_{p_2,v_2} \tag{4.2}$$

and

$$\left(\int_a^s g \right) \left\| \chi_{(s,b)}(x)\beta(x) \right\|_{q,\infty;w} \leq C \|\chi_{(a,s)} v_1^{-\frac{1}{p_1}} \|_{p_1'}^{-1} \|g\|_{p_2,v_2}, \tag{4.3}$$

with a constant C independent of s and g.

On one hand, since $q < p_2$, by Theorem C inequality (4.2) holds uniformly on s if and only if $C_1 < \infty$. On the other hand, a simple application of Holder's inequality shows that inequality (4.3) holds if and only if $B_4 < \infty$.

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