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Lorentzian Metrics Null-Projectively Related to Semi-Riemannian Metrics

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Abstract

We say that a Lorentzian metric and a semi-Riemannian metric on the same manifold M are null-projectively related if every null geodesic of the Lorentzian metric is an unparametrized geodesic of the semi-Riemannian one. This definition includes the case of conformally related Lorentzian metrics and the case of projectively equivalent metrics. We characterize the null-projectively relation by means of certain tensor and provide some examples. Then, we focus on the special case in which both metrics share parametrized null geodesics. In this case, it is said that they are null related. We show how to construct projectively equivalent metrics via a conformal transformation from null-related ones and conversely. The classical Levi-Civita theorem on projectively equivalent metrics is adapted to the case of null-related metrics and some results ensuring that two null-related metric are affinely equivalent are proven under curvature conditions.

Keywords Null-projectively related metrics · Null geodesics · Geodesically equivalent metrics · Affinely equivalent metrics

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1 Introduction

Two semi-Riemannian metrics g and g^* on an $n-$dimensional manifold M are affinely equivalent if they share their parametrized geodesics or, in other words, their

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Levi-Civita connections coincide. More generally, two semi-Riemannian metrics are geodesically equivalent (or projectively equivalent) if they share their unparametrized geodesics. The study of geodesically equivalent metrics goes back to E. Beltrami. The seminal work by Beltrami was motivated by cartography problems. He proved that a metric which is geodesically equivalent to a metric of constant curvature also has constant curvature, see [13] for a direct proof of this fact and [15] for a short history about the topic. Several years later and under some technical conditions, Levi-Civita obtained the local description of geodesically equivalent metrics [8], see also [12, pg.169] and references therein. The global aspects of geodesically equivalent metrics have also arisen the interest. For example, several rigidity theorems have been proven in [11, 15, 19]. These results led to sufficient conditions for two geodesically equivalent metrics being in fact affinely equivalent.

In this paper, we are mainly interested in the case that g is a Lorentzian metric (our convention is $(-, +, \ldots, +)$ for the signature of g). In this setting, there are different types of geodesics according with theirs causal characters. Namely, a geodesic γ is called spacelike if $g(\gamma', \gamma') > 0$, timelike if $g(\gamma', \gamma') < 0$ and null if $g(\gamma', \gamma') = 0$ and γ is not constant. It is well known that two Lorentzian metrics on M which are conformally related share their unparametrized null geodesics. This fact has leaded us to introduce the notion of null-projectively related metrics as a generalization of the conformal relation as follows. A Lorentzian metric g is said to be null-projectively related to a semi-Riemannian metric g^* if every null geodesic of g is (a non-necessary null) unparametrized geodesics of g^*, Definition 2. Observe that, unlike the conformally relation and the geodesic equivalence, the null-projectively relation is not a equivalence relation. Finally, we say that a Lorentzian metric g is null related to a semi-Riemannian metric g^* if every null geodesic of g is a geodesic of g^*, Definition 3.

Besides of the mathematical interest, the null-projectively relation also arises from a problem in geometrical optic. Roughly speaking, a nonimaging concentrator is a surface in $\mathbb{R}^3$ which reflects all input light rays (or at least most of them) into the exit aperture. We can think that it is essentially a funnel for light. The simplest example of nonimaging concentrator is a truncated cone, [20, pg. 49]. The edge-ray or string method is the classical one to design nonimaging concentrators. It basically consists in forcing the light rays emitted by the source to reach the exit aperture. The set formed by the reflection points of all light rays is the nonimaging concentrator, see details in [20, pg. 47]. But there are alternative methods in the literature. For example, one of them consists in finding a Lorentzian metric defined in an open set of $\mathbb{R}^3$ such that its unparametrized null geodesics are straight lines, [6], [20, pg. 144]. In our terminology, this Lorentzian metric is null-projectively related to the Euclidean metric of $\mathbb{R}^3$.

This paper deals with two main aims. Firstly, we find the characterization of null-projectively related metrics g and g^* by means of the difference tensor of the corresponding Levi-Civita connections, Theorem 6. Secondly, we focus on the so-called null-related metrics, which are a particular case of null-projectively related metrics. We look for conditions which permit to assure that two null-related metrics g and g^* are in fact affinely equivalent.

We have organized this paper as follows. Section 2 fixes some terminology and notations. We derive several formulas which involve the difference tensor of the Levi-Civita connections of two metrics g and g^* on the same manifold and relationships between the extrinsic geometry of a nondegenerate hypersurface L with respect to g and with respect to g^*, Lemma 4. It is worth pointing out that being a Lorentzian metric g affinely equivalent to a Riemannian metric g^* has strong consequences. Namely, (M, g) locally decomposes as $\left(\mathbb{R} \times L, -dt^2 + g_0\right)$ and (M, g^*) as $\left(\mathbb{R} \times L, c\,dt^2 + g_0^*\right)$, where g_0 and g_0^* are Riemannian metrics on L and c is a positive constant, Proposition 5.

Section 3 first deals with the above-mentioned Theorem 6 as follows. A Lorentzian metric g is null-projectively related to a semi-Riemannian metric g^* if and only if there are a vector field $N \in \mathfrak{X}(M)$ and a one form $\omega \in \mathfrak{X}^*(M)$ such that the difference tensor $D(X, Y) := \nabla^*_X Y - \nabla_X Y$ is given by

$$D(X, Y) = \omega(X)Y + \omega(Y) + g(X, Y)N$$

for all $X, Y \in \mathfrak{X}(M)$. We call the vector field $N \in \mathfrak{X}(M)$ the optical vector field and ω the projective form. These names are inspired by the above-mentioned geometrical optic problem and the classical projectively equivalent metrics theory. Namely, when the optical vector field N identically vanishes, the above formula reduces to the well-known relationship between the Levi-Civita connections of projectively equivalent metrics. On the other hand, the projective form ω vanishes if and only if g is null related to g^*. If M is orientable, the optical one form defined by $(n+1)\omega + \alpha$ is exact, where $\alpha := g(N, -)$. This allows us to prove that we can switch between null-projectively related, geodesically equivalent and null-related metrics using a suitable conformal change, see Proposition 11.

Section 4 is devoted to show a version of the Levi-Civita theorem [8] adapted to the case of null-related metrics, Theorem 14. It provides us a method to construct examples. For instance, there is a certain Lorentzian surface which is null related to an open subset of the Poincaré upper half-plane, Example 4.

Finally, we focus our attention on curvature relationships between null-related metrics in Sect. 5. Under certain assumptions, these relationships lead to assert that g and g^* are affinely equivalent. For example, Theorem 19 states that if g is null related to g^* and the optical vector field N satisfies $g(N, N) = 0$ and $Ric(N, N) \geq 0$, where Ric denotes the Ricci tensor of g, then they are affinely equivalent. The same conclusion is obtained in Theorem 27 if the Ricci tensors of g and g^* agree and in Theorem 29 when M is compact and g^* is a semi-Riemannian Ricci-flat metric.

We also give some obstructions to the existence of null-related metrics. For instance, when a Lorentzian metric g is null related to a Riemannian metric, the Ricci tensor of g must be diagonalizable, Corollary 22. Recall that in general, the Ricci tensor of a Lorentzian metric is not diagonalizable. Other obstructions are given in Theorem 23 for a Riemannian or Lorentzian manifold with constant curvature and in Corollary 30 for a compact, simply connected and Ricci-flat Riemannian manifold.

2 The Difference Tensor of Two Levi-Civita Connections

In this section, we suppose that g and g^* are two arbitrary semi-Riemannian metrics (unless otherwise stated) on a connected ($n \geq 2$)-dimensional manifold M with

Levi-Civita connections ∇ and ∇^*, respectively. The difference tensor is defined as

$$D(X,Y) = \nabla^*_X Y - \nabla_X Y \tag{1}$$

for every vector fields $X, Y \in \mathfrak{X}(M)$. The tensor D is symmetric, since ∇ and ∇^* are torsion free.

We consider the endomorphism fields $\varphi^*, \varphi : TM \to TM$ characterized by

$$g(X,Y) = g^*(\varphi^*(X), Y), \tag{2}$$

$$g^*(X,Y) = g(\varphi(X), Y) \tag{3}$$

for all $X, Y \in \mathfrak{X}(M)$. It is clear that φ^* and φ are isomorphisms with $\varphi^* = \varphi^{-1}$ and they are self-adjoint for both g and g^*. Therefore, if for example g^* is Riemannian, then φ and φ^* can be pointwise written in diagonal form with respect to a g^*-orthonormal basis at every point. The endomorphism fields φ and φ^* are closely related to the so-called Benenti tensor for projectively equivalent metrics. General facts and applications of Benenti tensors can be found in [3, 4, 9] and references therein.

Remark 1 Regardless on the signature of g and g^*, when φ^* can be written in diagonal form respect to a g^*-orthonormal basis $\{e_1, \ldots, e_n\}$ at $x \in M$ with $\varphi^*(e_i) = \lambda_i$, we get that

$$\left\{ \frac{1}{\sqrt{|\lambda_1|}} e_1, \ldots, \frac{1}{\sqrt{|\lambda_n|}} e_n \right\}$$

is also a g-orthogonal basis. Therefore, φ and φ^* are written in diagonal form with respect to the basis $\{e_1, \ldots, e_n\}$.

As the following lemma states, we can also ensure that φ and φ^* are diagonalizable if g and g^* are Lorentzian and they hold a causality condition. Recall that a vector u is called g-causal if it is g-timelike or g-null, i.e. $g(u,u) \leq 0$ with $u \neq 0$.

Definition 1 We say that a Lorentzian metric g^* has strictly wider null cones than another Lorentzian metric g if any g-causal vector is a g^*-timelike vector. In this case, we write $g < g^*$.

Lemma 1 *Let g and g^* be Lorentzian metrics with $g < g^*$ on a manifold M with* $\dim M \geq 3$. *Then φ and φ^* can be pointwise written in diagonal form respect to a g-orthonormal and a g^*-orthonormal basis.*

Proof If u is a g-null vector, then $g(\varphi(u), u) = g^*(u,u) < 0$, so from [7, p. 272] we conclude that φ can be written in diagonal form respect to a g-orthonormal basis. Since $\varphi^* = \varphi^{-1}$, it follows that φ^* is also written in diagonal form respect to this basis. From Remark 1, the same is true for a g^*-orthonormal basis. $\square$

We can easily check that

$$\begin{aligned}\left(\nabla_X g^*\right)(Y, Z) &= g^*(D(X, Y), Z) + g^*(Y, D(X, Z)) \\ &= g(D(X, Y), \varphi(Z)) + g(\varphi(Y), D(X, Z)). \end{aligned} \tag{4}$$

Using this, we can compute the covariant derivative of the tensors φ^* and φ in terms of the difference tensor as follows:

Lemma 2 *Given $X, Y, Z \in \mathfrak{X}(M)$, we have*

$$g((\nabla_X\varphi)(Y), Z) = g(D(X, Y), \varphi(Z)) + g(D(X, Z), \varphi(Y)), \tag{5}$$

$$g((\nabla_X\varphi^*)(Y), Z) = -g(D(X, \varphi^*(Y)), Z) - g(D(X, \varphi^*(Z)), Y). \tag{6}$$

Proof If we take the covariant derivative respect to $X \in \mathfrak{X}(M)$ in $g(\varphi(Y), Z) = g^*(Y, Z)$ and we use Equation (4), then

$$g((\nabla_X\varphi)(Y), Z) = g(D(X, Y), \varphi(Z)) + g(\varphi(Y), D(X, Z)),$$

but from $\varphi^* = \varphi^{-1}$, it is straightforward to check that

$$(\nabla_X\varphi)(Y) = -\varphi(\left(\nabla_X\varphi^*\right)(\varphi(Y))),$$

and therefore,

$$-g(\left(\nabla_X\varphi^*\right)(\varphi(Y)), \varphi(Z)) = g(D(X, Y), \varphi(Z)) + g(D(X, Z), \varphi(Y)).$$

If we replace $\varphi(Z)$ with Z and $\varphi(Y)$ with Y in above equation, then we obtain Equation (6). □

Lemma 3 *Suppose that M is oriented and call Ω and Ω^* the volume form of g and g^*, respectively. If $\varrho \in C^\infty(M)$ is the (necessarily positive) function such that $\Omega^* = \varrho\Omega$, then*

$$\mathtt{div}^*X = \mathtt{div}X + g(\nabla \ln \varrho, X)$$

for all $X \in \mathfrak{X}(M)$.

Proof A direct computation shows that

$$\mathtt{div}^*X \cdot \Omega^* = L_X\Omega^* = L_X(\varrho\Omega) = X(\varrho)\Omega + \varrho L_X\Omega = X(\varrho)\Omega + \varrho\mathtt{div}X \cdot \Omega.$$

Hence, we get the announced formula $\mathtt{div}^*X = X(\ln\varrho) + \mathtt{div}X$. □

Let us recall that a hypersurface L of a semi-Riemannian manifold (M, g) is called nondegenerate if L inherits a nondegenerate metric tensor from the ambient metric g. In the particular case that (M, g) is a Lorentzian manifold, a nondegenerate hypersurface

L is said to be spacelike when the induced metric on L is Riemannian and timelike when it is Lorentzian.

We have the following relations.

Lemma 4 *Let L be a hypersurface in M which is nondegenerate neither for (M, g) nor for (M, g^*) and denote by $\mathbb{I}$ and $\mathbb{I}^*$ the second fundamental forms of L in (M, g) and (M, g^*), respectively.*

1. *If $X, Y, Z \in \mathfrak{X}(L)$, then*
$$g^*(D(X,Y),Z) = g^*(D^L(X,Y),Z) - g^*(\mathbb{I}(X,Y),Z),$$
where D^L is the difference tensor of the induced connections on L.
2. *If E is g^*-orthogonal to L, then*
$$g^*(\mathbb{I}^*(X,Y) - \mathbb{I}(X,Y),E) = g^*(D(X,Y),E).$$

Proof The first assertion is a straightforward computation. For the second one, if E is a g^*-orthogonal vector field to L, then $\varphi(E)$ is g-orthogonal to L. From Lemma 2, we have

$$\begin{aligned}
&-g^*(\mathbb{I}(X,Y),E) = g(\nabla_X\varphi(E),Y) = g((\nabla_X\varphi)(E),Y) + g(\varphi(\nabla_X E),Y)\\
&\quad = g(D(X,E),\varphi(Y)) + g(D(X,Y),\varphi(E)) + g(\nabla^*_X E,\varphi(Y)) - g(D(X,E),\varphi(Y))\\
&\quad = g^*(D(X,Y),E) - g^*(\mathbb{I}^*(X,Y),E).
\end{aligned}$$

□

A classification result for Lorentzian metrics affinely equivalent was obtained in [10]. It can be stated as follows. If g and g^* are affinely equivalent Lorentzian metrics, then we have one of the following three possibilities.

1. g and g^* are proportional.
2. g and g^* admit a local decomposition as a direct product.
3. There is a null vector field K such that $g^* = g + \eta\otimes\eta$, being η the g-metrically equivalent one form to K.

Similarly, we have the following result for the case of a Riemannian metric g^* and a Lorentzian metric g which are affinely equivalent.

Proposition 5 *Let (M, g) be a Lorentzian manifold and g^* a Riemannian metric on M. If they are affinely equivalent, then locally (M, g) decomposes as $\left(\mathbb{R}\times L, -dt^2 + g_0\right)$ and (M, g^*) as $\left(\mathbb{R}\times L, c\,dt^2 + g_0^*\right)$, where g_0 and g_0^* are Riemannian metrics on L and c is a positive constant. In particular, if M is simply connected, then the above decompositions are global.*

Proof Call $g_s = sg^* + (1-s)g$ for $s \in \mathbb{R}$ and

$t_0 = sup\{t \in [0,1] : g_s$ is nondegenerate for each point of M and for all $0 \le s \le t\}$.

Obviously, we have $t_0 < 1$ because g_0 is Lorentzian and g^* is Riemannian. Since $\nabla = \nabla^*$, we have that $\nabla g_{t_0} = 0$ and so it is not difficult to show that dim $Rad(g_{t_0})$ is constant on M. Therefore, by continuity, we also have that $0 < t_0$.

If dim $Rad(g_{t_0}) = 0$, then g_{t_0} is nondegenerate for each point of M. Using again the continuity and that dim $Rad(g_{t_0})$ is constant, we can check that this contradicts the fact t_0 is the supremum. Therefore, g_{t_0} is degenerate and dim $Rad(g_{t_0}) > 0$. Moreover, we have that dim $Rad(g_{t_0}) = 1$ because for all $v \in Rad(g_{t_0})$ with $v \neq 0$ it holds $g(v, v) = -\frac{t_0}{1-t_0} g^*(v, v) < 0$.

Finally, taking into account that $Rad(g_{t_0})$ is g-parallel and applying the De Rham-Wu decomposition theorem [21], we have that (M, g) locally decomposes as $\left(\mathbb{R} \times L, -dt^2 + g_0\right)$ and (M, g^*) as $\left(\mathbb{R} \times L, c\, dt^2 + g_0^*\right)$, where $c = \frac{1-t_0}{t_0}$, L is a leaf of $Rad(g_{t_0})^\perp$ and g_0 and g_0^* are the restriction to L of g and g^*, respectively. □

3 Null-Projectively Related Metrics

From now on, let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric on M.

Definition 2 We say that g is *null-projectively related* to g^* if every null geodesic of g is a (non-necessarily null) unparametrized geodesic of g^*.

We can also define the following more restrictive notion.

Definition 3 We say that g is *null related* to g^* if every null geodesic of g is a (non-necessarily null) geodesic of g^*.

Obviously, if g is null related to g^* or geodesically equivalent to g^*, then g is also null-projectively related to g^*. On the other hand, recall that two Lorentzian metrics which share their null cones are conformally related, [2].

It is a well known that g and g^* are projectively equivalent if and only if there is a one form $\omega \in \mathfrak{X}^*(M)$ such that

$$D(X, Y) = \omega(X)Y + \omega(Y)X$$

for all $X, Y \in \mathfrak{X}(M)$, see for instance [18, pg. 273]. The following result characterizes null-projectively related metrics also in terms of the difference tensor.

Theorem 6 *Let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric on M.*

1. *g is null-projectively related to g^* if and only if there are a vector field $N \in \mathfrak{X}(M)$ and a one form $\omega \in \mathfrak{X}^*(M)$ such that*

$$D(X, Y) = g(X, Y)N + \omega(X)Y + \omega(Y)X, \tag{7}$$

for all $X, Y \in \mathfrak{X}(M)$.

2. *g is null related to g^* if and only if there is a vector field $N \in \mathfrak{X}(M)$ such that*

$$D(X,Y) = g(X,Y)N$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof Assume g is null-projectively related to g^*. Then, for every g-null tangent vector $u \in TM$, there exists a real number $\lambda(u)$ such that $D(u,u) = \lambda(u)u$. Fix a point p and take a g-orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM with e_n timelike. Since for each $k < n$, we have that $D(e_k + e_n, e_k + e_n)$ is proportional to $e_k + e_n$ and $D(e_k - e_n, e_k - e_n)$ to $e_k - e_n$, then

$$D(e_k, e_n) = \lambda_k e_k + \mu_k e_n,$$
$$D(e_k, e_k) + D(e_n, e_n) = 2\mu_k e_k + 2\lambda_k e_n$$

where $\lambda_k = \frac{1}{4}\left(\lambda(e_k + e_n) - \lambda(e_k - e_n)\right)$ and $\mu_k = \frac{1}{4}\left(\lambda(e_k + e_n) + \lambda(e_k - e_n)\right)$. Take $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $a \neq 0, b \neq 0$ and consider the g-null vector $u = ae_i + be_j + e_n$ for $i \neq j < n$. Then, we can compute

$$\begin{aligned}
\lambda(u)u = D(u,u) &= a^2 D(e_i, e_i) + 2abD(e_i, e_j) + b^2 D(e_j, e_j) + D(e_n, e_n) \\
&\quad + 2aD(e_i, e_n) + 2bD(e_j, e_n) \\
&= \left(2a^2\mu_i + 2a\lambda_i\right) e_i + \left(2b^2\mu_j + 2b\lambda_j\right) e_j \\
&\quad + \left(2a^2\lambda_i + 2b^2\lambda_j + 2a\mu_i + 2b\mu_j\right) e_n + 2abD(e_i, e_j).
\end{aligned}$$

Therefore, we get that $D(e_i, e_j) = Ae_i + Be_j + Ce_n$ for some $A, B, C \in \mathbb{R}$ and so

$$\lambda(u) = 2a\mu_i + 2\lambda_i + 2bA, \tag{8}$$
$$\lambda(u) = 2b\mu_j + 2\lambda_j + 2aB, \tag{9}$$
$$\lambda(u) = 2a^2\lambda_i + 2b^2\lambda_j + 2a\mu_i + 2b\mu_j + 2abC. \tag{10}$$

Subtracting Equation (8) from Equation (10), we get

$$\lambda_i + bA - a^2\lambda_i - b^2\lambda_j - b\mu_j - abC = 0$$

and using that $a^2 + b^2 = 1$ we arrive to

$$b(\lambda_j - \lambda_i) + aC = A - \mu_j,$$

for all $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$ and $a \neq 0, b \neq 0$. Thus, we necessarily obtain that $C = 0$, $A = \mu_j$ and $\lambda_i = \lambda_j$ for all $i \neq j < n$. Now, in a similar way, subtracting Equation (9) from Equation (10), we get $B = \mu_i$. Summarizing, we have

$$D(e_i, e_n) = \gamma e_i + \mu_i e_n,$$

$$D(e_i, e_i) + D(e_n, e_n) = 2\mu_i e_i + 2\gamma e_n,$$
$$D(e_i, e_j) = \mu_j e_i + \mu_i e_j,$$

for $i \neq j < n$, where $\gamma = \lambda_i$ for all $i < n$.

Now, we define the one form ω such that $\omega(e_i) = \mu_i$ for $i < n$ and $\omega(e_n) = \gamma$ and the vector

$$N = (2\gamma + g(D(e_n, e_n), e_n))e_n - \sum_{i=1}^{n-1} g(D(e_n, e_n), e_i)e_i.$$

It is a straightforward computation to check that $D(X, Y) = g(X, Y)N + \omega(X)Y + \omega(Y)X$ for all $X, Y \in \mathfrak{X}(M)$.

Finally, if g is null related to g^*, then $\lambda(u) = 0$ for all g-null tangent vectors $u \in TM$ and, thus, $\omega = 0$. □

Definition 4 In Equation (7), ω is called the projective form and N the optical vector field. The optical form is the g-metrically equivalent one form to N, and it is denoted by α. That is, $\alpha = g(N, -)$.

Observe that from Equations (2) and (3), we have $\alpha \circ \varphi = \alpha^*$ and $\alpha^* \circ \varphi^* = \alpha$, being α^* the one form g^* metrically equivalent to N.

Example 1 It is well known that two conformally related Lorentzian metrics are also null-projectively related. In fact, if $g^* = e^{2\Phi} g$, then we have

$$\nabla^*_X Y - \nabla_X Y = d\Phi(X)Y + d\Phi(Y)X - g(X, Y)\nabla\Phi. \tag{11}$$

So, in our terminology, the projective form is $\omega = d\Phi$ and the optical form is $\alpha = -d\Phi$.

Corollary 7 *Let g and g^* two Lorentzian metrics on a manifold M. If g is null related to g^* and g^* is null related to g, then g and g^* are affinely equivalent.*

Proof From Theorem 6 there are vector fields $N, N^* \in \mathfrak{X}(M)$ such that

$$\nabla^*_X Y - \nabla_X Y = g(X, Y)N,$$
$$\nabla_X Y - \nabla^*_X Y = g^*(X, Y)N^*$$

for all $X, Y \in \mathfrak{X}(M)$. We will show that N is identically zero. Suppose on the contrary that there is a point $p \in M$ with $N_p \neq 0$. In this case, there is a neighbourhood U of p where any g^*-null vector is also a g-null vector and so $g^* = e^{2\Phi} g$ for certain function Φ defined in U. Using Equation (11) we have that $d\Phi(u) = 0$ for all g-null vectors u and so Φ is constant. This means that g and g^* are homothetic in a neighbourhood of p and, therefore, $N_p = 0$, which is a contradiction. □

The following corollary will provide us with some examples of null-related metrics.

Corollary 8 *Let (M, g) be a Lorentzian manifold, $U \in \mathfrak{X}(M)$ a vector field with $g(U,U) \neq -\frac{1}{c}$ for some constant $c \neq 0$ and Ω its metrically equivalent one form. If U is closed (i.e. Ω is closed) and conformal, then g is null related to the semi-Riemannian metric $g^* = g + c\,\Omega \otimes \Omega$.*

Proof We first show that g^* is a metric tensor with index 0, 1 or 2. For each point $p \in M$, we can split $T_pM = \Pi \oplus \Pi^\perp$, where Π is a timelike plane which contains U_p. Since $g|_{\Pi^\perp} = g^*|_{\Pi^\perp}$, the index of g^* is equal to the index of $g^*|_\Pi$. If we write $\Pi = span\{v, U_p\}$ with v timelike and unitary, then the determinant of $g^*|_\Pi$ is

$$(1 + cg(U_p, U_p))(-g(U_p, U_p) - g(v, U_p)^2).$$

Taking into account that $0 < g(U_p, U_p) + g(v, U_p)^2$ because Π is timelike, we have that g^* is Lorentzian when $0 < 1 + cg(U,U)$ and g^* is Riemannian or it has index 2 when $1 + cg(U,U) < 0$.

Now, we can show as in [17, Proposition 2.3] that for the difference tensor holds

$$g^*(D(X,Y), Z) = \frac{c}{2}\big(\Omega(Z)\,(L_U g)\,(X,Y) + \Omega(X)d\Omega(Y,Z) + \Omega(Y)d\Omega(X,Z)\big)$$

for all $X, Y, Z \in \mathfrak{X}(M)$. Since U is closed and conformal, i.e. $d\Omega = 0$ and $L_U g = 2\lambda g$ for some $\lambda \in C^\infty(M)$, the above formula reduces to

$$g^*(D(X,Y), Z) = c\lambda g(X,Y)g(U,Z) = c\lambda g(X,Y)g^*(\varphi^*(U), Z),$$

and thus

$$D(X,Y) = c\lambda g(X,Y)\varphi^*(U).$$

Using Theorem 6, g is null related to g^* with optical vector field $N = c\lambda\varphi^*(U)$. □

Remark 2 In a general setting, although U is neither closed nor conformal, for a metric tensor $g^* = g + c\,\Omega \otimes \Omega$, we have that $\varphi^*(U) = \frac{c\lambda g(X,Y)}{1+cg(U,U)} U$.

Example 2 Consider the generalized Robertson-Walker space

$$\left(I \times F, -dt^2 + f(t)^2 g_0\right),$$

where $I \subset \mathbb{R}$, $f \in C^\infty(I)$ is a positive function and (F, g_0) is a Riemannian manifold [1]. The vector field $U = f\partial_t$ is timelike, closed and conformal with conformal factor $\lambda = \frac{f'}{f}$. If we suppose that $\frac{1}{\sqrt{c}} < f(t)$ for some constant $c > 0$ and for all $t \in I$, then g is null related to the Riemannian metric

$$g^* = g + c\Omega \otimes \Omega = \left(cf(t)^2 - 1\right)dt^2 + f(t)^2 g_0.$$

If we particularize to the case of the De Sitter space

$$\mathbb{S}_1^n = \left(\mathbb{R} \times \mathbb{S}^{n-1}, g = -dt^2 + \cosh^2(t) g_0\right),$$

then we get that g is null related to the family of Riemannian metrics

$$g^* = \left(c \cosh^2(t) - 1\right) dt^2 + \cosh^2(t) g_0$$

for all $c > 1$.

The following corollary permits to give more examples of null-related metrics. The proof relies on Lemma 4 and Theorem 6. It can also be proven using the remarkable property which states that a timelike hypersurface L is totally umbilical if and only if every null geodesic of L is a (necessarily null) geodesic of the ambient metric.

Corollary 9 *Let (M, g) be a Lorentzian manifold and L a timelike hypersurface. Suppose that g^* is a Riemannian metric such that g is null related to g^*.*

1. *If L is totally geodesic with respect to g^*, then L is totally umbilical with respect g.*
2. *If L is totally umbilical with respect to g, then (L, g) is null related to (L, g^*).*

Example 3 Let us consider the Minkowski spacetime $\mathbb{L}^{n+1} = (\mathbb{R}^{n+1}, g)$, which is null related (in fact affinely equivalent) to the euclidean space $\mathbb{E}^{n+1} = (\mathbb{R}^{n+1}, g^*)$. The De Sitter space $\mathbb{S}_1^n = \{x \in \mathbb{R}^{n+1} : g(x, x) = 1\}$ is a timelike and totally umbilical hypersurface in $\mathbb{L}^{n+1}$. From the above corollary, $(\mathbb{S}_1^n, g)$ is null related to $(\mathbb{S}_1^n, g^*)$, as it is well known. In this way, we recover Example 2 with $c = 2$.

Lemma 10 *Let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric on M. If g is null-projectively related to g^*, then the one form $(n + 1)\omega + \alpha$ is closed. Moreover, if M is orientable, then $(n + 1)\omega + \alpha$ is exact.*

Proof We can locally take volume n-forms Ω and Ω^* for g and g^*, respectively, compatible with a fixed (local) orientation on M. By construction, it holds $\Omega^* = \varrho\Omega$ for some positive function ϱ. For every $p \in M$, we take $E_1, \ldots, E_n$ a local positive $g-$orthonormal frame such that $\nabla_v E = 0$ for all $v \in T_p M$. Since $\nabla^* \Omega^* = \nabla\Omega = 0$, we have

$$\begin{aligned} d\varrho(v) = v\left(\Omega^*(E_1, \ldots, E_n)\right) &= \left(\nabla_v^* \Omega^*\right)(E_1, \ldots, E_n) \\ &\quad + \sum_{i=1}^n \Omega^*(E_1, \ldots, \nabla_v^* E_i, \ldots, E_n) \\ &= \sum_{i=1}^n \Omega^*(E_1, \ldots, D(v, E_i), \ldots, E_n). \end{aligned}$$

From Formula (7), we get

$$\begin{aligned} d\varrho(v) &= n\varrho\omega(v) + \sum_{i=1}^{n} \omega(E_i)\Omega^*(E_1, \ldots, v, \ldots, E_n) \\ &\quad + g(v, E_i)\Omega^*(E_1, \ldots, N, \ldots, E_n) \\ &= (n+1)\varrho\omega(v) + \varrho g(N, v). \end{aligned}$$

Thus, we have $d \ln \varrho = (n+1)\omega + \alpha$. □

The null-projectively relation can be reduced to the null relation via a conformal change as follows.

Proposition 11 *Let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric on M. Suppose that g is null-projectively related to g^* with projective form ω and optical form α.*

1. *If ω is exact, then there is a conformal metric to g which is null related to g^*. More concretely, if $\omega = d\Phi_1$, then $\widetilde{g} = e^{2\Phi_1} g$ is null related to g^* with optical form $\alpha + \omega$.*
2. *If α is exact, then there is a conformal metric to g which is geodesically equivalent to g^*. More concretely, if $\alpha = d\Phi_2$, then $\widetilde{g} = e^{2\Phi_2} g$ is geodesically equivalent to g^* with projective form $\alpha + \omega$.*

Proof Suppose that $\omega = d\Phi_1$. From Formulas (7) and (11), the Levi-Civita connections of g^*, $\widetilde{g} = e^{2\Phi_1} g$ and g are related by

$$\begin{aligned} \nabla^*_X Y - \nabla_X Y &= \omega(X)Y + \omega(Y)X + g(X, Y)N, \\ \widetilde{\nabla}_X Y - \nabla_X Y &= \omega(X)Y + \omega(Y)X - g(X, Y)\nabla\Phi_1. \end{aligned}$$

Therefore, we get

$$\nabla^*_X Y - \widetilde{\nabla}_X Y = g(X, Y)(N + \nabla\Phi_1) = \widetilde{g}(X, Y)e^{-2\Phi_1}(N + \nabla\Phi_1)$$

and thus, $\widetilde{g}$ is null related to g^* with optical form $\alpha + \omega$. For $\alpha = d\Phi_2$, we proceed in a similar way. □

Remark 3 Assume M is oriented. Then, from Lemma 10, the exactness of ω and α are equivalent. On the other hand, if g and g^* are geodesically equivalent, then $\alpha = 0$ and we get from Lemma 10 that $\omega = \frac{1}{n+1} d \ln \varrho$. On the contrary, if g is null related to g^*, then $\omega = 0$ and, thus, $\alpha = d \ln \varrho$ and $N = \nabla \ln \varrho$.

Using this remark, we can also link the null relation and the geodesic equivalence as follows.

Proposition 12 *Let (M, g) be an oriented Lorentzian manifod and g^* a semi-Riemannian metric on M.*

1. *If g is null related to g^* with optical form α, then there is a conformal metric to g which is geodesically equivalent to g^*. More concretely, if $\alpha = d\ln\varrho$, then g^* and $\widetilde{g} = \frac{1}{\varrho^2} g$ are geodesically equivalent with projective form α.*
2. *If g and g^* are geodesically equivalent with projective form ω, then there is a conformal metric to g which is null related to g^*. More concretely, if $\omega = \frac{1}{n+1} d\ln\varrho$, then $\widetilde{g} = \varrho^{\frac{2}{n+1}} g$ is null related to g^* with optical form ω.*

Proof 1. From Theorem 6 and Equation (11), we have for the Levi-Civita connections $\widetilde{\nabla}$ and ∇^* that

$$\begin{aligned}
\nabla^*_X Y - \nabla_X Y &= g(X, Y)\nabla \ln\varrho, \\
\widetilde{\nabla}_X Y - \nabla_X Y &= -\alpha(X)Y - \alpha(Y)X + g(X, Y)\nabla \ln\varrho.
\end{aligned}$$

Therefore, $\nabla^*_X Y - \widetilde{\nabla}_X Y = \alpha(X)Y + \alpha(Y)X$, and thus, g^* and $\widetilde{g}$ are geodesically equivalent with projective form α.

2. In a similar way, for the Levi-Civita connection of $\widetilde{g} = \varrho^{\frac{2}{n+1}} g$, we have

$$\begin{aligned}
\nabla^*_X Y - \nabla_X Y &= \omega(X)Y + \omega(Y)X, \\
\widetilde{\nabla}_X Y - \nabla_X Y &= \omega(X)Y + \omega(Y)X - \frac{1}{n+1} g(X, Y)\nabla \ln\varrho.
\end{aligned}$$

Therefore,

$$\nabla^*_X Y - \widetilde{\nabla}_X Y = \frac{1}{n+1} g(X, Y)\nabla \ln\varrho = \frac{\varrho^{-\frac{2}{n+1}}}{n+1} \widetilde{g}(X, Y)\nabla \ln\varrho$$

and, thus, $\widetilde{g}$ is null related to g^* with optical form ω.

□

4 On a Classical Levi-Civita Theorem

Levi-Civita characterized certain family of geodesically equivalent Riemannian metrics in [8]. A modern formulation can be found in [14], where the author generalizes this classical result to a wider family of geodesically equivalent Riemannian metrics. Moreover, in [12], the author points out that the Levi-Civita theorem remains true for two geodesically equivalent semi-Riemannian metrics provided that at least one of the metrics is Riemannian.

We can prove a similar result for null-related metrics. First, we recall the following statement of the classical Levi-Civita theorem adapted to the case of a Lorentzian metric g geodesically equivalent to a Riemannian metric g^*. Observe that in this case the field of endomorphisms φ and φ^* given in (2) and (3) have exactly one negative eigenvalue.

Theorem 13 *(Levi-Civita) Let (M, g^*) be a Riemannian manifold and g a geodesically equivalent Lorentzian metric to g^*. Suppose that at a point $x \in M$, the eigenvalues of φ^* are different and equal to $\rho_1(x) < 0$ and $0 < \rho_n(x) < \ldots < \rho_2(x)$. Then, there are smooth functions $\phi_1 < 0$ and $0 < \phi_2 < \ldots < \phi_n$ and a coordinate system $(x_1, \ldots, x_n)$ around $x \in M$ such that*

- *ϕ_i only depends on x_i.*
- *$\rho_i = \frac{-1}{\phi_1 \cdot \ldots \cdot \phi_i^2 \cdot \ldots \cdot \phi_n}$.*
- *$g^* = \sum_{i=1}^n \Pi_i dx_i^2$, where $\Pi_i = (\phi_i - \phi_1) \cdot \ldots \cdot (\phi_i - \phi_{i-1}) \cdot (\phi_{i+1} - \phi_i) \cdot \ldots \cdot (\phi_n - \phi_i)$.*
- *$g = \sum_{i=1}^n \rho_i \Pi_i dx_i^2$.*

Conversely, given a family of non-vanishing smooth functions ϕ_i as above, the corresponding metrics g and g^* are geodesically equivalent.

Note that our assumption on the Lorentzian signature (we are supposing that the timelike direction is always at the first place) yields to write a minus sign in the expression of ρ_i. Since $-g$ and g^* are also geodesically equivalent, this sign change is irrelevant.

If Ω^* and Ω are the locally defined volume forms around x of g^* and g, respectively, then $\Omega^* = \varrho\Omega$ where $\varrho = \frac{1}{\sqrt{|\rho_1 \cdot \ldots \cdot \rho_n|}}$. Hence, from Remark 3, the projective form is given by

$$\omega = \frac{1}{n+1} d \ln \varrho = \frac{1}{2} \sum_{i=1}^n \frac{\phi_i'}{\phi_i} dx_i,$$

where $\phi_i' = \frac{\partial \phi_i}{\partial x_i}$.

From the Levi-Civita theorem and Proposition 12, we have the following.

Theorem 14 *Let (M, g^*) be a Riemannian manifold and g a null-related Lorentzian metric to g^*. Suppose that at a point $x \in M$, the eigenvalues of φ^* are different and equal to $\lambda_1(x) < 0$ and $0 < \lambda_n(x) < \ldots < \lambda_2(x)$. Then, there is a coordinate system $(x_1, \ldots, x_n)$ such that*

- *λ_i only depends on x_i.*
- *$g^* = \sum_{i=1}^n \Pi_i dx_i^2$, where $\Pi_i = (\frac{1}{\lambda_i} - \frac{1}{\lambda_1}) \cdot \ldots \cdot (\frac{1}{\lambda_i} - \frac{1}{\lambda_{i-1}}) \cdot (\frac{1}{\lambda_{i+1}} - \frac{1}{\lambda_i}) \cdot \ldots \cdot (\frac{1}{\lambda_n} - \frac{1}{\lambda_i})$.*
- *$g = \sum_{i=1}^n \lambda_i \Pi_i dx_i^2$.*
- *The optical form is $\alpha = \frac{1}{2} \sum_{i=1}^n \frac{\lambda_i'}{\lambda_i} dx_i$.*

Conversely, given never vanishing smooth functions λ_i as above, the metric g is null related to g^*.

Proof Without lost of generality, we can assume that M is oriented. Let us consider $\widetilde{g} = \frac{1}{\varrho^2} g$, where, as always, ϱ is the positive function such that $\Omega^* = \varrho\Omega$ and Ω^*, Ω are the volume forms of g^* and g, respectively. From Proposition 12, g^* and $\widetilde{g}$ are geodesically equivalent. Now, taking into account that $\widetilde{g}(X, Y) = g^*(\varrho^2 \varphi^*(X), Y)$ for all $X, Y \in \mathfrak{X}(M)$, the functions ρ_i in the Levi-Civita theorem hold

$$\lambda_i = \varrho^2 \rho_i. \tag{12}$$

Therefore, the assumptions of the Levi-Civita theorem are satisfied and then locally

$$g^* = \sum_{i=1}^{n} \Pi_i dx_i^2, \quad g = \varrho^2 \widetilde{g} = \sum_{i=1}^{n} \varrho^2 \rho_i \Pi_i dx_i^2,$$

where $\Pi_i = (\phi_i - \phi_1) \cdot \ldots \cdot (\phi_i - \phi_{i-1}) \cdot (\phi_{i+1} - \phi_i) \cdot \ldots \cdot (\phi_n - \phi_i)$ and

$$\rho_i = \frac{-1}{\phi_1 \cdot \ldots \cdot \phi_i^2 \cdot \ldots \cdot \phi_n}, \tag{13}$$

for certain functions ϕ_i such that every ϕ_i only depends on x_i with $\phi_1 < 0$ and $0 < \phi_2 < \ldots < \phi_n$. If we call $\widetilde{\Omega}$ the volume form of $\widetilde{g}$, then $\Omega^* = \varrho^{n+1}\widetilde{\Omega}$ and, thus,

$$\varrho^{n+1} = \frac{1}{\sqrt{|\rho_1 \cdot \ldots \cdot \rho_n|}} = \left(\sqrt{|\phi_1 \cdot \ldots \cdot \phi_n|}\right)^{n+1}.$$

Using the above equation and Equations (12) and (13), we get $\lambda_i = \frac{1}{\Phi_i}$ and the announced result. $\square$

Example 4 Consider the Lorentzian surface (M, g) where $M = \{(x, y) : 0 < y < 1\}$ and

$$g = -\frac{1}{y^2} dx^2 + \frac{1}{1 - y^2} dy^2.$$

The Lorentzian metric g is null related to the Riemannian metric of constant negative curvature:

$$g^* = \frac{1}{y^2}(dx^2 + dy^2).$$

This assertion is a direct consequence of Theorem 14 taking $\lambda_1(x) = -1$ and $\lambda_2(y) = \frac{y^2}{1-y^2}$.

5 Curvature and Null-Related Metrics

In this section, we give some curvature conditions which prevent that a Lorentzian metric g is null related to a semi-Riemannian metric g^* or ensure that they are affinely equivalent. For the following lemma, we need the definition of the divergence of a symmetric (0, 2)-tensor, [16, pg. 86]. Concretely, the divergence of g^* with respect to g is

$$(\mathtt{div} g^*)(V) = \sum_i \epsilon_i \left(\nabla_{E_i} g^*)(E_i, V\right)$$

for all $V \in \mathfrak{X}(M)$, where $E_1, \cdots, E_n$ is a local $g-$orthonormal basis and $\epsilon_i = g(E_i, E_i)$.

Lemma 15 *Let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric. If g is null related to g^*, then*

1. $$g((\nabla_X \varphi)(Y), Z) = g(X, Y)\alpha^*(Z) + g(X, Z)\alpha^*(Y). \tag{14}$$

2. $$\left(\nabla_X \alpha^*\right)(Y) = g(X, Y)\alpha^*(N) + \alpha(X)\alpha^*(Y) + g(\nabla_X N, \varphi(Y)). \tag{15}$$

3. $$\alpha^* = \frac{1}{2} d\, \mathtt{tr}\varphi = \frac{1}{n+1}\mathtt{div} g^*.$$

Proof From Equation (5), Theorem 6 and the fact that $\alpha^* = \alpha \circ \varphi$, we get Equation (14). On the other hand, using Equation (4), we have

$$\begin{aligned}
\left(\nabla_X \alpha^*\right)(Y) &= \left(\nabla_X g^*\right)(Y, N) + g^*(\nabla_X N, Y) \\
&= g(X, Y)g(N, \varphi(N)) + g(X, N)g(\varphi(Y), N) + g(\nabla_X N, \varphi(Y)) \\
&= g(X, Y)\alpha^*(N) + \alpha(X)\alpha^*(Y) + g(\nabla_X N, \varphi(Y)),
\end{aligned}$$

which is Formula (15).

For the last point, given $p \in M$, we take a local g-orthonormal frame field $\{E_1, \ldots, E_n\}$ such that $\nabla_v E_i = 0$ for all $v \in T_p M$. Then, from Equation (14) we have

$$v(\mathtt{tr}\varphi) = v\left(\sum_{i=1}^{n} \epsilon_i g(\varphi(E_i), E_i)\right) = \sum_{i=1}^{n} \epsilon_i g\left(\left(\nabla_v \varphi\right)(E_i), E_i\right) = 2\alpha^*(v).$$

Finally, a straightforward computation from Equation (4) gives us

$$(\mathtt{div} g^*)(v) = \sum_i \epsilon_i \left(\nabla_{E_i} g^*\right)(E_i, v) = (n+1)\alpha^*(v).$$

□

Note that Equation (14) is similar to the well-known Sinjukov formula for geodesically equivalent metrics, see for instance [11].

Proposition 16 *Let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric on M. Suppose that g is null related to g^* and the optical vector field N is complete. If $g^*(N, N)$ is constant, then $g(N, N) = 0$ and N is both g-geodesic and g^*-geodesic.*

Proof We know from Lemma 15 that α^* is closed, so we have that

$$0 = d\alpha^*(N, X) = g^*(\nabla^*_N N, X) - g^*(\nabla^*_X N, N) = g^*(\nabla^*_N N, X)$$

for all $X \in \mathfrak{X}(M)$. Therefore, N is g^*-geodesic. Now, take γ an integral curve of N and call $y(t) = g(N, N)_{\gamma(t)}$. We have from Theorem 6 that

$$y'(t) = 2g(\nabla_N N, N)_{\gamma(t)} = -2y(t)^2.$$

Since N is complete, $y(t)$ is a solution of the differential equation $y' + 2y^2 = 0$ defined on all $\mathbb{R}$. A classical argument shows that $y(t) = 0$ and then $g(N, N) = 0$. But α is closed from Lemma 10, so as before we can conclude that N is also g-geodesic. □

Observe that if we only suppose that $g(N, N)$ is constant, then the same proof does not work to show that N is g^*-geodesic.

Let us recall that the Hessian of a function $f \in C^\infty(M)$ with respect to the metric g is the symmetric tensor field given by $\mathrm{Hess}_f(X, Y) = g(\nabla_X \nabla f, Y)$ for all $X, Y \in \mathfrak{X}(M)$.

Lemma 17 *Let (M, g) be an orientable Lorentzian manifold and g^* a semi-Riemannian metric. Suppose that g is null related to g^* with optical vector field $N = \nabla \ln \varrho$. Then, the following relation holds:*

$$\mathrm{Hess}_\varrho(X, \varphi(Y)) = \mathrm{Hess}_\varrho(\varphi(X), Y)$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof A direct computation shows

$$\mathrm{Hess}_{\ln \varrho}(X, Y) = -\alpha(X)\alpha(Y) + \frac{1}{\varrho}\mathrm{Hess}_\varrho(X, Y).$$

In particular, we have

$$\begin{aligned} \frac{1}{\varrho}\mathrm{Hess}_\varrho(X, \varphi(Y)) &= \alpha(X)\alpha(\varphi(Y)) + \mathrm{Hess}_{\ln \varrho}(X, \varphi(Y)) \\ &= \alpha(X)\alpha^*(Y) + \mathrm{Hess}_{\ln \varrho}(X, \varphi(Y)). \end{aligned}$$

On the other hand, from Equation (15), we get

$$\frac{1}{\varrho}\mathrm{Hess}_\varrho(\varphi(X), Y) = \left(\nabla_X \alpha^*\right)(Y) - g(X, Y)\alpha^*(N).$$

Since $\alpha^* = \frac{1}{2} d\,\mathrm{tr}_g \varphi$, we have that $(\nabla_X \alpha^*)(Y)$ is symmetric. Therefore, the result is a direct consequence of the above formula. □

We say that a symmetric $(0, 2)$-tensor T is diagonalizable with respect to g, if at every point $x \in M$, there is a g-orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x M$ such that $T(e_i, e_j) = 0$ for $i \neq j$. This is equivalent to say that the g-metrically equivalent endomorphism to T is diagonalizable respect to an orthonormal g-basis. Obviously, any symmetric $(0, 2)$-tensor is diagonalizable in the Riemannian case, but the same is not true in general in the Lorentzian setting.

Corollary 18 *Let (M, g) be an orientable Lorentzian manifold and g^* a Riemannian metric. If g is null related to g^* with optical vector field $N = \nabla \ln \varrho$, then* Hess_ϱ *is diagonalizable.*

Proof Since g^* is Riemannian, φ is diagonalizable and so there is a g^*-orthonormal basis $\{e_1, \ldots, e_n\}$ with $\varphi(e_i) = \lambda_i e_i$ for all $i = 1 \ldots n$. We can suppose that e_1 is g-timelike, $\lambda_1 < 0$ and $\lambda_i > 0$ for $i > 1$. Using Lemma 17, we get

$$\begin{aligned} \mathrm{Hess}_\varrho(\varphi(e_1), e_i) &= \mathrm{Hess}_\varrho(e_1, \varphi(e_i)), \\ (\lambda_1 - \lambda_i)\mathrm{Hess}_\varrho(e_1, e_i) &= 0 \end{aligned}$$

for $i > 1$ and, therefore, $\mathrm{Hess}_\varrho(e_1, e_i) = 0$ for all $i > 1$. Since e_1 is g-timelike, Hess_ϱ is diagonalizable. □

Next, we give a first curvature condition to ensure that two null related metrics are affinely equivalent.

Theorem 19 *Let (M, g) be an orientable null geodesically complete Lorentzian manifold and g^* a Riemannian metric. Suppose that g is null related to g^* with optical vector field N. If $g(N, N) = 0$ and $Ric(N, N) \geq 0$, then g and g^* are affinely equivalent.*

Proof We know that $N = \nabla \ln \varrho$ for a certain never vanishing function ϱ. We use the general formula:

$$Ric(\nabla\varrho, \nabla\varrho) = \mathtt{div}\nabla_{\nabla\varrho}\nabla\varrho - \nabla\varrho(\triangle\varrho) - ||\mathrm{Hess}_\varrho||^2,$$

where $||\mathrm{Hess}_\varrho||^2 = \sum_{i=1}^n \epsilon_i g(\nabla_{e_i}\nabla\varrho, \nabla_{e_i}\nabla\varrho)$ and $\{e_1, \ldots, e_n\}$ is a g-orthonormal basis. Since N is closed and $g(N, N) = 0$ we have that N and $\nabla\varrho$ are g-geodesic so

$$0 \leq -\nabla\varrho(\triangle\varrho) - ||\mathrm{Hess}_\varrho||^2. \tag{16}$$

We know from Corollary 18 that Hess_ϱ is diagonalizable; thus, it holds the inequality $\frac{1}{n}(\triangle\varrho)^2 \leq ||\mathrm{Hess}_\varrho||^2$, and we get

$$0 \leq -\nabla\varrho(\triangle\varrho) - \frac{1}{n}(\triangle\varrho)^2.$$

Taking into account that $\nabla\varrho$ is a complete vector field, a similar argument as in Proposition 16 shows that necessarily $\triangle\varrho = 0$. Then Inequality (16) reduces to $||\mathrm{Hess}_\varrho||^2 \leq 0$. Being Hess_ϱ diagonalizable, this ensures that $\mathrm{Hess}_\varrho = 0$, that is, $\nabla\varrho$ is g-parallel.

Suppose that there is $p \in M$ with $(\nabla \varrho)_p \neq 0$. Take $\gamma : \mathbb{R} \to M$ a null g-geodesic such that $g((\nabla \varrho)_p, \gamma'(0)) \neq 0$. Since $\nabla \varrho$ is parallel, then $\varrho(\gamma(t)) = at + b$ for some $a \neq 0$, but this is a contradiction because ϱ never vanishes. Hence, the vector field N identically vanishes. $\square$

We say that a Lorentzian manifold (M, g) holds the null convergence condition if $Ric(u, u) \geq 0$ for all null vector $u \in TM$. This is a mild physical condition frequently used in the literature. As a corollary of the above theorem, we can get the following.

Corollary 20 *Let (M, g) be an orientable Lorentzian manifold which obeys the null convergence condition and g^* a Riemannian metric. Assume (M, g) is null geodesically complete and g is null related to g^* with optical vector field N. If N is both g-geodesic and g^*-geodesic, then g and g^* are affinely equivalent.*

We can show that the endomorphism φ also commutes with the Ricci tensor, as in the case of geodesically equivalent metrics, [11]. For this, recall the formula

$$(\nabla_X \nabla_Y L)(Z) - (\nabla_Y \nabla_X L)(Z) = R_{XY} L(Z) - L(R_{XY} Z), \tag{17}$$

which holds for any (1, 1)-tensor L.

Proposition 21 *Let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric. If g is null related to g^*, then*

$$Ric(\varphi(X), Y) = Ric(X, \varphi(Y))$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof From Equation (14), we get

$$g((\nabla_X \nabla_Y \varphi)(Z), T) = \left(\nabla_X \alpha^*\right)(T) g(Y, Z) + \left(\nabla_X \alpha^*\right)(Z) g(Y, T)$$

for all $X, Y, Z, T \in \mathfrak{X}(M)$. Using Equation (17), we have

$$\begin{aligned} g(R_{XY}\varphi(Z), T) - g(R_{XY} Z, \varphi(T)) = {} & \left(\nabla_X \alpha^*\right)(T) g(Y, Z) + \left(\nabla_X \alpha^*\right)(Z) g(Y, T) \\ & - \left(\nabla_Y \alpha^*\right)(T) g(X, Z) - \left(\nabla_Y \alpha^*\right)(Z) g(X, T). \end{aligned} \tag{18}$$

Now, we take $\{e_1, \ldots, e_n\}$ a g-orthonormal frame, set $X = T = e_i$ in Equation (18) and sum over i. We obtain that

$$Ric(Y, \varphi(Z)) - \sum_i \epsilon_i g(R_{e_i Y} Z, \varphi(e_i)) = -n \left(\nabla_Y \alpha\right)(Z) + \mathtt{div}\, \alpha^* \cdot g(Y, Z). \tag{19}$$

Since α^* is closed, the right-hand-side term of Equation (19) defines a symmetric tensor. So we only have to check that $\sum_i \epsilon_i g(R_{e_i Y} Z, \varphi(e_i))$ is also symmetric. If we take $Z = T = e_i$ in Equation (18) and sum over i, we obtain

$$\sum_i \epsilon_i g(R_{XY}\varphi(e_i), e_i) = \sum_i \epsilon_i g(R_{XY} e_i, \varphi(e_i)).$$

Applying the first Bianchi identity, the above equation reads

$$\sum_i \epsilon_i g(R_{Y\varphi(e_i)}X, e_i) + \epsilon_i g(R_{\varphi(e_i)X}Y, e_i) = \sum_i \epsilon_i g(R_{Ye_i}X, \varphi(e_i)) + \epsilon_i g(R_{e_iX}Y, \varphi(e_i))$$

and using the symmetries of R, we get

$$\sum_i \epsilon_i g(R_{e_iX}Y, \varphi(e_i)) = \sum_i \epsilon_i g(R_{e_iY}X, \varphi(e_i)).$$

□

As a direct consequence of Proposition 21, we can prove as in Corollary 18 that Ric is also diagonalizable.

Corollary 22 *Let (M, g) be a Lorentzian manifold and g^* a Riemannian metric. If g is null related to g^*, then Ric is diagonalizable.*

Example 5 Consider the Lorentzian manifold (M, g) where $M = \mathbb{R}^2 \times \mathbb{R}^{n-2}$ and

$$g = 2dudv + Hdu^2 + \sum_{i=1}^{n-2} dx_i^2,$$

where H depends on $(u, x_1, \ldots, x_{n-2})$. This Lorentzian manifold is called a pp-wave, and its Ricci curvature is given by

$$Ric = -\frac{1}{2}\triangle_x H du \otimes du,$$

where $\triangle_x H$ is the Laplacian of H respect to $(x_1, \ldots, x_{n-2})$, [5]. If we take Φ the metrically equivalent $(1, 1)$-tensor to Ric, then we have

$$\begin{aligned} \Phi(\partial x_i) &= 0, \\ \Phi(\partial u) &= -\frac{1}{2}\triangle H_x \partial v, \\ \Phi(\partial v) &= 0. \end{aligned}$$

If $\triangle_p H \neq 0$ at some point p, then Φ is not diagonalizable at p, and thus, it cannot exist a Riemannian metric g^* such that g is null related to g^*.

If ∇ and ∇^* are two arbitrary connections on a manifold, then the curvature tensors are related by

$$\begin{aligned} R^*_{XY}Z - R_{XY}Z = {} & (\nabla_X D)(Y, Z) - (\nabla_Y D)(X, Z) \\ & + D(X, D(Y, Z)) - D(Y, D(X, Z)) \end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. If g is null-projectively related to g^*, then from Theorem 6, we get

$$(\nabla_X D)(Y, Z) = g(Y, Z)\nabla_X N + (\nabla_X \omega)(Y)Z + (\nabla_X \omega)(Z)Y$$

and we can check that

$$\begin{aligned}
&R^*_{XY}Y - R_{XY}Y \\
&\quad = g(Y, Y)\nabla_X N - g(X, Y)\nabla_Y N + d\omega(X, Y) \\
&\qquad + (\nabla_X \omega)(Y)Y - (\nabla_Y \omega)(Y)X + (g(Y, Y)g(X, N) - g(X, Y)g(Y, N))\, N \\
&\qquad + \left(g(Y, Y)\omega(N) + \omega(Y)^2\right) X - (g(X, Y)\omega(N) + \omega(X)\omega(Y))\, Y. \qquad (20)
\end{aligned}$$

Theorem 23 *Let (M, g) be an orientable geodesically null complete Lorentzian manifold, then*

- *g is not null related to any Riemannian metric g^* with positive constant sectional curvature.*
- *g is not null related to any Lorentzian metric g^* with positive constant sectional curvature and $g^* < g$.*
- *g is not null related to any Lorentzian metric g^* with negative constant sectional curvature and $g < g^*$.*

Proof By contradiction, suppose that g is null related to a (Riemannian or Lorentzian) metric g^* of constant sectional curvature $c \neq 0$. In this case, we have that $R^*_{XY}Z = c\,(g^*(Y, Z)X - g^*(X, Z)Y)$ for all $X, Y \in \mathfrak{X}(M)$. Take a g-null vector $u \in T_pM$ and a g-spacelike vector $v \in T_pM$ with $g(u, v) = 0$. Using Equation (20), we have

$$0 = g(R_{uv}v, u) + g(v, v)g(\nabla_u N, u) + g(v, v)g(u, N)^2$$

and

$$cg^*(u, u)g(v, v) = g(R_{vu}u, v).$$

Therefore, we get

$$-cg^*(u, u)g(v, v) = g(v, v)g(\nabla_u N, u) + g(v, v)g(u, N)^2$$

and taking into account that $N = \nabla \ln \varrho$, we also have

$$\mathrm{Hess}_\varrho(u, u) = -cg^*(u, u)\varrho$$

for all g-null vector u.

Therefore, for every $g-$null geodesic γ, the function $cg^*(\gamma'(t), \gamma'(t))$ is a positive constant whenever g^* is Riemannian and $c > 0$ or g^* is Lorentzian with $g^* < g$ and $c > 0$ or g^* is Lorentzian with $g < g^*$ and $c < 0$. Therefore, the function

$y(t) = \varrho(\gamma(t))$ holds the differential equation $y'' = -ky$ with $k > 0$. Since (M, g) is geodesically null complete, the function y vanishes somewhere, which is a contradiction. □

Remark 4 In [11], the authors proved that when a complete Einstein Riemannian metric g^* is geodesically equivalent to a complete Lorentzian metric g, then g^* and g are in fact affinely equivalent. In the situation of the first point of Theorem 23, the Riemannian metric g^* is Einstein and from Proposition 12, it is geodesically equivalent to the Lorentzian metric $\widetilde{g} = \frac{1}{\varrho^2} g$, but we cannot apply [11] because of the lack of completeness of $\widetilde{g}$, in general. The completeness of the metrics is an essential hypothesis, as pointed out in [11].

Taking trace in Equation (20), we can also obtain a relation between the Ricci curvatures as follows.

Lemma 24 *Let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric. If they are null-projectively related with optical form α and projective form ω, then*

$$\begin{aligned} Ric^*(v, v) &= Ric(v, v) + g(v, v) \left(\mathtt{div}\,\alpha + \alpha(N) + (n-1)\omega(N)\right) - (\nabla_v \alpha)(v) \\ &\quad - (n-1)(\nabla_v \omega)(v) - \alpha(v)^2 + (n-1)\omega(v)^2. \end{aligned} \tag{21}$$

In particular, if g is null related to g^* with optical vector field $N = \nabla \ln \varrho$, then

$$Ric^*(v, v) = Ric(v, v) + g(v, v)\frac{\triangle \varrho}{\varrho} - \frac{1}{\varrho}\mathrm{Hess}_{\varrho}(v, v) \tag{22}$$

Remark 5 If $g^* = e^{2\Phi} g$, then we know that g^* and g are null-projectively related with $\omega = d\Phi$ and $\alpha = -d\Phi$. Formula (21) reduces in this case to the well-known formula for the Ricci curvature under a conformal change

$$\begin{aligned} Ric^*(v, v) &= Ric(v, v) - g(v, v)\left(\triangle \Phi + (n-2)||\nabla \Phi||^2\right) \\ &\quad - (n-2)\left(\mathrm{Hess}_{\Phi}(v, v) - d\Phi(v)^2\right). \end{aligned}$$

Using Lemma 17, Proposition 21 and the above lemma, we immediately get the following.

Proposition 25 *Let (M, g) be a Lorentzian manifold and g^* a semi-Riemannian metric. If g is null related to g^*, then*

$$Ric^*(\varphi(X), Y) = Ric^*(X, \varphi(Y))$$

for all $X, Y \in \mathfrak{X}(M)$.

Corollary 26 *Let (M, g^*) be a complete orientable Riemannian manifold and g a Lorentzian metric. Suppose that g is null related to g^* with optical vector field N. If N is a complete vector field, $g^*(N, N)$ is constant and $Ric^*(N, N) \geq 0$, then g and g^* are affinely equivalent.*

Proof From Propostion 16, we know that N is g-geodesic and $g(N, N) = 0$. From Lemma 24, we have that $0 \leq Ric^*(N, N) = Ric(N, N)$ and then, Theorem 19 ensures that g and g^* are affinely equivalent. □

Theorem 27 *Let (M, g) be a null geodesically complete orientable Lorentzian manifold and g^* a semi-Riemannian metric. If g is null related to g^* and $Ric = Ric^*$, then g and g^* are affinely equivalent.*

Proof Taking trace in Equation (22), we get $\triangle\varrho = 0$ and therefore $\nabla\varrho$ is g-parallel. We can conclude as in the proof of Theorem 19 to show that ϱ is constant. □

The following lemma is a straightforward computation.

Lemma 28 *If $f \in C^\infty(M)$, then for all $v \in TM$, it holds*

$$\mathtt{div}\,\mathrm{Hess}_f(v) = Ric(\nabla f, v) + d(\triangle f)(v).$$

Theorem 29 *Let (M, g) be an orientable Lorentzian manifold and g^* a semi-Riemannian metric. Suppose that g is null related to g^*.*

1. *If g^* is Ricci-flat, then g has constant scalar curvature.*
2. *If g^* is Einstein with nonzero scalar curvature and g has constant scalar curvature, then g and g^* are affinely equivalent.*
3. *If M is compact and g^* is Ricci-flat, then g and g^* are affinely equivalent.*

Proof Assume $Ric^* = Ag^*$ for $A \in \mathbb{R}$. Then from Equation (22), we have

$$A\varrho g^* = \varrho Ric + \triangle\varrho g - \mathrm{Hess}_\varrho$$

and taking divergence with respect to g, we get

$$\begin{aligned} Ag^*(\nabla\varrho, v) + A\varrho(\mathtt{div} g^*)(v) &= Ric(\nabla\varrho, v) + \varrho(\mathtt{div}Ric)(v) + g(\nabla\triangle\varrho, v) \\ &\quad - \mathtt{div}\mathrm{Hess}_\varrho(v) \end{aligned}$$

for all $v \in TM$. From Lemmas 28 and 15, we have

$$\begin{aligned} Ag^*(\nabla\varrho, v) + (n+1)A\varrho\alpha^*(v) &= Ric(\nabla\varrho, v) + \varrho(\mathtt{div}Ric)(v) + g(\nabla\triangle\varrho, v) \\ &\quad - Ric(\nabla\varrho, v) - d(\triangle\varrho)(v). \end{aligned}$$

Since $g^*(\nabla\varrho, v) = \varrho g^*(N, v) = \varrho\alpha^*(v)$, the above formula reduces to

$$2A\varrho(n+2)\alpha^*(v) = \varrho dS(v),$$

where S is the scalar curvature of g

Now, if $A = 0$, then S is constant. If $A \neq 0$ and S is constant, then $\alpha^* = 0$ and, thus, $\nabla = \nabla^*$. Finally, if M is compact and $A = 0$, then taking trace in Equation (22), we get

$$0 = \varrho S + (n-1)\triangle\varrho.$$

Since S is constant from the first assertion, we have $S\int_M \varrho dg = 0$, but ϱ never vanishes so $S = 0$ and $\triangle\varrho = 0$. Therefore, ϱ is constant and $\nabla^* = \nabla$. □

Observe that, as in the Remark 4, we cannot use Proposition 12 and [11] to prove the third point of Theorem 29. Indeed, the metrics $\widetilde{g} = \frac{1}{\varrho^2} g$ and g^* are geodesically equivalent and g^* is Einstein, but they are not necessarily complete although M is compact.

If we combine Proposition 5 and Theorem 29, we get the following.

Corollary 30 *If (M, g^*) is a compact, simply connected, and Ricci-flat Riemannian manifold, then there is no Lorentzian metric g which is null related to g^*.*

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